2019-09-19

A project on Generalized SOR Iterative Method for a Class of Complex Symmetric Linear System of Equations

Deresse, Anteneh

Wisdom at the source of the Blue Nile

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS

A project

on

Generalized SOR Iterative Method for a Class of Complex Symmetric Linear System of Equations

By

Deresse Anteneh

June 2019

Bahir Dar, Ethiopia

BAHIR DAR UNIVERSITY
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

Generalized SOR Iterative Method for a Class of Complex Symmetric Linear System of Equations

A project submitted to department of mathematics in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

By

Deresse Anteneh

Advisor: Tesfaye Kebede (PhD)

June 2019

Bahir Dar, Ethiopia

BAHIR DAR UNIVERSITY
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

Approval of the project

I hereby certify that I have supervised, read and evaluated this project entitled "Generalized SOR iterative method for a class of complex symmetric linear system of equations" prepared by Deresse Anteneh under my guidance.

__________________________ _____________ ______________

Advisor Name Signature Date

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS

We hereby certify that we have examined the project entitled "Generalized SOR iterative method for a class of complex symmetric linear system of equations" by Deresse Anteneh, is approved for the degree of Master of Science in Mathematics.

Board of Examiners

	Name	Signature	Date
External examiner:	____________________	___________	___________
Internal examiner I:	____________________	___________	___________
Internal examiner II:	____________________	___________	___________

Acknowledgment

First, I would like to give my heart-felt gratitude to GOD for his unreserved help. In a sense, everything happens by a permission of GOD. Thus, my first thank is to our creature.

Secondly, my special and heart-felt thanks goes to my advisor Dr. Tesfaye Kebede for his constructive comments, important ideas, suggestions and corrections that were most vital for the success of this project. As well as, he has advised me as a fatherhood to have a successful life on my profession for the future.

Thirdly, I don't know how to thank her; she pays a great sacrify to me in addition to financial donate. In short, she is a reason for my life that has its own compass. So that she is the right one and shows me the right way. Totally this is the outcome of her. I'm very grateful to my best Addisie Ferede. Long live dear Madam.

Fourth, thanks a lot to my families for their financial and moral support.

Fifth, I would like to appreciate Mr. Gashaye Dessalew who is a PhD student in mathematics, as a result, he is so busy but he is voluntary to share his golden time. Moreover, he directs and supports me from practical computation to moral motivation.

At the last but not the least, I would like to thank all my teachers, classmates and friends who offer their contribution on this project.

Notations

GSOR= Generalized Successive Over Relaxation

HSS = Hermitian and Skew-Hermitian Splitting

MHSS = Modified Hermitian and Skew-Hermitian Splitting

A^T = Transpose of Matrix A

A^* = Conjugate Transpose of Matrix A

ρ (G) =Spectral Radius of Iteration Matrix G

SPD = Symmetric Positive Definite

PD = Positive Definite

z^* = The conjugate transpose of column vector or row vector z

A^{-1} =The inverse of matrix A

$\overline{A}$=The conjugate of matrix A

$\overline{z}$ =The conjugate of a complex number z

λ=Eigen value

det=determinant

α_{opt} =the optimal value of relaxation parameter α

$\mathbb{C}$=the set of complex number

$\mathbb{R}$=the set of real number

Abstract

In this project, generalized SOR iterative method is used to solve a class of complex symmetric linear system of equations as well as the investigation of its convergence analysis and determination of its optimal iteration parameter. Finally, some numerical results ensure that the validation of the theoretical results and compare the effectiveness of the GSOR iterative method with HSS and MHSS iterative method.

This project has two chapters. In the first chapter, the introduction and preliminary concepts are explained. Whereas in the second chapter, HSS and MHSS iterative methods be over viewed points, the effectiveness of GSOR iterative method and convergence analysis could be studied.

Table of Content

Contents page

List of Tables

Table Page

CHAPTER ONE

Introduction and Preliminaries

1.1 Introduction

In general, we define a linear equation that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $x_i's$ are variables, $a_i's$ and b are constants and $a's$ are not all zero.

A finite set of linear equations is called a system of linear equations or a linear system which used to represent physical problems that involve the interaction of various properties. The variables in the system represent the properties being studied, and the equations described the interaction between the variables [4].

These systems can be solved using direct or iterative methods. Starting with an initial approximation to the solution of the problem, if the approximate solution is improved using some rule (formula) repeatedly then such a method is called an iterative method. It is more attractive since they are very effective, requires less memory and arithmetic operations than direct methods.

The first iterative methods were the Jacobi (1824) and later the Gauss-Seidel (1848). After about 100 years the popular successive overrelaxation (SOR) method was discovered. This method introduces a parameter whose role is to minimize the spectral radius, the largest in modulus Eigen value, of their iteration matrix. The main result from the convergence analysis of the SOR method was the determination of the optimal value of the parameter for which the spectral radius is minimal and hence the rate of convergence of the iterative method becomes maximum and better, by an order of magnitude, than the Gauss-Seidel method [2]. Hereafter some other iterative methods, like HSS, MHSS, GSOR, etc. are appeared. The system of linear equations has of the form

$Az = b$, where $A \in \mathbb{C}^{n\times n}$; $z, b \in \mathbb{C}^n$ and A is a complex symmetric matrix of the form

$A = W + iT$, where i is imaginary unit and W, T $\in \mathbb{R}^{n\times n}$ are symmetric matrices with one of them being positive definite [5, 10, 11].

Then such system of linear equations can be solved by generalized successive over relaxation iterative method. As well as the HSS and MHSS iterative methods are also considered beside of that GSOR method as a comparison of it.

1.2 Preliminaries

1.2.1 Matrix

Definition1.1:- A matrix is a rectangular array of numbers or objects that we will enclose by a part of brackets [ ] or ( ) subject to the certain rules of operations. It is set of mn numbers arranged in a rectangular array of m rows and n columns. The matrix is said to be complex if the entries are in the set of complex numbers.

In general, the $m\times n$ matrix is usually written as

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

or A= (a_{ij}) ; where i=1: m and j=1: n the entries (elements) of the matrix A and $a_{11}, a_{22} \ldots a_{mn}$ are called the diagonal entries of the matrix A. If m= n, then the matrix is called a square matrix of order n.

1.2.2 Transpose of Matrix

Definition1.2:- Given a matrix A, the transpose of A denoted by A^T, is the matrix whose rows are columns of A (and whose columns are rows of A). That is, the entry a_{ij} becomes a_{ji}.

or if $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ then $A^T=\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$

1.2.3 Symmetric Matrix

Definition1.3:- A square matrix A is said to be symmetric if it is symmetric about its leading diagonal (top left to bottom right). That is $A = A^T$.

1.2.4 The Conjugate of Complex Number

Definition1.4:-The complex conjugate (or conjugate) of a complex number, $Z = x \pm iy$ denoted by $\overline{Z}$, is given by $\overline{Z} = x \mp iy$, where x, y $\in \mathbb{R}$.

1.2.5 Conjugate of Complex Matrix

Definition1.5:- The conjugate of a complex matrix A is denoted by $\overline{A}$, is a matrix obtained by taking the conjugate of each entry in A.

1.2.6 Conjugate Transpose of Matrix

Definition1.6:-The conjugate transpose of a matrix A denoted by A^* , is$A^* = (\bar{a}_{ji})$, where$\bar{a}$ is the complex conjugate of$a$. Thus if A is m by n then the conjugate transposeA^* is n by m with i, j element equal to the complex conjugate of the j, i element of A. Moreover, m=n then A is square complex matrix.

1.2.7 Hermitian and Skew- Hermitian Matrix

Definition 1.7:- A square matrix A is said to be Hermitian (or self-adjoint) if it is equal to its own conjugate transpose, i.e. $A = A^*$ .Whereas, if $A = -A^*$ then A is Skew- Hermitian matrix.

Example 1.1 a) $\begin{pmatrix} 3 & 1-2i & 4+7i \\ 1+2i & -4 & -2i \\ 4-7i & 2i & 0 \end{pmatrix}$ is Hermitian matrix.

b) $\begin{pmatrix} 4i & \sqrt{2}i & -1+i \\ \sqrt{2}i & 0 & -2-i \\ 1+i & 2-i & i \end{pmatrix}$ is skew Hermitian matrix.

