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Yenesew, Workineh

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BAHIR DAR UNIVERSITY
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

A project on

θ – filters in Almost Distributive Lattices

By

Yenesew Workineh

Advisor: Mihret Alemneh (Phd, Asso. prof)

June, 2019

Bahir Dar, Ethiopia

θ – filters in Almost Distributive Lattices



A project submitted to the department of mathematics in partial fulfillment of the requirements for the degree of “Master of Science in Mathematics”.

By

Yenesew Workineh

Advisor

Mihret Alamneh (Phd, Asso. Prof)

June 2019

BAHIR DAR UNIVERSITY
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

Approval of the project

I hereby certify that I have supervised, read and evaluated this project entitled θ -filters in almost distributive lattices by Yenesew Workineh prepared under my guidance.

Advisors name:

_____ Sign _____ Date _____

BAHIR DAR UNIVERSITY
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DEPARTMENT OF MATHEMATICS

We hereby certify that we have examined the project entitled θ -filters in almost distributive lattice by Yenesew Workineh is approved for the degree of “Master of Science in Mathematics”.

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	Name	Signature	Date
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Abstract

This project focused on θ -filters in ADL and characterize it in terms of almost distributive lattice congruence. A θ -Prime filters are also characterized in terms of prime filters, θ -filters and prime $-\theta$ filters in ADL. Some properties of θ -filters and θ -primes are also discussed. Furthermore a class of θ -filters be can made in to bounded distributive lattice.

SYMBOLS AND ACRONIM

ϱ	Binary relation
$(P; \leq)$	Partially ordered set or poset
Toset	Totally ordered set
$(L; \leq)$	Order lattice
$(L; \wedge, \vee)$	Algebraic lattice
ADL	Almost distributive lattice
$(L, \wedge, \vee, 0)$	Almost distributive lattice with zero
F_α	Family of filters
θ	Congruence relation
$[a]_\theta$	Congruence class containing a
$\frac{L}{\theta}$	Set of Congruence class of θ
$\mathfrak{F}_\theta(L)$	Set of θ -filters
J_α	Family of chain of θ -filters
Δ_L	Identity or diagonal relation read as delta L.
∇_L	Universal relation read as nabla L.

Chapter one

Introduction and Preliminary

1.1. Introduction

After Boole's axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [4] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other.

The notion of θ -filter in an Almost Distributive Lattices was introduced by N. Rafi, G.C. Rao and Ravi Kumar Bandaru [3] in 2013 and study some properties, and to gives some characterization of θ -filters. The usual lattice theoretic duality principle doesn't hold in ADLs. For example, in an ADL L , $\wedge$ is right distributive over $\vee$ but $\vee$ is not right distributive over $\wedge$. Any one of this converts ADL in to distributive lattice. In this project we discuss some properties of θ -filters in ADL, θ -prime and prime θ -filters in ADL. We will also characterize θ -prime filters in terms of θ -filter and prime θ -filters in ADL.

This project consists two chapters. In the first chapter we will deals with the basic concepts on lattice and ADL and in the second chapter we will look at θ - filter in ADL and some property of θ - filter in ADL. Finally, we will conclude the results.

1.2. Preliminary

In this section, we recall some basic definitions and results from [1, 3, 4, 5, 6] that lays the foundation for the second chapter.

Definition 1.2.1[6]: A binary relation ϱ defined on a non-empty set P is called an ordering relation or partial ordering relation on a set P if, for all $a, b, c \in P$, it satisfies the following axioms:

- | | |
|--|-----------------|
| (1) $(a, a) \in \varrho$ | (reflexivity) |
| (2) $(a, b) \in \varrho$ and $(b, a) \in \varrho$ imply $a = b$ | (anti-symmetry) |
| (3) $(a, b) \in \varrho$ and $(b, c) \in \varrho$ imply $(a, c) \in \varrho$. | (transitivity) |

A nonempty set P equipped with an ordering relation is called partially ordered set or simply a poset for short and denoted by $(P; \varrho)$.

Note: If P is a poset and if for every $a, b \in P$ either $a \leq b$ or $b \leq a$, then $(P; \leq)$ is called a totally ordered set (Toset) or a chain.

Definition 1.2.2[1]: If $(P; \leq)$ is a poset, then $\geq$ can also be regarded as a binary relation on P defined by $a \geq b$ iff $b \leq a$ and satisfies the axioms (1) – (3). Then $(P; \geq)$ is also a poset and is called the dual of the poset $(P; \leq)$. More precisely, if Φ is a statement about posets and if in Φ we replace all occurrences of $\leq$ by $\geq$ we get the dual of Φ .

Definition 1.2.3[6]:—Let P be a poset. Then we define the following:

- i. An element m of P is called a minimal element of P if P has no element x for which $x < m$ holds, and m is called the least (minimum) element of P if $m \leq x$ for all $x \in P$.
- (i) Dually maximal and greatest (maximum) elements of P can be defined.

Note: A poset P may have no, one, or many minimal (or maximal) elements. However; any poset possess at most one greatest element (which is maximal) and at most one least element (which is minimal). Minimal and maximal elements of poset are respectively the “top” and the “bottom” elements in the diagram. The greatest and least elements, when they exist are unique.

Example 1.2.1: Consider the non-empty set $P = \{2, 4, 5, 10, 12, 20, 25\}$ with integer division as the partial ordering relation. Then maximal and minimal elements of the poset $(P, |)$ are given in following diagram below:

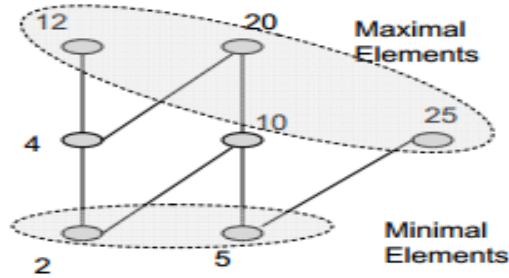


Fig 1.1

Definition 1.2.4[6]: Let R be a subset of a poset P .

- (i) An element u in P is an upper bound for R if $r \leq u$ for every $r \in R$.
- (ii) An element u in P is called the least upper bound of R if u is an upper bound of R and u is the least from all upper bounds of R .
- (iii) Dually lower bound and greatest lower bound of R can be defined.

Least upper bound of R is also called supremum and can be denoted by $\sup(R)$.

Greatest lower bound of R is also called infimum and can be denoted by $\inf(R)$.

Zorn's lemma 1.2.5[6]: If every sub-chain of a non-empty partly ordered set P has an upper bound, then P contains a maximal element.

Definition 1.2.6[1]: An algebra $(L; \vee, \wedge)$ of type $(2, 2)$ is called a lattice if for any $a, b, c \in L$, it satisfies the following properties:

- (1) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ (commutative law)
- (2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ (associative law)
- (3) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ (absorption law)
- (4) $a \wedge a = a$ and $a \vee a = a$ (Idempotent law)

Example 1.2.2: Let L be a collection of sets closed under intersection and union.

