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Muluken, Tamir

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Bahir Dar University

College of Science

Department of Mathematics

A project on

**Fuzzy Ideals and Fuzzy Filters of a Fuzzy
Lattice**

By

Muluken Tamir

July 10, 2019

Bahir Dar, Ethiopia

Fuzzy Ideals and Fuzzy Filters of a Fuzzy Lattice



A project submitted to the department of mathematics in partial fulfillment of the requirements for the degree of “Master of Science in mathematics”.

By

Muluken Tamir

Advisor : Bekalu Tarekegn (PhD)

July 10, 2019

Bahir Dar, Ethiopia

Bahir Dar University
College of Science
Department of Mathematics

Approval of the project

I hereby certify that I have supervised, read and evaluated this project entitled “**Fuzzy Ideals and Fuzzy Filters of a Fuzzy Lattice**” by [Muluken Tamir](#) prepared under my guidance.

Advisor's name: _____ : Signature : _____ Date : _____

Bahir Dar University
College of Science
Department of Mathematics

Fuzzy Ideals and Fuzzy Filters of a Fuzzy Lattice

By

Muluken Tamir

We hereby certify that we have examined this project entitled “**Fuzzy Ideals and Fuzzy Filters of a Fuzzy Lattice**” by **Muluken Tamir**. We recommend that Mr. **Muluken Tamir** is approved for the degree of “Master of science in mathematics”.

Board of Examiners

Name	Signature	Date
External Examiner : _____	_____	_____
Internal Examiner I: _____	_____	_____
Internal Examiner II: _____	_____	_____

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Muluken Tamir

Bahir Dar, Ethiopia

Abstract

In this project, using the fuzzy ideal and fuzzy filters of fuzzy lattice, defined by Mezzomo [9], we define α – level set of fuzzy set, and we prove some results based on Mezzomo’s definition, and we evaluate the necessity of those definitions based on general algebraic context of the notion of Lattice Theory.

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Symbols

χ – Universal set

μ – Membership function

$(P, \leq)$ – partial order set (poset)

$(L, \leq)$ – Order Lattice

$\tilde{X}$ – Fuzzy set

A – Fuzzy relation

$\mathcal{L} = (X, A)$ – Fuzzy lattice

$F_P(X)$ – Family of proper filters of a fuzzy lattice

$I_P(X)$ – Family of proper ideals of a fuzzy lattice

$I_P(\mathcal{L})$ – Family of proper fuzzy ideals of a fuzzy lattice

$F_P(\mathcal{L})$ – Family of proper fuzzy filters of a fuzzy lattice

$F(\mathcal{L})$ – Family of fuzzy filters of a fuzzy lattice

$I(\mathcal{L})$ – Family of fuzzy ideals of a fuzzy lattice

Chapter 1

Introduction and Preliminary concepts

1.1 Introduction

The development of lattice theory was started around the 1930's. Thereafter it was strengthened by Birkhoff [3], and he attempted on the one hand to introduce the most important concepts and most generally used simple methods of lattice theory, and on the other hand to point out the connections of lattice theory with other branches of Mathematics. Zadeh [12] was the first mathematician who introduced the concepts of fuzzy set to define and study fuzzy relation, and this concepts was adapted by Goguen [5] and Sanchaz [10] to define and study fuzzy relation. The concepts of fuzzy sub lattices and fuzzy ideals of a lattice was introduced by Yaua and Wu [11] . Fuzzy lattice was defined as a fuzzy algebra by Ajmal and Thomas [1] and they characterized fuzzy sub lattices a first time. In 2000, fuzzy ideals and fuzzy filters of a lattice was defined and characterized in terms of meet and join operation by Attallah [2]. In 2009 fuzzy partial ordering relation was characterized in terms of its level set by Chon [4]. Chon in the same paper defined a fuzzy lattice as a fuzzy relation and he developed basic properties and characterized a fuzzy lattice by its level set. As a continuation of these study in 2012 Mezzomo *et al.*[6] and [7] defined an ideals and filters of fuzzy lattice and more recently, Mezzomo [9] defined fuzzy ideals and fuzzy filters of a fuzzy lattice as in the sens of Chon [4] as a

crisp subset $Y \subseteq X$ endowed with the fuzzy order $A|_{Y \times Y}$. As a continuation of Mezzomo's work, we observed that, the definitions of some terms and its consequent propositions are found in tract forms in the Mezzomo [9], paper. This motivated us to do a project on the fuzzy ideals and fuzzy filters of a fuzzy lattice. So based on this idea in this project we define α – level set of a fuzzy set on a nonempty set X , and we prove some results based on Mezzomo's definition. As a result of this we have a paper which is accepted for a publication.

This project consists two chapters. The first chapter deals about the introductory and the preliminary concepts, of fuzzy ideals and fuzzy filters of a fuzzy lattice, and the second chapter deals about the definitions of fuzzy ideals and fuzzy filters of a fuzzy lattice, and set of equivalent conditions which will be characterize fuzzy ideals (filters) of a fuzzy lattice.

1.2 Preliminary

In this section, we recall some basic definitions, and results from [4, 6, 8, 9] and [12] that will be required in the main results of this project and readers who are familiar with these basic concepts may wish to proceed to the next section.

Definition 1.2.1. [9] A relation R between two nonempty sets X and Y are the mapping $R : X \times Y \longrightarrow \{0, 1\}$ meaning that:

1. If the value of relation is $R(x, y) = 1$ for some $x \in X$ and $y \in Y$, then x and y are related.
2. Otherwise $R(x, y) = 0$, we say that x and y are unrelated.

And its characteristic membership function is a mapping $\mu_R : X \times Y \longrightarrow \{0, 1\}$ which is defined by

$$\mu_R(x, y) = \begin{cases} 1, & \text{if } (x, y) \in R \\ 0, & \text{if } (x, y) \notin R. \end{cases}$$

Definition 1.2.2. [4] A partially ordered set (poset) is a pair $(P, \leq)$ where P is a non empty set and $\leq$ is a binary relation on P that satisfies the following axioms:

- (a) $x \leq x$ for all $x \in P$ (reflexive);
- (b) If $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in P$ (anti-symmetry);
- (c) If $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in P$ (transitive).

