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Fuzzy Heyting Algebras

Derebew, Nigussie

Wisdom at the source of the Blue Nile

Fuzzy Heyting Algebras

By

Derebew Nigussie Derso

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This doctoral dissertation entitled "**Fuzzy Heyting Algebras**" by Mr. Derebew Nigussie Derso is approved for the degree of "Doctor of Philosophy in Mathematics"
Supervisor of the Dissertation
Berhanu Assaye (PhD,Associate professor)

____

Signature

Co supervisor of the Dissertation

Mihret Alamneh Taye(PhD,Associate Professor)

____

Signature

External Examiner

____

Dr.Krishnamoorthy V. (PhD, Associate Professor)

Internal Examiner

____

Zelalem Teshome(PhD,Assistant professor)

Professor Ahmed Yousefian Darani is also an external examiner of the dissertation who gives his evaluation electronically

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Derebew Nigussie Derso

Declaration

I here by declare that the work presented in this dissertation entitled "**Fuzzy Heyting Algebras**" is based on the original work done by me under the supervision of Dr. Berhanu Assaye and Dr.Mihret Alamneh in the Department of Mathematics,College of Science, Bahir Dar University,and no part thereof has been presented for the award of any other degree or diploma.

Derebew Nigussie Signiture

Bahir Dar,Ethiopia Oct,2018

Dedication

This work is dedicated to:My kid, Kirubel Derebew,My beloved wife Bizualem Assefa, My beloved mother, Enanaw Gedefaw and My beloved father, Nigussie Derso

Certification

This is to certify that the dissertation entitled "**Fuzzy Heyting algebras**" is a bonafide record of the research work carried out by Mr. Derebew Nigussie Derso under our supervision in the Department of Mathematics,College of Science , Bahir Dar University.The result embodied in the dissertation have not been included in any other dissertations submitted previously for the award of any degree or diploma. (Supervisors of the Dissertation)

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Berhanu Assaye (PhD,Associate professor)

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Mihret Alamneh (PhD,Associate professor)

Department of Mathematics

College of Science

Bahir Dar University

Bahir Dar, Ethiopa.

List of Symbols

HA=Heyting Algebra

HAs= Heyting Algeras

FCR(H)=Fuzzy Congruence Relation on H

I(H)=The set of all ideals of H

F(H)=The set of all filters of H

FHA= Fuzzy Heyting Algebra

ADL= Almost Distributive lattice

HADL=Heyting Almost Distributive Lattices

HADFL=Heyting Almost Distributive Fuzzy Lattices

(H,A)=Fuzzy poset

A,B,C = Fuzzy relations on H

μ, θ, σ =Fuzzy subsets of H

H/A=The set of all congruence class of H

FS(H)=Fuzzy subset of H

$\subseteq$ =Subsets

$(a]$=Principal ideal genereted by a

Abstract

In this dissertation several results of the new notion fuzzy Heyting algebras are introduced based on the crisp theory.The new result of the concept of congruence relations on Heyting algebra using implicatively as well as multiplicatively closed subsets of H is introduced.Using the definition of homomorphism of Heyting algebras, we characterized and studied some important properties of quotient Heyting algebra by the congruence classes of it.

As a result of the new notion fuzzy Heyting Algebra (FHA), we further studied some important properties of fuzzy Heyting algebra using fuzzy relation and fuzzy poset defined by Chon.We also characterized fuzzy Heyting algebra using the directed above fuzzy poset and proved that any distributive fuzzy lattice is fuzzy Heyting algebra iff there exists a largest element c of H(Heyting Algebra) such that $A(a \wedge c, b) > 0$, for all a,b $\in$ H.

This dissertation aims to introduce fuzzy congruence relations over Heyting algebras (HA) and give constructions of quotient Heyting algebras induced by fuzzy congruence relations on HA. The fuzzy first,fuzzy second and fuzzy third isomorphism theorems of HA are established.Moreover, we investigate the relationships between fuzzy ideals and fuzzy congruence relations on HA.

The effect of a homomorphism on the join,product,and intersection of two fuzzy ideals of HA are discussed.The results obtained here will be useful in studying the algebraic nature of fuzzy prime (fuzzy maximal,fuzzy semiprime,fuzzy primary,fuzzy semiprimary) ideals under homomorphism.

The fuzzy prime ideals,fuzzy maximal ideals,fuzzy semi primary ideals of a Heyting algebra are also characterized with their level sets.We give a brief discussion on fuzzy prime ideals and fuzzy maximal ideals,fuzzy semiprime ideals and fuzzy primary ideals of of a Heyting algebra, cross product of fuzzy prime ideals and some characterizations. We concentrate on fuzzy prime ideals of Heyting algebra in such away that if μ is a fuzzy ideal of H and μ^* is a maximal ideal of H, then μ is a fuzzy

maximal ideal of H.We also proved fuzzy ideal $\mu \times \theta$ of $H \times H$ is said to be fuzzy semiprime iff the level ideals $(\mu \times \theta)_t, t \in im(\mu \times \theta)$ is semiprime ideal of $H \times H$.
We propose the notions of $\alpha-$ideals and $\alpha-$filters of a fuzzy Heyting algebra and characterize them by using its support and its level set.We characterize a fuzzy ideal on product between fuzzy Heyting algebras L and M and define fuzzy $\alpha-$ideals of fuzzy Heyting algebra.Here, we characterize a fuzzy $\alpha-$ideals of product between fuzzy Heyting algebras L and M.
This dissertation also has played great role on the study of Heyting almost distributive fuzzy lattices(HADFLs) based on FHA.After we define a Heyting Almost Distributive Fuzzy Lattices(HADFLs) as an extension of a fuzzy Heyting algebra, we give many equivalent conditions for FHAs to become an HADFL.From the definitions and results of the above concepts, many basic properties of HADFLs has been proved. We also introduce the concept of an implicative fuzzy filters in an HADL as a fuzzy filter of the same HADL and study the properties of implicative fuzzy filters of an HADL.Lastely,we define some particular implicative fuzzy filters of an HADL and prove that some of their properties are preserved under homomorphisms of HADLs.

List Of Publications

- Quotient Heyting Algebras Via Fuzzy Congruence Relations: International Journal of Mathematics And its Applications Volume 5, Issue 2C (2017), 371-378. ISSN: 2347-1557(Published)

- Fuzzy Heyting Algebra:Springer International Publishing AG 2018 (Published)

- Congruence Relation on Heyting Algebra :Bulletin of the international Mathematical Virtual Institute, Vol. 9(2019), 445-cd (Published)

- Heyting Almost Distributive Fuzzy Lattices: International Journal of Computing Science and Applied Mathematics, vol. 4, No. 1, February 2018.(Published)

- Fuzzy ideals and Fuzzy Filters of Heyting algebra: on communiction

- Implicative Fuzzy Filter of Heyting Almost Distributive Lattices: on communication.

* Published papers are attached at the end of the dissertation.

Contents

Introduction

In this dissertation several results of the new notion fuzzy Heyting algebras are introduced based on the crisp theory.We introduced the new result of the concept of congruence relations on Heyting algebra using implicatively as well as multiplicatively closed subsets [32] of H as it is important to characterize fuzzy congruence relation on H.Using the definition of homomorphism of Heyting algebras, we characterized and studied some important properties of quotient Heyting algebra [27] by the congruence classes of it.

We also give the definitions of ideal (prime ideal) and filters (prime filters) of Heyting algebra. Based on the concept of implicatively closed subset S of a Heyting algebra H, special congruence relation ψ^S which seems similar to [32] but quite different from [32] was introduced on a Heyting algebra H. Some properties of ψ^S,analogous to that for a distributive lattice proved in [32] are furnished.Further,we proved for any prime ideal P and a filter F of a Heyting algebra H, there exists an order preserving onto map between the set of all prime ideals of H/ψ^S and the set of all prime ideals of H disjoint with S.

The concept of fuzzy set was first introduced by Zadeh [12].This concept was adapted by Chon [8] to define and study fuzzy relations.G.Birkhoff [14] introduced the concept of Brouwerian lattice as a distributive lattice or Heyting algebra as a bounded distributive lattice in which for any two elements a,b there exists a largest element $a \to b$ such that $a \wedge (a \to b) \leq b$. Heyting algebra is a relatively pseudo complemented distributive lattice.It arises from non classical logic and was first investigated by Skolem T[6].It is named as Heyting algebra after the Dutch mathematician Arend Heyting [5].

As a result of the new notion fuzzy Heyting algebra (FHA), we further studied some important properties of fuzzy Heyting algebra using fuzzy relation and fuzzy poset defined by Chon [8].We also characterized fuzzy Heyting algebra using the directed above fuzzy poset and proved that any distributive fuzzy lattice is fuzzy Heyting algebra iff there exists a largest element c of H(Heyting Algebra) such that $A(a \wedge c, b) > 0$, for all a,b ∈ H.

This dissertation aims to introduce fuzzy congruence relations over Heyting algebras(HA) and give constructions of quotient Heyting algebras induced by fuzzy congruence relations on HA. The fuzzy first,fuzzy second and fuzzy third isomorphism theorems of HA are established.In this regrad,we also investigate the relationships between fuzzy ideals and fuzzy congruence relations on HA.

Fuzzy set theory, proposed by L.A. Zadeh [12], has been extensively applied to many scientific fields. In fact, the field grew enormously, and applications were found in areas by many authors see[[38],[39],[40],[41],[42],[43].] as medical diagnosis,decision making and other applications. Following the discovery of fuzzy sets, much attention has been paid to generalize the basic concepts of classical algebra in a fuzzy framework, and thus developed a theory of fuzzy algebras.

In recent years, much interest is shown to generalize algebraic structures of groups, rings, modules, etc.The notion of fuzzy ideals of a ring R was put forward and the operations on fuzzy ideals was discussed by several researchers (see, e.g., [44], [45],[46],[47]). Fuzzy congruence relations and fuzzy normal subgroups on groups was shown by N. Kuroki [44]. Later on, L. Filep and I. Maurer[45] and by V. Murali[47] further studied fuzzy congruence relations on universal algebras. Fuzzy isomorphism theorems of soft rings were shown by X.P. Liu [46],[47]. General algebraic structure, such as group and ring of congruence relations and ideals to depict the algebraic structure has played a very important role. The various constructions of quotient groups and quotient rings by fuzzy ideals was introduced by Y.L. Liu [47]. Moreover, N. Kuroki has been shown that there exists a one-to-one mapping from all fuzzy normal subgroups and all fuzzy congruence relations of groups. Naturally, the study of the definition and properties about fuzzy congruence relations on rings was a meaningful work.

From the properties of fuzzy set theory, we know that a fuzzy set defined on a set as follows: let H be a non-empty set, then $\mu : H \to [0, 1]$ is called a fuzzy set of H. In this dissertation ,H is always a Heyting algebra (HA) unless and otherwise specified. We also introduce the concept of ideals and filters of fuzzy Heyting algebra (FHA) and study some important properties of fuzzy Heyting algebra using fuzzy relation and fuzzy poset defined by Chon [8].We also characterize fuzzy Heyting algebra using the suport set and its level set.

With an idea of bringing common abstraction to most of the existing ring theoretic and lattice theoretic generalization of Boolean algebra, the concept of an Almost Distributive Lattice (ADL) was introduced by Swamy U.M. and Rao G.C. in [2].An Almost Distributive Lattice is an algebra $(H, \wedge, \vee)$ of the type (2, 2) which satisfies almost all the properties of a distributive lattice except possibly the commutativity of $\vee$, the commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$. It was also observed that any one of these three properties converts an ADL into a distributive lattice. The study of congruences is important both from theoretical stand point and for its applications in the field of logic based approaches to uncertainty. The concept of a filter congruence relation was introduced in an ADL analogous to that in a distributive lattice in [35]. Many existing properties of filter congruence relation in distributive lattices are extended to the class of ADLs.Some properties of filter congruence relations defined on an ADL are stated in [35].Based on the concept of multiplicatively and implicativelly closed subset S of an ADL special congruence relations ψ^S and ϕ^S, were introduced on an ADL.Now in this dissertation, we introduce one more congruence relation on HADL .That means we extend the concept of a congruence relation in Almost distributive lattice introduced by [32], to that for an HADL.

This dissertation is broadly divided into six chapters. Chapter one is devoted to collect all the necessary preliminaries which will be useful in our discussions in the main text of the dissertation. Even though these preliminaries are well known for those working in lattice theory, it will be convenient for others to have all these elementary notions and results in the beginning of the dissertation for the sake of ready reference. The proofs of most of the results presented in chapter 1 are either straight forward verifications or well known and hence we simply state the results and skip the proofs.Morover,we have introduced a new notion of congruence relation on Heyting algebra and we give characterizations of ideals and congruences.
Chapter two is divided into five sections namely: Fuzzy relations on Heyting algebra,ideals and filters of FHA,quotient Heyting algebra via fuzzy congruence relations,fuzzy congruence relation on Heyting algebra, ideal and homomorphism theorems on FHA . In section 2.1, we define the concept of a fuzzy relation on HA and prove a number of properties of fuzzy relations of HA on H. We introduce the concept of fuzzy Heyting algebra (FHA) as an extension of Heyting algebra. We also characterize fuzzy Heyting algebra using the properties of Heyting algebra(HA) and distributive fuzzy lattices.We state and prove some results on fuzzy Heyting algebra.

Most of the results in section 2.1 was included in the paper entitled "fuzzy Heyting algebra." and had got published in "Springer International Publishing AG 2018" J. Kacprzyk et al. (eds.), Advances in Fuzzy Logic and Technology 2017.

We also introduce the concept of ideals and filters of a FHA and give several characterizations.Most of the concepts of section 2.3 is included under the title "Quotient Heyting algebras Via Fuzzy Congruence Relations" had got published in the International Journal of Mathematics And its Applications Volume 5, Issue 2C (2017), 371- 378.

Chapter 3 is on fuzzy ideals and fuzzy filters of a Heyting algebra . Section 3.1 reflects the effect of a homomorphism on the implication, join ,product ,and intersection of two fuzzy ideals.The results obtained here is useful in studying the algebraic nature of fuzzy prime (fuzzy maximal,fuzzy semiprime,fuzzy primary,fuzzy semiprimary) ideals under homomorphism.

Chapter four is devoted to a brief discussion on fuzzy prime ideals and fuzzy maximal ideals,fuzzy semiprime ideals and fuzzy primary ideals of a Heyting algebra, cross product of fuzzy prime ideals and some characterizations. This concept of fuzzy ideal of a lattice was first introduced by Malik and Mordeson [48], [49] when the truth values are taken from the interval [0,1] .Here we extend these results to the case Heyting algebra when the truth values are taken from [0,1] and obtain certain comprehensive results on these.We concentrate on fuzzy prime ideals of Heyting algebra in such away that if μ is a fuzzy ideal of H and μ^* is a maximal ideal of H, then μ is a fuzzy maximal ideal of H.We also proved fuzzy ideal $\mu \times \theta$ of $H \times H$ is said to be fuzzy semiprime iff the level ideals $(\mu \times \theta)_t, t \in im(\mu \times \theta)$ is semiprime ideal of $H \times H$.

Fuzzy congruence on a product of lattices was defined by [24]. Quotient Heyting algebra via fuzzy congruences was characterized by the author [27]. The concepts of fuzzy congruence on the product L × K of Heyting algebras L and K, using fuzzy congruence on HAs(Heyting algebras) are discussed. It is proved that every fuzzy congruence A on the product $L \times K$ of HAs L and K is of the form $A \times B$ where A and B are fuzzy congruences on L and K respectively.The quotient HA corresponding to a fuzzy congruence on the product is isomorphic to the product of the quotient HAs of the component HAs is also obtained.Furthermore;we state the necessary and sufficient condition for direct product FHAs(fuzzy Heyting algebras) to be a HA and we obtain that for any homomorphic mapping on product FHA and image of FHA, there is a one to one correspondence beteween the set of all FHA on $X \times X$ and set of all FHAs on $Y \times Y$.Where X, Y are HAs. After that several researchers have applied the notion of fuzzy sets to the concept of congruence relation on general sets.[24] In particular Das [25] and Yijia [26] have introduced the concept of fuzzy congruences in the background of semigroups.Using a different definition for a fuzzy congruence on a HA [27], we consider the background of product $L \times K$ of HAs L and K. We define in this paper a fuzzy congruence relation $A \times B$ on $L \times K$ using fuzzy congruences A on L and B on K. As a converse, it is also proved that for every congruence relation A on $L \times K$ of HAs L and K, the congruences A_L and A_K can be defined on L and K respectively such that $A = A_L \times A_K$. Finally we show that the product of quotient HA $L/A \times K/B$ is isomorphic to the quotient HA $(L \times K)/(A \times B)$.

Chapter five dicusses about the results of the papers Mezzomo et al. [2013a] and Mezzomo et al. [2013d] where they use the fuzzy partial order relation notion defined by Zadeh [1971], and fuzzy lattices defined by Chon [2009]. Using the results Mezzomo et al. [2013a], we propose the notions of $\alpha-$ideals and $\alpha-$filters of a fuzzy Heyting algebra and characterize them by using its support and its level set. Observe that some definitions can be generalized in order to embrace the notions of ideals/filters with degree of possibility greater than or equal to α ; it is enough to generalize the first and third requirements to:"If $x \in X, y \in Y$ and $A(y, x) > \alpha$, then $x \in Y$, for $\alpha \in (0, 1]$." We characterize a fuzzy ideal on product between fuzzy Heyting algebra L and M and define fuzzy $\alpha-$ideals of fuzzy Heyting algebra.Here, we characterize a fuzzy $\alpha-$ideal on product between fuzzy Heyting algebras L and M.

Chapter six is devoted to the study of Heyting Almost Distributive fuzzy Lattices(HADFLs). In section 6.1, we define a Heyting Almost Distributive Fuzzy Lattices(HADFLs) as an extension of a fuzzy Heyting algebra. section 6.2 is about the characterization of HADFL.We give many equivalent conditions for FHAs to become an HADFL. In section 6.3, from the definitions and results of the above concepts, many basic properties of HADFLs has been proved.In section 6.4,congruence relation on HADL are presented using the ordinary theory,where as in section 6.5 and section 6.7 ordered fuzzy filter and implicative fuzzy filter are characterized.In section 6.6, QHADLs induced by FCR(H) is introduced. Finally,we introduce the concept of an implicative fuzzy filters in an HADL as a fuzzy filter of the same HADL and study the properties of implicative fuzzy filters of an HADL.Furthermore, we define some particular implicative fuzzy filters of an HADL and prove that some of their properties are preserved under homomorphisms of HADLs.Most of the contents of this chapter are included in the paper entitled "Heyting Almost Distributive fuzzy Lattices"[50].and had got published in International Journal of Computing Science and Applied Mathematics, Vol. 4, No. 1, February 2018. Almost all contents of section two and section three of this chapter are included in the paper entitled "implicative fuzzy filters of Heyting Almost Distributive Lattices" [51].

Chapter 1

Preliminaries

1.1 Heyting Algebra

The following important preliminary concepts are collected from [1],[2],[8],[18],[28],[29],[32]

Definition 1.1.1. An algebra $(H, \vee, \wedge \rightarrow, 0, 1)$ is called a Heyting algebra if it satisfies the following

1. $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice
2. $a \rightarrow a = 1$
3. $b \leq a \rightarrow b$
4. $a \wedge (a \rightarrow b) = a \wedge b$
5. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$
6. $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$,for all $a, b, c \in$ H

Theorem 1.1.2. *A bounded distributive lattice* $(H, \vee, \wedge, 0, 1)$ *is said to be a Heyting Algebra if there exist a binary operation "*$\rightarrow$*" on H such that,for any* $x, y, z \in$ *H,*$x \wedge z \leq y \Leftrightarrow z \leq x \rightarrow y$

Let H be a Heyting algebra,then we have the following properties: For $x \neq 0, x \rightarrow 0 = 0$.We use $x^{'}$ to denote $x \rightarrow 0$. Every Heyting algebra is a distributive lattice. The lattice of all open sets of a topological space is a Heyting algebra. We state without proof some elementary properties of Heyting algebras.

Theorem 1.1.3. *If a, b, and c are any elements of a Heyting algebra, then the following hold:*

(1) $1 \rightarrow a = a$.

(2) $a \rightarrow b \geq b$.

(3) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$.

(4) If $a \leq c$ *and* $a \rightarrow b = b$, *then* $c \rightarrow b = b$.

(5) $a \leq (a \rightarrow b) \rightarrow b$.

(6) $((a \rightarrow b) \rightarrow b) \rightarrow \ b = a \rightarrow b$.

(7) $((a \rightarrow b) \rightarrow b) \rightarrow (a \rightarrow b) = a \rightarrow b$.

(8) $(a \rightarrow b) \rightarrow ((a \rightarrow b) \rightarrow b)) = (a \rightarrow b) \rightarrow b$

(9) If $a \leq b$,*then* $a \wedge (b \rightarrow c) = a \wedge c$.

Theorem 1.1.4. *Let H be a Heyting algebra, then for any a,b,c* $\in$ *H,the following hold:*

(i) $a \wedge c \leq b \Leftrightarrow c \leq a \rightarrow b$

(ii) $a \leq b \Leftrightarrow a \rightarrow b = 1$

(iv) $a \rightarrow (b \rightarrow c) \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$

(v) $a \rightarrow c \leq (b \rightarrow c) \rightarrow ((a \vee b) \rightarrow c))$

Lemma 1.1.5. *In any Heyting algebra H, the following hold:*

(a) $a \leq b \Rightarrow x \rightarrow a \leq x \rightarrow b$

(b) $a \leq b \Rightarrow b \rightarrow x \leq a \rightarrow x$,*for all* $a, b, c, x \in H$

Theorem 1.1.6. *If* $(H, \vee, \wedge, \rightarrow, 0, 1)$ *is a Heyting Algebra and* $a,b \in H$*,then* $a \rightarrow b$ *is the largest element* c *of* H *such that* $a \wedge c \leq b$

Theorem 1.1.7. *The following are equivalent:*

1. *H is Heyting algebra*

2. *For any* $a,b,\ c \in H,\ a \wedge c \leq b \Leftrightarrow c \leq a \rightarrow b$

3. $b \leq a \rightarrow b$*, for all* $a, b \in H$

Definition 1.1.8. An Almost Distributive Lattice (ADL) is an algebra $(H, \vee, \wedge, 0)$ of type (2, 2, 0) satisfying the following axioms:

1. $a \vee 0 = a$,

2. $0 \wedge a = 0$,

3. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,

4. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

5. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

6. $(a \vee b) \wedge b = b$, for all $a, b, c \in H$.

Definition 1.1.9. Let $(H, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Suppose $\rightarrow$ is a binary operation on H satisfying the following conditions: for all $a, b, c \in H$,
(1) $a \rightarrow a = m$
(2) $(a \rightarrow b) \wedge b = b$
(3) $a \wedge (a \rightarrow b) = a \wedge b \wedge m$
(4) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$
(5) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$.
Then $(H, \vee, \wedge, \rightarrow, 0, m)$ is called a Heyting Almost Distributive Lattice (HADL).

Example 1.1.10. Every Heyting algebra $(H, \vee, \wedge, \rightarrow, 0, 1)$ is an HADL

Every non-empty set X can be regarded as an HADL with any arbitrarily preassigned element as its zero. This follows by the following example.

Example 1.1.11. Let H be a discrete ADL with 0 and with at least two elements. Fix x_0 and define, for any $x, y \in H$,

$$x \rightarrow y = \begin{cases} 0 & \text{if } x \neq 0, y = 0; \\ x_0 & \text{otherwise ;} \end{cases}$$

Then $(H, \vee, \wedge, \rightarrow, 0, x_0)$ is an HADL which is called a discrete HADL.

Definition 1.1.12. For any $a, b \in H$,where H is an HADL. define $a \leq b$ if and only if $a = a \wedge b$ or, equivalently, $a \vee b = b$, then $\leq$ is a partial ordering on H.

Definition 1.1.13. Let H be an ADL and $m \in H$. Then the following are equivalent:

1) m is the maximal with respect to $\leq$

2) $m \vee a = m$, for all $a \in H$,

3) $m \wedge a = a$, for all $a \in H$.

Definition 1.1.14. Let H be a nonempty set. Then a binary relation $\leq$ on H satisfying the following properties is called a partial order on H:

(1) Reflexivity: $a \leq a$

(2) Antisymmetric: $a \leq b$ and $b \leq a$ imply that $a = b$

(3) Transitivity: $a \leq b$ and $b \leq c$ imply that $a \leq c$ for all $a, b, c \in H$:

In this case $(H, \leq)$ is called a partial ordered set or simply a poset.

Definition 1.1.15. A poset $(H, \leq)$ with bottom element 0 and top element 1 is called a bounded poset

Definition 1.1.16. A subset S of H is said to be multiplicatively closed subset of H if $S \neq \emptyset$ and for any $a, b \in S \Rightarrow a \wedge b \in S$.

Definition 1.1.17. If P is a proper ideal of H, then we say that P is prime ideal if for any $a, b \in H, a \wedge b \in P \Rightarrow a \in P$ or $b \in P$.

Let I be an ideal and S be a multiplicatively closed subset of H such that $I \cap S = \emptyset$.Then there is a prime ideal M of H such that $I \subseteq M$ and $M \cap S = \emptyset$

1.2 Fuzzy Lattices

Definition 1.2.1. Let X be a set. A function A:$X \times X \to [0, 1]$ is called a fuzzy relation in X.The fuzzy relation A in X is reflexive iff $A(x, x) = 1$,for all $x \in X$.The fuzzy relation A in X is anti symmetric iff $A(x, y) > 0$ and $A(y, x) > 0 \Rightarrow x = y$.The fuzzy relation A in X is transitive iff $A(x, z) \geq Sup_{y \in X}(min(A(x, y), A(y, z)))$.A fuzzy relation A is fuzzy partial order relation if A is reflexive,symmetric and transitive. A fuzzy partial order relation A is fuzzy total order relation iff $A(x, y) > 0$ or $A(y, x) > 0$,for all $x, y \in X$. If A is a fuzzy partial order relation on a set X,then (X, A) is called a fuzzy partially ordered set or a fuzzy poset.If A is a fuzzy total order relation in a set X,then (X,A) is called a fuzzy totally ordered set or a fuzzy chain.

Definition 1.2.2. Let (X,A) be a fuzzy poset and B $\subseteq$ X.An element $u \in$ X is said to be an upper bound for a subset B iff $A(b, u) > 0, \forall b \in$ B. An upper bound u_0 for a subset B is least upper bound of B iff $A(u_0, u) > 0$ for every upper bound u of B.An element $v \in$ X is said to be an lower bound for a subset B iff $A(v, b) > 0, \forall b \in$ B. A lower bound v_0 for a subset B is the greatest lower bound of B iff $A(v, v_0) > 0$ for every lower bound v for B. We denote the lub of the set $\{x, y\} = x \vee y$ and glb of the set $\{x, y\} = x \wedge y$

Definition 1.2.3. Let (X,A) be a fuzzy poset.(X,A) is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exists for all $x, y \in X$.

Proposition 1.2.4. *Let (X,A) be a fuzzy lattice and $x, y, z \in X$.Then* *(i)* $A(x, x \vee y) > 0, A(y, x \vee y) > 0, A(x \wedge y, x) > 0, A(x \wedge y, y) > 0$

(ii) $A(x, z) > 0$ *and* $A(y, z) > 0 \Rightarrow A(x \vee y, z) > 0$

(iii) $A(z, x) > 0$ *and* $A(z, y) > 0 \Rightarrow A(z, x \wedge y) > 0$

(iv) $A(x, y) > 0$ *iff* $x \vee y = y$

(v) $A(x, y) > 0$ *iff* $x \wedge y = x$

(vi) *If* $A(y, z) > 0$,*then* $A(x \wedge y, x \wedge z) > 0$ *and* $A(x \vee y, x \vee z) > 0$

Proposition 1.2.5. *Let (X,A) be a fuzzy lattice and* $x, y, z \in X$.*Then*

1. $x \vee x = x, x \wedge x = x$
2. $x \vee y = y \vee x, y \wedge x = x \wedge y$
3. $(x \vee y) \vee z = x \vee (y \vee z), (x \wedge y) \wedge z = x \wedge (y \wedge z)$
4. $(x \vee y) \wedge x = x, (x \wedge y) \vee x = x$

Definition 1.2.6. Let (H,A) be a fuzzy lattice.(H,A) is distributive iff $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$,for all $x, y, z \in H$.

From distributive inequalities (H,A) is distributive iff $A(x \wedge (y \vee z), (x \wedge y) \vee (x \wedge z)) > 0$ and $A((x \vee y) \wedge (x \vee z), x \vee (y \vee z)) > 0$.

Definition 1.2.7. Let (H,A) be a fuzzy lattice.Then (H,A) is said to be a bounded fuzzy lattice iff $A(0, x) > 0$ and $A(x, 1) > 0$, for all $x \in H$

Definition 1.2.8. A fuzzy relation A on H is called a fuzzy equivalence relation if it satisfies the following conditions:

1. $A(x, x) = 1$ for all x of H (fuzzy relexive),
2. $A(x, y) = A(y, x)$ for all x, y of H (fuzzy symmetric),

3. $A(x,y) \geq Sup_{z\in H}(min[A(x,z), A(z,y)])$ for all x, y of H (fuzzy transitive).

Theorem 1.2.9. *Let H be a non empty set and A be fuzzy equivalence relation on H,then for $x, y \in H, A_x = A_y$ if and only if $A(x,y) = 1$,where $A_x = \{y \in H : A(x,y) = 1\}$*

Definition 1.2.10. Let f be a mapping from a set S to a set T;μ be FS(S) ; and σ be any FS (T).The image of μ under f,denoted by $f(\mu)$,is a fuzzy subset of T defined by $f(\mu)(y) = \begin{cases} Sup_{x\in f^{-1}(y)}\mu(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$ where $y \in T$

The preimage of σ under f ,symbolized by $f^{-1}(\sigma)$,is a fuzzy subset of S defined by $(f^{-1}(\sigma))(x) = \sigma(f(x))$, for all $x \in S$.

Definition 1.2.11. Let f be any function from a set S to a set T and let μ be any FS(S).Then μ is called f-invariant if $f(x) = f(y) \Rightarrow \mu(x) = \mu(y)$,where $x, y \in S$.

Definition 1.2.12. Let X and Y be sets and let $f : X \times X \to Y \times Y$ be a function. Let B be a fuzzy relation in Y . Then $f^{-1}(B)$ is a fuzzy relation in X defined by $f^{-1}(B)(x,y) = B(f(x,y))$. Let A be a fuzzy relation in X. Then f(A) is a fuzzy relation in Y defined by

$$f(A)(x,y) = \begin{cases} Sup\{A(a,b) : (a,b) \in X \times X, f(a,b) = (x,y)\} & \text{if } f^{-1}(x,y) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Chapter 2

Fuzzy Heyting Algebra

2.1 Congruence Relation on Heyting algebra

In this section we introduce the concept of congruence relations on Heyting algebra using implicatively as well as multiplicatively closed subsets [32] of H.Using the definition of homomorphism of Heyting algebras, we characterized and studied some important properties of quotient Heyting algebra [27] by the congruence classes of it.We also give the definitions of ideal (prime ideal) and filters (prime filters) of Heyting algebra.

Based on the concept of implicatively closed subset S of a Heyting algebra H, special congruence relation ψ^S which seems similar to [32] but quite different from [32] was introduced on a Heyting algebra H. Some properties of ψ^S,analogous to that for a distributive lattice proved in [32] are furnished.Further,we proved for any prime ideal P and a filter F of a Heyting algebra H, there exists an order preserving onto map between the set of all prime ideals of H/ψ^S and the set of all prime ideals of H disjoint with S.

Definition 2.1.1. Let H be a Heyting algebra and I be a non empty subset of H.Then I is said to be an ideal of H if it satisfies the following conditions.

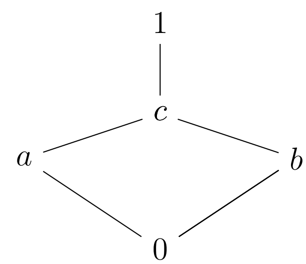
1. $a, b \in I \Rightarrow a \vee b \in I$

2. $a \in I, x \in H \Rightarrow a \wedge x, \in I$

3. $a \in I, x \in H$ for all $x \to a \neq 1 \Rightarrow x \to a \in I$

Clearly, we can see that $0 \in I$

Example 2.1.2. Let $H = \{0, a, b, c, 1\}$ and $I = \{0, a, b, c\}$ defined by the figure depicted below.Clearly $(H, \vee, \wedge, \to, 0, 1)$ is Heyting algebra,where $\wedge$, $\vee$, and $\to$ operators are defined by the following tables.



$\wedge$	0	a	b	c	1
0	0	0	0	0	1
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	1
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	c	c	1
b	b	c	b	c	1
c	c	c	c	c	1
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then I is an ideal of H.

Proof.

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

Difine the binary operation $' \rightarrow'$ on I as follows $a \rightarrow 0 = 0, b \rightarrow 0, c \rightarrow 0 = 0, a \rightarrow b = b, b \rightarrow a = a, c \rightarrow a = a, a \rightarrow c = c, b \rightarrow c = c, c \rightarrow b = b$, and

$\rightarrow$	0	a	b	c
1	0	a	b	c

So all the criterias of the definition are satisfied.Then I is an ideal of H. □

Definition 2.1.3. Let H be HA and $F \subseteq H$ and we call, F is a filter on H if it satisfies the following properties

1. $a \in F, x \in H \Rightarrow a \vee x, \in F$
2. $x, y \in F$ and $x \wedge y \neq 0 \Rightarrow x \wedge y \in F$
3. $a \in F, x \in H|\{0\} \Rightarrow a \rightarrow x \in F$

Clearly,we can see that $1 \in F$

Example 2.1.4. From example 2.1.2 above, let $H = \{0, a, b, c, 1\}$ and $F = \{a, b, c, 1\}$.Then F is a filter of H

Definition 2.1.5. Let H and H' be any two HAs. A mapping $f : H \rightarrow H'$ is called a homomorphism if it satisfies the following:
1. $f(a \vee b) = f(a) \vee f(b)$;
2. $f(a \wedge b) = f(a) \wedge f(b)$;
3. $f(a \rightarrow b) = f(a) \rightarrow f(b)$;for all $a, b \in H$.
4.$f(0) = 0'$

Remark 2.1.6. $f(1) = 1'$

Definition 2.1.7. An equivalence relation A on H is called a congruence relation if for all $a, b, c, d \in H, a \equiv b(A), c \equiv d(A) \Rightarrow a \wedge c \equiv b \wedge d(A), a \vee c \equiv b \vee d(A)$ and $a \rightarrow c \equiv b \rightarrow d(A)$

For any congruence relation A on H, we denote the congruence class containing $x \in H$ by $[x]^A$ and the set of all congruence classes of H is denoted by H/A. The set H/A is a HA, under the binary operations $\wedge, \vee$ and $\rightarrow$ defined by $[x]^A \vee [y]^A = [x \vee y]^A$, $[x]^A \wedge [y]^A = [x \wedge y]^A$,$[x]^A \rightarrow [y]^A = [x \rightarrow y]^A$ for all $[x]^A$ and $[y]^A \in H/A$

Definition 2.1.8. A subset S of H is said to be implicatively closed subset of H if $S \neq \emptyset$ and for any $a, b \in S \Rightarrow a \rightarrow b \in S$.

Define a relation on H by $a \equiv b(\psi^S) \Leftrightarrow a \rightarrow t = b \rightarrow t$, for some t$\in S$, $a, b \in H$
Note:$a \rightarrow b \rightarrow c = (a \wedge b) \rightarrow c$
$(a \wedge b) \wedge (a \rightarrow b \rightarrow c) = b \wedge a \wedge (a \rightarrow b \rightarrow c)$ [$\wedge$ ia commmutative]
$= b \wedge a \wedge (b \rightarrow c)$=$a \wedge b \wedge (b \rightarrow c)$=$a \wedge b \wedge c$.
Similarly,$(a \wedge b) \wedge ((a \wedge b) \rightarrow c)$ gives the same result.

Theorem 2.1.9. ψ^S *is a congruence relation*

Proof. Clearly, ψ^S is reflexive and symmetric.To show transitive property, we use Theorem 1.1.3 (3).Let $a \equiv b(\psi^S)$ and $b \equiv c(\psi^S)$.Then $a \rightarrow s = b \rightarrow s$ and $b \rightarrow t = c \rightarrow t$ $s, t \in S$.As $a \rightarrow b \rightarrow c = (a \wedge b) \rightarrow c$, let we consider,$a \rightarrow s \rightarrow t = b \rightarrow s \rightarrow t = (b \wedge s) \rightarrow t = (s \wedge b) \rightarrow t = s \rightarrow (b \rightarrow t) = s \rightarrow (c \rightarrow t) = (s \wedge c) \rightarrow t = c \rightarrow (s \rightarrow t)$. Since S is implicatively closed subset of H,we have $a \equiv c(\psi^S)$.Therefore, S is transitive.
To show ψ^S is a congruence relation,we will show that the three operations hold for the given relation.Suppose $a \equiv b(\psi^S)$ and $c \equiv d(\psi^S)$.Then $a \rightarrow s = b \rightarrow s$ and $c \rightarrow t = d \rightarrow t$,$t, s \in S$. Consider $a \rightarrow c \rightarrow s \rightarrow t = (a \wedge c) \rightarrow s \rightarrow t = (c \wedge a) \rightarrow s \rightarrow t = c \rightarrow a \rightarrow s \rightarrow t = c \rightarrow s \rightarrow b \rightarrow t = b \rightarrow s \rightarrow d \rightarrow t$=$b \rightarrow d \rightarrow s \rightarrow t$. Hence,$a \rightarrow c \equiv b \rightarrow d(\psi^S)$.
From this result it follows that $(a \wedge c) \rightarrow s \rightarrow t = (b \wedge d) \rightarrow s \rightarrow t$,$s, t \in S$
Hence $a \wedge c \equiv b \wedge d(\psi^S)$.Finally,$(a \vee c) \rightarrow s \rightarrow t = (a \rightarrow s \rightarrow t) \wedge (c \rightarrow s \rightarrow t)$= $(b \rightarrow s \rightarrow t) \wedge (s \rightarrow c \rightarrow t) = (b \rightarrow s \rightarrow t) \wedge (s \rightarrow d \rightarrow t) = (b \rightarrow s \rightarrow t) \wedge (d \rightarrow$

$s \to t) = (b \vee d) \to s \to t$. Hence,$a \vee c \equiv b \vee d(\psi^S)$.
Therefore, ψ^S is a congruence relation on H. □

Theorem 2.1.10. *H/ψ^S is a Heyting algebra and the operation "$\to$" is commutative*

Proof. Let $x, y \in H$. Since $S \neq \emptyset$, we can choose $a \in S$. But then $x \to y \to a = y \to x \to a$ implies $x \to y \equiv y \to x(\psi^S)$. Hence $[x]^{\psi^S} \to [y]^{\psi^S} = [x \to y]^{\psi^S} = [y \to x]^{\psi^S} = [y]^{\psi^S} \to [x]^{\psi^S}$. Thus the operation "$\to$"is commutative on H/ψ^S. □

Theorem 2.1.11. *Let S and T be any multiplicatively and implicatively closed subsets of H_1 and H_2 resp.Then for any homomorphism ϕ:$H_1 \to H_2$ such that $\phi(S) \subseteq T$, there exists a homomorphism $f : H_1/\psi^S \to H_2/\psi^T$ such that $f \circ h = k \circ \phi$ where $h : H_1 \to H_1/\psi^S$ and $k : H_2 \to H_2/\psi^T$ denote the canonical epimorphisms.Further if ϕ is a monomorphism and if $\phi(S) = T$,then f is a monomorphism.*
If ϕ is an epimorphism,then f is an epimorphism :

Proof. Define $f : H_1/\psi^S \to H_2/\psi^T$ by $f([x]^{\psi^S}) = [\phi(x)]^{\psi^T}$.
Let $[x]^{\psi^S} = [y]^{\psi^S}, x, y \in H_1$.
Then $x \equiv y(\psi^S)$
$\Rightarrow x \to s = y \to s, s \in S$
$\Rightarrow \phi(x \to s) = \phi(y \to s)$
$\Rightarrow \phi(x) \to \phi(s) = \phi(y) \to \phi(s)$
$\Rightarrow \phi(x) \equiv \phi(y)^{\psi^T}$,as $\phi(s) \in T$
$[\phi(x)]^{\psi^T} = [\phi(y)]^{\psi^T}$.
$\Rightarrow f([x]^{\psi^S}) = f([y]^{\psi^S})$.
Hence,f is well defined.
Let $x, y, \in S.f([x]^{\psi^S}) \to [y]^{\psi^S}) = f([x \to y]^{\psi^S}) = [\phi(x \to y)]^{\psi^T} = [\phi(x) \to \phi(y)]^{\psi^T} = [\phi(x)]^{\psi^T} \to [\phi(y)]^{\psi^T} = f([x]^{\psi^S}) \to f([y]^{\psi^S})$.
Similarly, we can prove the congruence relations with respect to $\vee$ and $\wedge$ that is, $f([x]^{\psi^S}) \vee [y]^{\psi S}) = f([x]^{\psi^S}) \vee f([y]^{\psi^S})$ and $f([x]^{\psi^S}) \wedge [y]^{\psi S}) = f([x]^{\psi^S}) \wedge f([y]^{\psi^S})$

for all $x, y \in H_1$. Hence f is a homomorphism.

Now $f \circ h : H_1 \to H_2/\psi^T$ and for any $x \in H$, we have $[f \circ h](x) = f(h(x)) = f([x]^{\psi^S}) = [\phi(x)]^{\psi^T}$.

Again $k \circ \phi : H_1 \to H_2/\psi^T$ and for any $x \in H_1$, we have $[k \circ \phi](x) = k(\phi(x)) = [\phi(x)]^{\psi^T}$.Hence $[f \circ h](x) = [k \circ \phi](x), \forall x \in H_1$. This shows that $f \circ h = k \circ \phi$.

i) Let ϕ be a monomorphism and let $\phi(S) =$ T. Let $f([x]^{\psi^S}) = f([y]^{\psi^S})$ for some $x, y \in H_1$.Then $[\phi(x)]^{\psi^T} = [\phi(y)]^{\psi^T} \Rightarrow \phi(x) \equiv \phi(y)(\psi^T) \Rightarrow \phi(x) \to t = \phi(y) \to t$, for some $t \in T$.

$\Rightarrow \phi(x) \to \phi(s) = \phi(y) \to \phi(s)$, for some $s \in S$ (since $\phi(S) = T$.)

$\Rightarrow \phi(x \to s) = \phi(y \to s)$ (since ϕ is a homomorphism).

$\Rightarrow x \to s = y \to s. \Rightarrow x \equiv y(\psi^S) \Rightarrow [x]^{\psi S} = [y]^{\psi^S}$.

