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A PROJECT ON APPROXIMATION OF FIXED POINTS OF MULTI-VALUED SELF MAPPINGS IN HILBERT SPACES

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BY

TSEGANESH GETACHEW

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BAHIR DAR, ETHIOPIA

APPROXIMATION OF FIXED POINTS OF MULTI-VALUED SELF MAPPINGS IN HILBERT SPACES

A project Submitted to the Department of Mathematics in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics.

By

Tseganesh Getachew

Advisor: Mollalgn Haile (PhD)

June, 2019

Bahir Dar, Ethiopia

BAHIR DAR UNIVERSITY

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Approval of the project for defense

I hereby certify that I have supervised, read and evaluated this project entitled "Approximation of fixed points of multi-valued self mappings in Hilbert space" prepared by Tseganesh Getachew under my guidance. I recommend that this project is submitted for oral defense.

Mollalgn Haile (PhD)	_____________	_________
Advisor name	Signature	Date

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Approved by the Board of Examiners

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Internal Examiner I:	________________________	__________	____________
Internal Examiner II:	________________________	__________	____________

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Abstract

In this project let H be a real Hilbert space and K a nonempty closed convex subset of H. Suppose $T: K \to CB(K)$ is a multi-valued Lipschitzian pseudocontractive mapping such that $F(T) \neq \emptyset$. An Ishikawa-type iterative algorithm was constructed and it was shown that, for the corresponding sequence $\{x_n\}$, under appropriate conditions on the iteration parameters, $\lim_{n\to\infty} inf\ d(x_n, Tx_n) = 0$ holds. Finally, convergence theorems were proved under approximate additional conditions. Djitte and Sene Theorems were significant improvement on important recent results of Panyanak and Sastry and Babu.

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Chapter 1

Introduction

1.1 General Introduction

One of the most important instruments to treat nonlinear problems with the aid of functional analytic methods is the fixed point approach. This approach is an important part of nonlinear functional-analysis and is deeply connected to geometric methods of topology.

The theory itself is a beautiful combination of analysis (pure and applied), topology and geometry. Over the last fifty years or so, the theory of fixed points has been revealed as a major, powerful and important tool in the study of nonlinear phenomena.

Fixed Point Theory has been applied in such diverse fields as Differential Equations, Topology, Economics, Biology, Chemistry, Engineering, Game Theory, Physics, Dynamics, Optimal Control, and Functional Analysis. Recent rapid development of efficient techniques for computing fixed points has enormously increased the usefulness of the theory of fixed points for applications. Fixed points are of interest in themselves but they also provide a way to establish the existence of a solution to a set of equations.

Existence theorems for fixed point of multi-valued contractions and nonexpansive mappings using the Hausdorff metric have been proved by several authors Lim (1974); Markin (1973) & Nadler (1969). Later an interesting and rich fixed point theory for such maps and more general maps was developed which has application on Control theory, Convex Optimization, Differential inclusion, and Economics Gorniewicz (1999).

Several theorems have been proved on the approximation of fixed points of multi-valued nonexpansive mappings and their generalization. The Mann and Ishikawa iteration schemes of multi-valued mapping T with fixed point p converge to a fixed point of T under certain conditions.

The Mann iteration process and an Ishikawa iteration processes were introduced by Sastry and Babu (2005). They proved in Sastry and Babu that the Mann and Ishikawa iteration schemes for a multi-valued map T with fixed point P converges to a fixed point of T under certain conditions. However, Panyanak (2007) extended the result of Sastry and Babu to uniformly convex real Banach spaces. Panyanak (2007) also modified the iteration schemes of Sastry and Babu (2005). In Addition to this Song and Wang (2007 & 2008) modified the iteration process of Panyanak (2007) and improved the results.

Moreover, Shahzad and Zegeye (2009) extended and improved the results of Sastry and Babu (2005); Panyanak (2007) and Song and Wang (2007 & 2008) to multi-valued quasi-nonexpansive mapping. Also, in an attempt to remove the restriction $Tp = \{p\}\ \forall p \in F(T)$, they introduced a new iteration. Browder and Petryshyn (1967) introduced and studied the class of strictly pseudocontractive mappings as an important generalization of the class of nonexpansive mappings. It is trivial to see that every nonexpansive mapping is strictly pseudocontractive. Motivated by Browder and Petryshyn (1967), Chidume et al. (2013) introduced the class of multi-valued strictly pseudocontractive mappings defined on a real Hilbert space H.

1.2 Basic Notions and Definitions

Definition 1.2.1: A norm on a real or complex vector space X is a real-valued function on X whose value at an $x \in X$ is defined by $\|x\|$ which has the properties

(N_1) $\|x\| \geq 0$,

(N_2) $\|x\| = 0$ if and only if $x = 0$,

(N_3) $\|\alpha x\| = |\alpha|\|x\|$,

(N_4) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$ and scalar α.

A normed space X is a vector space with a norm defined on it. A Banach space is a complete normed space (Kreyszing, 1978).

Definition 1.2.2: A mapping $\langle\, ,\rangle : X \times X \to K$ is an inner product space in X if for any $x, y, z \in X$ and scalar α in the scalar field K the following conditions are satisfied

(i) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,

(ii) $\langle \alpha x, y\rangle = \alpha\langle y, x\rangle$,

(iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$,

(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

An inner product space is a vector space X with an inner product defined on X.

An inner product space is called Pre-Hilbert space. Every inner product space is also normed vector space, with norm given by $\|x\| = \langle x, x\rangle$ (Kreyszing, 1978).

Definition 1.2.3: A complete inner product space H is called a Hilbert space (Kreyszing, 1978).

Definition 1.2.4: A mapping T on an inner product space X into itself is said to have a fixed point if there exists $x \in X$ such that $Tx = x$ (Siddiqi, 2003).

Definition 1.2.5: Let X and Y be nonempty sets. T is said to be a multi-valued or set-valued mapping from X to Y if T is a function from X to the power set of Y. We denote a multi-valued map by $T: X \to 2^Y$. A point $x \in X$ is said to be a fixed point of the multi-valued mapping T if $x \in Tx$. The set $F(T) = \{x \in D(T): x \in Tx\}$ is the fixed point set of T. A point $x \in D(T)$ is called a strict fixed point of T if $Tx = \{x\}$ (Maria and Ramganesh, 2013).

Definition 1.2.6: Let E be a normed space. A multi-valued mapping $T: D(T) \subseteq E \to 2^E$ is said to be

(a) Lipschitzian mapping, if there exists $L > 0$ such that

$$H(Tx, Ty) \leq L\|x - y\| \text{, for all } x, y \in D(T). \quad (1)$$

(b) Contraction mapping, if there is $0 \leq L < 1$ such that

$$H(Tx, Ty) \leq L\|x - y\| \text{, for all } x, y \in D(T).$$

(c) Non-expansive mapping, if

$$H(Tx, Ty) \leq \|x - y\| \text{, for all } x, y \in D(T)$$ (Djitte and Sene, 2014).

Also Aunyarat and Suthep (2012) defined Quasi-Nonexpansive Mapping as follows:

A multi-valued mapping $T: D(T) \subseteq E \to 2^E$ is said to be

(d) Quasi-Nonexpansive Mapping, if $F(T) = \{x \in D(T): x \in Tx\} \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|, \text{ for all } p \in F(T).$$

Definition 1.2.7: Let K be a nonempty subset of a Normed space E. The set K is called proximinal if for each $x \in E$ there exists $u \in K$ such that

$$d(x, u) = inf\{\|x - y\|: y \in K\}$$

$$= d(x, K) \qquad (2)$$

where $d(x, y) = \|x - y\| \ \forall x, y \in E$ (Djitte and sene, 2014).

It is known that every nonempty closed convex subset of a real Hilbert space is proximinal.

Definition 1.2.8: The Hausdorff distance is the greatest of all the distances from a point in one set to the closest point in the other set. Let $CB(K)$ and $P(K)$ denote the families of nonempty closed bounded subsets and nonempty proximinal bounded subset of K, respectively.

The Hausdorff metric on $CB(K)$ is defined by

$$H(A, B) = max\left\{\sup_{a \in A} d(a, B)\ ,\ \sup_{b \in B} d(b, A)\right\} \qquad (3)$$

for all $A, B \in CB(K)$. where $d(a, B) = \inf\{d(a, b): b \in B\}$ is the distance from the point a to the set B. In general the Hausdorff metric is a metric (Djitte and Sene, 2014).

Example 1.2.8.1: Let $A = \{1,3\}$, and $B = \{2, 6\}$ be two sets. Then, determine the Hausdorff distance from A to B (Hausdorff, 1914).

Solution: Let $a \in A = \{1,3\}$ and $b \in B = \{2,6\}$.