1.2.8 Positive Definite Matrix

Definition1.8:-A matrix A is said to be positive definite (PD) if $z^* Az > 0$ or $zAz^* > 0$ for all $0 \neq z \in \mathbb{C}^{n\times 1}$ and $z \in \mathbb{C}^{1\times n}$respectively.

As well as it is said to be semi-positive definite if $z^*Az \geq 0$ or $zAz^* \geq 0$, wherez^* stands for the conjugate transpose of z.

Example 1.2 a) $\begin{pmatrix} 100 & 1-2i & 3+5i \\ 1+2i & 100 & -9+i \\ 3-5i & -9-i & 100 \end{pmatrix}$ is positive definite matrix.

b) $\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$ is positive semi definite matrix.

1.2.9 Invertible Matrix

Definition1.9:- A square matrix is said to be invertible if there exists a matrix A^{-1} such that $A^{-1}A$=I=AA^{-1}, where I is an identity matrix.

1.2.10 Eigen value of the matrix

Definition 1.10:- If A is an n×n matrix, then the polynomial p defined by $p(\lambda) = det(A - \lambda I)$ is called the characteristic polynomial of A and the zeros of p are called eigenvalues of A, where I is an identity matrix.

1.2.11 Spectral Radius of the Iteration Matrix

Definition 1.11:- Let $\lambda_1, \lambda_{2,}, \ldots \lambda_n$ be the (real or complex) Eigen values of iteration matrix

$G \in \mathbb{C}^{n\times n}$ then its spectral radius denoted by $\rho(G)$ is defined as $\rho(G)=\max|\lambda_i|$, where i=1, 2…n.

CHAPTER TWO

Solving a Class of Complex Symmetric Linear System of Equations by Using Generalized SOR Iterative Method

2.1 HSS Iterative Method

The Hermitian and skew-Hermitian Splitting (HSS) iteration method was first introduced by Bai, Golub and Ng for the solution of a broad class of non-Hermitian linear systems Az = b [10]. To solve the systems of linear equations of the form

$$Az = b \;; \quad A \in \mathbb{C}^{n\times n} \text{and } z, b \in \mathbb{C}^{n} \tag{2.1}$$

and, A is a complex symmetric matrix of the form

$$A = W + iT \tag{2.2}$$

where the imaginary unit $i = \sqrt{-1}$ and $W, T \in \mathbb{R}^{n\times n}$are real symmetric positive semi-definite matrices such that at least one of them being positive definite and without loss of generality, we assume that W is positive definite (PD). The Hermitian and skew-Hermitian splitting (HSS) method to solve the system of linear equations with the complex symmetric coefficient matrix is known that the matrix A possesses the Hermitian/skew-Hermitian (HS) splitting as follows

$$A = H + S \tag{2.3}$$

where $H = \frac{1}{2}\left(A + A^*\right) = W$ and $S = \frac{1}{2}\left(A - A^*\right) = iT$

where A^* stands for the conjugate transpose of A.Thus, when we add an additive identity $(\alpha I - \alpha I)$at the right side from equation (2.2) and substitute in equation (2.1) then we obtain

$$(W + iT + \alpha I - \alpha I)z = b$$

$$(W + \alpha I + iT - \alpha I)z = b$$

$$(W + \alpha I)z + (iT - \alpha I)z = b$$

$$(\alpha I + W)z = (\alpha I - iT)z + b \tag{2.4}$$

and also$(W + iT + \alpha I - \alpha I)z = b$

$$(W - \alpha I + iT + \alpha I)z = b$$

$$(W - \alpha I)z + (iT + \alpha I)z = b$$

$$(\alpha I + iT)z = (\alpha I - W)z + b \qquad (2.5)$$

From equation (2.4) and (2.5) with a systematic iteration index, we get the two sub-systems, and given as follows.

$$\begin{cases}(\alpha I + W)z^{(k+\frac{1}{2})} = (\alpha I - iT)z^{(k)} + b \\ (\alpha I + iT)z^{(k+1)} = (\alpha I - W)z^{(k+\frac{1}{2})} + b\end{cases} \qquad (2.6)$$

$$z^{(k+\frac{1}{2})} = (\alpha I + W)^{-1}(\alpha I - iT)z^{(k)} + (\alpha I + W)^{-1}b$$

$$z^{(k+1)} = (\alpha I + iT)^{-1}(\alpha I - W)z^{(k+\frac{1}{2})} + (\alpha I + iT)^{-1}b$$

$$z^{(k+1)} = (\alpha I + iT)^{-1}(\alpha I - W)\left[(\alpha I + W)^{-1}(\alpha I - iT)z^{(k)} + (\alpha I + W)^{-1}b\right] + (\alpha I + iT)^{-1}b \qquad (2.7)$$

$z^{(k+1)} = G_H z^{(k)} + C$ which is called HSS iterative scheme.

Where G_H= $(\alpha I + iT)^{-1}(\alpha I - W)(\alpha I + W)^{-1}(\alpha I - iT)$

C=$(\alpha I + iT)^{-1}(\alpha I - W)(\alpha I + W)^{-1}b$+$(\alpha I + iT)^{-1}b$

α is positive constant,$z^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess, I is an identity matrix and k = 0, 1, 2... until the sequence of iterations $\{z^{(k)}\}$ converges, compute the next iteration $z^{(k+1)}$ according to the this iterative scheme.

Since W $\in \mathbb{R}^{n\times n}$ issymmetricpositivedefinite and T is positive semi definite, the HSS iteration method is convergent for $\alpha > 0$ [10]. That is the sequence $\{z^{(k)}\}$ converges to the solution $\tilde{z}$ = A^{-1}b as k →∞ for any choice $z^{(0)}$. Of course, at each step of the HSS iteration we need to solve two linear sub-systems with their coefficient matrices being the symmetric positive definite one $\alpha I + W$ and the shifted skew-Hermitian one $\alpha I + iT$, respectively. The matrix $\alpha I + W$ can be treated in real arithmetic so the iteration $z^{(k+\frac{1}{2})}$ in the first-half step may be computed.

However, the matrix $\alpha I + iT$ is complex and non-Hermitian, although it is positive definite and symmetric. These necessities the use of complex arithmetic, and the iteration $z^{(k+1)}$ in the second-half step be computed.

A potential difficulty with the HSS iteration approach is the need to solve the shifted skew-Hermitian sub-system of linear equations at each iteration step. Thus, it should be modified.

A considerable advantage of the modified HSS (MHSS) iteration method consists in the fact that the solution of linear system with coefficient matrix $\alpha I + iT$ is avoided and only two linear sub-systems with coefficient matrices $\alpha I + W$ and $\alpha I + T$ both being real and symmetric positive definite need to be solved at each step [11]. Hence the following MHSS iterative method simplifies this matter of HSS.

2.2 MHSS Iterative Method

By making use of the special structure of the coefficient matrix $A \in \mathbb{C}^{n\times n}$of the complex symmetric linear system$(2.1-2.2)$, we derive a modification of the HSS iteration method that was initially proposed in [10].Thus, the complex symmetric linear system $(2.1-2.2)$ be rewritten into the system of fixed-point equations

$$(\alpha I + W)z = (\alpha I - iT)z + b \tag{2.8}$$

whereα is positive parameter and I is an identity matrix which has the same order of A.

Now the complex symmetric linear system (2.1–2.2) is also equivalent to

$$-iAz = -ib$$

$$-i(W + iT)z = -ib$$

$$(T - iW)z = -ib$$

$$(\alpha I + T - iW - \alpha I)z = -ib$$

$$(\alpha I + T)z - (iW + \alpha I)z = -ib$$

$$(\alpha I + T)z = (\alpha I + iW)z - ib \tag{2.9}$$

By alternate iterating between the two systems of equations (2.8) and (2.9), we obtain the following iterative method which is modified HSS iteration method for solving the complex symmetric linear system (2.1–2.2) in an analogous fashion to the HSS iteration scheme in [10].