Definition 1.2.3. [4] Let $(P, \leq)$ be a poset and let $Y \subseteq P$. An element $u \in P$ is said to be an upper bound for a subset Y if and only if $b \leq u$ for all $b \in Y$. An upper bound u_0 for Y is the least upper bound of Y if and only if $u_0 \leq u$ for every upper bound u of Y . An element $v \in P$ is said to be a lower bound for a subset Y if and only if $v \leq b$ for all $b \in Y$. A lower bound v_0 for Y is the greatest lower bound of Y if and only if $v \leq v_0$ for every lower bound v of Y .

Definition 1.2.4. [9] Let L be nonempty set and $(L, \leq)$ be a poset, if $x \vee y$ and $x \wedge y$ exist $\forall x, y \in L$, then L is called a **lattice**.

Definition 1.2.5. [8] Let L be a nonempty set and $(L, \leq)$ be a lattice. A nonempty subset I of L is called an **ideal of a lattice** if it satisfies the following conditions:

- (i) If $y \in I$, $x \in L$ and $x \leq y$, then $x \in I$;
- (ii) If $x, y \in I$ implies $x \vee y \in I$.

On the other hand a nonempty subset F of L is called **filter** of a lattice if it satisfies the following conditions:

1. If $y \in F$, $x \in L$ and $y \leq x$, then $x \in F$;
2. If $x, y \in F$ implies $x \wedge y \in F$.

An ideal and a filter is called **proper** if it does not coincide with L . A proper ideal I of L is prime iff $x, y \in L$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$. Dually if $x \vee y \in I$ implies that either $x \in I$ or $y \in I$, then I is called a proper filter of a lattice.

Definition 1.2.6. [12] A fuzzy set $\tilde{X}$ in χ is characterized by a membership function $\mu_{\tilde{X}}$ maps the elements of the domain, space or universal set χ to the unit interval $[0, 1]$, that is,

$$\mu_{\tilde{X}} : \chi \longrightarrow [0, 1]$$

where $[0, 1]$ means a real number between 0 and 1 including them and a set χ is a universal set it includes all possible elements related with the given problem.

The fuzzy set $\tilde{X}$ in χ is represented as a set of order pairs of an element $x \in \chi$ and its grade of membership, i.e, $\tilde{X} = \{(x, \mu_{\tilde{X}}(x)) : x \in \chi\}$.

Definition 1.2.7. [9] Let X and Y be two nonempty sets. A fuzzy relation A is a mapping from the cartesian product $X \times Y$ to the interval $[0, 1]$, i.e, $A : X \times Y \longrightarrow [0, 1]$. If $X = Y$, then we say that A is a binary fuzzy relation on X .

The strength of the relation between ordered pairs of two universes is not measured with the characteristic function, but rather with a membership function expressing various degrees of strength of relation on the unit interval $[0, 1]$.

N.B For the simplicity we use $A(x, y)$ instead of $\mu_A(x, y)$.

Remark 1.2.1. ([9] Remark 2.2.1) The α – **level set** of a fuzzy relation $A : X \times Y \rightarrow [0, 1]$ is defined as:

$$A_{\alpha} = \{(x, y) \in X \times Y : A(x, y) \geq \alpha\} \text{ for all } x \in X, y \in Y \text{ and } \alpha \in (0, 1].$$

In the same way, we define the support of a fuzzy relation A as:

$$S(A) = \{(x, y) \in X \times Y : A(x, y) > 0\}$$

Definition 1.2.8. [8] Let X be a non-empty set. A binary fuzzy relation A in X is:

1. fuzzy reflexive if $A(x, x) = 1$, for all $x \in X$,
2. fuzzy anti-symmetric if $A(x, y) > 0$ and $A(y, x) > 0$ implies $x = y$, for all $x, y \in X$,

3. fuzzy transitive if $A(x, z) \geq \sup_{y \in X} \min\{A(x, y), A(y, z)\}$, for all $x, y, z \in X$.

Definition 1.2.9. [9] A fuzzy relation A on a nonempty set X is a fuzzy partial order relation if:

1. A is fuzzy reflexive;
2. A is fuzzy anti-symmetric and
3. A is fuzzy transitive for all $x, y, z \in X$.

If A is a fuzzy partial order relation on a set X , then (X, A) is called a fuzzy partial ordered set or a fuzzy poset.

According to the ([9] Definition 3.1.1) Let (X, A) be a fuzzy poset and let $\emptyset \neq B \subseteq X$. Then an element $u \in X$ is said to be an upper bound for a subset B if $A(b, u) > 0$ for all $b \in B$. An upper bound t for B is the least upper bound of B if $A(t, u) > 0$ for every upper bounds u of B . Dually an element $a \in X$ is said to be a lower bound for a subset B if $A(a, b) > 0$ for all $b \in B$ and a lower bound y for B is the said to be greatest lower bound of B if $A(a, y) > 0$ for every lower bounds a of B . We denote the least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \wedge y$.

Definition 1.2.10. ([4] Definition 3.2) A fuzzy poset (X, A) on the fuzzy relation A is a fuzzy lattice if and only if it satisfies the following axioms for all $x, y \in X$.

- (i) $x \wedge y$ ($\inf\{x, y\}$) is exist and;
- (ii) $x \vee y$ ($\sup\{x, y\}$) is also exist.

Where $\wedge$ and $\vee$ are meet and join operator on the set X respectively.

Example 1.2.1. ([8] Example 2.1) Let $X = \{x, y, z, w\}$ and let $A : X \times X \rightarrow [0, 1]$ be a fuzzy relation such that $A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1.0$, $A(x, y) = A(x, z) = A(x, w) = A(y, z) = A(y, w) = A(z, w) = 0.0$, $A(y, x) = 0.3$, $A(z, x) = 0.5$, $A(w, x) = 0.8$, $A(z, y) = 0.2$, $A(w, y) = 0.4$ and $A(w, z) = 0.1$. Then (X, A) is fuzzy poset and also $x \vee y = x$, $x \vee z = x$, $x \vee w = x$, $y \vee z = y$, $y \vee w = y$, $z \vee w = z$ and $x \wedge y = y$, $x \wedge z = z$, $x \wedge w = w$, $y \wedge z = z$, $y \wedge w = w$ and $z \wedge w = w$ is exist then (X, A) is a fuzzy lattice.

Proposition 1.2.2. [9] Let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation. If A is fuzzy partial ordering relation on X , then for each $\alpha \in (0, 1]$ the α -level set A_α is a partial order relation on X .