Then $H(A, B) = max\left\{\sup_{a \in A} d(a, B)\ ,\ \sup_{b \in B} d(b, A)\right\}$. But $d(a, B) = \inf\{\mathrm{d(a, b): b \in B}\}$.

So that

$$d(1, B) = \inf\{d(1, b): b \in B\} = \inf\{1, 5\} = 1$$

and

$$d(3, B) = \inf\{d(3, b): b \in B\} = \inf\{1,3\} = 1.$$

Then

$$\sup_{a \in A} d(a, B) = \sup\{1,1\} = 1.$$

And also $d(b, A) = \inf\{d(b, a): a \in A\}$.

So that

$$d(2, A) = \inf\{d(2, a): a \in A\} = \inf\{1, 1\} = 1$$

and

$$d(6, A) = \inf\{d(6, a): a \in A\} = \inf\{5, 3\} = 3.$$

Then

$$\sup_{b \in B} d(b, A) = \sup\{1, 3\} = 3.$$

Therefore, $H(A,B) = max\left\{\begin{matrix}\sup d(a,B) \\ a \in A\end{matrix}, \begin{matrix}sup\, d(b,A) \\ b \in B\end{matrix}\right\} = \max\{1, 3\} = 3.$

Example 1.2.8.2: Determine the distance between A and B, where

(a) In $\mathbb{R}^2$: $A = [0,1] \times [0,1]$, and $B = \{0\} \times [0,3]$ (Hausdorff, 1914).

Solution: Since $d(0,B) = \inf\{d(0,b): b \in B\} = \inf\{0,3\} = 0$ and $d(1,B) = \inf\{1,2\} = 1$. Then

$$\sup_{a\in A} d(a,B) = \sup\{0,1\} = 1.$$

Similarly

$$\sup_{b\in B} d(b,A) = \sup\{0,2\} = 2.$$

Therefore

$$H(A,B) = max\left\{\begin{matrix}\sup d(a,B) \\ a \in A\end{matrix}, \begin{matrix}sup\, d(b,A) \\ b \in B\end{matrix}\right\} = \max\{1,2\} = 2.$$

(b) On $\mathbb{R}$: $A = \{0\}$, and $B = \{-10,10\}$ (Hausdorff, 1914).

Solution: Since $d(0,B) = \inf\{d(0,b): b \in B\} = \inf\{10,10\} = 10$.

Then

$$\sup_{a\in A} d(a,B) = \sup\{10,10\} = 10.$$

Similarly

$$d(b,A) = \inf\{d(\{-10,10\},\{0\})\} = \inf\{10,10\} = 10.$$

Then

$$\sup_{b\in B} d(b,A) = 10.$$

Therefore,

$$H(A,B) = max\left\{\begin{matrix}\sup d(a,B) \\ a \in A\end{matrix}, \begin{matrix}sup\, d(b,A) \\ b \in B\end{matrix}\right\} = \max\{10,10\} = 10.$$

(c) On $\mathbb{R}$: $A = \{1,2,7\}$, and $B = \{6,100\}$ (Hausdorff, 1914).

Solution: $d(a,B) = \inf\{\mathrm{d(a,b)}: \mathrm{b} \in \mathrm{B}\}$

Then

$$d(1,B) = \inf\{5,99\} = 5; \quad d(2,B) = \inf\{4,98\} = 4 \text{ and } d(7,B) = \inf\{1,93\} = 1.$$

Thus

$$\sup_{a\in A} d(a,B) = \sup\ \{5,4,1\} = 5.$$

Again $d(b,A) = \inf\ \{\mathrm{d(b,a)}: \mathrm{a} \in \mathrm{A}\}$;

This implies that

$$d(6,A) = \inf\ \{\mathrm{d(6,a)}: \mathrm{a} \in \mathrm{A}\} = \inf\ \{5,4,1\} = 1$$

and

$$d(100,A) = \inf\ \{\mathrm{d(100,a)}: \mathrm{a} \in \mathrm{A}\} = \inf\ \{99,98,93\} = 93.$$

Then

$$\sup_{b\in B} d(b,A) = \sup\ \{1,93\} = 93.$$

Therefore

$$H(A,B) = max\begin{Bmatrix} sup d(a,B), & sup\, d(b,A) \\ a \in A & b \in B \end{Bmatrix} = \max\ \{5,93\} = 93.$$

Definition 1.2.9: A Banach space X is said to be uniformly convex if for any $\varepsilon, 0 < \varepsilon \leq 2,$ the inequalities $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) > 0$ such that $\left\|\frac{(x+y)}{2}\right\| \leq 1 - \delta$. This says that if x and y are in the closed unit ball $B_x \coloneqq \{x \in X: \|x\| \leq 1\}$ with $\|x - y\| \geq \varepsilon > 0$, the midpoint of x and y lies inside the unit ball B_x at a distance of at least δ from the unit sphere S_x (Agarwal et al., 2009).

Example 1.2.9.1: Every Hilbert space H is uniformly convex space. In fact, the parallelogram low gives us

$$\|x + y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x - y\|^2 \text{ for all } x, y \in H.$$

Suppose $x, y \in B_H$ with $x \neq y$ and $\|x - y\| \geq \varepsilon$. Then $\|x + y\|^2 \leq 4 - \varepsilon^2$, so it follows that

$$\left\|\frac{(x+y)}{2}\right\| \leq 1 - \delta(\varepsilon), \text{ where } \delta(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}.$$

Therefore, H is uniformly convex (Agarwal et al., 2009).

Definition 1.2.10: A sequence $\{x_n\}$ in a normed space X is said to be strongly convergent if there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0.$

This is written

$$\lim_{n\to\infty} x_n = x \text{ or simply } x_n \to x.$$

The element x is called the strong limit of $\{x_n\}$, and we say that $\{x_n\}$ converges strongly to x (Kreyszing, 1978).

Weak convergence is defined in terms of bounded linear functional as follows.

Definition 1.2.11: A sequence $\{x_n\}$ in a normed space X is said to be weakly convergent if there is $x \in X$ such that for every $f \in X'$,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

This is written as $x_n \overset{w}{\to} x$. The element x is called weak limit of $\{x_n\}$, and we say that $\{x_n\}$ converges weakly to x (Kreyszing, 1978).

Now let us see the relation of strong convergence and weak convergence as follows:

Theorem 1.2.12: Let $\{x_n\}$ be a sequence in a normed space X. Then strong convergence implies weak convergence with the same limit (Kreyszing, 1978).

Proof: By definition, $x_n \to x$ means $\|x_n - x\| \to 0$ and implies that for every $f \in X'$

$$\begin{aligned} |f(x_n) - f(x)| &= |f(x_n - x)| \\ &\leq \|f\|\|x_n - x\| \to 0. \end{aligned}$$

This shows that $x_n \overset{w}{\to} x$. ■

But, the converse of this theorem is not true.

Theorem 1.2.13:

In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded (Kreyszing, 1978).

Chapter 2

Fixed Point Iterative Methods

In 1930, R.Caccioppoli remarked on Banach contraction principle that the contraction condition may be replaced by the assumption of the convergence of the sequence of iterates, which lead to open another direction of studying fixed point theory, known as approximation of fixed point of an operator. So, to speak on iterative sequence, Picard iteration scheme has a wide range of applications in different branches of sciences; nevertheless, it has found to have some crucial drawback that the iterative sequence obtained by this method may not always converge, which was modified by Mann (1953) by introducing a new type of iteration scheme, called the Mann iterative process, formed by certain regular type of infinite matrices, with which he proved some theorems on approximation of fixed point for continuous mapping. Consequently, the Ishikawa iteration scheme was introduced.

2.1 Iteration Process

In recent years, several works have been done on the approximation of fixed points of multi-valued nonexpansive mappings by many authors. Different iterative schemes have been introduced by several authors to approximate the fixed points of nonexpansive mappings (see for example Sastry and Babu (2005); Panyanak (2007); and Song and Wang (2007 & 2008)). Among the iterative schemes, Sastry and Babu (2005) introduced Mann and Ishikawa iteration as follows:

2.1.1 Mann Iteration Process

Let E be a normed space, K be nonempty, closed and convex subset of E and $T: E \to P(E)$ be a multi-valued mapping and let p be a fixed point of T. The sequence of Mann iterates is given for $x_0 \in K$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \forall n \geq 0, \qquad y_n \in Tx_n$$
$$\|y_n - p\| = \mathrm{d}(p, Tx_n), \tag{4}$$

where α_n is a real sequence in (0,1) satisfying the following conditions:

$(i) \sum_{n=1}^{\infty} \alpha_n = \infty$; $(ii) \lim \alpha_n = 0$.