This shows that f is one-one.

ii) Let ϕ be an epimorphism. Let $[y]^{\psi^T} \in H_2/\psi^T$.As $\phi : H_1 \to H_2$ is onto and $y \in H_2, \phi(x) = y$ for some $x \in H_1$. Thus $[x]^{\psi^S} \in H_1/ \psi^S$ and $f([x]^{\psi^S}) = [\phi(x)]^{\psi^T}$ $= [y]^{\psi^T}$. This shows that f is an epimorphism. For any two congruence relation ψ^S and ψ^T induced by two implicativelly closed subsets S and T of H with $S \subseteq T$,we have the following theorem. □

Theorem 2.1.12. *Let H be HA and let S, T be any two implicatively closed subsets of H with $S \subseteq T$. Then following are equivalent:*

i) The mapping $f : H/\psi^S \to H/\psi^T$ defined by $f([x]^{\psi^S}) = [x]^{\psi^T}$ for each $x \in H, \psi^T$ is an isomorphism.

ii) For each $t \in T$, there exists $s \in S$ such that $t \to s \in S$.

iii) For any prime ideal P of H, $P \cap T \neq \emptyset \Rightarrow P \cap S \neq \emptyset$.

Proof. i)⇒ ii) Obviously f is a well defined map. Let $x, y \in H$. Then $f([x]^{\psi^S}) = f([y]^{\psi^S}) \Rightarrow [x]^{\psi^S} = [y]^{\psi^S}$ (since f is one-one).

$\Rightarrow x \equiv y(\psi^S)$. Again $f([x]^{\psi^S}) = f([y]^{\psi^S}) \Rightarrow [x]^{\psi^T} = [y]^{\psi^T}$.

$\Rightarrow x \equiv y(\psi^T)$.Therefore, $x \equiv y(\psi^T) \Rightarrow x \equiv y(\psi^S) \Rightarrow \psi^T \subseteq \psi^S$.As $S \subseteq T, \psi^S \subseteq \psi^T$. Hence $\psi^S = \psi^T$.Hence any $t \in T$ must be congruent to some $s_1 \in S$. i.e. $t \equiv s_1(\psi^S)$.

Therefore $t \to s = s_1 \to s$ for some $s \in S$. As $s_1 \to s \in S$, we get $t \to s \in S$.

$ii) \Rightarrow iii)$ Let P be a prime ideal in H such that $P \cap T \neq \emptyset$. Select any $t \in P \cap T$. As $t \in T$ there exists $s \in S$ such that $t \to s \in S$. As P is prime ideal $t \in P$ and $s \in S \Rightarrow t \wedge s \in P$ which implies $t \to (t \wedge s) \in P$.As $t \to t \neq 1$,this gives $t \to s \in P$. Thus $t \to s \in P \cap S$. This shows that $P \cap S \neq \emptyset$.

$iii) \Rightarrow i)$

Claim : $\psi^S = \psi^T$. As $S \subseteq T \Rightarrow \psi^S \subseteq \psi^T$ .To prove that $\psi^T \subseteq \psi^S$. Let $a \equiv b(\psi^T)$. Hence, $a \to t = b \to t$ for any $t \in T$.Suppose $S \cap (t] = \emptyset$.Then there is a prime ideal P such that $(t] \subseteq P$ and $P \cap S = \emptyset$, (by Definition 1.1.17) which contradicts the assumption. (iii) as $t \in P \cap T \Rightarrow P \cap S \neq \emptyset$. Hence $S \cap (t] \neq \emptyset$. Therefore $\exists s \in S \cap (t]$. Hence $s = x \to t$ for some $x \in H$. Now $a \to s = a \to (x \to t) = (a \to x) \to t = (b \to x) \to xt = b \to (x \to t) = b \to s$. But this shows that $a \equiv b(\psi^S)$. Thus $\psi^T \subseteq \psi^S$. Combining both the inclusions we get $\psi^T = \psi^S$ and the implication follows. □

Theorem 2.1.13. *Let H be a HA with maximal elements and F be implicatively closed subset of H and let $h : H \to H/\psi^F$ be the canonical epimorphism. Then we have*

(I) If P' is a prime ideal in H/ψ^S , then $h^{-1}(P')$ is a prime ideal in H disjoint with F.

(II) Let $\theta : P(H/\psi^F) \to \{Q \in P(H)|Q \cap F = \emptyset\}$ be defined by $\theta(P') = h^{-1}(P')$. Then θ is an order preserving onto map, where $P(H)$ and $P(H/\psi^F)$ denote the set of all prime ideals of H and H/ψ^F respectively.

Proof. (I) As $h : H \to H/\psi^F$ is an epimorphism, we get $h^{-1}(P')$ is a prime ideal in H.Only to prove that $h^{-1}(P') \cap F = \emptyset$. Let $s \in h^{-1}(P') \cap F$. If m is a maximal element, then m, $s \in F$ and hence $m \equiv s(\psi^F)$.Therefore $h(m) = h(s) \in P$.A contradiction since $h(m)$ is a maximal element in H/ψ^F .Hence, $h^{-1}(P') \cap F = \emptyset$. *Thus*, $h^{-1}(P') \in \{Q \in P(H)|Q \cap F = \emptyset\}$.Let $P', Q' \in P[H/\psi F]$ such that $P \subseteq Q'$. Let $\theta(P') = P$ and $\theta(Q') = Q$. If $P' \subseteq Q'$ then $h^{-1}(P') \subseteq h^{-1}(Q')$ and hence

$\theta(P^{'}) \subseteq \theta(Q^{'})$.Then θ is order preserving. Let $P \in P(H)$ be such that $P \cap F = \emptyset$. $P \subseteq h^{-1}(h(P))$ always. To prove that $h^{-1}(h(P)) \subseteq P$.Let $x \in h^{-1}(h(P))$. Then as $h(x) \in h(P), [x]^{\psi^F} = [p]^{\psi^F}$for some $p \in P$. This means $x \equiv p(\psi^F)$. Therefore $x \to s = p \to s$ for some $s \in F$. As $P \cap F = \emptyset, s \notin P$. Again $p \to s \in P$, but then $x \to s \in P$ implies $x \in P$ as $s \notin P$. This shows that $h^{-1}(h(P)) \subseteq P$. Combining both the inclusions we get $h^{-1}(h(P)) = P$. Hence θ is onto. Now we prove the following theorem. □

Theorem 2.1.14. *Let S denote an implicatively closed subset of a HA H with maximal elements. Let $P^{'}$ be a prime ideal in H/ψ^S. Define $h^{-1}(P^{'}) = P$, where $h : H \to H/\psi^S$ is the canonical epimorphism. Then the mapping $\alpha : H/\psi^T \to H/\psi^S/\psi^T$ defined by $\alpha([x]^{\psi^T}) = [[x]^{\psi^S}]^{\psi^{T^{'}}}$ is an isomorphism, where $T = H\backslash P$ and $T^{'} = [H/\psi^S]\backslash P^{'}$ are the filters in the HAs H and H/ψ^S respectively.*

Proof. Let $[x]^{\psi^T} = [y]^{\psi T}$. Then $x \to t = y \to t$ for some $t \in T$ as $x \equiv y(\psi^T)$. But t $\notin P$ implies $h(t) = [t]^{\psi^S} \in P^{'}$ and hence $[t]^{\psi^S} \in [H/\psi^S]\backslash P^{'} = T^{'}$. Further; $[x]^{\psi^S} \to [t]^{\psi^S} = [y]^{\psi^S} \to [t]^{\psi^S}$ implies $[x]^{\psi^S} \equiv [y]^{\psi^S}(\psi^T)$.

Therefore, $[[x]^{\psi^S}]^{\psi^{T^{'}}} = [[y]^{\psi^S}]^{\psi^{T^{'}}}$.Hence $\alpha([x]^{\psi^T}) = \alpha([y]^{\psi^T})$.

This shows that α is well defined.

To prove that α is one-one.

Claim : $P \cap S = \emptyset$.

As $P^{'}$ is a prime ideal in H/ψ^S.$P^{'}$ is a proper ideal in H/ψ^S.Hence $[m]^{\psi^S} \nsubseteq P^{'}$ for any maximal element in H.But $[m]^{\psi^S} = S$ for all maximal elements m in H.Hence $S \nsubseteq P^{'}$.Let $s_1 \in P \cap S$.Then $s_1 \in P \Rightarrow s_1 \in h^{-1}(P^{'})$.

$\Rightarrow h(s_1) \in P^{'} \Rightarrow [s_1]^{\psi^S}. \Rightarrow [s_1]^{\psi^S} \subseteq P^{'} \Rightarrow S \subseteq P^{'}$, a contradiction.Hence $P \cap S = \emptyset$.

Let $\alpha([x]^{\psi^T}) = ([y]^{\psi^T})$.Then $[[x]^{\psi^S}]^{\psi^T} = [[y]^{\psi^S}]^{\psi^{T^{'}}}$ implies $[x]^{\psi^S} \equiv [y]^{\psi^S}(\psi^{T^{'}})$.

Hence $[x]^{\psi^S} \to [t]^{\psi^S} = [y]^{\psi^S} \to [t]^{\psi^S}$ for some $[t]^{\psi^S} \in T^{'} = [H/\psi^S]\backslash P^{'}$.

Hence $[x \to t]^{\psi^S} = [y \to t]^{\psi^S}$ for some $[t]^{\psi^S} \in T^{'} = [H/\psi^S]\backslash P^{'}$.

Therefore, $(x \to t \to s) = (y \to t \to s)$ for some $s \in S$.

As $P^{'}$ is prime ideal in H/ψ^S,by claim $P \cap S = \emptyset$.Hence $t \in T$ and $s \in T$ imply

$t \to s \in T$. But then $x \to (t \to s) = y \to (t \to s)$ for $t \to s \in T \Rightarrow x \equiv y(\psi T) \Rightarrow [x]^{\psi^T} = [y]^{\psi^T}$.But this shows that α is one-one .
Now to prove that α is a homomorphism. For any $x, y \in H$, we have $\alpha([x]^{\psi^T} \to [y]^{\psi^T}) = [[x \to y]^{\psi^S}]^{\psi^{T'}} = [[x]^{\psi^S} \to [y]^{\psi^S}]^{\psi^{T'}} = [[x]^{\psi^S}]^{\psi^{T'}} \to [[y]^{\psi^S}]^{\psi T} = \alpha([x]^{\psi^T}) \to \alpha([y]^{\psi^T})$.Similarly, we can prove that $\alpha([x]^{\psi^T} \wedge [y]^{\psi^T})$=$\alpha([x]^{\psi^T}) \wedge \alpha([y]^{\psi^T})$ and $\alpha([x]^{\psi^T} \vee [y]^{\psi^T})$=$\alpha([x]^{\psi^T}) \vee \alpha([y]^{\psi^T})$. Obviously α being an onto map, we get α is an isomorphism and hence the result. □

From now onwards by H, we mean Heyting algebra unless otherwise specified. In this section, we introduced the concept of fuzzy Heyting algebra (FHA) and studied some important properties.

Definition 2.1.15. A bounded distributive fuzzy lattice (H,A) is said to be a fuzzy Heyting algebra if there exists a binary operation $' \to'$ such that ,for any $x, y, z \in H, A(x \wedge z, y) > 0 \Leftrightarrow A(z, x \to y) > 0$

Theorem 2.1.16. *Let (H,A) be a bounded distributive fuzzy lattice,then (H,A) is called a fuzzy Heyting algebra if it satisfies the following axioms:*

1. $A(1, a \to a) > 0$
2. $A(b, a \to b) > 0$
3. $A(a \wedge (a \to b), a \wedge b) = A(a \wedge b, a \wedge (a \to b)) = 1$
4. $A(a \to (b \wedge c), (a \to b) \wedge (a \to c)) = A((a \to b) \wedge (a \to c), a \to (b \wedge c)) = 1$
5. $A((a \vee b) \to c), (a \to c) \wedge (b \to c)) = A((a \to c) \wedge (b \to c), (a \vee b) \to c)) = 1$ *for all a,b,c∈ H*

Example 2.1.17. Let $(B, \vee, \wedge, ', 0, 1)$ be a Boolean algebra and a,b ∈ B and A:$B \times B \to$ [0,1] is a fuzzy relation.define $a \to b = a' \vee b$.Then (B,A) is a fuzzy Heyting algebra

Proof. Clearly, $(B, \vee, \wedge, \rightarrow, 0, 1)$ is a Heyting algebra and (B, A) is a bounded distributive fuzzy lattice.

1. $A(a \rightarrow a, a^{'} \vee a) = A(a^{'} \vee a, a \rightarrow a) = 1$
2. $A((a \rightarrow b) \wedge b, (a^{'} \vee b) \wedge b) = A((a^{'} \vee b) \wedge b, (a \rightarrow b) \wedge b)) = 1$
3. $A(a \wedge (a \rightarrow b), a \wedge b) = A(a \wedge b, a \wedge (a \rightarrow b)) = 1$
4. $A(a \rightarrow (b \wedge c), a^{'} \vee (b \wedge c)) = A(a \rightarrow b) \wedge (a \rightarrow c), a \rightarrow (b \wedge c)) = 1$
5. $A((a \vee b) \rightarrow c, (a \rightarrow c) \wedge (b \rightarrow c)) = A((a \rightarrow c) \wedge (b \rightarrow c), (a \vee b) \rightarrow c)$,for all $a, b, c \in B$. Thus,(B,A) is a fuzzy Heyting algebra

□

Lemma 2.1.18. *Let (H,A) be a bounded distributive fuzzy lattice.Then* $(H, \vee, \wedge, \rightarrow, 0, 1)$ *is a Heyting Algebra iff (H,A) is a fuzzy Heyting algebra.*

From the definition of Heyting algebra and fuzzy lattice property,we have the following lemma.

Lemma 2.1.19. $A(b, a \rightarrow b) > 0$ *iff* $b \wedge (a \rightarrow b) = b$ *or equivalently* $b \vee (a \rightarrow b) = a \rightarrow b$,$\forall a, b \in H$

Lemma 2.1.20. *In any fuzzy Heyting algebra the following holds:*

(i) $A(a \rightarrow (b \wedge a), a \rightarrow b) = 1$

(ii) $A(a, b) > 0 \Rightarrow A(x \rightarrow a, x \rightarrow b) > 0$

(iii) $A(a, b) > 0 \Rightarrow A(b \rightarrow x, a \rightarrow x) > 0$

Proof. (i) $A(a \rightarrow (b \wedge a), a \rightarrow b)$= $A((a \rightarrow b) \wedge (a \rightarrow a), a \rightarrow b)$=$A((a \rightarrow b) \wedge 1, a \rightarrow b)$=$A(a \rightarrow b, a \rightarrow b)$=1

(ii) $A(a,b) > 0$ iff $a \wedge b = a$ or $a \vee b = b$ [By proposition 1.2.4]

$A(x \to a, x \to b) = A(x \to (a \wedge b), x \to b)$ − − − − − − − − [Since a $\wedge\, b = a$]

$=A((x \to a) \wedge (x \to b), x \to b) > 0$ [Since (x$\to a) \wedge (x \to b) \leq x \to b$]

(iii) $A(a,b) > 0$ iff $a \vee b = b$ [Proposition 1.2.4].

Now,$A(b \to x, a \to x) = A((a \vee b) \to x, a \to x) = A((a \to x) \wedge (b \to x), a \to x) > 0$.

Hence ,$A(b \to x, a \to x) > 0$.

□

Theorem 2.1.21. *If (H, A) is a FHA and $a, b \in H$,then $a \to b$ is the largest element of the set S=$\{c \in H : A(a \wedge c, b) > 0\}$*

Proof. Let H be a FHA. We shall show that $a \to b \in$ S.Let $a, b \in$ H.Then $A(a \wedge (a \to b), a \wedge b) > 0$.Clearly, $A(a \wedge b, b) > 0$.

This implies, $A(a \wedge (a \to b), b) \geq sup_{a \wedge b \in H}(min A(a \wedge (a \to b), a \wedge b), A(a \wedge b, b)) > 0$.

$\Rightarrow A(a \wedge (a \to b), b) > 0$.

$\Rightarrow a \to b \in S$. Let d be such that $a \wedge d \leq b$.Then $A(a \wedge d, b) > 0$.

$\Rightarrow A(a \to (a \wedge d), a \to b) > 0$

$\Rightarrow A(a \to d, a \to b) > 0$.

$\Rightarrow (d, a \to b) > 0$

$\Rightarrow A(d, a \to b) \geq Sup_{a \to d \in H}(min(A(d, a \to d), A(a \to d, a \to b))) > 0$

$\Rightarrow A(d, a \to b) > 0$

$\Rightarrow a \to b$ is an upper bound of d.

Thus, $a \to b$ is the largest element of S. □

Lemma 2.1.22. *Let (H,A) be a fuzzy Heyting algebra,then for any a,b,c $\in$ H,we have $A(a,b) > 0 \Leftrightarrow a \to b = 1$*

Proof. $A(a,b) > 0$.Then $A(a \to a, a \to b) > 0$.

$\Rightarrow A(1, a \to b) > 0$.But a$\to b \leq 1$,as 1 is the largest element.

$\Rightarrow A(a \to b, 1) > 0 \Rightarrow a \to b = 1$.

Hence the result.

Conversely,assume $A(a \to b, 1) = A(1, a \to b)$.Then,$a \wedge (a \to b) = a \wedge 1$

$\Rightarrow a \wedge (a \to b) = a$

$\Rightarrow a \wedge b = a$

$\Rightarrow a \leq b$.Hence,$A(a, b) > 0$. □

Theorem 2.1.23. *Let (H,A) be a fuzzy Heyting algebra,then the following are equivalent.*

1. $A(a \wedge c, b) > 0$
2. $A(a \to c, a \to b) > 0$
3. $A(c, a \to b) > 0$,*for* $a, b, c \in H$

Proof. straightforward □

Theorem 2.1.24. *Every fuzzy Heyting algebra is a distributive fuzzy lattice*

Proof. Since $A(x, x \vee y) > 0$,we have $A(y \wedge x, (y \wedge x) \vee (z \wedge x)) > 0$.

Hence $A(y, x \to (y \wedge x) \vee (z \wedge x)) > 0$.Similarly,$A(z, x \to (y \wedge x) \vee (z \wedge x)) > 0$.

This implies $A(y \vee z, x \to (y \wedge x) \vee (z \wedge x)) > 0$.

$\Rightarrow A(x \wedge (y \vee z), x \wedge (x \to (y \wedge x) \vee (z \wedge x))) > 0$.

$\Rightarrow A(x \wedge (y \vee z), x \wedge (y \wedge x) \vee (z \wedge x)) > 0$.

$\Rightarrow A(x \wedge (y \vee z), (y \wedge x) \vee (z \wedge x)) > 0$. *

From $A(y, y \vee z) > 0$ and $A(y \wedge x, y) > 0$ and $A(y \wedge x, x) > 0$, We have

$A(y \wedge x, (y \vee z) \wedge x) > 0$. Similarly,$A(z \wedge x, (y \vee z) \wedge x) > 0$.

Thus,$A((y \wedge x) \vee (z \wedge x), (y \vee z) \wedge x) > 0$. **

From * and ** we have the result.Hence the theorem follows. □

Definition 2.1.25. The fuzzy poset (H,A) is said to be directed above if $\forall a, b, c \in H, A(a, c) > 0$ and $A(b, c) > 0$,then $\exists x \in$ H such that $A(x, c) > 0$.

Theorem 2.1.26. *Let (H,A) be a fuzzy lattice.Then the following statements are equivalent:*

1. *(H,A) is a fuzzy Heyting algebra*

2. *The fuzzy poset (H,A) is directed above*

3. *(H,A) is a distributive fuzzy lattice*

Proof. (1) $\Rightarrow$ (2). Let $a, b \in$ H.Then $\exists c \in H$ such that $A(a, c) > 0$ and $A(b, c) > 0$
$\Rightarrow A(a \vee b, c) > 0$.Take $x = a \vee b \in H$.Hence $A(x, c) > 0$
(2) $\Rightarrow$ (3). Suppose (2) holds.
Then,$A(a \vee b, c) > 0$.
$\Rightarrow A(c \wedge (a \vee b), c \wedge c) > 0$
We claim to show $A(a\wedge(b\vee c), (a\wedge b)\vee(a\wedge c)) > 0$ and $A(a\vee(b\wedge c), (a\vee b)\wedge(a\vee c)) > 0$.
We know that $A(a \vee b, c) > 0, A(a, c) > 0, A(b, c) > 0, A(c, c) > 0$.
$\Rightarrow A(b \vee c, c) > 0$.
$\Rightarrow A(a \wedge (b \vee c), a \wedge c) > 0$.
It is clear that $A((a \vee b) \wedge (a \vee c), a \vee c) > 0$ and $A((a \wedge b), (a \wedge b) \vee (a \wedge c)) > 0$.
Also $A((a \wedge c), (a \wedge b) \vee (a \wedge c)) > 0$.
Thus, we have $A(a \wedge (b \vee c), (a \wedge b) \vee (a \vee c)) \geq Sup_{a\wedge c\in H}(min(A(a \wedge (b \vee c), a \wedge c), A((a \wedge c), (a \wedge c) \vee (a \wedge b))) > 0$.Hence, $A(a \wedge (b \vee c), (a \wedge b) \vee (a \vee c)) > 0$.
Similarly $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) > 0$.
Therefore,(H, A) is a distributive fuzzy lattice
(3) $\Rightarrow$ (1). Suppose (H, A) is a distributive fuzzy lattice such that $A(a \wedge c, b) > 0$
We need to show $A(c, a \rightarrow b) > 0$. Clearly,$A(a \wedge c, a \wedge b) > 0$
$\Rightarrow A(a \rightarrow (a \wedge c), a \rightarrow (a \wedge b)) > 0$
$\Rightarrow A(a \rightarrow c, a \rightarrow b) > 0$,but $A(c, a \rightarrow c) > 0$
$\Rightarrow A(c, a \rightarrow b) \geq sup_{a\rightarrow c\in H}(min(A(c, a \rightarrow c), A(a \rightarrow c, a \rightarrow b))) > 0$
Hence,(H,A)is a fuzzy Heyting Algebra □

Lemma 2.1.27. *Let (H,A) be fuzzy Heyting algebra and if $A(a,c) > 0$ and $A(b,c) > 0$,for a, b and $c \in H$,then we have the following.*

1. $A(a \wedge b, b \wedge a) > 0$
2. $A(a \vee b, b \vee a) > 0$
3. $A((a \to c) \wedge (b \to c, 1) > 0$
4. $A((c \to a) \wedge (c \to b), 1) > 0$
5. $A(((a \to c) \wedge (b \to c)) \vee ((c \to a) \wedge (c \to b)), 1) > 0$

Proof. Straightforward □

Theorem 2.1.28. *Let (H,A) be a distributive fuzzy lattice.Then (H,A) is a fuzzy Heyting algebra iff for any $a, b \in H$,there exists a largest element $c \in H$ such that $A(a \wedge c, b) > 0$.*

Proof. $(\Rightarrow)$.*Clearly*, $A(a \wedge (a \to b), a \wedge b) > 0$ and $A(a \wedge b, b) > 0$.
This implies $A(a \wedge c, b) > 0$. Let $d \in$ H such that $A(a \wedge d, b) > 0$.
We shall prove that $A(d, c) > 0.A(a \wedge d, b) > 0$.
$\Rightarrow A(a \to d, a \to b) > 0$,but $A(d, a \to d) > 0$
$\Rightarrow A(d, a \to b) > 0$.Taking $c = a \to b$,we have $A(d, c) > 0$.
Therefore,there is a largest element $c \in$ H such that $A(a \wedge c, b) > 0$.
Conversely,suppose the given conditions hold.Define a binary operation $\to$ on H such that $a \to b$ is the largest element of the set$\{c \in H : A(a \wedge c, b) > 0\}$.
we prove that (H,A) is a fuzzy Heyting algebra

1. Clearly $A(b, a \to b) > 0$.For $b \in$H,since $A(a \wedge b, a) > 0$, we have
   $A(a \to (a \wedge b), a \to a) > 0$
   $\Rightarrow A(b, a \to a) > 0$
   $\Rightarrow a \to a$ is an upper bound of b.

But $a \to a = 1$

$\Rightarrow A(a \to a, 1) = A(1, a \to a) = 1$

2. Since $A(a \wedge b, b) > 0$.Then $A(a \to b, a \to b) > 0$

   $\Rightarrow A(b, a \to b) > 0$

3. Since $A(a \wedge (a \to b), b) > 0$,we have $A(a \wedge (a \wedge (a \to b)), a \wedge b) > 0$

   $\Rightarrow A(a \wedge (a \to b), a \wedge b) > 0$, on the other hand,$A(a \wedge b, b) > 0$.

   $\Rightarrow A(b, a \to b) > 0$

   $\Rightarrow A(a \wedge b, a \wedge (a \to b)) > 0$ from anti symmetry we have $a \wedge (a \to b) = a \wedge b$

   Thus,we have $A(a \wedge (a \to b), a \wedge b)$=$A(a \wedge b, a \wedge (a \to b) = 1$

4. From Heyting algebra,we have $a \wedge (a \to (b \wedge c)) = a \wedge a \wedge (b \wedge c) \leq b$,we have

   $A(a \wedge (a \to (b \wedge c), b) > 0$

   $\Rightarrow A(a \wedge (b \wedge c), b) > 0$

   $\Rightarrow A(a \to (b \wedge c), a \to b) > 0$. Similarly,$A(a \to (b \wedge c), a \to c) > 0$

   $\Rightarrow a \to (b \wedge c)$ is a lower bound of $\{a \to b, a \to c\}$

   $\Rightarrow A(a \to (b \wedge c), (a \to b) \wedge (a \to c)) > 0$.

   On the other hand $A(a \wedge (a \to b) \wedge (a \to c), a \wedge b \wedge (a \to c)) > 0$

   $\Rightarrow A(a \wedge (a \to b) \wedge (a \to c), b \wedge a \wedge (a \to c)) > 0$

   $\Rightarrow A(a \wedge b \wedge c, b \wedge a \wedge c) > 0$

   $\Rightarrow A((a \to b) \wedge (a \to c), (a \to b) \wedge (a \to c)) > 0$

   $\Rightarrow A((a \to b) \wedge (a \to c), a \to (b \wedge c)) > 0$

   Therefore,$A((a \to b) \wedge (a \to c), a \to (b \wedge c)) = A(a \to (b \wedge c), (a \to b) \wedge (a \to c)) = 1$

5. Consider $A((a \vee b) \wedge (a \to c) \wedge (b \to c), (a \wedge (a \to c) \wedge (b \to c)) \vee (b \wedge (a \to c) \wedge (b \to c)) > 0$

   $\Rightarrow A((a \vee b) \wedge (a \to c) \wedge (b \to c), (a \wedge c) \vee (b \wedge c)) > 0$

   $\Rightarrow A((a \vee b) \wedge (a \to c) \wedge (b \to c), (a \vee b) \wedge c) > 0$ but $A((a \vee b) \wedge c, c) > 0$

   $\Rightarrow A((a \vee b) \wedge (a \to c) \wedge (b \to c), c) > 0$.

By Theorem 1.1.7, we have $A((a \to c) \wedge (b \to c), (a \vee b) \to c) > 0$.

On the other hand $A(a, a \vee b) > 0 \Rightarrow A((a \vee b) \to c, a \to c) > 0$.

Similarly,$A((a \vee b) \to c, b \to c) > 0$

$\Rightarrow A((a \vee b) \to c, (a \to c) \wedge (b \to c)) > 0$

Hence,$A((a \vee b) \to c, (a \to c) \wedge (b \to c)) = A((a \to c) \wedge (b \to c), (a \vee b) \to c)$

Therefore (H,A) is a fuzzy Heyting Algebra

□

Definition 2.1.29. Let (H,A) be a distributive fuzzy lattice.Then the fuzzy Heyting algebra (H,A) satisfies the infinite meet distributive fuzzy law if $A(a \wedge (\vee_{i \in I} s_i), \vee_{i \in I}(a \wedge s_i))) = A(\vee_{i \in I}(a \wedge s_i), a \wedge (\vee_{i \in I} s_i)) = 1$ where

$$\{s_i : i \in I\} \subseteq H$$

Theorem 2.1.30. *Let (H,A) be a distributive fuzzy lattice.Then (H,A)is a fuzzy Heyting algebra iff it satisfies the infinite meet distributive fuzzy law.That is for any family*

$$\{s_i : i \in I\} \subseteq H$$

if $\vee_{i \in I} s_i$ exists,then $\vee_{i \in I}(a \wedge s_i)$ exists for any $a \in H$ and it is equal to

$$a \wedge (\vee_{i \in I} s_i).$$

Proof. Let (H, A) be a distributive fuzzy lattice and $a, b \in$ H.

Define $a \to b = \vee_{s \in S_{ab}} s$, where $S_{ab} = \{s \in H : A(a \wedge s, b) > 0\}$. Now,let $a, b, c \in H$.Then

1. $S_{aa} = \{s \in H : A(a \wedge s, a) > 0\} = (H, A)$

   $\Rightarrow a \to a = \vee H = 1$

   Thus,$A(a \to a, 1) = A(1, a \to a) = 1$

2. Since $A(a \wedge b, b) > 0$,we have $b \in S_{ab}$.This implies $A(b, a \to b) > 0$.Thus, $A((a \to b) \wedge b, b) = A(b, (a \to b) \wedge b) = 1$

3. $A(a \wedge (a \to b), a) > 0$ and $A(a \wedge (a \to b), a \wedge (\vee_{s \in S_{ab}} s)) > 0$

   $\Rightarrow A(\vee_{s \in S_{ab}}(a \wedge s), b) > 0$

   $\Rightarrow A(a \wedge (a \to b), \vee_{s \in S_{ab}}(a \wedge s)) > 0$

   $\Rightarrow A(a \wedge (a \to b), b) > 0.$

   Hence $a \wedge (a \to b)$ is a lower bound of $\{a, b\}$

   $\Rightarrow A(a \wedge (a \to b), a \wedge b) > 0$.On the other hand,we have

   $A(a \wedge (a \wedge b), b) > 0.$

   $\Rightarrow a \wedge b \in S_{ab}$

   $\Rightarrow A(a \wedge b, a \to b) > 0$

   $\Rightarrow A(a \wedge (a \wedge b), a \wedge (a \to b)) > 0$

   $\Rightarrow A(a \wedge b, a \wedge (a \to b)) > 0.$

   Thus,$A(a \wedge b, a \wedge (a \to b)) = A(a \wedge (a \to b), a \wedge b) = 1$

4. Since $A(a \wedge (a \to (b \wedge c)), b) > 0.$

   $\Rightarrow A(a \to (b \wedge c), a \to b) > 0$

   Similarly, $A(a \to (b \wedge c), a \to c) > 0.$

   $\Rightarrow a \to (b \wedge c)$ is a lower bound of $\{a \to b, a \to c\}$

   $\Rightarrow A(a \to (b \wedge c), (a \to b) \wedge (a \to c)) > 0$. On the other hand,$A(a \wedge (a \to b) \wedge (a \to c), a \wedge b \wedge (a \to c)) > 0$

   $\Rightarrow A(a \wedge (a \to b) \wedge (a \to c), a \wedge b \wedge c) > 0$

   $\Rightarrow A(a \wedge (a \to b) \wedge (a \to c), b \wedge c) > 0$

   $\Rightarrow A((a \to b) \wedge (a \to c), a \to (b \wedge c)) > 0$. Hence, $A((a \to b) \wedge (a \to c), a \to (b \wedge c)) = A(a \to (b \wedge c), (a \to b) \wedge (a \to c)) = 1$

5. Consider,$A((a \vee b) \wedge (a \to c) \wedge (b \to c), (a \wedge (a \to c) \wedge (b \wedge c)) \vee (b \wedge (a \to c) \wedge)(b \wedge c)) > 0$

   $\Rightarrow A((a \to) \wedge (b \to c), (a \vee b) \to c) > 0$. Since $A(a, a \vee b) > 0, A(b, a \vee b) > 0$.This implies $A((a \vee b) \to c, a \to c) > 0$ and $A((a \vee b) \to c, b \to c) > 0$

   $(a \vee b) \to c$ is a lower bound of $\{a \to c, b \to c\}$.

$\Rightarrow A((a \vee b) \rightarrow c, (a \rightarrow c) \wedge (b \rightarrow c)) > 0$. Hence,$A((a \vee b) \rightarrow c, (a \rightarrow c) \wedge (b \rightarrow c)) = A((a \rightarrow) \wedge (b \rightarrow c), (a \vee b) \rightarrow c) = 1$.

Therefore,(H,A) is fuzzy Heyting algebra.

Conversely,suppose (H,A) be a fuzzy Heyting algebra.Let a $\in H, \{s_i : i \in I\}$ $\subseteq$H.Then $a \wedge s_i \in$ H.Since $A(s_i, \vee s_i) > 0$, we have $A(\vee_{i \in I}(a \wedge s_i), a \wedge (\vee_{i \in I} s_i)) > 0$.On the other hand $A(a \wedge s_i, \vee_{i \in I}(a \wedge s_i)) > 0$.

$\Rightarrow A(s_i, a \rightarrow \vee_{i \in I}(a \wedge s_i)) > 0, \forall i \in I$

$\Rightarrow A(\vee_{i \in I} s_i, a \rightarrow \vee_{i \in I}(a \wedge s_i)) > 0$

$\Rightarrow A(a \wedge (\vee_{i \in I} s_i), a \wedge (a \rightarrow \vee_{i \in I}(a \wedge s_i))) > 0$

but,$A(a \wedge \vee_{i \in I}(a \wedge s_i), \vee_{i \in I}(a \wedge s_i)) > 0$

$\Rightarrow A(a \wedge (\vee_{i \in I} s_i), \vee_{i \in I}(a \wedge s_i))) > 0$

$A(a \wedge (\vee_{i \in I} s_i), \vee_{i \in I}(a \wedge s_i))) = A(\vee_{i \in I}(a \wedge s_i), a \wedge (\vee_{i \in I} s_i)) = 1$

□

2.2 Fuzzy Relations on Heyting Algebra

Definition 2.2.1. Let A and B be any two fuzzy relations of a Heyting algebra H.Then A is said to be contained in B,denoted by $A \subseteq B$, if $A(x, y) \leq B(x, y)$ for all $(x, y) \in H \times H$.

Definition 2.2.2. The union of two fuzzy relations A and B of the HA H,denoted by $A \cup B$ is a fuzzy relation of H defined by $(A \cup B)(x, y) = max(A(x, y), B(x, y))$.The intersection of A and B,symbolized by $A \cap B$ is a fuzzy relation on H,defined by $(A \cap B)(x, y) = min(A(x, y), B(x, y))$, for all $x, y \in H$. More generally,the union and intersection of any family $\{A_i : i \in I\}$ of fuzzy relations of a set H,are defined by $(\cup_{i \in I} A_i)(x, y) = Sup_{i \in I} A_i(x, y)$ and $(\cap_{i \in I} A_i)(x, y) = inf_{i \in I} A_i(x, y)$ for all $x, y \in H$

Definition 2.2.3. A fuzzy relation A of a set S is said to have sup property if there exist $(a_0, b_0) \in S \times S$ such that $A(a_0, b_0) = Sup_{(a,b) \in S \times S} A(a, b)$.

Definition 2.2.4. Let A be a fuzzy relation on a Heyting algebra H and let $t \in [0,1]$.The fuzzy relation A_t=$\{(x,y) \in H \times H : A(x,y) \geq t\}$ is called the level relation of A. Clearly, $A_\alpha \subseteq A_\beta$ whenever $\alpha > \beta$

Theorem 2.2.5. *Two level relations A_t and A_s with $(s < t)$ of a fuzzy relation A of a Heyting algebra H are equal iff, there is no $(x,y) \in H \times H$ such that $s \leq A(x,y) \leq t$*

Proof. Suppose $A_s = A_t$ with $s < t$.We calim to show that no $(x,y) \in H \times H$ such that $s \leq A(x,y) \leq t$. Suppose there is $(x,y) \in H \times H$ such that $s \leq A(x,y) \leq t$.Then $s \leq A(x,y)$ which implies $(x,y) \in A_s = A_t$.This implies that $(x,y) \in A_t$ which is a contradiction.Hence the forward part follows.
Conversely,suppose there is no $(x,y) \in H \times H$ such that $s \leq A(x,y) \leq t$.
We claim to show that $A_s = A_t$.Let $(x,y) \in A_t$.Then $A(x,y) \geq t > s$.
$\Rightarrow A(x,y) \geq s \Rightarrow (x,y) \in A_t \subseteq A_s$.For the other inclusion, let $(x,y) \in A_s$.Then $A(x,y) \geq s$.Since there is no $(x,y) \in H \times H$ such that $s \leq A(x,y) < t$.Then $A(x,y) \geq t$ which gives $(x,y) \in A_t$.Hence $A_s \subseteq A_t$.Therefore, the theorem follows. □

Theorem 2.2.6. *A fuzzy relation A of a Heyting algebra H is a FHA iff the level relations $A_t, t \in im(A)$ is Heyting algebra(In particular (H, χ_H) is FHA H iff H is a Heyting algebra)*

Remark 2.2.7. The family F_A of level relations of A is precisely $\{A_t | t \in imA\}$,where (H,A) is any fuzzy Heyting algebra. Moreover,if $imA = \{t_0, t_1, ..., t_n\}$ with $t_0 > t_1 > ... > t_n$,then the level relations of A form the following chain: $A_{t_0} \subset A_{t_1} \subset A_{t_2} \subset ... \subset A_{t_n} = H$

Theorem 2.2.8. *Two FHAs (H,A) and (H,B) such that $cardImA < \infty$ and $cardImB < \infty$ of a HA H are equal iff $ImA = ImB$ and $F_A = F_B$*

Proof. Let A and B be any two fuzzy HAs of H.Let $cardImA < \infty$,$cardImB < \infty$ and $A = B$.

We need to show

a) $ImA = ImB$

b) $F_A = F_B$. That is $ImA = \{t_0, t_1, ..., t_n\}$ such that $t_0 > t_1 > t_2 > ... > t_n$ and $ImB = \{l_o, l_1, l_2, ..., l_n\}$ such that $l_0 > l_1 > l_2 > ... > l_m$. By the above remark,we have $A_{t_o} \subset A_{t_1} \subset A_{t_2} \subset ... \subset A_{t_n} = H$, $B_{l_o} \subset B_{l_1} \subset ... \subset B_{l_m} = H$

Hence $F_A = \{A_{t_0}, A_{t_1}, ..., A_{t_n}\}$ and

$F_B = \{B_{l_0}, B_{l_1}, ..., B_{l_m}\}$

a) Suppose $A = B \Rightarrow A(x, y) = B(x, y)$,for all $x, y \in H$

Hence, $ImA = ImB \Rightarrow n = m$

b) To prove $F_A = F_B$

Consider F_A=$\{A_{t_i} | t_i \in ImA\}$

=$\{(x, y) \in H \times H \in H | A(x, y) \geq t_i, t_i \in ImA\}$

=$\{(x, y) \in H \times H | B(x, y) \geq l_i, l_i \in ImB\}$

=$\{B_{l_i} | l_i \in ImB\}$

=F_B

Hence $F_A = F_B$

Conversely ,Suppose ,$ImA = ImB$ and$F_A = F_B$

Claim: $A = B$ that is $A(x, y) = B(x, y)$,for all $x, y \in H)$

(a) Let $x, y \in H$ and let $A(x, y) = l\in ImA = ImB$

$\Rightarrow (x, y) \in A_l = B_l$,$F_A = F_B$

$\Rightarrow (x, y) \in B_l \Rightarrow B(x, y) \geq l$

Hence $A(x, y) \leq B(x, y)$ (1)

Let $(x, y) \in A_t\Rightarrow A(x, y) \geq t$

Hence $B(x, y) \leq A(x, y)$ (2)

Thus $A(x, y) = B(x, y)$,for all $x, y \in H$

$\Rightarrow A = B$ by (1) and (2). □

Theorem 2.2.9. *A homomorphic image or preimage of a fuzzy Heyting algebra is fuzzy Heyting algebra,provided that the sup property holds in the former case.*

Proof. Let H and $H^{'}$ be HAs and $f : H \to H^{'}$ epimorphism.
Let A be a fuzzy Heyting algebra on H.
we shall show that $f(A)$ is a fuzzy Heyting algebra of $H^{'}$.
a) (i) Let $t \in imf(A)$ .Then for some $(y, z) \in H^{'} \times H^{'}$,

$$f(A)(y, z) = sup_{(a,b)\in f^{-1}(y,z)}A(a, b) = t \leq 1$$

Since $A(u, v) \leq 1, \forall(u, v) \in H \times H$,we have if $t = 0$,then $(f(A))_t = H^{'} \times H^{'}$. Let $t > 0$ and $\epsilon > 0$ be any real numbers. We will show that $(f(A))_t = f(A_\alpha), \alpha = t-\epsilon$.
Let $(y, z) \in (f(A))_t$.Then $t \leq f(A)(y, z) = sup_{(a,b)\in f^{-1}(y,z)}A(a, b)$
$\Rightarrow \alpha + \epsilon \leq sup_{(a,b)\in f^{-1}(y,z)}A(a, b)$
$\Rrightarrow \alpha < A(a, b)$ for some $(a, b) \in f^{-1}(y, z)$
$\Rightarrow (a, b) \in A_\alpha$ for some $f(a, b) = (y, z)$
$\Rightarrow f(a, b) \in f(A_\alpha)$ and $(y, z) \in f(A_\alpha)$
Hence,$(f(A))_t \subseteq f(A_\alpha)$
On the other hand, $(y, z) = f(a, b)$ for some (a, b) such that $\alpha \leq A(a, b)$
$\alpha \leq (f(A))(y, z)$ since $(a, b) \in f^{-1}(y, z)$
$\Rightarrow t \leq f(A)(y, z)$,since $\epsilon > 0$ is arbitrary
$\Rightarrow (y, z) \in (f(A))_t$.Since $\alpha \leq 1, A_\alpha$ is a HA on H by Definition 2.2.4 $f(A_\alpha)$ and so $(f(A))_t$ is Heyting algebra. Hence $(H^{'}, f(A))$ is a fuzzy Heyting algebra, for the pre -image is easy to prove by theorem 2.2.6 □

2.3 Ideals and Filters of Fuzzy Heyting Algebra

In this section ,we introduce the concept of ideals and filters of fuzzy Heyting algebra (FHA).We also characterize ideals and filters of fuzzy Heyting algebra using the support and level sets of fuzzy Heyting algebra (FHA) .We,finally, state and prove some results on ideals and filters of fuzzy Heyting algebra.

Remark 2.3.1. $([0,1], \vee, \wedge, \rightarrow)$ is a Heyting algebra

Definition 2.3.2. Let H be a Heyting algebra. A function A:$H \times H \rightarrow [0,1]$ is called a fuzzy relation on H. The fuzzy relation A in H is reflexive iff $x \rightarrow x = 1$,for all $x \in H$.The fuzzy relation A in H is anti symmetric iff $x \rightarrow y = 1$ and $y \rightarrow x = 1$ $\Rightarrow x = y$.The fuzzy relation A in H is transitive iff $Sup_{y \in H}(min(A(x,y), A(y,z))) \rightarrow A(x,z) = 1$.

