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# A PROJECT ON SOME MATHEMATICAL MODELS INVOLVING FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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**MATHEMATICS DEPARTMENT**

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BY

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# SOME MATHEMATICAL MODELS INVOLVING FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

A project submitted to department of mathematics in partial fulfillment of the requirements for the degree of “Master of Science in mathematics”.

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Approval of the project for defense

I hereby certify that I have supervised, read and evaluated this project entitled "Some mathematical models involving first order ordinary differential equations" by Tigist Yitayew prepared under my guidance. I recommend that the project is submitted for oral defense.

| | | |
|---|---|---|
| Birilew Belayneh (PhD) | ___________ | _________ |
| Advisor's name | Signature | Date |

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

MATHEMATICS DEPARTMENT

Approval of the project for defense result

We hereby certify that we have examined this project entitled "Some mathematical models involving first order ordinary differential equations" by Tigist Yitayew. We recommend that this project is approved for the degree of "Master of Science in Mathematics".

Board of examiners

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| Internal examiner's name | signature | date |
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| Chair person's name | signature | date |

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# ABSTRACT

The major purpose of this project is to show the applications of some mathematical models involving first order ordinary differential equations in describing some biological processes and mixing problems.

Applications of first order ordinary differential equations in modeling some biological phenomena such as logistic and non-logistic population growth and decay and prey-predator interaction for three species in linear food chain system has been analyzed. And also its application in substance mixing problems in both single and multiple tank system have been demonstrated in this project.

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# INTRODUCTION

Many real life problems in science and engineering, when formulated mathematically give rise to differential equation. In order to understand the physical behavior of the mathematical representation, it is necessary to have some knowledge about the mathematical character, properties and the solution of the governing differential equation (Sharma, 1985).

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms the relations are equations and the rates are derivatives (zill, 2013).

If we want to solve a real life problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, and equations. Such an expression is known as a mathematical model of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other term is called mathematical modeling (kreyszing, 2011).

Generally, a mathematical model is an evolution equation which can potentially describe the evolution of some selected aspects of the real life problem. The description is obtained by solving mathematical problems generated by the application of the model to the description of real physical behaviors (Bellomo, Angelis and Delitala, 2007).

Many physical problems concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more derivatives. Such equations are differential equations (Trench, 2013).

Different physical problems are described by single ordinary differential equations depending on the nature of the problem (kreyszing, 2011). Many applications, however, require the use of two or more dependent variables, each a function of a single independent variable (typically time) such a problem leads naturally to a system of simultaneous ordinary differential equation (Edwards and Penney, 2008).

Real life problems such as population growth or decay, and single tank mixing problems, are modeled by single first order ordinary differential equations; whereas problems like prey predator model and multiple tank mixing problems are described by system of first order ordinary differential equations (kreyszing, 2011).

The mathematical model for an applied problem is almost always simpler than the actual situation being studied, since simplifying assumptions are usually required to obtain a mathematical problem that can be solved. For example in modeling the motion of a falling object, we might neglect air resistance and gravitational pull of celestial bodies other than the earth, or in modeling population growth we might assume that the population grows continuously rather than in discrete steps (Trench, 2013).

Modeling physical phenomena or process mathematically follows the following steps:

1) The formulation of the physical phenomena or process in mathematical terms; that is, the construction of a mathematical model,
2) Solution of the mathematical problem,
3) The interpretation of the mathematical results in the context of the original physical phenomena or process.

