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Almost Boolean Fuzzy Rings

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A Project on

Almost Boolean Fuzzy Rings

by

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Bahir Dar

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A Project on

Almost Boolean Fuzzy Rings

A Project Submitted to the Department of Mathematics in Partial Fulfillment of the Requirements for the Degree of “Master of Science in Mathematics ’’.

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I hereby certify that I have supervised, read and evaluated this project entitled “ Almost Boolean Fuzzy Rings ’’ by Mequanint Sharew prepared under my guidance. I recommend that the project is submitted for oral defense.

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Date___________________________

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We hereby certify that we have examined this project entitled " Almost Boolean Fuzzy Rings '' by Mequanint Sharew . We recommend that Mr. Mequanint Sharew is approved for the degree of "Master of Science in Mathematics ''.

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Abstract

In this project we discuss the concept of Almost Distributive Lattice and Almost Boolean Ring and the concept of fuzzy set and fuzzy partial order relations, fuzzy lattices.In this project we also study the class of Relatively Complemented Almost Distributive Fuzzy Lattice in detail. We study the concept of an Almost Boolean Fuzzy Rings as a generalization of a Boolean Fuzzy Rings. We also study a one to one correspondence between Relatively Complemented Almost Distributive Fuzzy Lattice and Almost Boolean Fuzzy Rings.

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Chapter One
Introduction and Preliminaries
1.1. Introduction
The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy, U.M. and Rao, G.C. [9] as a common abstraction of almost all the existing ring generalizations of a Boolean Algebra. Again the concept of relatively complemented ADLs and an Almost Boolean Rings are introduced by Rao, G.C. [7]. A relatively complemented ADL is an ADL in which every interval is a complemented lattice. An Almost Boolean Ring is a triple (R ,+, . ,0) satisfying all the properties of a Boolean Ring except possibly the associativity of +. On the other hand the concept of a fuzzy set was first introduced by Zadeh, L.A. [10] and this concept was adapted by Goguen, J.A. [6] and Sanchez ,E. [8] to define a fuzzy lattice and study fuzzy relations. In 1994, Ajmal, N. and Thomas, K.V. [1] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sub lattices. In 2009, Chon, I. [5] considering the notion of fuzzy order of Zadeh, L.A. [10] introduced a new notion fuzzy lattice and studied the level set of fuzzy lattice. In this project we study a new mathematical notion relatively complemented Almost Distributive Fuzzy Lattice and an Almost Boolean Fuzzy Ring and characterized some properties of them using the fuzzy partial order relations and fuzzy lattice defined by Chon, I. and establish the process of obtaining an Almost Boolean Fuzzy Rings from a given relatively complemented Almost Distributive Fuzzy Lattice and the process of obtaining a relatively complemented Almost Distributive Fuzzy Lattice from a given Almost Boolean Fuzzy Ring.

1.2. Preliminaries

This section is consisting of some definitions and results that will be used in the next chapter. We simply list these in the form of lemma and theorems and no proofs are included.

1.2.1. Possets, Lattice, and Distributive Lattices

The definitions and results mentioned in this section are taken from Birkhoff , G. and Gratzer,G. [4].

Definition 1.2.1.1 Let P be a non-empty set. Then a binary relation $\leq$ on P is called a partial order on P if it satisfies the following properties;

(1) Reflexive: $a \leq a$

(2) Antisymmetric: $a \leq b$ and $b \leq a$ imply that $a = b$.

(3) Transitive: $a \leq b$ and $b \leq c$ imply that $a \leq c$ for all $a , b , c \in P$.

In this case $(P, \leq)$ is called a partially order set or simply a poset.

In a poset $(P, \leq)$, if $a \leq b$ and $a \neq b$, then we write $a < b$.

Definition 1.2.1.2 Let $(P , \leq)$ be a poset and $a , b \in P$.Then we say that a and b are comparable if either $a \leq b$ or $b \leq a$. Otherwise we say that a and b are incomparable.

Definition 1.2.1.3 An algebra $(R , \vee , \wedge)$ of type (2 , 2) is called a lattice if it satisfies the following identities:

(1) Idempotency: $a \wedge a = a$ and $a \vee a = a$.

(2) Commutativity: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.

(3) Absorption: $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

(4) Associativity: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$.

In any lattice (R , ∨ , ∧), the following identities are equivalent:

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$.

Definition1.2.1.4 A lattice (R , ∨ , ∧) satisfying any one of the above four identities is called a Distributive Lattice.

If (R , ∨ , ∧) is a lattice ,then an element a of R is called zero element or least element of R if $a \wedge x = a$,∀ x ∊ R, then it is unique and it is denoted by 0.Similarly an element a of R is called one element or greatest element of R if $a \wedge x = x$,∀ x ∊ R. If R has a greatest element , then it is unique and it is denoted by 1.

1.2. 2.Almost Distributive Lattice

In this section we recall the definition of an Almost Distributive Lattice (ADL) and an Almost Boolean Ring (ABR) taken from Swamy ,U.M. and Rao, G.C.[9] and Rao,G.C.[7].