Let $z^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For k $= 0,1,2,\ldots$ until the sequence of iteration $\{z^{(k)}\}$ converges, compute the next iteration $z^{(k+1)}$ according to the following procedure:

$$\begin{cases}(\alpha I + W)z^{(k+\frac{1}{2})} = (\alpha I - iT)z^{(k)} + b \\ (\alpha I + T)z^{(k+1)} = (\alpha I + iW)z^{(k+\frac{1}{2})} - ib\end{cases} \tag{2.10}$$

$$z^{(k+\frac{1}{2})} = (\alpha I + W)^{-1}(\alpha I - iT)z^{(k)} + (\alpha I + W)^{-1}b$$

$$z^{(k+1)} = (\alpha I + T)^{-1}(\alpha I + iW)z^{(k+\frac{1}{2})} - (\alpha I + T)^{-1}b$$

$$z^{(k+1)} = (\alpha I + T)^{-1}(\alpha I + iW)\left[(\alpha I + W)^{-1}(\alpha I - iT)z^{(k)} + (\alpha I + W)^{-1}b\right] - (\alpha I + T)^{-1}b \tag{2.11}$$

$z^{(k+1)} = G_{MH}z^{(k)} + C$ whichis called MHSS iterative method

where $G_{MH}=(\alpha I + T)^{-1}(\alpha I + iW)(\alpha I + W)^{-1}(\alpha I - iT)$

C$=(\alpha I + T)^{-1}(\alpha I + iW)(\alpha I + W)^{-1}b - (\alpha I + T)^{-1}b$

α is positive constant and I is an identity matrix. Each iteration of the MHSS iteration alternates between the two symmetric matrices W and T. As W $\in \mathbb{R}^{n\times n}$ is symmetric positive definite andT$\in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite, we see that matrices $\alpha I + W$ and $\alpha I + T$ are symmetric positive definite. Hence, the two linear sub-systems involved in each step of the MHSS iteration can be solved. The use of the above modified Hermitian and skew-Hermitian splitting (MHSS) method is more efficient than the HSS iteration method for solving the complex symmetric linear system (2.1) [7, 11]. The computation of the iteration $z^{(k+\frac{1}{2})}$ requires only real arithmetic, whereas the computation of $z^{(k+1)}$ requires a modest amount of complex arithmetic due to the fact that the right-hand side in the corresponding system is complex.

It is necessary to mention that a potential difficulty with the HSS and MHSS iteration methods is the need to use complex arithmetic [1, 8]. Moreover, Axelsson et al. [6] have presented a comparison of iterative methods to solve the complex symmetric linear system of equation (2.1).

Rather than solving the original complex linear system (2.1), Salkuyeh et al. [3, 9] solved the real equivalent system by the generalized successive overrelaxation (GSOR) iterative method.

By some numerical experiments, they have shown that the performance of the GSOR method is much better than the MHSS method.

2.3 Generalized SOR Iterative Method

Let$z = x + iy$ and $b = f + ig$ where $x, y, f, g \in \mathbb{R}^n$. Thus, the complex linear system (2.1) can be written as 2-by-2 block real equivalent formulation as follows.

$$Az = b$$

$$(W + iT)(x + iy) = f + ig$$

$$(Wx + iWy + iTx - Ty) = f + ig$$

$$(Wx - Ty) + i(Tx + Wy) = f + ig$$

$$\begin{cases} Wx - Ty = f \\ Tx + Wy = g \end{cases}$$

$$\begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

We split the coefficient matrix A as

$$A = D - L - U$$

$where\ D = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix},\ L = \begin{pmatrix} 0 & 0 \\ -T & 0 \end{pmatrix}\ and\ U = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$

Then the equation $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$ (2.12)

$$(D - L - U)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\alpha(D - L - U)\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\begin{pmatrix} f \\ g \end{pmatrix}$$

$$D\begin{pmatrix} x \\ y \end{pmatrix} + \alpha(D-L-U)\begin{pmatrix} x \\ y \end{pmatrix} = D\begin{pmatrix} x \\ y \end{pmatrix} + \alpha\begin{pmatrix} f \\ g \end{pmatrix}$$

$$D\begin{pmatrix} x \\ y \end{pmatrix} - \alpha L\begin{pmatrix} x \\ y \end{pmatrix} + \alpha(D-U)\begin{pmatrix} x \\ y \end{pmatrix} = D\begin{pmatrix} x \\ y \end{pmatrix} + \alpha\begin{pmatrix} f \\ g \end{pmatrix}$$

$$D\begin{pmatrix} x \\ y \end{pmatrix} - \alpha L\begin{pmatrix} x \\ y \end{pmatrix} = -\alpha(D-U)\begin{pmatrix} x \\ y \end{pmatrix} + D\begin{pmatrix} x \\ y \end{pmatrix} + \alpha\begin{pmatrix} f \\ g \end{pmatrix}$$

$$(D-\alpha L)\begin{pmatrix} x \\ y \end{pmatrix} = [D-\alpha(D-U)]\begin{pmatrix} x \\ y \end{pmatrix} + \alpha\begin{pmatrix} f \\ g \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = (D-\alpha L)^{-1}[D-\alpha(D-U)]\begin{pmatrix} x \\ y \end{pmatrix} + \alpha(D-\alpha L)^{-1}\begin{pmatrix} f \\ g \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = (D-\alpha L)^{-1}[(1-\alpha)D+\alpha U]\begin{pmatrix} x \\ y \end{pmatrix} + \alpha(D-\alpha L)^{-1}\begin{pmatrix} f \\ g \end{pmatrix}$$

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = (D-\alpha L)^{-1}[(1-\alpha)D+\alpha U]\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \alpha(D-\alpha L)^{-1}\begin{pmatrix} f \\ g \end{pmatrix}$$

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = G\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + C \qquad (2.13)$$

This is called the GSOR iterative method.

where G= $(D-\alpha L)^{-1}[(1-\alpha)D+\alpha U]$ is called the iteration matrix of GSOR method and

$$C=\alpha(D-\alpha L)^{-1}\begin{pmatrix} f \\ g \end{pmatrix}$$

As it has been seen that from equation (2.13) the iteration matrix can be written as follows:

$$G=\begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix}^{-1}\left[(1-\alpha)\begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} + \alpha\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}\right]$$

$$G=\begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix}^{-1}\begin{pmatrix} (1-\alpha)W & \alpha T \\ 0 & (1-\alpha)W \end{pmatrix}$$

Then equation (2.13) can be written as follows:

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix}^{-1} \begin{pmatrix} (1-\alpha)W & \alpha T \\ 0 & (1-\alpha)W \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \alpha \begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{f} \\ \mathrm{g} \end{pmatrix}$$

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix}^{-1} \begin{pmatrix} (1-\alpha)W & \alpha T \\ 0 & (1-\alpha)W \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix}^{-1} \begin{pmatrix} \alpha\mathrm{f} \\ \alpha\mathrm{g} \end{pmatrix}$$

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix}^{-1} \left[\begin{pmatrix} (1-\alpha)W & \alpha T \\ 0 & (1-\alpha)W \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} \alpha\mathrm{f} \\ \alpha\mathrm{g} \end{pmatrix} \right]$$

$$\begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} (1-\alpha)W & \alpha T \\ 0 & (1-\alpha)W \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} \alpha\mathrm{f} \\ \alpha\mathrm{g} \end{pmatrix}$$

$$\begin{pmatrix} Wx^{(k+1)} \\ \alpha Tx^{(k+1)} + Wy^{(k+1)} \end{pmatrix} = \begin{pmatrix} (1-\alpha)Wx^{(k)} + \alpha Ty^{(k)} \\ (1-\alpha)Wy^{(k)} \end{pmatrix} + \begin{pmatrix} \alpha\mathrm{f} \\ \alpha\mathrm{g} \end{pmatrix}$$

$$\begin{pmatrix} Wx^{(k+1)} \\ \alpha Tx^{(k+1)} + Wy^{(k+1)} \end{pmatrix} = \begin{pmatrix} (1-\alpha)Wx^{(k)} + \alpha Ty^{(k)} + \alpha\mathrm{f} \\ (1-\alpha)Wy^{(k)} + \alpha\mathrm{g} \end{pmatrix}$$

$$\begin{cases} Wx^{(k+1)} = (1-\alpha)Wx^{(k)} + \alpha Ty^{(k)} + \alpha f \\ \alpha Tx^{(k+1)} + Wy^{(k+1)} = (1-\alpha)Wy^{(k)} + \alpha\mathrm{g} \end{cases}$$

Thus equation (2.13) is equivalent to the following iterative scheme.

$$\begin{cases} Wx^{(k+1)} = (1-\alpha)Wx^{(k)} + \alpha Ty^{(k)} + \alpha f \\ Wy^{(k+1)} = -\alpha Tx^{(k+1)} + (1-\alpha)Wy^{(k)} + \alpha g \end{cases} \quad (2.14)$$

where$x^{(0)}$ and $y^{(0)}$are initial approximations for x and y respectively. As we mentioned the iterative method (2.14) is real valued and the coefficient matrix of both of the sub-systems is W.