2.1.2 Ishikawa Iteration Process

Let E be a normed space, K be nonempty, closed and convex subset of E. Let $T: E \to P(E)$ be a multi-valued mapping and let p be a fixed point of T. The sequence of Ishikawa iterates is given for $x_0 \in K$ by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n, \qquad z_n \in TX_n, \\ \|z_n - p\| &= d(p, Tx_n), \\ X_{n+1} &= (1-\alpha_n)x_n + \alpha_n u_n, \qquad u_n \in Ty_n, \\ \|u_n - p\| &= d(p, Ty_n\,), \end{aligned} \tag{5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences satisfying the following conditions:

$(i)\ 0 \le \alpha_n,\ \beta_n < 1$; $(ii)\ \lim_{n\to\infty} \beta_n = 0$; and $(iii) \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

2.2 Basic Concepts and Lemmas

In this section we will see some basic lemmas, which are used to prove the following theorems which are presented on chapter three.

Lemma 2.2.1: Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences such that $(i)\ 0 \le \alpha_n,\ \beta_n \le 1$; $(ii)\ \beta_n \to 0$ as $n \to \infty$ and $(iii) \sum \alpha_n \beta_n = \infty$. Let $\{\alpha_n\}$ be a non-negative real sequence such that $\sum \alpha_n \beta_n (1-\beta_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero (Sastry and Babu, 2005).

Proof: Since $\lim_n \beta_n = 0$ and $\sum \alpha_n \beta_n = \infty$, then $\sum \alpha_n \beta_n (1-\beta_n) = \infty$. we shall show that $lim_n\, inf\, \gamma_n = 0$. Suppose not, i.e. there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\gamma_n > \varepsilon$ for all $n \ge \mathbb{N}$. This implies

$$\varepsilon \sum_{n=N}^{\infty} \alpha_n \beta_n (1-\beta_n) \le \sum_{n=N}^{\infty} \alpha_n \beta_n (1-\beta_n)\, \gamma_n < \infty,$$

which is a contradiction, and hence the conclusion follows. ∎

Kreyszing (1978) defined compactness in a normed space as follows:

Let X and Y be normed spaces and $T: X \to Y$ a linear operator. Then T is compact if and only if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{x_n\}$ in Y which has a convergent subsequence.

Lemma 2.2.2: Let X be a Banach space. Then X is uniformly convex if and only if for any given number $\rho > 0$, the square norm $\|.\|^2$ of X is uniformly convex on B_ρ, the closed ball centered at the origin with radius ρ; namely, there exists a continuous strictly increasing function $\varphi: [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\varphi(\|x - y\|),$$

for all $x, y \in B_\rho$, $\alpha \in [0,1]$ (Xu, 1991).

Lemma 2.2.3: Let H be a real Hilbert space. Then

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \qquad (6)$$

for all $x, y \in H,\ and\ \lambda \in [0,1]$ (Djitte and Sene, 2014).

Proof: $\|\lambda x + (1-\lambda)y\|^2 = \langle \lambda x + (1-\lambda)y, \lambda x + (1-\lambda)y \rangle$

$$= \langle \lambda x, \lambda x \rangle + \langle \lambda x, (1-\lambda)y \rangle + \langle (1-\lambda)y, \lambda x \rangle + \langle (1-\lambda)y, (1-\lambda)y \rangle$$

$$= \lambda^2 \langle x, x \rangle + \lambda(1-\lambda)\langle x, y \rangle + (1-\lambda)\lambda\langle y, x \rangle + (1-\lambda)(1-\lambda)\langle y, y \rangle$$

$$= \lambda * \lambda\langle x, x \rangle + \lambda(1-\lambda)\langle x, y \rangle + (1-\lambda)\lambda\langle x, y \rangle + (1-\lambda)\langle y, y \rangle - \lambda(1-\lambda)\langle y, y \rangle$$

$$= [1 - (1-\lambda)]\lambda\langle x, x \rangle + 2\lambda(1-\lambda)\langle x, y \rangle + (1-\lambda)\langle y, y \rangle - \lambda(1-\lambda)\langle y, y \rangle$$

$$= \lambda\langle x, x \rangle - \lambda(1-\lambda)\langle x, x \rangle + 2\lambda(1-\lambda)\langle x, y \rangle - \lambda(1-\lambda)\langle y, y \rangle + (1-\lambda)\langle y, y \rangle$$

$$= \lambda\langle x, x \rangle - \lambda(1-\lambda)[\langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle] + (1-\lambda)\langle y, y \rangle$$

$$= \lambda\langle x, x \rangle - \lambda(1-\lambda)[\langle x, x \rangle - \langle x, y \rangle - \langle x, y \rangle + \langle y, y \rangle] + (1-\lambda)\langle y, y \rangle$$

$$= \lambda\langle x, x \rangle - \lambda(1-\lambda)[\langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle] + (1-\lambda)\langle y, y \rangle$$

$$= \lambda\langle x, x \rangle - \lambda(1-\lambda)[\langle x - y, x - y \rangle] + (1-\lambda)\langle y, y \rangle$$

$$= \lambda\|x\|^2 - \lambda(1-\lambda)\|x - y\|^2 + (1-\lambda)\|y\|^2. \qquad \blacksquare$$

Chapter 3

Iterative Methods for Multi-valued Self Mapping

Sastry and Babu (2005) called the process defined by equation (4) Mann iteration process and the process defined by equation (5) where the iteration parameters α_n, β_n satisfy conditions $(i), (ii)\ and\ (iii)$ an Ishikawa iteration process. They proved that the Mann and Ishikawa iteration schemes for a multi-valued map with fixed point converge to a fixed point of T under certain conditions. More precisely, they proved the following result for a multi-valued nonexpansive map with compact domain.

Theorem 3.1: Let H be real Hilbert space, K be a nonempty compact convex subset of H, and $T: K \to P(K)$ a multi-valued nonexpansive map with a fixed point p. Assume that $(i)\ 0 \leq \alpha_n, \beta_n < 1$; $(ii)\ \beta_n \to 0$; and $(iii) \sum \alpha_n\beta_n = \infty$. Then, the sequence defined by equation (5) converges strongly to a fixed point of T (Sastry and Babu, 2005).

Proof: By using Lemma 2.2.3, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n u_n - p\|^2\\ &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2\\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n H^2(Ty_n, Tp) - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2\\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2, \qquad (7)\end{aligned}$$

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n z_n - p\|^2\\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - z_n\|^2\\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n H^2(Tx_n, Tp) - \beta_n(1-\beta_n)\|x_n - z_n\|^2\\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - z_n\|^2\\ &= \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - z_n\|^2. \qquad (8)\end{aligned}$$

From (7) and (8), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - z_n\|^2\\ - \alpha_n(1-\alpha_n)\|x_n - u_n\|^2.\end{aligned}$$

Therefore,

$$\alpha_n\beta_n(1-\beta_n)\|x_n - z_n\|^2 + \alpha_n(1-\alpha_n)\|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

This implies

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n)\|x_n - z_n\|^2 \leq \|x_1 - p\|^2 < \infty.$$

Hence, by Lemma 2.2.1, there exists a subsequence

$$\{x_{n_l} - z_{n_l}\} \text{ of } \{x_n - z_n\} \text{ such that } \|x_{n_l} - z_{n_l}\| \to 0 \text{ as } l \to \infty.$$

Since $z_{n_l} \in Tx_{n_l}$, $d(Tx_{n_l}, x_{n_l}) \leq \|x_{n_l} - z_{n_l}\| \to 0$ as $l \to \infty$ and $\{x_{n_l}\} \subseteq K$, K being compact, without loss of generality, we may assume that $x_{n_l} \to q$ as $l \to \infty$. Now

$$d(Tx_{n_l}, q) \leq d(Tx_{n_l}, x_{n_l}) + \|x_{n_l} - q\| \to 0 \text{ as } l \to \infty.$$

Also $H(d(Tx_{n_l}, Tq)) \to 0\ as\ l \to \infty$. Hence

$$d(q, Tq) \leq d(q, Tx_{n_l}) + H(Tx_{n_l}, Tq) \to 0 \text{ as } l \to \infty.$$

This show that $q \in Tq$. Hence, the theorem follows. ■

Panyanak (2007) extended the above result of Sastry and Babu (2005) to uniformly convex real Banach spaces. He proved the following result.

Theorem 3.2: Let E be a uniformly convex real Banach space, K a nonempty compact convex subset of E, and $T: K \to P(K)$ a multi-valued nonexpansive map with a fixed point p. Assume that (i) $0 \leq \alpha_n,\ \beta_n < 1$; (ii) $\beta_n \to 0$; and $(iii) \sum \alpha_n \beta_n = \infty$. Then, the sequence defined by equation (5) converges strongly to a fixed point of T (Song and Wang, 2008).