Definition1.2.2.1 An algebra (R , ∨ , ∧ , 0) of type (2 , 2 , 0) is called Almost Distributive Lattice if it satisfies the following axioms:

(L1) $a \vee 0 = a$

(L2) $0 \wedge a = 0$

(L3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(L4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(L5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(L6) $(a \vee b) \wedge b = b$ ∀ a , b , c ∊ R.

It can be seen directly that every distributive lattice is an ADL.

Theorem1.2.2.2 For any a , b , c , d ∈ R ,we have

(1) $a \wedge 0 = 0$ and $0 \vee a = a$

(2) $a \wedge a = a = a \vee a$

(3) $(a \wedge b) \vee b = b$, $a \vee (b \wedge a) = a$ and $a \wedge (a \vee b) = a$

(4) $a \wedge b = a \Leftrightarrow a \vee b = b$ and $a \wedge b = b \Leftrightarrow a \vee b = a$

(5) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ whenever $a \leq b$

(6) $a \wedge b \leq b$ and $a \leq a \vee b$

(7) $\wedge$ is associative in R

(8) $a \wedge b \wedge c = b \wedge a \wedge c$

(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(10) $a \vee (b \vee a) = a \vee b$

(11) $(a \vee b) \vee a = a \vee b$

(12) $a \vee (b \vee a) = (a \vee b) \vee a$

(13) $\{ a \vee (b \vee c) \} \wedge d = \{ (a \vee b) \vee c \} \wedge d$

Definition1.2.2.3 An ADL (R , ∨ , ∧) is said to be a relatively complemented ADL if for any a , b ∈ R with $a < b$,the interval [a , b] is a complemented lattice.

Lemma1.2.2.4 An ADL (R , ∨ , ∧ , 0) is relatively complemented if and only if, given a , b ∈ R there exists, x ∈ R such that $a \vee b = a \vee x$ and $a \wedge x = 0$ and in this case, x is unique which we denote by a^b.

Lemma 1.2.2.5 If R is relatively complemented and a , b ∈ R, then $a^b \leq b$.

Lemma1.2.2.6 If R is a relatively complemented ADL, then for any a , b , c ∈ R , we havethe following:

(1) $a^a = 0 = a^0$ and $0^a = a$

(2) $b^a \wedge a = b^a$

(3) $a^b \wedge a = 0$

(4) $a^c = (a \wedge c)^c$

(5) $(a \vee b)^c = a^c \wedge b^c$

(6) $(a \wedge b)^c = a^c \vee b^c$

(7) $(a \wedge c)^{(b \wedge c)} = a^b \wedge c$

(8) $(c \wedge a)^{(c \wedge b)} = c \wedge a^b$

(9) $a \leq b \Leftrightarrow b^a = 0$

(10) $a \wedge b = 0 \Longrightarrow a^b = b$ and $b^a = a$

(11) $a^b \vee b^a = b^a \vee a^b$

(12) $a^b \wedge b^a = b^a \wedge a^b$

Next we introduce the concept of Almost Boolean Rings as a generalization of that of Boolean Rings.

Definition 1.2.2.7 An algebra (R, +, . , 0) of type (2 , 2 , 0) is called a Boolean ring if it satisfies the following axioms:

(R1) (x + y) + z = x + (y + z)

(R2) x + 0 = x

(R3) x + x = 0

(R4) (x y) z = x (y z)

(R5) $x^2 = x$

(R6) x (y + z) = x y + x z

(R7) (x + y) z = x z + y z , ∀ x , y , z ∈ R.

Lemma 1.2.2.8 If R is a Boolean Ring, then

i. x + x = 0
ii. x . y = y . x for all x , y ∈ R.

Definition 1.2.2.9 An algebra (R , + , . , 0) of type (2 , 2 , 0) is called an Almost Boolean Ring if it satisfies the following axioms:

(R1) x + 0 = x

(R2) x + x = 0

(R3) $x^2 = x$

(R4) (x y) z = x (y z)

(R5) x (y + z) = x y + x z

(R6) (x + y) z = x z +y z

(R7) { (x + y) + z } t = { x + (y + z) } t ,∀ x , y ,z , t ∈ R.
Remark: An Almost Boolean Ring is a triple (R , + , . , 0) satisfying all the properties of a Boolean Ring except possibly the associativity of +.

In the rest of this section by R we mean an ABR (R , + , . , 0).

1.2.3 Fuzzy Partial Order Relations and Fuzzy Lattices

Here we give some properties and definitions of Fuzzy Partial Order Relations, Fuzzy Lattices and Distributive Fuzzy Lattices from Chon , I.[5] and [3].

Definition 1.2.3.1 Let X be a non empty set.

(1)A function A : X x X → [0 , 1] is

called a fuzzy relation in X.

(2) The fuzzy relation A in X is:

• Reflexive if and only if A (x , x) = 1 ∀ x ∈ X

• Antisymmetry if and only if A (x , y) > 0 and A (y , x) > 0 implies x = y.

•Transitive if and only if A (x , z) ≥ $sup_{y \in X}$ min (A (x , y) , A (y, z)) and (3) A fuzzy relation A is fuzzy partial order relation if A is reflexive, antisymmetry and transitive.

(4) A fuzzy partial order relation is a fuzzy total order relation if and only if A (x , y) > 0 or A (y , x) > 0 , ∀ x , y ∈ R.