A fuzzy relation A is fuzzy partial order relation if A is reflexive,symmetric and transitive. A fuzzy partial order relation A is fuzzy total order relation iff $x \rightarrow y = 1$ or $y \rightarrow x = 1$,for all $x, y \in$ H. If A is a fuzzy partial order relation on a set H,then (H,A) is called a fuzzy partially ordered set or a fuzzy poset.If A is a fuzzy total order relation in a set H,then (H,A) is called a fuzzy totally ordered set or a fuzzy chain.

Definition 2.3.3. Let (H,A) be a fuzzy poset and B $\subseteq$ H.An element u $\in$H is said to be an upper bound for a subset B iff $b \rightarrow u = 1, \forall b \in$ B. An upper bound u_0 for a subset B is least upper bound of B iff $u_0 \rightarrow u = 1$ for every upper bound u for B.An element $v \in$ H is said to be a lower bound for a subset B iff $v \rightarrow b = 1, \forall b \in$ B. A lower bound v_0 for a subset B is the greatest lower bound of B iff $v \rightarrow v_0 = 1$ for every lower bound v for B. We denote the lub of the set $\{x, y\} = x \vee y$ and glb of the set $\{x, y\} = x \wedge y$

Remark 2.3.4. Since A is anti symmetric,then the LUB and GLB is unique.

Proof. Suppose $u_0$and u_1 are LUBs in a subset Y $\subseteq$ H.Then $u_0 \rightarrow u_1 = 1$ and $u_1 \rightarrow u_0 = 1$.Then $u_0 = u_1$ □

Definition 2.3.5. Let (H,A) be a fuzzy poset.(H,A) is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exists for all $x, y \in$ H.

Proposition 2.3.6. *Let (H,A) be a fuzzy Heyting algebra and x,y,z* $\in H$*.Then*

(i) $x \rightarrow (x \vee y) = 1, y \rightarrow (x \vee y) = 1, (x \wedge y) \rightarrow x = 1, (x \wedge y) \rightarrow y = 1$

(ii) $x \to z = 1$ *and* $y \to z = 1 \Rightarrow x \vee y \to z = 1$

(iii) $z \to x = 1$ *and* $z \to y = 1 \Rightarrow z \to (x \wedge y) = 1$

(iv) $x \to y = 1$ *iff* $x \vee y = y$

(v) $x \to y = 1$ *iff* $(x \wedge y) = x$

(vi) *If* $y \to z = 1$,*then* $x \wedge y \to x \wedge z = 1$ *and* $x \vee y \to x \vee z = 1$

(vii) $x \to y = 1$ *and* $x \to z = 1$, *then* $x \to (y \wedge z) = 1$*and* $x \to (y \vee z) = 1$

(Viii) $x \to z = 1, y \to z = 1 \Rightarrow (x \vee y) \to z = 1$

Lemma 2.3.7. *Let (H,A) be a FHA ,if* $x \to y = 1$ *and* $y \to z = 1$,*then* $x \to z = 1$

Lemma 2.3.8. *Let (H,A) be a FHA ,if* $(a \wedge c) \to b = 1 \Leftrightarrow c \to (a \to b) = 1$

From now onwards by H, we mean Heyting algebra unless otherwise specified.

Definition 2.3.9. Let (H,A) be a fuzzy lattice.Then (H,A) is said to be a bounded fuzzy lattice if $x \to 1 = 1$,for all $x \in H$

From the definition of Heyting Algebra and fuzzy lattice property,we have the following lemma.

Lemma 2.3.10. *Let* $Y \subseteq H$ *and* (H, A) *be a FHA,then if* $A(x, a) > 0$,*then either* $x \to a = 1$ *or* $x \to a \in Y$

Proof. Suppose $x \to a \neq 1$ and $x \to a$.This implies $x \nleq a$.This gives $x > a$.Which is a contradiction.Also if $x \to a \notin Y$.
Then from the properties of Heyting algebra,$a \leq x \to a \notin Y$. Thus, $a \notin Y$ □

Lemma 2.3.11. *Let (H,A) be a fuzzy Heyting algebra,then for any a,b,c* $\in$ *H,we have* $A(a, b) > 0 \Leftrightarrow a \to b = 1$

Proof. Suppose $A(a,b) > 0$.Then $A(a \to a, a \to b) > 0$.

$\Rightarrow A(1, a \to b) > 0$.But a$\to b \leq 1$,as 1 is the largest element.

$\Rightarrow (a \to b, 1) > 0 \Rightarrow a \to b = 1$.Hence the result.

Conversely,assume $A(a \to b, 1) = A(1, a \to b)$.Then,$a \wedge (a \to b) = a \wedge 1$

$\Rightarrow a \wedge (a \to b) = a$

$\Rightarrow a \wedge b = a$

$\Rightarrow a \leq b$.Hence,$A(a,b) > 0$. □

Definition 2.3.12. Because a fuzzy relation is a fuzzy set,then the p -level sets and support of fuzzy relations is defined in fuzzy sets ,then the p- level set of a fuzzy relation$A : H \times H \to [0,1]$ is defined as ,for all $x \in H$,$y \in$ H. $A_p = \{(x,y) : A(x,y) \geq p\}$.In the same way ,we define the support of a fuzzy relation S(A) as S(A)=$\{(x,y) \in H \times H : A(x,y) > 0\}$

Proposition 2.3.13. *Let $A : H \times H \to [0,1]$ be a fuzzy relation.Then,A is fuzzy partial order relation on H iff for each $p \in (0,1]$,the p-level set A_p is a partial order relation in H.*

Proof:Let A be a fuzzy partial order relation on a Heyting algebra H.we shall show that A_p is a partial order relation on H,$p \in (0,1]$.since by hypothesis $x \to x = 1$.Then $(x,x) \in A_p, p \in (0,1]$.Thus $p \to A(x,x) = 1$.Therefore A_P is reflexive.Next, suppose $(x,y) \in A_p$ and $(y,x) \in A_p$.We shall show that x=y.from the hypothesis we have $p \to A(x,y) = 1$ and $p \to A(y,x) = 1$.From the properties of the HA, $1 \wedge 1$ =$(p \to A(x,y)) \wedge (p \to A(y,x))$ =$p \to (A(x,y)) \wedge A(y,x))$.Since (H,A) is a fuzzy poset $(A(x,y)) \wedge A(y,x)) = A(x,x) = 1$.This implies both $A(x,y)$ and $A(y,x) > 0$.Thus $x = y$.Finally,suppose $(x,y) \in A_p$ and $(y,z) \in A_p$. We shall show that $(x,z) \in A_p$. From hypothesis $p \to A(x,y) = 1$ and $p \to A(y,z) = 1$.Clearly, $1 \wedge 1$ =$(p \to A(x,y)) \wedge (p \to A(y,z))$.Thus 1 =$p \to A(x,z)$.Therefore ,The result follows.

Lemma 2.3.14. *Let $A : H \times H \to [0,1]$ be a fuzzy relation.Then,if A is fuzzy partial order relation on H,then S(A) is a partial order relation on H.*

Proof. Since $A(x,x) = 1 > 0$.then $x \to x = 1$.Thus $(x,x) \in$ S(A).Suppose $(x,y) \in S(A)$ and $(y,x) \in S(A)$.Then $x \to y = 1$ and $y \to x = 1$.Thus $x = y$.Suppose $(x,y) \in S(A)$ and $(y,z) \in S(A)$.We need to show that$(x,z) \in S(A)$.This implies $x \to y = 1$ and $y \to z = 1$.Thus $x \to z = 1$.Therefore ,S(A) is a partial order relation on a Heyting algebra H. ☐

Definition 2.3.15. Let $L = (H,A)$ be FHA and $Y \subseteq H.Y$ is an ideal of L if

(i) $x \in H$,$y \in Y, A(x,y) > 0 \Rightarrow x \in Y$

(ii) $x, y \in Y \Rightarrow x \vee y \in Y$

(iii) $x \to a \neq 1$,$a \in Y, x \in H \Rightarrow x \to a \in Y$

Definition 2.3.16. Let $L = (H,A)$ be FHA and $Y \subseteq H.Y$ is a filter of L if

(i) $x \in H$,$y \in Y, A(y,x) > 0 \Rightarrow x \in Y$

(ii) $x, y \in Y, x \wedge y \neq 0 \Rightarrow x \wedge y \in Y$

(iii) $a \in F, x \in H|\{0\} \Rightarrow a \to x \in Y$

Lemma 2.3.17. *Let Y be an ideal of $L = (H,A)$,then there exists an element t such that $y \wedge (t \to x) = y \wedge x$,for every $x, y \in Y$ or $x \wedge (t \to y) = x \wedge y$,for every $x, y \in Y$*

Proof. Since Y is an ideal of L,then $Y \neq \emptyset$. Now,take $x, y \in Y$ such that $x \vee y \in Y$.Let $x \vee y = t$,for some $t \in Y$.Since $Y \subseteq$ H and H is a Heyting algebra,then

$(x \vee y) \to x = t \to x$

$y \to x = t \to x$

$y \wedge (y \to x) = y \wedge (t \to x)$

$y \wedge x = y \wedge (t \to x)$
interchanging the roles of x into y,we have $x \wedge (t \to y) = x \wedge y$. □

Proposition 2.3.18. *1. Let Y be an ideal of $L = (H, A)$,then $x \wedge (t \to y) = y \wedge (t \to x)$,$t, x, y \in Y$*

2. Let Y be an ideal of (H, A) such that $a \in Y$ and $x \in H$ if $A(x, a) > 0$,then $x \to a \in Y$

Proof. 1. Straightforward from the above lemma.

2. Suppose $A(x, a) > 0 \Rightarrow x \in Y$
$\Rightarrow x \to a \in Y$. Since Y is an ideal $x \to a \neq 1$.

□

Lemma 2.3.19. *Let Y be filter of (H, A),then the following holds.*
1.$x \to t = x \to y$,*for some* $t \in Y$,*for all* $x, y \in Y$
2.$x \wedge t = x \wedge y$
3.$y \wedge t = y \wedge x$

Proposition 2.3.20. *Let (H, A) be a fuzzy poset (or chain) and $Y \subseteq H$. If $B = A|Y \times Y$, that is, B is a fuzzy relation on Y such that for all $x, y \in Y, B(x, y) = A(x, y)$, then (Y, B) is a fuzzy poset (or chain).*

In previous sections, we have defined an ideal of a FHA (H,A). We have also defined the support set S(A) of a fuzzy relation A in a set H as well as p-level set A_p of a fuzzy relation A in a set H and characterize a relation on H. Then, we can think of a set of ideals from a p-cut, that is, the set of ideals with degree greater than or equal to p or, the set of elements $x \in H$ and y ∈ Y such that $p \to A(x, y) = 1$ with $p \in (0, 1]$.

Proposition 2.3.21. *(Y, B) is a ideal (filter) of FHA H iff Y is an ideal (filter) of $(H, S(A))$.*

Proof.(⇒) Let (Y, B) be an ideal of L and let $y \in Y$. Then,

(i) If $(x, y) \in S(A)$, then $A(x, y) > 0$. So, by Definition 2.3.15 item (i) $x \in Y$.

(ii) If $x \in Y$ and $y \in Y$, then by Definition 2.3.15 item (ii), $x \vee y \in Y$.

Also item (iii) $B(x, a) > 0$ by Proposition 2.3.18 (ii) $x \to a \in Y$.

(⇐)(i) Let $x \in H$and y $\in Y$ and suppose that $A(x, y) > 0$, then $(x, y) \in S(A)$ and $x \in Y$.

(ii) Suppose $B(x, y) > 0$.Then $(x, y) \in S(B)$.Then $x, y \in Y$ which implies $x \vee y \in Y$

(iii) $x \to a \neq 1, x \to a \in Y$. Similarly, we can proof that (Y, B) is a fuzzy filter of L iff Y is a filter of (H , S(A)).

Theorem 2.3.22. *(Y, B) is a ideal (filter) of FHA iff for each $p \in (0, 1], B_p$ is an ideal (filter) of (Y, A_p).*

Proof. (⇒) Let (Y , B) be an ideal of L and let $y \in Y$. Then, (i) If $(x, y) \in B_p$, then $p \to A(x, y) = 1$. So, by Definition 2.3.15 item (i) x $\in Y$. (ii) If $x \in Y$ and $y \in Y$, then by Definition 2.3.15 item (ii), $x \vee y \in Y$.

(⇒) (i) Let $x \in H$ and $y \in Y$ and suppose that $A(x, y) > 0$ then $(x, y) \in B_p$ and $x \in Y$.

(ii) Trivially.Similarly, we can proof that (Y , B) is a fuzzy filter of L. iff Y is a filter of (X, A_p). □

Definition 2.3.23. A fuzzy poset (H,A) is called fuzzy sup-HA if each pair of element has supremum on Y. Dually, a fuzzy poset (Y,A) is called fuzzy inf- HA if each pair of element has infimum on Y.

Notice that a structure is a complete fuzzy Heyting Algebra iff it is simultaneously fuzzy sup-HA and fuzzy inf-HA. We define supremum and infimum of a fuzzy set I on H as follows.

Definition 2.3.24. Let (H, A) be a fuzzy poset and I be a fuzzy set on H. sup I is an element of H such that if $x \in H$ and $\mu_I(x) > 0$, then $A(x, supI) > 0$ and if

$u \in H$ is such that $A(x, u) > 0$ when $\mu_I(x) > 0$, then $A(supI, u) > 0$. Similarly, inf I is an element of H such that if $x \in H$ and $\mu_I(x) > 0$, then $A(infI, x) > 0$ and if $v \in H$ is such that $A(v, x) > 0$ when $\mu_I(x) > 0$, then $A(v, infI) > 0$.

Proposition 2.3.25. *Let (H,A) be a sup-complete (inf-complete) FHA and I be a fuzzy set on H. Then, sup I (inf I) exists and it is unique.*

Proposition 2.3.26. *If (H,A) is a complete FHA, then (H,S(A)) is a complete crisp HA.*

Proof. Let (H,A) be a complete FHA and $Y \subseteq H$. Since, for each $x, y \in Y$, if $A(x, y) > 0$ then we have that $(x, y) \in S(A)$. So, by Proposition 1.2.4 (iv) and (v), all $Y \subseteq H$ has supremum and infimum. Therefore, (H,S(A)) is a complete HA. □

Proposition 2.3.27. *Let (H,A) be a fuzzy Heyting algebra, then (H, A) be a fuzzy sup-HA and $Y \subseteq H$. The set $\downarrow Y = \{x \in H : A(x, y) > 0$ for some $y \in Y\}$ is an ideal of (H,A).*

Proof. *(i) Let $z \in \downarrow Y$ and $w \in H$ such that $A(w, z) > 0$. If $z \in Y$, then exists $x \in Y$ such that $A(z, x) > 0$, and by transitivity, $A(w, x) > 0$. Therefore, $w \in \downarrow Y$.*
(ii) Suppose $x, y \in \downarrow Y$, then exist $z_1, z_2 \in Y$ such that $A(x, z_1) > 0$ and $A(y, z_2) > 0$. So, $A(x, z_1 \vee z_2) > 0$ and $A(y, z_1 \vee z_2) > 0$.
By hypothesis (Y, A) is a fuzzy sup-HA, then $z_1 \vee z_2 \in Y$ and $A(x \vee y, z_1 \vee z_2) > 0$.Also,Therefore, $x \vee y \in \downarrow Y$.
(iii) Let $a \in \downarrow Y$ and $x \in H$ such that $x \to a \neq 1$.Then there exist $z \in H$ such that $A(a, z) > 0$.But$A(x \to a, x \to z) > 0$.Since $x, z \in H$,Then $x \to z \in H$ Thus,$x \to a \in \downarrow Y$

Proposition 2.3.28. *Let (H,A) be a FHA (Y,A) be a fuzzy inf-HA and $Y \subseteq H$. The set $\uparrow Y = \{x \in H : A(y, x) > 0$ for any $y \in Y\}$ is a filter of (H,A).*
Proof. Analogously the proposition 2.2.28

Proposition 2.3.29. *Let (H,A) be a FHA and* $Y \subseteq H$*, then* $\downarrow Y$ *satisfies the following properties: (i)* $Y \subseteq \downarrow Y$ *(ii)* $Y \subseteq W \Rightarrow \downarrow Y \subseteq \downarrow W$ *(iii)* $\downarrow\downarrow Y = \downarrow Y$

Proof. (i) If $y \in$Y and how $A(y, y) > 0$, then $y \in \downarrow Y$.
(ii) Suppose $Y \subseteq W$ and $y \in \downarrow Y$, then by definition, exists $z \in Y$ such that $A(z, y) > 0$. If $Y \subseteq W$, then $z \in W$ and $A(z, y) > 0$. So $y \in \downarrow W$.
(iii)($\Rightarrow$) $\downarrow\downarrow Y \subseteq \downarrow Y$. Suppose $y \in \downarrow\downarrow Y$, then exists $x \in \downarrow Y$. Then there exists $z \in Y$ such that $A(x, z) > 0$. So, $A(y, z) > 0$. Therefore, $y \in \downarrow Y$.
($\Leftarrow$)Straightforward from (i). Similarly, we prove the same properties for $\uparrow Y$ □

Proposition 2.3.30. *Let (H,A) be a FHA and* $Y \subseteq H$*, then* $\uparrow Y$ *satisfies the following properties: (i)* $Y \subseteq \downarrow Y$ *(ii)* $Y \subseteq W \subseteq \uparrow Y \subseteq \uparrow W$ *(iii)* $\uparrow\uparrow Y = \uparrow Y$

Proof. Analogous to the above proposition

Corollary 2.3.31. $\downarrow Y (\uparrow Y)$ *is the lowest ideal (filter) containing* Y *.*

An important kind of ideal is called principal ideal generated by $x \in H$ and defined by:

Definition 2.3.32. Let (H,A) be a FHA and $x \in H$. Then, the set defined by $\downarrow x = \{y \in H : A(y, x) > 0\}$ is called principal ideal of (H,A) generated by x.

And, dually, we define principal filter by:

Definition 2.3.33. Let (H,A) be a FHA and $x \in$ H. Then, the set defined by $\uparrow x = \{y \in H : A(x, y) > 0\}$ is called principal filter of (H,A) generated by x

Remark 2.3.34. Obviously, $\downarrow x = \downarrow \{x\} and \uparrow x = \uparrow \{x\}$.

The family of all ideals of a FHA (H,A) will be denoted by I(H). Duality, will be denoted by F(H) the family of all filters of a FHA (H,A). This families are subsets of parts of (H,A), denoted by P(H), that is, I(H) $\subseteq$ P(H) and F(H) $\subseteq$ P(H).

Proposition 2.3.35. *Let Z be a subset of I(H) and W be a nonempty set of I(H), then(i)* $H \in I(H)$; *(ii)* $\cap Z \in I(H)$; *(iii)* $\cup W \in I(H)$.

Proof. (i) Straightforward. (ii) Let Z ⊆ I(H). Suppose $x \in \cap Z$ and $A(y, x) > 0$, then $x \in Z_j$ for all $Z_j \in Z$. If $A(y, x) > 0$, then $y \in Z_j$ for each $Z_j \in Z$. So y ∈ Z and therefore, $\cap Z \in I(H)$. Notice that if Z is an empty set then $\cap Z = H$. (iii) Let $W \subseteq I(H)$. Suppose $x \in \cup W$ and $A(y, x) > 0$, then exists $W_j \in W$ such that $x \in W_j$, and if $W_j \in I(H)$, then $y \in W_j$. So $y \in \cup W$ and therefore, $\cup W \in I(H)$. □

Proposition 2.3.36. *Let Z be a subset of F(H) and W be a nonempty set of F(H), then (i)* $H \in F(H)$ *(ii)* $\cap Z \in F(H)$ *(iii)* $\cup W \in F(H)$.

Proof. Analogous to the proposition 2.3.36. The following proposition prove the relation between the ideal $\downarrow Y$ and the principal ideal $\downarrow y$. □

Proposition 2.3.37. *For all* $Y \in P(H)$, $\downarrow Y = \cup_{y \in Y} \downarrow y$

Proof. Let $Y \in P(H)$. Then, $x \in \downarrow Y$ iff exists $y \in Y$ such that $A(x, y) > 0$ iff exists $y \in Y$ such that $x \in \downarrow y$ iff $x \in \cup_{y \in Y} \downarrow y$.By duality $\uparrow Y = \cup_{y \in Y} \uparrow y$ □

Proposition 2.3.38. *If (Y,B) is a complete FHA, then* $\downarrow Y \subseteq \downarrow supY$.

Proof. In fact, suppose $x \in \downarrow Y$, then exists $y \in Y$ such that $A(x, y) > 0$. Therefore, because $A(Y, supY) > 0$, then $x \in \downarrow supY$. □

Remark 2.3.39. $\downarrow supY \subseteq \downarrow Y$ only if $supY \in Y$. Similarly, we prove that $\downarrow Y \subseteq \downarrow infY$.

An ideal Y of (H,A) such that $Y \neq H$ is called proper ideal of (H,A). Duality, a filter Y of (H,A) such that $Y \neq H$ is called proper filter of (H,A). Before to define prime ideal and prime filter of FHA we will prove important results involving proper ideals and proper filters. Consider $I_P(H)$ the family of all proper ideals of a FHA and $F_p(H)$ the family of all proper filters of a FHA.

Proposition 2.3.40. *Let $Z \subseteq I_p(H)$. Then,$\cup Z \neq H$.*

Proof. If all proper ideals not containing the top, then $\cup Z \neq H$ □

Corollary 2.3.41. *The union of proper ideals is a proper ideal.*

Proof. Straightforward from the above Proposition □

Corollary 2.3.42. *Let $Z \subseteq I_p(H)$. Then, $\cap Z \neq H$.*

Proof. Suppose $x \in \cap Z$, then $x \in Z_j$ for all $Z_j \in Z$. By definition exists $y \in H$ such that $y \notin Z_j$ for some $Z_j \in Z$. So, $y \notin \cap Z_p$. Therefore, $\cap Z \neq H$. The proof of Propositions 2.3.35,2.3.36,2.3.37,2.3.38 and 2.3.40 are analogously for filters. □

Now, we define a prime ideal of a FHA as follow:

Definition 2.3.43. Let Y be a proper ideal of (H,A). We say that Y is a prime ideal of (H,A) if $A((x \wedge y), z) > 0$, then either $x \in Y$ or $y \in Y$, for all $x, y \in H$ and $z \in Y$.

Definition 2.3.44. Let Y be a proper filter of (H,A). We say that Y is a prime filter of (H,A) if $A(z, x \vee y) > 0$, then either $x \in Y$ or $y \in Y$, for all x, $y \in H$ and $z \in Y$.

We will denote by $I_{pr}(H)$ the family of all prime ideals of a FHA and $F_{pr}(H)$ the family of all prime filters of a FHA.

Proposition 2.3.45. *Let $Y \subseteq I_{pr}(H)$. Then, $\cap Y$ is a prime ideal.*

Proof. By Corollary above we know $\cap Y$ is a proper ideal. So, we can only prove the primality of $\cap Y$. Suppose x, $y \in X$and $z \in \cap Y$ such that $A(x \wedge y, z) > 0$. If $z \in \cap Y$, then $z \in Y_j$ for all $Y_j \in Y$. If $A(x \wedge y, z) > 0$, then $x \wedge y \in Y_j$ for each $Y_j \in Y$. By hypothesis, Y_j is a prime ideal of (H,A), then either $x \in Y_j$ or $y \in Y_j$ for each $Y_j \in Y$. Therefore, $x \in \cap Y$ or $y \in \cap Y$. □

Proposition 2.3.46. *Let $F \subseteq F_{pr}(H)$. Then, $\cap F$ is a prime filter*

Proof. Analogous to the above proposition □

another kind of ideals (filters) of a FHA is the maximal ideal (maximal filter), defined by:

Definition 2.3.47. Let Y and Z be an ideals of (H,A). We say that a proper ideal Y is a maximal ideal(filter) of (H,A) if $Y \subseteq Z \subseteq H$, then either Y = Z or Z = H.

2.4 Quotionet Heyting Algebra Via Fuzzy Congruence Relations

Definition 2.4.1. A relation A on the set H is called left compatible if $(a, b) \in A$ implies $(x \vee a, x \vee b) \in A$, $(x \wedge a, x \wedge b) \in A$ and $(x \to a, x \to b) \in A$, for all a, b, x of H, and is called right compatible if $(a, b) \in A$ implies $(a \vee x, b \vee x) \in A$, $(a \wedge x, b \wedge x) \in A$, and $(a \to x, b \to x) \in A$ for all a, b, x of H. A relation A on the set H is called compatible if it is both right and left compatible.

Remark 2.4.2. A compatible equivalence relation on H is called a congruence relation on H.

Fuzzy Congruence Relation.

In this section, we introduce the notion of fuzzy congruence relations on HA and give some properties about fuzzy congruence relations.We also introduce the notion of quotient HA's by fuzzy congruence relations and give the fuzzy first,fuzzy second and fuzzy third isomorphism theorems of HA's by means of fuzzy congruence relations.

Definition 2.4.3. A fuzzy equivalence relation A on H is called a fuzzy congruence relation if the following conditions are satisfied,for all x, y, z, t of H.

1. $A(x \vee y, z \vee t) \geq min(A(x, z), A(y, t))$

2. $A(x \wedge y, z \wedge t) \geq min(A(x, z), A(y, t))$

3. $A(x \rightarrow y, z \rightarrow t) \geq min(A(x, z), A(y, t))$

Lemma 2.4.4. *Let (H,A) be FHA,then a fuzzy equivalence relation A is called a fuzzy congruence relation on H if*

(1) $A(a \vee x, b \vee x) \geq A(a, b)$

(2) $A(a \wedge x, b \wedge x) \geq A(a, b)$

(3) $A(a \rightarrow x, b \rightarrow x) \geq A(a, b)$

(4) $A(x \rightarrow a, x \rightarrow b) \geq A(a, b)$

Proposition 2.4.5. *Let A and B be any fuzzy compatible relations on H. Then A* $\circ$ *B is also a fuzzy compatible relation on H.*

Proof. Let $a, b, x \in H$.
Since A and B are fuzzy compatible equivalence relations on H,
$A \circ B(x \vee a, x \vee b) = sup_{z \in H}(min(A(x \vee a, z), B(z, x \vee b)))$
$\geq min[A(x \vee a, x \vee z), B(x \vee z, x \vee b)]$
$\geq min(A(a, z), B(z, b))$
$\geq sup_{z \in H}(min(A(a, z), B(z, b))$
$= A \circ B(a, b)$
Also $(A \circ B)(x \wedge a, x \wedge b)$
$= sup_{z \in H}(min(A(x \wedge a, z), B(z, x \wedge b))$
$\geq min[A(x \wedge a, x \wedge z), B(x \wedge z, x \wedge b)]$
$\geq min(A(a, z), B(z, b))$

$\geq sup_{z\in H}(min(A(a,z),B(z,b))$

$= A\circ B(a,b)$

Finally,

$(A\circ B)(x\to a, x\to b)$

$= sup_{x\to z\in H}(min(A(x\to a, x\to z), B(x\to z, x\to b))$

$\geq min(A(x\to a, x\to z), B(x\to z, x\to b)$

$\geq min(A(a,z),B(z,b))$

$\geq A\circ B(a,b)$.

Therefore, $A\circ B$ is a fuzzy left compatible relation.Similarly $A\circ B$ is a fuzzy right compatible relation.Thus,we obtain $A\circ B$ is a fuzzy compatible relation. □

Example 2.4.6. Let $H = [a,b], a,b \in H$ be an interval.Then H is a HA with its operations.Then the fuzzy relation A on H defined by

$$A(x,y) = \begin{cases} 1 & \text{if } x = y; \\ 0.5 & \text{if } x \neq y. \end{cases}$$

is a fuzzy congruence relation on H.

Proposition 2.4.7. *Let A and B be fuzzy congruence relations on H. Then $A\circ B$ is a fuzzy congruence relation on H if and only if $A\circ B = B\circ A$.*

Let A be a fuzzy relation on H. For each $\alpha \in [0,1]$, we put $H_A(\alpha) = \{(a,b) : (a,b) \in H\times H, A(a,b) \geq \alpha\}$.This set is called the α - level set of A.

Theorem 2.4.8. *A fuzzy relation A is a fuzzy congruence relation on H if and only if for each $\alpha \in [0,1]$, the α level set $H_A(\alpha)$ is a congruence relation on H.*

2.5 Ideals and Homomorphisms of Fuzzy Heyting Algebras

Definition 2.5.1. Let $L = (X, A)$ and $M = (Y, B)$ FHAs . A mapping $h : X \to Y$ is a fuzzy homomorphism from L into M if, for all $x, y \in X$:
(i) $h(x \wedge y) = h(x) \wedge h(y)$;
(ii) $h(x \vee y) = h(x) \vee h(y)$;
(iii) $h(x \to y) = h(x) \to h(y)$;
(iv) $h(0) = 0$;

Like in crisp algebra, fuzzy homomorphisms can be classified as: (see [23])
(i) fuzzy monomorphism - injective fuzzy homomorphism;
(ii) fuzzy epimorphism - surjective fuzzy homomorphism;
(iii) fuzzy isomorphism - bijective fuzzy homomorphism.

Definition 2.5.2. The HAs L and K are isomorphic and the map $\phi : L \to K$ is an isomorphism if ϕ is one-to-one, onto and if $\phi(a \wedge b) = \phi(a) \wedge \phi(b), \phi(a \vee b) = \phi(a) \vee \phi(b)$ and $\phi(a \to b) = \phi(a) \to \phi(b)$, for all $a, b \in L$.

Proposition 2.5.3. *Let $L = (X, A)$ and $M = (Y, B)$ be FHAs and let a mapping $h : X \to Y$ be a fuzzy homomorphism.For all $x, y \in X, if A(x, y) > 0$, then $B(h(x), h(y)) > 0$.*

Proof. ($\Rightarrow$) By [Mezzomo et.al, proposition 5.1], each fuzzy homomorphism is a fuzzy order homomorphism. So, $A(x, y) > 0 \Rightarrow (h(x), h(y)) > 0$.
$(\Leftarrow) B(h(x), h(y)) > 0 \Rightarrow h(x) \vee h(y) = h(y)$ (By [Proposition 1.2.4]) $\Rightarrow h(x \vee y) = h(y) \Rightarrow x \vee y = y$ (Because h is injective) $\Rightarrow A(x, y) > 0$ (By Proposition 1.2.4) □

Definition 2.5.4. [36] Let X and Y be sets and $h : X \to Y$ be a map. So, for all $Z \subseteq X$, the set defined by $h(Z) = \{h(x) : x \in Z\}$ is called image of Z from Y induced by h.

On the other hand, for each $W \subseteq Y$, the set $h(W) = \{x \in X : h(x) \in W\}$ is called inverse image of W from X induced by h.

Notice that some fuzzy homomorphisms do not preserve ideals, i.e. if h is a fuzzy homomorphism and I is an ideal of L, then h(I) is not necessarily an ideal of M.The example below illustrates this case.

Example 2.5.5. Let $L = (X, A)$ and $M = (Y, B)$ be the FHA and let $h : L \to M$ be the fuzzy homomorphism defined by $h(x) = x^{'}, h(y) = y^{'}, h(z) = z^{'}$ and $h(w) = w^{'}$. The set $I = \{y, z, w\}$ is an ideal of L, but their image $h(I) = \{y^{'}, z^{'}, w^{'}\}$ is not an ideal of M because $y^{'} \in h(I)$ and $B(v^{'}, y^{'}) > 0$, but $v^{'} \in h(I)$. Therefore, I is an ideal of L and $h(I)$ is not an ideal of M.[see.[36]]

Proposition 2.5.6. *[36] Let $L = (X, A)$ and $M = (Y, B)$ be bounded fuzzy lattices, $I \subseteq X$ and $h : X \to Y$ a fuzzy homomorphism. Then, if $h(I)$ is an ideal of M, then I is an ideal of L.*

Proposition 2.5.7. *Let $L = (X, A)$ and $M = (Y, B)$ be FHA, $I \subseteq X$ and $h : X \to Y$ a fuzzy isomorphism. Then, $h(I)$ is an ideal of M iff I is an ideal of L.*

Proof. Suposse $h(I) = \{h(x) : x \in I\}$ is an ideal for M. Let $x^{'} = h(x)$ and $y^{'} = h(y)$ such that $B(x^{'}, y^{'}) = B(h(x), h(y)) > 0$.Since $h(I)$ is an ideal $x^{'} = h(x) \in h(I)$.This implies $x \in I$. Let $x^{'} = h(x)$ and $y^{'} = h(y)$ such that $x^{'} \vee y^{'} \in h(I)$.Then $x^{'} \vee y^{'} = h(x) \vee h(y) = h(x \vee y) \in h(I)$. This implies $x \vee y \in I$. Finally, let $x^{'} \in H$ and $a^{'} \in h(I)$ such that $x^{'} \to a^{'} \neq 1$.Then $x^{'} \to a^{'} = h(x) \to h(a) = h(x \to a) \in h(I)$.This implies $x \to a \in I$.Therefore, I is an ideal of L.Converse is easy to prove.

□

Proposition 2.5.8. *Let $L = (X, A)$ and $M = (Y, B)$ be FHA and let $h : X \to Y$ be a fuzzy homomorphism. If the inverse image induced by h is always finite, then the inverse image of all principal ideals of M are principal ideals of L.*

Definition 2.5.9. Let A be a fuzzy congruence relation on H. For every element $x \in H$, we define a subset $A_x = \{y \in H : A(x, y) = 1\}$ of H and the quotient HA of H is $H/A = \{A_x : x \in H\}$

Theorem 2.5.10. *If A is a fuzzy congruence relation of H,then H/A is a HA under the binary operations defined by* $A_x \vee A_y = A_{x\vee y}$, $A_x \wedge A_y = A_{x\wedge y}$ *and* $A_x \to A_y = A_{x\to y}$ $x, y \in H$

Proof. First we show that the above binary operations are well defined.In fact,if $A_x = A_{x'}$ and $A_y = A_{y'}$,then $A(x, x') = 1$ and $A(y, y') = 1$. Since $A(x, x') \leq A(x \vee y, x' \vee y)$ and $A(y, y') \leq A(x' \vee y, x' \vee y')$.
Also, $A(x \vee y, x' \vee y') \geq Sup_{x'\vee y\in H}(min(A(x \vee y, x' \vee y), A(x' \vee y, x' \vee y')))$

$$\geq Sup(min(A(x, x'), A(y, y'))) = 1$$

, which implies $A(x \vee y, x' \vee y') = 1$.Thus, $A_{x\vee y} = A_{x'\vee y'}$.
Therefore, the operation $'\vee'$ is well defined.Similarly,the operation $'\wedge'$ is also well defined.
Now,by lemma 2.4.4,we have $A(x, x') \leq A(x \to y, x' \to y)$ and $A(y, y') \leq A(x' \to y, x' \to y')$, in similar approach,$A(x \to y, x' \to y') \leq 1$.
$A(x \to y, x' \to y') \geq 1$.
By Theorem 1.2.9,we have $A_{x\to y} = A_{x'\to y'}$.Thus,$' \to'$ is well defined.
We shall show that $H/A = \{A_x : x \in H\}$
(i) Let $A_x \in H/A$.Then $A_x \to A_x = A_{x\to x} = A_1$
(ii) Let $A_x, A_y \in H/A, x, y \in H$.Then $A_y \wedge (A_x \to A_y)$=$A_y \wedge (A_{x\to y})$=$A_{y\wedge(x\to y)}$=A_y.
$\Rightarrow A_y \leq A_x \to A_y$.
(iii)Let A_x, A_y and $A_z \in H/A$.Then
$A_x \to (A_y \wedge A_z) = A_x \to A_{y\wedge z}$
=$A_{x\to(y\wedge z)} = A_{(x\to y)\wedge(x\to z)}$
=$(A_x \to A_y) \wedge (A_x \to A_z)$

(iv) $A_x \wedge (A_x \rightarrow A_y)$

$= A_x \wedge (A_x \rightarrow y))$=$A_{x\wedge(x\rightarrow y)} = A_{x\wedge y}$

$= A_x \wedge A_y$

(v) $(A_x \vee A_y) \rightarrow A_z = A_{x\vee y} \rightarrow A_z$

=$A_(x \vee y) \rightarrow z$=$A_{(x\rightarrow z)\wedge(y\rightarrow z)}$

$= (A_x \rightarrow A_z) \wedge (A_y \rightarrow A_z)$. Hence,$(H/A, \wedge, \vee, \rightarrow, A_0, A_1)$ is a HA which is called a quotient HA. □

Lemma 2.5.11. *Let H and H' be HA'S and f be a homomoriphism from H to H'.If A' is a fuzzy congruence relation on H',then the map defined by* $f^{-1}(A')(x,y) = A'(f(x), f(y))$*,for all* $x, y \in H$ *is a fuzzy congruence relation on H.*

Proof. For all $x, y, z \in H$, $f^{-1}(A')(x,x) = 1, f^{-1}(A')(x,y) = A'(f(x), f(y) = A'(f(y), f(x))$ which means $f^{-1}(A')$ is fuzzy reflexive and fuzzy symmetric relation on H.

Since $f^{-1}(A')(x,y) = A'(f(x), f(y)) \geq Sup_{f(z)\in H'}(min(A'(f(x), f(z)), A'(f(z), f(y)))$

$\geq min(A'(f(x), f(z)), A'(f(z), f(y))$

$= min(f^{-1}(A')(x,z), f^{-1}(A')(z,y))$

$\geq sup_{z\in H}(min(f^{-1}(A')(x,z), f^{-1}(A')(z,y)))$.

Therefore,$f^{-1}(A')$ is a transitive relation of H.So $f^{-1}(A')$ is a fuzzy equivalence relation.

Again,$f^{-1}(A')(z \vee x, z \vee y)$

$= A'(f(z \vee x), f(z \vee y))$=$A'(f(z) \vee f(x), f(z) \vee f(y))$

$\geq A'(f(x), f(y)) = f^{-1}(A')(x,y)$.

Similarly, $f^{-1}(A')(z \wedge x, z \wedge y) \geq f^{-1}(A')(x,y)$.

Further,$f^{-1}(A')(z \rightarrow x, z \rightarrow y)$=$A'(f(z \rightarrow x), f(z \rightarrow y)) \geq A'(f(z) \rightarrow f(x), f(z) \rightarrow f(y)) \geq A'(f(x), f(y) = f^{-1}(A')(x,y)$.

This means that $f^{-1}(A')$ is a fuzzy left compatible relation on a Heyting algebra H.By the same argument,we can see that $f^{-1}(A')$ is a fuzzy right compatible relation of H.Therefore,$f^{-1}(A')$ is a fuzzy congruence relation on H'. □

Theorem 2.5.12. *[Fuzzy First Isomorphism Theorem.] Let H ,$H^{'}$ be HA's, f be an epimorphism from H to $H^{'}$,and $A^{'}$ be a fuzzy congruence relation on $H^{'}$.Then* $\frac{H}{f^{-1}(A')} \cong \frac{H'}{A'}$

Proof. It follows from definition 2.5.9,Theorem 2.5.10 and Lemma 2.5.11 $H/f^{-1}(A')$ and H'/A' are both quotient HA's.We define a map h from $H/f^{-1}(A')$ to H'/A' by $h(f^{-1}(A')_x) = A'_x, x \in H$.By Definition 2.5.9,

(i) h is well defined.

Suppose $f^{-1}(A')_x = f^{-1}(A')_y$,then $f^{-1}(A')(x, y) = 1$.

$\Rightarrow A'(f(x), f(y)) = 1$

$\Rightarrow A'_{f(x)} = A'_{f(y)}$[Theorem 1.2.9].

Therefore h is well defined.

(ii) h is homomorphism.

(a). $h(f^{-1}(A')_x \vee f^{-1}(A')_y) = h(f^{-1}(A')_{x\vee y})$=$A'_{f(x\vee y)}$

=$A'_{f(x)\vee f(y)}$=$A'_{f(x)} \vee A'_{f(y)}$=$h(f^{-1}(A')_x) \vee h(f^{-1}(A')_y)$

(b). $h(f^{-1}(A')_x \wedge f^{-1}(A')_y) = h(f^{-1}(A')_{x\wedge y})$=$A'_{f(x\wedge y)}$

=$A'_{f(x)\wedge f(y)}$=$A'_{f(x)} \wedge A'_{f(y)}$=$h(f^{-1}(A')_x) \wedge h(f^{-1}(A')_y)$

(c). $h(f^{-1}(A')_x \to f^{-1}(A')_y) = h(f^{-1}(A')_{x\to y})$=$A'_{f(x\to y)}$

=$A'_{f(x)\to f(y)}$=$A'_{f(x)} \to A'_{f(y)}$=$h(f^{-1}(A')_x) \to h(f^{-1}(A')_y)$

Hence,from a, b and c, we have h is a homomorphism.

(iii) h is an epimorphism: For $A'_y \in H'/A', y \in H'$.Since f is onto,there exists $x \in H$,such that $f(x) = y$ so $h(f^{-1}(A')_x) = A'_{f(x)} = A'_{y'}$

(iv) h is monomorphism.Suppose $h(f^{-1}(A')_x) = h(f^{-1}(A')_y)$,then $A'_{f(x)} = A'_{f(y)} \Rightarrow A'(f(x), f(y)) = 1 \Rightarrow f^{-1}(A')(x, y) = 1$ Hence,$f^{-1}(A')_x = f^{-1}(A')_y$ this means h is a monomorphism. In conclusion, $\frac{H}{f^{-1}(A')} \cong \frac{H'}{A'}$.)

□

Corollary 2.5.13. *Let A be a fuzzy congruence relation on H.Then the mapping $\pi : H \to H/A$ defined by $\pi(x) = A_x, \forall x \in H$,is a homomorphism.*

Proof. $\pi(a \wedge b) = A_{a\wedge b} = A_a \wedge A_b = \pi(a) \wedge \pi(b)$. $\pi(a \vee b) = A_{a\vee b} = A_a \vee A_b = \pi(a) \vee \pi(b)$ and $\pi(a \to b) = A_{a\to b} = A_a \to A_b = \pi(a) \to \pi(b)$.Hence the corollary. □

Lemma 2.5.14. *Let A be a fuzzy congruence relation of H.Let $H_A = \{y \in H : A(y, 1) = 1\}$. Then H_A is an ideal of H.*

Proof. (i)Let $x \in H$ and $y \in H_A$.Then $A(y, 1) = 1.A(x, 1)= A(x \wedge y, 1) \geq min\{A(x, 1), A(y, 1)\}$ $min(1, 1) = 1$. Thus,$x \wedge y \in H_A$

(ii) Let $x \in H_A$ and $y \in H_A$.Then $A(x, 1) = 1, A(y, 1) = 1$. Then $A_1 = A_x$ and $A_1 = A_y$. $\Rightarrow A_1 \vee A_1 = A_x \vee A_y \Rightarrow A_{1\vee 1} = A_{x\vee y}$. $\Rightarrow A_1 = A_{x\vee y} \Rightarrow A(1, x \vee y) = 1.Thus, x \vee y \in H_A$.