Proof. Let A be a fuzzy partial ordering relation on X and $\alpha \in (0, 1]$. Since $A(x, x) = 1$ for all $x \in X$, $(x, x) \in A_\alpha$ for all $\alpha \in (0, 1]$, and hence A_α is reflexive. Suppose (x, y) and (y, x) is in A_α . Then $A(x, y) \geq \alpha > 0$ and $A(y, x) \geq \alpha > 0$ this implies that, $A(x, y) > 0$ and $A(y, x) > 0$ since A is fuzzy anti-symmetric, $x = y$. Again suppose that (x, y) and (y, z) is in A_α , then $A(x, y) \geq \alpha$ and $A(y, z) \geq \alpha$. Since $A(x, z) \geq \sup_{y \in X} \min\{A(x, y), A(y, z)\}$, then $A(x, z) \geq \min\{A(x, y), A(y, z)\} \geq \alpha$

$$\Rightarrow A(x, z) \geq \alpha \quad (\text{by transitivity property}).$$

$$\Rightarrow (x, z) \in A_\alpha, \text{ and hence } A_\alpha \text{ is partial ordering relation on } X. \quad \square$$

Proposition 1.2.3. [9] Let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation. If A is fuzzy partial order relation on X , then $S(A)$ is a partial order relation on X .

Proof. Let A be fuzzy partial order relation on X . Since $A(x, x) = 1 > 0 \forall x \in X$, implies that $(x, x) \in S(A)$. Suppose $(x, y), (y, x) \in S(A)$ then $A(x, y) > 0$ and $A(y, x) > 0$, because A is fuzzy anti-symmetric we have $x = y$. Again suppose $(x, y), (y, z) \in S(A)$. Then, $A(x, y) > 0$ and $A(y, z) > 0$. Since

$$A(x, z) \geq \sup_{y \in X} \min\{A(x, y), A(y, z)\} > 0.$$

Implies that $(x, z) \in S(A)$, hence $S(A)$ is partial order relation on X . $\square$

Proposition 1.2.4. [9] Let $L = (X, A)$ be a fuzzy poset, then $x \vee y$ and $x \wedge y$ coincides with $x \vee y$ and $x \wedge y$ in $(X, S(A))$.

Lemma 1.2.5. [9] Let (X, A) be a fuzzy poset. (X, A) is a fuzzy lattice iff $(X, S(A))$ is a lattice.

Proof. $(\Rightarrow)$ suppose (X, A) is a fuzzy lattice, then A is fuzzy partial relation on X . Hence by the proposition 1.2.3 $S(A)$ is partial order relation on X . Therefore $(X, S(A))$ is poset.

And thus by proposition 1.2.4 $(X, S(A))$ is a lattice.

($\Leftarrow$) Straightforward from the proposition 1.2.4. $\square$

Definition 1.2.11. ([9]) Let $\mathcal{L} = (X, A)$ be a fuzzy lattice. Then $\mathcal{L}$ is said to be bounded fuzzy lattice, if there exist least or ($\perp$) and the greatest or ($\top$) elements in X such that $A(\perp, x) > 0$ and $A(x, \top) > 0$ for all $x \in X$.

Definition 1.2.12. ([6] Definition 3.1) Let $\mathcal{L} = (X, A)$ be a fuzzy lattice and $\emptyset \neq Y \subseteq X$. Then Y is said to be **ideal** of $\mathcal{L}$ if it satisfies the following conditions:

- (i) If $x \in X, y \in Y$ and $A(x, y) > 0$, then $x \in Y$;
- (ii) If $x, y \in Y$, then $x \vee y \in Y$.

On the other hand, Y is **filter** of $\mathcal{L}$ if:

- 1. $x \in X, y \in Y$ and $A(y, x) > 0$, then $x \in Y$;
- 2. $x, y \in Y$, then $x \wedge y \in Y$.

In both cases, if $Y \neq X$, then Y is called **proper ideal** (**proper filter**) of $\mathcal{L}$.

Notation: The class of all proper ideals (filters) of $\mathcal{L}$ are denoted by $I_p(F_p)(X)$.

Example 1.2.6. Let $X = \{0, a, b, c\}$ and define two binary operations $\vee$ and $\wedge$ in X as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	b
c	c	c	b	c

and

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c
c	0	a	c	c

Again define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ as follows: $A(0, 0) = A(a, a) = A(b, b) = A(c, c) = 1$, $A(a, 0) = A(b, 0) = A(c, 0) = A(b, a) = A(b, c) = A(c, a) = 0$, $A(0, a) = 0.3$, $A(0, b) = 0.5$, $A(0, c) = 0.8$, $A(a, b) = 0.2$, $A(c, b) = 0.1$, and $A(a, c) = 0.4$. Now let $\mathcal{L} = (X, A)$. A fuzzy relation A in this example on nonempty set X is a fuzzy partial order relation and hence $\mathcal{L}$ is a fuzzy poset, and easily we can show that from above tables both sup and inf are exist for each pair of elements in X , then $\mathcal{L}$ is a fuzzy lattice.

Now let us consider $I = \{0, a\}$ and $F = \{a, b, c\}$. I and F are two non empty subsets of X . It is a simple task to show I and F satisfy all the axioms of ideals and filters of a fuzzy lattice respectively and hence, I is an ideal of L and F is a filter of L .

Definition 1.2.13. ([9] Definition 5.2.1) Let Y be a proper ideal (filter) of a fuzzy lattice $\mathcal{L} = (X, A)$. Then Y is said to be a **prime ideal (prime filter)** of $\mathcal{L}$ if for all $x, y \in X$ and $z \in Y$, $A(x \wedge y, z) > 0$ implies either $x \in Y$ or $y \in Y$ ($A(x \vee y, z) > 0$ implies either $x \in Y$ or $y \in Y$).

Chapter 2

Fuzzy Ideals and Fuzzy Filters of a Fuzzy Lattice

In this chapter, we recall some definitions and basic results from Mezzomo [9], and we evaluate the necessity of those definitions based on general algebraic context of the notion of Lattice Theory.

2.1 Definitions of Fuzzy Ideals and Fuzzy Filters of a Fuzzy Lattice

In this section, we recall the definitions of fuzzy ideals and fuzzy filters of a fuzzy lattice, and we redefine the definition of α -level set of a fuzzy set, and in addition to this we investigate some basic properties that coincides with the properties of the usual ideals and filters of a lattice.

First, let us recall the definitions of the fuzzy ideals and fuzzy filters of a fuzzy lattice, from [Mezzomo [9] Definition 6.1.1 and 6.1.2] respectively.