Then $(L; \cup, \cap)$ forms a lattice.

Definition 1.2.7[6]: (Duality principle for lattices): Let Φ be a statement about lattices expressed in terms of $\vee$ and $\wedge$. The dual of Φ is the statement Φ^δ we get from Φ by interchanging $\vee$ and $\wedge$. If Φ is true for all lattices, then Φ^δ is also true for all lattices.

Definition 1.2.8 [6]: We define a partial order $\leq$ on a lattice L by $a \leq b$ if $a \vee b = b$ or $a \wedge b = a$. An alternative way we can also define a lattice as a poset as follow.

Definition 1.2.9[6]: An order lattice is a poset $(L; \leq)$ in which $a \wedge b = \inf(a, b)$ and $a \vee b = \sup(a, b)$ exist for any pair of elements a and b in L .

For example, any totally ordered set $(C; \leq)$ is a lattice. Since $\inf(a, b) = \min\{a, b\}$ and $\sup(a, b) = \max\{a, b\}$ for any $a, b \in C$.

Definition 1.2.10[6]: Let L and K be a lattice and define $\wedge$ and $\vee$ on $L \times K$ by

$$\begin{aligned}(a_0, b_0) \vee (a_1, b_1) &= ((a_0 \vee a_1), (b_0 \vee b_1)) \\ (a_0, b_0) \wedge (a_1, b_1) &= ((a_0 \wedge a_1), (b_0 \wedge b_1))\end{aligned}$$

For any $a_0, b_0 \in L$ and $a_1, b_1 \in K$ this makes $L \times K$ into a lattice called product lattice.

Definition 1.2.11[1]: A lattice L is called distributive if, for all $a, b, c \in L$, it satisfies either of the following conditions:

- (1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- (2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Otherwise, L is called *non-distributive*.

Note that by the principle of duality the condition (1) holds if and only if (2) holds.

Examples 1.2.3: The lattices show in the fig 1.2 and fig 1.3 below are distributive and non-distributive lattices, respectively.



Fig 1.2

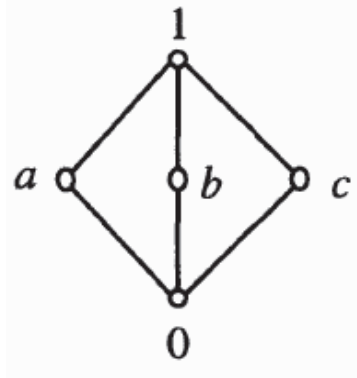


Fig 1.3

Definition 1.2.12[6]: Let $(L; \vee, \wedge)$ be a lattice and it has an element 0 and 1 such that for any $x \in L$ it satisfies the inequality $0 \leq x$ and $x \leq 1$.i. e $0 \wedge x = 0$ and $x \vee 1 = 1$, then 0 and 1 are the least and greatest element of the lattice respectively, and are called bounded elements. Such types of lattices are called bounded lattices, denoted by $(L; \vee, \wedge, 0, 1)$.

Example 1.2.4: Let $L = \{a_1, a_2, \dots, a_n\}$ be a finite lattice. Then $a_1 \wedge a_2 \wedge \dots \wedge a_n$ and $a_1 \vee a_2 \vee \dots \vee a_n$ are the least and greatest elements of L respectively. Hence L is a bounded lattice.

Definition 1.2 13: An algebra $(L; \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following axioms for all $a, b, c \in L$.

- (L1) $a \vee 0 = a$
- (L2) $0 \wedge a = 0$
- (L3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
- (L4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- (L5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- (L6) $(a \vee b) \wedge b = b$

Note: An Almost distributive lattice is an Algebra $(L; \vee, \wedge)$ of type $(2, 2)$ satisfying the all axioms of distributive lattice except the following properties.

- (C; $\wedge$) Commutativity of the operation “meets”
- (C; $\vee$) Commutativity of the operation “joins”

(RD; $\vee$) Right distributive of the operation “join” over “meet”

(A; $\vee$) Associativity of the operation “join”

Any one of these properties make an ADL to a distributive lattice.

Note: Every distributive lattice is an ADL, but converse need not be true.

Definition 1.2.14[5]: Let L be a non-empty set. Fix $x_0 \in L$. For any $x, y \in L$, define $x \wedge y = y$, $x \vee y = x$ if $x \neq x_0$, $x_0 \wedge y = x_0$ and $x_0 \vee y = y$. Then $(L; \vee, \wedge, x_0)$ is an ADL and it is called a Discrete ADL with x_0 as its zero elements.

Alternatively, Discrete ADL is defined as an ADL in which every non-zero element is maximal in the poset $(L; \leq)$.

Theorem 1.2.15[4]: Let L be an almost distributive lattice with zero. Then for any element $a, b, c \in L$, we have the following:

$$(1) a \vee a = a \text{ and } a \wedge a = a$$

$$(2) a \wedge b \wedge c = b \wedge a \wedge c$$

$$(3) (a \wedge b) \wedge c = a \wedge (b \wedge c)$$

$$(4) a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Notation: Throughout this project, L stands for an ADL with zero unless otherwise mentioned.

Definitions 1.2.16[4]: A non-empty subset I of L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $x \in L$ and $a, b \in I$. An ideal I of L is said to be the proper ideal if $I \neq L$.

Definitions 1.2.17[4]: A non-empty subset F of L is called a filter of L if $a \wedge b \in F$ and $a \vee x \in F$ whenever $a, b \in F$ and $x \in L$. A filter F is called proper if $F \neq L$.

Lemma 1.2.18[4]: If L has maximal element, then the set of all maximal elements of L forms a filter.

Theorem 1.2.19[5]. For any sub set S of L the smallest ideal containing S is given by

$$(S) = \{(\bigvee_{i=1}^n s_i) \wedge x \mid x \in L, s_i \in S \text{ and } n \in \mathbb{N}\}.$$

If $S = \{a\}$, then we write (a) instead of writing $(\{a\})$, we call (a) , the principal ideal of L generated by a .

Theorem 1.2.20[5]: Let S be a non-empty sub set of L then $[S] = \{x \vee (\bigwedge_{i=1}^n s_i) \mid x \in L, s_i \in S \text{ for } 1 \leq i \leq n \text{ and } n \text{ is positive integer}\}$ is the smallest filter of L containing S .

If $S = \{a\}$, then we write $[a]$ instead of writing $(\{a\})$, we call $[a]$, the principal filter of L generated by a .

Example 1.2.5: In the lattice L described in fig 1.4 below, $[c] = \{0, a, c\}$ is a proper and principal ideal of L . Also, $[b] = \{b, d, 1\}$ is a proper and principal filter of L .