(5) If A is a fuzzy partial order relation in a set X , then (X , A) is called a fuzzy partially ordered set or a fuzzy poset.

(6) If B is a fuzzy total order relation in a set X, then (X , B) is called a fuzzy totally ordered set or a fuzzy chain.

Definition 1.2.3.2 Let (X , A) be a fuzzy poset. **(** X , A **)** is a fuzzy lattice if and only if x ∨ y and x ∧ y exists for all x , y ∈ X.

Proposition 1.2.3.4 Let (X , A) be a fuzzy lattice and let x , y , z ∈ X .Then ,

(1) x ∨ x = x , x ∧ x = x

(2) x ∨ y = y ∨ x , x ∧ y = y ∧ x

(3) (x ∨ y) ∨ z = x ∨ (y ∨ z) , (x ∧ y) ∧ z = x ∧ (y ∧ z)

(4) (x ∨ y) ∧ x = x , (x ∧ y) ∨ x = x.

Definition 1.2.3.5 Let (X , A) be a fuzzy lattice. (X , A) is distributive fuzzy lattice if and only if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.

Definition 1.2.3.6 Let (R , ∨ , ∧ , 0) be an algebra of type (2 , 2 , 0) and (R , A) be a fuzzy poset. Then we call (R , A) is an Almost Distributive Fuzzy Lattice (ADFL) if the following axioms are satisfied:

(1) $A (a , a \vee 0) = A (a \vee 0 , a) = 1$

(2) $A (0 , 0 \wedge a) = A (0 \wedge a , 0) = 1$

(3) $A (a \vee b) \wedge c , (a \wedge c) \vee (b \wedge c) = A ((a \wedge c) \vee (b \wedge c) , (a \vee b) \wedge c) = 1$

(4) $A (a \wedge (b \vee c) , (a \wedge b) \vee (a \wedge c) = A ((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$

(5) $A (a \vee (b \wedge c) , (a \vee b) \wedge (a \vee c) = A ((a \vee b) \wedge (a \vee c) , a \vee (b \wedge c)) = 1$

(6) $A ((a \vee b) \wedge b , b) = A (b , (a \vee b) \wedge b) = 1 \ \forall a , b , c \in R$.

Definition 1.2.3.7 Let (R , A) be an ADFL. Then for any $a , b \in R$, $a \leq b$ if and only if $A (a , b) > 0$.

Chapter Two

Almost Boolean Fuzzy Rings (ABFRs)

In the first section we have seen the concept of relatively complemented Almost Distributive Fuzzy Lattices and in the next section we study the concept of an Almost Boolean Fuzzy Rings (ABFR) as a generalization of Boolean Fuzzy Rings(BFR) and give an example of ABFR which is not BFR. We establish the process of obtaining an ABFR from the given relatively complemented Almost Distributive Fuzzy Lattice and the process of obtaining a relatively complemented ADFL from a given ABFR.

2.1 Relatively Complemented ADFL

In this section we introduce a new mathematical notion relatively complemented ADFLs and we investigate and prove some results.

Definition2.1.1 An ADFL (R , A) is said to be relatively complemented if every interval in R is a Boolean Algebra.

Definition 2.1.2 Let (R , A) be an ADFL. For a , b ∈ R with A (a , b) > 0, and x ∈ [a . b] , then y is the complement of x in [a , b] if and only if A (x ∧ y , a) > 0 and A (b , x ∨ y) > 0 where a is the least element and b is the greatest element.

Lemma 2.1.3 An ADFL (R , A) is said to be relatively complemented if and only if for any a , b ∈ R , there exists x ∈ R , such that A (a ∨ x , a ∨ b) = A (a ∨ b , a ∨ x) = 1 and A (a ∧ x , 0) > 0, in this case , x is unique which we denote by a^b.

Proof

(⟹) Suppose an ADFL is relatively complemented .

Claim: For any a , b ∈ R, there exists a unique x ∈ R, such that A (a ∨ x , a ∨ b) = A (a ∨ b , a ∨ x) = 1 and A (a ∧ x , 0) > 0.

Now, an ADFL (R , A) is relatively complemented if every interval in R is a Boolean algebra.

Hence [0 , a] is a Boolean algebra.

Which implies [0 , a] is complemented lattice.

Let a , b $\in$ R, such that A (a , b) $>$ 0. Then the interval [0 , a $\vee$ b] is complemented and a $\in$ [0 , a $\vee$ b].

If x is the complement of a in [0 , a $\vee$ b] , then A (a $\vee$ b , a $\vee$ x) $>$ 0 and A (a $\wedge$ x , 0) $>$ 0 , as a $\vee$ x $\leq$ a $\vee$ b, A (a $\vee$ x , a $\vee$ b) $>$ 0……………………………………….(i)

To show uniqueness , for y $\in$ R, let y $\in$ [0 , a $\vee$ b] satisfying

A (a $\vee$ b , a $\vee$ y) $>$ 0 and A (a $\wedge$ y , 0) $>$ 0 , as a $\vee$ y $\leq$ a $\vee$ b , A (a $\vee$ y , a $\vee$ b) $>$ 0…..…………………………………………………………………………………..(ii)

Claim: A (x , y) = A (y , x) =1.