Definition 2.1.1. [9] Let $\mathcal{L} = (X, A)$ be a fuzzy lattice and I be a fuzzy set on X . Then I is said to be a **fuzzy ideal** of $\mathcal{L}$ if it satisfies the following axioms for all $x, y \in X$.

1. If $\mu_I(y) > 0$ and $A(x, y) > 0$, then $\mu_I(x) > 0$;

2. If $\mu_I(x) > 0$ and $\mu_I(y) > 0$, then $\mu_I(x \vee y) > 0$

Similarly, at the Definition 6.1.2 Mezzomo defined the notion of fuzzy filters of fuzzy lattice as follows:

Definition 2.1.2. [9] Let F be a fuzzy set on the set X . Then F is said to be **fuzzy filters** of fuzzy lattice (X, A) if it satisfies the following conditions for all $x, y \in X$.

1. If $\mu_F(y) > 0$ and $A(y, x) > 0$, then $\mu_F(x) > 0$;
2. If $\mu_F(x) > 0$ and $\mu_F(y) > 0$, then $\mu_F(x \wedge y) > 0$.

Example 2.1.1. Let $\mathcal{L} = (X, A)$ be a fuzzy lattice defined at **Example 1.2.1**. Consider a fuzzy sets $I = \{(x, 0.0), (y, 0.2), (z, 0.4), (w, 0.7)\}$ and $F = \{(x, 0.2), (y, 0.5), (z, 0.4), (w, 0.0)\}$ on X . Then I and F satisfies the conditions of fuzzy ideals and fuzzy filters of fuzzy lattice respectively. So I is fuzzy ideal of $\mathcal{L}$ and F is fuzzy filter of $\mathcal{L}$.

Notation: We denote the family of fuzzy ideals (filters) of a fuzzy lattice by $I(\mathcal{L})(F(\mathcal{L}))$ respectively.

Here in the above definition, we can understand that, the definitions of fuzzy ideals and fuzzy filters of a fuzzy lattice is analogous to the definitions of ideals and filters of a fuzzy lattice respectively. In the definitions of ideals and filters of a fuzzy lattice, the conditions of a given nonempty subset Y of X to be an ideal (filter) of a fuzzy lattice (X, A) are formulated based on the memberships of elements of X in Y . On the other hand, the definitions of fuzzy ideals and fuzzy filters of a fuzzy lattice, the conditions of a given fuzzy set I on X to be fuzzy ideal (filter) of a fuzzy lattice (X, A) are formulated based on the degree of memberships of elements of X in a fuzzy set I rather than being a members of a given set only.

Now, let us recall the definition of support sets of a given fuzzy set I on a set X . In [9] Mezzomo, defined a support set of a fuzzy set I on a set X denoted by $S(I)$ by $S(I) = \{x \in X : \mu_I(x) > 0\}$. Consequently, we can easily verify that a fuzzy set I on a set

X is nonempty iff $S(I)$ is a non empty subset of a set X . From the definition of the support set of a fuzzy set I , and the definitions of fuzzy ideal (filter) of a fuzzy set I (F) on X , we can verify that an element x of X is in the support set $S(I)$ iff $\mu_I(x) > 0$. Since the support set $S(A)$ of a fuzzy relation A on a set X is a crisp relation, and the support set $S(I)$ of a fuzzy set I on X is a subset of a set X , we can characterize a fuzzy ideal (filter) of a fuzzy lattice (X, A) via a support set $S(I)$ in a lattice $(X, S(A))$. Mezzomo in [9], characterizes a fuzzy ideal (filter) of a fuzzy lattice as follows.

Proposition 2.1.2. [9] *A fuzzy set I on X is a fuzzy ideal of a fuzzy lattice (X, A) iff the support set $S(I)$ of I is an ideal of a lattice $(X, S(A))$.*

Proof. $(\Rightarrow)$ Let I be a fuzzy ideal of (X, A) .

Claim1: $S(I)$ is an ideal of $(X, S(A))$.

(i) If $y \in S(I)$ and $(x, y) \in S(A)$, then $\mu_I(y) > 0$ and $A(x, y) > 0$. So by hypothesis I is a fuzzy ideal of (X, A) , then by Definition 2.1.1 (1) $\mu_I(x) > 0$ and hence $x \in S(I)$. This implies that $x \wedge y = x \in S(I)$.

(ii) Suppose $x, y \in S(I)$, then $\mu_I(x) > 0$ and $\mu_I(y) > 0$. Since I is fuzzy ideal of (X, A) , then by Definition 2.1.1 (2), $\mu_I(x \vee y) > 0$. This implies that $x \vee y \in S(I)$. Therefore, from (i) and (ii) $S(I)$ is an ideal of $(X, S(A))$.

$(\Leftarrow)$ Suppose that $S(I)$ is an ideal of $(X, S(A))$ and let $x, y \in X$.

Claim2: I is a fuzzy ideal of (X, A) .

(i) If $\mu_I(y) > 0$ and $A(x, y) > 0$, then $y \in S(I)$ and $(x, y) \in S(A)$. Since $S(I)$ is an ideal of $(X, S(A))$ then by definition of classical ideal, $x \in S(I)$ and hence $\mu_I(x) > 0$.

(ii) Suppose $\mu_I(x) > 0$ and $\mu_I(y) > 0$, then $x \in S(I)$ and $y \in S(I)$. Since $S(I)$ is ideal of $(X, S(A))$, then by the definition of classical ideal $x \vee y \in S(I)$. Thus, $\mu_I(x \vee y) > 0$ and hence I is a fuzzy ideal of (X, A) . $\square$

Dually also we have the following proposition for the fuzzy filter.

Proposition 2.1.3. [9] *Let F be a fuzzy set on X . F is fuzzy filter of a fuzzy lattice (X, A) iff $S(F)$ is a filter of $(X, S(A))$.*

Proof. ($\Rightarrow$) Let F be a fuzzy filter of (X, A) .

Claim1: $S(F)$ is a filter of $(X, S(A))$.

(i) If $y \in S(F)$ and $(y, x) \in S(A)$, then $\mu_F(y) > 0$ and $A(y, x) > 0$. So by hypothesis F is a fuzzy filter of (X, A) , then by Definition 2.1.2 (1) $\mu_F(x) > 0$ and hence $x \vee y = x \in S(F)$.

(ii) Suppose $x, y \in S(F)$, then $\mu_F(x) > 0$ and $\mu_F(y) > 0$. Since F is fuzzy filter of (X, A) , then by Definition 2.1.2 (2), $\mu_F(x \wedge y) > 0$. This implies that $x \wedge y \in S(F)$.