2.3.1. Convergence Property and Optimum parameter of GSOR Method

A sequence of GSOR iteration for solving (2.12) is given as:

$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = (D-\alpha L)^{-1}[(1-\alpha)D + \alpha U] \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \alpha(D-\alpha L)^{-1} \begin{pmatrix} f \\ g \end{pmatrix}$is said to be convergent if the sequence approaches to the solution$\tilde{z} = A^{-1}b$with in the required accuracy or the difference of two successive iterations is less than the tolerance error when the tolerance error is well known. Moreover, the spectral radius of the iteration matrix,$(D-\alpha L)^{-1}[(1-\alpha)D + \alpha U]$ is less than one then the GSOR iterative scheme converges.

The following theorems (theorem 2.1 and theorem 2.2) are stated for convergence property and optimum parameter of GSOR method respectively.

Theorem 2.1:- Let W, T$\in\mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric respectively. then the GSOR method to solve equation (2.12) is convergent if and only if

$0<\alpha<\frac{2}{1+\rho(S)}$ where $\rho(S)$ is the spectral radius of $S=W^{-1}T$.

Proof: -Let $\lambda\neq 0$ be an Eigen value of G corresponding to the eigenvector V $=\begin{pmatrix}v_1\\v_2\end{pmatrix}$. Note that for $\lambda = 0$ there is nothing to investigate. Then we have

$$GV=\lambda V \qquad (2.15)$$

$$\begin{pmatrix}W & 0\\ \alpha T & W\end{pmatrix}^{-1}\begin{pmatrix}(1-\alpha)W & \alpha T\\ 0 & (1-\alpha)W\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\lambda\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

$$\begin{pmatrix}(1-\alpha)W & \alpha T\\ 0 & (1-\alpha)W\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\lambda\begin{pmatrix}W & 0\\ \alpha T & W\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

$$\begin{pmatrix}(1-\alpha)I & \alpha W^{-1}T\\ 0 & (1-\alpha)I\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\lambda\begin{pmatrix}I & 0\\ \alpha W^{-1}T & I\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

$$\begin{pmatrix}(1-\alpha)I & \alpha S\\ 0 & (1-\alpha)I\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\lambda\begin{pmatrix}I & 0\\ \alpha S & I\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

$$\begin{pmatrix}(1-\alpha)v_1+\alpha Sv_2\\ (1-\alpha)v_2\end{pmatrix}=\begin{pmatrix}\lambda v_1\\ \lambda\alpha Sv_1+\lambda v_2\end{pmatrix}$$

$$\begin{cases}(1-\alpha)v_1+\alpha Sv_2=\lambda v_1\\ (1-\alpha)v_2=\lambda\alpha Sv_1+\lambda v_2\end{cases}$$

$$\begin{cases}\alpha Sv_2=\lambda v_1-(1-\alpha)v_1\\ (1-\alpha)v_2-\lambda v_2=\lambda\alpha Sv_1\end{cases}$$

$$\begin{cases}\alpha Sv_2=(\lambda+\alpha-1)v_1\\ -(\lambda+\alpha-1)v_2=\lambda\alpha Sv_1\end{cases} \quad (2.16)$$

If $\lambda=1-\alpha$, for convergence of the GSOR method we must have

$$|\lambda|<1$$

$$|1-\alpha|<1$$

$$|\alpha - 1| < 1$$

$$-1 < \alpha - 1 < 1$$

$$0 < \alpha < 2 \qquad (2.17)$$

If $\lambda \neq 1-\alpha$, from the first equation in equation (2.16), we have:

$$v_1 = \frac{1}{\alpha+\lambda-1}\alpha S v_2 \qquad (2.18)$$

Now, substitute equation (2.18) into the second equation of (2.16), we obtain:

$$-(\lambda + \alpha - 1)v_2 = \lambda\alpha S(\alpha S v_2)\frac{1}{\lambda+\alpha-1}$$

$$(\lambda + \alpha - 1)^2 v_2 = -\lambda\alpha^2 S^2 v_2$$

$$(1 - \lambda - \alpha)^2 v_2 = -\lambda\alpha^2 S^2 v_2$$

This shows that for every Eigenvalue $\lambda \neq 0$ of G, there is an Eigen value μ of S that satisfies

$$(1 - \lambda - \alpha)^2 = -\lambda\alpha^2\mu^2 \qquad (2.19)$$

$$(1 - \lambda - \alpha)^2 + \lambda\alpha^2\mu^2 = 0$$

$$\lambda^2 - 2\lambda + 2\lambda\alpha + \lambda\alpha^2\mu^2 + \alpha^2 - 2\alpha + 1 = 0$$

$$\lambda^2 + (-2\lambda + 2\lambda\alpha + \lambda\alpha^2\mu^2) + (\alpha^2 - 2\alpha + 1) = 0$$

$$\lambda^2 + (\alpha^2\mu^2 + 2\alpha - 2)\lambda + (\alpha - 1)^2 = 0 \qquad (2.20)$$

We know that the roots of the real quadratic equation $x^2 - bx + c = 0$ are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$. Thus, from E.q (2.20) $|\lambda| < 1$ if and only if

$$\begin{cases} |\alpha - 1|^2 < 1\,, \\ |\alpha^2\mu^2 + 2\alpha - 2| < 1 + (\alpha - 1)^2 \end{cases} \qquad (2.21)$$

$$\begin{cases} |\alpha - 1| < 1\,, \\ |\alpha^2\mu^2 + 2\alpha - 2| < \alpha^2 - 2\alpha + 2 \end{cases}$$

$$\begin{cases} |\alpha - 1| < 1\,, \\ -\alpha^2 + 2\alpha - 2 < \alpha^2\mu^2 + 2\alpha - 2 < \alpha^2 - 2\alpha + 2 \end{cases}$$

$$\begin{cases} |\alpha - 1| < 1\,, \\ -\alpha^2 < \alpha^2\mu^2 < \alpha^2 - 4\alpha + 4 \end{cases}$$

$$\begin{cases} |\alpha - 1| < 1\,, \\ \alpha^2\mu^2 < \alpha^2 - 4\alpha + 4 \end{cases}$$

$$\begin{cases} |\alpha - 1| < 1\,, \\ \alpha^2\mu^2 < (\alpha - 2)^2 \end{cases} \tag{2.22}$$

The first equation in (2.22) is equal to (2.17) and the second equation in (2.22) is equal to

$$(\alpha - 2)^2 > \alpha^2\mu^2$$

$$(\alpha - 2)^2 > \alpha^2\rho(S)^2$$

$$|\alpha - 2| > \alpha\rho(S)$$

$$\alpha - 2 < -\alpha\rho(S) \text{ or } \alpha - 2 > \alpha\rho(S)$$

We consider the inequality:

$$\alpha - 2 < -\alpha\rho(S)$$

$$\alpha + \alpha\rho(S) < 2$$

$$\alpha(1 + \rho(S)) < 2$$

$$\alpha < \frac{2}{1+\rho(S)} \tag{2.23}$$

From (2.17) we have $\alpha > 0$ thus, equation (2.23) can be written as follows

$$0 < \alpha < \frac{2}{1+\rho(S)}$$ this completes the proof.