Theorem 1. 2.21[5]: For any x, y in L the following are equivalent:

- 1). $[x] \subseteq [y]$
- 2). $y \wedge x = x$
- 3). $y \vee x = y$
- 4). $[y] \subseteq [x]$

Theorem 1.2.22[5]: For any $a, b \in L$, we have the following:

1. $[a] \cap [b] = [a \vee b] = [b \vee a]$
2. $a \in [b] \Leftrightarrow a = a \vee b$

Definition 1.2. 23 [1]. A proper ideal I of L is called a prime ideal if for any $a, b \in L$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Definition 1.2.24 [1]: A proper filter P of L is called a prime filter if for any $a, b \in L$, $a \vee b \in P$ implies $a \in P$ or $b \in P$.

Definition 1.2.25[4]: An ideal of L said to be maximal ideal of L if I is proper and if there exists no proper ideal J of L such that $I \subset J$.

Definition 1.2.26[4]: A filter F of L said to be maximal filter of L if F is proper and if there exists no proper ideal P of L such that $F \subset P$.

Example1.2.6: Consider the lattice $L = \{0, a, b, c, d, 1\}$ depicted in fig 1.4 below.

Let $I = \{0, a, b, c, d\}$ and $F = \{a, c, d, 1\}$. Here I is a prime and maximal ideal of L . On the other hand F is a prime filter of L .

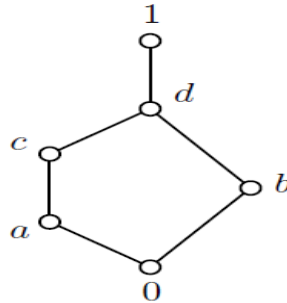


Fig 1.4

Definition 1.2.27 [6]: A binary relation θ on a non-empty set L is called an equivalence relation on L if, for any a, b, c in L , if it satisfies:

- (1) $(a, a) \in \theta$ (reflexivity)
- (2) $(a, b) \in \theta$ implies $(b, a) \in \theta$ (symmetry)
- (3) $(a, b) \in \theta$ and $(b, c) \in \theta$ imply $(a, c) \in \theta$ (transitivity).

Definition 1.2.28 [4]: An equivalence relation θ on L is called a congruence relation on L if $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \theta$, for all $(a, b), (c, d) \in \theta$.

Note: The smallest congruence relation on a non-empty set L is the identity or the diagonal relation $\Delta_L = \{(a, a) \mid a \in L\}$, read as delta L . The largest congruence relation is universal relation $\nabla_L = L \times L$ read as nabla L .

Definition 1.2.29[4]: For any congruence relation θ on L and $a \in L$, we define

$[a]_\theta = \{b \in L \mid (a, b) \in \theta\}$ and it is called the congruence class containing a .

Remark: For any congruence relation θ on L , we denote the congruence class containing $x \in L$ by $[x]_\theta$ and the set of all congruence classes of L is denoted by L/θ . The set L/θ is an ADL, under the binary operations $\wedge$ and $\vee$ defined by $[x]_\theta \vee [y]_\theta = [x \vee y]_\theta$ and $[x]_\theta \wedge [y]_\theta = [x \wedge y]_\theta$ for all $[x]_\theta$ and $[y]_\theta \in L/\theta$.

Theorem 1.2.30[4]: An equivalence relation θ on L is a congruence relation if and only if for any $(a, b) \in \theta, x \in L, (a \wedge x, b \wedge x), (x \wedge a, x \wedge b), (a \vee x, b \vee x), (x \vee a, x \vee b)$ are all in θ .

Chapter two

θ -filters in Almost Distributive Lattice

In this section, we discuss some definitions and results of θ -filters in ADL, proper θ -filters and characterization of smallest congruence relation in terms of θ -filters in ADL [3]. Also we discuss θ -prime filters in ADL and characterizations of θ -prime filters in terms of prime filters, θ -filters and prime- θ filters. We further discuss the class of θ -filter of an ADL can be made in to bounded distributive lattice.

First let us recall the definition of θ -filters in ADL as follows:

Definition 2.1: Let θ be a congruence relation on L . A filter F of L is called a θ -filter of L , if for any $a \in F \Rightarrow [a]_{\theta} \subseteq F$.

Example 2.1: Let $D = \{0', a'\}$ be a discrete ADL and $R = (0, a, b, c, 1)$ be a distributive lattice whose Hasse diagram is given in the figure 1.5 below. Then $L = D \times R = \{(0', 0), (0', a), (0', b), (0', c), (0', 1), (a', 0), (a', a), (a', b), (a', c), (a', 1)\}$. Clearly L is an ADL.

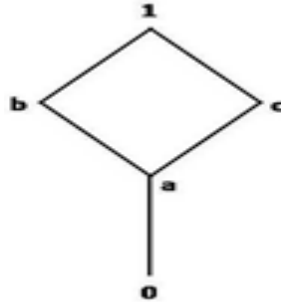


Figure 1.5

Take $\theta = \{((0', 0), (0', 0)), ((0', a), (0', a)), ((0', b), (0', b)), ((0', c), (0', c)), ((0', 1), (0', 1)), ((a', 0), (a', 0)), ((a', a), (a', a)), ((a', b), (a', b)), ((a', c), (a', c)), ((a', 1), (a', 1)), ((0', c), (0', 1)), ((0', 1), (0', c)), ((0', 0), (0', a)), ((0', a), (0', 0)), ((0', a), (0', b)), ((0', b), (0', a)), ((0', 0), (0', b)), ((0', b), (0', 0)), ((a', 0), (a', a)), ((a', a), (a', 0))\}$. Clearly θ is congruence relation on L .

Define $[(0', c)]_{\theta} = \{(0', c), (0', 1)\}$, $[(0', a)]_{\theta} = [(0', b)]_{\theta} = [(0', 0)]_{\theta} = \{(0', a), (0', 0), (0', b)\}$, $[(a', 1)]_{\theta} = \{(a', 1)\}$, $[(a', b)]_{\theta} = \{(a', b)\}$, $[(a', c)]_{\theta} = \{(a', c)\}$, $[(a', a)]_{\theta} = [(a', 0)]_{\theta} = \{(a', a), (a', 0)\}$, $[(0', 1)]_{\theta} = \{(0', 1)\}$

Consider the filter $F = \{(a', c), (a', 1)\}$. Clearly F is filter of L .

Now $(a', c) \in F \Rightarrow [(a', c)]_\theta = \{(a', c)\} \subseteq F$ and $(a', 1) \in F \Rightarrow [(a', 1)]_\theta = \{(a', 1)\} \subseteq F$.

Therefore F is a θ -filter of L .

Consider $F_1 = \{(a', a), (a', b), (a', c), (a', 1)\}$.