Proof: By using Lemma 2.2.2, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n u_n - p\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|u_n - p\|^2 - \alpha_n(1-\alpha_n)\varphi(\|x_n - u_n\|) \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n H^2(Ty_n, Tp) - \alpha_n(1-\alpha_n)\varphi(\|x_n - u_n\|) \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|y_n - p\|^2 - \alpha_n(1-\alpha_n)\varphi(\|x_n - u_n\|), \qquad (9)\end{aligned}$$

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n z_n - p\|^2 \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n \|z_n - p\|^2 - \beta_n(1-\beta_n)\varphi(\|x_n - z_n\|) \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n H^2(Tx_n, Tp) - \beta_n(1-\beta_n)\varphi(\|x_n - z_n\|) \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n \|x_n - p\|^2 - \beta_n(1-\beta_n)\varphi(\|x_n - z_n\|) \\ &= \|x_n - p\|^2 - \beta_n(1-\beta_n)\varphi(\|x_n - z_n\|). \qquad (10)\end{aligned}$$

From (9) and (10), we get

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n \beta_n (1-\beta_n)\varphi(\|x_n - z_n\|). \qquad (11)$$

Therefore,

$$\alpha_n\beta_n(1-\beta_n)\varphi(\|x_n - z_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

This implies

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)\varphi(\|x_n - z_n\|) \le \|x_1 - p\|^2 < \infty.$$

Hence, by Lemma 2.2.1, there exist a subsequence $\{x_{n_k} - z_{n_k}\}$ of $\{x_n - z_n\}$ such that $\varphi(\|x_{n_k} - z_{n_k}\|) \to 0$ as $k \to \infty$ and hence $\|x_{n_k} - z_{n_k}\| \to 0$, by the continuity and strictly increasing nature of φ. By the compactness of K, we may assume that $x_{n_k} \to q$ for some $q \in K$. Thus,

$$d(q, Tq) \le \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq)$$
$$\le \|q - x_{n_k}\| + \|x_{n_k} - z_{n_k}\| + \|x_{n_k} - q\| \to 0 \text{ as } k \to \infty.$$

Hence q is a fixed point of T. Now on taking q in place of p, we get that $\{\|x_n - q\|\}$ is a decreasing sequence by equation (11). Since $\|x_{n_k} - q\| \to 0$ as $k \to \infty$, it follows that $\{\|x_{n_k} - q\|\}$ decreases to 0, so that the conclusion of the theorem follows. ■

Panyanak (2007) also modified the iteration schemes of Sastry and Babu (2005). Let K be a nonempty closed convex subset of a real Banach space and let $T: K \to P(K)$ be a multi-valued nonexpansive map with $F(T)$ a nonempty proximinal subset of K.

The sequence of Mann iteration is defined by $x_0 \in K$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n,$$
$$\alpha_n \in [a, b], \quad 0 < a < b < 1, \quad n \ge 0, \qquad (12)$$

where $y_n \in Tx_n$ such that $\|y_n - u_n\| = d(u_n, Tx_n)$ and $u_n \in F(T)$ such that

$$\|x_n - u_n\| = d(x_n, F(T)).$$

The sequence of Ishikawa iteration is defined by $x_0 \in K$,

$$y_n = (1-\beta_n)x_n + \beta_n z_n \qquad (13)$$
$$\beta_n \in [a, b], \quad 0 < a < b < 1, \quad n \ge 0$$

where $z_n \in Tx_n$ such that $\|z_n - u_n\| = d(u_n, Tx_n)$ and $u_n \in F(T)$ such that

$$\|x_n - u_n\| = d(x_n, F(T)).$$

Consider

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n z_n{}',$$

$$\alpha_n \in [a,b], \quad 0 < a < b < 1, \quad n \geq 0, \tag{14}$$

where $z_n' \in Ty_n$ such that $\|z_n' - v_n\| = d(v_n, Ty_n)$ and $v_n \in F(T)$ such that

$$\|y_n - v_n\| = d(y_n, F(T)).$$

Before we state Panyanak theorem (2007), we need the following definition.

Definition 3.3: A mapping $T: K \to CB(K)$ is said to satisfy condition (I) if there exists a strictly increasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$d\big(x, T(x)\big) \geq f\Big(d\big(x, F(T)\big)\Big), \forall x \in D(T) \text{ (Senter and Dotson, 1974).} \tag{15}$$

Theorem 3.4: Let E be a uniformly convex real Banach space, K a nonempty closed bounded convex subset of E, and $T: K \to P(K)$ be a multi-valued nonexpansive map that satisfies condition (I). Assume that (i) $0 \leq \alpha_n < 1$ and $(ii) \sum \alpha_n = \infty$. Suppose that $F(T)$ is a nonempty proximinal subset of K. Then, the sequence $\{x_n\}$ defined by equation (12) converges strongly to a fixed point of T (Panyanak, 2007).

Proof: The generalization of Theorem 3.2.

Lemma 3.5 Let $A, B \in CB(X)$ and $a \in A$. For every $\gamma > 0$, there exists $b \in B$ such that

$$d(a,b) \leq H(A,B) + \gamma \text{ (Nadler, 1969)} \qquad (16).$$

Song and Wang (2007 & 2008) modified the iteration process by Panyanak (2007) and improved the results therein. They made the important observation that generating the Mann and Ishikawa sequence in Panyanak (2007) and is in some sense dependent on the knowledge of fixed point. They gave their iteration scheme as follows:

Let K be a nonempty closed, convex subset of a real Banach space and let $T: K \to CB(K)$ be a multi-valued map. Let $\alpha_n, \beta_n \in [0,1]$ and $\gamma_n \in (0,\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Choosing $x_0 \in K$,

$$y_n = (1-\beta_n)x_n + \beta_n z_n,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \tag{17}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ such that

$$\|z_n - u_n\| \leq H(Tx_n, Ty_n) + \gamma_n,$$

and

$$\|z_{n+1} - u_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_{n.} \tag{18}$$

Then they prove the following result.

Theorem 3.6: Let K be a nonempty compact, convex subset of a uniformly convex real Banach space E. Let $T: K \to CB(K)$ be a multi-valued nonexpansive mapping with $F(T) \neq \emptyset$ satisfying $T(p) = \{p\}$ for all $p \in F(T)$. Assume that (i) $0 \leq \alpha_n,\ \beta_n < 1$; (ii) $\beta_n \to 0$; and $(iii) \sum \alpha_n \beta_n = \infty$. Then, the sequence defined by equation (17) converges strongly to a fixed point of T (Song and Wang, 2007).

Proof: Take $p \in F(T)$, that is $T(p) = \{p\}$, and $\|z_n - p\| = d(z_n, Tp)$. Using a similar proof of Theorem 3.2, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\varphi(\|x_n - u_n\|) \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n (H(Ty_n, Tp))^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[(1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 \\
&\quad -\beta_n(1-\beta_n)\varphi(\|x_n - z_n\|)] \\
&\leq \|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)\varphi(\|x_n - z_n\|)
\end{aligned}$$

Therefore,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$$

and

$$\alpha_n\beta_n(1-\beta_n)\varphi(\|x_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{19}$$

Then $\{\|x_n - p\|\}$ is a decreasing sequence and $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F(T)$. It follows from (19) that

$$\sum \alpha_n\beta_n(1-\beta_n)\varphi(\|x_n - z_n\|) \leq \|x_1 - p\|^2.$$

The remainder proof is the same as that of Theorem 3.4. ■

Shahzad and Zegeye (2009) extended and improved the results of Sastry and Babu (2005); Panyanak (2007); and Song and Wang (2007 & 2008) to multi-valued quasi-nonexpansive mappings. Also in attempt to remove the restriction $T(p) = \{p\}$ for all $p \in F(T)$ in Song and Wang theorem (2007 & 2008), they introduced a new iteration scheme as follows.

Let K be a nonempty closed convex subset of a real Banach space, $T: K \to P(K)$ be a multi-valued map, and $P_T x = \{y \in Tx: \|x - y\| = d(x, Tx)\}$. Let $\alpha_n, \beta_n \in [0,1]$. Choose $x_0 \in K$ and define $\{x_n\}$ as follows:

$$y_n = (1 - \beta_n)x_n + \beta_n z_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \quad (20)$$

where $z_n \in P_T x_n$, $u_n \in P_T y_n$. Then they prove the following result.

Theorem 3.7: Let X be a uniformly convex real Banach space, K a nonempty convex subset of X, and let $T: K \to P(K)$ be a multi-valued map with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the Ishikawa iterates defined by equation (20). Assume that T satisfies condition (I) and α_n, $\beta_n \in [a, b] \subset (0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of T (Shahzad and Zegeye, 2009).