Theorem2.2:- Let W,T $\in\mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric respectively. then the optimal value of the relaxation parameter for the GSOR iterative method (2.14) is given by

$\alpha_{opt} = \frac{2}{1+\sqrt{1+(\rho(S))^2}}$,where $S = W^{-1}T$ and $\rho(S)$ is the spectral radius of S.

Proof: In the proof of Theorem 2.1, we have seen that for every Eigen value $\lambda \neq 0$ of G, there is an Eigenvalue μ of S that satisfies (2.19). When we use the roots of this quadratic equation to determine the optimal parameter α_{opt}.

$$i.e. \qquad (1-\lambda-\alpha)^2 = -\lambda\alpha^2\mu^2$$

$$(1-\lambda-\alpha)^2 + \lambda\alpha^2\mu^2 = 0$$

$$\lambda^2 - 2\lambda + 2\lambda\alpha + \lambda\alpha^2\mu^2 + \alpha^2 - 2\alpha + 1 = 0$$

$$\lambda^2 + (-2\lambda + 2\lambda\alpha + \lambda\alpha^2\mu^2) + (\alpha^2 - 2\alpha + 1) = 0$$

$$\lambda^2 + (\alpha^2\mu^2 + 2\alpha - 2)\lambda + (\alpha-1)^2 = 0$$

$$\lambda_{1,2} = \frac{-(\alpha^2\mu^2 + 2\alpha - 2) \pm \sqrt{(\alpha^2\mu^2 + 2\alpha - 2)^2 - 4(\alpha-1)^2}}{2}$$

$$\lambda_{1,2} = \frac{-(\alpha^2\mu^2 + 2\alpha - 2) \pm \sqrt{\alpha^2\mu^2(\alpha^2\mu^2 + 4\alpha - 4)}}{2}$$

We can write the equation, $(1-\lambda-\alpha)^2 = -\lambda\alpha^2\mu^2$ as

$$(\lambda + \alpha - 1)/\alpha = \pm\mu\sqrt{-\lambda}$$

Let us define the following functions

$$f(\lambda) = \frac{\lambda+\alpha-1}{\alpha} \qquad \text{and} \quad g(\lambda) = \pm\mu\sqrt{-\lambda}$$

Then $f(\lambda)$ is a straight line through the point (1,1), whose slope increases monotonically with decreasing α. Thus, the largest abscissa of the two points of intersection decreases until $f(\lambda)$ becomes tangent to $g(\lambda)$. In this case, we have $\lambda_1 = \lambda_2$. As a result, from the above we get

$$\alpha^2\mu^2(\alpha^2\mu^2 + 4\alpha - 4) = 0$$

$$\alpha^2\mu^2 = 0 \ \text{ or } \ \alpha^2\mu^2 + 4\alpha - 4 = 0$$

$$\mu = 0 \ \text{ or } \alpha^2\mu^2 + 4\alpha - 4 = 0$$

If $\mu \neq 0$ then $\alpha^2\mu^2 + 4\alpha - 4 = 0$

$$\alpha = \frac{-4\pm\sqrt{16+16\mu^2}}{2\mu^2} = \frac{-4\pm\sqrt{16(1+\mu^2)}}{2\mu^2} = \frac{-2\pm2\sqrt{1+\mu^2}}{\mu^2}$$

Hence the nonnegative root of $\alpha^2\mu^2 + 4\alpha - 4 = 0$ is equal to

$$\alpha = (-2+2\sqrt{1+\mu^2})/\mu^2$$

Now, multiply $\frac{-2+2\sqrt{1+\mu^2}}{\mu^2}$ by $\frac{-2-2\sqrt{1+\mu^2}}{-2-2\sqrt{1+\mu^2}}$ then we obtain

$\alpha_{opt} = \frac{2}{1+\sqrt{1+\mu^2}}$ is the optimum parameter.

2.4 Numerical Examples

Example 2.1: Solve the following linear system of equation using:

I. HSS method.

II. MHSS method.

III. GSOR method

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2+3i & 1 \\ 0 & 1 & 3+i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 3+4i \\ 3i \end{pmatrix}$$

Solution: The coefficient matrix A $=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2+3i & 1 \\ 0 & 1 & 3+i \end{pmatrix}$ is symmetric. Now we can find W, T&I

. Thus, W=$\frac{1}{2}(A + A^*)$⟹W= $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$,

iT=$\frac{1}{2}(A - A^*)$ ⟹iT= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3i & 0 \\ 0 & 0 & i \end{pmatrix}$ ⟹T= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and I= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

I Apply the HSS iterative method

$$z^{(k+1)} = (\alpha I + iT)^{-1}(\alpha I - W)(\alpha I + W)^{-1}(\alpha I - iT)z^{(K)} + (\alpha I + iT)^{-1}(\alpha I - W)$$

$$(\alpha I + W)^{-1}b + (\alpha I + iT)^{-1}\mathrm{b}$$

where the initial guess$z^{(0)}$= $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\alpha = 1.7$

Method	Spectral radius
HSS	0.38

Thus, the following table shows the numerical result of the given system.

Table 1.1 HSS iterative method result

k	x	y	z
0	-1.2538+0.5649i	1.2273-0.0758i	0.1484+0.7446i
1	-0.9722 - 0.1176 i	1.0744 + 0.0800i	0.0623 + 1.1147i
2	-0.9464 - 0.0189i	0.9746 - 0.0093i	-0.0590 + 1.0355i
3	-1.0098 + 0.0059i	0.9970 - 0.0139i	0.0010 + 0.9765i
4	-1.0070 - 0.0010i	1.0025 + 0.0023i	0.0103 + 1.0010i
5	-0.9988 - 0.0003i	0.9993 + 0.0019i	-0.0019 + 1.0028i
6	-0.9992 + 0.0007i	0.9998 - 0.0004i	-0.0012 + 0.9991i
7	-0.0012 + 0.9991i	1.0002 - 0.0002i	0.0004 + 0.9998i
8	-1.0000 - 0.0001i	1.0000 + 0.0001i	0.0001 + 1.0002i
9	-1.0000 + 0.0000i	1.0000 + 0.0000i	-0.0001 + 1.0000i
10	-1.0000 + 0.0000i	1.0000 - 0.0000i	0.0000 + 1.0000i
11	-1.0000 - 0.0000i	1.0000+0.0000i	0.0000+1.0000i

The required solution is obtained at k=10. or $z^{(11)}$.i.e $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ i \end{pmatrix}$

II Apply the MHSS iterative method

$$z^{(k+1)} = (\alpha I + T)^{-1}(\alpha I + iW)(\alpha I + W)^{-1}(\alpha I - iT)z^{(K)} + (\alpha I + T)^{-1}(\alpha I + iW)$$

$$(\alpha I + W)^{-1}b - i(\alpha I + T)^{-1}\mathrm{b}$$

where the initial guess $z^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and α=1.7

Method	Spectral radius
MHSS	0.72

Hence the following table shows the numerical result of the given system.