A (x , y $\wedge$ x) = A ((a $\vee$ x) $\wedge$ x , y $\wedge$ x)

= A ((a $\vee$ y) $\wedge$ x , y $\wedge$ x)……[from (i) and (ii) , a $\vee$ x = a $\vee$ y = a $\vee$ b]

= A ((a $\wedge$ x) $\vee$ (y $\wedge$ x) , y $\wedge$ x)

= A (0 $\vee$ (y $\wedge$ x) , y $\wedge$ x)………[A (a $\wedge$ x , 0) $>$ 0 , A (0 , a $\wedge$ x) $>$ 0]

= A (y $\wedge$ x , y $\wedge$ x)

= 1.

A (x $\wedge$ y , y) = A (x $\wedge$ y) , (a $\vee$ y) $\wedge$ y)

= A (x $\wedge$ y , (a $\vee$ x) $\wedge$ y)………………[from (i) and (ii) above]

= A (x $\wedge$ y , (a $\wedge$ y) $\vee$ (x $\wedge$ y))

$= A (x \wedge y , 0 \vee (x \wedge y))$………[$A (a \wedge y, 0) > 0$, $A (0, a \wedge y) > 0$]

$= A (x \wedge y , x \wedge y)$

$= 1$

Since both $x , y \in [0 , a \vee b]$ and $[0 , a \vee b]$ is a Boolean algebra, then $x \wedge y = y \wedge x$

$\Longrightarrow A (y \wedge x , y) = 1$.

Now, $A (x , y) \geq sup_{c \in R} \min (A (x , c), A (c , y))$

$\geq \min (A (x , y \wedge x) , A (y \wedge x , y))$

$= \min (1 , 1)$

$= 1$

Hence , $A (x , y) = 1$. Similarly $A (y , x) = 1$.

Therefore $A (x , y) = A (y , x) = 1$. Hence the complement is unique

Conversely suppose for any $a , b \in R$, there exists a unique x in R such that

$A (a \vee x , a \vee b) = A (a \vee b , a \vee x) = 1$ and $A (a \wedge x , 0) > 0$.

Claim : An ADFL (R , A) is relatively complemented.

WTS: The interval [a , b] in R is a Boolean algebra.

Since every interval in an ADL is bounded distributive lattice , it suffices to show the interval is complemented lattice.

Now, let $a , b \in R$ such that $A (a , b) > 0$ and $x \in [a , b]$.

Then by hypothesis there exists a unique $y \in R$, such that

$A (x \vee y , b) = A (b , x \vee y) = 1$ and $A (x \wedge y , 0) > 0$.

Since $x \leq b$, then $x \vee y = x \vee b = b$……………………………………….[iii]

Now , from ADL we have $y \wedge (y \vee x) = y$. Then

$y \leq y \vee x$

$= x \vee y$

$= b$……………………...[iv]

Hence, $A (y , b) > 0$………………….. [Since by Definition1.2.3.7. i. e , $a \leq b$ if and only if $A (a , b) > 0$ for each a and b in R]……………………………………….[v]

Now, we prove the element $a \vee y \in [a , b]$ and it is the complement of x in [a , b].

From $A (y , b) > 0$………………………………………………..[Since from (v) above]

$\Longrightarrow A (a \vee y , a \vee b) > 0$

Hence , $A (a \vee y , b) > 0$ as $A (a , b) > 0$,we have $a \vee b = b$ .Then $a \leq a \vee y \leq b$.

Therefore $a \vee y \in [a , b]$.

Now , $A (x \wedge (a \vee y) , a) = A ((x \wedge a) \vee (x \wedge y) , a)$

$= A (a \vee 0 , a)$……………………………….[$A (a , x) > 0$ and $A (x \wedge y , 0) > 0$, $A (0 , x \wedge y) > 0$]

$= A (a , a)$

$= 1$

Hence , $A (x \wedge (a \vee y) , a) > 0$ and

$A (b , x \vee (a \vee y)) = A (b , (x \vee (a \vee y)) \wedge b)$………………………….[$a \vee y \leq b \Longrightarrow x \vee (a \vee y) \leq x \vee b \leq b$ ($x \leq b$)]

= A (b , (x ∨ (a ∨ y)) ∧ (x ∨ y))......[Since from (iii) above , i .e, x ∨ y = b]

= A (b , x ∨ ((a ∨ y) ∧ y))

= A (b , x ∨ y)

= A (b , b) $>$ 0

On the other hand as x ∨ (a ∨ y) ≤ b, we have A (x ∨ (a ∨ y) , b) $>$ 0.Therefore, A (x ∨ (a ∨ y) , b) = A (b , x ∨ (a ∨ y)) = 1.

Hence , a ∨ y is complement of x. Therefore, the interval [a , b] in R is a Boolean algebra.

Lemma 2.1.4 If an ADFL (R , A) is relatively complemented and a , b ∈ R , then A ($a^{b} \vee b^{a}$, $b^{a} \vee a^{b}$) = 1.

Proof : Suppose (R , A) is an ADFL and a , b ∈ R.

A ($a^{b} \wedge b^{a}$, 0) = A ($a^{b} \wedge b^{a} \wedge$ a , 0)…………………[Since $b^{a} \wedge$ a = b^{a}]

= A ($b^{a} \wedge a^{b} \wedge$ a , 0)

= A ($b^{a} \wedge$ 0 , 0)

= A (0 , 0)

= 1.