Clearly F_1 is filter of L . Now $(a', b) \in F_1 \Rightarrow [(a', b)]_\theta = \{(a' \cdot b)\} \subseteq F_1$

$(a', c) \in F_1 \Rightarrow [(a', c)]_\theta = \{(a', c)\} \subseteq F_1$

$(a', a) \in F_1 \Rightarrow [(a', a)]_\theta = \{(a', a), (a', 0)\} \not\subseteq F_1$ since

$(a', 0) \in [(a', a)]_\theta$ and $(a', 0) \notin F_1$.

Hence F_1 is not a θ -filter of L .

Note: From the above example we understood that every θ -filters be a filter but the converse is not true. For any congruence relation θ on L with maximal element m the filter $\{m\}$ is a θ -filter if and only if $[m]_\theta = \{m\}$.

Definition 2.2: Let θ be a congruence relation on L with maximal element m . A θ -filter F of L is called proper θ -filter of L if $F \neq L$ and if F contains m then $F \neq \{m\}$.

Example 2.2: In the lattice L depicted in the above example 2.1, consider the filter $F = \{(a', c), (a', 1)\}$ and $F_1 = \{(a', a), (a', b), (a', 0), (a', c), (a', 1)\}$ are proper θ -filters of L .

Lemma 2.3: Let L be a lattice with a maximal element m and θ be congruence relation on L . For any filter F of L , the following holds:

1. $\{m\}$ is a θ -filter if and only if $[m]_\theta \subseteq \{m\}$
2. If F is a θ -filter, then $[m]_\theta \subseteq F$
3. If F is a proper θ -filter, then $F \cap [0]_\theta = \emptyset$

Proof: 1. Assume that $\{m\}$ is θ -filter and m be maximal elements of L .

Let $m \in [m]_\theta$ since $[m]_\theta \subseteq \{m\}$, then $m \in \{m\}$. Thus $\{m\}$ is θ -filter of L .

Conversely suppose that $[m]_\theta \subseteq \{m\}$ holds:

Claim: To show that $\{m\}$ is θ -filter

By the lemma 1.2.17 $\{m\}$ is filter of L . Let $a \in [m]_\theta \subseteq \{m\}$. Then we get $a \in \{m\}$.

Since θ reflexive $a \in [a]_\theta$ so that $[m]_\theta \subseteq [a]_\theta$. (*)

Again θ is symmetric, $(a, m) \in \theta$, then we have $m \in [a]_\theta$ and hence θ also reflexive we get $m \in [m]_\theta$. Thus $[a]_\theta \subseteq [m]_\theta$ (**)

Then from (*) and (**) $[a]_\theta = [m]_\theta \subseteq \{m\}$. This implies that $[a]_\theta \subseteq \{m\}$.

Therefore $\{m\}$ is θ -filter of L .

2. Suppose that F is a θ -filter of L . We have always that $[m]_\theta \subseteq F$ for any $m \in F$ by definition.

3. Let F be a proper θ -filter of L such that $F \cap [0]_\theta = \emptyset$.

Suppose that $F \cap [0]_\theta \neq \emptyset$.

Let $x \in F \cap [0]_\theta$ then $x \in [0]_\theta$ and $x \in F$. So that $(0, x) \in \theta$. This implies that $(x, 0) \in \theta$ since θ is symmetric.

Now $(x, 0) \in \theta$ implies that $0 \in [x]_\theta$. Then $0 \in [x]_\theta \subseteq F$ since F is θ -filter. We get $0 \in F$.

So that $0 \in F \cap [0]_\theta$ which is contradiction. Therefore $F \cap [0]_\theta = \emptyset$. ■

Theorem 2.4: Let θ be a congruence relation on L . Then for any filter F of L , the following conditions are equivalent:

1. F is a θ -filter
2. For any $x, y \in L$, $(x, y) \in \theta$ and $x \in F \Rightarrow y \in F$
3. $F = \bigcup_{x \in F} [x]_\theta$

Proof: (1) $\Rightarrow$ (2): Assume that F is a θ -filter of L .

Let for any $(x, y) \in \theta$ and $x \in F$. This implies $y \in [x]_\theta \subseteq F$, since F is θ -filter. So that $y \in F$.

(2) $\Rightarrow$ (3): Assume that (2) holds. Let $a \in F$. Then $a \in [x]_\theta$ for all $x \in F$.

Thus we get $F \subseteq \bigcup_{x \in F} [x]_\theta$ (*)

Let $a \in \bigcup_{x \in F} [x]_\theta$, then $a \in [y]_\theta$, for some $y \in F$. This implies that $(a, y) \in \theta$. By our assumption we get $a \in F$. Thus $\bigcup_{x \in F} [x]_\theta \subseteq F$ (**)

Therefore from (*) and (**) $F = \bigcup_{x \in F} [x]_\theta$.

(3) $\Rightarrow$ (1): Assume that $F = \bigcup_{x \in F} [x]_\theta$ holds.

Let $a \in F$. Then $a \in [y]_\theta$, for some $y \in F$. We prove that $[a]_\theta \subseteq F$. Let $t \in [a]_\theta$.

Then $(a, t) \in \theta$ and hence $(t, y) \in \theta$ by transitivity of θ . This implies that $t \in [y]_\theta \subseteq F$.

Therefore $[a]_\theta \subseteq F$. Hence F is θ -filter of L . ■

Theorem 2.5: If θ is the smallest congruence relation on L , then every filter of L is a θ -filter.

Proof: Suppose that F be any filter of L and θ is the smallest congruence relation on L .

Let $a \in F$. Since θ is the smallest congruence relation on L we get $\{a\} = [a]_\theta$. Then $\{a\} \subseteq F$. This implies that $[a]_\theta = \{a\} \subseteq F$. Thus for any $a \in F \Rightarrow [a]_\theta \subseteq F$. Thus every filter of L is θ -filter. ■

Theorem 2.6[2]: Let θ be congruence on a lattice L . Then the following conditions are equivalent:

- (1) θ is the smallest congruence on L .
- (2) Every filter of L is a θ -filter.
- (3) Every principal filter of L is a θ -filter.

Proof: (1) $\Rightarrow$ (2) hold by Theorem 2.5.

(2) $\Rightarrow$ (3). Assume that (2) holds. Let $[x]$ be principal filter of L such that $[x]$ is the smallest filter of L containing of x . Then by our assumption we get that $[x]$ is a θ -filter of L .