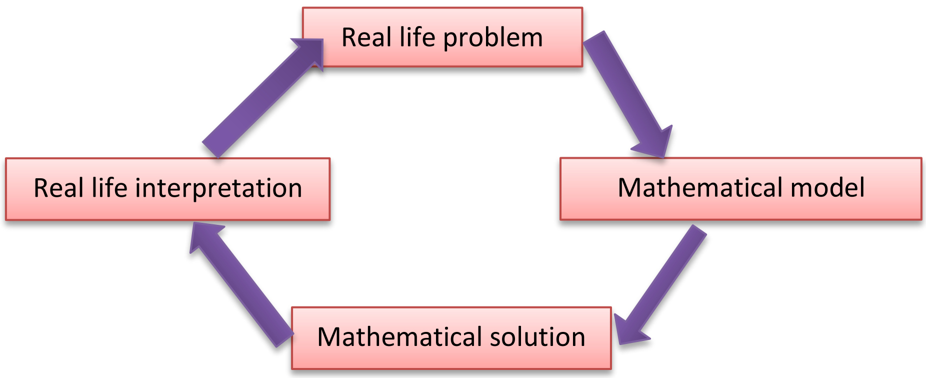


Figure 1: The Process of Mathematical Modeling (Edwards and Penney, 2018)

Biological systems that we would be interested in modeling will, of course, usually involve more than one variable. This will result in a system of ordinary differential equations. If we get lucky and this set happens to be a set of linear differential equations, we can apply techniques similar to those we studied for linear differential equations (Haberman, 1998).

Mathematical model of the interaction between predator and host populations have been expressed as systems of nonlinear ordinary differential equations. The study of species population has long been of interest in the biological sciences. Mathematical models have proven useful in describing how such populations vary over time. Knowledge about the various rates of growth, death and interaction of species naturally leads to models involving differential equations (Paullet et al., 2002).

A typical mixing problem investigates the behavior of a mixed solution of some substances. Typically, the solution is being mixed in a large tank. A solution (or solutions) of a given concentration enters the mixture at some fixed rate and is thoroughly mixed in the tank. The tank is also being drained at some fixed rate. Setting up a model for this situation typically involves a differential equation (Slavik, 2013).

The purpose of this project is to show the applications of some mathematical models involving first order ordinary differential equations in describing some biological process and mixing problems.

This project consists of two chapters. In the first chapter we will deal with basic preliminary concepts on mathematical modeling and some governing principles and laws of physical phenomena. Fundamental techniques of solving non-linear systems and concepts of their stability analysis will be examined.

In the second chapter we will study the applications of first order ordinary differential equation as a mathematical modeling in the study of physical phenomena or process particularly having biological nature (population growth and decay and prey predator model). The modeling of mixing problem in single and multiple tank system also considered in this chapter.

# CHAPTER ONE

# INTRODUCTION AND PRILIMINARY CONCEPTS

## 1.1 Introduction to modeling and systems

Modeling is a general process in engineering, physics, computer science, biology, medicine, environmental science, chemistry, economics and other fields that translate a physical situation or some other observations in to a “mathematical model”. Numerous examples from engineering (e.g., mixing problem), physics (e.g., Newton’s law of cooling), biology (e.g., Predator prey model), chemistry (e.g., radio carbon dating), environmental science (e.g., population control), etc. are modeled by first order ordinary differential equations. (Kreyszing, 2011)

Mathematical models essentially find the relationship between certain variables. A variable represents a concept or an item whose magnitude can be represented by a number. Mathematical models are excellent methods of conceptualizing knowledge about a process and to convey it to other people. An accurate model of a process allows us to predict the behavior for different conditions and thereby we can optimize and control a process for a specific purpose of our choice (Bellomo et al., 2007).

Almost all of the known laws of physics and chemistry are actually differential equations and differential equation models are used extensively in biology to study bio-chemical reactions, population dynamics, organism growth and the spread of diseases. The most common use of differential equations in science is to model dynamical systems, i.e. systems that change in time according to some fixed rule. A mathematical model is dynamic if the state variable depends on the time variable. Otherwise it is static. All real systems can be observed and represented at different scales by mathematical equations (Zill, 2013).

## 1.2 Basic principles and laws of modeling

Usually in modeling certain physical phenomena or process, we think of some information which helps us to formulate the desired model. Among these the following are the common once:

1. Notations and variable representation of quantities,
2. Pre-conditions of the initial phase,
3. Assumptions to be considered,
4. Governing principles and universal laws.