Similarly , A (0 , $a^{b} \wedge b^{a}$) = 1. Hence , we have ($a^{b} \wedge b^{a}$) = 0.

Therefore, A ($a^{b} \vee b^{a}$, $b^{a} \vee a^{b}$) =1.

Lemma 2.1.5 If an ADFL (R , A) is relatively complemented and a , b ∈ R , then A (a^{b} , b) $>$ 0.

Proof: Suppose an ADFL (R , A) is relatively complemented and a , b ∈ R.

Now , A ($a^b \wedge$ b , a^b) = A (0 ∨ ($a^b \wedge$ b) , a^b)

= A (($a^b \wedge$ a) ∨ ($a^b \wedge$ b) , a^b)………..[Since $a^b \wedge$ a = 0]

= A ($a^b \wedge$ (a ∨ b) , a^b)

= A ($a^b \wedge$ ($a^b \vee$ a) , a^b)………………[Since $a^b \vee$ a = a ∨ b]

= A (a^b , a^b)

=1

Similarly , A (a^b , $a^b \wedge$ b) = 1.

Hence , A ($a^b \wedge$ b , a^b) = A (a^b, $a^b \wedge$ b) = 1.

Then we have $a^b \leq$ b.

Therefore, A (a^b , b) $>$ 0.

2.2 Almost Boolean Fuzzy Rings

In this section we study the concept of Almost Boolean Fuzzy Rings as a generalization of Boolean Fuzzy Rings and we also observe an example of Almost Boolean Fuzzy Ring which is not Boolean Fuzzy Ring.

Definition 2.2.1 Let (R , + , . , 0) be an algebra of type (2 , 2 ,0) and (R , A) be a fuzzy poset. Then (R , A) is a Fuzzy Ring if it satisfies the following axioms :

(1) A ((a + b) + c , a + (b + c) = A (a + (b + c) , (a + b) + c) = 1

(2) A (a + 0 , a) = A (a , a + 0) = 1

(3) A (a + (-a) , 0) = A (0 , a + (-a)) = 1

(4) A (a + b , b + a) = A (b + a , a + b) = 1

(5) A ((a b) c , a (b c)) = A (a (b c) , (a b) c) = 1

(6) A (a (b + c) , a b + a c) = A (a b + a c , a (b + c)) = 1

(7) A ((b + c) a , b a + c a) = A (b a + c a , (b + c) a) = 1 ∀ a , b , c , -a ∈ R.

Definition 2.2.2 Let (R , A) be a fuzzy ring. Then (R , A) is a Boolean Fuzzy Ring (BFR) if and only if A (a^2 , a) = A (a , a^2) = 1 ∀ a ∈ R.

Definition 2.2.3 Let (R , + , . , 0) be an algebra of type (2 , 2 , 0) and (R , A) be a fuzzy poset.Then we call (R , A) is an Almost Boolean Fuzzy Rings (ABFRs) if the following axioms are satisfied:

(RF1) A (a + 0 , a) = A (a , a + 0) = 1

(RF2) A (a + a , 0) = A (0 , a + a) = 1

(RF3) A ((a b) c , a (b c)) = A (a (b c) , (a b) c) = 1

(RF4) A (a^2 , a) = A (a , a^2) =1

(RF5) A (a (b + c) , a b + a c) = A (a b + a c , a (b + c)) = 1

(RF6) A ((a + b) c , a c + b c) = A (a c + b c , (a + b) c) = 1

(RF7) A ({ a + (b + c) } d , { (a + b) + c } d) = A ({ (a + b) + c } d , { a + (b + c) } d) = 1 ∀ a , b , c , d ∈ R.

Remark: Almost Boolean Fuzzy Ring is a generalization of Boolean Fuzzy Ring that satisfies all the properties of a Boolean Fuzzy Rings except possibly the associativity of " + ''.

Example 2.2.4 Let R = {0, a , b, c , d } and define two binary operations + and . in R as follows.

+	0	a	b	c	d
0	0	a	b	c	d
a	a	0	0	0	0
b	b	0	0	0	0
c	c	0	0	0	0
d	d	0	0	0	0

And

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	c	d
b	0	a	b	c	d
c	0	a	b	c	d
d	0	a	b	c	d

Define a fuzzy relation :

$A : R \times R \rightarrow [0 , 1]$ as follows;

A (0 , 0) = A (a , a) = A (b , b) = A (c , c) =1,

A (a , 0) = A (b , 0) = A (c , 0) = A (b , a) = A (b , c) = A (c , a) = 0 ,

A (0 , a) = 0.4 , A (0 , b) = 0.5 , A (0 , c) = 0.7 , A (a , b) = 0.9 , A (a , c) = 0.1 and

A (c , b) = 0.3.

Since (R , A) is a fuzzy poset ,

(RF1) A (a + 0 , a) = A (a , a) = 1 , and A (a , a + 0) = A (a , a) = 1,

Hence, A (a + 0 , a) = A (a , a + 0) = 1.

(RF2) A (a + a , 0) = A (0 , 0) = 1 , and A (0 , a + a) = (0 , 0) = 1 ,

Hence , A (a + a , 0) = A (0 , a + a) = 1.