Table 1.2 MHSS iterative method result

k	x	y	z
0	-0.3445 + 0.9093i	0.6241 + 0.1318i	0.1707 + 0.3531i
1	-1.0077 + 0.9323i	0.9568 + 0.1625i	0.1621 + 0.6661i
2	-1.3888 + 0.4718i	1.0724 + 0.1518i	0.1059 + 0.8648i
3	-1.3732 + 0.0014i	1.0722 + 0.1162i	0.0554 + 0.9682i
4	-1.1478 - 0.2248i	1.0368 + 0.0686i	0.0221 + 1.0118i
5	-0.9386 - 0.2072i	1.0092 + 0.0249i	0.0032 + 1.0233i
6	-0.8577 - 0.0789i	0.9996 - 0.0034i	-0.0058 + 1.0200i
7	-0.8917 + 0.0321i	1.0013 - 0.0137i	-0.0082 + 1.0120i
8	-0.9682 + 0.0708i	1.0050 - 0.0116i	-0.0067 + 1.0046i
9	-1.0250 + 0.0501i	1.0056 - 0.0050i	-0.0037 + 1.0001i
10	-1.0402 + 0.0106i	1.0033 + 0.0002i	-0.0010 + 0.9985i
11	-1.0255 - 0.0162i	1.0002 + 0.0021i	0.0006 + 0.9988i
12	-1.0037 - 0.0212i	0.9984 + 0.0016i	0.0010 + 0.9996i
13	-0.9904 - 0.0119i	0.9982 + 0.0002i	0.0007 + 1.0001i
14	-0.9890 - 0.0004i	0.9990 - 0.0007i	0.0002 + 1.0003i
15	-0.9945 + 0.0058i	0.9999 - 0.0007i	-0.0001 + 1.0002i
16	-1.0005 + 0.0058i	1.0004 - 0.0003i	-0.0002 + 1.0000i
17	-1.0033 + 0.0025i	1.0004 + 0.0001i	-0.0001 + 0.9999i
18	-1.0029 - 0.0006i	1.0002 + 0.0003i	0.0000 + 0.9999i
19	-1.0011 - 0.0019i	1.0000 + 0.0002i	0.0001 + 1.0000i
20	-0.9995 - 0.0015i	0.9999 + 0.0001i	0.0000 + 1.0000i
21	-0.9990 - 0.0004i	0.9999 - 0.0000i	0.0000 + 1.0000i
22	-0.9993 + 0.0003i	1.0000 - 0.0001i	-0.0000 + 1.0000i
23	-0.9998 + 0.0006i	1.0000 - 0.0001i	-0.0000 + 1.0000i
24	-1.0002 + 0.0004i	1.0000 - 0.0000i	-0.0000 + 1.0000i
25	-1.0003 + 0.0000i	1.0000 + 0.0000i	-0.0000 + 1.0000i
26	-1.0002 - 0.0001i	1.0000 + 0.0000i	0.0000 + 1.0000i
27	-1.0000 - 0.0002i	1.0000 + 0.0000i	0.0000 + 1.0000i
28	-0.9999 - 0.0001i	1.0000 + 0.0000i	0.0000 + 1.0000i
29	-0.9999 + 0.0000i	1.0000 - 0.0000i	-0.0000 + 1.0000i
30	-1.0000 + 0.0000i	1.0000 - 0.0000i	-0.0000 + 1.0000i
31	-1.0000 + 0.0000i	1.0000 - 0.0000i	-0.0000 + 1.0000i

Thus, the solution is obtained at k=30 or $z^{(31)}$. i.e $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ i \end{pmatrix}$

III Solve the above example 2.1 by GSOR iterative method with α=0.5.

Now apply GSOR iterative method

$$\begin{cases} Wx^{(k+1)} = (1-\alpha)Wx^{(k)} + \alpha Ty^{(k)} + \alpha f \\ Wy^{(k+1)} = -\alpha Tx^{(k+1)} + (1-\alpha)Wy^{(k)} + \alpha g \end{cases}$$

where $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $y^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $f = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$

Method	Spectral radius
GSOR	0.5

Thus, the following table shows the numerical result of the given system.

Table 1.3: GSOR iterative method result

k	x_1	x_2	x_3	y_1	y_2	y_3
0	-0.4286	0.6429	-0.2143	0.2143	0.4286	0.3929
1	-0.3954	1.4592	-0.4209	-0.2181	-0.4362	0.9834
2	-0.9769	0.6711	-0.0271	0.1004	0.2009	0.8566
3	-0.8491	1.1143	-0.1304	-0.0326	-0.0652	1.0052
4	-0.9668	0.9726	-0.0361	-0.0013	-0.0026	0.9986
5	-0.9849	0.9832	-0.0173	0.0089	0.0179	0.9958
6	-0.9807	1.0151	-0.0172	-0.0065	-0.0130	1.0081
7	-0.9993	0.9897	-0.0013	0.0033	0.0066	0.9999
8	-0.9954	1.0013	-0.0035	-0.0007	-0.0015	1.0021
9	-0.9988	0.9995	-0.0007	-0.0001	-0.0001	1.0010
10	-0.9996	0.9994	-0.0001	0.0003	0.0007	1.0002
11	-0.9994	1.0005	-0.0003	-0.0002	-0.0004	1.0004
12	-1.0000	0.9997	0.0001	0.0001	0.0002	1.0001
13	-0.9999	1.0001	-0.0000	-0.0000	-0.0000	1.0001
14	-1.0000	1.0000	0.0000	-0.0000	-0.0000	1.0000
15	-1.0000	1.0000	0.0000	0.0000	0.0000	1.0000

Thus, the required solution is obtained at k=14. i. e $x^{(15)}$ and $y^{(15)}$.

We know that $z = x + iy$ hence the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ i \end{pmatrix}$

Example 2.2 Solve the following linear system of equations using:

I HSS method
II MHSS method
III GSOR method

$$\begin{pmatrix} 3 & 0 & 1 & 1 \\ 0 & 3+2i & 2 & 2i \\ 1 & 2 & 4+3i & 1 \\ 1 & 2i & 1 & 3+3i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} -4+i \\ 3+8i \\ -1+4i \\ 4+9i \end{pmatrix}$$

Solution: $W=\begin{pmatrix} 3 & 0 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 4 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$, $T=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \end{pmatrix}$, $I=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $z^{(0)}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

I Apply HSS iterative method

$$z^{(k+1)} = (\alpha I + iT)^{-1}(\alpha I - W)(\alpha I + W)^{-1}(\alpha I - iT)z^{(K)} + (\alpha I + iT)^{-1}$$
$$(\alpha I - W)(\alpha I + W)^{-1}b + (\alpha I + iT)^{-1}\mathrm{b}$$, where α =2.6 .

Method	Spectral radius
HSS	0.43

Hence the following table shows the numerical result.

Table 1.4: HSS iterative method result

k	z_1	z_2	z_3	z_4
0	-1.6267 - 0.2038i	1.4705 + 0.2765i	-0.3843 + 0.3101i	1.9462 - 0.1023i
1	-1.8163 + 0.0968i	1.2809 - 0.1258i	-0.0187 + 1.1386i	1.9468 - 0.0838i
2	-1.9805 - 0.0039i	0.9560 - 0.0187i	0.0366 + 1.0669i	1.9713 + 0.0194i
3	-2.0155 - 0.0145i	0.9785 + 0.0126i	-0.0098 + 0.9856i	2.0021 + 0.0114i
4	-1.9999 - 0.0017i	1.0057 - 0.0023i	-0.0046 + 0.9948i	2.0011 - 0.0022i
5	-1.9979 + 0.0008i	1.0016 - 0.0017i	0.0021 + 1.0014i	1.9995 - 0.0007i
6	-2.0002 + 0.0002i	0.9994 + 0.0006i	0.0005 + 1.0005i	2.0000 + 0.0003i
7	-2.0003 - 0.0001i	0.9999 + 0.0002i	-0.0003 + 0.9999i	2.0001 + 0.0000i
8	-2.0000 - 0.0000i	1.0000 - 0.0001i	-0.0000 + 0.9999i	2.0000 - 0.0000i
9	-2.0000 + 0.0000i	1.0000 - 0.0000i	0.0000 + 1.0000i	2.0000 - 0.0000i
10	-2.0000 + 0.0000i	1.0000 + 0.0000i	0.0000 + 1.0000i	2.0000 + 0.0000i

The required solution is obtained at k = 9

II Apply the MHSS iterative method

$z^{(k+1)} = (\alpha I + T)^{-1}(\alpha I + iW)(\alpha I + W)^{-1}(\alpha I - iT)z^{(K)} + (\alpha I + T)^{-1}(\alpha I + iW)$
$(\alpha I + W)^{-1}b - i(\alpha I + T)^{-1}$b

where, $W = \begin{pmatrix} 3 & 0 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 4 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $z^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\alpha = 2.6$

Method	Spectral radius
MHSS	0.71

Thus, the following table shows the numerical result.