(3) $\Rightarrow$ (1) Assume that every principal filter is a θ -filter of L . Let $x, y \in L$ such that $(x, y) \in \theta$. Then $[x]_{\theta} = [y]_{\theta}$. Since $[y]$ is a θ -filter of L , we get $x \in [x]_{\theta} = [y]_{\theta} \subseteq [y]$.

$$\begin{aligned} &\Rightarrow x \in [y] \\ &\Rightarrow x = x \vee y \quad (\text{Theorem 1.2.20(2)}) \\ &\Rightarrow y \leq x \end{aligned} \tag{1}$$

Similarly since $[x]$ is a θ -filter of L , we have $y \in [y]_{\theta} = [x]_{\theta} \subseteq [x]$

$$\begin{aligned} &\Rightarrow y \in [x] \\ &\Rightarrow y = y \vee x \quad (\text{Theorem 1.2.20(2)}) \\ &\Rightarrow x \leq y \end{aligned} \tag{2}$$

From (1) and (2) $x = y$. Hence θ is the smallest congruence on L . ■

Now we discuss θ -Prime filters in ADL and we characterize it in terms of prime filters θ -filters and prime θ -filters in ADL. First of all let as define θ -Prime filters in ADL.

Definition 2.7: Let θ be a congruence relation on an L with any maximal element m . Proper θ -filter P of L is called a θ -prime filter of L if for any $a, b \in L$ with $a \vee b \in [m]_{\theta}$ implies either $a \in P$ or $b \in P$.

Example 2.8: In the example 2.1 above, the filter $F = \{(a', 0), (a', a), (a', b), (a', c), (a', 1)\}$ is θ -prime filter. Analogously F is also maximal θ -filter.

Lemma 2.9: If θ is the smallest congruence relation on L with maximal elements, then every prime filter of L is a θ -prime filter of L .

Proof: Let L be an almost distributive lattice with maximal element m and θ be the smallest congruence relation on L .

Suppose that P is a prime filter of L . Then by the theorem 2.5 P is a θ -filter of L . Let $a, b \in L$

with $a \vee b \in [m]_\theta$, where m is maximal element of L . Then $[m]_\theta = [a \vee b]_\theta$. Since θ is the smallest congruence relation on L $a \vee b = m \in P$. We get $a \vee b \in P \Rightarrow a \in P$ or $b \in P$.

Therefore P is θ -prime filter of L . ■

Definition 2.10: Let θ be a congruence relation on L . A proper θ -filter P of L is called a prime θ -filter of L , if for any $a, b \in L$, $a \vee b \in P \Rightarrow a \in P$, or $b \in P$.

Example 2.2: In example 2.1 above, consider the filter $F_1 = \{(a', 0), (a', a), (a', b), (a', c), (a', 1)\}$ and $F = \{(a', c), (a', 1)\}$. Here F_1 is prime θ -filter and θ -prime filter of L but F is not θ -prime filter and a prime θ -filter of L . To verify this let $(0', c), (a', b) \in L$, then $(a', b) \vee (0', c) \in [(a', 1)]_\theta$ implies $(a', b) \notin F$ and $(0', c) \notin F$. Thus F is neither a θ -prime filter nor a prime θ -filter of L .

Lemma 2.11: Let θ be a congruence relation on L with maximal elements. Then every prime θ -filter of L is a θ -prime filter of L .

Proof: Suppose that P be a prime θ -filter and θ be a congruence relation on L with maximal element m . Let $x, y \in L$, with $x \vee y \in [m]_\theta$, where m is any maximal element of L , since $m \in P$ and P is θ -filter of L by theorem 2.5, we get $[m]_\theta \subseteq P$ and hence $x \vee y \in P$, this implies either $x \in P$ or $y \in P$. Thus P is a θ -prime filter of L . ■

Theorem 2.12: Let θ be a congruence relation on L with maximal element m and P be θ -filter of L , if $[a]_\theta = [m]_\theta \Rightarrow (a) \subseteq [m]_\theta$ for all $a \in L$ Then the following conditions are equivalent:

1. P is a θ -prime filter of L
2. For any filters I, J of L with $I \cap J \subseteq [m]_\theta \Rightarrow I \subseteq P$ or $J \subseteq P$
3. For any $a, b \in L, [a]_\theta \vee [b]_\theta = [m]_\theta \Rightarrow$ either $a \in P$ or $b \in P$

Proof: (1) $\Rightarrow$ (2): Assume that P is a θ -prime filter of L . Let I and J be any filters of L with $I \cap J \subseteq [m]_\theta$. Let $a \in I$ and $b \in J$. Then $a \vee b \in I \cap J \subseteq [m]_\theta$. This implies that $a \vee b \in [m]_\theta$ by our assumption, we have either $a \in P$ or $b \in P$. This implies either $I \subseteq P$ or $J \subseteq P$.

(2) $\Rightarrow$ (3): Assume that for any filters I, J of L with $I \cap J \subseteq [m]_\theta \Rightarrow I \subseteq P$ or $J \subseteq P$.

Let $a, b \in L$, with $[a]_\theta \vee [b]_\theta = [m]_\theta$. Then we prove that $[a \vee b]_\theta = [a]_\theta \vee [b]_\theta$.

To show $[a \vee b]_\theta = [a]_\theta \vee [b]_\theta$ it is enough to show that $[a \vee b]_\theta = [m]_\theta$. Since P is θ -prime filter we have $a \vee b \in [m]_\theta$ and hence $a \vee b \in [a \vee b]_\theta$ since θ equivalence relation.

Thus $[m]_\theta \subseteq [a \vee b]_\theta$. (*)

Again since θ is equivalence relation on L $(a \vee b, m) \in \theta$, this implies that $m \in [a \vee b]_\theta$ and $m \in [m]_\theta$. So that $[a \vee b]_\theta \subseteq [m]_\theta$. (**)

Therefore from (*) and (**) we get $[a \vee b]_\theta = [m]_\theta = [a]_\theta \vee [b]_\theta$. Then $[a \vee b]_\theta = [m]_\theta$.

This implies that $(a \vee b) \subseteq [m]_\theta$ and by theorem 1.2.19 we get that $[a \vee b] \subseteq [m]_\theta$ and hence $[a] \cap [b] \subseteq [m]_\theta$ since by theorem 1.2.20(1). Then by our assumption we get either $[a] \subseteq P$ or $[b] \subseteq P$. Thus $a \in P$ or $b \in P$.

(3) $\Rightarrow$ (1): Assume that condition (3). Let $a \vee b \in [m]_\theta$. Then $[a \vee b]_\theta = [a]_\theta \vee [b]_\theta = [m]_\theta$ by our assumption we have $[a]_\theta \subseteq P$ or $[b]_\theta \subseteq P$. This implies that $a \in P$ or $b \in P$. Hence P is a θ -prime filter of L . ■

Lemma 2.13: Let θ be a congruence relation on L . Then every minimal prime filter disjoint from $[0]_\theta$ is a θ -filter of L .

Proof: Suppose that M be a minimal prime filter of L and $M \cap [0]_\theta \neq \phi$.

Let $x, y \in L$, with $(x, y) \in \theta$ and $x \in M$. We need to show that $y \in M$.