Therefore, $S(F)$ is a filters of $(X, S(A))$.

($\Leftarrow$) Suppose that $S(F)$ is an ideal of $(X, S(A))$ and let $x, y \in X$.

Claim2: F is fuzzy filters of a fuzzy lattice (X, A) .

(i) If $\mu_F(y) > 0$ and $A(y, x) > 0$, then $y \in S(F)$ and $(y, x) \in S(A)$. Since $S(F)$ is a filter of $(X, S(A))$ then by the definition of classical filter, $x \in S(F)$ and hence $\mu_F(x) > 0$.

(ii) Suppose $\mu_F(x) > 0$ and $\mu_F(y) > 0$, then $x \in S(F)$ and $y \in S(F)$. Since $S(F)$ is ideal of $(X, S(A))$ then by the definition of classical filter $x \wedge y \in S(F)$. Thus, $\mu_F(x \wedge y) > 0$ and hence F is a fuzzy filters of (X, A) . $\square$

In [9], Mezzomo defined a subset $I_\alpha(F_\alpha)$ of a fuzzy set $I(F)$ on X , where $\alpha \in (0, 1]$ as follows:

Let A_α be the α – level set where $A_\alpha = \{(x, y) \in X \times X : A(x, y) \geq \alpha\}$ for all $\alpha \in (0, 1]$ and let $I_\alpha = \{x \in I : A(x, y) \geq \alpha \text{ for some } y \in I\}$ be an ideal of (X, A_α) , and $F_\alpha = \{x \in F : A(y, x) \geq \alpha \text{ for some } y \in F\}$ be a filter of (X, A_α) . As a results of these definition, Mezzomo in [9], state and prove the following propositions:

1. Let I be a fuzzy set on X . I is a fuzzy ideal of fuzzy lattice $\mathcal{L} = (X, A)$ iff for each $\alpha \in (0, 1]$, I_α is an ideal of (X, A_α) .
2. Let F be a fuzzy set on X . F is a fuzzy filter of fuzzy lattice $\mathcal{L} = (X, A)$ iff for each $\alpha \in (0, 1]$, F_α is a filter of (X, A_α) .

Now, if we look at the way Mezzomo [9] defined the notions of $I_\alpha(F_\alpha)$, and the statements of the above consequent propositions, there is a confusion between them. From

the notions of $I_\alpha(F_\alpha)$, $I_\alpha(F_\alpha)$ is defined in terms of a given fuzzy relation A , and it is guaranteed that $I_\alpha(F_\alpha)$ is not arbitrary subset rather a subset which is dependent on the degree of relation of an order pair $(x, y) \in I \times I$. On the other hand, when we look at the above two consequent propositions, the statement started by letting a fuzzy set $I(F)$ on X without expressing what kinds of a fuzzy set is it, and its relation with $I_\alpha(F_\alpha)$, and a given fuzzy relation A . In this sense, one may take any arbitrary fuzzy set $I(F)$, and generate $I_\alpha(F_\alpha)$ in terms of A . Thus, definitions of $I_\alpha(F_\alpha)$ given above is not clearly stated whether a fuzzy set $I(F)$ on X is defined by $I_\alpha(F_\alpha)$ in terms of a fuzzy relation A , or $I_\alpha(F_\alpha)$ is defined for any fuzzy set $I(F)$ in terms of a fuzzy relation A .

To make it more clear, let us consider two distinct cases according to the above Explanation.

Case I: Assume a fuzzy set $I(F)$ is defined by $I_\alpha(F_\alpha)$ in terms of a fuzzy relation A . In this case, $I(F)$ can not be any arbitrary fuzzy set on a set X rather a fuzzy set on X defined by $I_\alpha(F_\alpha)$ in terms of a fuzzy relation A , and it is clear that the degree of memberships of $x \in X$ in a fuzzy set $I(F)$ is depend on the degree of relations of (x, y) $((y, x)) \in X \times X$.

Now, let us consider the above propositions. If it is so, the statements of the proposition could be stated as follows.

1. Let I be a fuzzy set on X such that $I_\alpha = \{x \in I : A(x, y) \geq \alpha \text{ for some } y \in I\}$. I is a fuzzy ideal of fuzzy lattice $\mathcal{L} = (X, A)$ iff for each $\alpha \in (0, 1]$, I_α is an ideal of (X, A_α) .
2. Let F be a fuzzy set on X such that $F_\alpha = \{x \in F : A(y, x) \geq \alpha \text{ for some } y \in F\}$. F is a fuzzy filter of fuzzy lattice $\mathcal{L} = (X, A)$ iff for each $\alpha \in (0, 1]$, F_α is a filter of (X, A_α) .

Because of these statements, one can understand that the assumed fuzzy set $I(F)$ is generated by $I_\alpha(F_\alpha)$ in terms of a fuzzy relation A . Based on our assumption and the modified

statements of the two propositions, $I(F)$ is defined by $I_\alpha(F_\alpha)$, and for any $\alpha \in (0, 1]$, $A(x, y) (A(y, x)) \geq \alpha$ for some $y \in I(F)$. Here also, when we look at the definitions of $I_\alpha(F_\alpha)$, it says $y \in I(F)$ such that $A(x, y) \geq \alpha$. But $I(F)$ is a fuzzy set, and it does not contain an element of X , rather it contains an ordered pair in the form of $(x, \mu_I(x))((x, \mu_F(x)))$, where $x \in X$. Thus, definition of $I_\alpha(F_\alpha)$ is not clearly stated. Now, based on the Mezzomo's definition of I_α and F_α , we have the following corollaries.

Corollary 2.1.4. *Let (X, A) be a fuzzy lattice, and I be a fuzzy set on X such that $I_\alpha = \{x \in I : A(x, y) \geq \alpha \text{ for some } y \in I\}$, where $\alpha \in (0, 1]$. Then $I_\alpha = I$ for all $\alpha \in (0, 1]$.*

Proof. Let $\alpha \in (0, 1]$, and I be a fuzzy set such that $I_\alpha = \{x \in I : A(x, y) \geq \alpha \text{ for some } y \in I\}$, and x be an arbitrary element in I , then $A(x, x) = 1 \geq \alpha$;
 $\Rightarrow A(x, x) \geq \alpha$ this implies that $x \in I_\alpha$. Thus $I \subseteq I_\alpha$. And from the definition of I_α , we can understand that I_α is nonempty subset of I , and hence $I_\alpha = I$, for all $\alpha \in (0, 1]$. $\square$

Analogously, we have the following corollary.