Proof: Let $p \in F(T)$. Then $p \in p_T(p) = \{p\}$ and thus, we obtain

$$\begin{aligned}
\|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n z_n - p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n d(z_n, p_T(p)) \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n H(p_T(x_n),\ p_T(p)) \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned}$$

And

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n)x_n + \alpha_n u_n - p\| \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|u_n - p\| \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n d(u_n, T(p)) \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n H(p_T(y_n),\ p_T(p)) \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned}$$

Consequently, the sequence $\{\|x_n - p\|\}$ is decreasing and bounded below and thus $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F(T)$.

Applying Lemma 2.2.2, we have

$$\|x_{n+1}-p\|^2 = \|(1-\alpha_n)x_n + \alpha_n u_n - p\|^2$$
$$\leq (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|u_n-p\|^2 - \alpha_n(1-\alpha_n)\varphi(\|x_n-u_n\|)$$
$$\leq (1-\alpha_n)\|x_n-p\|^2 + \alpha_n(H(p_T(y_n),\, p_T(p))^2 - \alpha_n(1-\alpha_n)\varphi(\|x_n-u_n\|)$$
$$\leq (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|y_n-p\|^2.$$

From Lemma 2.2.2, it follows that

$$\|y_n-p\|^2 = \|(1-\beta_n)x_n + \beta_n z_n - p\|^2$$
$$\leq (1-\beta_n)\|x_n-p\|^2 + \beta_n\|z_n-p\|^2 - \beta_n(1-\beta_n)\varphi(\|x_n-z_n\|)$$
$$\leq (1-\beta_n)\|x_n-p\|^2 + \beta_n(H(p_T(x_n),\, p_T(p))^2 - \beta_n(1-\beta_n)\varphi(\|x_n-z_n\|)$$
$$\leq (1-\beta_n)\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 - \beta_n(1-\beta_n)\varphi(\|x_n-z_n\|)$$
$$\leq \|x_n-p\|^2 - \beta_n(1-\beta_n)\varphi(\|x_n-z_n\|).$$

So

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|x_n-p\|^2 - \alpha_n\beta_n(1-\beta_n)\varphi(\|x_n-z_n\|).$$

This implies that

$$a^2(1-b)\varphi(\|x_n-y_n\|) \leq \alpha_n\beta_n(1-\beta_n)\varphi(\|x_n-z_n\|)$$
$$\leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2.$$

And so

$$\sum_{n=1}^{\infty} a^2(1-b)\varphi(\|x_n-y_n\|) < \infty.$$

Thus, $\lim_{n\to\infty}\varphi(\|x_n-z_n\|)=0$. Since φ is continuous at 0 and strictly increasing, we have $\lim_{n\to\infty}\|x_n-z_n\|=0$. Also $d(x_n,T(x_n)) \leq \|x_n-z_n\| \to 0$ as $n\to\infty$. Since T satisfies condition (I), we have $\lim_{n\to\infty} d(x_n,F(T))=0$. Thus, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k}-p_k\| \leq \frac{1}{2^k}$ for some $\{p_k\}\subset F(T)$ and all k. And $\{p_k\}$ is a Cauchy sequence in K and thus, converges to $q\in K$.

Since $d(p_k,T(q)) \leq d(p_k,\, p_T(q)) \leq H(p_T(p_k),\, p_T(q)) \leq \|p_k-q\|$ and $p_k\to q$ as $k\to\infty$, it follows that $d(q,T(q))=0$ and so $q\in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty}\|x_n-q\|$ exists, it follows that $\{x_n\}$ converges strongly to q. This completes the proof. ■

Remark: In recursion formula (4), the authors take $y_n\in Tx_n$ such that $\|y_n-p\|=d(p,Tx_n)$. The existence of y_n satisfying this condition is guaranteed by the assumption that Tx_n is proximinal. In general such $a\, y_n$ is extremely difficult to pick. If Tx_n is proximinal, it is not difficult to prove that it is closed. If in addition, it is a convex subset of a real Hilbert space, then y_n is unique and is characterized by

$$\langle p - y_n, y_n - u_n \rangle \geq 0 \quad \forall u_n \in Tx_n. \tag{21}$$

One can see from this inequality that it is not easy to pick $y_n \in Tx_n$ satisfying

$$\|y_n - p\| = d(p, Tx_n) \tag{22}$$

at every step of the iteration process. So, recursion formula (4) is not convenient to use in any possible application. Also, the recursion formulas defined in (13) and (14) are not convenient in any possible application. The sequences $\{z_n\}\ and\ \{z_n'\}$ are not known precisely. The restrictions

$$z_n \in Tx_n, \|z_n - u_n\| = d(u_n, Tx_n)\,,\ u_n \in F(T),$$

and

$$z_n' \in Ty_n, \|z_n' - v_n\| = d(v_n, Ty_n)\,,\ v_n \in F(T)\,.$$

make them difficult to use. These restrictions on z_n and z_n' depend on $F(T)$, the fixed points set. So, the recursions formula (13) and (14) are not easily usable.

Definition 3.8: Let K be a nonempty subset of a real Hilbert space H. A map $T: K \to H$ is called k-strictly pseudocontractive if there exists $k \in (0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \ \ \forall x, y \in K. \tag{23}$$

If $k = 1$ in (23), the map T is said to be pseudocontractive (Djitte and sene, 2014).

Browder and Petryshyn (1967) introduced and studied the class of strictly pseudocontractive maps as an important generalization of the class of nonexpansive maps. A mapping $T: K \to K$ is said to be nonexpansive mapping satisfies $\|Tx - Ty\| \leq \|x - y\| \ \ \forall x\,, y \in K$. It is trivial to see that every nonexpansive map is strictly pseudocontractive.

Motivated by Browder and Petryshyn (1967); Chidume et al. (2013) introduced the class of multi-valued strictly pseudocontractive maps defined on a real Hilbert space H as follows.

Definition 3.9: A multi-valued map $T: D(T) \subseteq H \to 2^H$ is called k-strictly pseudocontractive if there exists $k \in (0,1)$ such that for all $x, y \in D(T)$,

$$\big(H(Tx, Ty)\big)^2 \leq \|x - y\|^2 + k\|x - y - (u - v)\|^2 \tag{24}$$

$$\forall u \in Tx, v \in Ty.$$

If $K = 1$ in (24), then T is said to be pseudocontractive (Djitte and sene, 2014).

Example 3.9.1: Let $T: [0,1] \to 2^R$ be the multi-valued map defined by

$$Tx = \begin{cases} \{2\} & if x = 0 \\ \{0, x\} & if x \neq 0. \end{cases} \tag{25}$$

Then,

(i) T Satisfies for all $x, y \in D(T)$,

$$\big(H(Tx,Ty)\big)^2 \leq |x-y|^2 + k|x-u-(y-v)|^2, \tag{26}$$
$$\forall u \in Tx, v \in Ty.$$

(ii) T is not nonexpansive (Djitte and sene, 2014).

Proof of $(\boldsymbol{i})$:

Inequality (26) is obvious for $x = y = 0$. Now for $(x, y) \neq (0, 0)$, we proceed as follows.

Case1. Assume that $x, y \in (0,1]$. In this case, $Tx = \{0, x\}$ and $Ty = \{0, y\}$.Therefore, we have

$$H(Tx,Ty) = max\left\{\begin{matrix} \sup d(a, \{0,x\}) , & supd(b,\{0,y\}) \\ a \in Ty & b \in Tx \end{matrix}\right\}$$

$$= max\{min\{x, |x-y|\}, min\{y, |x-y|\}\}$$

$$= \begin{cases} max\{|x-y|, min\{y, |x-y|\}\}, & if\ x \geq y \\ max\{min\{x, |x-y|\}, |x-y|\}, & if\ x \leq y \end{cases}$$

$$=|x-y|. \tag{27}$$

Hence, for all $x, y \in (0,1]$, we have

$$\big(H(Tx,Ty)\big)^2 \leq |x-y|^2 + |x-u-(y-v)|^2, \tag{28}$$
$$\forall u \in Tx, v \in Ty.$$

Case2. Assume that $x \in (0,1]$ and $y = 0$. In this case, we have $Tx = \{0, x\}$ and $Ty = T0 = \{2\}$.

Therefore, $H(Tx,Ty) = max\left\{\begin{matrix} supd(a,\{0,x\}), & supd(|2-b|) \\ a \in \{2\} & b \in \{0,x\} \end{matrix}\right\}$

$$= \sup_{b \in \{0,x\}} |2-b|$$

$$= sup\{2, -x+2\} \tag{29}$$

$$= 2.$$

On the other hand, let $u \in Tx = \{0, x\}$ and $v \in Ty = \{2\}$.