Table 1.5: MHSS iterative method result

k	z_1	z_2	z_3	z_4
0	-0.9152 + 0.7115i	0.8065 + 0.3547i	-0.2032 + 0.1413i	0.9020 + 0.2037i
1	-1.7397 + 0.7240i	1.1614 + 0.3935i	-0.3183 + 0.4228i	1.4403 + 0.0957i
2	-2.1613 + 0.3798i	1.2052 + 0.3171i	-0.3040 + 0.6577i	1.7666 - 0.0470i
3	-2.2065 + 0.0441i	1.1184 + 0.2024i	-0.2191 + 0.8129i	1.9564 - 0.1108i
4	-2.0767 - 0.1099i	1.0278 + 0.0900i	-0.1253 + 0.9059i	2.0440 - 0.0960i
5	-1.9548 - 0.1000i	0.9839 + 0.0080i	-0.0560 + 0.9587i	2.0593 - 0.0486i
6	-1.9159 - 0.0258i	0.9821 - 0.0319i	-0.0177 + 0.9861i	2.0370 - 0.0091i
7	-1.9434 + 0.0304i	0.9973 - 0.0362i	-0.0022 + 0.9977i	2.0082 + 0.0073i
8	-1.9882 + 0.0422i	1.0091 - 0.0218i	0.0014 + 1.0005i	1.9906 + 0.0057i
9	-2.0157 + 0.0242i	1.0109 - 0.0054i	0.0012 + 0.9997i	1.9871 - 0.0017i
10	-2.0189 + 0.0021i	1.0061 + 0.0042i	0.0007 + 0.9985i	1.9921 - 0.0063i
11	-2.0088 - 0.0092i	1.0004 + 0.0061i	0.0006 + 0.9981i	1.9984 - 0.0061i
12	-1.9985 - 0.0087i	0.9974 + 0.0036i	0.0006 + 0.9986i	2.0020 - 0.0031i
13	-1.9944 - 0.0029i	0.9973 + 0.0006i	0.0005 + 0.9993i	2.0024 - 0.0003i
14	-1.9958 + 0.0018i	0.9988 - 0.0010i	0.0002 + 0.9998i	2.0012 + 0.0011i
15	-1.9991 + 0.0032i	1.0001 - 0.0010i	-0.0000 + 1.0000i	2.0000 + 0.0010i
16	-2.0013 + 0.0019i	1.0006 - 0.0004i	-0.0001 + 1.0000i	1.9995 + 0.0004i
17	-2.0016 + 0.0002i	1.0005 + 0.0001i	-0.0001 + 0.9999i	1.9996 - 0.0002i
18	-2.0008 - 0.0008i	1.0001 + 0.0003i	-0.0000 + 0.9999i	1.9999 - 0.0003i
19	-1.9999 - 0.0008i	0.9999 + 0.0002i	0.0000 + 1.0000i	2.0001 - 0.0002i
20	-1.9995 - 0.0003i	0.9999 + 0.0000i	0.0000 + 1.0000i	2.0002 - 0.0000i
21	-1.9996 + 0.0001i	0.9999 - 0.0001i	0.0000 + 1.0000i	2.0001 + 0.0001i
22	-1.9999 + 0.0003i	1.0000 - 0.0001i	0.0000 + 1.0000i	2.0000 + 0.0001i

23	-2.0001 + 0.0002i	1.0000 - 0.0000i	-0.0000 + 1.0000i	2.0000 + 0.0000i
24	-2.0001 + 0.0000i	1.0000 + 0.0000i	-0.0000 + 1.0000i	2.0000 - 0.0000i
25	-2.0001 - 0.0001i	1.0000 + 0.0000i	-0.0000 + 1.0000i	2.0000 - 0.0000i
26	-2.0000 - 0.0001i	1.0000 + 0.0000i	0.0000 + 1.0000i	2.0000 - 0.0000i
27	-2.0000 - 0.0000i	1.0000 + 0.0000i	0.0000 + 1.0000i	2.0000 - 0.0000i
28	-2.0000 + 0.0000i	1.0000 - 0.0000i	0.0000 + 1.0000i	2.0000 + 0.0000i

The required solution is obtained at k=27. i. e. $z^{(28)}$

III Apply the GSOR iterative scheme;

$$\begin{cases} Wx^{(k+1)} = (1-\alpha)Wx^{(k)} + \alpha T y^{(k)} + \alpha f \\ Wy^{(k+1)} = -\alpha T x^{(k+1)} + (1-\alpha)Wy^{(k)} + \alpha g \end{cases}$$

where, $W = \begin{pmatrix} 3 & 0 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 4 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \end{pmatrix}$, $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $y^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $g = \begin{pmatrix} 1 \\ 8 \\ 4 \\ 9 \end{pmatrix}$, $f = \begin{pmatrix} -4 \\ 3 \\ -1 \\ 4 \end{pmatrix}$, α=0.47

Method	Spectral radius
GSOR	0.53

Hence the following table shows the numerical result.

Table 1.6: GSOR iterative method numerical result

k	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
0	-0.7773	0.9038	-0.6508	1.1027	-0.1644	0.3248	0.4497	0.5134
1	-1.3008	1.6912	-1.0642	2.0902	-0.2421	-0.7448	1.7165	-0.2713
2	-1.7154	0.4579	0.3211	1.4205	-0.1225	0.4143	0.6933	0.5253
3	-1.8400	1.4082	-0.4315	2.2671	-0.1109	-0.3680	1.4014	-0.1477
4	-1.9645	0.7386	0.2456	1.8152	-0.0352	0.1763	0.8654	0.1984
5	-1.9750	1.1438	-0.1172	2.1309	-0.0269	-0.1210	1.1213	-0.0629
6	-2.0003	0.9149	0.0934	1.9546	-0.0034	0.0535	0.9491	0.0495
7	-1.9960	1.0383	-0.0272	2.0401	-0.0030	-0.0303	1.0241	-0.0213
8	-1.9999	0.9792	0.0229	1.9900	0.0011	0.0121	0.9851	0.0083
9	-1.9983	1.0075	-0.0061	2.0079	0.0002	-0.0054	1.0026	-0.0051
10	-1.9991	0.9966	0.0029	1.9981	0.0004	0.0015	0.9972	0.0006
11	-1.9991	1.0006	-0.0011	2.0005	0.0001	-0.0003	0.9997	-0.0005
12	-1.9995	1.0000	-0.0004	1.9998	-0.0000	-0.0002	1.0000	-0.0002
13	-1.9997	0.9997	-0.0000	1.9997	-0.0000	0.0002	0.9998	0.0002
14	-1.9998	1.0002	-0.0004	2.0001	-0.0001	-0.0002	1.0003	-0.0001
15	-1.9999	0.9998	0.0001	1.9998	-0.0000	0.0001	0.9999	0.0002
16	-2.0000	1.0001	-0.0001	2.0001	-0.0000	-0.0001	1.0001	-0.0001
17	-2.0000	0.9999	0.0001	2.0000	-0.0000	0.0001	1.0000	0.0001
18	-2.0000	1.0000	-0.0000	2.0000	-0.0000	-0.0000	1.0000	-0.0000
19	-2.0000	1.0000	0.0000	2.0000	-0.0000	0.0000	1.0000	0.0000

Summary

Solving a class of complex symmetric linear system of equation by GSOR iterative method and comparing it with HSS and MHSS iterative methods is the major task of this project.

A potential difficulty with the HSS and MHSS iteration methods is the need to use complex arithmetic. Thus, rather than solving the original complex symmetric linear system (2.1), it is better to solve it as the real equivalent system by the generalized successive overrelaxation (GSOR) iterative method.

We have observed that the generalized successive overrelaxation (GSOR) iterative method to solve the equivalent real formulation of complex linear system (2.12), where W is symmetric positive definite and T is symmetric positive semidefinite. Convergence properties of the method have been also investigated.

The theoretical analysis is supported by some numerical examples, we have shown that the GSOR method is much better than the MHSS method regarding to their spectral radius and number of iteration.