Assume that that $y \notin M$. Then $M \vee [y] = L$. Then $a \wedge y = 0$, for some $a \in M$. Since $a \in M$ and for any $(x, y) \in \theta$ we have $(a \wedge x, a \wedge y) \in \theta$. Then we have $(a \wedge x, 0) \in \theta$ since $a \wedge y = 0$. This implies $a \wedge x \in [0]_\theta$ and $a \wedge x \in M$, since M is a filter for any $x \in L$ and $a \in M$, $a \wedge x \in M$.

Thus $a \wedge x \in M \cap [0]_\theta$ which is a contradiction.

Therefore $y \in M$ and $M \cap [0]_\theta = \phi$. Hence M is a θ -filter of L . ■

Now we have the following definition.

Definition 2.14: Let θ be a congruence relation on L . For any filter F of L , define the set F^θ as given by $F^\theta = \{x \in L \mid (x, a) \in \theta, \text{ for some } a \in F\}$.

Lemma 2.15: Let θ be a congruence relation on L . Then For any filter F of L the set F^θ is a filter of L .

Proof: $F^\theta \neq \phi$ since $F \neq \phi$. Let $x, y \in F^\theta$. Then $(x, a) \in \theta$ and $(y, b) \in \theta$ for some $a, b \in F$. This implies that $(x \wedge y, a \wedge b) \in \theta$ and $a \wedge b \in F$, since θ is congruence relation. Therefore $x \wedge y \in F^\theta$.

Let $x \in F^\theta$ and $r \in L$. Then $(x, a) \in \theta$ for some $a \in F$. Then $(r \vee x, r \vee a) \in \theta$ and $r \vee a \in F$ since F is a filter of L . Hence $r \vee x \in F^\theta$. Thus F^θ is a filter of L . ■

Lemma 2.16: Let θ be a congruence relation on L . For any two filters I, J of L we have the following.

1. $I \subseteq I^\theta$
2. If $I \subseteq J$ then $I^\theta \subseteq J^\theta$
3. $(I \cap J)^\theta = I^\theta \cap J^\theta$
4. $(I^\theta)^\theta = I^\theta$

Proof: (1) Let $a \in I$. We have $(a, a) \in \theta$ and hence $a \in I^\theta$. Therefore $I \subseteq I^\theta$.

(2) Suppose that $I \subseteq J$. Let $x \in I^\theta$. Then $(x, a) \in \theta$ for some $a \in I$. Since $I \subseteq J$ we get $(x, a) \in \theta$ and $a \in J$. Thus $x \in J^\theta$. Therefore $I^\theta \subseteq J^\theta$.

(3) Let $x \in (I \cap J)^\theta$. Then $(x, a) \in \theta$ for some $a \in I \cap J$. This implies $(x, a) \in \theta$ and $a \in I, a \in J$. Therefore $x \in I^\theta \cap J^\theta$. Thus $(I \cap J)^\theta \subseteq I^\theta \cap J^\theta$. (*)

Let $x \in I^\theta \cap J^\theta$. This implies $(x, a) \in \theta$ and $(x, b) \in \theta$, for some $a \in I$ and $b \in J$. Since θ is congruence relation $(x, a \vee b) \in \theta$ and $a \vee b \in I \cap J$.

This implies that $x \in (I \cap J)^\theta$. Thus $(I \cap J)^\theta \subseteq I^\theta \cap J^\theta$ (**)

Therefore from (*) and (**) we get that $(I \cap J)^\theta = I^\theta \cap J^\theta$

(4) Let $x \in (I^\theta)^\theta$. Then $(x, a) \in \theta$ for some $a \in I^\theta$. Since $a \in I^\theta$ we have $(a, b) \in \theta$ for some $b \in I$. This implies $(x, b) \in \theta$ and $b \in I$ and hence $x \in I^\theta$.

Therefore $(I^\theta)^\theta \subseteq I^\theta$. (1)

Let $x \in I^\theta$ then $(x, a) \in \theta$ for some $a \in I$. Since $a \in I$, we have $a \in I^\theta$ since $I \subseteq I^\theta$.

This implies that $(x, a) \in \theta$ for some $a \in I^\theta$. Thus $x \in (I^\theta)^\theta$

Therefore $I^\theta \subseteq (I^\theta)^\theta$. (2)

Hence from (1) and (2) we get that $(I^\theta)^\theta = I^\theta$. ■

Lemma 2.17: Let θ be a congruence relation on L $\{F_\alpha\}_{\alpha \in \Delta}$ be the indexed family of filters of L . Then we have the following:

$$(\cap_{\alpha \in \Delta} F_\alpha)^\theta = \cap_{\alpha \in \Delta} (F_\alpha)^\theta$$

Proof: Let $x \in (\cap_{\alpha \in \Delta} F_\alpha)^\theta$. Then there exists $a \in \cap_{\alpha \in \Delta} F_\alpha$ such that $(x, a) \in \theta$. This implies that $(x, a) \in \theta$ and $a \in F_\alpha$ for each $\alpha \in \Delta$. Therefore $x \in (F_\alpha)^\theta$, for all $\alpha \in \Delta$. Hence $x \in \cap_{\alpha \in \Delta} (F_\alpha)^\theta$.

Thus $(\cap_{\alpha \in \Delta} F_\alpha)^\theta \subseteq \cap_{\alpha \in \Delta} (F_\alpha)^\theta$. (*)

Conversely: Let $x \in \bigcap_{\alpha \in \Delta} (F_\alpha)^\theta$. Then $x \in (F_\alpha)^\theta$ for all $\alpha \in \Delta$. Since $x \in (F_\alpha)^\theta$, there exists an element $a \in \{F_\alpha\}$ such that $(x, a) \in \theta$. This implies $(x, a) \in \theta$ and $a \in \bigcap_{\alpha \in \Delta} (F_\alpha)$.

Thus $x \in \bigcap_{\alpha \in \Delta} (F_\alpha)^\theta$. Therefore $\bigcap_{\alpha \in \Delta} (F_\alpha)^\theta \subseteq (\bigcap_{\alpha \in \Delta} F_\alpha)^\theta$ (**)

Hence from (*) and (**) we get that $(\bigcap_{\alpha \in \Delta} F_\alpha)^\theta = \bigcap_{\alpha \in \Delta} (F_\alpha)^\theta$. ■

Proposition 2.18: Let θ be a congruence relation on a lattice L . For any filters F of L . F^θ is the smallest θ -filter of L such that $F \subseteq F^\theta$.