(iii) $x, y \in H_A$ Then $A(y, 1) = 1$. Then $A_1 = A_y \Rightarrow A(x \to y, 1) \geq A(1, x) \wedge A(y, 1) = 1.Thus, x \to y \in H_A$.

Hence H_A is an ideal of H. □

Lemma 2.5.15. *Let A be a fuzzy congruence relation of H.Let $H_A = \{y \in H : A(0, y) = 1\}$. Then H_A is an not ideal of H.*

Proof. (i) Let $x \in H$ and $y \in H_A$ such that $A(x, y) > 0$ .Then $A(0, y) = 1.A(0, x)=$ $A(x \wedge 0, x \wedge y) \geq min\{A(x, x), A(0, y)\} = min(1, 1) = 1$. Thus,$x \in H_A$

(ii) Let $x \in H_A$ and $y \in H_A$.Then $A(0, x) = 1, A(0, y) = 1$. Then $A_0 = A_x$ and $A_0 = A_y$. $\Rightarrow A_0 \vee A_0 = A_x \vee A_y \Rightarrow A_{0\vee 0} = A_{x\vee y}$. $\Rightarrow A_0 = A_{x\vee y} \Rightarrow A(0, x \vee y) = 1.Thus, x \vee y \in H_A$.

(iii)$x \in H_A$ and $y \in H_A$. Then $A(0, x) = 1, A(0, y) = 1$. Then $A_0 = A_x$ and $A_0 = A_y \Rightarrow A_0 \to A_0 = A_x \to A_y \Rightarrow A_{0\to 0} = A_{x\to y}$. $\Rightarrow A_1 = A_{x\to y} \Rightarrow A(1, x \to y) = 1.Thus, x \to y \notin H_A$. □

Lemma 2.5.16. *Let I be an ideal of H, A and B are fuzzy congruence relations of H. (i) If A is restricted to I, then A is a fuzzy congruence relation of I*
(ii) $A \cap B$ is fuzzy congruence relation of H
(iii) I/A is an ideal of H/A.

Proof. (i) is clear (ii) For any $x, y \in H$,since $(A \cap B)(x, y) = min(A(x, y), B(x, y))$. Then $(A \cap B)(x, y)$is fuzzy reflexive and fuzzy symmetric relations.We only show that $A \cap B$ is a fuzzy transitive relations.Since $(A \cap B)(x, y)$
$= min(A(x, y), B(x, y))$
$\geq min(A(x, z), A(z, y), B(x, z), B(z, y))$
$=min(A(x, z), B(x, z), A(z, y), B(z, y))$
$\geq min((A \cap B)(x, z), (A \cap B)(z, y)))$
$\geq Sup_{z\in H}(min((A \cap B)(x, z), (A \cap B)(z, y)))$
Hence, $A \cap B$ is a fuzzy transitive relation on H.
Furthermore; for every $a \in$ H, $(A\cap B)(a\vee x, a\vee y) = min(A(a\vee x, a\vee y), B(a\vee x, a\vee y))$
$\geq min(A(x, y), B(x, y))$ $=(A \cap B)(x, y)$.
Similarly, $(A \cap B)(a \wedge x, a \wedge y) \geq (A \cap B)(x, y)$.
$(A \cap B)(a \to x, a \to y) = min(A(a \to x), B(a \to y))$
$= min(A(a \to x, a \to y), B(a \to x, a \to y))$
$\geq min(A(x, y), B(x, y)) = (A \cap B)(x, y)$. This means that $A \cap B$ is a fuzzy left compatible relation.Similarly, $A \cap B$ is a fuzzy right compatible relation.
Hence ,$A \cap B$ is a fuzzy congruence relation.
(iii) First,we show that $\{A_a : a \in I\}$ is an ideal of H/A.
For any $A_a, A_b \in \{A_a : a \in I\}$.
Since I is an ideal,then $a \vee b \in$ I,hence $A_a \vee A_b = A_{a\vee b} \in \{A_a : a \in H\}$. For any A_a $\in \{A_a : a \in I\}$,$A_x \in H/A$,$a \in I, x \in H$,then $a \wedge x, x \wedge a, x \to a \in I$.
Hence $A_a \wedge A_x = A_{a\wedge x}$ and $A_x \to A_a = A_{x\to a} \in \{A_a : a \in I\}$.
Thus,$\{A_a : a \in I\}$ is an ideal of H/A. □

Theorem 2.5.17 (Fuzzy second isomorphism theorem)**.** *Let A and B be two fuzzy* *congruence relations of a Heyting algebra H with $A_1 \subseteq B_1$. Then $H_A \vee H_B/B \cong H_A/A \cap B$.*

Proof. By lemma 2.5.16,B is a fuzzy congruence relation of $H_{A\vee B}$ and $A \cap B$ is a fuzzy congruence relation of H_A.Thus $H_{A\vee B}/B$ and $H_A/A \cap B$ are both HA's.For any $x \in H_{A\vee B}$,then $x = a \vee b$,where $a \in H_A$,$b \in H_B$,it is implies $A(1, a) = 1$ and $B(1, b) = 1$. Define $f : (H_A \vee H_B)/B \to H_A/A \cap B$ by $f(B_x) = (A \cap B)_a$.If $B_x = B_{x'}$,then $x' = a' \vee b'$ where $a' \in H_A, b' \in H_B$,then,we have $A(1, a') = 1, B(1, b') = 1, B(x, x') = B(a \vee b, a' \vee b') = 1$. Since $A(a, a') \geq A(a, 1) \vee A(1, a') = 1$,and so $A(a, a') = 1$.Similarly $B(b, b') = 1$.

From definition 2.5.9 and lemma 2.5.14 with $A_1 \subseteq B_1$,we have $H_A \subseteq H_B$. Therefore,$a, a' \in H_B$ which implies $B(1, a) = 1, B(1, a') = 1$.Since $B(a, a') \geq min(B(a, 1), B(1, a')) = 1$.This gives $B(a, a') = 1$.From definition,$(A \cap B)(a, a') = min(A(a, a'), B(a, a')) = 1$.$(A \cap B)(a, a') = 1$.Hence, $(A \cap B)_a = (A \cap B)_{a'}$ which means f is well defined.For any $B_x, B_y \in H_A \vee H_B/B$,where $x = a \vee b, y = a_1 \vee b_1, a, a_1 \in H_A, b, b_1 \in H_B$.Then $x \vee y$ and $x \wedge y \in H_A \vee H_B$.,We have $f(B_x \vee B_y) = f(B_{x\vee y}) = (A \cap B)_{a\vee a_1} = (A \cap B)_a \vee (A \cap B)_{a_1}$=$f(B_x) \vee f(B_y)$.

$f(B_x \wedge B_y) = f(B_{x\wedge y}) = (A \cap B)_{a\wedge a_1}$=$f(B_x) \wedge f(B_y)$. $f(B_x \to B_y) = f(B_{x\to y}) = (A \cap B)_{a\to a_1} = f(B_x) \to f(B_y)$. □

Theorem 2.5.18 (Fuzzy Third Isomorphism Theorem)**.** *Let A,B be two fuzzy congruence relations of a Heyting algebra H with $A \subseteq B$. Then $(H/A)/(H_B/A) \cong H/B$.*

Proof. Clearly,H_B/A is an ideal of H/A. Define $f : H/A \to H/B$ by $f(A_x) = B_x$ for all $x \in H$.If $A_x = A_y$,then $A(x, y) = 1$.Since $A \subseteq B$,so $B(x, y) \geq A(x, y) = 1$,thus $B(x, y) = 1, B_x = B_y$.Hence f is well-defined.

$f(A_x \vee A_y) = f(A_{x\vee y}) = B_{x\vee y} = B_x \vee B_y = f(A_x) \vee f(A_y)$.

Similarly,$f(A_x \wedge A_y) = f(A_x) \wedge f(A_y)$. Moreover; $f(A_x \to A_y)$=$f(A_{x\to y})$=$B_{x\to y}$=$B_x \to B_y$. For any $B_x \in H/B$,there exists $A_x \in H/A$ such that $f(A_x) = B_x$,so f is an epimorphism.Now,we show that ker $f = H_B/A$.$kerf = \{A_x \in H/A : f(A_x) =$

$B_1\}$.=$\{A_x \in H/A : B(1,x) = 1\} = \{A_x \in H/A : x \in H_B\} = H_B/A$.
Therefore,$(H/A)/(H_B/A) \cong H/B$.This complets the proof. □

We denote by χ_f the characteristic function of the binary relation f on H.Then we have the following conclusions.

Proposition 2.5.19. *Let f be a binary relation on a HA H.Then ker f is equivalence(a congruence) on H iff χ_{kerf} is a fuzzy equivalence(a fuzzy congruence) on H.*

Let A be a fuzzy equivalence relation on H,and let $a \in H$,We define a fuzzy subset A_a on H as follows. $A_a(x) = A(a,x), \forall a \in H$.Then we have the following.

Proposition 2.5.20. *Let A be a fuzzy congruence relation on H.Then*

$$A^{-1}(1) = \{A(a,b) = 1, a, b \in H\}$$

is a congruence relation on H.

proof: Since $A_a = A_a$.Then $A(a,a) = 1$,hence $A^{-1}(1)$ is reflexive.Since A is $FCR(H)$,then $A(a,b) = A(b,a) = 1$,Hence $A^{-1}(1)$ is symmetric. Let $A(a,b) = 1$ and $A(b,c) = 1$.Then $A(a,b) \wedge A(b,c) \leq A(a,c), \forall a,b,c \in H$. Therefore $A(a,c) \geq Sup_{b\in H}(min(A(a,b), A(b,c)))$.Hence, $A^{-1}(1)$ is transitive.Moreover from the definition of $FCR(H)$,We have $A(a \to x, b \to x) \geq A(a,b) = 1$.This implies $A(a \to x, b \to x) = 1$.Similarly $A(x \to a, x \to b) \geq A(a,b) = 1$.Also $\wedge$ and $\vee$ are left and right compatibles.Hence the result follows.

Homomorphism Theorems

In this section,we declare that FCR(H) means fuzzy congruence relation on H. Let H_1 and H_2 be two HAs and f be a homomorphism of H_1 to H_2.Then the relation $ker(f) = \{(a,b) : f(a) = f(b), a, b \in H_1\}$ is a congruence relation on H_2.The characteristic function χ_f defined by

$$\chi_{kerf}(a,b) = \begin{cases} 1 & \text{if } f(a) = f(b); \\ 0 & \text{if } f(a) \neq f(b). \end{cases}$$

is a fuzzy congruence relation on H_2.

Theorem 2.5.21. *Let $H_1$and H_2 be two HAs and $f : H_1 \to H_2$ a homomrphism.Then $\chi_{ker(f)}$ is a fuzzy congruence on H_1 and there is a homomorphism $g : H_1/\chi_{ker(f)} \to H_2$ such that $f = g \circ (\chi_{ker(f)})$*

Proof. Define $g : H_1/\chi_{ker(f)} \to H_2$ by $g((\chi_{ker(f))_a}) = f(a), \forall a \in H_1$.
Let $a, b \in H_1$.Then $(\chi_{ker(f)})_a = (\chi_{ker(f)})_b$
$\Rightarrow \chi_{ker(f)}(a,b) = 1$
so $(a,b) \in ker(f)$.Thus,we have $g((\chi_{ker(f)})_a) = f(a) = f(b) = g((\chi_{ker(f)})_b)$
Therefore, g is well defined.
If $f(a) = f(b)$,then $(a,b) \in ker(f)$
$\Rightarrow \chi_{ker(f)}(a,b) = 1$
$\Rightarrow (\chi_{ker(f)})_a = (\chi_{ker(f)})_b$
Hence g is one to one. Let $a, b \in H_1, g((\chi_{ker(f)_a}) \to (\chi_{ker(f)_b}))$
$= g((\chi_{ker(f)})_{a \to b})$
$= f(a \to b) = f(a) \to f(b)$
$= g((\chi_{ker(f)})_a) \to g((\chi_{ker(f)})_b)$
Again let $a, b \in H_1, g((\chi_{ker(f)_a}) \wedge (\chi_{ker(f)_b}))$
$= g((\chi_{ker(f)})_{a \wedge b})$
$= f(a \wedge b) = f(a) \wedge f(b)$
$= g((\chi_{ker(f)})_a) \wedge g((\chi_{ker(f)})_b)$
Let $a, b \in H_1, g((\chi_{ker(f)_a}) \vee (\chi_{ker(f)_b}))$
$= g((\chi_{ker(f)})_{a \vee b})$
$= f(a \vee b) = f(a) \vee f(b)$
$= g((\chi_{ker(f)})_a) \vee g((\chi_{ker(f)})_b)$

Therefore,g is a homomorphism.

Let $a \in H_1, g((\chi_{ker(f)})_a) = f(a)$. So we obtain that $g \circ (\chi_{ker(f)}) = f$ □

Theorem 2.5.22. *Let A and B be $FCR(H)$ such that $A \subseteq B$.then there is one unique homomorphisms $g : H/A \to H/B$ such that $g \circ \pi_A = \pi_B$ and $H/A/\chi_{ker(g)} \cong H/B$*

Proof. Define $g : H/A \to H/B$ by setting $g(A_a) = B_a, a \in H$.

Assume that $A_a = A_b$,then $1 = A(a, b) \leq B(a, b)$,So $B(a, b) = 1$ That is,$B_a = B_b$,then g is well defined. □

Definition 2.5.23. Let H be HA and $A, B \in FCR(H)$ with $A \leq B$.Then the fuzzy relation B/A on H/A defined by $(B/A)(A_x, A_y) = B(x, y), \forall x, y \in H$ is a $FCR(H/A)$

Proof. B/A is well defined Let $x, x_1, y, y_1 \in H$ with $A_x = A_{x_1}$ and $B_y = B_{y_1}$.Now,

$(B/A)(A_x, A_y) = B(x, y) \geq min\{B(x, x_1), B(x_1, y_1), B(y_1, y)\}$

$\geq min\{B(x, x_1), B(x_1, y_1), A(y_1, y)\}$

$=B(x_1, y_1)$

$= (B/A)(A_{x_1}, A_{y_1})$.

Interchanging x, x_1 and y, y_1,we get similarly that $(B/A)(A_{x_1}, A_{y_1}) \geq (B/A)(A_x, A_y)$ Hence the result follows. It is is evident that from the above definition $B/A \in FCR(H/A)$ and the following propositions are easy to prove. □

Proposition 2.5.24. *Let H be a HA $B, A \in FCR(H)$ and $A \leq B$,then $\pi_A^{-1}(B/A) = B$*

Proposition 2.5.25. *Let A be a fuzzy congruence relation on H and B be Fuzzy congruence relation on H/A,then $\pi_A^{-1}(B)/A = B$.*

Now,we have the following analog to the correspondence theorem.

Theorem 2.5.26. *Let H be a HA and $A \in FC(H), B \in FC(H/A), \pi_A^{-1}(B) \geq A$. Let $(L, \leq)$ be the sublattice of FC(H),where $L = \{C \in FC(H) : C \geq A\}$. Then the map $\alpha : FCR(H/A) \to L$ defined by $\alpha(A) = \pi_A^{-1}(B)$ is a lattice isomorphism.*

Define $\beta : L \to FC(H/A)$ by $\beta(C) = C/A, C \in L$.Applying the above propositions,we get $(\alpha \circ \beta)(C) = \alpha(C/A) = \pi_A^{-1}(C/A) = C$ and $(\beta \circ \alpha)(B) = B(\pi_A^{-1}(B)) = \pi_A^{-1}(B)/A = B$,and $B \in FC(H/A)$ with $\pi_A^{-1}(B) \geq A$.Thus both the mappings are one to one and onto.Moreover,if $A_1, A_2 \in FC(H/A)$ such that $A_1 \leq A_2$,then it is easy to see that $\alpha(A_1) \leq \alpha(A_2)$ and conversely,if $C_1, C_2 \in L$ such that $C_1 \leq C_2$,then $\beta(C_1) \leq \beta(C_2)$ Therefore,α is a lattice isomorphism. The following theorem is fuzzy analog of the second isomorphism theorem.

Theorem 2.5.27. *Let H be a HA, $A, B \in FC(H), A \leq B$. Then $H/A \cong (H/A)/(B/A)$.*

Proof. Define $\alpha : (H/A)/(B/A) \to H/B$ by $\alpha((B/A)_{A_x}) = B_x$
$(B/A)_{A_x} = (B/A)_{A_y}$
$\Leftrightarrow (B/A)(A_x, A_y) = 1 \Leftrightarrow B(x, y) = 1 \Leftrightarrow B(x, y) = 1 \Leftrightarrow B_x = B_y$. Thus,$\alpha$ is well defined and one to one.It is clear that α is onto and easy to see that α is a homomorphism. $\square$

Chapter 3

Fuzzy Ideals and Fuzzy Filters of Heyting Algebra

Throughout this chapter, $H = (H, \vee, \wedge, \rightarrow, 0, 1)$ denotes a Heyting lattice and FS(H) denotes the set of fuzzy subsets of H (i.e. of maps from H into $([0,1], \vee, \wedge, \rightarrow)$, where $[0,1]$ is the set of reals between 0 and 1 and $x \vee y = max(x,y), x \wedge y = min(x,y)$). This section reflects the characterization of fuzzy ideals and fuzzy filters of Heyting algebras interms of their level sets.

Definition 3.0.28. A fuzzy subset μ of H is called fuzzy ideal of H if the following conditions are satisfied.

1. $\mu(0) = 1$
2. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$
4. $\mu(x \rightarrow y) \geq \mu(x) \vee \mu(y), \forall x, y \in H$

Remark 3.0.29. If μ is a fuzzy ideal of H.Then $\mu(x \wedge y) \wedge \mu(x \rightarrow y) \geq \mu(x) \vee \mu(y)$

Proof. Follows from 3 and 4 □

Example 3.0.30. Let $H = \{0, a, b, 1\}$ with $0 < a < b < 1$. Then $\vee, \wedge$ and $\rightarrow$ are defined as follows:

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	b	1
b	0	a	1	1
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Then $(H, \wedge, \vee, \rightarrow, 0, 1)$ is a HA.define μ as follows $\mu(0) = 1, \mu(a) = 0.8 = \mu(b) = \mu(1)$.Then μ is a fuzzy ideal of H

Lemma 3.0.31. *Let μ be a fuzzy ideal of H.Then $\mu(x) \leq \mu(0)$ for all $x \in H$*

Proof. Suppose the condition, $\mu(x) \leq \mu(0)$ not true.Then $\mu(x) > \mu(0) = 1$. which is a contradiction.Hence the result holds. □

Definition 3.0.32. Let μ and θ be fuzzy subsets of H.Then $\mu \circ \theta$ defined by $\mu \circ \theta(x) = Sup_{x=y\wedge z}\{min(\mu(y), \theta(z))\}$ for all $x \in H$

Definition 3.0.33. Let H be HA and μ fuzzy subset of H and we call μ is a fuzzy filter of H if it satisfies the following properties

1. $\mu(1) = 0$
2. $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(x \rightarrow y) \geq \mu(x) \vee \mu(y)$
4. $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$,$\forall x, y \in H$,

Clearly, $\mu(1) \leq \mu(x)$,$\forall x \in H$

Remark 3.0.34. If μ is a fuzzy filter of H.Then $\mu(x \rightarrow y) \wedge \mu(x \vee y) \geq \mu(x) \vee \mu(y)$,$\forall x, y \in H$,

Lemma 3.0.35. *A fuzzy subset μ of H is said to be a fuzzy ideal(fuzzy filter) of H iff $\mu_t, t \in im\mu$,is an ideal(filter)*

Proof. Suppose μ is a fuzzy ideal of H.We shall show that μ_t is an ideal of H. Since $\mu(0) = 1, 0 \in \mu_t$.Let $x, y \in \mu_t$.Then $\mu(x) \geq t$ and $\mu(y) \geq t$.This implies $\mu(x \vee y) \geq \mu(x) \wedge \mu(y) \geq t$.Hence $x \vee y \in \mu_t$.Next let $x \in \mu_t$ and $y \in$ H,we claim to show that $x \wedge y \in \mu_t$.By hypothesis $\mu(x \wedge y) \geq \mu(x) \vee \mu(y) \geq \mu(x) \geq t$.Hence $x \wedge y \in \mu_t$.Finally,Let $x \in \mu_t$ and $y \in H$.Then $\mu(x) \geq t$.We shall show that $y \to x \in \mu_t$.By hypothesis,$\mu(y \to x) \geq \mu(y) \vee \mu(x) \geq t$.Hence the result.
Conversely,suppose μ_t is an ideal of H. Let $x, y \in H$ and $\mu(x) = t_1$ and $\mu(y) = t_2$.Then Take $t = min(t_1, t_2)$.This implies $\mu(x) = t_1 \geq t$ and $\mu(y) = t_2 \geq t$
$\Rightarrow x \in \mu_t$ and $y \in \mu_t$. Then $x \vee y \in \mu_t \Rightarrow \mu(x \vee y) \geq t = min(\mu(x), \mu(y))=\mu(x) \wedge \mu(y)$.Next, let $\mu(x) = t$.Then $x \in \mu_t$ which implies $x \wedge y \in \mu_t$ for $y \in H$.This gives $\mu(x \wedge y) \geq t = \mu(x)$.Similarly,$\mu(x \wedge y) \geq t = \mu(y)$.Hence $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.Finally, let $\mu(x) = t$.Then for $y \in H$,we have $y \to x \in \mu_t$.This gives $\mu(y \to x) \geq t = \mu(x)$.Let $\mu(y) = s$.
Assume $s \leq t.\mu_t \subseteq \mu_s$.Since $x \in \mu_t$ and $y \in \mu_s$,
we have $y \to x \in \mu_s$.
$\Rightarrow \mu(y \to x) \geq s = \mu(y)$.
$\Rightarrow \mu(y \to x)$ is an upper bound of $\{s, t\}$. For $s > t$,the result also holds true.
Thus,$\mu(y \to x) \geq \mu(y) \vee \mu(x)$.Therefore,$\mu$ is a fuzzy ideal of H. For the filter case,the forward part is clear and we only prove the backward case. Take $\mu(x) = t_1$ and $\mu(y) = t_2$ and $t = t_1 \wedge t_2$.This gives $x \in \mu_{t_1} \subseteq \mu_t$. Similarly, $y \in \mu_t$.
Hence, $x \wedge y \in \mu_t$ .Therefore,$\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.
Let again $x \in \mu_t, y \in H$ with $\mu(y) = s$,$t \leq s$.Then $x \to y \in \mu_t$.This gives $\mu(x \to y) \geq \mu(x)$ and similarly $\mu(x \to y) \geq \mu(y)$.Therefore,the result follows. □

Let μ and θ be a FS(H).The cartesian product of μ and θ is defined by $(\mu \times \theta)(x, y) = min(\mu(x), \theta(y)), \forall x, y \in H$

Theorem 3.0.36. *Let μ and θ be a fuzzy ideal(fuzzy filter) of a HA H,then $\mu \times \theta$ is a fuzzy ideal (fuzzy filter) of $H \times H$*

Proof. Since $(0,0) \in H \times H$, $\mu \times \theta((0,0) \wedge (0,0))$

$= \mu \times \theta(0 \wedge 0, 0 \wedge 0)$

$=\mu(0 \wedge 0) \wedge \theta(0 \wedge 0)$

$\geq (\mu(0) \vee \mu(0)) \wedge (\theta(0) \vee \theta(0))$

$=(\mu(0) \wedge \theta(0)) \vee (\mu(0) \wedge \theta(0))$

$=\mu \times \theta(0,0) \vee \mu \times \theta(0,0).$

Therefore,$\mu \times \theta((0,0) \wedge (0,0)) \geq \mu \times \theta(0,0) \vee \mu \times \theta(0,0).$

Let $(a,b),(c,d) \in H \times H$.Then

$\mu \times \theta((a,b) \wedge (c,d))$

$= \mu \times \theta(a \wedge c, b \wedge d)$

$=\mu(a \wedge c) \wedge \theta(b \wedge d)$

$\geq (\mu(a) \vee \mu(c)) \wedge (\theta(b) \vee \theta(d))$

$=(\mu(a) \wedge \theta(b)) \vee (\mu(c) \wedge \theta(d))$

$=\mu \times \theta(a,b) \vee \mu \times \theta(c,d).$

Therefore,$\mu \times \theta((a,b) \wedge (c,d)) \geq \mu \times \theta(a,b) \vee \mu \times \theta(c,d).$

$\mu \times \theta((a,b) \rightarrow (c,d))$

$= \mu \times \theta(a \rightarrow c, b \rightarrow d)$

$=\mu(a \rightarrow c) \wedge \theta(b \rightarrow d)$

$\geq (\mu(a) \vee \mu(c)) \wedge (\theta(b) \vee \theta(d))$

$=(\mu(a) \wedge \theta(b)) \vee (\mu(c) \wedge \theta(d))$

$=\mu \times \theta(a,b) \vee \mu \times \theta(c,d).$

Therefore,$\mu \times \theta((a,b) \rightarrow (c,d)) \geq \mu \times \theta(a,b) \vee \mu \times \theta(c,d).$

Similarly it is easy to prove that,$\mu \times \theta((a,b) \vee (c,d)) \geq \mu \times \theta(a,b) \wedge \mu \times \theta(c,d)$

Hence,$\mu \times \theta$ is fuzzy ideal of $H \times H$.

For fuzzy filter, $\mu \times \theta((x,y) \wedge (z,r))$

$= \mu(x \wedge z) \wedge \theta(y \wedge r).$

Similarly it is easy for the other criteria to verify.Hence it is fuzzy filter. □

Lemma 3.0.37. *A fuzzy subset $\mu\times\theta$ of $H\times H$ is said to be fuzzy ideal(fuzzy filter) iff the level ideal $(\mu\times\theta)_t, t\in im(\mu\times\theta)$ is an ideal(filter)of $H\times H$.*

By theorem ,we know that $\mu\times\theta$ is a fuzzy ideal of $H\times H$.So we must show that the level ideals are prime ideals.Let $t\in im(\mu\times\theta)$ and $(x,y)\wedge(z,r)\in(\mu\times\theta)_t$.Then,$\mu\times\theta((x,y)\wedge(z,r))\geq t$.
$\Rightarrow \mu(x\wedge z)\wedge\theta(y\wedge r)\geq t$
$\Rightarrow \mu(x\wedge z)\geq t$ and $\theta(y\wedge r)\geq t$
$\Rightarrow x\wedge z\in\mu_t$ and $y\wedge r\in\theta_t$
Since μ_t and θ_t are prime ideals of H,we have $(x\wedge z, y\wedge r)\in\mu_t\times\theta_t=(\mu\times\theta)_t$.
Therefore,$(x,z)\wedge(y,r)\in(\mu\times\theta)_t$.Hence, the result follows.Converse is easy to prove.

Definition 3.0.38. Let μ , θ be any two fuzzy ideals of H.Then the join,the meet and the arrow operators $\mu\vee\theta$, $\mu\wedge\theta$,$\mu\rightarrow\theta$ of μ and θ is defined respectively, by $(\mu\vee\theta)(x)=Sup_{x=y\vee z}(min(\mu(y),\theta(z)))$, $(\mu\wedge\theta)(x)=Sup_{x=y\wedge z}(min(\mu(y),\theta(z)))$and $(\mu\rightarrow\theta)(x)=Sup_{x=y\rightarrow z}(min(\mu(y),\theta(z))), x,y,z\in H$.

Lemma 3.0.39. *Let f be a function from a set S to a set S' μ, θ be any two FS (S) and μ',θ' be any two FS(S').Then the following statements are true:*
(i) $f(f^{-1}(\mu'))=\mu'$, $\mu\subseteq f^{-1}(f(\mu))$
(ii) $f^{-1}(f(\mu))=\mu$,provided that μ is f- invariant;
(iii) $\mu\subseteq\theta\Rightarrow f(\mu)\subseteq f(\theta)$;
(iv)$\mu'\subseteq\theta'\Rightarrow f^{-1}(\mu')\subseteq f^{-1}(\theta')$

Proof. i) Let y be any arbitrary element of S'.
Then $(f(f^{-1}(\mu'))(y)=\begin{cases} Sup(f^{-1}(\mu'),(x)_{x\in f^{-1}(y)} & \text{if } f^{-1}(y)\neq\emptyset \\ 0 & otherwise\end{cases}, y\in T$
$=Sup\mu'(f(x)),f^{-1}(y)\neq\emptyset$

$=Sup\mu^{'}(y)$,since $f(x) = y, forall x \in f^{-1}(y)$

$=\mu^{'}(y)$. for all $y \in S^{'}$ such that $f(x) = y$.

Hence $f(f^{-1}(\mu^{'}))=\mu^{'}$

Claim: $\mu \subseteq f^{-1}(f(\mu))$

Let $x \in S$, such that $f^{-1}(f(\mu))(x)=f(\mu)f(x) = \begin{cases} sup\mu(x)_{x\in f^{-1}(f(x))} & \text{if } f^{-1}(f(x)) \neq \emptyset \\ 0 & otherwise \end{cases}$

$=sup\mu(x)$, for all $x \in f^{-1}(f(x))$ since $f^{-1}(f(x)) = x \neq \emptyset$

$\geq \mu(x)$,for all $x \in S$.

Hence $\mu \subseteq f^{-1}(f(\mu))$

(ii) Let μ be f-invariant.

Claim: $f^{-1}(f(\mu)) = \mu$. Let $x \in S$,then $(f^{-1}(f(\mu))(x)$

$=f(\mu)f(x) = \begin{cases} sup_{x\in f^{-1}(f(x))}\mu(x) & \text{if } f^{-1}(f(x)) \neq \emptyset \\ o & otherwise \end{cases}$

$=sup\mu(x)$, for all $x \in f^{-1}(f(x)$

$=\mu(x)$, for all $x \in S$ .Since $z_1, z_2 \in S$, $f(z_1) = f(z_2)$,we get $\mu(z_1) = \mu(z_2)$ .As μ is f-invariant.

Hence $f^{-1}(f(\mu)) = \mu$

(iii) Suppose $\mu \subseteq \theta$ then $\mu(x) \leq \theta(x)$, for all $x \in S$.

Claim: $f(\mu) \subseteq f(\theta)$.

Let $y \in S^{'}$,then $(f(\mu))(y) = \begin{cases} sup\mu(x)_{x\in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$

$\leq \begin{cases} sup\theta(x)_{x\in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$

$\Rightarrow (f(\mu))(y) \leq (f(\theta))(y)$, fo all $y \in S^{'}$

Hence $f(\mu) \subseteq f(\theta)$

(iv) Suppose $\mu^{'} \subseteq \theta^{'}$,then $\mu^{'}(y) \leq \theta^{'}(y)$, for all $y \in S^{'}$

Let $x \in S$, then $f^{-1}(\mu^{'}))(x) = \mu^{'}(f(x))$,$f(x) = y$

$\leq \theta'(f(x)), f(x) = y$

$= f^{-1}(\theta'))(x)$

Hence $f^{-1}(\mu') \subseteq f^{-1}(\theta')$

□

3.1 Fuzzy Ideals and Homomorphism on Heyting Algebra

This section reflects the effect of a homomorphism on the join ,product ,and intersection of two fuzzy ideals.The results obtained here will be useful in studying the algebraic nature of fuzzy prime (fuzzy maximal,fuzzy semiprime,fuzzy primary, and fuzzy semiprimary) ideals under homomorphism.

Theorem 3.1.1. *Let f be a homomorphism from a HA H onto a HA H'. If μ and σ are fuzzy ideals of H,then following are true.*

i) $f(\mu \vee \sigma) = f(\mu) \vee f(\sigma)$.

ii) $f(\mu \wedge \sigma) = f(\mu) \wedge f(\sigma)$ *and*

iii) $f(\mu \cap \sigma) \subseteq f(\mu) \cap f(\sigma)$ *with equality if at least one μ or σ f-invariant.*

iv) $f(\mu \rightarrow \sigma) = f(\mu) \rightarrow f(\sigma)$

Proof. Let $y \in H'$ and $\epsilon > 0$ be given.

i) Set $\alpha = f(\mu \vee \sigma)(y)$ and $\beta = (f(\mu) \vee f(\sigma))(y)$,then,$\alpha - \epsilon < sup_{x \in f^{-1}(y)}(\mu \vee \sigma)(x)$

$\Rightarrow \alpha - \epsilon < (\mu \vee \sigma)(x_0)$ for some $x_0 \in H$ such that $f(x_0) = y$

$= Sup_{x_0 = a \vee b}(min(\mu(a), \sigma(b)))$,where $a, b \in H$

$\Rightarrow \alpha - \epsilon < min(\mu(a_0), \sigma(b_0))$ for some $a_0, b_0 \in H$ such that $x_0 = a_0 \vee b_0$

Now,$\beta = Sup_{y = y_1 \vee y_2}(min(f(\mu))(y_1), (f(\sigma))(y_2))$,where $y_1, y_2 \in H'$

$\Rightarrow \beta \geq min(((f(\mu))(f(a_0), (f(\sigma))(f(b_0))$,Since $y = f(x_0) = f(a_0) \vee f(b_0)$

$= min(f^{-1}(f(\mu))(a_0), (f^{-1}(f(\sigma))(b_0))$

$\geq min(\mu(a_0), \sigma(b_0))$ in view of Lemma 3.0.41(i)

$> \alpha - \epsilon$

$\Rightarrow \beta > \alpha - \epsilon$

$\Rightarrow \beta \geq \alpha$ since $\epsilon > 0$ is arbitrary.

Next,$\beta \leq \alpha$,since

$\beta - \epsilon < sup_{y=y_1 \vee y_2}(min((f(\mu))(y_1), ((f(\sigma))(y_2))$ where ,$y_1, y_2 \in H^{'}$

$\beta - \epsilon < (f(\mu))(y_1)$ and $\beta - \epsilon < (f(\sigma))(y_2)$ for some $y_1, y_2 \in H^{'}$ such that $y = y_1 \vee y_2$

$\Rightarrow \beta - \epsilon < \mu(x_1)$ and $\beta - \epsilon < \sigma(x_2)$ for some $x_1, x_2 \in H$ such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ by [Definition 3.0.39]

$\Rightarrow \beta - \epsilon < min(\mu(x_1), \sigma(x_2)) \leq (\mu \vee \sigma)(x_1 \vee x_2)$ by [Definition 3.0.38]

$\leq sup_{x \in f^{-1}(y)}((\mu \vee \sigma)(x))$,since $x_1 \vee x_2 \in f^{-1}(y)$

$=(f(\mu \vee \sigma))(y) = \alpha$

$\Rightarrow \beta - \epsilon < \alpha$

Hence,$\beta \leq \alpha$

Thus, $\beta = \alpha$ showing that $f(\mu \vee \sigma) = f(\mu) \vee f(\sigma)$

ii) Let $\alpha = (f(\mu \wedge \sigma))(y)$ and $\beta = (f(\mu) \wedge f(\sigma))(y)$.Then $\alpha \leq \beta$ follows from the following arguments.

$\alpha - \epsilon < sup_{z \in f^{-1}}(y)(\mu \wedge \sigma)(z)$

$\Rightarrow \alpha - \epsilon < (\mu \wedge \sigma)(x), x \in f^{-1}(y)$

$\Rightarrow \alpha - \epsilon < min(\mu(x_1), \sigma(x_2))$ for some $x_1, x_{2 \in H}$ such that $x = x_1 \wedge x_2$ [by Definition 3.0.29]

$\leq min((f^{-1}(f(\mu))(x_1), (f^{-1}f(\sigma))(x_2)))$ [by Lemma 3.0.38]

$=min((f(\mu))(f(x_1)), ((f(\sigma))(f(x_2)))$

$\leq (f(\mu) \wedge f(\sigma)(f(x_1), f(x_2))$

$=(f(\mu) \wedge f(\sigma))(f(x)) = \beta$.

Next,$\beta \leq \alpha$ because $\beta - \epsilon < (f(\mu) \wedge f(\sigma)(y)$

$sup_{y=y_1 \wedge y_2}(min((f(\mu))(y_1), (f(\sigma))(y_2)))$,where $y_1, y_2 \in H^{'}$

$=sup_{y=y_1 \wedge y_2}(min((sup_{z \in f^{-1}}(y_1)\mu(z), sup_{z \in f^{-1}}(y_2)\sigma(z)))$

$\Rightarrow \beta - \epsilon < min((sup_{z \in f^{-1}}(y_1)\mu(z), sup_{z \in f^{-1}}(y_2)\sigma(z)))$ for some $y_1, y_2 \in H^{'}$ such that

$y = y_1 \wedge y_2$

$\Rightarrow \beta - \epsilon < min(\mu(x_1), \sigma(x_2))$ for some $x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)$

$\leq (\mu \wedge \sigma)(x_1 \wedge x_2)$ [by Definition 3.0.38]

$\leq sup_{x \in f^{-1}(y)}((\mu \wedge \sigma)(x))$,since $y = y_1 \wedge y_2 = f(x_1 \wedge x_2)$

$=(f(\mu \wedge \sigma))(y) = \alpha$

Hence $\beta \leq \alpha$.

Thus $\beta = \alpha$,and we completed proof of (ii)

(iii)$f(\mu \cap \sigma) \subseteq f(\mu) \cap f(\sigma)$ follows immediately by applying Lemma 3.0.39(iii) to the trivial facts $\mu \cap \sigma \subseteq \mu$ and $\mu \cap \sigma \subseteq \sigma$

Next assume that σ is f-invariant.Then $f^{-1}(f(\sigma)) = \sigma$ by Lemma 3.0.39 (ii).

put $\alpha = (f(\mu) \cap f(\sigma))(y)$and $\beta = (f(\mu \cap \sigma))(y)$, then $\alpha - \epsilon < min((f(\mu))(y), (f(\sigma))(y))$

$= min(sup_{x \in f^{-1}(y)}\mu(x), (f(\sigma))(y))$

$\Rightarrow \alpha - \epsilon < \mu(z)$ forsome $z \in f^{-1}(y)$ and $\alpha - \epsilon < (f(\sigma))(y)$

$\Rightarrow \alpha - \epsilon < \mu(z)$ and $\alpha - \epsilon < (f(\sigma))(f(z)) = (f^{-1}((f(\sigma)))(z) = \sigma(z)$

$\Rightarrow \alpha - \epsilon < min(\mu(z), \sigma(z)) = (\mu \cap \sigma)(z)$

$\Rightarrow \alpha - \epsilon < sup_{z \in f^{-1}(y)}((\mu \cap \sigma)(z))$,since $z \in f^{-1}(y)$

$= f((\mu \cap \sigma))(y) = \beta$

Hence,$f(\mu) \cap f(\sigma) \subseteq f(\mu \cap \sigma)$ and the equality follows.

iv)Let $y \in H'$ and $\epsilon > 0$ be given.

i) Set $\alpha = (f(\mu \to \sigma))(y)$ and $\beta = (f(\mu) \to f(\sigma))(y)$,then,$\alpha - \epsilon < sup_{x \in f^{-1}(y)}(\mu \to \sigma)(x)$

$\Rightarrow \alpha - \epsilon < (\mu \to \sigma)(x_0)$ for some $x_0 \in H$ such that $f(x_0) = y$

$=Sup_{x_0 = a \to b}(min(\mu(a), \sigma(b)))$,where $a, b \in H$

$\Rightarrow \alpha - \epsilon < min(\mu(a_0), \sigma(b_0))$ for some $a_0, b_0 \in H$ such that $x_0 = a_0 \to b_0$

Now,$\beta = Sup_{y = y_1 \to y_2}(min(f(\mu))(y_1), (f(\sigma))(y_2))$,where $y_1, y_2 \in H'$

$\Rightarrow \beta \geq min(((f(\mu))(f(a_0), (f(\sigma))(f(b_0))$,Since $y = f(x_0) = f(a_0) \to f(b_0)$

$= min(f^{-1}(f(\mu))(a_0), (f^{-1}(f(\sigma))(b_0))$

$\geq min(\mu(a_0), \sigma(b_0))$ in view of Lemma 3.0.39(i)

$> \alpha - \epsilon$

$\Rightarrow \beta > \alpha - \epsilon$

$\Rightarrow \beta \geq \alpha$ since $\epsilon > 0$ is arbitrary.

Next,$\beta \leq \alpha$,since

$\beta - \epsilon < sup_{y=y_1 \rightarrow y_2}(min((f(\mu))(y_1), ((f(\sigma))(y_2))$ where ,$y_1, y_2 \in H^{'}$

$\beta - \epsilon < (f(\mu))(y_1)$ and $\beta - \epsilon < (f(\sigma))(y_2)$ for some $y_1, y_2 \in H^{'}$ such that $y = y_1 \rightarrow y_2$

$\Rightarrow \beta - \epsilon < \mu(x_1)$ and $\beta - \epsilon < \sigma(x_2)$ for some $x_1, x_2 \in H$ such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ by definition 3.0.39

$\Rightarrow \beta - \epsilon < min(\mu(x_1), \sigma(x_2)) \leq (\mu \rightarrow \sigma)(x_1 \rightarrow x_2)$ by definition 3.0.38

$\leq sup_{x \in f^{-1}(y)}((\mu \rightarrow \sigma)(x))$,since $x_1 \rightarrow x_2 \in f^{-1}(y)$

$=(f(\mu \rightarrow \sigma))(y) = \alpha$

$\Rightarrow \beta - \epsilon < \alpha$

Hence $\beta \leq \alpha$

Thus $\beta = \alpha$. showing that $f(\mu \rightarrow \sigma) = f(\mu) \rightarrow f(\sigma)$

□

Theorem 3.1.2. *Let f be a homomorphism from a HA H onto a HA* $H^{'}$. *If* $\mu^{'}$ *and* $\theta^{'}$ *are any two fuzzy ideals of* $H^{'}$,*then the following holds:*
$f^{-1}(\mu^{'}) \wedge f^{-1}(\theta^{'}) \subseteq f^{-1}(\mu^{'} \wedge \theta^{'})$

Proof. Let $x \in H$ and let $\epsilon > 0$ be given. For convenience,set

$\alpha = (f^{-1}(\mu^{'}) \wedge f^{-1}(\theta^{'})(x)$ and $\beta = (f^{-1}(\mu^{'} \wedge \theta^{'})(x)$. Then

$\alpha - \epsilon < sup_{x=x_1 \wedge x_2}(min((f^{-1}(\mu^{'}))(x_1), (f^{-1}(\theta^{'}))(x_2)))$,$x_1, x_2 \in H$,

$=sup_{x=x_1 \wedge x_2}(min(\mu^{'}(f(x_1)), \theta^{'}(f(x_2)))$

$\Rightarrow \alpha - \epsilon < min(\mu^{'}(f(x_1), \theta^{'}(f(x_2)))$ for some $x_1, x_2 \in H$ such that $x = x_1 \wedge x_2$

$\leq (\mu^{'} \wedge \theta^{'})(f(x_1 \wedge x_2) = (f^{-1}(\mu^{'} \wedge \theta^{'}))(x) = \beta$

$\Rightarrow \alpha \leq \beta$, since $\epsilon > 0$ is arbitrary.