(RF3) A ((a b) c , a (b c)) = A (b c , a c) = A (c , c) = 1 , and

A (a (b c) , (a b) c) = A (a c , b c) = A (c , c) = 1;

Hence , A ((a b) c , a (b c) = A (a (b c) , (a b) c) = 1.

(RF4) A (a^2 , a) = A (a , a) = 1 , and

A (a , a^2) = A (a , a) = 1 ,

Hence , A (a^2 , a) = A (a , a^2) = 1.

(RF5) A (a (b + c) , a b + a c) = A (a 0 , b + c) = A (0 , 0) = 1 , and

A (a b + a c , a (b + c) = A (b + c , a 0) = A (0 , 0) = 1,

Hence , A (a (b + c) , a b + a c) = A (a b + a c , a (b+ c) = 1.

(RF6) A ((a + b) c , a c + b c) = A (0 c , c + c) = A (0 , 0) = 1 , and

A (a c + b c , (a + b) c) = A (c + c , 0 c) = A (0 , 0) = 1 ,

Hence , A ((a + b) c , a c + b c) = A (a c + b c) , (a + b) c) = 1 .

(RF7) A ({ a + (b + c) } d , { (a + b) + c } d) = A ({ a + 0 } d , { 0 + c } d)

= A (a d , c d)

= A (d , d)

= 1 , and

A ({ (a + b) + c } d , { a + (b + c) } d) = A ({ 0 + c } d , { a + 0 } d)

= A (c d , a d)

= A (d , d)

= 1 ,

Hence , A ({ a + (b + c) } d , { (a + b) + c } d) = A ({ (a + b) + c } d , { a + (b + c) } d) = 1.

Therefore ,(R , A) is an ABFR, but (R , A) is not BFR , [since A (a + (b + c) , (a + b) + c) = A (a + 0 , 0 + c) = A (a , c) = 0.1 and A ((a + b) + c , a + (b + c)) = A (0 + c , a + 0) = A (c , a) = 0 , Hence , A (a + (b + c) , (a + b) + c) ≠ A ((a + b) + c , a + (b + c)) ≠ 1.]

Therefore ABFR is a generalized BFR except possibly the associativity of " + ''.

Lemma 2.2.5 Let (R , A) be an ABFR. For any a , b , c ∊ R, we have

(1) A (a 0 , 0) = 1

(2) A (0 , 0 a) = 1

(3) A (a 0 , 0 a) = 1.

Proof: Suppose (R , A) is an ABFR and a , b , c ∊ R.

(1) A (a 0 , 0) = A (a (a + a) , 0)---------------------[Since a + a = 0 in ABR]

= A (a^2 + a^2 , 0)

= A (a + a , 0)----------------------------[Since a^2 = a in ABR]

= A (0 , 0)

= 1

Hence , A (a 0 , 0) = 1.

(2) A (0 , 0 a) = A (0 , (a + a) a)

= A (0 , a^2 + a^2)

= A (0 , a + a)

= A (0 , 0)

= 1

Hence, A (0 , 0 a) = 1.

Therefore A (0 , 0 a) = 1.

(3) A (a 0 , 0 a) ≥ $Sup_{c \in R}$ min (A (a 0 , c) , A (c , 0 a)

≥ min (A (a 0 , 0) , A (0 , 0 a)

≥ min (A (0 , 0) , A (0 , 0))

= min (1 , 1)

= 1

Hence , A (a 0 , 0 a) = 1.

Theorem 2.2.6 Let (R , A) be a relatively complemented ADFL. Define a binary operations + on R by a + b = $a^b \vee b^a$. Then (R , A) is an ABFR.

Proof: Suppose (R , A) is relatively complemented ADFL.

Claim: (R , A) is ABFR , where a + b = $a^b \vee b^a$.

(1) A (a + 0 , a) = A ($a^0 \vee 0^a$, a)

= A (0 ∨ a , a)

= A (a , a)

= 1, similarly A (a , a + 0) = 1.

Therefore, A (a + 0 , a) = A (a , a + 0) = 1.

(2) A (a + a , 0) = A ($a^a \vee a^a$, 0)

= A (0 ∨ 0 , 0)------[since a^a = 0 in relatively complemented ADFL]

= A (0 , 0)

=1 similarly A (0 , a + a) = 1,

Therefore , A (a + a , 0) = A (0 , a + a) = 1.

(3) A ((a ∧ b) ∧ c , a ∧ (b ∧ c)) = A (a ∧ (b ∧ c) , a ∧ (b ∧ c)) = 1

Similarly , A (a ∧ (b ∧ c) , (a ∧ b) ∧ c) = 1.

Therefore, A ((a ∧ b) ∧ c , a ∧ (b ∧ c)) = A (a ∧ (b ∧ c) , (a ∧ b) ∧ c) = 1.

(4) A (a ∧ a , a) = A (a , a) = 1 = A (a , a ∧ a).