(i) $if\ u = 0$, then

$$|(x-u)-(y-v)|^2 = (x+2)^2 \tag{30}$$

$$\geq (H(Tx,Ty))^2.$$

(ii) $if\ u = x$, we have

$$|(x-u)-(y-v)|^2 = 4 = (H(Tx,Ty))^2. \tag{31}$$

Therefore,

$$\left(H(Tx,Ty)\right)^2 \leq |x-y|^2 + |(x-u)-(y-v)|^2, \tag{32}$$
$$\forall u \in Tx, v \in Ty.$$

This completes the proof of (i)**.**

Proof of (ii)

If $x \in (0,1]$ and $y = 0$, we have that $H(Tx,Ty) = 2$ and $|x-y| = x$. so,

$$H(Tx,Ty) > |x-y|. \tag{33}$$

This proves that T is not nonexpansive. ■

We see from equation (24) that every nonexpansive mapping is strict pseudocontractive and hence the class of pseudocontractive mappings is a more general class of mappings. Then, they prove strong convergence theorems for this class of mappings. The recursion formula used is of the Krasnosel'ski (1955).

Theorem 3.10: Let K be a nonempty closed, convex subset of a real Hilbert space H. Suppose that $T: K \to CB(K)$ is a multi-valued k-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. Assume that $T(p) = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be the sequence defined by $x_0 \in K$,

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \qquad n \geq 0, \tag{34}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ (Chidume et al., 2013).

Proof: Let $p \in F(T)$, we have the following known identity:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{35}$$

which holds for all $x, y \in H$ and for all $\lambda \in [0,1]$. Using inequality (20) and the assumption that $T(p) = \{p\}$ for all $p \in F(T)$, we obtain the following estimates:

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\lambda)x_n + \lambda y_n - p\|^2 \\
&= \|(1-\lambda)(x_n-p) + \lambda(y_n-p)\|^2 \\
&= (1-\lambda)\|x_n-p\|^2 + \lambda\|y_n-p\|^2 - \lambda(1-\lambda)\|x_n-y_n\|^2 \\
&\leq (1-\lambda)\|x_n-p\|^2 + \lambda(H(Tx_n,Tp))^2 - \lambda(1-\lambda)\|x_n-y_n\|^2 \qquad (36) \\
&\leq (1-\lambda)\|x_n-p\|^2 + \lambda(\|x_n-p\|^2 + k\|x_n-y_n\|^2) - \lambda(1-\lambda)\|x_n-y_n\|^2
\end{aligned}$$

$$= \|x_n - p\|^2 - \lambda\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda k\|x_n - y_n\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2$$
$$= \|x_n - p\|^2 + \lambda k\|x_n - y_n\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2$$
$$= \|x_n - p\|^2 + \lambda(k - (1-\lambda))\|x_n - y_n\|^2$$
$$= \|x_n - p\|^2 - \lambda(1 - k - \lambda)\|x_n - y_n\|^2 \quad (37)$$

It then follows that

$$\lambda(1 - k - \lambda)\sum_{n=1}^{\infty}\|x_n - y_n\|^2 \le \|x_0 - p\|^2 < \infty \quad (38)$$

which implies that

$$\sum_{n=1}^{\infty}\|x_n - y_n\|^2 < \infty. \quad (39)$$

Hence, $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Since $y_n \in Tx_n$ we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. ■

Definition 3.11: A mapping $T: K \to CB(K)$ is called hemicompact if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \to p \in K$. we note that if K is compact, then every multi-valued mapping $T: K \to CB(K)$ is hemicompact (Aunyarat and Suthep, 2012).

Example 3.11.1: The real line is Hemi-compact.

Theorem 3.12: Let K be a nonempty compact convex subset of a real Hilbert space H and let $T: K \to CB(K)$ be a multi-valued k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ such that $T(p) = \{p\}, \forall p \in F(T)$. Suppose that T is continuous. Let $\{x_n\}$ be the sequence defined by $x_0 \in K$,

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \qquad n \ge 0, \quad (40)$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T (Chidume et al., 2013).

Proof: Observing that if K is compact, every mapping $T: K \to CB(K)$ is hemicompact. So the proof becomes as follows: From Theorem 3.10, we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is hemicompact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \to q$ as $k \to \infty$ for some $q \in K$. Since T is continuous, we also have $d(x_{n_k}, Tx_{n_k}) \to d(q, Tq)$ as $k \to \infty$. Therefore, $d(q, Tq) = 0$ and so $q \in F(T)$. Setting $p = q$ in the proof theorem 3.10, it follows from inequality (37) that $\lim_{n\to\infty}\|x_n - q\|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. ■

We see that, for the more general situation of approximating a fixed point of a multi-valued Lipschitz pseudocontractive map in a real Hilbert space, an example of Chidume and Mutangadura (2001) shows that, even in the single-valued case, the Mann iteration method does not always converge in the setting of Theorem 3.12.

Djitte and sene (2014) proved strong convergence theorems for the class of multi-valued Lipschitz pseudocontractive maps in real Hilbert spaces. They use the recursion formula (17), dispending with the second restriction on the sequences $\{z_n\}$ and $\{u_n\}$:

$$\|z_{n+1} - u_{n-1}\| \le H(Tx_{n+1}, Ty_n) + \gamma_n \text{ , } n \ge 1.$$

This class of maps is much larger than that of multi-valued nonexpansive maps used in theorem Song and Wang (2007 & 2008). So, in the setting of real Hilbert spaces, a result of Djitte and Sene (2014) improves and extends the result of Song and Wang (2007 & 2008).

Lemma 3.13: (Daffer and Kaneko, 1995).

Let $\{a_n\}$ *and* $\{\gamma_n\}$ be sequences of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \le a_n + \gamma_n \qquad \forall n \ge n_0, \tag{41}$$

where n_0 is a nonnegative integer. If $\sum \gamma_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Proof: For $n, m \ge 1$, we have

$$\begin{aligned} a_{n+m+1} &\le a_{n+m} + \gamma_{n+m} \\ &\;\;\vdots \\ &\le a_n + \sum_{j=n}^{n+m} \gamma_j. \end{aligned}$$

Hence, $\overline{lim}_m\, a_m \le a_n + \sum_{j=n}^{n+m} \gamma_j$, which implies that $\overline{lim}_m\, a_m \le \underline{lim}_n\, a_n$.

This completes the proof. ■

3.1 Main Result

Djitte and Sene (2014) use the following iteration scheme.

Let K be a nonempty closed convex subset of a real Hilbert space H and α_n, β_n and γ_n real sequences in $(0,1]$. Let $\{x_n\}$ be the sequence defined from arbitrary $x_1 \in K$ by

$$y_n = (1-\beta_n)x_n + \beta_n u_n, \qquad n \geq 1,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n w_{n,} \qquad n \geq 1, \tag{42}$$

Where $u_n \in Tx_n \; and \; w_n \in Ty_n$ are such that

$$\|u_n - w_n\| \leq H(Tx_n, Ty_n) + \gamma_n. \tag{43}$$

Then Djitte and Sene (2014) first prove the following Theorem.

Theorem 3.14: Let K be a nonempty closed convex subset of a real Hilbert space H and $T: K \to CB(K)$ a multi-valued L-lipschitz pseudocontractive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ $\forall p \in F(T)$. let $\{x_n\}$ be the sequence defined by equation (42) and (43). Assume that (i) $0 \leq \alpha_n, \; \beta_n < 1 \;\; \forall n \geq 0$; $(ii) \lim \beta_n = 0$; $(iii) \sum \alpha_n \beta_n = \infty$ and $\sum {\gamma_n}^2 < \infty$.
Then $\lim inf_{n\to\infty} d(x_n, Tx_n) = 0$ (Djitte and sene, 2014).

Proof: Let $p \in F(T)$. Using Lemma 2.2.3, the fact that T is pseudocontractive, and the assumption $Tp = \{p\}$ $\forall p \in F(T)$, we have

$$\|x_{n+1} - p\|^2 = \|(1-\alpha_n)x_n + \alpha_n w_n - p\|^2$$
$$= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|w_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - w_n\|^2$$
$$\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\big(H(Ty_n, Tp)\big)^2 - \alpha_n(1-\alpha_n)\|x_n - w_n\|^2 \tag{44}$$
$$\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(\|y_n - p\|^2 + \|y_n - w_n\|^2) - \alpha_n(1-\alpha_n)\|x_n - w_n\|^2$$
$$= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 + \alpha_n\|y_n - w_n\|^2 - \alpha_n(1-\alpha_n)\|x_n - w_n\|^2$$

Observing that

$$\|y_n - w_n\|^2 = \|(1-\beta_n)x_n + \beta_n u_n - w_n\|^2$$
$$= (1-\beta_n)\|x_n - w_n\|^2 + \beta_n\|u_n - w_n\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2 \tag{45}$$

Then, from inequality (44) and identity (45), we have that

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 + \alpha_n((1-\beta_n)\|x_n - w_n\|^2$$
$$+\beta_n\|u_n - w_n\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2) - \alpha_n(1-\alpha_n)\|x_n - w_n\|^2. \tag{46}$$