References

[1] D.Hezari, D. K. Salkuyeh, V. Edalatpour, A new iterative method for solving a class of complex symmetric system linear of equations, Numer. Algor. 73 (2016) 927-955.

[2] D.M.Young, iterative solution of large linear systems, Academic press, New York, 1971.

[3] D. K. Salkuyeh, D. Hezari and V. Edalatpour, Generalized SOR iterative method for a class of complex symmetric linear system of equations, International Journal of Computer Mathematics, 2014, DOI: 10.1080/00207160.2014.912753.

[4] J. Douglas Fairs and Richard L.Burden, Numerical methods 3^{rd} edition, June 18, 2002.

[5] M.Benzi, A generalization of the Hermitian and Skew-Hermitian splitting iteration. STAM.J Matrix Anal.Appl. 31 (2009) 360-374.

[6] O. Axelsson, M. Neytcheva and B. Ahmad, A comparison of iterative methods to solve complex valued linear algebraic systems, Numer. Algor. DOI 10.1007/s11075-013-9764-1.

[7] S. L. Wu, Several variants of the Hermitian and skew-Hermitian splitting method for a classof complex symmetric linear system, Numer. Linear Algebra Appl. 22 (2015) 338-356.

[8] Z.Z. Bai, B.N. Parlett, Z.Q. Wang, A generalization successive over relaxation methods for augmented linear systems, Numer. Math. 102 (2005) 1-38.

[9] Z.Z. Bai, G.H. Golub, L.Z. Lu and J.F. Yin, Block triangular and skew-Hermitian splitting methods for positive-definite linear systems, SIAM J. Sci. Compute. 26 (2005)844-863.

[10] Z.Z. Bai, G.H. Golub and M.K. Ng, Hermitian and skew-Hermitian splitting methods fornon-Hermitian positive definite linear systems, SIAM. J. Matrix Anal. Appl. 24 (2003) 603-626.

[11] Z.Z. Bai, M. Benzi and F. Chen, Modified HSS iteration methods for a class of complex symmetric linear systems, Computing 87 (2010) 93–111.

Appendixes

Appendix I

% MAT LAB Code for HSS method

```
 function z=HSS (A, b, α, I,z0,n);

disp ('Enter system of equations in the form of Az=b')
% Calling matrix A
A=input ('Enter matrix A : \n');
% Check for matrix A
% It should be square matrix
[ na , ma ]= size (A);
if na ~= ma
disp ('Error: matrix A should be square matrix');
return
end
%Calling matrix b
b=input('Enter matrix b :\n');
% Check for matrix b
[ nb , mb ] = size (b);
ifnb ~= na || mb ~= 1
disp ('Input error: please re-enter data')
return
end
α=input ('Enter the relaxation parameter\n');
% A= W+iT
W=1/2*(A+A');
T=-i/2*(A-A');
% calling identity matrix
I=input ('Enter identity matrix I: \n');
%It should be the same order to matrix A
[ni , mi]=size (I);
ifni~=na && mi~=ma
disp('Error: its order is not equals to the order of matrix A')
return
end
% Check for convergence of system
e=eig (inv (a*I+i*T)*(a*I-W)*inv (a*I+W)*(a*I-i*T));
ifmax( abs (e)) >=1
disp ('Since the modulus of the largest eigen value of the
iterative matrix is not less than 1')
disp ('Thus, it is not convergent.Please try some other
processes')
return
end
```

```
% Asking for initial guess
z0=input ('please enter initial guess for z:\n');
[ nz ,mz ] = size (z0);
ifnz ~= na || mz ~= 1
disp ('Error: please check input')
return
end
%The expected result
n=input('Enter the maximum iteration:\n')
k=1;
z(: , 1) = z0;
for k=1:n
z(: , k+1)= inv (a*I+i*T)*(a*I-W)*inv (a*I+W)*(a*I-i*T)*z (: ,
k)+inv(a*I+i*T)*(a*I-W)*inv(a*I+W)*b+ inv(a*I+i*T)*b
end
fprintf ('The final answer obtained after %d iteration is
\n',k+1)
```

Appendix II

% MATLAB Code for MHSS method

```
 function z=MHSS (A, b, α, I,z0,n);
disp ('Enter system of equations in the form of Az=b')
% Calling matrix A
A=input ('Enter matrix A : \n');
% Check for matrix A
% It should be square matrix
[ na , ma ]= size (A);
if na ~= ma
disp ('Error: matrix A should be square matrix');
return
end
%Calling matrix b
b=input('Enter matrix b :\n');
% Check for matrix b
[ nb , mb ] = size (b);
ifnb ~= na || mb ~= 1
disp ('Input error: please re-enter data')
return
end
α=input ('Enter the relaxation parameter:\n');
% A= W+iT
W=1/2*(A+A');
T=-i/2*(A-A');
% calling identity matrix
I=input ('Enter identity matrix I: \n');
%It should be the same order to matrix A
[ni , mi]=size (I);
ifni~=na && mi~=ma
disp('Error: its order is not equals to the order of matrix A')
return
end
% Check for convergence of system
e=eig (inv (a*I+T)*(a*I+i*W)*inv (a*I+W)*(a*I-i*T));
ifmax( abs (e)) >=1
disp ('Since the modulus of the largest eigen value of the
iterative matrix is not less than 1')
disp ('Thus, it is not convergent.Please try some other
processes')
return
end
% Asking for initial guess
z0=input ('please enter initial guess for z:\n');
[ nz ,mz ] = size (z0);
```

```
ifnz ~= na || mz ~= 1
disp ('Error: please check input')
return
end
%The expected result
n=input('Enter the maximum iteration:\n')
k=1;
z(: , 1) = z0;
for k=1:n
z(: , k+1)= inv (a*I+T)*(a*I+i*W)*inv (a*I+W)*(a*I-i*T)*z (: ,
k)+inv(a*I+T)*(a*I+i*W)*inv(a*I+W)*b-i* inv(a*I+T)*b
end
fprintf ('The final answer obtained after %d iteration is
\n',k+1)
```

Appendix III

%MATLAB Code for GSOR Method

```
function z=GSOR (A, f,g, a, I,x0,y0,n);
disp ('Enter system of equations in the form of Az=b')
% Calling matrix A
A=input ('Enter matrix A : \n');
% Check for matrix A
% It should be square matrix
[ na , ma ]= size (A);
if na ~= ma
disp ('Error: matrix A should be square matrix');
return
end
%Calling collumnvectors(f and g)
%b=f+ig
f=input('Enter matrix f :\n');
% Check for matrix f
[ nf , mf ] = size (f);
ifnf ~= na || mf ~= 1
disp ('Input error: please re-enter data')
return
end
g=input('Enter matrix g :\n');
% Check for matrix g
[ ng , mg ] = size (g);
if ng ~= na || mg ~= 1
disp ('Input error: please re-enter data')
return
end
a=input ('Enter the relaxation parameter:\n');
% A= W+iT
W=1/2*(A+A');
T=-i/2*(A-A');
%z=x+iy
%asking for initial guesses (x0)
x0=input('please enter initial guess for x:\n');
%check for initial guess
[nx , mx]=size(x0);
ifnx~=ma || mx~=1
disp('Error: please check input')
return
end
%asking for initial guesses (y0)
y0=input('please enter initial guess for y:\n');
%check for initial guess
```

```
[ny , my]=size(y0);
ifny~=ma || my~=1
disp('Error: please check input')
return
end
%The expected result
n=input('Enter the maximum iteration:\n');
k=1;
x(:,1)=x0;
y(:,1)=y0;
for k=1:n
x(:,k+1)=(1-a)*x(:,k)+a*inv(W)*T*y(: , k)+a*inv(W)*f
y(:,k+1)=-a*inv(W)*T*x(:,k+1)+(1-a)*y(:,k)+a*inv(W)*g
end
fprintf ('The final answer obtained after kth iteration =%2.4f
\n',x(:,k+1))
fprintf ('The final answer obtained after kth iteration =%2.4f
\n',y(:,k+1))
```