(5) A (a ∧ (b + c) , (a ∧ b) + (a ∧ c)) = A (a ∧ ($b^c \vee c^b$) , (a ∧ b) + (a ∧ c))

= A ((a ∧ b^c) ∨ (a ∧ c^b) , (a ∧ b) + (a ∧ c))

= A ($(a \wedge b)^{a \wedge c} \vee (a \wedge c)^{a \wedge b}$, (a ∧ b) + (a ∧ c))

[Since ,$(a \wedge b)^{a \wedge c}$ = a ∧ b^c and $(a \wedge c)^{a \wedge b}$ = a ∧ c^b by lemma 1.2.2.6.]

= A ((a ∧ b) + (a ∧ c) , (a ∧ b) + (a ∧ c)

= 1.

Similarly , A ((a ∧ b) + (a ∧ c) , a ∧ (b + c)) = 1.

Therefore, A (a ∧ (b + c) , (a ∧ b) + (a ∧ c)) = A ((a ∧ b) + (a ∧ c) , a ∧ (b + c)) = 1.

(6) A ((a + b) ∧ c , (a ∧ c) + (b ∧ c)) = A (($a^b \vee b^a$) ∧ c , (a ∧ c) + (b ∧ c))

= A ((a^b∧ c) ∨ (b^a∧ c) , (a ∧ c) + (b ∧ c))

= A ($(a \wedge c)^{b \wedge c} \vee (b \wedge c)^{a \wedge c}$, (a ∧ c) + (b ∧ c))

= A ((a ∧ c) + (b ∧ c) , (a ∧ c) + (b ∧ c))

= 1

Similarly , A ((a ∧ c) + (b ∧ c) , (a + b) ∧ c) = 1.

Therefore , A ((a + b) ∧ c , (a ∧ c) + (b ∧ c)) = A ((a ∧ c) + (b ∧ c) , (a + b) ∧ c) = 1.

(7) A ({ a + (b + c) } ∧ d , { (a + b) + c } ∧ d) = A ({ a + ($b^c \vee c^b$)} ∧ d , {(a + b) + c } d) = A ({$a^{b^c \vee c^b} \vee (b^c \vee c^b)^a$ } ∧ d , { (a + b) + c } ∧ d)

= A ({ $a^{b^c \vee c^b}$ ∧ d } ∨ { ($b^c \vee c^b)^a$ } ∧ d , { (a + b) + c } ∧ d)

= A ((a ∧ d) + { ($b^c \vee c^b) \wedge d$ } , { (a + b) + c } ∧ d)

=A ($(a \wedge d)^{\{(b^c \vee c^b) \wedge d\}} \vee \{(b^c \vee c^b) \wedge d\}^{(a \wedge d)}$, { (a + b) + c } ∧ d)

= A ((a ∧ d) + { (b^c∧ d) ∨ (c^b∧ d) } , { (a + b) + c } ∧ d)

= A ((a ∧ d) + [$(b \wedge d)^{c \wedge d} \vee (c \wedge d)^{b \wedge d}$] , { (a + b) + c } ∧ d)

= A ((a ∧ d) + [(b ∧ d) + (c ∧ d)] , { (a + b) + c } ∧ d)

= A ([(a ∧ d) + (b ∧ d)] + (c ∧ d) , { (a + b) + c } ∧ d)

= A ([$(a \wedge d)^{b \wedge d} \vee (b \wedge d)^{a \wedge d}$] + (c ∧ d) , { (a + b) + c } ∧ d)

= A ({ (a^b∧ d) ∨ (b^a∧ d)} + (c ∧ d) , { (a + b) + c } ∧ d)

= A (({ $a^b \vee b^a$} ∧ d) + (c ∧ d) , { (a + b) + c } ∧ d)

= A{ $(a^b \vee b^a \wedge d\}^{c \wedge d} \vee (c \wedge d)^{\{(a^b \vee b^a) \wedge d\}}$, { (a + b) + c} ∧ d)

= A ([(($a^b \vee b^a)^c$ ∧ d] ∨ [$c^{a^b \vee b^a}$∧ d] , { (a + b) + c } ∧ d)

= A ([($a^b \vee b^a)^c \vee (c^{a^b \vee b^a}$)] ∧ d , { (a + b) + c } ∧ d)

= A ([$(a + b)^c \vee c^{a+b}$] ∧ d , { (a + b) + c } ∧ d)

= A ({ (a + b) + c } ∧ d , { (a + b) + c } ∧ d)

= 1.

Similarly , A ({ (a + b) + c } ∧ d , { a + (b + c) } d) = 1.

$\therefore$ $A(\{a+(b+c)\}\wedge d, \{(a+b)+c\}\wedge d) = A(\{(a+b)+c\}\wedge d, \{a+(b+c)\}\wedge d) = 1$.Hence , (R , A) is an ABFR.

Theorem 2.2.7 Let (R , A) be an ABFR. Define the operation $\vee$ on R by $a \vee b = a + (b + ab)$,then (R , A) is relatively complemented ADFL.

Proof: Let (R , A) be an ABFR and $a \vee b = a + (b + ab)$.

Claim: (R , A) is relatively complemented ADFL.

First we need to show (R , A) is ADFL.

(1) $A(a \vee 0, a) = A(a + (0 + a0), a)$……….[since $a \vee 0 = a + (0 + a0)$]

$= A(a + 0 + 0, a)$

$= A(a + 0, a)$

$= A(a, a)$

$= 1.$

Similarly , $A(a, a \vee 0) = 1$.