Using again Lemma 2.2.3, the fact that T is pseudocontractive, and the assumption $Tp = \{p\}$, $\forall p \in F(T)$, we obtain the following estimates:

$$\|y_n - p\|^2 = \|(1-\beta_n)x_n + \beta_n u_n - p\|^2$$
$$= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|u_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2$$
$$\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n(H(Tx_n, Tp))^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2 \quad (47)$$
$$\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n(\|x_n - p\|^2 + \|u_n - x_n\|^2) - \beta_n(1-\beta_n)\|x_n - u_n\|^2$$
$$= \|x_n - p\|^2 + {\beta_n}^2\|x_n - u_n\|^2.$$

Therefore, inequalities (46) and (47) and condition (I) imply that

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\big(\|x_n - p\|^2 + {\beta_n}^2\|x_n - u_n\|^2\big) +$$
$$\alpha_n((1-\beta_n)\|x_n - w_n\|^2 + \beta_n\|u_n - w_n\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2) -$$
$$\alpha_n(1-\alpha_n)\|x_n - w_n\|^2$$
$$\leq \|x_n - p\|^2 + \alpha_n{\beta_n}^2\|x_n - u_n\|^2 + \alpha_n(1-\beta_n)\|x_n - w_n\|^2 + \alpha_n\beta_n\|u_n - w_n\|^2$$
$$-\alpha_n\beta_n(1-\beta_n)\|x_n - u_n\|^2 - \alpha_n(1-\alpha_n)\|x_n - w_n\|^2$$
$$= \|x_n - p\|^2 - \alpha_n\beta_n(1-2\beta_n)\|x_n - u_n\|^2 - \alpha_n(\beta_n - \alpha_n)\|x_n - w_n\|^2$$
$$+ \alpha_n\beta_n\|u_n - w_n\|^2$$
$$\leq \|x_n - p\|^2 - \alpha_n\beta_n(1-2\beta_n)\|x_n - u_n\|^2 + \alpha_n\beta_n\|u_n - w_n\|^2. \quad (48)$$

Using inequality (43), the fact that T is L-Lipschitzian, and the recursion formula (42), we have

$$\|u_n - w_n\|^2 \leq 2{\beta_n}^2L^2\|x_n - u_n\|^2 + 2{\gamma_n}^2. \quad (49)$$

Therefore, from inequalities (48) and (49), we obtain

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2\alpha_n\beta_n\big(1 - 2\beta_n - 2L^2{\beta_n}^2\big) \times \|x_n - u_n\|^2 + 2{\gamma_n}^2. \quad (50)$$

Observing that condition (ii) yields that $\beta_n + L^2{\beta_n}^2 \leq 1/4$, for all $n \geq N$ for some N, it then follows that

$$\frac{1}{2}\sum_{n=N}^{\infty} \alpha_n\beta_n\|x_n - u_n\|^2 \leq \|x_N - p\|^2 + 2\sum_{n=N}^{\infty} {\gamma_n}^2 < \infty, \quad (51)$$

which implies, by condition (iii), that $\liminf_{n\to\infty}\|x_n - u_n\| = 0$. Since $u_n \in Tx_n$, it follows that $d(x_n, Tx_n) \leq \|x_n - u_n\|$. Therefore, $\liminf_{n\to\infty} d(x_n, Tx_n) = 0$. ∎

Example 3.14.1: Let $T: [-1,1] \to CB[-1,1]$ be the multi-valued map defined by

$$Tx = \begin{cases} \{0\}, & if\ x \in [-1,0] \\ [-x, 0], & if\ x \in [0,1] \end{cases}.$$

Then T satisfies for all $x, y \in D(T)$,

$$H(Tx, Ty) \leq \|x - y\|$$

and (52)

$$\big(H(Tx,Ty)\big)^2 \leq \|x-y\|^2 + \|(x-u)-(y-w)\|^2,$$
$$\forall u \in Tx\,, \forall w \in Ty.$$

Proof: Inequality (52) is obvious for $x = y = 0$. Now, we proceed as follows.

Case 1: Assume that $x, y \in [0,1]$. In this case $Tx = [-x, 0]$ and $Ty = [-y, 0]$. Therefore, we have

$$\begin{aligned} H(Tx,Ty) &= \max\left\{\sup_{a\in Tx} d(a,Ty), \sup_{b\in Ty} d(b,Tx)\right\} \\ &= max\{|x-y|, |x-y|\} \\ &= |x-y| \\ &\leq \|x-y\|. \end{aligned}$$

Hence, for all $x, y \in [0,1]$, we have

$$\big(H(Tx,Ty)\big)^2 \leq \|x-y\|^2 + \|(x-u)-(y-w)\|^2,$$
$$\forall u \in Tx\,, \forall w \in Ty.$$

Case 2: Assume that $x, y \in [-1,0]$. In this case, we have $Tx = \{0\}$ and $Ty = \{0\}$. Therefore, we have

$$H(Tx,Ty) = 0 \leq \|x-y\|.$$

Hence, for all $x, y \in [-1,0]$, we have

$$\big(H(Tx,Ty)\big)^2 \leq \|x-y\|^2 + \|(x-u)-(y-w)\|^2,$$
$$\forall u \in Tx\,, \forall w \in Ty.$$

Case 3: Assume that $x \in [0,1]$ and $y \in [-1,0]$. In this case, we have

$$Tx = [-x, 0] \text{ and } Ty = \{0\}.$$

Therefore, we have

$$\begin{aligned} H(Tx,Ty) = H(Tx,0) &= max\left\{\underset{a\in\{0\}}{sup}\, d(a,Tx), \underset{b\in Tx}{sup}\, d(|0-b|)\right\} \\ &= \underset{b\in Tx}{sup}|b| \\ &= sup|x,0| \\ &= |x| \\ &\leq \|x-y\|. \end{aligned}$$

Hence, for $x \in [0,1]$ and $y \in [-1,0]$, we have

$$\big(H(Tx,Ty)\big)^2 \le \|x-y\|^2 + \|(x-u)-(y-w)\|^2,$$

$$\forall u \in Tx\,, \forall w \in Ty. \qquad \blacksquare$$

Therefore, from three cases: If T is nonexpansive, then it is Lipschitzian and Pseudocontractive.

Now let us continue the iteration process for the above example.

Let $x_1 = 0.5$. Take $\beta_n = (n+1)^{-1/2}$, $\alpha_n = (n+1)^{-1/2}$ and $\gamma_n = (n+1)^{-1/2}$.

Step 1: For $n = 1$

$$y_1 = (1-\beta_1)x_1 + \beta_1 u_1, \quad \text{where } u_1 \in Tx_1 = T(0.5) = [-0.5, 0].$$

$$= \left(1-\tfrac{1}{\sqrt{2}}\right)(0.5) + \tfrac{1}{\sqrt{2}}(-0.4), \qquad u_1 = -0.4.$$

$$= 0.146447 - 0.282843$$

$$= -0.136396,$$

$$x_2 = (1-\alpha_1)x_1 + \alpha_1 w_1, \quad \text{where } w_1 \in Ty_1 = T(-0.136396) = 0.$$

$$= \left(1-\tfrac{1}{\sqrt{2}}\right)(0.5) + \tfrac{1}{\sqrt{2}}(0), \qquad w_1 = 0.$$

$$= 0.146447$$

such that $\|u_1 - w_1\| \le H(Tx_1, Ty_1) + \gamma_1$.

This implies that $\|u_1 - w_1\| = \|-0.4 - 0\| = 0.4$,

$$H(Tx_1,Ty_1) = H(Tx_1,0)$$

$$\le \|x_1 - y_1\|$$

$$= \|0.5 + 0.136396\|$$

$$= 0.636396$$

and

$$\gamma_1 = \frac{1}{\sqrt{2}} = 0.707107.$$

Therefore, $0.4 \le 0.636396 + 0.707107 = 1.343503$.

Step 2: For $n = 2$, $x_3 \to 0.061319$

Step 3: For $n = 3$, $x_4 \to 0.028660$

Step 4: For $n = 4$, $x_5 \to 0.013607$

Step 5: For $n = 5$, $x_6 \to 0.007236$

Step 6: For $n = 6$, $x_7 \to 0.003367$

Step 7: For $n = 7$, $x_8 \to 0.00200$

Step 8: For $n = 8,\ x_9 \to 0.001300$

Step 9: For $n = 9,\ x_{10} \to 0.000699$

Step 10: For $n = 10,\ x_{11} \to 0.000398$

Step 11: For $n = 11,\ x_{12} \to 0.000254$

Step 12: For $n = 12,\ x_{13} \to 0.000181$

Step 13: For $n = 13,\ x_{14} \to 0.000130$

Step 14: For $n = 14,\ x_{15} \to 0.000075$

Proceeding in similar way, it converges to zero.