Proof: By lemma 2.15 F^θ is a filter of L and $F \subseteq F^\theta$ by lemma 2.16(1). Let $x \in F^\theta$ then $(x, a) \in \theta$ for some $a \in F$. We need prove that $[x]_\theta \subseteq F^\theta$. Let $t \in [x]_\theta$. Then $(t, x) \in \theta$. Since $(x, a) \in \theta$ and $a \in F$, we get $(t, a) \in \theta$, since θ is congruence relation and hence $t \in F^\theta$. Therefore F^θ is θ -filter of L containing F . Let K be any θ -filter containing of F . Now we prove that $F^\theta \subseteq K$.

Let $x \in F^\theta$. Then $(x, a) \in \theta$, for some $a \in F$. Since $F \subseteq K$, we have $a \in K$. Since K is θ -filter of L we get $x \in K$. Therefore F^θ is the smallest θ -filter of L such that $F \subseteq F^\theta$. ■

Notation: The set of all θ -filters of L is denoted by $\mathfrak{F}_\theta(L)$.

Remark: Let F and G be two arbitrary filters of L . Then for any congruence θ on L , we define the set $(F \vee G)^\theta = \{a \in L \mid [a]_\theta = [x \wedge y]_\theta \text{ for some } x \in F, y \in G\}$.

Theorem 2.19: Let θ be congruence relation on L . Then the set $\mathfrak{F}_\theta(L)$ of all of all θ -filters of L forms bounded distributive lattice under the set inclusion in which the glb and lub of any F, G in $\mathfrak{F}_\theta(L)$ are given by:

$$F \wedge G = F \cap G \text{ and } (F \vee G)^\theta = F \sqcup G$$

Proof: Let $F, G \in \mathfrak{F}_\theta(L)$, since F and G are θ -filters of L . Then $F \cap G$ is non-empty set and clearly $F \cap G$ is filter of L .

Claim1: The poset $(\mathfrak{F}_\theta(L), \subseteq)$ is lattice

Sub-claim 1: $F \cap G$ is θ -filter of L .

Let $a \in F \cap G = F \cap G$

$\Rightarrow a \in F \text{ and } a \in G$

$\Rightarrow [a]_\theta \subseteq F \text{ and } G$ since F, G are θ -filters of L .

$\Rightarrow [a]_\theta \subseteq F \cap G = F \wedge G$

Hence $F \wedge G$ is a θ -filter of L .

Sub-claim 2: $F \cap G$ is glb of F and G in $(\mathfrak{F}_\theta(L); \subseteq)$

Since $F \cap G \subseteq F$ and $F \cap G \subseteq G$, then $F \cap G$ is lower bound of F and G . Assume that H is lower bound of F and G . Then $H \subseteq F$ and $H \subseteq G$ this implies $H \subseteq F \cap G$. Therefore $F \cap G$ is the greatest lower bound of F and G in $(\mathfrak{F}_\theta(L); \subseteq)$.

Sub-claim 3: $(F \vee G)^\theta$ is θ -filter of L

Clearly $(F \vee G)^\theta$ is nonempty and by the lemma 2.15 is $(F \vee G)^\theta$ filter of L and hence by the proposition 2.18 above $(F \vee G)^\theta$ is θ -filter of L .

Sub-claim 4: $(F \vee G)^\theta$ is lub of F and G in $(\mathfrak{F}_\theta(L); \subseteq)$

Since $F, G \subseteq F \vee G$, then $F, G \subseteq (F \vee G)^\theta$ and $(F \vee G)^\theta$ is upper bound of F and G . Assume that a θ – filter H of L be upper bound of F and G . Now we need to show $(F \vee G)^\theta \subseteq H$.

Let $a \in (F \vee G)^\theta$, then $[a]_\theta = [x \wedge y]_\theta$ for some $x \in F$ and $y \in G$ by the above remark since $F, G \subseteq H$, then $x, y \in H$. This implies $x \wedge y \in H$. Since H is θ -filter of L $[x \wedge y]_\theta \subseteq H$. Then we get $a \in H$. Thus $(F \vee G)^\theta \subseteq H$.

Hence $(F \vee G)^\theta$ is lub of F and G in $(\mathfrak{F}_\theta(L); \subseteq)$. (2)

From (1) and (2) the poset $(\mathfrak{F}_\theta(L); \subseteq)$ is a lattice under the binary operations $\sqcup$ and $\cap$ defined on it.

Claim2: $(\mathfrak{F}_\theta(L), \sqcup, \cap)$ is distributive lattice

Let F, G and H be any three θ -filters in $\mathfrak{F}_\theta$. Then to show $(\mathfrak{F}_\theta(L), \sqcup, \cap)$ is distributive lattice it is enough to show that $F \wedge (G \sqcup H) = (F \wedge G) \sqcup (F \wedge H)$.

Clearly $(F \cap G) \sqcup (F \cap H) \subseteq F \cap (G \sqcup H)$, since $F \cap G \subseteq F$ and $F \cap H \subseteq H \subseteq (G \vee H)^\theta = G \sqcup H$.

Then $H \subseteq G \sqcup H$

Let $a \in F \wedge (G \vee H)^\theta = F \cap (G \vee H)^\theta$

$\Rightarrow a \in F$ and $a \in (G \vee H)^\theta$

$\Rightarrow a \in F$ and $[a]_\theta = [x \wedge y]_\theta$, for some $x \in G$ and $y \in H$ by the remark above

$\Rightarrow x, y \in F, x \in G$ and $y \in H$

$\Rightarrow [x]_\theta, [y]_\theta \subseteq F, [x]_\theta \subseteq G$ and $[y]_\theta \subseteq H$, since F, G and H are θ -filters of L

$\Rightarrow [x]_\theta \subseteq F \cap G$ and $[y]_\theta \subseteq F \cap H$

$\Rightarrow [a]_\theta = [x \wedge y]_\theta = [x]_\theta \wedge [y]_\theta \subseteq ((F \wedge G) \vee (F \wedge H))^\theta = (F \cap G) \sqcup (F \cap H)$,

for some $[x \wedge y]_\theta \subseteq F \cap G$ and $[x \wedge y]_\theta \subseteq F \cap H$

$\Rightarrow a \in (F \cap G) \sqcup (F \cap H)$, for some $[x \wedge y]_\theta \subseteq F$ and $[x \wedge y]_\theta \subseteq G$

$\Rightarrow F \cap (G \sqcup H) \subseteq (F \cap G) \sqcup (F \cap H)$

Hence $F \wedge (G \vee H)^\theta = F \cap (G \sqcup H) = (F \cap G) \sqcup (F \cap H)$

Therefore $(\mathfrak{F}_\theta(L); \sqcup, \cap)$ is distributive lattice.

Clearly $\{0\}$ and L are the bounds of the poset $(\mathfrak{F}_\theta(L); \subseteq)$.

Thus the lattice $(\mathfrak{F}_\theta(L); \sqcup, \cap)$ is bounded distributive lattice. ■

Theorem 2.20: Let θ be a congruence relation on a lattice L with a maximal elements. For any proper θ -filter F of L we have $F = \bigcap \{P \mid P \text{ is } \theta\text{-prime filter and } F \subseteq P\}$.