Corollary 2.1.5. *Let (X, A) be a fuzzy lattice, and F be a fuzzy set on X such that $F_\alpha = \{x \in F : A(y, x) \geq \alpha \text{ for some } y \in F\}$, where $\alpha \in (0, 1]$. Then $F_\alpha = F$ for all $\alpha \in (0, 1]$.*

Proof. Let $\alpha \in (0, 1]$, and F be a fuzzy set such that $F_\alpha = \{x \in F : A(y, x) \geq \alpha \text{ for some } y \in F\}$, and x be an arbitrary element in F , then $A(x, x) = 1 \geq \alpha$;
 $\Rightarrow A(x, x) \geq \alpha$, this implies that $x \in F_\alpha$. Thus $F \subseteq F_\alpha$. And from the definition of F_α , we can understand that F_α is nonempty subset of F , and hence $F_\alpha = F$, for all $\alpha \in (0, 1]$. $\square$

From the above two corollary's, we proved that, $I_\alpha = I$ and $F_\alpha = F$, for all $\alpha \in (0, 1]$ respectively. Therefore, according to our first Case, the definitions of I_α and F_α are not necessary.

Thus, the significance of the definitions of I_α and F_α , and there consequent propositions are not clear.

CaseII: Assume $I(F)$ is any arbitrary fuzzy set on a set X , and its subset $I_\alpha(F_\alpha)$ is defined in terms of a fuzzy relation A , and a given fuzzy set $I(F)$.

In this case, we can take any arbitrary fuzzy set $I(F)$ on a set X , and we can define $I_\alpha(F_\alpha)$ in terms of a fuzzy relation A on X , and a fuzzy set $I(F)$. But according to our assumption, we have arbitrary fuzzy set $I(F)$ on X and a newly defined fuzzy set $I'(F')$ in terms of a fuzzy relation A . As we have seen here, the two fuzzy sets $I(F)$ and $I'(F')$ are independent of each other. As a result, based on Mezzomo's definition in [9], we may have two different degree of membership values for each elements x of X . One value in terms of the arbitrary fuzzy set $I(F)$ and second value in terms of the newly defined fuzzy set $I'(F')$ which is defined in terms of a fuzzy relation A . To make this more clear, let us justify it by Example.

Example 2.1.6. Let $X = \{a, b, c, d\}$ and let $A : X \times X \rightarrow [0, 1]$ be a fuzzy relation such that $A(a, a) = A(b, b) = A(c, c) = A(d, d) = 1.0$, $A(b, a) = A(c, a) = A(d, a) = A(c, b) = A(d, b) = A(d, c) = 0.0$, $A(c, d) = 0.3$, $A(b, d) = 0.5$, $A(a, d) = 0.8$, $A(b, c) = 0.2$, $A(a, c) = 0.4$ and $A(a, b) = 0.1$. One can easily show that (X, A) is a fuzzy lattice.

Now, let us consider a fuzzy set $I = \{(a, 0.7), (b, 0.4), (c, 0.2), (d, 0.0)\}$. As we have seen here, the degree of memberships of all elements of X in I is not expressed in terms of the degree of relations of all elements of $X \times X$ in A . If we define I_α based on the notions of I_α stated in [9], the degree of relation of an element is different from their degree of memberships in I . Hence we can generalize that, the notions of $I_\alpha(F_\alpha)$ stated above dose not holds true.

Without lose of generality, let us modify the definitions of $I_\alpha(F_\alpha)$ by taking element y from X such that $(y, \mu_I(y))((y, \mu_F(y))) \in I(F)$ as follows:

Definition 2.1.3. Let I be a fuzzy set on X , then α - level set of I denoted by I_α is defined as:

$$I_\alpha = \{x \in X : A(x, y) \geq \alpha \text{ for some } (y, \mu_I(y)) \in I\}$$

2.2 Proper Fuzzy Ideals and Proper Fuzzy Filters of a Fuzzy Lattice

In this section, we recall definitions of proper fuzzy ideals and proper fuzzy filters of a fuzzy lattice, from Mezzomo [9], which is closed under the fuzzy intersection and it lays the foundation for the next section.

Definition 2.2.1. ([9] Definition 6.2.4) Let $\mathcal{L} = (X, A)$ be a fuzzy lattice. A nonempty fuzzy set Z on X is called **proper fuzzy set** if $\mu_Z(x) = 0$ for at least one $x \in X$. A fuzzy set Z is called **improper** if $\mu_Z(x) \neq 0 \forall x \in X$. Fuzzy ideals (filters) Z of $\mathcal{L}$ are **proper (improper)** fuzzy ideals (filters) of $\mathcal{L}$, if Z is proper (improper) fuzzy set on X .

Example 2.2.1. Let $\mathcal{L} = (X, A)$ be the fuzzy lattice defined in the Example 1.2.6. Then the proper fuzzy set $I = \{(0, 0), (a, 0), (b, 0.2), (c, 0.5)\}$ is a proper fuzzy ideal of $\mathcal{L}$.

Proposition 2.2.2. ([9] Proposition 6.2.21) Let I be fuzzy ideals of a bounded fuzzy lattice (X, A) . Then I is a proper fuzzy ideal iff $\mu_I(\top) = 0$.

Proof. $(\Rightarrow)$ Let I be a proper fuzzy ideal of (X, A) .

Claim: $\mu_I(\top) = 0$.

By Definition 2.2.1, there exists at least one $x \in X$ such that $\mu_I(x) = 0$. Since (X, A) is a bounded fuzzy lattice, then by Definition 1.2.11 there exist a greatest element $\top \in X$ such that $A(x, \top) > 0$. Suppose $\mu_I(\top) > 0$. Then by the Definition 2.1.1 (1) $\mu_I(x) > 0$ which is contradiction to the hypothesis of $\mu_I(x) = 0$.

$\therefore \mu_I(\top) = 0$.

$(\Leftarrow)$ Let $\mu_I(\top) = 0$ implies that there exists at least one $\top \in X$ such that $\mu_I(\top) = 0$.

Thus, I is a proper fuzzy ideal. $\square$

Proposition 2.2.3. [9] Let F be a fuzzy filter of bounded fuzzy lattice (X, A) . Then F is proper fuzzy filter iff $\mu_F(\perp) = 0$.

Proof. $(\Rightarrow)$ Let F be proper fuzzy filter of bounded fuzzy lattice (X, A) .