Hence , $f^{-1}(\mu^{'}) \wedge f^{-1}(\theta^{'}) \subseteq f^{-1}(\mu^{'} \wedge \theta^{'})$

□

Chapter 4

Fuzzy Prime Ideals,Fuzzy Maximal Ideals, Fuzzy Semi Prime Ideals on HA and Fuzzy Congruences on FHAs

Fuzzy subgroup and its important properties were defined and studied by Rosenfeld (1971).Then several authors have studied about it.The notion fuzzy ideal of a ring was introduced by Malik and Mordeson (1998).They also studied about fuzzy relation on rings.It is now necessary to extend this concept to the concept of Heyting lattice which is Heyting algebra. In this chapter we have studied about the fuzzy prime ideals, fuzzy maximal ideal and fuzzy semiprime ideals on Heyting algebra.It is also discussed that the cross product of two fuzzy prime ideals,fuzzy primary ideals,fuzzy semiprime ideals respectivelly is fuzzy prime ideals,fuzzy primary ideals,fuzzy semiprime ideals iff each of the level ideals is prime ideals, primary ideals, semiprime ideals of $H \times H$.

Fuzzy Prime Ideals,Fuzzy Maximal Ideals, Fuzzy Semi Prime Ideals on HAs.

Here,we recall that an ideal I of a lattice L is called semiprime whenever $x \wedge y \in I$ and $x \wedge z \in I$,then $x \wedge (y \vee z) \in I$.Dually, a filter F is semiprime if $x \vee y \in F$ and $x \vee z \in F$,then $x \vee (y \wedge z)$ for all $x, y, z \in L$[52].This concept can also be extended into Heyting algebra and the fuzzy version can also be fuzzfied.

Definition 4.0.3. A fuzzy ideal μ of H is called a fuzzy prime ideal of H if $\mu(a \wedge b) = \mu(a)$ or $\mu(a \wedge b) = \mu(b)$

Theorem 4.0.4. *If μ and θ are fuzzy prime ideals of H,then $\mu \times \theta$ is a fuzzy prime ideal of $H \times H$*

Proof. $\mu \times \theta((a, b)\wedge, (c, d))$
$=\mu \times \theta(a \wedge c, b \wedge d)$
$= \mu(a \wedge c) \wedge \theta(b \wedge d)$
$= \mu(a) \wedge \theta(b)$
$= \mu \times \theta(a, b)$
or $\mu \times \theta(a \wedge c, b \wedge d)$
$= \mu(a \wedge c) \wedge \theta(b \wedge d)$
$= \mu(c) \wedge \theta(d)$
$= \mu \times \theta(c, d)$
Hence,$\mu \times \theta$ is a fuzzy prime ideal of $H \times H$ □

Definition 4.0.5. A fuzzy ideal μ is called fuzzy semi prime ideal of H if $\mu(x \wedge (y \vee z)) \geq \mu(x \wedge y) \wedge \mu(x \wedge z) = \mu(x) \vee (\mu(y) \wedge \mu(z))$,for all $x, y, z \in H$

Definition 4.0.6. A proper ideal (filter) P of H is said to be maximal if, there is no proper ideal (filter) Q of H such that P ⊆ Q.

Definition 4.0.7. A fuzzy ideal μ of H is called fuzzy semiprimary if $\forall a, b \in H$,either $\mu(a \wedge b) \leq \mu(a^n)$ for some $n \in \mathbb{Z}+$ or $\mu(a \wedge b) \leq \mu(b^m)$ for some $m \in \mathbb{Z}+$

Definition 4.0.8. A fuzzy ideal μ of H is called fuzzy primary if $\forall a, b \in H$,either $\mu(a \wedge b) \leq \mu(a)$ or $\mu(a \wedge b) \leq \mu(b^m)$ for some $m \in \mathbb{Z}+$.

Theorem 4.0.9. *If μ and θ are fuzzy semiprime ideals of H,then $\mu \times \theta$ is a fuzzy semiprime ideal of $H \times H$*

Proof. Since the cartesian product of fuzzy ideals of R is a fuzzy ideal(Mordson and Malik on ring R) analogosly works.It is enough to show that $\forall x, y$ and $z \in H$
$\mu \times \theta((x \wedge (y \vee z)) = \mu((x \wedge (y \vee z)) \wedge \theta((x \wedge (y \vee z))$
$\geq \mu(x \wedge y) \wedge \mu(x \wedge z) \wedge \theta(x \wedge y) \wedge \theta(x \wedge z)$
$= \mu \times \theta(x \wedge y) \wedge \mu \times \theta(x \wedge z)$. [Since μ and θ are fuzzy semiprime ideals.]
Hence,$\mu \times \theta$ is fuzzy semiprime ideal □

Corollary 4.0.10. *A fuzzy ideal $\mu \times \theta$ of $H \times H$ is said to be fuzzy semiprime iff the level ideals $(\mu \times \theta)_t, t \in im(\mu \times \theta)$ is semiprime ideal of $H \times H$.*

Proof. Proof is similar to the above theorem □

Theorem 4.0.11. *If μ and θ are fuzzy primary ideals of H,then $\mu \times \theta$ is a fuzzy primary ideal of $H \times H$.*

Proof. Since μ and θ are fuzzy primary ideals of H,then $\forall a, b, c, d \in H$ either $\mu(a \wedge b) = \mu(a)$ or else $\mu(a \wedge b) \leq \mu(b^n), n \in \mathbb{Z}+$ and either $\theta(c \wedge d) = \theta(c)$ or else $\theta(c \wedge d) \leq \theta(d^n), n \in Z+$.Then we have the following cases.
(1.) $\mu(a \wedge b) = \mu(a)$ and $\theta(c \wedge d) = \theta(c)$. Consider;

$$\begin{aligned}\mu \times \theta((a, c) \wedge (b, d)) &= \mu(a \wedge b) \wedge \theta(c \wedge d) \\ &= \mu(a \wedge b) = \mu(a) = (\mu \times \theta)(a, c) \text{ and} \\ \mu \times \theta((a, c) \wedge (b, d)) &= \mu(a \wedge b) \wedge \theta(c \wedge d) \\ &= \theta(c \wedge d) = \theta(c) = (\mu \times \theta)(a, c)\end{aligned}$$

(2.) If $\mu(a \wedge b) \leq \mu(b^m)$ and $\theta(c \wedge d) \leq \theta(d^n), n, m \in \mathbb{Z}+$
$(\mu \times \theta)((a, c) \wedge (b, d)) = (\mu \times \theta)(a \wedge b, c \wedge d)$

$=\mu(a \wedge b)$

$\leq \mu(b^{max(m,n)})$

$(\mu \times \theta)((b,d)^{max(m,n)})$

Again $(\mu \times \theta)((a,c) \wedge (b,d)) = (\mu \times \theta)(a \wedge b, c \wedge d)$

$=\mu(c \wedge d)$

$\leq \mu(d^{max(m,n)})$

$(\mu \times \theta)((b,d)^{max(m,n)})$

Therefore,$\mu \times \theta$ is a fuzzy primary ideal of $H \times H$. □

Theorem 4.0.12. *If μ and θ are fuzzy semiprimary ideals of H,then $\mu \times \theta$ is a fuzzy semiprimary ideal of $H \times H$*

Proof. We know that the cartesian product of any two fuzzy ideals is fuzzy. Since μ and θ are fuzzy semiprimary ideals of H,then $\forall a,b,c,d \in H$ either $\mu(a \wedge b) \leq \mu(a^n)$ or else $\mu(a \wedge b) \leq \mu(b^m)$ for some $m,n \in \mathbb{Z}+$ and either $\theta(c \wedge d) \leq \theta(c^k)$ or else $\theta(c \wedge d) \leq \theta(d^l)$ for some $k,l \in \mathbb{Z}+$

Let $t = max(m,n)$ and $s = max(k,l)$.Then $\mu \times \theta((a,c) \wedge (b,d)) = \mu \times \theta(a \wedge b, c \wedge d))$=$\mu(a \wedge b) \wedge \theta(c,d)$

$= \mu(a \wedge b) \leq \mu(a^l) = \mu \times \theta((a,c)^l)$ or else $\mu \times \theta((a,c) \wedge (b,d)) = \mu \times \theta(a \wedge b, c \wedge d))$=$\mu(a \wedge b) \wedge \theta(c,d)$

$= \theta(c \wedge d) \leq \mu(d^s) = \mu \times \theta((b,d)^s)$.

Therefore,$\mu \times \theta$ is is a fuzzy semi primary ideal of $H \times H$ □

Corollary 4.0.13. *A fuzzy ideal $\mu \times \theta$ of $H \times H$ is said to be fuzzy semiprimary iff the level ideals $(\mu \times \theta)_t, t \in im(\mu \times \theta)$ is semiprimary ideals of $H \times H$.*

Proof. Let μ be a fuzzy ideal of H. Then for any $0 \leq t \leq \mu(0), \mu_t$ is a ideal of H with respect to μ.Let μ be a fuzzy subset of H. We denote by μ^*, the set $\mu^* = \{x \in H : \mu(x) = \mu(0)\}$. □

Lemma 4.0.14. *If μ is a fuzzy ideal of H, then μ^* is an ideal of H.*

Proof. Suppose $\mu(x)$ is fuzzy ideal of H.Then we shall show that μ^* is an ideal of H.

Clearly, $0 \in \mu^*$. Let $x, y \in \mu^*$.Then $\mu(x) = \mu(0)$and $\mu(y) = \mu(0)$.

$\mu(x \wedge y) \geq \mu(x) \vee \mu(y) = \mu(0) \vee \mu(0) = \mu(0) = 1$.

Hence,for $\mu(x \wedge y) = 1 = \mu(0)$

Which gives, $x \wedge y$ and $x \vee y \in \mu^*$.

Similarly,we have $x \to y \in \mu^*$. Then, μ^* is an ideal of H.

□

Proposition 4.0.15. *Let $\{\mu_i : i \in \Lambda\}$ be a family of fuzzy ideals of H, $\cap_{i\in\lambda}\mu_i$ is a fuzzy ideal of H.*

Lemma 4.0.16. *Let μ and θ be fuzzy ideals of H. Then $\mu^* \cap \theta^* \subseteq (\mu \cap \theta)^*$*

Proof. Let $x \in \mu^* \cap \theta^*$.Then $\mu(x) = \mu(0)$ and $\theta(x) = \theta(0)$.Now $(\mu \cap \theta)(x) = \mu(x) \wedge \theta(x) = \mu(0) \wedge \theta(0) = (\mu \cap \theta)(0)$.Thus,$x \in (\mu \cap \theta)^*$ □

In general the equality in the above lemma need not hold, as shown by the following example.

Example 4.0.17. Let H be a HA. Let μ and θ be fuzzy subsets of H such that $\mu(x) = 0$ for all $x \in H$,$x \neq 0$ and $\theta(x) = 0$ if $x \neq 0, \theta(0) = 1$.Then μ and θ are fuzzy ideals of H. Now $\mu^* \cap \theta^* = H \cap \{0\} = \{0\}$ and $(\mu \cap \theta)^* = H$.

Lemma 4.0.18. *Let μ and θ be fuzzy ideals of H such that $\mu(0) = 1 = \theta(0)$.Then $\mu^* \cap \theta^* = (\mu \cap \theta)^*$.*

Proof. Let $x \in (\mu \cap \theta)^*$. Then $(\mu \cap \theta)(x) = (\mu \cap \theta)(0)$. Thus, $min(\mu(x), \theta(x)) = min(\mu(0), \theta(0)) = 1$. Hence $\mu(x) = 1 = \theta(x)$. Then $x \in \mu^* \cap \theta^*$. Thus $(\mu \cap \theta) \subseteq \mu^* \cap \theta^*$.Also $\mu^* \cap \theta^* \subseteq (\mu \cap \theta)^*$. by Lemma 4.0.16. Hence,the lemma follows. □

Lemma 4.0.19. *Let $\{\mu_i : i \in \Lambda\}$ be a family of fuzzy ideals of H such that $\mu_i(0) = 1$.for all $i \in \Lambda$. Then $\cap_{i\in\Lambda}(\mu_i^*)$=$(\cap_{i\in\Lambda}\mu_i)^*$.*

Fuzzy Maximal Ideal

Let μ and θ be fuzzy subsets of a nonempty set H. Let $I \subseteq H$ and let $0 \leq t \leq 1$. Let χ_I , be a fuzzy subset of H such that $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = t$ if $x \notin I$. then χ_I is the characteristic function of I.
Note that if H is a HA and I is an ideal of H, then $\chi_I(0) = 1$, $(\chi_I)^* = 1$, $Im(\chi_I) = \{t, 1\}$ and χ_I, is a fuzzy ideal of H. The following result shows that a fuzzy maximal ideal μ of H cannot be defined as a fuzzy ideal $\mu \neq \chi_H$, such that if θ is a fuzzy ideal of H for which $\mu \subseteq \theta \subseteq \chi_H$, then $\mu = \theta$.

Theorem 4.0.20. *Let μ be a fuzzy ideal of H such that $\mu(x) \neq 1$ for some $x \in H$. Then there exists a fuzzy ideal θ of H such that $\theta(y) \neq 1$ for some $y \in H$ and $\mu \subset \theta$.*

Case 1: $\mu(0) \neq 1$. Let $\mu(0) < t < 1$. Let μ be a fuzzy subset of H such that $\theta(x) = t$ for all $x \in H$. Then θ is a fuzzy ideal of H such that $\mu \subset \theta$ and $\theta(x) \neq 1$ for all $x \in H$.
Case 2: $\mu(0) = 1$. By the hypothesis there exists $x \in H$ such that $\mu(x) \neq 1$. Let $\mu(x) < t < \mu(0)$. The μ_t, is a ideal of H. Let θ be a fuzzy subset of H such that $\theta(u) = 1$ if $u \in \mu_t$, and $\theta(t) = t$ if $u \in \mu_t$. Then θ is a fuzzy ideal of H. Since $x \notin \mu_t, \theta(x) = t \neq 1$. Also it can be easily checked that $\mu \subset \theta$.

Definition 4.0.21. Let μ be a fuzzy ideal of H. Then μ is called a fuzzy maximal ideal of H if μ is not constant and for any fuzzy ideal θ of H, if $\mu \subseteq \theta$ then either $\mu^* = \theta^*$ or $\theta = \chi_H$

Theorem 4.0.22. *Let μ be a fuzzy maximal ideal of H. Then card $Im(\mu) = 2$.*

Since μ is an fuzzy ideal, $\mu(0) = 1$.We claim that for any $0 \leq t < 1$, if $t \in Im(\mu)$ then $\mu_t = H$. Let $0 \leq t < 1$ and $t \in Im(\mu)$.Now μ, is an ideal of H, and since $t < 1$, $\mu^* \subset \mu_t$. Let θ be a fuzzy subset of H such that $\theta(x) = 1$ if $x \in \mu_t$, and $\theta(x) = t$ if $x \notin \mu_t$. Then θ is a fuzzy ideal of H, and $\theta^*, = \mu_t$. Clearly $\mu \subseteq \theta$. Since μ is fuzzy maximal and $\mu^* \subseteq \mu_t = \theta^*$. we have $\theta = \chi_H$. Thus $\theta(x) = 1$ for all $x \in H$. Hence

$\mu_t = \theta^* = H$. This proves our claim. Now for any $t_1, t_2 \in Im(\mu), 0 < t_1, t_2 < 1$, we have $\mu_{t_1} = H = \mu_{t_2}$. This implies $t_1 = t_2$ Thus μ is two-valued.

Lemma 4.0.23. *Let μ be a fuzzy ideal of H. If μ^* is a maximal ideal of H, then μ is two-valued.*

Since μ^* is a maximal ideal of H, $\mu^* \neq H$. Thus there exists $x \in H$ such that $\mu(x) \neq \mu(0)$. Hence μ is at least two-valued. Let $0 < t < \mu(0)$ and $t \in Im(\mu)$. Then μ_t, is an ideal of H such that $\mu^* \subseteq \mu_t$. Since μ^* is a maximal ideal, $\mu_t = H$. Thus if $t_1, t_2 \in Im(\mu)$ and $t_1 \neq \mu(0), t_2 \neq \mu(0)$, then $\mu_{t_1} = H = \mu_{t_2}$.This gives $t_1 = t_2$. Thus μ is two-valued.

Theorem 4.0.24. *Let μ be a fuzzy ideal of H. If μ^* is a maximal ideal of H, then μ is a fuzzy maximal ideal of H.*

Proof. By Lemma 4.0.23, μ is two-valued. Let $Im(\mu) = \{t, 1\}$ where $0 \leq t < 1$. Let θ be a fuzzy ideal of H such that $\mu \subseteq \theta$. Then $\theta(0) = 1$. Let $x \in \mu^*$. Then $1 = \mu(0) = \mu(x) \leq \theta(x)$. Thus $\theta(x) = 1 = \theta(0)$ and hence $x \in \theta^*$. Hence $\mu^* \subseteq \theta^*$. Since μ^* is a maximal ideal of $H, \mu^* = \theta^*$ or $\mu^* = H$. If $\theta^* = H$, then $\theta = \chi_H$. Hence μ is a fuzzy maximal ideal of H. □

4.1 Fuzzy Congruence Relations on Heyting Algebras

An element $m \in H$ is called maximal if it is a maximal element in the partially ordered set $(H, \leq)$. That is for any $a \in H$, $m \leq a \Rightarrow m = a$.
For any fuzzy subset μ of H, it is clear that $\mu(x) =Sup\{\alpha \in [0,1] : x \in \mu_\alpha\}$ for all $x \in H$.
In the following theorem, we characterize a fuzzy ideal induced by fuzzy sets. In the remaining part of this section we define fuzzy homomorphisms on HAs and

we present some results on fuzzy homomorphisms in connection with fuzzy ideals. Recall from Chon definition that, for any sets H_1 and H_2 a mapping $f : H_1 \times H_2 \to [0, 1]$ is called a fuzzy relation of H_1 into H_2. A fuzzy relation f of H_1 into H_2 is called a fuzzy mapping if for each $x \in H_1$ there exists a unique element $y_x \in H_2$ such that $f(x, y_x) = 1$.

In this case we call this unique element y_x a fuzzy image of x under f.

We write f : $H_1 \to H_2$; for a fuzzy mapping f of H_1 into H_2. Image of f is the set $f = \{y_x : x \in H_1\} = \{y \in H_2 : f(x, y) = 1\}$. As usual, f is said to be onto, if for each $y \in H_1$; there exists $x \in H_2$ such that $y_x = y$ and f is said to be one-one, if for each $a, b \in H_1, y_a = y_b \Rightarrow a = b :$

Definition 4.1.1. Let H_1 and H_2 be HAs. A fuzzy mapping $f : H_1 \to H_2$ is called a fuzzy homomorphism of HAs, if the following conditions are satisfied.

(i) $y_0 = 0$ (a zero element in H_2)

(ii) $f(x_1 \vee x_2, y) \geq sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y = y_1 \vee y_2, y_1, y_2 \in H_2\}$

(iii) $f(x_1 \wedge x_2, y) \geq sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y = y_1 \wedge y_2, y_1; y_2 \in H_2\}$

(iv)$f(x_1 \to x_2, y) \geq sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y = y_1 \to y_2, y_1, y_2 \in H_2\}$

Lemma 4.1.2. *Let $f : H_1 \to H_2$ be a fuzzy homomorphism of HAs. Then we have the following:*

(1) $y_{(a \vee b)} = y_a \vee y_b$;

(2) $y_{(a \wedge b)} = y_a \wedge y_b$;

(3)$y_{(a \to b)} = y_a \to y_b$; *for all* $a; b \in H_1 :$

Proof. We have y_a and y_b are the unique elements in H_2 such that $f(a, y_a) = 1$ and $f(b, y_b) = 1$. We show that $f(a \vee b, y_a \vee y_b) = 1$. Put $z = y_a \vee y_b$.

Then $f(a \vee b, z) = Sup\{f(a, z_1) \wedge f(b, z_2) : z = z_1 \vee z_2, z_1, z_2 \in H_2\} \geq f(a, y_a) \wedge f(b, y_b) = 1$. Since $y_{(a \vee b)}$ is the unique element in H_2 such that $f(a \vee b, y_{(a \vee b)}) = 1$, we get that $y_{(a \vee b)} = y_a \vee y_b$.

For (3) We show that $f(a \to b, y_a \to y_b) = 1$. Put $z = y_a \to y_b$. Then $f(a \to b, z) =$

$Sup\{f(a,z_1) \wedge f(b,z_2) : z = z_1 \to z_2, z_1, z_2 \in H_2\} \geq f(a,y_a) \wedge f(b,y_b) = 1.$

Since $y_{(a\to b)}$ is the unique element in H_2 such that $f(a \to b, y_{(a\to b)}) = 1$, we get that $y_{(a\to b)} = y_a \to y_b$

Similarly, it can be verifed that $y_{(a\wedge b)} = y_a \wedge y_b$.

Hence,the result follows. □

Theorem 4.1.3. *Let H_1 and H_2 be HAs. f is a fuzzy hommomorphism from H_1 to H_2 and μ be fuzzy ideal of H_1 and θ be the fuzzy ideal of H_2.Then*
i) $f(\mu)$ is a fuzzy ideal of H_2
ii)$f^{-1}(\theta)$ is a fuzzy ideal of H_1

Proof. (i) Since $f(\mu)(y_1 \wedge y_2) = Sup(\mu(x_1 \wedge x_2))$ for some $x_1 \wedge x_2$ such that $y_1 \wedge y_2 = f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$.This implies $f(\mu)(y_1 \wedge y_2) \geq (Sup\mu(x_1)$ for some x_1 such that $y_1 = f(x_1)) \vee (Sup\mu(x_2)$ for some x_2 such that $f(x_2) = y_2) \geq f(\mu)(y_1) \vee f(\mu)(y_2)$. Similarly,it is easy to check the other criterias.
(ii) $f^{-1}(\theta)(x \to y) = \theta(f(x \wedge y)) = \theta(f(x)) \to \theta(f(y)) \geq \theta(f(x)) \vee \theta(f(y)) \forall x, y \in H_1$. The other criterias are similar. □

Theorem 4.1.4. *Let f be a fuzzy homomorphism of H_1 into H_2,then a subset $f_x = \{x \in H_1 : f(x,1) \geq 0\}$ is an ideal of H_1.*

Proof. Clearly,since $f(0,1) \geq 0$, $0 \in f_x$.Let a and $b \in f_x$.Then $f(a,1) \geq 0$ and $f(b,1) \geq 0$.
We show that (i.)$f(a \vee b, 1) \geq 0$
Consider $f(a \vee b, 1) \geq Sup\{f(a,y_1) \wedge f(b,y_2) : 1 = y_1 \vee y_2$ and $y_1, y_2 \in H_2\}$
$= Sup\{f(a,1) \wedge f(b,1)\}$
$\geq f(a,1) \wedge f(b,1) \geq 0$
Hence, $f(a \vee b, 1) \geq 0$
$\Rightarrow a \vee b \in f_x$
(ii.)Let $a \in f_x, x \in H$,we show that $a \wedge x \in f_x$
$f(a \wedge x, 1 \wedge 1) \geq Sup\{f(a,y_1) \wedge f(x,y_2) : 1 = y_1 \wedge y_2$ and $y_1, y_2 \in H_2\} \geq f(a,1) \wedge$

$f(x,1) \geq 0$

$f(a \wedge x, 1) \geq 0$

Hence,$a \wedge x \in f_x$

(iii.) Let $a \in f_x, x \in H$. we show that $x \to a \in f_x$

$f(x \to a, 1 \to 1)$

$\geq Sup\{f(a,1) \wedge f(x,1) : 1 = y_1 \to y_2$ and $y_1, y_2 \in H_2\}$

$\geq f(a,1) \wedge f(x,1) \geq 0$

Hence, $x \to a \in f_x$.

Therefore,from $(i),(ii),(iii)$,we have f_x is an ideal of H_1 □

Theorem 4.1.5. *Let f be a fuzzy homomorphism of HAs H into L.Then a fuzzy subset $\mu_f(x) = f(x,1), \forall x \in H$ satisfies the following properties.*

1. $\mu_f(a \vee b) \geq \mu_f(a) \wedge \mu_f(b)$

2. $\mu_f(a \wedge b) \geq \mu_f(a) \wedge \mu_f(b)$

3. $\mu_f(a \to b) \geq \mu_f(a) \wedge \mu_f(b)$

Proof. Clearly,$\mu_f(0) = f(0,1) = 1$.

$\mu_f(a \vee b) = f(a \vee b, 1) = f(a \vee b, 1 \vee 1) = Sup\{f(a,y_1) \wedge f(b,y_2) : 1 = y_1 \vee y_2$ and $y_1, y_2 \in H_2\} \geq f(a,y_1) \wedge f(b,y_2)$.

In particular,for $y_1 = 1$ and $y_2 = 1$, $\mu_f(a \vee b) \geq f(a,1) \wedge f(b,1) = \mu_f(a) \wedge \mu_f(b)$.

$\mu_f(a \wedge b) = f(a \wedge b, 1) = f(a \wedge b, 1 \wedge 1) = Sup\{f(a,y_1) \wedge f(b,y_2) : 1 = y_1 \wedge y_2$ and $y_1, y_2 \in H_2\} \geq f(a,y_1) \wedge f(b,y_2)$.

In particular,for $y_1 = 1$ and $y_2 = 1$, $\mu_f(a \wedge b) \geq f(a,1) \wedge f(b,1) = \mu_f(a) \wedge \mu_f(b)$

$\mu_f(a \to b) = f(a \to b, 1) = f(a \to b, 1 \to 1) = Sup\{f(a,y_1) \wedge f(b,y_2) : 1 = y_1 \to y_2$ and $y_1, y_2 \in H_2\} \geq f(a,y_1) \wedge f(b,y_2)$.

In particular,for $y_1 = 1$ and $y_2 = 1$, $\mu_f(a \to b) \geq f(a,1) \wedge f(b,1) = \mu_f(a) \wedge \mu_f(b)$

Hence,μ_f is a fuzzy ideal of H □

Theorem 4.1.6. *Let A be a fuzzy congruence relation on H such that the condition $a \wedge (x \rightarrow y) = (a \wedge x) \rightarrow y$,for some fixed $a \in H, x, y \in H$.A fuzzy subset $\theta(x) = inf\{A(a \wedge x, x) : a \in H\}$ for all $x \in H$ is a fuzzy ideal of H.*

Proof. $\theta(0) = inf\{A(a \wedge 0, 0) : a \in H\}$ $=inf\{A(0, 0)\} = 1$.

$\theta(x \vee y) = inf\{A(a \wedge (x \vee y), x \vee y) : a \in H\}$

$\theta(x \vee y) = inf\{A((a \wedge x) \vee (a \wedge y), x \vee y) : a \in H\}$

$\geq inf\{A(a \wedge x, x) \wedge A(a \wedge y, y) : a \in H\}$

$\geq inf\{A(a \wedge x, x) : a \in H\} \wedge inf\{A(a \wedge y, y) : a \in H\}$

$= \theta(x) \wedge \theta(y)$

$\theta(x \rightarrow y) = inf\{A((a \wedge (x \rightarrow y), x \rightarrow y) : a \in H\}$

$\theta(x \rightarrow y) = inf\{A(a \wedge x) \rightarrow y, x \rightarrow y) : a \in H\}$

$\geq inf\{A(a \wedge x, x) \wedge A(y, y) : a \in H\}$

$\geq inf\{A(a \wedge x, x)\}a \in H\}$

$= \theta(x)$.

Similarly,$\theta(x \rightarrow y) \geq \theta(y)$

Hence,$\theta(x \rightarrow y) \geq \theta(x) \vee \theta(y)$

$\theta(x \wedge y) = inf\{A(a \wedge (x \wedge y), x \wedge y) : a \in H\}$

$\theta(x \wedge y) = inf\{A((a \wedge x) \wedge y), x \wedge y) : a \in H\}$

$\geq inf\{A(a \wedge x, x) \wedge A(y, y) : a \in H\}$

$\geq inf\{A(a \wedge x, x)\} : a \in H\}$

$= \theta(x)$. In similar way, $\theta(x \wedge y) \geq \theta(y)$

Hence,$\theta(x \wedge y) \geq \theta(x) \vee \theta(y)$. □

Theorem 4.1.7. *Let A be a fuzzy congruence relation on H as defined above. Then $\theta(x) = \mu(x)$,where $\mu(x) = inf\{A(a, x) : a \in H\}$*

Proof. For any fuzzy congruence relation on H, we claim to show that $\theta(x) = \mu(x)$. For any $x \in H$; we have $\theta(x) = Inf\{A(a \wedge x, x) : a \in H\}$. Then $\theta(x) \leq A(a \wedge x, x)$; for all $a \in H$. In particular, for $a = 0, \theta(x) \leq A(0 \wedge x, x) = A(0, x) = A(x, 0) = \mu(x)$.

On the other hand, for any $a \in H$.

Consider, $\theta(x) = Inf\{A(a \wedge x, x) : a \in H\} \geq inf\{A(a, x) \wedge A(x, x) : a \in H\} = \mu(x).Hence, \theta(x) = \mu(x).$ □

Theorem 4.1.8. *Let $f : H \to [0, 1]$ be a fuzzy homomorphism.Define a fuzzy kernel of f denoted by $K_f : H \times H \to [0, 1]$ as follows.*

$$K_f(a, b) = \begin{cases} 1 & \textit{if } f(a) = f(b); \\ 0 & \textit{otherwise}, \forall a, b \in H. \end{cases}$$

Then K_f is a fuzzy congruence relation on H

Definition 4.1.9. A fuzzy subset μ of H is said to be implicatively (multiplicatively) closed resp.if $\mu(x \to y) \geq \mu(x) \wedge \mu(y)$ $(\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)), \forall x, y \in H.$

Let μ be both multiplicatively and implicatively closed fuzzy subsets of H and $S \subseteq H$ with $Sup\{\mu(x) : x \in S\} = 1$.

Define fuzzy relation $\psi^{\mu_a}(x, y) = Sup\{\mu(a) : x \to a = y \to a, a \in S\}$

Then we have the following theorem.

Theorem 4.1.10. *ψ^{μ_a} is a FCR(H) and H/ψ^{μ_a} is HA*

Proof. We first show that ψ^{μ_S} is a fuzzy congruence relation on H.For $x, y, z \in H$

(1) $\psi^{\mu_a}(x, x) = Sup\{\mu(a) : x \to a = x \to a, a \in S\}$

(2) $\psi^{\mu_a}(x, y) = Sup\{\mu(a) : y \to a = x \to a, a \in S\}$=$\psi^{\mu_a}(y, x)$

(3) For $a, b \in S$ if $x \to a = y \to a$ and $y \to b = z \to b, x, y \in H$,then we get

$x \to (a \to b) = (x \wedge a) \to b$=$y \to a \to b$=$a \to y \to b$

=$a \to z \to b$=$z \to a \to b$. Since $a \to b \in S$ and Now consider $\psi^{\mu_s}(x, y) \wedge \psi^{\mu_s}(y, z)$

$= Sup\{\mu(a) : x \to a = y \to a, a \in S\} \wedge Sup\{\mu(a) : y \to b = z \to b, b \in S\}$

$= Sup\{\mu(a) \wedge \mu(b) : x \to a = y \to a, a \in S$ and $y \to b = z \to b, b \in S\}$

$\leq Sup\{\mu(a \to b) : x \to a = y \to a, a \in S$ and $y \to b = z \to b, b \in S\}$

$= Sup\{\mu(a \to b) : x \to a \to b = z \to a \to b, a, b \in S\}$

$= Sup\{\mu(c) : x \to c = z \to c, c = a \to b \in S\} = \psi^{\mu_s}(x, z)$

Hence ψ^{μ_s} is a fuzzy equivalence relation.

Similarly, it can be proved that $\psi^{\mu_s}(x_1 \vee x_2, y_1 \vee y_2) \geq \psi^{\mu_s}(x_1, y_1) \wedge \psi^{\mu_s}(x_2, y_2)$

$\psi^{\mu_s}(x_1 \wedge x_2, y_1 \wedge y_2) \geq \psi^{\mu_s}(x_1, y_1) \wedge \psi^{\mu_s}(x_2, y_2)$

$\psi^{\mu_s}(x_1 \rightarrow x_2, y_1 \rightarrow y_2) \geq \psi^{\mu_s}(x_1, y_1) \wedge \psi^{\mu_s}(x_2, y_2)$

Thus,the theorem follows. □

4.2 Fuzzy Congruence Relation on Products of Fuzzy Heyting Algebras

In this section, we introduce the notion of fuzzy congruence relations on products of FHA and we give some properties about fuzzy congruence relations.

Definition 4.2.1. The HAs L and K are isomorphic and the map $\phi : L \rightarrow K$ is an isomorphism if ϕ is one-to-one, onto and if $\phi(a \wedge b) = \phi(a) \wedge \phi(b), \phi(a \vee b) = \phi(a) \vee \phi(b)$ and $\phi(a \rightarrow b) = \phi(a) \rightarrow \phi(b)$, for all $a, b \in L$.

Proposition 4.2.2. *Let* (H, A) *be a fuzzy poset.*$x, y, z \in H$*.If* $A(x, y) > 0$ *and* $A(y, z) > 0$*,then* $A(x, z) > 0$

Definition 4.2.3. [29] Let (H,A) be a fuzzy lattice and let $x, y, z \in H$. If $A((x \wedge y) \vee (x \wedge z), x \wedge (y \vee z)) > 0$ and $A(x \vee (y \wedge z), (x \vee y) \wedge (x \vee z)) > 0$.Then (H,A) is a distributive fuzzy lattice

Definition 4.2.4. Let (H,A) is distributive fuzzy lattice.Then (H,A) is bounded if for any $x \in H$, we have that $A(0, x) > 0$ and $A(x, 1) > 0$.

Proposition 4.2.5. *Let* (H, A) *be a fuzzy poset.*(H, A) *bounded fuzzy lattice iff* $(H, S(A))$ *is a bounded crisp lattice*

Example 4.2.6. Every interval $[a, b]$ on Heyting algebra H is FHA,$a, b \in R$ (real numbers)

Definition 4.2.7. Let L and K be HAs. Define $\wedge$,$\leq$, $\vee$ and $\to$ in $L \times K$ by $(a, b) \wedge (a_1, b_1) = (a \wedge a_1, b \wedge b_1)$, $(a, b) \vee (a_1, b_1) = (a \vee a_1, b \vee b_1)$,$(a, b) \leq (a_1, b_1) = (a \leq a_1, b \leq b_1)$ and $(a, b) \to (a_1, b_1) = (a \to a_1, b \to b_1)$.
This makes $L \times K$ into a HA called the direct product of L and K.

Theorem 4.2.8. *The direct product of a two bounded distributive lattices is bounded.*

Proof: Let H and K be HAs two bounded distributive lattices.Then $H \times K$ is bounded, as $(H \times K, \vee, \wedge, \to, (0, 0), (1, 1))$ is bounded with bottom element (0,0) and top element (1,1).

Definition 4.2.9. [25] Let A be a fuzzy relation on H.Then,for any $(a, b), (c, d) \in H \times H$,$A(a, b) \leq A(c, d)$ whenever $(a, b) \leq (c, d)$,$\forall a, b, c, d \in H$.

From now onwards by H we mean a Heyting algebra unless otherwise stated.

4.3 Direct Product of Fuzzy Heyting Algebras

Theorem 4.3.1. *[29] Let (P, A) and (Q, B) be fuzzy posets. The direct product $P \times Q$ of P and Q is defined by $(PQ, A \times B)$, where $A \times B : PQ \to [0, 1]$ is a fuzzy relation defined by $(A \times B)((p_1, q_1), (p_2, q_2)) = min[A(p_1, p_2), B(q_1, q_2)]$.*

Theorem 4.3.2. *[29] Let (P, A) and (Q, B) be fuzzy lattices. The the direct product $(PQ, A \times B)$ of (P, A) and (Q, B) is a fuzzy lattice.*

Definition 4.3.3. Let (H, A) and (K, B) be two FHAs and A,B be fuzzy relations on H and K respectivelly.The product $A \times B$ of A and B is a fuzzy relation on $H \times K$ defined by $(A \times B)((p_1, q_1), (p_2, q_2)) = min[A(p_1, p_2), B(q_1, q_2)]$.,where $p_1, p_2 \in H$,$q_1, q_2 \in K$

Theorem 4.3.4. *Let P and Q be two Heyting algebras (HAs),then the direct product $P \times Q$ is also a Heyting algebra (HA) under pointwise operation defined above.*

Proof. Clearly,$(p \times Q, \vee, \wedge, (0,0), (1,1))$ is a bounded distributive lattices.
Let $(a,b),(c,d),(e,f) \in P \times Q$ such that $(a,b) \wedge (c,d) \leq (e,f)$. $(a,b) \wedge (c,d) \leq (e,f)$.
$\Rightarrow a \wedge c \leq e$ in P and $b \wedge d \leq f$ in Q
$\Rightarrow c \leq a \rightarrow e$ and $d \leq b \rightarrow f$[since P and Q are HAs]
$\Rightarrow (c,d) \leq (a \rightarrow e, b \rightarrow f)$
Hence,$P \times Q$ is a Heyting algebra □

Corollary 4.3.5. *The direct product of a bounded distributive fuzzy lattices is bounded distributive fuzzy lattice.*

Proof: Follows from Theorems 4.3.1, 4.3.2,4.3.4 ,Definition 4.3.3

4.4 Direct Product of Fuzzy Congruences on Fuzzy Heyting Algebras

Theorem 4.4.1. *Let L and K be HAs, A be a fuzzy congruences on L and B be a fuzzy congruence on K. Define the fuzzy relation $A \times B$ on $L \times K$ by $(A \times B)((a,b),(c,d)) = A(a,c) \wedge B(b,d)$. Then $A \times B$ is a fuzzy congruence on $L \times K$. Conversely every fuzzy congruence relation on $L \times K$ is of this form.*

Proof. First we show that $A \times B$ is a fuzzy congruence on $L \times K$. Since A and B are fuzzy congruences,
(1) $(A \times B)((a,b),(a,b)) = A(a,a) \wedge B(b,b) = 1$.
(2) $(A \times B)((a,b),(c,d)) = A(a,c) \wedge B(b,d) = A(c,a) \wedge B(d,b)$
$= (A \times B)((c,d),(a,b))$.
(3) $(A \times B)((a,b),(z_1,z_2)) = A(a,z_1) \wedge B(b,z_2)$
$\geq sup_{c \in L}\{A(a,c) \wedge A(c,z_1)\} \wedge sup_{d \in K}\{B(b,d) \wedge B(d,z_2)\}$
$= sup_{(c,d) \in L \times K}\{A(a,c) \wedge A(c,z_1) \wedge B(b,d) \wedge B(d,z_2)\}$
$= sup_{(c,d) \in L \times K}\{(A(a,c) \wedge B(b,d)) \wedge (A(c,z_1) \wedge B(d,z_2))\}$
$= sup_{(c,d) \in L \times K}\{(A \times B)((a,b),(c,d)) \wedge (A \times B)((c,d),(z_1,z_2))\}$

$= sup_{(c,d)\in L\times K} min\{(A\times B)((a,b),(c,d)),(A\times B)((c,d),(z_1,z_2))\}$.

Thus, $A\times B$ is reflexive, symmetric and transitive and hence a fuzzy equivalence relation.

Also,$(A\times B)((a,b)\wedge(t_1,t_2),(c,d)\wedge(t_1,t_2))$

$=(A\times B)((a\wedge t_1,b\wedge t_2),(c\wedge t_1,d\wedge t_2))$

$=A(a\wedge t_1,c\wedge t_1)\wedge B(b\wedge t_2,d\wedge t_2)\geq A(a,c)\wedge B(b,d)$ [by the meet compatibility of A and B]

$=(A\times B)((a,b),(c,d))$.

Similarly, $(A\times B)((a,b)\vee(t_1,t_2),(c,d)\vee(t_1,t_2))\geq(A\times B)((a,b),(c,d))$.

Then $A\times B$ is meet and join compatible.

Again, $A\times B((a,b)\to(t_1,t_2),(c,d)\to(t_1,t_2))$

$=(A\times B)((a\to t_1,b\to t_2),(c\to t_1,d\to t_2))$

$=A(a\to t_1,c\to t_1)\wedge B(b\to t_2,d\to t_2)$

$\geq A(a,c)\wedge B(b,d)$

$(A\times B)((a,b),(c,d))$.

Therefore,"$\to$" is compatible.

Hence, $A\times B$ is a fuzzy congruence on $L\times K$. □

Remark 4.4.2. If A is a fuzzy congruence relation on $L\times K$, then for $c\in K, A((a,c),(b,c))=A((a,y),(b,y))$ for all $y\in K$ and for $x\in L, A((x,c),(x,d))=A((y,c),(y,d))$ [proved in [24]]

Now,we prove the converse part of the theorem.

Let A be a fuzzy congruence relation on $L\times K$.For $a,b\in L$,define A_L on L by $A_L(a,b)=A((a,y),(b,y)),y\in K$ and for $c,d\in K$, define $A_K(c,d)=((x,c),(x,d)),x\in L$. By above A_K and A_L are well defined (proved in [24]).Let us show the congruence with respect to "$\to$".