Proof: Take $F_0 = \bigcap \{P \mid P \text{ is } \theta\text{-prime filter and } F \subseteq P\}$. $F \subseteq F_0$ since $F \subseteq P$. Let $a \notin F$.

Consider $\mathfrak{F} = \{J \mid J \text{ is a } \theta\text{-filter, } F \subseteq J \text{ and } a \notin J\}$. Clearly $F \in \mathfrak{F}$. Let $\{J_\alpha\}_{\alpha \in \Delta}$ be a chain of θ -filters in $\mathfrak{F}$. Since $\{J_\alpha\}_{\alpha \in \Delta} \subseteq \mathfrak{F}$ for each $\alpha \in \Delta$. Then $\bigcup_{\alpha \in \Delta} J_\alpha \subseteq \mathfrak{F}$. This implies $\bigcup_{\alpha \in \Delta} J_\alpha$ is a θ -filter of L such that $F \subseteq \bigcup_{\alpha \in \Delta} J_\alpha$ and $a \notin \bigcup_{\alpha \in \Delta} J_\alpha$. Since $\{J_\alpha\}_{\alpha \in \Delta} \subseteq \bigcup_{\alpha \in \Delta} J_\alpha$, then $\{J_\alpha\}_{\alpha \in \Delta}$ has upper bound $\bigcup_{\alpha \in \Delta} J_\alpha$. Hence by the Zorn's lemma, $\mathfrak{F}$ has a maximal element M . That is M is θ -filter,

$F \subseteq M$ and $a \notin M$. Let $x, y \in L$ with $x \vee y \in [m]_\theta$, where m is any maximal element of L .

Suppose $x \notin M$ and $y \notin M$. Then $M \subset M \vee [x]$ and $M \vee [x] \subseteq (M \vee [x])^\theta$ by lemma 2.16(1) and similarly $M \subset M \vee [y]$ and by lemma 2.16(1) $M \vee [y] \subseteq (M \vee [y])^\theta$. By the maximality of M , we get that $a \in (M \vee [x])^\theta \cap (M \vee [y])^\theta = ((M \vee [x]) \cap (M \vee [y]))^\theta$ (Lemma 2.16(3))

$$= ((M \vee [x]) \cap [y])^\theta \quad (\text{Theorem 1.2.15(4)})$$

$$= (M \vee [x \vee y])^\theta \quad (\text{Theorem 1.2.20(1)})$$

Since $x \vee y \in [m]_\theta \subseteq M$, so that $x \vee y \in M$ since M is θ -filter of L . Then $a \in (M \vee [x \vee y])^\theta$ implies that $a \in (M)^\theta = M$. This implies $a \in M$. This is contradiction. Hence M is θ -prime filter of L . Therefore for any $a \notin F$ there exists a θ -prime filter M of L such that $F \subseteq M$ and $a \notin M$.

Thus $a \notin F_0$. Hence $F_0 \subseteq F$. Therefore $F_0 = F$. ■

Theorem 2.21: Let L be a lattice with a maximal element m and θ be a congruence relation on L . Suppose that F is θ -filter and I is an ideal of L such that $F \cap I = \phi$. Then exists a θ -prime filter P of L such that $F \subseteq P$ and $F \cap P = \phi$.

Proof: Let F be a θ -filter and I be an ideal of L with $I \cap F = \phi$. Consider $\mathfrak{F} = \{J \mid J \text{ is a } \theta\text{-filter, } F \subseteq J \text{ and } J \cap I = \phi\}$. Since $F \subseteq J$, then $F \in \mathfrak{F}$. Let $\{J_\alpha\}_{\alpha \in \Delta}$ be a chain of θ -filters in $\mathfrak{F}$. Since $\{J_\alpha\}_{\alpha \in \Delta} \subseteq \mathfrak{F}$ for each $\alpha \in \Delta$. Then $\bigcup_{\alpha \in \Delta} J_\alpha \subseteq \mathfrak{F}$, so that $\bigcup_{\alpha \in \Delta} J_\alpha$ is a θ -filter of L such that $F \subseteq \bigcup_{\alpha \in \Delta} J_\alpha$ and $(\bigcup_{\alpha \in \Delta} J_\alpha) \cap I = \phi$. Then $\{J_\alpha\}_{\alpha \in \Delta}$ has upper bound $\bigcup_{\alpha \in \Delta} J_\alpha$. Hence by the Zorn's lemma, $\mathfrak{F}$ has a maximal element M .

Now we prove that M is θ -prime filter. Let $x, y \in L$ with $x \vee y \in [m]_\theta$. To show that M is θ -prime filter it is enough to show that $x \in M$ or $y \in M$. Suppose that $x \notin M$ and $y \notin M$.

Then $M \subset M \vee [x] \subseteq (M \vee [x])^\theta$ and $M \subset M \vee [y] \subseteq (M \vee [y])^\theta$ by lemma 2.16(1).

By the maximality of M , we get that $(M \vee [x])^\theta \cap F \neq \phi$ and $(M \vee [y])^\theta \cap F \neq \phi$ since $(M \vee [x])^\theta, (M \vee [y])^\theta$ and F are nonempty sets. Let $a \in (M \vee [x])^\theta \cap F$ and $b \in (M \vee [y])^\theta \cap F$.

Since F is a filter $a \vee b \in F$ and

$$\begin{aligned} a \vee b &\in (M \vee [x])^\theta \cap (M \vee [y])^\theta = ((M \vee [x]) \cap (M \vee [y]))^\theta && \text{(Lemma 2.16(3))} \\ &= ((M \vee [x]) \cap [y])^\theta && \text{(Theorem 1.2.15(4))} \\ &= (M \vee [x \vee y])^\theta && \text{(Theorem 1.2.20(1))} \end{aligned}$$

Since $x \vee y \in [m]_\theta$, where m is maximal element of L , then $m \in M$ such that $[m]_\theta \subseteq M$. we get that $x \vee y \in M$. Then $a \vee b \in (M)^\theta = M$. Thus $a \vee b \in M$. Since $a \vee b \in F$, M we get that $a \vee b \in M \cap F$, which is contradiction. Therefore M is θ -prime filter of L . ■

Conclusion

In this project, we observe that θ -filters are defined in terms of filters in an almost distributive lattice and almost distributive lattice congruence. Some properties like proper θ -filters and prime θ -filters are also defined in terms of θ -filters in ADL. We described that if θ is the smallest congruence relation on ADL, every filter is θ -filter and every principal θ -filter is θ -filter of ADL. We also discussed that the set of all θ -filters in ADL forms a bounded distributive lattice. We also observed that the intersection and union of any arbitrary θ -filter is also θ -filter of ADL.

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