Claim: $\mu_F(\perp) = 0$.

By Definition 2.2.1, there exist at least one $x \in X$ such that $\mu_F(x) = 0$. Since (X, A) is a bounded fuzzy lattice, then by Definition 1.2.11, there exists an element $\perp \in X$ such that $A(\perp, x) > 0$. Assume that $\mu_F(\perp) > 0$. Then by Definition 2.1.2 (1), we have $\mu_F(x) > 0$ which is contradiction to the hypothesis $\mu_F(x) = 0$. Thus, $\mu_F(\perp) = 0$.

($\Leftarrow$) Straightforward from the definition of proper fuzzy filters. $\square$

Notation: We denote the family of proper fuzzy ideals of a fuzzy lattice by $I_p(\mathcal{L})$ and family of proper fuzzy filters of a fuzzy lattice by $F_p(\mathcal{L})$.

Proposition 2.2.4. ([9] Proposition 6.2.23) Let $Z \subseteq I_p(\mathcal{L})$ and $\tilde{X} \in I(\mathcal{L})$. Then $\bigcup Z \neq \tilde{X}$.

Proof. Since $I_p(\mathcal{L})$ be a family of a proper fuzzy ideals then by Proposition 2.2.1, $\mu_{I_p}(\top) = 0$. Then $\mu_{\bigcup Z}(\top) = \sup\{\mu_{Z_j}(\top) : Z_j \in Z\} = \sup\{0\} = 0$. Therefore, $\bigcup Z \neq \tilde{X}$. $\square$

This proposition also holds for proper fuzzy filters of a fuzzy lattice.

Remark 2.2.1. ([9]) The union and intersections of a proper fuzzy ideals and a proper fuzzy filters of a fuzzy lattice are a proper fuzzy ideal and proper fuzzy filter of fuzzy lattice.

2.3 Prime Fuzzy Ideals and Prime Fuzzy Filters of a Fuzzy Lattice

In this section, we recall the definitions of a prime fuzzy ideals (filters) of a fuzzy lattice, which agrees with a prime ideals and prime filters of a lattice, and in addition, we investigate and prove some basic properties analogous to the properties of prime ideals(filters) of a lattice and we correct some proposition from Mezzomo's [9] result.

Definition 2.3.1. [9] A proper fuzzy ideal I of (X, A) is called **prime fuzzy ideal**:

if $\mu_I(x \wedge y) > 0$, then either $\mu_I(x) > 0$ or $\mu_I(y) > 0$, for all $x, y \in X$.

Definition 2.3.2. [9] A proper fuzzy filter F of (X, A) is called **prime fuzzy filter**:

if $\mu_F(x \vee y) > 0$, then either $\mu_F(x) > 0$ or $\mu_F(y) > 0$, for all $x, y \in X$.

Example 2.3.1. Let $X = \{v, x, y, z, w\}$ be a set and (X, A) be a fuzzy lattice such that $A(v, v) = A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1.0$, $A(v, x) = A(v, y) = A(v, z) = A(v, w) = A(x, y) = A(x, z) = A(x, w) = A(y, z) = A(z, y) = A(y, w) = A(z, w) = 0.0$, $A(x, v) = 0.2$, $A(y, v) = 0.4$, $A(z, v) = 0.7$, $A(w, v) = 0.9$, $A(y, x) = 0.3$, $A(z, x) = 0.5$, $A(w, x) = 0.8$, $A(w, y) = 0.4$ and $A(w, z) = 0.1$.

Consider the proper fuzzy ideal $I = \{(v, 0.0), (x, 0.3), (y, 0.7), (z, 0.6), (w, 1.0)\}$ on (X, A) and hence I is fuzzy ideal of (X, A) . But I is not a fuzzy filter since $\mu_I(x) = 0.3$ and $A(x, v) = 0.2 > 0$ but $\mu_I(v) = 0.0$. Then let us check is it prime fuzzy ideal or not.

$\mu_I(v \wedge x) = \mu_I(x) = 0.3 > 0$, $\mu_I(v) = 0.0$ and $\mu_I(x) = 0.3$;

$\mu_I(v \wedge y) = \mu_I(y) = 0.7 > 0$, $\mu_I(v) = 0.0$ and $\mu_I(y) = 0.7$;

$\mu_I(v \wedge z) = \mu_I(z) = 0.6 > 0$, $\mu_I(v) = 0.0$ and $\mu_I(z) = 0.6$;

$$\begin{aligned}
\mu_I(v \wedge w) &= \mu_I(w) = 1.0 > 0, \mu_I(v) = 0.0 \text{ and } \mu_I(w) = 1.0; \\
\mu_I(x \wedge y) &= \mu_I(y) = 0.7 > 0, \mu_I(x) = 0.3 \text{ and } \mu_I(y) = 0.7; \\
\mu_I(x \wedge z) &= \mu_I(z) = 0.6 > 0, \mu_I(x) = 0.3 \text{ and } \mu_I(z) = 0.6; \\
\mu_I(x \wedge w) &= \mu_I(w) = 1.0 > 0, \mu_I(x) = 0.3 \text{ and } \mu_I(z) = 1.0; \\
\mu_I(y \wedge z) &= \mu_I(w) = 1.0 > 0, \mu_I(y) = 0.7 \text{ and } \mu_I(z) = 0.6; \\
\mu_I(y \wedge w) &= \mu_I(w) = 1.0 > 0, \mu_I(y) = 0.7 \text{ and } \mu_I(w) = 1.0; \\
\mu_I(z \wedge w) &= \mu_I(w) = 1.0 > 0, \mu_I(z) = 0.6 \text{ and } \mu_I(w) = 1.0;
\end{aligned}$$

Therefore, I is a prime fuzzy ideal of (X, A) . Consequently, I is not a proper fuzzy filter because $\mu_I(w) = \mu_I(\perp) \neq 0$ and therefore, I is not a prime fuzzy filter.

Proposition 2.3.2. [9] *The intersection of a family of any prime fuzzy ideals are not prime fuzzy ideal.*

Proof. Let I and J be an arbitrary prime fuzzy ideals and $x, y \in X$ with $x \wedge y = w$. If $\mu_{I \cap J}(x \wedge y) = \mu_{I \cap J}(w) > 0$, then we do not conclude that either $\mu_{I \cap J}(x) > 0$ or $\mu_{I \cap J}(y) > 0$ since both of them may be equal to zero. In this case it falls the definition of a prime fuzzy ideals.