Since $A_L(a\to c,b\to c)=A((a\to c,1),(b\to c,1))$,$1\in K$

$=A((a\to c,y\to y),(b\to c,y\to y))$,$y\in K$

$\geq A((a,y),(b,y))$

$=A_L(a,b)$. In similarl way $'\rightarrow'$ is compatible for A_K. Thus, A_L is a fuzzy congruence on L and A_K a fuzzy congruence on K . $A = A_L \times A_K$ [proved in [24]]

Corollary 4.4.3. *Let A be fuzzy congruence relation on H,Then $A_{(a,c)} = A_{(a,y)}$,if and only if $A(c,y) = 1$,$\forall a, c, y \in H$*

Example 4.4.4. The fuzzy relation A defined on a HA H by $A(x,y) = 1$ if $x = y$ and 0,otherwise is fuzzy congruence relation on H

Theorem 4.4.5. *If L,K,A, B and $A \times B$ are as in theorem 4.3.1, then the quotient HA $(L \times K)/(A \times B)$ corresponding to $A \times B$ is isomorphic to the product of the corresponding quotient HAs L/A and K/B.*

Proof. By Definition 2.5.9,we have $L/A = \{A_a : a \in L\}, K/B = \{B_b : b \in K\}$ and $(L \times K)/(A \times B) = \{(A \times B)_{(a,b)} : (a,b) \in L \times K\}$.Define a map $\varphi : L/A \times K/B \rightarrow (L \times K)/(A \times B)$ by $\varphi(A_a, B_b) = (A \times B)_{(a,b)}$. Clearly, φ is one to one , onto and $\vee$ and $\wedge$ homomorphism. see [24],it remains to show that φ is $\rightarrow$ hommomorphism.
$\varphi((A_a, B_b) \rightarrow (A_c, B_d))$
$=\varphi(A_a \rightarrow A_c, B_b \rightarrow B_d)$
$= \varphi(A_{a\rightarrow c}, B_{b\rightarrow d})$
$= (A \times B)_{(a\rightarrow c, b\rightarrow d)}$
$= (A \times B)_{((a,b)\rightarrow(c,d))}$
$= (A \times B)_{(a,b)} \rightarrow (A \times B)_{(c,d)}$
$= \varphi(A_a, B_b) \rightarrow \varphi(A_c, B_d)$.
Thus,φ is a homomorphism which completes the proof. □

Theorem 4.4.6. *Let H be a HA and A, B be fuzzy congruences relation on H such that $A \subseteq B$.Then the relation A/B on H/A defined by on defined by $(A/B)(A_x, A_y) = A(x,y), \forall x, y \in H$ is fuzzy congruence on H*

Theorem 4.4.7. *Let (L, A) and (K, B) be two FHA's,then the direct product $((L \times K), (A \times B))$ is also a FHA under point wise operations.*

Proof. Suppose $(A \times B)((a,b) \wedge (c,d),(e,f)) > 0$.We need to show that $(A \times B)((c,d),(a,b) \to (e,f)) > 0$.

Consider $(A \times B)((a,b) \wedge (c,d),(e,f)) > 0$.

$\Rightarrow (A \times B)((a \wedge c, b \wedge d),(e,f)) > 0$

$\Rightarrow A(a \wedge c, e) \wedge B(b \wedge d, f) > 0$

$\Rightarrow A(a \wedge c, e) > 0$ and $B(b \wedge d, f) > 0$

$\Leftrightarrow A(c, a \to e) > 0$ and $B(d, b \to f) > 0$ [Since A is FHA on L and B is a FHA on K]

$\Leftrightarrow A(c, a \to e) \wedge B(d, b \to f) > 0$,$\forall a,c,e \in L$ and $\forall b,d,f \in K$

$\Rightarrow A((c,d),(a \to e, b \to f)) > 0$.

$\Leftrightarrow A((c,d),(a,b) \to (e,f)) > 0$.

Thus,$A \times B$ is a FHA on $L \times K$ □

Theorem 4.4.8. *If $A \times B$ is FHA on $L \times K$,then A is FHA on L and B is FHA on K.*

Proof. Suppose $A \times B((a,b) \wedge (c,d),(e,f)) > 0$.

Then $A((a \wedge c, b \wedge d),(e,f)) > 0$.

$\Rightarrow (A \times B)((c,d),((a \to e),(b \to f)) > 0$.

$\Rightarrow A(c, a \to e) \wedge B(d, b \to f) > 0$.

$\Rightarrow A(c, a \to e) > 0$ and $B(d, b \to f) > 0$,$\forall a,c,e \in L$ and $\forall b,d,f \in K$.

Hence the result follows. □

Theorem 4.4.9. *Let A and B be FHAs of L and K respectivelly,then the cartesian product of A and B is FHA of $L \times K$ iff $(A \times B)_t$ is a HA ,$\forall t \in [0,1]$,is also a FHA of $L \times K$*

Proof. ($\Rightarrow$) Assume $A \times B$ is FHA on $L \times K$.we claim to show that $(A \times B)_t$ is a Heyting algebra.Now $(A \times B)_t$=$\{(a,c),(c,d) \in L \times K : (A \times B)((a,b),(c,d)) \geq t\}$.

Suppose $((a,b),(c,d)),((e,f),(h,g)),((k,l),(m,n)) \in (A \times B)_t$.

From hypothesis,$((a,b),(c,d)) \wedge ((e,f),(h,g)) \leq ((k,l),(m,n))$

$\Rightarrow ((a,b) \wedge (e,f), (c,d) \wedge (h,g)) \leq ((k,l),(m,n))$

$\Rightarrow (a,b) \wedge (e,f) \leq (k,l)$ and $(c,d) \wedge (h,g) \leq (m,n)$

Since $L \times K$ is a HA,we have $(e,f) \leq (a,b) \rightarrow (l,k)$ and $(g,h) \leq (c,d) \rightarrow (m,n)$

And so $((e,f),(g,h)) \leq ((a,b) \rightarrow (k,l),(c,d) \rightarrow (m,n))$.

Thus,$(A \times B)_t$ is a FHA on $L \times K$,for all $a,c,e,g,m \in L$ and $b,d,f,h,n \in K$. In similar way one can prove that the converse holds. □

Definition 4.4.10. Let A be a fuzzy relation on H.Then f is said to f-invariant of A whenever $f(x,y) = f(u,v) \Rightarrow A(x,y) = A(u,v)$ for all $x,y,u,v \in H$.

Theorem 4.4.11. *[29] Let X and Y be sets and let B be a fuzzy partial order relation in Y . Let $f : X \times X \rightarrow Y \times Y$ be a map such that (1)$f_1(x,x) = f_2(x,x)$ for all $x \in X$,*

(2) $f_1(x,y) = f_1(x,z)$ for all $x,y,z \in X$,

(3) $f_2(p,q) = f_2(r,q)$ for all $p,q,r \in X$,

(4) $p \neq q$ implies $f_1(p,q) \neq f_1(q,p)$ (or $p \neq q$ implies $f_2(p,q) \neq f_2(q,p)$), where $f(x,y) = (f_1(x,y), f_2(x,y))$. Then $(X, f^{-1}(B))$ is a fuzzy poset

Definition 4.4.12. [29] Let X and Y be sets let A be a fuzzy partial order relation in X. Let $f : X \times X \rightarrow Y \times Y$ be a map such that

(1) for each $y \in Y$, there exists $x \in X$ such that $f(x,x) = (y,y)$,

(2) for each $x,z \in X$, there exists $y \in Y$ such that $f(x,z) = (y,y)$. Then $(Y, f(A))$ is a fuzzy poset.

Theorem 4.4.13. *Let $f : X \times X \rightarrow Y \times Y$ be a function.And A be f-invariant of $X \times X$ and $Y \times Y$.If A is a FHA on X,then f(A) is a FHA on Y. where X and Y are assumed to be Heyting algebras.*

Proof. Clearly ,f(A) is bounded distributive fuzzy lattice. Suppose (A,X) is a FHA satisfying the f -invariant property.We claim to show that f(A) is FHA.Suppose $f(A)(z \wedge a, u) > 0, \forall a,u,z \in Y$.Now

$$f(A)(a, z \to u) = \begin{cases} Sup\{A(b, x \to y) : f(b, x \to y) = (a, z \to u)\} & \text{if } f^{-1}(a, z \to u) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

$\geq A(b, x \to y)$,where $f(b, x \to y) = (a, z \to u)$

> 0 [by hypothesis] Thus,(f(A),Y) is a FHA. □

Theorem 4.4.14. *Let f be a function from X into Y. Then the following assertions hold.*

(1) For all A_i of family of fuzzy relations on X , $i \in I$, $f(\cup A_i) = \cup f(A_i)$, in particular,

if $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2), \forall A_1 and A_2$ of family fuzzy relations on X

(2) For all family of fuzzy posets $B_j, j \in J$, where J is a nonempty index set,

$f^{-1}(\cup_{j\in J} Bj) = \cup_{j\in J} f^{-1}(B_j), f^{-1}(\cap_{j\in J} B_j) = \cap_{j\in J} f^{-1}(B_j)$

and therefore $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2), \forall B_1, B_2$ of fuzzy posets on Y.

Proof: Using theorem 4.4.13 it is straight forward.

Theorem 4.4.15. *Let $f : X \times X \to Y \times Y$ be a function.Let $A \times B$ be f-invariant of $X \times X$ and $Y \times Y$.If $A \times B$ is FHA on $X \times X$,then $f(A \times B)$ is a FHA on $Y \times Y$. where X and Y are assumed to be Heyting algebras.*

Proof:Clearly $f(A \times B)$ is FHA by Theorem 4.4.13

Theorem 4.4.16. *Let $f : X \times X \to Y \times Y$ be a onto homomorphism.Let $A \times B$ be f-invariant of $X \times X$ and $Y \times Y$ and .If $A' \times B'$ is a FHAs on $Y \times Y$,then $f^{-1}(A' \times B')$ is a FHA on $X \times X$. where X and Y are assumed to be Heyting algebras.*

Proof. Proof: Follows from Theorem 4.4.15 and 4.4.15 □

Theorem 4.4.17. *$f : X \times X \to Y \times Y$ is an homomorphism and $A \times B$ is an f invariant of $X \times X$ and $Y \times Y$. The mapping $(A \times B) \to f(A \times B)$ defines a* *one-one correspondence between the set of all FHA of $X \times X$ and the the set of all FHA of $Y \times Y$*

Proof. Follows from Theorems 4.4.16 and 4.4.17 □

Theorem 4.4.18. *Let $L = (X, A)$ and $M = (Y, B)$ be complete fuzzy Heyting algebras. Then, $L \times M = (X \times Y, C)$ is a complete fuzzy Heyting algebra.*

Proof. First we will prove that $L \times M$ is a sup-complete FHA. For that, let $L = (X, A)$ and $M = (Y, B)$ be complete FHA and let I be a nonempty set on $X \times Y$. Let $I_x = \{x \in H : (x, y) \in I$ for some $y \in Y\}$ and $I_y = \{y \in Y : (x, y) \in I$ for some $x \in X\}$. By hypothesis L and M are complete FHA, then there exist sup I_x and supI_y. We will prove that $(supI_x, supI_y)$ is the supremum of I. Clearly, $(supI_x, supI_y)$ is an upper bound of I. Suppose $(x_2, y_2) \in X \times Y$ is also an upper bound of I. Then, $A(supI_x, x_2) > 0$ and $B(supI_y, y_2) > 0$ and so, $C((supI_x, supI_y),(x_2, y_2)) > 0$. Therefore, $(supI_x,\ supI_y)$ is the supremum of I and $L \times M$ is a sup-complete FHA. In the same manner, we prove that $(infI_x, infI_y)$ is the infimum of I and $L \times M$ is a inf-complete FHA. Therefore, together with proposition 2.3.26 and the above results we can conclude that $L \times M$ is a complete FHA. □

Chapter 5

α Ideal and Fuzzy α Ideals of Fuzzy Heyting Algebra

In this chapter we show the results of the papers Mezzomo et al. [2013a] and Mezzomo et al. [2013d] where they use the fuzzy partial order relation notion defined by Zadeh [1971], and fuzzy lattices defined by Chon [2009]. In Mezzomo et al. [2013a], we propose the notions of $\alpha-$ideals and $\alpha-$filters of a fuzzy Heyting algebra and characterize them by using its support and its level set. Observe that Definition 2.3.15 and 2.3.16 can be generalized in order to embrace the notions of ideals/filters with degree of possibility greater than or equal to α ; it is enough to generalize the first and third requirements to:"If $x \in X, y \in Y$ and $A(y,x) > \alpha$, then $x \in Y$, for $\alpha \in (0,1]$." We also characterize a fuzzy ideal on operation of product between fuzzy Heyting algebra L and M and define fuzzy $\alpha-$ideals of fuzzy Heyting algebra.Here, we characterize a fuzzy $\alpha-$ ideal on product between fuzzy Heyting algebras L and M

5.1 α Ideal and α Filters of a Fuzzy Heyting Algebra

In this section, we propose the notions of $\alpha-$ideals and $\alpha-$ filters of a fuzzy Heyting algebra and characterize them by using its support and its level set.

Definitions and Some Results

We define $\alpha-$ideals and $\alpha-$ filters of a fuzzy Heyting algebra as follows:

Definition 5.1.1. Let $L = (H, A)$ be FHA and $Y \subseteq H.Y$ is an α- ideal of L if

(i) $x \in H$,$y \in Y, A(x, y) \geq \alpha \Rightarrow x \in Y$

(ii) $x, y \in Y \Rightarrow x \vee y \in Y$

(iii) For $x \to a \neq 1$, $a \in I, x \in H \Rightarrow x \to a \in I$

Definition 5.1.2. Let $L = (H, A)$ be FHA and $Y \subseteq H.Y$ is an α-filter of L if

(i) $x \in H$,$y \in Y, A(y, x) \geq \alpha \Rightarrow x \in Y$

(ii) $x, y \in Y, x \wedge y \neq 0 \Rightarrow x \wedge y \in Y$

(iii) $a \in F, x \in H|\{0\} \Rightarrow a \to x \in Y$

Proposition 5.1.3. *If $\alpha \leq \beta$, then any $\alpha-$ ideal Y is a $\beta-$ ideal.*

Let Y be a $\alpha-$ideal and $\alpha \leq \beta$. Then for any $x \in H$, if $A(x, y) \geq \beta$, then $A(x, y) \geq \alpha$, so by Definition 5.1.1 (i), $x \in Y$. On the other hand, if $x, y \in Y$, then by Definition 5.1.1 (ii), $x \vee y \in Y$ .Moreover $x, y \in Y$,by Definition 5.1.1 $x \to y \in Y$. Therefore, Y is a $\beta-$ideal of (H,A). Dually, we prove that if $\alpha \leq \beta$, then any $\beta-$filter is an $\alpha-$filter

Remark 5.1.4. Notice that the set H of a FHA (H,A) is an $\alpha-$ideal, for all $\alpha \in (0, 1]$. Dually, the set H of a fuzzy Heyting algebra (H,A) is an $\alpha-$filter, for all $\alpha \in (0, 1]$

Corollary 5.1.5. *All ideal (filter) in the sense of Definition 2.3.15 and 2.3.16 is an $\alpha-$ideal ($\alpha-$filter)*

Proof. Straightforward from proposition 5.1.3 □

Definition 5.1.6. Let $\alpha \in (0, 1]$. If Y is an ideal (filter) of the Heyting algebra $(H, S(A))$, then for all $\alpha \in (0, 1]$, Y is an $\alpha-$ideal ($\alpha - filter$) of the fuzzy Heyting algebra (H,A).

Proof. Let Y be an ideal of $(H, S(A))$ and $y \in Y$. Consider α is fixed. If $A(x, y) \geq \alpha$, then $(x, y) \in S(A)$ and so, because Y is an ideal, $x \in Y$. So, trivially satisfy the condition (i) of Definition 5.1.1 and the conditions (ii) and (iii) is satisfied because it does not depend on the value of α .Analogously, we prove that if Y is a filter of the Heyting algebra $(H, S(A))$, then for all $\alpha \in (0, 1]$, Y is an $\alpha-$ filter of fuzzy Heyting algebra (H,A). □

Proposition 5.1.7. *Let (H,A) be a fuzzy Heyting algebra,$\alpha \in (0, 1]$ and $Y \subseteq H$. If $(Y, A|Y \times Y)$ is a fuzzy sup-HA,then the set $\downarrow Y_\alpha = \{x \in H : A(x, y) \geq \alpha$ for some $y \in Y\}$ is an $\alpha-$ ideal of (H, A)*

Proof. (i) Let $\alpha \in (0, 1], z \in \downarrow Y_\alpha$ and $w \in H$ such that $A(w, z) \geq \alpha$.Because $z \in \downarrow Y_\alpha$, then exists $x \in Y$ such that $A(z, x) \geq \alpha$, and by proposition 2.3.29, $A(w, x) \geq \alpha$.Therefore, $w \in Y_\alpha$.
(ii)Let $\alpha \in (0, 1]$.Suppose $x, y \in \downarrow Y_\alpha$, then exist $z_1, z_2 \in Y$ such that $A(x, z_1) \geq \alpha$ and $A(y, z_2) \geq \alpha$. So, $A(x, z_1 \vee z_2) \geq \alpha$ and $A(y, z_1 \vee z_2) \geq \alpha$ and by proposition 1.2.4 (ii) $A(x \vee y, z_1 \vee z_2) \geq \alpha$. By hypothesis $(Y, A|Y \times Y)$ is a fuzzy sup-HA, then $z_1 \vee z_2 \in Y$. Hence, $A(x \vee y, z) \geq \alpha$, for some $z \in Y$,and therefore, $x \vee y \in \downarrow Y_\alpha$.
(iii) Let $a \in \downarrow Y_\alpha$ and $x \in H$ for $x \to a \neq 1$,we claim to show that $x \to a \in \downarrow Y_\alpha$.Let $a \in \downarrow Y_\alpha$.Then there exists $z \in Y$ such that $A(a, z) \geq \alpha$.This implies that by Lemma 2.1.20 $A(x \to a, x \to z) \geq \alpha$.Thus, $x \to a \in \downarrow Y_\alpha$ for some $x \to z \in Y$
Hence, the result. □

Theorem 5.1.8. *Let μ be a fuzzy subset of H. Define* $(\downarrow \mu)(x)$=$sup_{y\in H}\{\mu(y) : A(x, y) > 0\}$. *Then* $(\downarrow \mu)_t =\downarrow \mu_t$

Proof. $(\downarrow \mu)_t = \{x \in H : (\downarrow \mu)(x) \geq t\}$
$= \{x \in H : sup_{y\in H}\{\mu(y) : A(x, y) > 0\} \geq t\}$
$= \{x \in H : \mu(y) \geq t$ for some y such that $A(x, y) > 0\}$
$= \{x \in H : y \in \mu_t$ for some y such that $A(x, y) > 0\}$
$=\downarrow \mu_t$
Hence, the result follows □

Corollary 5.1.9. *Let μ be a fuzzy subset of H. Then $\mu \subseteq (\downarrow \mu)$*

Proof. $(\downarrow \mu)(x)$=$sup_{y\in H}\{\mu(y) : A(x, y) > 0\}$
$\geq \mu(x), x \in H$ [Since the supremum is taken overall $y \in H$]
Hence,the result follows. □

Proposition 5.1.10. *i)Let $A^* = \{y : A(x, y) > 0\}, B^* = \{y : A(x\wedge y, y) > 0\}$. Then $A^* \subseteq B^*$*
ii)Let A be FCR(H), $A^* = \{y : A(x, y) > 0\}, B^* = \{y : A(x \to y, y) > 0\}$. Then $A^* \subseteq B^*$

Proof. (i)Let $y \in A^*$.Then $A(x, y) > 0$.
$\Rightarrow A(x \wedge y, y) > 0$.
$\Rightarrow y \in B^*$.Hence, $A^* \subseteq B^*$
(ii) Let A be FCR(H) $y \in A^*$.Then $A(x, y) > 0$.
$\Rightarrow A(x \to y, y \to y) > 0$.
$\Rightarrow A((x \to y) \wedge (x \to y), y \to y) > 0$.
$\Rightarrow A(x \to y, y) \wedge A(x \to y, y) > 0$
. $\Rightarrow A(x \to y, y) > 0$.Hence, $y \in B^*$.Hence, $A^* \subseteq B^*$

□

Remark 5.1.11. From the above proposition, If $A \subseteq B$,then $sup\{\mu(a) : a \in A\} \leq sup\{\mu(b) : b \in B\}$ and $\downarrow \mu(x) \leq \downarrow \mu(x \to y)$.

Proof. Let $t = sup\{\mu(b) : b \in B\}$ and $s = sup\{\mu(a) : a \in A\}$
$\Rightarrow \mu(b) \leq t$, for all $b \in B$
$\Rightarrow \mu(a) \leq t$,for all $a \in A$
$\Rightarrow sup\{\mu(a) : a \in A\} \leq t$
$\Rightarrow s \leq t$
Hence,the result follows □

Theorem 5.1.12. *Let μ be a fuzzy subset of H and A be a fuzzy congruence relation,then $\downarrow \mu$ is a fuzzy ideal of H.*

Proof. $(\downarrow \mu)(0)$=$sup_{y \in H}\{\mu(y) : A(0, y) > 0\}$
Since the supremum is taken overall $y \in H$,we take $0 \in H$ in particular.
$\Rightarrow (\downarrow \mu)(0) \geq \mu(0) = 1$
$\Rightarrow (\downarrow \mu)(0) = 1$
$(\downarrow \mu)(x) = sup_{y \in H}\{\mu(y) : A(x, y) > 0\}$
$\leq sup_{y \in H}\{\mu(y) : A(x \wedge y, y) > 0\}$
$=(\downarrow \mu)(x \wedge y)$
$(\downarrow \mu)(y) = sup_{y \in H}\{\mu(y) : A(y, y) > 0\}$
$\leq sup_{y \in H}\{\mu(y) : A(x \wedge y, y) > 0\}$[By Proposition 5.1.10]
$=(\downarrow \mu)(x \wedge y)$
$(\downarrow \mu)(x \wedge y)$ is an upper bound of $\{(\downarrow \mu)(x), (\downarrow \mu)(y)\}$
Hence,$(\downarrow \mu)(x \wedge y) \geq (\downarrow \mu(x) \vee (\downarrow \mu)(y))$
Now,consider $(\downarrow \mu)(x \vee y)$=$sup_{y \in H}\{\mu(y) : A(x \vee y, y) > 0\}$
$\geq sup_{y \in H}\{\mu(y) : A(x, y) \wedge A(y, y) > 0\}$
$= sup_{y \in H}\{\mu(y) : A(x, y) > 0\}$
$\Rightarrow (\downarrow \mu)(x \vee y) \geq (\downarrow \mu)(x) \wedge (\downarrow \mu)(y)$

By proposition 5.1.10, $(\downarrow \mu)(x) = Sup\{\mu(z) : A(x,z) > 0\} \leq sup_{y\in H}\{\mu(z) : A(x \to y, z) > 0\} = \downarrow \mu(x \to y)$

Similarly, $(\mu)(y) = Sup\{\mu(z) : A(y,z) > 0\} \leq sup_{y\in H}\{\mu(z) : A(x \to y, z) > 0\} = \downarrow \mu(x \to y)$

$\geq (\downarrow \mu)(x) \vee (\downarrow \mu)(y)$.

Hence, By Definition 3.0.28 $\downarrow \mu$ is fuzzy ideal of H. □

Remark 5.1.13. Let $A = \{z \in H : A(z,x) > 0\}, B = \{z \in H : A(z, x \to y) > 0\}$.Then $A \subseteq B$

Theorem 5.1.14. *Let μ be a fuzzy subset of H and A be a fuzzy congruence relation.Define $\uparrow \mu(x) = sup\{\mu(y) : A(y,x) > 0\}$,then $\uparrow \mu$ is a fuzzy filter of H.*

Proof. 1. $\uparrow \mu(1) = sup\{\mu(y) : A(y,1) > 0\} \geq \mu(y), \forall y \in H$
 $\geq \mu(1)$,since $1 \in H$
 =0 as μ is a fuzzy filter.

2. $\mu(x \wedge y) = sup\{\mu(z) : A(z, x \wedge y) > 0\}$
 $= sup\{\mu(z) : A(z,x) \wedge A(z,y) > 0\}$
 $\geq sup\{\mu(z) : A(z,x) > 0\} \wedge sup\{\mu(z) : A(z,y) > 0\}$
 $= \uparrow \mu(x) \wedge \uparrow \mu(y)$

3. $(\uparrow \mu)(x \to y) = sup\{\mu(z) : A(z, x \to y) > 0\}$
 $\geq sup\{\mu(z) : A(z,y) > 0\}$ [By remark 5.1.13]
 $= \uparrow \mu(y)$.
 Similarly,$(\uparrow \mu)(x \to y) \geq (\uparrow \mu)(x)$.
 Hence ,$(\uparrow \mu)(x \to y) \geq (\uparrow \mu)(x) \vee \uparrow \mu(y)$.

4. $\mu(x \vee y) = sup\{\mu(z) : A(z, x \vee y) > 0\}$
 $\geq sup\{\mu(z) : A(z,y) > 0\}=$
 $= \uparrow \mu(x)$.
 Hence,$(\uparrow \mu)(x \vee y) \geq (\uparrow \mu)(x) \vee (\uparrow \mu)(y)$

Therefore, $\uparrow\mu$ is a fuzzy filter of H ∎

Proposition 5.1.15. *Let (H, A) be a FHA and μ be fuzzy subset of H . Then $\downarrow\mu$ and $\uparrow\mu$ satisfy the following properties:*

1. $\mu \subseteq \downarrow\mu$
2. $\mu \subseteq \uparrow\mu$
3. $\theta \subseteq \mu \Rightarrow \downarrow\theta \subseteq \downarrow\mu$
4. $\theta \subseteq \mu \Rightarrow \uparrow\theta \subseteq \uparrow\mu$
5. $\downarrow\downarrow\mu = \downarrow\mu$
6. $\uparrow\uparrow\mu = \uparrow\mu$.

Proof. (1.) Let $t \in im\mu$,$t \in (0, 1]$.Then there exists $x \in H$ such that $\mu(x) = t$.Since $A(x, x) > 0$,then $\downarrow\mu(x) = sup\{\mu(y) : A(x, y) > 0\}$ for all $y \in H$.In particular, $\downarrow\mu(x) = sup\{\mu(x) : A(x, x) > 0\}$.Hence $t \in \downarrow\mu$.Therefore, $\mu \subseteq \downarrow\mu$

(3.) Suppose $\theta \subseteq \mu$.Now consider $\downarrow\mu(x) = sup\{\mu(y) : A(x, y) > 0\}$,for all $y \in H$
$\geq sup\{\theta(y) : A(x, y) > 0\}$,for all $y \in H$[since $\theta \subseteq \mu$]
$= \downarrow\theta(x)$.Hence, the result follows.

(6.) $(\uparrow\uparrow\mu)(z) = sup\{(\uparrow\mu)(x) : A(x, z) > 0\}$ for all $z \in H$
$= sup\{sup\{\mu(y) : A(y, x) > 0\}, A(x, z) > 0\}$for all $y, z \in H$
$= sup\{\mu(y) : A(y, x) > 0, A(x, z) > 0\}$
$= sup\{\mu(y) : A(y, z) > 0\}$
$= (\uparrow\mu)(z)$.In similar manner,we prove the rest of the proposition.

∎

Corollary 5.1.16. *Let (H, A) be a FHA and μ is fuzzy subset of H. $\downarrow\mu(\uparrow\mu)$ is the least fuzzy ideal (fuzzy filter) containing μ .*

Proof. Suppose there exists an ideal θ such that $\mu \subseteq \theta \subseteq\downarrow \mu$.
Let $t \in im(\downarrow \mu)$ and $t \notin im\theta$. Then there exist $x \in H$ such that $t =\downarrow \mu(x)$
$= sup\{\mu(y) : A(x, y) > 0\}$
$\geq \mu(y)$ for all y such that $A(x, y) > 0$.
$\geq \mu(x)$ for $x \in H$ such that $A(x, x) > 0$
$\Rightarrow t \in im\mu$ which directly implies $t \in im\theta$.
A contradiction arises.Hence the result follows. □

5.2 Fuzzy Ideals and Fuzzy Filters of a Fuzzy Heyting Algebra

In this section we show the results of the paper Mezzomo et al. [2013c] where we use the same notion of fuzzy Heyting algebra (H,A), defined by [30], used in the papers Mezzomo et al. [2012a] and Mezzomo et al. [2012b] for fuzzy ideal and filters in fuzzy lattice. We define a fuzzy ideals and fuzzy filters of fuzzy Heyting algebra (H,A) as a fuzzy set I on set H as follows:

Definition 5.2.1. Let (H,A) be a FHA and fuzzy set I on H.Then we call μ_I is a fuzzy ideal of (H,A) if, for all $x, y \in H$, the following conditions are verified

1. If $\mu_I(y) > 0$ and $A(x, y) > 0$, then $\mu_I(x) > 0$;

2. If $\mu_I(x) > 0$ and $\mu_I(y) > 0$, then $\mu_I(x \vee y) > 0$;

3. If $\mu_I(x) > 0$ and $\mu_I(y) > 0$, then $\mu_I(x \rightarrow y) > 0$;

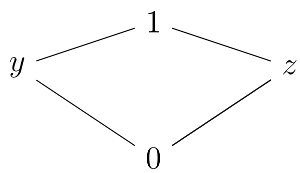
Definition 5.2.2. Let (H,A) be a FHA. A fuzzy set F on H is a fuzzy filter of (H,A) if, for all $x, y \in H$, the following conditions are verified

1. If $\mu_F(y) > 0$ and $A(y, x) > 0$, then $\mu_F(x) > 0$;

2. If $\mu_F(x) > 0$ and $\mu_I(y) > 0$, then $\mu_F(x \wedge y) > 0$;

3. If $\mu_F(x) > 0$ and $\mu_F(y) > 0$, then $\mu_F(y \to x) > 0$;

Example 5.2.3. Let $H = \{0, y, z, 1\}$ defined by the figure depicted below.



And let $A : H \times H \to [0, 1]$ be a fuzzy relation such that $A(0, y) = A(y, z) = A(z, z) = A(1, 1) = 1, A(y, 0) = A(z, 0) = A(1, 0) = A(z, y) = A(1, y) = A(1, z) = 0.8, A(y, z) = 0.3, A(0, 1) = 0.7, A(0, z) = 0.4$ and $A(0, y) = 0.1$ Clearly, (H,A) is FHA.Then, the fuzzy set $I = (1, 0.1), (z, 0.2), (y, 0.4), (0, 1)$ is a fuzzy ideal of L.

Let A_α be the α-level set $A_\alpha = \{(x, y) \in H \times H : A(x, y) \geq \alpha\}$ for some $\alpha \in (0, 1]$ and let $I_\alpha = \{x \in I : A(x, y) \geq \alpha$ for some $y \in I\}$ be an ideal of (H, A_α).
Clearly, $x \in I_\alpha \Rightarrow \mu_I(x) \geq \alpha$

Theorem 5.2.4. *Let I be a fuzzy set on H. I is a fuzzy ideal of FHA (H,A) iff for each $\alpha \in (0, 1], I_\alpha$ is an ideal of (H, A_α).*

Proof. $(\Rightarrow)$

1. Let $y \in I_\alpha$ and $A(x, y) \geq \alpha$.Then $\mu_I(y) \geq \alpha$.Since I is a fuzzy ideal,then $\mu_I(x) \geq \alpha$,Hence $x \in I_\alpha$

2. Let $x, y \in I_\alpha$ for some $\alpha \in (0, 1]$.Then $\mu_I(x) \geq \alpha$ and $\mu_I(y) \geq \alpha$.
   Then $\mu_I(x \vee y) \geq \alpha$. $\Rightarrow x \vee y \in I_\alpha$

3. Let $a \in I_\alpha$ and $x \in H$.Then $\mu_I(a) \geq \alpha$ and $\mu_I(x) \geq \alpha$. $\Rightarrow \mu_I(x \to a) \geq \alpha$ $\Rightarrow x \to a \in I_\alpha$. Conversely

1. If $\mu_I(y) > 0$ and $A(x, y) > 0$, then $y \in I_\alpha$ for $\alpha = A(x, y)$ and so, $(x, y) \in A_\alpha$. Because I_α is an ideal of (H, A_α), then by definition of classical ideal, $x \in I_\alpha$. Therefore, $\mu_I(x) \geq A(x, y) > 0$.

2. Suppose $\mu_I(x) > 0, \mu_I(y) > 0$ and $\alpha = min\{\mu_I(x)), \mu_I(y)\}$. Then, $x \in I_\alpha$ and $y \in I_\alpha$. Because I_α is an ideal of (H, I_α), then by definition of classical ideal, $x \vee y \in I_\alpha$. Therefore, $\mu_I(x \vee y) \geq min\{\mu_I(x), \mu_I(y)\} > 0$.

3. Suppose $\mu_I(x) > 0, \mu_I(y) > 0$ and $\alpha = min\{\mu_I(x)), \mu_I(y)\}$. Let $x \in H$ and $y \in I_\alpha$. Because I_α is an ideal of (H, I_α), then by definition of classical ideal, $x \rightarrow y \in I_\alpha$. Therefore, $\mu_I(x \rightarrow y) \geq min\{\mu_I(x), \mu_I(y)\} > 0$.

□

Let A_α be the α-level set $A_\alpha = \{(x, y) \in H \times H : A(x, y) \geq \alpha\}$ for some $\alpha \in (0, 1]$ and let $F_\alpha = \{x \in F : A(y, x) \geq \alpha$ for some $y \in F\}$ be an α filter of (H, A_α). Then we have the following result.

Theorem 5.2.5. *Let F be a fuzzy set on H. F is a fuzzy filter of (H,A) iff for each* $\alpha \in (0, 1], F_\alpha$ *is a filter of* (H, A_α).

Proof. Similar to the above theorem □

Theorem 5.2.6. *Let* $L = (H, A)$ *and* $M = (G, B)$ *be FHAs, I and J be fuzzy ideals of L and M, respectively. The fuzzy set defined by* $\mu_{I \times J}(x, y) = \mu_I(x) \wedge \mu_J(y)$ *on* $H \times G$ *is a fuzzy ideal of* $L \times M$.

Proof. 1. Let $x_1, x_2 \in H$ and $y_1, y_2 \in G$ such that $\mu_{I \times J}(x_2, y_2) > 0$ and $A \times B((x_1, y_1), (x_2, y_2)) > 0$.

$\Rightarrow \mu_I(x_2) \wedge \mu_J(y_2) > 0$, and $A(x_1, x_2) \wedge B(y_1, y_2) > 0$.

$\Rightarrow \mu_I(x_2) > 0$, $\mu_J(y_2) > 0$ and $A(x_1, x_2) > 0.$, $B(y_1, y_2) > 0$.

$\Rightarrow \mu_I(x_2) > 0$,$A(x_1, x_2) > 0$ and $\mu_I(y_2) > 0$, $B(y_1, y_2) > 0$.

$\Rightarrow \mu_I(x_1) > 0$ and $\mu_J(y_1) > 0$

$\Rightarrow \mu_I(x_1) \wedge \mu_J(y_1) > 0$

$\Rightarrow \mu_{I \times J}(x_1, y_1) > 0$

2. Let $x_1, x_2 \in H$ and $y_1, y_1 \in G$ such that $\mu_{I\times J}(x_1, y_1) > 0$ and $\mu_{I\times J}(x_2, y_2) > 0$
   $\Rightarrow \mu_I(x_1) \wedge \mu_J(y_1) > 0$ and $\mu_I(x_2) \wedge \mu_J(y_2) > 0$
   $\Rightarrow \mu_I(x_1) > 0, \mu_I(x_2) > 0$ and $\mu_J(y_1) > 0, \mu_J(y_2) > 0$
   $\Rightarrow \mu_I(x_1 \to x_2) \wedge\ \mu_J(y_1 \to y_2) > 0$
   $\mu_{I\times J}(x_1 \to x_2, y_1 \to y_2) > 0$

3. Similarly,$\mu_{I\times J}(x_1 \vee x_2, y_1 \vee y_2) > 0$
   $\mu_{I\times J}(x_1 \vee x_2, y_1 \vee y_2) > 0.$
   Hence, $\mu_{I\times J}$ is a fuzzy ideal of $L \times M$

□

Definition 5.2.7. Let (H,A) be a FHA and let I be a fuzzy set on H. The fuzzy set $\Downarrow I$ is defined by $\mu_{\Downarrow I}(x) = Sup_{y\in H}\{min(\mu_I(y), A(x, y))\}$ for all $x \in H$. Dually,let F be a fuzzy set on H. The fuzzy set $\Uparrow F$ is defined by $\mu_{\Uparrow F}(x) = sup_{y\in H}\{min(\mu_F(y), A(y, x))\}$ for all $x \in H$.

Theorem 5.2.8. *Let (H,A) be a FHA, I a fuzzy ideal of H and A be FCR(H). The fuzzy set $\mu_{\Downarrow I}$ of H is a fuzzy ideal of (H,A).Dually,Let F a fuzzy set on H. The fuzzy set $\mu_{\Uparrow F}$ of H is a fuzzy filter of (H,A).*

Proof. 1. Let $\mu_{\Downarrow I}(y) > 0$ and $A(x, y) > 0$.We need to show that $\mu_{\Downarrow I}(x) > 0$.By definition $\mu_{\Downarrow I}(x) = sup_{y\in H}\{min(\mu_I(y), A(x, y))\}$ and $\mu_{\Downarrow I}(y) = sup_{z\in H}\{min(\mu_I(z), A(y, z))\}$. Since A is FCR(H) and I is fuzzy ideal of H,then there exists $z \in H$ such that $\mu_I(z) > 0$ and $A(x, z) > 0$.This implies $min(\mu_I(z), A(x, z)) > 0$. Taking the sup,we have $Sup\{min(\mu_I(z), A(x, z))\} > 0$. Hence, $\mu_{\Downarrow I}(x) > 0$

2. $\mu_{\Downarrow I}(x) > 0$ and $\mu_{\Downarrow I}(y) > 0$.We claim to show that $\mu_{\Downarrow I}(x \to y) > 0$.Then there are $z, w \in H$ such that $\mu_I(z) > 0$, $A(x, z) > 0$ and $\mu_I(w) > 0$ $A(y, w) > 0$.
   As I is a fuzzy ideal of H,we have $\mu_I(z \to w) > 0$, $A(x, z \to w) > 0$ and $\mu_I(z \to w) > 0$, $A(y, z \to w) > 0$.
   $\mu_{\Downarrow I}(z \to w) = Sup(min(\mu_I(t), A(z \to w, t)))$

$A(x,z) > 0$ and $A(y,w) > 0$

$\Rightarrow A(x,z) \wedge A(y,w) > 0$

$A(x \to y, z \to w) \geq A(x,z) \wedge A(y,w) > 0$. [As A is fuzzy congruence relation.] This gives $A(x \to y, z \to w) > 0$, there exists $z \to w \in H$ such that $min(\mu_I(z \to w), A(x \to y, z \to w) > 0$.Hence,taking sup of min over all $x \to y \in H$ $Sup\{min(\mu_I(z \to w), A(x \to y, z \to w))\} > 0$. Thus ,$\mu_{\Downarrow I}(x \to y) > 0$.

3. For $\mu_{\Downarrow I}(x \vee y) > 0$ is Similar to 2.

□

5.3 Fuzzy α Ideals of Fuzzy Heyting Algebra

In this section we define a fuzzy $\alpha-$ ideals of fuzzy Heyting algebra and fuzzy $\alpha-$ ideals on product of FHAs. In addition, we prove that $\alpha-$ideals of the product are equal to the product of $\alpha-$ideals on FHAs

Definition 5.3.1. Let (H, A) be fuzzy Heyting algebra and $\alpha \in (0, 1]$. A fuzzy set I_α on H is a fuzzy $\alpha-$ ideal of (H,A) if, for all $x, y \in H$

(i) If $\mu_{I_\alpha}(y) \geq \alpha$ and $A(x,y) > 0$, then $\mu_{I_\alpha}(x) \geq \alpha$

(ii) If $\mu_{I_\alpha}(x) \geq \alpha$ and $\mu_{I_\alpha}(y) \geq \alpha$, then $\mu_{I_\alpha}(x \vee y) \geq \alpha$.

(iii)If $\mu_{I_\alpha}(x) \geq \alpha$ and $\mu_{I_\alpha}(y) \geq \alpha$, then $\mu_{I_\alpha}(x \to y) \geq \alpha$.

Proposition 5.3.2. *Let (H,A) be a FHA, $\alpha \in (0, 1]$ and I_α be a fuzzy set on H. If A is a fuzzy congruence relation on H and μ_{I_α} is a fuzzy subset of H, then the fuzzy set $\mu_{\Downarrow I_\alpha}(x) = sup\{\mu_{I_\alpha}(y) : A(x,y) > 0$ and $\mu_{I_\alpha}(y) \geq \alpha\}$ is a fuzzy $\alpha-$ideal of (H, A).*

Proof. (i) $x, y \in H$. If $\mu_{\Downarrow I_\alpha}(y) \geq \alpha$ and $x \in H$ such that $A(x,y) > 0$. Then, by definition, $\mu_{\Downarrow I_\alpha}(y) = sup_{z \in H}\{\mu_{I_\alpha}(z) : A(y,z) > 0$ and $\mu_{I_\alpha}(z) \geq \alpha\} \geq \alpha$. So, there exists $z \in H$ such that $\mu_{\Downarrow I_\alpha}(z) \geq \alpha$ and $A(y,z) > 0$. Since $A(x,y) > 0$

and $A(y,z) > 0$, then by transitive property, we have that $A(x,z) > 0$. Thus, $sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x,z) > 0 \text{ and } \mu_{I_\alpha}(z) \geq \alpha\} \geq \alpha$. Therefore,$\mu_{I_\alpha}(x) \geq \alpha$.

(ii) Suppose $\mu_{I_\alpha}(x) \geq \alpha$ and $\mu_{I_\alpha}(y) \geq \alpha$.Then, $\mu_{I_\alpha}(x) \wedge \mu_{I_\alpha}(y) \geq \alpha$.By definition, $\mu_{\Downarrow I_\alpha}(x) = sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x,z) > 0 \text{ and } \mu_{I_\alpha}(z) \geq \alpha\} \geq \alpha$ and $\mu_{\Downarrow I_\alpha}(y) = sup_{w\in H}\{\mu_{I_\alpha}(w) : A(y,w) > 0 \text{ and } \mu_{I_\alpha}(w) \geq \alpha\} \geq \alpha$.

Consider $\mu_{\Downarrow I_\alpha}(x \vee y) = sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x\vee y, z) > 0 \text{ and } \mu_{I_\alpha}(z) \geq \alpha\} \geq \alpha$.

$\geq Sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x,z) \wedge A(y,z) > 0 \text{ and } \mu_{I_\alpha}(z) \geq \alpha\}$[Since A is a congruence]

$= sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x,z) > 0 \text{ and } \mu_{I_\alpha}(z) \geq \alpha\}$ and $sup_{z\in H}\{\mu_{I_\alpha}(z) : A(y,z) > 0\}$ and $\mu_{I_\alpha}(z) \geq \alpha\}$ $=\mu_{\Downarrow I_\alpha}(x) \wedge \mu_{\Downarrow I_\alpha}(y) \geq \alpha$

(iii.) Similrly, Suppose $\mu_{I_\alpha}(x) \geq \alpha$ and $\mu_{I_\alpha}(y) \geq \alpha$.Then, $\mu_{I_\alpha}(x) \wedge \mu_{I_\alpha}(y) \geq \alpha$. By definition, $\mu_{\Downarrow I_\alpha}(x) = sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x,z) > 0 \text{ and } \mu_{I_\alpha}(z) \geq \alpha\}$

Consider $\mu_{\Downarrow I_\alpha}(x \to y) = sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x \to y, z) > 0 \text{ and } \mu_{I_\alpha}(z) \geq \alpha\}$

$\geq sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x,z) \wedge A(y,z) > 0$ and

$\mu_{I_\alpha}(z) \geq \alpha\}$[Since A is a congruence]

$= sup_{z\in H}\{\mu_{I_\alpha}(z) : A(x,z) > 0 \text{ and } \mu_{I_\alpha}(z) \geq \alpha\}$ and $sup_{z\in H}\{\mu_{I_\alpha}(z) : A(y,z) > 0\}$ and $\mu_{I_\alpha}(z) \geq \alpha\}$ $=\mu_{\Downarrow I_\alpha}(x) \wedge \mu_{\Downarrow I_\alpha}(y) \geq \alpha$.