We now prove the following corollaries of Theorem 3.14.

Corollary 3.15: Let K be a nonempty closed convex subset of a real Hilbert space H and $T: K \to CB(K)$ a multi-valued Lipschitz pseudocontractive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}\ \forall p \in F(T)$. Let $\{x_n\}$ be the sequence defined by (42) and (43). Assume that T is hemicompact, and $(i)\ 0 \leq \alpha_n \leq \beta_n < 1\ \forall n \geq 0$; $(ii)\ lim\beta_n = 0$; and $(iii)\ \sum \alpha_n \beta_n = \infty$; and $(iv) \sum {\gamma_n}^2 < \infty$. Then, $\{x_n\}$ converges strongly to a fixed point of T (Djitte and Sene, 2014).

Proof: From Theorem 3.14, we have that $\liminf_{n\to\infty} d(x_n, Tx_n) = 0$. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the $\liminf_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$. Using the fact that T is hemicompact, the sequence $\{x_{n_k}\}$ has a subsequence denoted again by $\{x_{n_k}\}$ that converges strongly to some $q \in K$. Since T is continuous, we have $d\left(x_{n_k}, Tx_{n_k}\right) \to d(q, Tq)$. Therefore, $d(q, Tq) = 0$ and so $q \in F(T)$. Now setting $p = q$ in inequality (50) and using condition (ii) we have that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + 2{\gamma_n}^2, \tag{53}$$

for all $n \geq \mathbb{N}$ for some $\mathbb{N} \geq 1$. Therefore, lemma 3.13 implies that $\lim_{n\to\infty}\left\|x_{n_k} - q\right\|$ exists. Since $\lim_{n\to\infty}\left\|x_{n_k} - q\right\| = 0$, it then follows that $\{x_n\}$ converges strongly to $q \in F(T)$, completing the proof. ■

We can easily observe that if T is nonexpansive, then it is Lipschitzian and pseudocontractive. Therefore, the following corollary generalizes Song and Wang theorem (2007 & 2008) in the setting of Hilbert spaces.

Corollary 3.16: Let K be a nonempty compact convex subset of a real Hilbert space H and $T: K \to CB(K)$ a Multi-valued Lipschitz Pseudocontractive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ $\forall p \in F(T)$. Let $\{x_n\}$ be the sequence defined by (42) and (43). Assume that (i) $0 \leq \alpha_n \leq \beta_n < 1$ $\forall n \geq 0$; (ii) $lim\beta_n = 0$; and (iii) $\sum \alpha_n \beta_n = \infty$; and (iv) $\sum \gamma_n{}^2 < \infty$. Then, $\{x_n\}$ converges strongly to a fixed point of T (Djitte and Sene, 2014).

Proof: Since K is compact, it follows that T is hemicompact. So, the proof follows from corollary 3.15.

Corollary 3.17: Let K be a nonempty compact convex subset of a real Hilbert space H and $T: K \to CB(K)$ a Multi-valued Lipschitz Pseudocontractive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ $\forall p \in F(T)$. Let $\{x_n\}$ be the sequence defined by (42) and (43). Assume that T satisfies condition (I) and (i) $0 \leq \alpha_n \leq \beta_n < 1$ $\forall n \geq 0$; (ii) $lim\beta_n = 0$; and (iii) $\sum \alpha_n \beta_n = \infty$; and (iv) $\sum \gamma_n{}^2 < \infty$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T (Djitte and Sene, 2014).

Proof: From Theorem 3.14, we have that $\liminf_{n\to\infty} d(x_n, Tx_n) = 0$. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\liminf_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$. Since T satisfies condition (I) we have $\liminf_{k\to\infty} d(x_{n_k}, F(T)) = 0$. Thus there exists a subsequence of $\{x_{n_k}\}$ denoted again by $\{x_{n_k}\}$ and a sequence $\{p_k\} \subset F(T)$ such that

$$\left\|x_{n_k} - p_k\right\| \leq \frac{1}{2^k} \qquad \forall k. \tag{54}$$

By setting $p = p_k$ in inequality (47) and using condition (ii), we have

$$\begin{aligned}\left\|x_{n_{k+1}} - p_k\right\|^2 &\leq \left\|x_{n_k} - p_k\right\|^2 + 2\gamma_{n_k}{}^2 \\ &\leq \frac{1}{2^k} + 2\gamma_{n_k}{}^2 \end{aligned} \tag{55}$$

For all $k \geq k_0$, for some $k_0 \geq 1$.

We now show that $\{p_k\}$ is a Cauchy sequence in K. Notice that, for $k \geq k_0$,

$$\begin{aligned}\|p_{k+1} - p_k\|^2 &\leq 2\left\|p_{k+1} - x_{n_{k+1}}\right\|^2 + 2\left\|x_{n_{k+1}} - p_k\right\|^2 \\ &\leq 2\left(\frac{1}{2^{2k+1}} + \frac{1}{2^k} + 2\gamma_{n_k}{}^2\right) \qquad (56) \\ &\leq 2\left(\frac{1}{2^{k-1}} + 2\gamma_{n_k}{}^2\right).\end{aligned}$$

From condition (iv), it follows that $\{p_k\}$ is a Cauchy sequence in K and thus converges strongly to some $q \in K$. Using the fact that T is L-Lipschitzian and $p_k \to q$, we have

$$d(p_k, Tq) \leq H(Tp_k, Tq) \leq L\|p_k - q\|, \tag{57}$$

So that $d(q, Tq) = 0$ and thus $q \in Tq$. Therefore, $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Now setting $p = q$ in inequality (50) and using condition (ii) we have that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + 2{\gamma_n}^2, \tag{58}$$

for all $n \geq N$ for some $N \geq 1$. Therefore, Lemma 3.13 implies that $\lim_{n\to\infty}\|x_n - q\|$ exists. Since $\lim_{k\to\infty}\|x_{n_k} - q\| = 0$. It then follows that the sequence $\{x_n\}$ converges strongly to $q \in F(T)$, completing the proof. ■

Remark: The result of Djitte and Sene (2014) theorem and corollaries improve convergence theorems for multi-valued nonexpansive mappings in Sastry and Babu (2005); Panyanak (2007); Song and Wang (2007 & 2008); Daffer and Kaneko (1995) and Shahzad and Zegeye (2009) in the following sense.

(i) In Djitte and Sene algorithm, $u_n \in Tx_n$, $w_n \in Ty_n$ do not have to satisfy the restrictive conditions $\|u_n - p\| = d(p, Tx_n)$ and $\|w_n - p\| = d(p, Ty_n)$ in the recursion formula (5) and similar restrictions in the recursion formulas (13) and (14). These restrictions on u_n and w_n depend on p, a fixed point that is being approximated or the fixed points set $F(T)$. Also in our algorithm, the second restriction on the sequences

$$\{z_n\} \text{ and } \{u_n\}: \|z_{n+1} - u_{n-1}\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n, n \geq 1,$$

in the recursion formula (17) is removed.

(ii) Djitte and Sene theorems and corollaries are proved for the class of multi-valued Lipschitz pseudocontractive mapping which is much more general than that of multi-valued nonexpansive mappings.

(iii) Corollary (3.16) is an extension of the theorem of Ishikawa (1974) from single-valued to multi-valued Lipschitz pseudocontractive mappings. And also Real sequences that satisfy the hypothesis of Theorems 3.14 are

$$\alpha_n = (n+1)^{-1/2}, \ \beta_n = (n+1)^{-1/2}, \text{ and } \gamma_n = (n+1)^{-1/2}, n \geq 0.$$

3.2 Summery

In this project Sastry and Babu introduced the Mann and Ishikawa iteration schemes for a multi-valued map T with a fixed point p converge to a fixed point T under certain conditions. More precisely, they proved their result for a multi-valued nonexpansive map with compact domain.

However, Panyanak (2007) extended the result of Sastry and Babu (2005) to a uniformly convex real Banach spaces and modified the iteration schemes. In addition Song and Wang (2007 & 2008) modified the iteration process by Panyanak (2007) and improved the results therein.

Moreover, Shahzad and Zegeye (2009) extended and improved the results of Sastry and Babu, Panyanak and Song and Wang to a multi-valued quasi-nonexpansive mappings. All these authors which mentioned above has been done for a multi-valued nonexpansive mappings.

Browder and Petryshyn (1967) introduced and studied the class of strictly pseudocontractive maps as an important generalization of the class of nonexpansive maps. Motivated by Browder and Petryshyn, Chidume et al. (2013) introduced the class of multi-valued strictly pseudocontractive maps on a real Hilbert space.

The most important conclusion from this project work is to prove strong convergence theorems for the class of multi-valued Lipschitz pseudocontractive mappings in Hilbert spaces by Djitte and Sene (2014). This class of maps is much larger than that of multi-valued nonexpansive maps.

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