Let us clarify it by example. Let $X = \{x, y, z, w\}$ be a set and (X, A) be a fuzzy lattice such that $A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1.0$, $A(x, y) = A(x, z) = A(x, w) = 0.0$, $A(y, x) = 0.3$, $A(z, x) = 0.5$, $A(w, x) = 0.8$, $A(w, y) = 0.4$ and $A(w, z) = 0.2$. Consider an arbitrary prime fuzzy ideals $I = \{(x, 0.0), (y, 0.5), (z, 0.0), (w, 1.0)\}$ and $J = \{(x, 0.0), (y, 0.0), (z, 0.3), (w, 1.0)\}$ on (X, A) . The fuzzy intersection $I \cap J = \{(x, 0.0), (y, 0.0), (z, 0.0), (w, 1.0)\}$ is a fuzzy ideal. So $\mu_{I \cap J}(y \wedge z) = \mu_{I \cap J}(w) = 1.0$ but $\mu_{I \cap J}(y) = 0.0$ and $\mu_{I \cap J}(z) = 0.0$. Therefore, $I \cap J$ is not prime fuzzy ideal. Thus, I and J are an arbitrary so it holds for a family of prime fuzzy ideals. $\square$

Dually, we can show that the intersection of a prime fuzzy filters of a fuzzy lattice is not a prime fuzzy filter.

The following proposition was transcribed from Mezzomo's [9] Proposition 6.2.24, but the statement of the proposition and its proof is not agree since the statement of the proposition is not correct. Then the following proposition corrects it as:

Proposition 2.3.3. *Let I be a fuzzy set on X . I is a prime fuzzy ideal of a fuzzy lattice (X, A) iff $S(I)$ is a prime ideal of $(X, S(A))$.*

Proof. $(\Rightarrow)$ Let I be a prime fuzzy ideal of (X, A) and $x, y \in X$. Then we **claim** that, $S(I)$ is a prime ideal of $(X, S(A))$.

By hypothesis I is prime fuzzy ideal of (X, A) , then by Definition 2.3.1, if $\mu_I(x \wedge y) > 0$, then either $\mu_I(x) > 0$ or $\mu_I(y) > 0$, this implies that if $x \wedge y \in S(I)$, then either $x \in S(I)$ or $y \in S(I)$.

Therefore, $S(I)$ is a prime ideal of $(X, S(A))$.

$(\Leftarrow)$ suppose that $S(I)$ is a prime ideal of $(X, S(A))$.

Again we **claim** that I is a fuzzy prime ideal of (X, A) . Since by hypothesis $S(I)$ is a prime ideal then by definition of classical prime ideal we have if $x \wedge y \in S(I)$ then either $x \in S(I)$ or $y \in S(I)$, this implies that if $\mu_I(x \wedge y) > 0$, then either $\mu_I(x) > 0$ or $\mu_I(y) > 0$ from this we can conclude that I is a prime fuzzy ideal of (X, A) . $\square$

Dually in the same fashion we have the following proposition for the prime fuzzy filters of the fuzzy lattice.

Proposition 2.3.4. *Let F be a fuzzy set on X . F is a prime fuzzy filter of a fuzzy lattice (X, A) iff $S(F)$ is a prime filter of $(X, S(A))$.*

Proof. $(\Rightarrow)$ Let F be a prime fuzzy filter of (X, A) and $x, y \in X$. Then we **claim** that $S(F)$ is a prime filter of $(X, S(A))$.

By hypothesis F is prime fuzzy filter of (X, A) , then by Definition 2.3.2, if $\mu_F(x \vee y) > 0$, then either $\mu_F(x) > 0$ or $\mu_F(y) > 0$, this implies that if $x \vee y \in S(F)$, then either $x \in S(F)$ or $y \in S(F)$.

Therefore, $S(F)$ is a prime filter of $(X, S(A))$.

$(\Leftarrow)$ suppose that $S(F)$ is a prime filter of $(X, S(A))$. Then we **claim** that F is a prime fuzzy

filter of (X, A) . Since by hypothesis $S(F)$ is a prime filter, then by definition of classical prime filter we have, if $x \vee y \in S(F)$ then either $x \in S(F)$ or $y \in S(F)$, this implies that if $\mu_F(x \vee y) > 0$, then either $\mu_F(x) > 0$ or $\mu_F(y) > 0$ from this we can conclude that F is a prime fuzzy filter of (X, A) . $\square$

Conclusion

The definitions of fuzzy ideal and fuzzy filter of a fuzzy lattice is analogous to the definitions of ideal and filter of a fuzzy lattice respectively. In the definitions of ideals and filters of fuzzy lattice, the conditions of a given nonempty subset Y of X to be an ideal (filter) of a fuzzy lattice (X, A) are formulated based on the memberships of elements of X in Y . On the other hand, the definitions of fuzzy ideals and fuzzy filters of a fuzzy lattice, the conditions of a given fuzzy set I on X to be fuzzy ideal (filter) of fuzzy lattice (X, A) are formulated based on the degree of memberships of elements of X in a fuzzy set I rather than being a members of a given set only.

Fuzzy ideals and fuzzy filters of fuzzy lattice is characterized via its support set.

The union and intersections of proper fuzzy ideals and proper fuzzy filters of a fuzzy lattice are a proper fuzzy ideal and proper fuzzy filter of a fuzzy lattice.

In [9], Mezzomo defined a subset $I_\alpha(F_\alpha)$ of a fuzzy set I (F) on X , where $\alpha \in (0, 1]$ as:

- Let $I_\alpha = \{x \in I : A(x, y) \geq \alpha \text{ for some } y \in I\}$ be an ideal of (X, A_α) , and $F_\alpha = \{x \in F : A(y, x) \geq \alpha \text{ for some } y \in F\}$ be a filter of (X, A_α) for all $\alpha \in (0, 1]$.

From this one can understand that there is an element which is contained in I (F). But I (F) is a fuzzy set, and it does not contains a single element, rather it contains an ordered pair in the form of $(x, \mu_I(x))((x, \mu_F(x)))$, where $x \in X$. And also, if we take an arbitrary $x \in I$ we have, $A(x, x) = 1 \geq \alpha$ for all $\alpha \in (0, 1]$, this implies that $I \subseteq I_\alpha$, and from its definition, I_α is nonempty subset of I . From this we can conclude that $I = I_\alpha$.

By this result we conclude that, according to the Mezzomo's, definition, the definition of I_α is not necessary and also not correctly defined.

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