Therefore , $A(a \vee 0, a) = A(a, a \vee 0) = 1$

(2) $A(0a, 0) = A(0, 0) = 1$

Similarly , $A(0, 0a) = 1$.

Therefore , $A(0a, 0) = A(0, 0a) = 1$.

(3) $A((a \vee b)c, ac \vee bc) = A((a + (b + ab))c, ac \vee bc)$

$= A(ac + (b + ab)c, ac \vee bc)$

$= A(ac + (bc + abc), ac \vee bc)$

$= A (a c + (b c + a b c^2) , a c \vee b c)$

$= A (a c + (b c + a c b c) , a c \vee b c)$

$= A (a c \vee b c , a c \vee b c)$

$= 1.$

Similarly , $A (a c \vee b c , (a \vee b) c) = 1.$

Therefore , $A ((a \vee b) c , a c \vee b c) = A (a c \vee b c , (a \vee b) c) = 1.$

(4) $A (a (b \vee c) , a b \vee a c) = A (a (b + (c + b c)) , a b \vee a c)$

$= A (a b + a (c + b c) , a b \vee a c)$

$= A (a b + (a c + a b c) , a b \vee a c)$

$= A (a b + (a c + a^2 b c) , a b \vee a c)$

$= A (a b + (a c + a b a c) , a b \vee a c)$

$= A (a b \vee a c , a b \vee a c) = 1.$

Similarly , $A (a b \vee a c , a (b \vee c)) = 1.$

Therefore , $A (a (b \vee c) , a b \vee a c) = A (a b \vee a c , a (b \vee c)) = 1.$

(5) $A (a \vee (b c) , (a \vee b) (a \vee c)) = A (a \vee (b c) , (a + b + a b) (a + c + a c))$

$= A (a \vee (b c) , a (a + c + a c) + b (a + c + a c) + a b (a + c + a c))$

$= A (a \vee (b c) , a^2 + a c + a^2 c + b a + b c + b a c + a b a + a b c + a b a c)$

$= A (a \vee (b c) , a + a c + a c + b a + b c + a b c + a^2 b + a b c + a^2 b c)$

$= A (a \vee (b c) , a + a c + a c + b a + b c + a b c + a b + a b c + a b c)$

$= A (a \vee (b c) , a + 0 + b a + a b + b c + a b c + a b c + a b c)$

$= A (a \vee (b c) , a + a b + a b + b c + 0 + a b c)$

$= A (a \vee (b c) , a + 0 + b c + a b c)$

$= A (a \vee (b c) , a + b c + a b c)$

$= A (a \vee (b c) , a \vee (b c))$…………[since , $a \vee b = a + b + a b c$]

$=1$.

Similarly, $A ((a \vee b) (a \vee c) , a \vee (b c)) = 1$.

Therefore , $A (a \vee (b c) , (a \vee b) (a \vee c)) = A ((a \vee b) (a \vee c) , a \vee (b c)) = 1$.

(6) $A ((a \vee b) b , b) = A ((a + b + a b) b , b)$……..[Since $a \vee b = a + b + a b$]

$= A (a b + b^2 + a b^2 , b)$

$= A (a b + b + a b , b)$

$= A (a b + a b + b , b)$

$= A (0 + b , b)$

$= A (b , b)$

$= 1$.

Similarly , $A (b , (a \vee b) b) = 1$.

Therefore , $A ((a \vee b) b , b) = A (b , (a \vee b) b) = 1$.

Hence , (R , A) is an ADFL.

To show (R , A) is relatively complemented ADFL.

Let $a , b \in R$, then $A (a (b + a b) , 0) = A (a b + a^2 b , 0)$

$= A (a b + a b , 0)$

$= A(0,0)$

$= 1 > 0$, and

$$A(a \vee (b + ab), a \vee b) = A(a + ((b + ab) + a(b + ab)), a \vee b)$$

$$= A(a + b + ab + a^2, a \vee b)$$

$$= A(a + b + 0 + ab, a \vee b)$$

$$= A(a + b + ab, a \vee b)$$

$$= A(a \vee b, a \vee b)$$

$$= 1.$$

Similarly , $A(a \vee b, a \vee (b + ab)) = 1$. Hence , $A(a \vee (b + ab), a \vee b) = A(a \vee b, a \vee (b + ab)) = 1$,therefore , by lemma 2.1.3…………[An ADFL (R , A) is relatively complemented if and only if for any a , b ∈ R ,there exist x ∈ R , such that $A(a \vee x, a \vee b) = A(a \vee b, a \vee x) = 1$ and $A(a \wedge x, 0) > 0$] , (R , A) is relatively complemented ADFL in which for any a , b ∈ R , $a^b = b + ab$.

Conclusion

In this project by using properties of Almost Boolean Ring we characterize Almost Boolean Fuzzy Ring as a Boolean Fuzzy Ring that is an Almost Boolean Fuzzy Ring is an algebra (R , + , . , 0) of type (2 , 2 , 0) satisfying all the properties of a Boolean Fuzzy Rings except possibly the associativity of “ + ’’. Again I have seen the difference and similarity between a relatively complemented Almost Distributive Fuzzy Lattice and an Almost Boolean Fuzzy Ring .

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