Hence, the result follows. □

Theorem 5.3.3. *Let L = (H,A) and M = (G,B) be FHAs, μ_{I_α} and μ_{J_α} be fuzzy $\alpha-$ideals of L and M respectively.The fuzzy set $\mu_{I_\alpha\times J_\alpha} = \mu_{I_\alpha} \wedge \mu_{J_\alpha}$ is a fuzzy $\alpha-$ideal of $L \times M$, denoted by $I_\alpha \times J_\alpha$.*

Proof. 1. Let $x_1, x_2 \in H$ and $y_1, y_2 \in G$ such that $\mu_{I_\alpha\times J_\alpha}(x_2, y_2) > \alpha$ and $A \times B((x_1, y_1), (x_2, y_2)) > 0$.

$\Rightarrow \mu_{I_\alpha}(x_2) \wedge \mu_{J_\alpha}(y_2) \geq \alpha$, and $A(x_1, x_2) \wedge B(y_1, y_2) > 0$.

$\Rightarrow \mu_{I_\alpha}(x_2) \geq \alpha$, $\mu_{J_\alpha}(y_2) \geq \alpha$ and $A(x_1, x_2) > 0.$, $B(y_1, y_2) \geq \alpha$.

$\Rightarrow \mu_{I_\alpha}(x_2) \geq \alpha$,$A(x_1, x_2) > 0$ and $\mu_{I_\alpha}(y_2) \geq \alpha$, $B(y_1, y_2) > 0$.

$\Rightarrow \mu_{I_\alpha}(x_1) \geq \alpha$ and $\mu_{J_\alpha}(y_1) \geq \alpha$

$\Rightarrow \mu_{I_\alpha}(x_1) \wedge \mu_{J_\alpha}(y_1) \geq \alpha$

$\Rightarrow \mu_{I_\alpha\times J_\alpha}(x_1, y_1) \geq \alpha$

2. Let $x_1, x_2 \in H$ and $y_1, y_1 \in G$ such that $\mu_{I_\alpha \times J_\alpha}(x_1, y_1) \geq \alpha$ and $\mu_{I_\alpha \times J_\alpha}(x_2, y_2) \geq \alpha \Rightarrow \mu_{I_\alpha}(x_1) \wedge \mu_{J_\alpha}(y_1) \geq \alpha$ and $\mu_{I_\alpha}(x_2) \wedge \mu_{J_\alpha}(y_2) \geq \alpha$
   $\Rightarrow \mu_{I_\alpha}(x_1) \geq \alpha, \mu_{I_\alpha}(x_2) \geq \alpha$ and $\mu_{J_\alpha}(y_1) \geq \alpha, \mu_{J_\alpha}(y_2) \geq \alpha$
   $\Rightarrow \mu_{I_\alpha}(x_1 \rightarrow x_2) \wedge \mu_{J_\alpha}(y_1 \rightarrow y_2) \geq \alpha$
   $\mu_{I_\alpha \times J_\alpha}(x_2 \rightarrow x_2, y_1 \rightarrow y_2) \geq \alpha$

3. Similarly, it is easy to show that $\mu_{I_\alpha \times J_\alpha}(x_1 \vee x_2, y_1 \vee y_2) \geq \alpha$. Hence, the result follows.

$\square$

Chapter 6

Heyting Almost Distributive Fuzzy Lattices

6.1 Definition and Some Results

Definition 6.1.1. [4] Let $(H, \vee, \wedge, 0)$ be an algebra of type (2,2,0) and (H,A) be a fuzzy poset.Then we call L = (H,A) is an Almost Distributive Fuzzy Lattice (ADFL) if the following axioms are satisfied:

1) $A(a, a \vee 0) = A(a \vee 0, a) = 1$;

2) $A(0, 0 \wedge a) = A(0 \wedge a, 0) = 1$;

3) $A((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)) = A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c) = 1$;

4)$A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$;

5) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \wedge c), a \vee (b \wedge c)) = 1$;

6) $A((a \vee b) \wedge b, b) = A(b, (a \vee b) \wedge b) = 1$; for all $a, b, c \in H$.

Definition 6.1.2. [4] Let L be an ADFL . Then for any $a, b \in H, a \leq b$ if and only if $A(a, b) > 0$.

Definition 6.1.3. An ADFL L=(H,A) with maximal element m is said to be Heyting almost distributive fuzzy lattices if to each $a \in H$,the interval $[0, a]$ is fuzzy Heyting algebra.

Example 6.1.4. Every FHA is an HADFL since every interval in FHA is itself FHA

Theorem 6.1.5. *Let (H,A) be an HADFL with a maximal element m. Then the following are equivalent.*

(1) H is an HADFL

(2) [0,m] is a Fuzzy Heyting algebra.

(3) There exists a binary operation $\rightarrow$ on H such that the following conditions hold: for all $x, y, z \in H$

(i) $A(m, x \rightarrow x) > 0$

(ii) $A(y, (x \rightarrow y) \wedge y) > 0$

(iii) $A(x \wedge y \wedge m, x \wedge (x \rightarrow y)) > 0$

(iv) $A((x \rightarrow (y \wedge z), (x \rightarrow y) \wedge (x \rightarrow z)) = A((x \rightarrow y) \wedge (x \rightarrow z), x \rightarrow (y \wedge z)) = m$

(v) $A((x \vee y) \rightarrow z, (x \rightarrow z) \wedge (y \rightarrow z)) = A((x \rightarrow z) \wedge (y \rightarrow z)), (x \vee y) \rightarrow z) = m$

Proof. Let H be an HADFL with 0 and a maximal element m. Let $\rightarrow$ be a binary operation on H.

(1)$\Rightarrow$ (2) is trivial .

(2)$\Rightarrow$(3) : Assume that $[0, m]$ is a fuzzy Heyting algebra in which the binary operation ($\rightarrow$) is denoted by $\rightarrow_m$.For $x, y \in$ H, define $x \rightarrow y = x \wedge m \rightarrow_m y \wedge m$.Let $x, y, z \in$ H.

Now (i) $A(m, x \rightarrow x)$=$A(m, x \wedge m \rightarrow_m x \wedge m) = A(m, m)$ >0

(ii) $A(y, (x \rightarrow y) \wedge y)$=$A(y, (x \wedge m \rightarrow_m y \wedge m) \wedge y)$= $A(y, (x \wedge m \rightarrow_m y \wedge m) \wedge y \wedge m)$=$A(y, y \wedge m) > 0$ routenly we can show others. (3) $\Rightarrow$ (1).

Assume the condition hold on H,then for $a \in H$.

We know that $[0, a]$ is a distributive fuzzy lattice.Define a binary operation $\rightarrow_a$ on $[0, a]$ by $x \rightarrow_a y = (x \rightarrow y) \wedge a, \forall x, y \in [0, a]$ Let $x, y, z \in [0, a]$ and $A(x \wedge z, y)$> 0.Then $A(x \rightarrow (x \wedge z), x \rightarrow y)$> 0.

$\Rightarrow A(m \wedge (x \to z), x \to y) > 0$

$\Rightarrow A(x \to z, x \to y) > 0$

$\Rightarrow A((x \to z) \wedge a, (x \to y) \wedge a) > 0$

$\Rightarrow A((x \to z) \wedge a, x \to_a y) > 0$

$\Rightarrow A(z \wedge (x \to z) \wedge a, z \wedge (x \to_a y)) > 0$

$\Rightarrow A((x \to z) \wedge z \wedge a, z \wedge (x \to_a y)) > 0$

$\Rightarrow A(z \wedge (x \to z) \wedge a, x \to_a y) > 0$[since $z \wedge (x \to_a y) \leq x \to_a y$]

$\Rightarrow A(z \wedge a, x \to_a y) > 0$

Thus, $A(z, x \to_a y) > 0$[Since $z \leq a$]

Conversely ,assume that $z \in [0, a]$ and $A(z, x \to_a y) > 0$.

$\Rightarrow A(x \wedge z, x \wedge (x \to y) \wedge a) > 0$.

$\Rightarrow A(x \wedge z, x \wedge y \wedge a) > 0$.

$\Rightarrow A(x \wedge z, x \wedge y) > 0$.

$\Rightarrow A(x \wedge z, y) > 0$. [Since $A(x \wedge y, y) > 0$]

Thus,$[0, a]$ is a fuzzy Heyting algebra Therefore,H is an HADFL.

Through out this section the symbol H stands for an HADFL (H,A) unless otherwise specified.In the following lemma,we give some important properties of HADFL. □

Lemma 6.1.6. *Let $x, y, a \in H$ and $A(x, y) > 0$.Then the following hold:*

1. $A(a \to x, a \to y) > 0$

2. $A(y \to a, x \to a) > 0$

3. $A(m, (a \wedge b) \to b) > 0$

Proof. 1. Let $x, y, a \in H$ and $A(x, y) > 0$. Since H is an HADFL $A((a \to x) \wedge (a \to y), a \to (x \wedge y)) = 1$

$A((a \to x) \wedge (a \to y), a \to x) = 1$. [Since, $A(x, y) > 0 \Leftrightarrow x \wedge y = x$].This implies $a \to x = (a \to x) \wedge (a \to y)$

$\Rightarrow A(a \to x, a \to y) > 0$

2. $A((y \to a) \wedge (x \to a), (y \vee x) \to a) = 1$

   $A((y \to a) \wedge (x \to a), y \to a) = 1$

   $\Rightarrow A(y \to a, x \to a) > 0$

3. $A(a \wedge b, b) > 0$

   $\Rightarrow A((a \wedge b) \to (a \wedge b), (a \wedge b) \to b) > 0$

   $\Rightarrow A(m, (a \wedge b) \to b) > 0.$

□

6.2 Properties of Heyting Almost Distributive Fuzzy Lattices

Theorem 6.2.1. *For any $x, y \in H$,we have the following.*

1. $A(y, x \to 0) > 0 \Rightarrow x \wedge y = 0$
2. $A(y \to 0, x \to 0) > 0 \Leftrightarrow x \wedge (y \to 0) = 0$
3. $A(m \to x, x) > 0$
4. $A(x \to m, m) > 0$

Proof. 1. $A(y, x \to 0) > 0$

   $\Rightarrow A(x \wedge y, x \wedge (x \to 0)) > 0$

   $\Rightarrow A(x \wedge y, x \wedge 0) > 0$.This implies $x \wedge y \leq 0$.But $0 \leq x \wedge y$

   Thus, $x \wedge y = 0$

2. $(\Rightarrow) A(x \wedge (y \to 0), x \wedge (x \to 0)) > 0$

   $\Rightarrow A(x \wedge (y \to 0), x \wedge 0 \wedge m) > 0$

   $\Rightarrow x \wedge (y \to 0) = 0.$

   Conversely,assume $x \wedge (y \to 0) = 0$.Then $A(y \to 0, x \to 0)$

$= A(y \to 0, x \to (x \wedge (y \to 0)))$

$=A(y \to 0, m \wedge (x \to (y \to 0)) \wedge (y \to 0))$

$=A(y \to 0, y \to 0) = 1 > 0$

3. $A(m \to x, m \wedge (m \to x)) > 0$

   $\Rightarrow A(m \to x, m \wedge x \wedge m) > 0$

   $\Rightarrow A(m \to x, x \wedge m) > 0.$

   $\Rightarrow A(m \to x, x) > 0$.But $A(x, m \to x) > 0$.Therefore,the result holds.

4. $A(m, x \to m) > 0$

   $A(x \to m, m) > 0$

   $\Rightarrow A(m \wedge (x \to m), m) > 0$

   $\Rightarrow A((m \to m) \wedge (x \to m), m) > 0$

   $\Rightarrow A((m \vee x) \to m, m) > 0$

   Clearly, $A(m, m) = 1 > 0$

   but $A(x \to m, m) > 0$

   Thus,antisymmetry gives $m = x \to m$

□

Lemma 6.2.2. $A(x \wedge y, x) = 1 \Leftrightarrow A(x, y) > 0$

Theorem 6.2.3. *Let m be the maximal element in H. Then for any $a, b, c \in H$,the following holds.*

1. $A(b \wedge m, (a \to b) \wedge m)) > 0$

2. $A(a \wedge m, b \wedge m) > 0 \Leftrightarrow (a \to b) \wedge m = m$

3. $A(a \wedge b \wedge m, a \wedge c \wedge m) = 1 \Leftrightarrow A((a \to b) \wedge m, (a \to c) \wedge m) = 1$

4. $A(a \wedge c \wedge m, b \wedge m) > 0 \Leftrightarrow A(c \wedge m, (a \to b) \wedge m) > 0$

5. $A(a \wedge m, [(a \to b) \to b] \wedge m) > 0$

6. $A(a \wedge m, (b \to c) \wedge m) > 0 \Leftrightarrow A(b \wedge m, (a \to c) \wedge m) > 0$

Proof. 1. Since$A(b, a \to b) > 0, A(m, m) > 0$. Then,$A(b \wedge m, (a \to b) \wedge m) > 0$.

2. $A(a \wedge m, b \wedge m) > 0$

$\Rightarrow A(a \to (a \wedge m), a \to (b \wedge m)) > 0$

$\Rightarrow A((a \to a) \wedge (a \to m), (a \to b) \wedge (a \to m)) > 0.$

$\Rightarrow A(m \wedge (a \to m), a \to (b \wedge m)) > 0$

$\Rightarrow A(m, a \to (b \wedge m)) > 0$[Since $a \to m = m$,[By Theorem 6.2.1]

But $A((a \to b) \wedge m, m) > 0$.

Anti symmetry property gives $(a \to b) \wedge m = m$.

Conversely,$A((a \to b) \wedge m, m) = 1, A(a, a) > 0$.

$\Rightarrow A(a \wedge (a \to b) \wedge m), a \wedge m) = 1$

$\Rightarrow A(a \wedge b \wedge m, a \wedge m) = 1.$

$\Rightarrow A(a \wedge m \wedge b \wedge m, a \wedge m) = 1.$

$\Rightarrow A(a \wedge m, b \wedge m) > 0.$[Lemma 6.2.2]

3. Assume that $A(a \wedge b \wedge m, a \wedge c \wedge m) = 1 > 0$.

By (2) $A(a \to (a \wedge b \wedge m), a \to (a \wedge c \wedge m)) > 0$. Hence $A((a \to b) \wedge m, (a \to c) \wedge m.) > 0$

Conversely, assume that $A((a \to b) \wedge m, (a \to c) \wedge m) > 0$.

$A(a \wedge (a \to b) \wedge m, a \wedge (a \to c) \wedge m) > 0.$

Hence $A(a \wedge b \wedge m, a \wedge c \wedge m) > 0$.

4. Assume $A(a \wedge c \wedge m, b \wedge m) > 0$.

$\Rightarrow A(a \to (a \wedge c \wedge m), a \to (b \wedge m)) > 0.$

$\Rightarrow$. A($a \to (c \wedge m), (a \to b) \wedge (a \to m)) > 0.$

$\Rightarrow A((a \to c) \wedge m, (a \to b) \wedge m) > 0$ [Theorem 6.2.3]

$\Rightarrow A(c \wedge m, (a \to b) \wedge m) > 0$[By 1 above]

This gives $A(a \wedge (c \wedge m), a \wedge (a \to b) \wedge m) > 0$.

And hence $A(a \wedge c \wedge m, a \wedge b \wedge m) > 0$.

Thus, $A(a \wedge c \wedge m, b \wedge m) > 0$.

5. Now $A(a \wedge (a \to b) \wedge m, a \wedge b \wedge m) > 0$

   $\Rightarrow A(a \wedge (a \to b) \wedge m, b \wedge m) > 0$

   $\Rightarrow A((a \to b) \wedge a \wedge m, b \wedge m) > 0$

   $\Leftrightarrow A(a \wedge m, [(a \to b) \to b] \wedge m) > 0$

6. Assume that $A(a \wedge m, (b \to c) \wedge m) > 0$.

   $\Rightarrow A(b \wedge a \wedge m, b \wedge (b \to c) \wedge m) > 0$

   $\Rightarrow A(b \wedge a \wedge m, b \wedge c \wedge m) > 0$ and $A(b \wedge c \wedge m, c \wedge m) > 0$

   $\Rightarrow A(b \wedge a \wedge m, c \wedge m) > 0$

   Therefore, $A(b \wedge m, (a \to c) \wedge m) > 0$.

   Conversely,assume $A(b \wedge m, (a \to c) \wedge m) > 0$

   $\Rightarrow A(a \wedge b \wedge m, a \wedge (a \to c) \wedge m) > 0$

   $\Rightarrow A(a \wedge b \wedge m, a \wedge c \wedge m) > 0$

   $\Rightarrow A(a \wedge b \wedge m, c \wedge m) > 0$

   $\Rightarrow A(b \wedge a \wedge m, c \wedge m) > 0$

   Thus, $A(a \wedge m, (b \to c) \wedge m) > 0$

□

6.3 Charaterization of HADFL

Definition 6.3.1. Let (H, A) be an HADFL and $a \in H$,then the principal ideal generated by a is denoted by $(a]_A$ and is equal to $\{x \in H : A(x, a \wedge x) > 0\}$

Lemma 6.3.2. *If* $(a] \subseteq (b]$,*then* $(a]_A \subseteq (b]_A$,*for all* $a, b \in H$.

Lemma 6.3.3. $a \in (b] \Leftrightarrow A(a, b \wedge a) > 0$.

Lemma 6.3.4. *Let $a, b \in H$ and (H,A) be an HADFL,then the following are equivalent.*

1. $(a]_A \subseteq (b]_A$
2. $A(a, b \wedge a) > 0$
3. $A(a \wedge x, b \wedge x) > 0, \forall x \in H$

Proof. Follows from the above two lemmas □

Theorem 6.3.5. *Let (H,A) be an ADFL with 0 and a maximal element m,then (H,A) is an HADFL iff (PI(H),A) is a FHA .*

Proof. Suppose (H,A) be an HADFL.Then (PI(H),A) is a distributive fuzzy lattice.For any $x, y \in H$,define $(x] \to (y] = (x \to y]$.If $(a] = (b]$,and $(c] = (d]$.Then $A(b, a \wedge b) > 0, A(a, b \wedge a) > 0, A(d, c \wedge d) > 0, A(c, d \wedge c) > 0$.
Consider $A(b \to d, (b \to c) \wedge (b \to d))$
$= A(b \to d, ((b \vee a) \to c) \wedge (b \to d))$
$= A(b \to d, (b \to c) \wedge (a \to c) \wedge (b \to d))$
> 0.Again $A((b \to c) \wedge (a \to c) \wedge (b \to d), (a \to c) \wedge (b \to d)) > 0$
$\Rightarrow A(b \to d, (a \to c) \wedge (b \to d)) > 0$ and $A((a \to c) \wedge (b \to d), b \to d)) > 0$.
$\Rightarrow A(b \to d, (a \to c) \wedge (b \to d)) > 0$.By lemma 6.3.3,we have $(a \to c] \subseteq (b \to d]$,by symmetry,$(b \to d] \subseteq (a \to c]$.Thus, $(a \to c] = (b \to d]$ Therefore,"$\to$" is well defined on PI(H).By theorem on FHA,a bounded distributive fuzzy lattice is a FHA.Conversely,assume PI(H) is a FHA.For $a, b \in H$,define $a \to b = c \wedge m$,where $(a] \to (b] = (c]$ for some $c \in H$.Let $(s] = (t]$ for some $s, t \in H$.Then $A(s \wedge t, t) > 0, A(t, s \wedge t) > 0, A(s, t \wedge s) > 0$ and $A(t \wedge s, s) > 0$
$\Rightarrow A(s \wedge m, t \wedge s \wedge m) > 0$
$\Rightarrow A(s \wedge m, s \wedge t \wedge m) > 0$.
But $A(t \wedge s \wedge m, s \wedge m) > 0$
$\Rightarrow t \wedge s \wedge m = s \wedge m$[since antisymmetry]

$\Rightarrow s \wedge t \wedge m = s \wedge m$

$\Rightarrow t \wedge m = s \wedge m$

Thus the binary operation " $\rightarrow$ " is well defined.Let $a, b, c \in H$. We prove that (H, A) is HADFL.

(1) Since $(a] \rightarrow (a] = (m]$.Then we get $a \rightarrow a = m \wedge m$

$\Rightarrow a \rightarrow a = m$

$\Rightarrow A(m, a \rightarrow a) > 0$

(2) Let $(a] \rightarrow (b] = (c]$ .Then $(a \rightarrow b) \wedge b = c \wedge m \wedge b = c \wedge b = b$.

Thus,$A((a \rightarrow b) \wedge b, b) > 0$.

(3) Since $(a] \rightarrow (b] = (c]$.Then $(a] \wedge (c] = (a] \wedge ((a] \rightarrow (b])$

$\Rightarrow (a \wedge c] = (a \wedge b]$.Now $(a \wedge (a \rightarrow b)) = a \wedge c \wedge m = a \wedge b \wedge m$

$\Rightarrow A(a \wedge (a \rightarrow b), a \wedge b \wedge m) > 0$. And $A(a \wedge b \wedge m, a \wedge (a \rightarrow b)) > 0$

(4) Let $(a] \rightarrow (c] = (t]$ and $(b] \rightarrow (c] = (s]$.Then $a \rightarrow c = t \wedge m$ and $b \rightarrow c = s \wedge m$.Consider,$(a \vee b] \rightarrow (c]$=$((a] \vee (b]) \rightarrow (c] = ((a] \rightarrow (c]) \wedge ((b] \rightarrow (c])$=$(t] \wedge (s] = (t \wedge s]$.

$\Rightarrow (a \vee b) \rightarrow c = t \wedge s \wedge m = t \wedge m \wedge s \wedge m = (a \rightarrow c) \wedge (b \rightarrow c)$

$A((a \vee b) \rightarrow c, (a \rightarrow c) \wedge (b \rightarrow c)) > 0$ and $A((a \rightarrow c) \wedge (b \rightarrow c), (a \vee b) \rightarrow c) > 0$

(5) similar to (4) we can prove that $A(a \rightarrow (b \wedge c), (a \rightarrow c) \wedge (a \rightarrow c)) > 0$ and $A((a \rightarrow c) \wedge (a \rightarrow c), a \rightarrow (b \wedge c)) > 0$.Thus,(H,A) is an HADFL.

Now, we give another characterization for an HADFL to become a FHA. □

Theorem 6.3.6. *Let (H,A) be an HADFL.Then (H,A) is a FHA iff for any* $a \in H, \theta_a = \{(x, y) \in H \times H : A(a, (x \rightarrow y) \wedge (y \rightarrow x)) > 0\}$ *is a congruence relation on H.*

Proof. Assume that (H,A) is a FHA and $a \in H$.

1. θ_a is reflexive. Since$A(a, (a \rightarrow a) \wedge (a \rightarrow a)) = A(a, m) > 0$ [as m maximal in H] Thus,$(a, a) \in \theta_a$

2. θ_a is symmetric. Let $(x, y) \in \theta_a$.

   Then $A(a, (x \to y) \wedge (y \to x)) > 0$.

   $\Rightarrow A(a, (y \to x) \wedge (x \to y)) > 0$.[Hypothesis]

3. θ_a is transitive.Let $x, y, z \in H$ such that $(x, y) \in \theta_a, (y, z) \in \theta_a$,then $A(a, (x \to y) \wedge (y \to x)) > 0$ and $A(a, (y \to z) \wedge (z \to y)) > 0$.

   $\Rightarrow A(a, (x \to y) \wedge (y \to x) \wedge (y \to z) \wedge (z \to y)) > 0$. *[Lemma 1.2.4]

   θ_a is transitive Since , $x \wedge (x \to y) \wedge (y \to z) = x \wedge y \wedge z \leq z$,we have $A(x \wedge (x \to y) \wedge (y \to z), z) > 0$.

   $\Rightarrow A((x \to y) \wedge (y \to z), x \to z) > 0$. **[From Definition of FHA]

   Similarly,Again $z \wedge (z \to y) \wedge (y \to x) = z \wedge y \wedge x \leq x$,we have $A(z \wedge (z \to y) \wedge (y \to x), x) > 0$.

   $\Rightarrow A((z \to y) \wedge (y \to x), z \to x) > 0$ ***

   From ** and ***, we have $A((x \to y) \wedge (y \to z) \wedge (z \to y) \wedge (y \to x), (x \to z) \wedge (z \to x)) > 0$ ****.

   Thus, from * and **** we have $A(a, (x \to z) \wedge (z \to x)) > 0$.

   Therefore,$(x, z) \in \theta_a$

   From (1),(2)and (3) θ_a is an equivalence relation on H.

4. θ_a is a congruence relation.

   Since $A(x \wedge d, x) > 0$ and $A(y \wedge d, y) > 0$,we have $A(x \to y, (x \wedge d) \to y) > 0$ and $A(y \to x, (y \wedge d) \to x) > 0$.

   $\Rightarrow A((x \to y) \wedge (y \to x), (x \wedge d) \to y) \wedge (y \wedge d) \to x)) > 0$.But $A(a, (x \to y) \wedge (y \to x)) > 0$. This gives $A(a, (x \wedge d) \to y) \wedge (y \wedge d) \to x)) > 0$. Thus,$(x \wedge d, y \wedge d) \in \theta_a$.

   By similar argument,we can show that $(x \vee d, y \vee d) \in \theta_a$.Now, $A(x \wedge (x \to y) \wedge (y \to d), d) > 0$.

   This implies $A((x \to y) \wedge (y \to d), (x \to d)) > 0$.

   $\Rightarrow A((x \to y), (y \to d) \to (x \to d)) > 0$.

Also by symmetry,$A((y \to x),(x \to d) \to (y \to d)) > 0$.

$\Rightarrow A((x \to y) \wedge (y \to x),((y \to d) \to (x \to d)) \wedge ((x \to d) \to (y \to d))) > 0$...
*1

Using $A(a,(x \to y) \wedge (y \to x)) > 0$ and *1 we have $A(a,(y \to d) \to (x \to d)) \wedge ((x \to d) \to (y \to d))) > 0$.Thus,$(x \to d, y \to d) \in \theta_a$.Similarly one can show that $(d \to x, d \to y) \in \theta_a$.Hence the result.

Conversely,assume that H is an HADFL in which θ_a is a congruence relation on H for all $a \in H$.Now for any $a \in H,(a,a) \in \theta_a$,we get $A(a,a \to a) > 0$.This implies $A(a,m) > 0$

(H,A) is a distributive fuzzy lattice and hence it is a FHA.(By Theorem 2.1.26).Finally summing up all the characterization of an HADFL (H,A) to become a FHA.We state the following theorem.

□

Theorem 6.3.7. *In an HADFL (H,A),the follwing are equivalent*

1. *(H,A) is FHA.*

2. *(H,A) is a distributive fuzzy lattice.*

3. *The fuzzy poset (H,A) is directed above fuzzy poset.*

4. *For* $a,b,c \in H, A(a \wedge c,b) > 0 \Leftrightarrow A(c,a \to b) > 0$

5. $A(b,a \to b) > 0$,*for all* $a,b \in H$

6. $\theta_a = \{(x,y) \in H \times H : A(a,(x \to y) \wedge (y \to x)) > 0\}$ *is a congruence relation on* $H, \forall a \in H$

6.4 Congruence Relation on Heyting Almost Distributive Lattice

In this section,to each implicative filter F of H,we define a congruence relation θ_F on H and prove that for any implicative filter F of H,the quotient H/θ_F is always an HADL with least and greatest element.Moreover; we also proved that if F and $F^{'}$ be any two filters of HADLs H_1 and H_2 respectively,then for any homomorphism ϕ:$H_1 \to H_2$ such that $\phi(F) \subseteq F^{'}$, there exists a homomorphism $f : H_1/\theta_F \to H_2/\theta_{F^{'}}$ such that $f \circ h = k \circ \phi$ where $h : H_1 \to H_1/\theta_F$ and $k : H_2 \to H_2/\theta_{F^{'}}$ denote the canonical epimorphisms.Further if ϕ is a monomorphism and if $\phi(F) = F^{'}$,then f is a monomorphism. If ϕ is an epimorphism,then f is an epimorphism

Lemma 6.4.1. *[37] Let $(H, \vee, \wedge, \to 0, m)$ be an HADL. Then, for $x, y, z, c \in H$, the following hold.*
(i)$(x \to z) \wedge c = (x \wedge c \to z \wedge c) \wedge c$
(ii) $x \wedge c = y \wedge c \Rightarrow (x \to z) \wedge c = (y \to z) \wedge c$
(iii)$x \wedge c = y \wedge c \Rightarrow (z \to x) \wedge c = (z \to y) \wedge c.$

Theorem 6.4.2. *Let $(H, \vee, \wedge, \to, 0, m)$ be an HADL and F be an implicative filter of H. Define $\theta_F = \{(x, y) \in H \times H : x \wedge c = y \wedge c;$ for some $c \in F\}$. Then θ_F is a congruence relation on H and is the smallest congruence on H containing F in a single equivalent class.*

Proof. Clearly $\theta_F$is an equivalence relation on H. Let $(x, y) \in \theta_F$ and $d \in H$. Then $x \wedge c = y \wedge c$ for some $c \in F$ and $x \wedge d \wedge c = d \wedge x \wedge c = d \wedge y \wedge c = y \wedge d \wedge c$. This implies $(x \wedge d, y \wedge d) \in \theta_F$. Also $(d \wedge x, d \wedge y) \in \theta_F$ by a similar argument.Now from lemma 6.4.1 we have $(x \to d) \wedge c = [x \wedge c \to d \wedge c] \wedge c = [y \wedge c \to d \wedge c] \wedge c = (y \to d) \wedge c$. This implies $(x \to d, y \to d) \in \theta_F$. Similarly $(d \to x, d \to y) \in \theta_F$
Let α be a congruence relation on H containing F in a single equivalent class.Let $(x, y) \in \theta_F$.then there exists $a \in F$ such that $x \wedge a = y \wedge a$.Now $a \in F$ implies

$x \vee a \in F$ so that $(a, x \vee a) \in \alpha$ which shows that $(x \wedge a, x) \in \alpha$.Similarly $(y \wedge a, y) \in \alpha$ and hence $(x, y) \in \alpha$. Thus the theorem follows. □

Corollary 6.4.3. *For any $a \in H, \theta_a = \{(x, y) \in H \times H : x \wedge a = y \wedge a\}$ is a congruence relation on H*

Proof. Observe that $\theta_a = \theta_{[a)}$ □

Notation: $H/\theta_F = \{\theta_F[x] : x \in H\}$ where $\theta_F[x]$ =$\{y : (x, y) \in \theta_F\}$

Theorem 6.4.4. *H/θ_F is a HADL under the binary operations defined by $\theta_F[x \rightarrow y] = \theta_F[x] \rightarrow \theta_F[y]$, $\theta_F[x \wedge y] = \theta_F[x] \wedge \theta_F[y]$,$\theta_F[x \vee y] = \theta_F[x] \vee \theta_F[y]$.*

Proof. First let us show that the above binary operations are well defined. Assume $\theta_F[x] = \theta_F[x']$and $\theta_F[y] = \theta_F[y'], x, x', y, y' \in H$.We claim to show that $\theta_F[x \vee y] = \theta_F[x' \vee y']$,$\theta_F[x \wedge y] = \theta_F[x' \wedge y']$,$\theta_F[x \rightarrow y] = \theta_F[x' \rightarrow y']$.Clearly,$x \in \theta_F(x)$ which implies $x \in \theta_F[x']$ and then $(x, x') \in \theta_F$.Similarly $(y, y') \in \theta_F$.Hence $(x \wedge y, x' \wedge y') \in \theta_F$, $(x \vee y, x' \vee y') \in \theta_F$, $(x \rightarrow y, x' \rightarrow y') \in \theta_F$.This implies that $\theta_F[x \wedge y] = \theta_F[x' \wedge y']$,$\theta_F[x \vee y] = \theta_F[x' \vee y']$,$\theta_F[x \rightarrow y] = \theta_F[x' \rightarrow y']$ Hence,the binary operations are well defined.It can be easily verified that H/θ_F is HADL with least element $\theta_F[0]$ and maximal element $\theta_F[m], 0, m \in H$. □

Proposition 6.4.5. *For any filter F of H,the quotient HADL H/θ_F is always a distributive lattice with greatest element.*

Proof. Let $a \in F$.Then,for any $x \in H, x \vee a \in F$ so that $(a, x \vee a) \in \theta_F$.Thus,$\theta_F(x) \leq \theta_F(a)$ for all $x \in H$.Hence ,H/θ_F is a distributive lattice with greatest element $\theta_F(a)$ □

Lemma 6.4.6. *Let H be a HADL.Then the canonical map $h : H \rightarrow H/\theta_F$ defined by $h(x) = \theta_F[x], x \in H$ is a homomorphism.*

Theorem 6.4.7. *Let F and $F^{'}$ be any two filters of HADLs H_1 and H_2 resp.Then for any homomorphism ϕ:$H_1 \to H_2$ such that $\phi(F) \subseteq F^{'}$, there exists a homomorphism $f : H_1/\theta_F \to H_2/\theta_{F'}$ such that $f \circ h = k \circ \phi$ where $h : H_1 \to H_1/\theta_F$ and $k : H_2 \to H_2/\theta_{F'}$ denote the canonical epimorphisms.Further*
(i.)if ϕ is a monomorphism and if $\phi(F) = F^{'}$,then f is a monomorphism.
(ii.)If ϕ is an epimorphism,then f is an epimorphism.

Proof. Define $f : H_1/\theta_F \to H_2/\theta_{F'}$ by $f(\theta_F[x]) = \theta_{F'}[\phi(x)]$.
Let $\theta_F[x] = \theta_F[y], x, y \in H_1$.
Then $(x, y) \in \theta_F$
$\Rightarrow x \wedge c = y \wedge c, c \in F$
$\Rightarrow \phi(x \wedge c) = \phi(y \wedge c)$
$\Rightarrow \phi(x) \wedge \phi(c) = \phi(y) \wedge \phi(c)$
$\Rightarrow (\phi(x), \phi(y)) \in \theta_{F'}$,as $\phi(c) \in F^{'}$
$\theta_{F'}[\phi(x)] = \theta_{F'}[\phi(y)]$.
$\Rightarrow f(\theta_F[x]) = f(\theta_F[y])$.
Hence,f is well defined.
Let $x, y, \in F. f(\theta_F[x]) \to \theta_F[y]) = f(\theta_F[x \to y]) = \theta_{F'}[\phi(x \to y)] = \theta_{F'}[\phi(x) \to \phi(y)] = \theta_{F'}[\phi(x)] \to \theta_{F'}[\phi(y)] = f(\theta_F[x]) \to f(\theta_F[y])$ Similarly, we can prove congruence relations $\theta_{F'}[\phi(x)] \wedge \theta_{F'}[\phi(y)] = f(\theta_F[x]) \wedge f(\theta_F[y])$,$\theta_{F'}[\phi(x)] \vee \theta_{F'}[\phi(y)] = f(\theta_F[x]) \vee f(\theta_F[y])$ for all $x, y \in H_1$. Hence f is a homomorphism.Now $f \circ h : H_1 \to H_2/\theta_{F'}$ and for any $x \in H_1$. we have $[f \circ h](x) = f(h(x)) = f(\theta_F[x]) = \theta_{F'}[\phi(x)]$. Again $k \circ \phi : H_1 \to H_2/\theta_{F'}$ and for any $x \in H_1$, we have $[k \circ \phi](x) = k(\phi(x)) = \theta_{F'}[\phi(x)]$.Hence $[f \circ h](x) = [k \circ \phi](x), \forall x \in H_1$. This shows that $f \circ h = k \circ \phi$.
(i.) Let ϕ be a monomorphism and let $\phi(F) = F^{'}$. Let $f(\theta_F[x]) = f(\theta_F[y]\ )$ for some $x, y \in H_1$. Then $\theta_{F'}[\phi(x)] = \theta_{F'}[\phi(y)] \Rightarrow (\phi(x), \phi(y)) \in \theta_{F'} \Rightarrow \phi(x) \wedge t = \phi(y) \wedge t$, for some $t \in F^{'} \Rightarrow \phi(x) \wedge \phi(c) = \phi(y) \wedge \phi(c)$, for some $s \in F$ (since $\phi(F) = F^{'}$) $\Rightarrow \phi(x \wedge c) = \phi(y \wedge c)$ (since ϕ is a monomorphism) $\Rightarrow x \wedge c = y \wedge c. \Rightarrow (x, y) \in \theta_F \Rightarrow \theta_F[x] = \theta_F[y]$. This shows that f is one-one.

(ii.) Let ϕ be an epimorphism. Let $\theta_{F'}[y] \in H_2/\theta_{F'}$.As $\phi : H_1 \to H_2$ is onto and $y \in H_2, \phi(x) = y$ for some $x \in H_1$. Thus,$\theta_F[x] \in H_1/\ \theta_F$ and $\theta_F[f(x)] = \theta_{F'}[\phi(x)]$ $= \theta_{F'}[y]$. This shows that f is an epimorphism. □

6.5 Ordered Fuzzy Filter of HADL

Definition 6.5.1. [19] Let m be a maximal element of H and F a non-empty subset of H. Then F is an ordered filter of H if and only if it satisfies the following properties.
(i) $m \in F$
(ii) $x \in F, x \to y \in F$ imply $y \in F$ for all $x, y \in H$

Definition 6.5.2. [19] Let m be a maximal element of a HADL H. For any $a, b, c \in H$, the following conditions hold:
(1)$(a \to (b \to c)) \wedge m = ((a \to b) \to (a \to c)) \wedge m$.
(2)$(a \to b) \wedge m = (b \to c) \wedge m = m$ implies that $(a \to c) \wedge m = m$.

Let F be a nonempty subset of H. Then the smallest filter containing F is called the filter generated by F and denoted by $< F >$. Then the following theorem explains about the description of elements of $< F >$.

Theorem 6.5.3. *[19] Let m be a maximal element of H. For any nonempty subset F of H, we have* $< F >= \{x \in H : ((a_1 \wedge a_2 \wedge ... \wedge a_n) \to x) \wedge m = m; a_1, a_2, ..., a_n \in F\}$.

Example 6.5.4. Let H be a discrete ADL with 0 and with at least two elements. Fix $m(\neq 0) \in H$ and define for any $x, y \in H$.

$$x \to y = \begin{cases} 0 & \text{if } x \neq 0, y = 0; \\ m & \text{if } x \neq y; \end{cases}$$

Then clearly $(L, \vee, \wedge, \to, 0, m)$ is an HADL and $\{m\}$is an ordered filter in H.

Lemma 6.5.5. *Let H be a HADL with a maximal element m. A non-empty subset F of H is called an ordered filter if it satisfies the following conditions for all $x, y \in H$:*
(1) $x, y \in F$ implies $x \wedge y \in H$
(2) $x \in F$ and $x \wedge m \leq y \wedge m$ imply $y \in F$

Let F be an ordered filter of an HADL H. Then choose $x \in F$. Since $x \wedge m \leq m = m \leq m$ by the condition (ii), we get that $m \in F$. The following Lemma can help us to understand the relation between a filter and an ordered filter of an HADL.

Lemma 6.5.6. *[4] Let m be a maximal element of a HADL H. Then every filter of H is an ordered filter.*

Definition 6.5.7. Let μ be a fuzzy subset of H and m a maximal element.Then μ is called an ordered fuzzy filter iff it satisfies the following conditions for all $x, y \in H$.
(i)$\mu(m) \geq \mu(x), \forall x \in H$
(ii)$\mu(y) \geq \mu(x \to y) \wedge \mu(x)$

Proposition 6.5.8. *Let H be an HADL and μ ordered fuzzy filter of H,then for any $x, y \in H, x \leq y \Rightarrow \mu(x) \leq \mu(y)$*
Proof:Suppose $x \leq y$ by definition 6.5.7,we have $\mu(y) \geq \mu(m) \wedge \mu(x) = \mu(x)$. Hence, $\mu(x) \leq \mu(y)$

Lemma 6.5.9. *μ be an ordered fuzzy filter of H iff μ_t=$\{x \in H : \mu(x) \geq t\}$ is a level ordered filter of μ,$t \in [0, 1]$*

Proof. Suppose μ is an ordered fuzzy filter and $\mu(x) = t, t \in im\mu$.S By hypothesis $\mu(m) \geq \mu(x) \geq t$.Hence, $m \in \mu_t$. Suppose $x \in \mu_t$ and $x \to y \in \mu_t$.Then $\mu(x) \geq t$ and $\mu(x \to y) \geq t$ and so $\mu(y) \geq \mu(x \to y) \wedge \mu(x) \geq t$.Thus, $y \in \mu_t$.Conversely,assume μ_t is an ordered filter.Then $m \in \mu_t$,$t \in im\mu$.Take $t = \mu(x)$,then $\mu(m) \geq \mu(x)$ for all $x \in \mu_t$. Let $r = \mu(x)$ and $\mu(x \to y)$= s. Take

$t = r \wedge s$.Since $y \in \mu_t$, we have $\mu(y) \geq t = \mu(x) \wedge \mu(x \to y)$. Hence, the result follows. □

Definition 6.5.10. Let χ_S denote the characteristic function of any subset S of an HADL H.i.e

$$\chi_S(x) = \begin{cases} m & \text{if } x \in S; \\ 0 & \text{if } x \in S; \end{cases}$$

Theorem 6.5.11. *A non empty subset F of H is an ordered filter iff χ_F is an ordered fuzzy filter.*

Proof. Suppose that F is an ordered filter of H,then $\chi_F(m) = m$ and $\chi_F(x) = m$ for all $x \in F$

$m \geq m \Rightarrow \chi_F(m) \geq \chi_F(x)$ .For any $x, y \in H$, $x \to x = m$. Since $x, x \to y \in F \Rightarrow y \in F$, $\chi_F(x) = m$ and $\chi_F(x \to y) = m$. But $m \geq m \wedge m$

$\Rightarrow \chi_F(y) \geq \chi_F(x) \wedge \chi_F(x \to y)$.Conversely, assume χ_F is an ordered fuzzy filter.We claim to show that F is an ordered filter.$\chi_F(m) \geq \chi_F(x)$ if $x \in F, \chi_F(x) = m$.This implies $m \leq \chi_F(m) \leq m$.Thus $\chi_F(m) = m$.Therefore, $m \in F$

Since $x \in F$ and $x \to y \in F$,we have $x \wedge (x \to y) \in F$.This gives $x \wedge y \in F$.But in an HADL,we have $x \wedge y \leq y$.By proposition 6.5.8,we have $\chi_F(x \wedge y) \leq \chi_F(y)$.This gives $m \leq \chi_F(y) \leq m$.Thus,$\chi_F(y) = m$.Therefore, $y \in F$

□

Theorem 6.5.12. *Let μ and θ be two ordered fuzzy filters of an HADL H.Then $\mu \cap \theta$ is also an ordered fuzzy filter.*

Proof. $(\mu \cap \theta)(m) = \mu(m) \wedge \theta(m) \geq \mu(x) \wedge \theta(x) = \mu \cap \theta(x)$.

Also $(\mu \cap \theta)(y) = \mu(y) \wedge \theta(y) \geq \mu(x \to y) \wedge \mu(x) \wedge \theta(x \to y) \wedge \theta(x)=$

$\mu(x \to y) \wedge \theta(x \to y) \wedge \mu(x) \wedge \theta(x)$=$(\mu \cap \theta)(x \to y) \wedge (\mu \cap \theta)(x)$

Hence,the result follows □

Lemma 6.5.13. *Let H be an HADL and μ be an ordered fuzzy filter of H. If $x \leq y$, then $\mu(x) \leq \mu(y)$. Moreover $\mu(x \wedge y) \leq min\ \{\mu(x), \mu(y)\}$ for all $x, y \in H$*

Proof. If $x \leq y$, then $x \rightarrow y = m$. $\mu(y) \geq min\{\mu(x \rightarrow y), \mu(x)\}$
$=min\{\mu(m), \mu(x)\} = \mu(x) \Rightarrow \mu(x) \leq \mu(y)$.
Since $x \wedge y \leq x$ implies $\mu(x \wedge y) \leq \mu(x)$ and $x \wedge y \leq y$ implies $\mu(x \wedge y) \leq \mu(y)$,thus $\mu(x \wedge y) \leq \min\ \{\mu(x), \mu(y)\}$. □

Lemma 6.5.14. *Let μ be a fuzzy subset of H in the sense of the above lemma.Then every ordered fuzzy filter of an HADL H is fuzzy prime ideal of H if $\mu(x \wedge y) = Max$ $\{\mu(x), \mu(y)\}$ for all $x, y \in H$.*

Proof. $\mu(x \wedge y) \geq \mu(x \rightarrow (x \wedge y)) \wedge \mu(x)$
$\geq \mu(m) \wedge \mu(x)$
$= \mu(x)$.But from the above,we have $\mu(x \wedge y) \leq \mu(x)$ Thus,$\mu(x \wedge y) = \mu(x)$
Similarly,$\mu(x \wedge y) = \mu(y)$. Hence,$\mu(x \wedge y) = \mu(x) \vee \mu(y)$
Hence, the Definition 4.0.3 of fuzzy prime ideal is satisfied. □

Let μ and θ be any two ordered fuzzy filters of a HADL H. The product $\mu o \theta$ of μ and θ is defined by $(\mu o \theta)(x)$=$sup_{x=\vee(y_i \rightarrow z_i)}, i < \infty(min(min(\mu(y_i), \theta(z_i))))$.
It can be verfied that $\mu o \theta$ is an ordered fuzzy filter of H.

Theorem 6.5.15. *If f is an homomorphism from an HADL H onto an HADL $H^{'}$,then for each ordered fuzzy filters μ of H,$f(\mu)$ is an ordered fuzzy filter of $H^{'}$; and for each ordered fuzzy filter $\mu^{'}$ of H,$f^{-1}(\mu^{'})$ is an ordered fuzzy filter of H.*

Proof. Similar to Theorem 4.1.3 □

Definition 6.5.16. Let μ, and θ be any two ordered fuzzy filters of an HADL H.The join $(\mu \vee \theta)$ of μ and θ is defined by $(\mu \vee \theta)(x)$=$sup_{x=y \vee z}(min(\mu(y), \theta(z)))$,where $x, y, z \in H$.

Theorem 6.5.17. *The fuzzy subset $\mu(x) = Sup\{k \in [0, 1] : x \in \mu_k\}$ is an ordered fuzzy filter*

Proof. Straightforward from the definition of ordered fuzzy filter □

Let μ and θ be a fuzzy subsets of H.The cartesian product of μ and θ is defined by $(\mu \times \theta)(x, y) = min(\mu(x), \theta(y)), \forall x, y \in H$.

Theorem 6.5.18. *Let μ and θ be an ordered fuzzy filters of an HADL H,then $\mu \times \theta$ is an ordered fuzzy filters of $H \times H$*

Proof. Since μ and θ are ordered fuzzy filters,we have,

(i) $\mu \times \theta(m, m) = \mu(m) \wedge \theta(m) \geq \mu(x) \wedge \theta(y) = \mu \times \theta(x, y)$ for all $x, y \in H$

(ii) $\mu(y) \wedge \theta(y) \geq \mu(x \to y) \wedge \mu(x) \wedge \theta(x \to y) \wedge \theta(x) = \mu \times \theta(x \to y) \wedge (\mu \times \theta)(x).,x, y \in H$.

Hence, it is an ordered fuzzy filter. □

Let $< \mu > (x) = Sup\{\mu(x) : A((a_1 \wedge a_2 \wedge a_3 \wedge ... \wedge a_n) \to x, m) > 0)\}$ for all $x, m, a_1, ..., a_n \in H$.

Lemma 6.5.19. *Let A be a fuzzy relation of H and μ is an ordered fuzzy filter,then $< \mu >$ is an ordered fuzzy filter*

Proof. (i).By definition,6.5.7, we have

$< \mu > (m) = sup\{\mu(m) : A((a_1 \wedge a_2 \wedge a_3 \wedge ... \wedge a_n) \to m, m) > 0)\}$ for all $x, m, a_1, ..., a_n \in H$.

$=Sup\{\mu(m) : A(m, m) > 0)\} \geq sup\{\mu(x) : A((a_1 \wedge a_2 \wedge a_3 \wedge ... \wedge a_n) \to x, m) > 0)\} \geq < \mu > (x)$ for all $x, m, a_1, ..., a_n \in H$.

(ii)$y \leq x \to y \Rightarrow \mu(x \to y) \geq \mu(y) \geq \mu(x \to y) \wedge \mu(x)$

Taking sup overall $y \in H$, $Sup\{\mu(y) : A((a_1 \wedge a_2 \wedge a_3 \wedge ... \wedge a_n) \to y, m) > 0)\} \geq Sup\{\mu(x \to y) : A((a_1 \wedge a_2 \wedge a_3 \wedge ... \wedge a_n) \to (x \to y), m) > 0) \wedge Sup\{\mu(x) : A((a_1 \wedge a_2 \wedge a_3 \wedge ... \wedge a_n) \to x, m) > 0)\}\}$

Hence,$< m > (y) \geq < \mu > (x \to y) \wedge < \mu > (x)$

Therefore,$< \mu >$ is an ordered fuzzy filter

□

6.6 Qutioent HADL Induced by Fuzzy Congruence Relations

In this section, we introduce the concept of implicative fuzzy filters in an HADL and study some important properties of implicative fuzzy filters. This concept was studied in the class of Heyting algebra by J. Picardo, A. Pultur and A. Tozzi in [12]under the name ideals. The following definition of left(right) filter of an HADL is direct from defnition of left(right) filter of an ADL in [18]. Let us collect some of the definitions from [1]

Definition 6.6.1. Let $(H, \vee, \wedge, \rightarrow, 0, m)$ be an HADL and F be a non-empty subset of H. Then F is called a right(left) filter of H if
(1) $x, y \in F \Rightarrow x \wedge y \in F$
(2) $x \in F, a \in H \Rightarrow x \vee a \in F, (a \vee x \in F)$

Definition 6.6.2. A non-empty subset F of H is said to be an implicative filter if
(1) $s, t \in F \Rrightarrow s \wedge t \in F$
(2) $s \in F, a \in H \Rightarrow a \rightarrow s \in F.$

Definition 6.6.3. Let $(H, \vee, \wedge, \rightarrow, 0, m)$ be an HADL.A fuzzy equivalence relation on H is called a fuzzy congruence relation if $\forall, x, y, z, t \in H.$ satisfies definition 2.1.7. If A is a FCR of H,then H/A is a HADL under binary operations defined by $A_x \vee A_y = A_{x \vee y}$,$A_x \wedge A_y = A_{x \wedge y}$ and $A_x \rightarrow A_y = A_{x \rightarrow y}$.The structure $(H/A, \vee, \wedge, \rightarrow, A_0, A_m)$ is a Quotient HADL.

Definition 6.6.4. Let $(H/A, \vee, \wedge, \rightarrow, A_0, A_m)$ and $(H^{'}/A^{'}, \vee, \wedge, \rightarrow, A^{'}_0, A^{'}_m)$ is said to be a homomorphism of H/A, $H^{'}/A^{'}$.Then a mapping $\alpha : H/A \rightarrow H^{'}/A^{'}$ is said to be a homomorphism of H/A into $H^{'}/A^{'}$ iff for any $A_x, A_y \in H/A$ the following hold.

1. $\alpha(A_x \wedge A_y) = \alpha(A_x) \wedge \alpha(A_y)$

2. $\alpha(A_x \vee A_y) = \alpha(A_x) \vee \alpha(A_y)$

3. $\alpha(A_x \rightarrow A_y) = \alpha(A_x) \rightarrow \alpha(A_y)$

4. $\alpha(A_0) = A^{'}_{0'}$

If $\alpha : H/A \rightarrow H^{'}/A^{'}$ is a homomorphism,then $\alpha(A_m) = A^{'}{}_{m'}$.An onto homomorphism $\alpha : H/A \rightarrow H^{'}/A^{'}$ is said to be an epimorphism. is called an isomorphism from H/A onto $H^{'}/A^{'}$. 4 is the consequence of 1.
If we define $\alpha : H/A \rightarrow H^{'}/A^{'}$ by $\alpha(A_x) = A^{'}_{m'}$,then α satisfies (1),(2),(3) but not (4).

Definition 6.6.5. Let $(H/A, \vee, \wedge, \rightarrow, A_0, A_m)$ and $(H^{'}/A^{'}, \vee, \wedge, \rightarrow, A^{'}_0, A^{'}_m)$ be two QHADLs.Let $\alpha : H/A \rightarrow H^{'}/A^{'}$ be a homomorphism from H/A into $H^{'}/A^{'}$.Then we define the kernel of α by ker $\alpha = \{A_x \in H/A : \alpha(A_x) = A^{'}_{m'}\}$.Now we have the following theorem.

Theorem 6.6.6. *Let $(H/A, \vee, \wedge, \rightarrow, A_0, A_m)$ and $(H^{'}/A^{'}, \vee, \wedge, \rightarrow, A^{'}_0, A^{'}_m)$ be two QHADLs.Let $\alpha : H/A \rightarrow H^{'}/A^{'}$ be a homomorphism from H/A into $H^{'}/A^{'}$.Then the kernel of α is a quotient right filter as well as quotient implicative filter induced by A and $A^{'}$=*

Proof. Let A_x and $A_y \in Ker\alpha$. Then by definition of Ker α we have, $\alpha(A_x) = \alpha(A_y) = A^{'}_{m'}$.This implies $\alpha(A_x \wedge A_y) = \alpha(A_x) \wedge \alpha(A_y)$=$A^{'}_{m'} \wedge A^{'}_{m'} = A^{'}_{m'}$ and hence $\alpha(A_x \wedge A_y) = \alpha(A_{x\wedge y}) = A^{'}_{m'}$.Thus $A_{x\wedge y} \in ker\alpha$.
Assume $A_x \in$ ker α and $A_a \in H/A$, then $\alpha(A_x) = A^{'}_{m'}$ and $\alpha(A_x \vee A_a) = \alpha(A_x) \vee \alpha(A_a)$=$A^{'}_{m'}$.$\alpha(A_x \vee A_a) = \alpha(A_{x\vee a}) = A^{'}_{m'}$. Hence $A_{x\vee a} = A_x \vee A_a \in Ker\alpha$. Also, $\alpha(A_a \rightarrow A_x) = \alpha(A_a) \rightarrow \alpha(A_x)$=$\alpha(A_a) \rightarrow A^{'}_{m'} = A^{'}_{m'}$ [Since $A_x \rightarrow A_m = A_{x\rightarrow m} = A_m$.]
Hence, $A_a \rightarrow A_x \in ker\alpha$ □

Theorem 6.6.7. *Let $\phi : H/A \to H^{'}/A^{'}$ be an epimorphism from an QHADL H/A onto an QHADL $H^{'}/A^{'}$.*

(1) If F/A is quotient implicative filter in H/A, then $\phi(F/A)$ is a quotient implicative filter in $H^{'}/A^{'}$

(2) If F/A is a quotient right filter in H/A, then $\phi(F/A)$ is a quotient right filter in $H^{'}/A^{'}$

(3) If G is is a quotient right filter(quotient implicative filter) in $H^{'}/A'$, then $\phi^{-1}(G)$ is a quotient right filter(quotient implicative filter) in H/A.

Proof. 1.Clearly,$\phi(F/A) \neq \emptyset$.Since $F/A \neq \emptyset$. Let $\phi(A_x)$ and $\phi(A_y) \in \phi(F/A)$.Then $\phi(A_x) \wedge \phi(A_y)$=$\phi(A_{x\wedge y}) \in \phi(F/A)$.Since $A_{x\wedge y} \in F/A$.

Let $\phi(A_x) \in \phi(F/A)$ and $A_y \in H^{'}/A^{'}, y \in H^{'}$.Then $\exists A_t \in H/A, t \in H$ such that $\phi(A_t) = A_y.\phi(A_t) \to \phi(A_x) = \phi(A_t \to A_x) = \phi(A_{t\to x}) \in \phi(F/A)$.since $A_{t\to x} \in F/A$.Then $\phi(F/A)$ is a Quotient implicative filter in $H^{'}/A^{'}$.

2. Assume that F/A is a Quotient right filter in H/A,then for any A_x and $A_y \in \phi(F/A)$,then there exists $A_a, A_b \in F/A$ such that $A_x = \phi(A_a), A_y = \phi(A_b)$.Now $A_{x\wedge y} = \phi(A_a) \wedge \phi(A_b) = \phi(A_a \wedge A_b) \in \phi(F/A)$.

Now,$A_x \vee A_y = \phi(A_a) \vee \phi(A_b)$

$= \phi(A_a \vee A_b) = \phi(A_{a\vee b}) \in \phi(F/A)$.Since F/A is quotient right filter in H/A.Thus,$\phi(F/A)$ is quotient right filter in $H^{'}/A^{'}$.

3. Assume that $F^{'}/A^{'}$ is quotient right filter in $H^{'}/A^{'}$,then for $A_x, A_y \in \phi^{-1}(F^{'}/A^{'})$,we get $\phi(A_x), \phi(A_y) \in F^{'}/A^{'}$.Now $\phi(A_x \wedge A_y) = \phi(A_x) \wedge \phi(A_y) \in F^{'}/A^{'}$.Thus $A_x \wedge A_y \in \phi^{-1}(F^{'}/A^{'})$.Let $A_x \in \phi^{-1}(F^{'}/A^{'})$ and $A_a \in H/A$.Then $\phi(A_x) \in F^{'}/A^{'}$ and $A_y = \phi(A_a) \in H^{'}/A^{'}$.Now $\phi(A_x) \vee \phi(A_a) = \phi(A_{x\vee a}) \in F^{'}/A^{'}$ and $\phi(A_a) \to \phi(A_x) = \phi(A_{a\to a}) \in F^{'}/A^{'} \in F^{'}/A^{'}$.Since $F^{'}/A^{'}$ is a quotient right filter and a quotient implicative filter in $H^{'}/A^{'}$.Thus $A_x \vee A_a \in \phi^{-1}(F^{'}/A^{'})$. $A_a \to A_x \in \phi^{-1}(F^{'}/A^{'})$.Therefore $\phi^{-1}(F/A)$ is quotient right filter and quotient implicative filter $H^{'}/A^{'}$ is a quotient implicative filter.

□

Theorem 6.6.8. *Let A_m be a maximal element in H/A. Then for any*

$$A_a \in H/A, A_a^* = \{(A_x \to A_a) \wedge A_m : A_x \in H/A\}$$

is a quotient implicative filter.

Proof. Let $A_x, A_y \in H/A$. Then $(A_x \to A_a) \wedge A_m, (A_y \to A_a) \wedge A_m \in A_a^*$.
Now,$(A_x \to A_a) \wedge A_m \wedge (A_y \to A_a) \wedge A_m = (A_x \to A_a) \wedge (A_y \to A_a) \wedge A_m$ $((A_x \vee A_y) \to A_a) \wedge A_m \in A_a^*$.
Let $(A_x \to A_a) \wedge A_m \in A_a^*, A_y \in H/A$. Then $A_y \to [(A_x \to A_a) \wedge A_m]$ $=[A_y \to (A_x \to A_a)] \wedge [A_y \to A_m]$= $[A_y \wedge (A_x \to A_a)] \wedge [A_y \to A_m]$=$[(A_y \wedge A_x) \to A_a)] \wedge A_m \in A_a^*$.
Thus,A_a^* quotient implicative filter of H/A. □

Definition 6.6.9. Let H/A is a QHADL with maximal element A_m Then for any $A_a \in H/A$, we define $F_{A_a} = \{A_x \in H/A : (A_a \to A_x) \wedge A_m = A_x \wedge A_m\}$
$F^{A_a} = \{A_x \in H/A : A_a \wedge A_m \subseteq A_x \wedge A_m\}$

Lemma 6.6.10. *F_{A_a} is Quotient implicative filter of H/A and F^{A_a} is a Quotient filter.*

Proof. Trivially, one can show that F_{A_a} is a quotient filter of H/A.We claim to show that F_{A_a} is Quotient implicative filter.Let A_x and A_y such that
$(A_a \to A_x) \wedge A_m = A_{x \wedge m}$, $(A_a \to A_y) \wedge A_m = A_{y \wedge m}$.
Now,$A_x \wedge A_y \wedge A_m = A_{x \wedge y \wedge m \wedge m}$=$A_{x \wedge m \wedge x \wedge m}$=$A_{x \wedge m} \wedge A_{y \wedge m}$=$(A_a \to A_x) \wedge A_m \wedge (A_a \to A_y) \wedge A_m$=$(A_a \to A_x) \wedge (A_a \to A_y) \wedge A_m$=$(A_{a \to x}) \wedge (A_{a \to y}) \wedge A_m$=$(A_a \to (A_{x \to y})) \wedge A_m$.
Thus, $A_x \wedge A_y \in F_{A_a}$
Let $A_x \in F_{A_a}$ and $A_y \in H/A$.Then by we have,$(A_a \to (A_{y \to x}) \wedge A_m = (A_y \to A_{(a \to x)}) \wedge A_m = A_{(y \to (a \to x) \wedge m)}$=$A_{y \to (a \to x) \wedge m}$=$A_{y \to x} \wedge A_m$ and hence $A_{y \to x} \in F_{A_a}$ and hence F_{Aa} is a quotient implicative filter of H.

□

6.7 Implicative Fuzzy Filters of HADL

In this section,we introduce the concept of implicative fuzzy filters in an HADL and study some important properties of implicative fuzzy filter.The concept Implicative filter was studied in the class of Heyting ADL(HADL) in. [1]

Remark 6.7.1. The definitions of ideal(filter) fuzzy ideal(fuzzy filter) of a Heyting algebra H is also similar to that of an HADL.

Definition 6.7.2. A fuzzy subset μ of H is said to be an implicative fuzzy filter if

1. $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$
2. $\mu(x \rightarrow y) \geq \mu(x) \vee \mu(y), \forall x, y \in H$.

Lemma 6.7.3. *If μ is a implicative fuzzy filter of H,then $\mu_t, t \in [0,1]$ is an implicative filter.*

Proof. Suppose μ is an implicative fuzzy filter and let $x, y \in \mu_t$.Then $\mu(x) \geq t$ and $\mu(y) \geq t$. $\mu(x) \wedge \mu(y) \geq t$. Since μ is an implicative fuzzy filter $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y) \geq t$.This implies $\mu(x \wedge y) \geq t$.Hence $x \wedge y \in \mu_t$.Next,$x \in \mu_t$, Let $a \in H, \mu(a) \geq s, s \in [0,1]$.Since $a, x \in H$,then $a \rightarrow x \in H$.Also Let $\mu(x) = s$ and $\mu(a) = r$. Take $t = min(s, r)$.
Then by hypothesis $\mu(a \rightarrow x) \geq \mu(a) \vee \mu(x) \geq t, t = min(s, r) \in [0,1]$.
Hence, $a \rightarrow x \in \mu_t$ is an implicative filter.
Conversely,Suppose μ_t is implicative filter.We claim to show that μ is implicative fuzzy filter. Let $x, y \in H$. Then $\mu(x) = t_1$ and $\mu(y) = t_2$.Take $t = min(t_1, t_2)$. Then $x \wedge y \in \mu_t$.This implies $\mu(x \wedge y) \geq t = \mu(x) \wedge \mu(y)$.Similarly , Let $\mu(x) = r$ and $\mu(y) = s$.Then $a \rightarrow x \in \mu_t$,$t = max(r, s)$.This implies $\mu(x \rightarrow y) \geq t$.Hence $\mu(x \rightarrow y) \geq \mu(x) \vee \mu(y)$. □

Lemma 6.7.4. *Every ordered fuzzy filter is an implicative fuzzy filter*

Example 6.7.5. $\{m\}$ is an implicative fuzzy filter.

Example 6.7.6. Let μ be a fuzzy subset of an HADL H with 0 and with atleast two elements.Fix m $(\neq 0) \in H$.define $a, b \in H$

$$\mu(a \to b) = \begin{cases} m & \text{if } a \neq 0, b = 0; \\ \mu(m) & \text{otherwise.} \end{cases}$$

Then μ is an implicative fuzzy filter

Proof. $\mu(a \to b) = m \geq \mu(a) \vee \mu(b)$
$\mu(0 \wedge m) \geq \mu(m)$
Hence,μ implicative fuzzy filter. □

But every fuzzy implicative filter can't be an fuzzy filter in H.

Definition 6.7.7. Let H and $H^{'}$ be two HADLs. Let $f : H \to H^{'}$.then we say that f is a homomorphism from H into $H^{'}$ if the following conditions are satisfied.
(1) $f(x \wedge y) = f(x) \wedge f(y)$
(2) $f(x \vee y) = f(x) \vee f(y)$
(3) $f(x \to y) = f(x) \to f(y)$
(4) $f(0) = 0$.

If μ is a fuzzy subset of H,then $f(\mu)$ is a fuzzy subset of $H^{'}$. Define kernel of f,denoted by ker $f = \{y \in H^{'} : f(\mu)(y) = m^{'}\}$

Lemma 6.7.8. *Let f be an epimorphism from HADLs H into $H^{'}$ and μ is a fuzzy filter of H.Then $f(\mu)$, Ker f is an implicative fuzzy filter.Moreover,$f^{-1}(\mu)$ is an implicative fuzzy filter.*

Proof. (i.) Let $x, y \in H^{'}$.$(f^{-1}(\mu))(x \wedge y)$ =$\mu(f(x \wedge y)) = \mu(f(x) \wedge f(y)) \geq$
$f^{-1}(\mu)(x) \wedge f^{-1}(\mu(y)$.
(ii.) Let $x, y \in H^{'}$.$(f^{-1}(\mu))(x \to y)$ =$\mu(f(x \to y)) \geq \mu(f(x) \vee f(y)) \geq$
$f^{-1}(\mu)(x) \vee f^{-1}(\mu(y)$.
(iii.) Let $x, y \in H^{'}$.$(f^{-1}(\mu))(x \vee y)$ =$\mu(f(x \vee y)) \geq \mu(f(x) \vee f(y)) \geq$

$(f^{-1}(\mu)(x) \vee f^{-1}(\mu(y)).$

Hence,$(f^{-1}(\mu))$ is a fuzzy filter it is an implicative fuzzy filter □

Theorem 6.7.9. *Let $f : H \to H'$ be an epimorphism from an HADL H onto H', then we have the following.*

(i) If μ is ordered fuzzy filter (implicative fuzzy filter) of H,then $f(\mu)$ is a ordered fuzzy filter(implicative fuzzy filter) of H'

(ii)if θ is an implicative fuzzy filter (an ordered fuzzy filter) of H',then $f^{-1}(\theta)$ is an implicative fuzzy filter (ordered fuzzy filter) H.

Proof. (i.) Clearly,$f(\mu) \neq 0$,Since $a, b \in H$.

$$(f(\mu))(0') = \begin{cases} Sup_{0 \in f^{-1}(0')}(\mu(0)) & \text{if } f^{-1}(0') \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

$f^{-1}(0') \neq \emptyset$ implies $0 \in f^{-1}(0') = f(0) = 0'$.

ii.Let x' and $y' \in H'$.Then Since f is epimorphism there are $x, y \in H$ such that $f(x) = x'$,$f(y) = y'$

$$(f(\mu))(x' \wedge y') = \begin{cases} Sup_{x \wedge y \in f^{-1}(x' \wedge y')}(\mu(x \wedge y)) & \text{if } f^{-1}(x' \wedge y') \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} Sup(\mu(x \wedge y)) & \text{if } f(x \wedge y) = x' \wedge y'; \\ 0 & \text{otherwise.} \end{cases}$$

$$\geq \begin{cases} Sup(\mu(x) \wedge \mu(y)) & \text{if } f(x) = x' \text{ and} f(y) = y'; \\ 0 & \text{otherwise.} \end{cases}$$

$$\geq \begin{cases} Sup\mu(x) & \text{if } f(x) = x' \\ 0 & \text{otherwise.} \end{cases} \wedge \begin{cases} Sup\mu(y) & \text{if } f(y) = y' \\ 0 & \text{otherwise.} \end{cases}$$

$$= f(\mu)(x') \wedge f(\mu)(y')$$

$$ii.(f(\mu))(x^{'} \to y^{'}) = \begin{cases} Sup_{x \to y \in f^{-1}(x^{'} \to y^{'})}(\mu(x \to y)) & \text{if } f^{-1}(x^{'} \to y^{'}) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} Sup(\mu(x \to y)) & \text{if } f(x \to y) = x^{'} \to y^{'}; \\ 0 & \text{otherwise.} \end{cases}$$

$$\geq \begin{cases} Sup(\mu(x) \vee \mu(y)) & \text{if } f(x) = x^{'} \text{ and} f(y) = y^{'}; \\ 0 & \text{otherwise.} \end{cases}$$

$$\geq \begin{cases} Sup\mu(x) & \text{if } f(x) = x^{'} \\ 0 & \text{otherwise.} \end{cases} \vee \begin{cases} Sup\mu(y) & \text{if } f(y) = y^{'} \\ 0 & \text{otherwise.} \end{cases}$$

$$= f(\mu)(x^{'}) \vee f(\mu)(y^{'})$$

Hence,the theorem follows. □

Theorem 6.7.10. *Let m be a maximal element in H.Then for any $a \in S$,where S is a multiplicatively as well as implicatively closed subset of H.Then,$S_a = \{(x \to a) \wedge m : x \in H\}$ is an implicative filter*

Let $\mu : S_a \to [0,1]$ define $\psi^{\mu} = Sup\{\mu(a) : (x \to a) \wedge m = (y \to a) \wedge m\}$ such that $(x \to s) \vee [(x \to t) \wedge m] \leq [x \to (t \to s)] \wedge m$ for all $s, t \in S$,then we have the following result.

Lemma 6.7.11. *If μ is an ordered fuzzy filter(implicative fuzzy filter),then ψ^{μ} is a (an ordered fuzzy filter) implicative fuzzy filter.*

Proof. First we shall show that ψ^{μ} is implicative fuzzy filte.Assume $s, t \in S \subseteq H$.Then $\psi^{\mu}(s) = Sup\{\mu(s) : (x \to s) \wedge m = (y \to s) \wedge m\}$
and $\psi^{\mu}(t) = Sup\{\mu(t) : (x \to t) \wedge m = (y \to t) \wedge m\}$
$\psi^{\mu}(s) \wedge \psi^{\mu}(t) = Sup\{\mu(s) : (x \to s) \wedge m = (y \to s) \wedge m\} \wedge Sup\{\mu(t) : (x \to t) \wedge m = (y \to t) \wedge m\}$
$=Sup\{\mu(s) \wedge \mu(t) : (x \to s) \wedge m \wedge (x \to t) \wedge m = (y \to s) \wedge m \wedge (y \to t) \wedge m\}$
$\leq Sup\{\mu(s \wedge t) : (x \to s) \wedge m \wedge (x \to t) \wedge m = (y \to s) \wedge m \wedge (y \to t) \wedge m\}$

$=Sup\{\mu(s\wedge t):(x\to s)\wedge(x\to t)\wedge m=(y\to s)\wedge(y\to t)\wedge m\}$

$=Sup\{\mu(s\wedge t):(x\to(s\wedge t))\wedge m=(y\to(s\wedge t))\wedge m\}$.[Since S is a multiplicatively closed subset of H,$s\wedge t\in S$]

$=\psi^{\mu}(s\wedge t)$

Similarly,let $s,t\in S\subseteq H$.Then $\psi^{\mu}(s)=Sup\{\mu(s):(x\to s)\wedge m=(y\to s)\wedge m\}$ and $\psi^{\mu}(t)=Sup\{\mu(t):(x\to t)\wedge m=(y\to t)\wedge m\}$

$\psi^{\mu}(s)\vee\psi^{\mu}(t)=Sup\{\mu(s):(x\to s)\wedge m=(y\to s)\wedge m\}\vee Sup\{\mu(t):(x\to t)\wedge m=(y\to t)\wedge m\}$

$=Sup\{\mu(s)\vee\mu(t):[(x\to s)\wedge m]\vee[(x\to t)\wedge m]=[(y\to s)\wedge m]\vee[(y\to t)\wedge m]\}$

$\leq Sup\{\mu(s\to t):[(x\to s)\vee((x\to t)\wedge m)]\wedge[(x\to t)\wedge m]=[(y\to s)\vee((x\to t)\wedge m)]\wedge[(y\to t)\wedge m]\}$ [Since $(x\to s)\vee[(x\to t)\wedge m]\leq[x\to(t\to s)]\wedge m$ and S is implicatively closed]

$\leq Sup\{\mu(s\to t):(x\to(s\to t))\wedge m=(y\to(s\to t))\wedge m\}=\psi^{\mu}(s\to t)$.

Here we have $\psi^{\mu}(s)\wedge\psi^{\mu}(t)=Sup\{\mu(s):(x\to s)\wedge m=(y\to s)\wedge m\}\wedge Sup\{\mu(t):(x\to t)\wedge m=(y\to t)\wedge m\}$

$=Sup\{\mu(s)\wedge\mu(t):(x\to s)\wedge m\wedge(x\to t)\wedge m=(y\to s)\wedge m\wedge(y\to t)\wedge m\}$

$\leq Sup\{\mu(s\wedge t):(x\to s)\wedge m\wedge(x\to t)\wedge m=(y\to s)\wedge m\wedge(y\to t)\wedge m\}$

$=Sup\{\mu(s\wedge t):(x\to(s\wedge t))\wedge m=(y\to(s\wedge t))\wedge m\}=\psi^{\mu}(s\wedge t)$.

Hence,ψ^{μ} is an implicative fuzzy filter.

Now,suppose μ is an ordered filter we claim to show that ψ^{μ} is an ordered fuzzy filter.

Consider,$\psi^{\mu}(m)$=$\{\mu(m):(x\to m)\wedge m=(x\to m)\wedge m\}$

$\geq Sup\{\mu(x):(x\to x)\wedge m=(x\to x)\wedge m\}=\psi^{\mu}(x)$ for all $x\in$ H.

$\psi^{\mu}(s\to t)=Sup\{\mu(s\to t):(x\to(s\to t))\wedge m=(x\to(s\to t))\wedge m\}$

$\psi^{\mu}(s)\wedge\psi^{\mu}(s\to t)=Sup\{\mu(s):(x\to s)\wedge m=(x\to s)\wedge m\}\wedge Sup\{\mu(s\to t):(x\to(s\to t))\wedge m=(x\to(s\to t))\wedge m\}$

$\leq Sup\{\mu(s)\wedge\mu(s\to t):(x\to s)\wedge(x\to(s\to t))\wedge m=(y\to s)\wedge(y\to(s\to t))\wedge m\}$

$=Sup\{\mu(s)\wedge\mu(s\to t):(x\to s)\wedge(x\wedge s)\to t))\wedge m=(y\to s)\wedge(y\wedge s)\to t))\wedge m\}$

$= Sup\{\mu(s) \wedge \mu(s \to t) : (x \to s) \wedge ((s \wedge x) \to t)) \wedge m = (s \to y) \wedge ((s \wedge y) \to t)) \wedge m\}$[Since in HADL $[(a \wedge b) \to c] \wedge m = [(b \wedge a) \to c] \wedge m]$

$\leq Sup\{\mu(s) \wedge \mu(s \to t) : (x \to s) \wedge (x \to t) \wedge m = (y \to s) \wedge (y \to t) \wedge m\}$[Since $s \wedge y \leq y]$

$= Sup\{\mu(s) \wedge \mu(s \to t) : (x \to (s \wedge t)) \wedge m = (y \to (s \wedge t)) \wedge m\}$

$\leq Sup\{\mu(t) : (x \to t) \wedge m = (y \to t) \wedge m\}$

$= \psi^{\mu}(t)$.

Thus,$\psi^{\mu}(t) \geq \psi^{\mu}(s \to t) \wedge \psi^{\mu}(s)$.

ψ^{μ} is an ordered fuzzy filter.

□

Remarks and Future Works

In this section,we make a brief discussion of the results achieved in this dissertation and the prospects for future work.From the crisp theory HA,we got several properties of HA and the fuzzy set theory introduced by Zadeh(1965) helps us to come up with the new concept FHA.This gives a way to have several characterizations of both the crisp and fuzzy theory.For example congruence relation on HA was discussed.Using this concept the fuzzy version of HA (FHA) was introduced.

Using the definition of Chon (2009),family of fuzzy relation on HA was characterized.The image of FHA under the homomorphic map is preserved.Moreover, the concept of ideals and filters on FHA was defined and characterized.

It is possible to have a conclusion that FCR(H) played especial role on the introduction of the fuzzy version of isomorphism theorems.

It has also been discussed that the effect of homomorphism on the join,product and intersection of two fuzzy ideals.This was made simple by the Malik and Moderson approach.This approach also play a great role on the result fuzzy prime ideals and fuzzy semiprime ideals. Both chon and Malik Moderson approach FRC(H) approach contribute several results of fuzzy theory.Chon approach has also been applied to study FCR(H) products of FHAs and ;therefore,several characterizations had been deduced.

Another important result was α ideal and α filter of FHA.This result was very important to study fuzzy α ideals of FHA.It also give the way to study the concept of fuzzy ideals and fuzzy filters of FHA.

Using FHA and ADFL(which was introduced by Berhanu,Yohannes an Bekalu,2017) HADFL and its properties was introduced and studied.Moreover, charaterization of HADFL using principal ideals of HADFL was critically examined.Finally,It is possible to say that FCR on HADL was a base to introduce QHADL.

In the future,so many open problems are there and now is definitely opened.One is the application of fuzzy theory in real life and other related fields.The characterization of Generalized HADFL,Heyting fuzzy algebra,fuzzy HADL etc. and the properties of each subtopic in this dissertation can be extended to other fuzzy as well as crisp concepts. In general,fuzzy theory can be associated with the crisp theory to formulate a new theory.

Bibliography

[1] G. C. Rao, Berhanu Assaye and R. Prasad L-Almost Distributive Lattices,Asian-European Journal of Mathematics Vol. 04, No. 01, pp. 171-178 (2011)

[2] Swamy, U. M., Rao, G. C. (1981). Almost distributive lattices. Society, 31(August 1980), 77-91. https://doi.org/10.1017/S1446788700018498

[3] Alaba, B. A., Bitew, B. T. (2017). The Fuzzy Lattice of Ideals and Filters of an Almost distributive lattices, 3(1), 1-4.

[4] Alaba B.A, Wondifraw.Y.G and Bitew.B.T Almost distributive fuzzy Lattice ,International Journal of Mathematics And its Applications Volume 5, Issue 1-C (2017), 307-316. ISSN: 2347-1557

[5] Arend Heyting, An Introduction to Intuitionism, North-Holland Publishing Co., Amsterdam, (1956).

[6] Thoralf Skolem, Logico-combinatorial investigations in the satisfiability or provability of mathematical propositions, Harvard University Press, Cambridge,(1920).

[7] AJMAL, N., THOMAS, K.V.; Fuzzy Lattices, Information Sciences 79 (1994), 271- 291.

[8] I. Chon, Fuzzy partial order relations and fuzzy lattices, Korean J. Math. 17 (2009) 361 - 374.

[9] Dorien Zwaneveld,Subdirectly irreducible algebras in varieties of universal algebra.University of Amsterdam(2014)

[10] A. Rosenfeld, Fuzzy groups, d. Math. Anal. Appl. 35 (1977) 512-517.

[11] Rao, G.C., Berhanu Assaye and M. V. Ratna Mani, Heyting Almost Distributive Lattices(HADL), to appear in the International Journal of Computational Cognition

[12] ZADEH, L.A. Fuzzy Sets. Information and Control 8 (1965), 338-353

[13] C.S. Hoo, Fuzzy ideals of BCI and MV-algebras, Fuzzy Sets and Systems 62 (1994) 111-114.

[14] Birkhoff, G.,Lattice Theory, Amer. Math. Soc. Colloq. Publ. XXV, Providence (1967), U.S.A.

[15] GOGUEN, J.A.; L-fuzzy sets, J. Math Anal. Appl 18 (1967), 145-174.

[16] B. Yuan, Homomorphisms and isomorphisms of fuzzy subalgebras, J. Shanghai Teachers' Univ. 2 (1987) 1-9.

[17] M. S. Rao, "Some Topological Properties of the Prime Spectrum of Heyting Almost Distributive Lattices," pp. 267-278, 2018.

[18] Rao G. C., B. Ravi Kumar and N. Rafi: Generalized Almost Distributive Lattices-II(GADL-II).

[19] M. S. Rao, "The Space of Prime Ordered Filters of Heyting Almost Distributive Lattices," pp. 865-875, 2015.

[20] S.Burries and H.P.Sankappanavar, A Course in Univesal algebra,Springer-Verlag,1981

[21] Yuan Bo, Wu Wangming, "Fuzzy ideals on a distributive lattices,"pp.1-10,1988

[22] Erik Palmgren, Semantics of intuitionistic propositional logic:Lecture Notes for Applied Logic, Fall 2009 ,Department of Mathematics, Uppsala University

[23] H. Sherwood, Products of fuzzy subgroups, Fuzzy Sets and Systems 11 (1983) 79-89

[24] Rani, M. J., Mangalambal, N. R. (2011). Fuzzy Congruence on a Product Lattice, 5(22), 1049-1057.

[25] P. Das, Lattice of fuzzy congruences in inverse semigroups, Fuzzy Sets and Systems, 91 (1997) 399-408.

[26] T. Yijia, Fuzzy congruences on a regular semigroup, Fuzzy Sets and Systems, 117 (2001), 447-453

[27] Journal, I. (2017). Quotient Heyting Algebras Via Fuzzy Congruence Relations, 5(2), 371-378.

[28] Mezzomo, I., Rn, N.,Mezzomo, I. (2013). On Fuzzy Ideals and Fuzzy Filters of Fuzzy Lattices

[29] Math, K. J. (2009). fuzzy partial order relation and fuzzy lattices. Inheung Chon, 17(4), 361-374.

[30] B. A. Alaba and D. N. Derso, "Fuzzy Heyting Algebra," Springer international publishing, 2018.

[31] Rao GC, Assaye B, Mani MVR. Heyting Almost Distributive Lattices. 2010;8(3):89-93.

[32] Pawar YS. Congruence Relations on Almost Distributive Lattices. 2012:519-527.

[33] G.C. Rao, S. Ravikumar,Minimal prime ideals in almost distributive lattices, Int. J. Math. Sciences 10 (2009) 475-484.

[34] T.P. Speed, Two congruences on distributive lattices, Bulletin de la Soci' e Royale et' des Sciences de Li' ege, 38e ann' ee 3-4 (1969) 86-96.

[35] U.M. Swamy, S. Ramesh, Birkhoff center of an almost distributive lattices, Int. Jour. Algebra 11 (2009) 539-546

[36] Mezzomo I. Fuzzy Homomorphism in Fuzzy Lattices Preserving Ideals. 2014;(October).

[37] Rao GC, Assaye B. Closed Elments in Heyting Almost Distributive Lattices. 2011;9(2):56-60.

[38] G. Licata,"Employing fuzzy logic in the diagnosis of a clinical case," vol. 2, no. 3, pp. 211-224, 2010.

[39] S. Engineering, "THE USE OF FUZZY LOGIC IN DECISION-MAKING," 2004.

[40] J. B. Awotunde, O. E. Matiluko, and O. W. Fatai,"Medical Diagnosis System Using Fuzzy Logic," vol. 7, no. 2, pp. 0-7, 2014.

[41] P. R. Innocent, R. I. John, and J. M. Garibaldi,"Fuzzy Methods and Medical Diagnosis," pp. 1-28, 2004.

[42] K. Shang and Z. Hossen,"Applying Fuzzy Logic to Risk Assessment and Decision-Making Sponsored by CAS / CIA / SOA Joint Risk Management Section Prepared by," pp. 1-59, 2013.

[43] Zimmermann, H.-J. (Hans-Jiirgen), 1934- Fuzzy set theory and its applications / 4th ed

[44] Y. Tan,"Fuzzy congruences on a regular semigroup," vol. 117, pp. 447-453, 2001.

[45] F.sets,"Department of Mathematics, Shanghai Teachers' University, Shanghai, People's Republic of China," vol. 35, pp. 231-240, 1990.

[46] K. Math,"Fuzzy normal subgroups in fuzzy subgroups d.s. m," vol. 29, no. 1, pp. 1-7, 1992.

[47] J. N. Mordeson, K. R. Bhutani, A. Rosenfeld, and L. W. Eds, Studies in Fuzziness and Soft Computing , Volume 182.

[48] D.S. Malik and J.N. Mordeson, Fuzzy prime ideals of a ring, Fuzzy Sets and Systems 37 (1990) 93-98.

[49] Malik, D.S. and Mordeson, J.N., Extensions of fuzzy subrings and fuzzy ideals, Fuzzy sets and Systems, 45 (1992), 245-251.

[50] B. A. Alaba and D. N. Derso, "Heyting Almost Distributive Fuzzy Lattices," vol. 4, no. 1, pp. 23-26, 2018.

[51] B. A. Alaba and D. N. Derso, "implicative fuzzy filter of Heyting almost distributive lattices," on communication.

[52] Yehada Rev,Semiprime ideals in General lattices ,Department of mathematics,University of Paris XI,Vol. 56,pp.105-118,1989.