Among these, the following all are very important in dealing the problems in this project.

**Population law of mass action**

The rate of change of one population due to interaction with another is proportional to the product of the two populations at a given time t, that is

$$\frac{dx}{dt} = axy,$$

where $a$ is the proportionality constant, $x(t)$ denotes one species population at a time $t$ and $y(t)$ denotes another species population at a time $t$. We will apply this law in particular in the prey predator model.

**Balance law for population**

The net rate of change of the population is equal to the rate of change of a population in to the ecosystem minus the rate of change of population out of the ecosystem at a time $t$, that is

$$\frac{dp}{dt} = \left(\frac{dp}{dt}\right)_{in} - \left(\frac{dp}{dt}\right)_{out},$$

where $p(t)$ is the number of population in that ecosystem at a time $t$. We will use this law in particular in the population growth and prey predator interaction model.

**First order rate law**

The rate at which a population grows or decays in a first order process is proportional to its population at that time. That is

$$\frac{dp}{dt} = \gamma p(t),$$

where $p(t)$ is the population at a time $t$, and $\gamma$ is proportionality constant. We will use this law in the population growth model and prey predator model.

**Law of conservation of mass**

The law of conservation of mass states that mass can neither be created nor destroyed. That is

$$\frac{dm}{dt} = 0,$$

where $m(t)$ is the mass of a substance at a time $t$. We will use this law in mixing problems.

**Definition 1.2.1:** We say that a surface is said to be invariant if it remains unchanged when transformation of a certain type are applied to the surface.

**Theorem** [1.2.1] (Paullet. et al., 2002): Let $S$ be a smooth closed surface without boundary in $R^3$ and

$$\begin{cases} \frac{dx}{dt} = f(x,y,z), \\ \frac{dy}{dt} = g(x,y,z), \\ \frac{dz}{dt} = h(x,y,z), \end{cases} \tag{1.1.1}$$

where f, g and h are continuously differentiable. Suppose that $\hat{n}$ is a normal vector to the surface $S$ at $(x_0, y_0, z_0)$ and for all $(x, y, , z) \in S$ we have that

$$\hat{n}.\langle\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle = 0,$$

Then $S$ is invariant with respect to the system (1.1.1)

## 1.3 Linearization of nonlinear system

**Definition** 1.3.1: Linearization is the process of finding the linear approximation of a nonlinear function (system) at a given point. In the study of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point of a system of nonlinear differential equations or discrete dynamical system.

Consider a nonlinear system of $m$ first order ordinary differential equations with $n$ variables

$$\frac{dX_i(t)}{dt} = f_i(x_1, x_2, \dots, x_n), i = 1,2,3 \dots, m \qquad (1.1.2)$$

**Definition** 1.3.2: The Jacobian matrix of the system (1.1.2) is the matrix of all first-order partial derivatives of a vector-valued function, $f_i(x_1, x_2, \dots, x_n)$. It is denoted by $J$ and defined as:

$$J = \frac{\partial(f_1, f_2, \dots, f_m)}{\partial(x_1, x_2, \dots, x_n)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

**Definition** 1.3.3: we say that the point $x_0 = ({x_1}^0, {x_2}^0, \dots, {x_n}^0)$ is an equilibrium point if

$$f_i({x_1}^0, {x_2}^0, \dots, {x_n}^0) = 0, \quad \forall i$$

Its importance lies in the fact that it represents the best linear approximation to a differentiable function near a given point.

The linearization form of the non-linear system (1.1.2) (Jordan and Smith, 2007) is given by

$$\frac{dx(t)}{dt} = Ju(t) \qquad (1.1.3)$$

where

$$J_f(x_0) = \begin{bmatrix} \frac{\partial f_1(x_0)}{\partial x_1} & \frac{\partial f_1(x_0)}{\partial x_2} & \cdots & \frac{\partial f_1(x_0)}{\partial x_n} \\ \frac{\partial f_2(x_0)}{\partial x_1} & \frac{\partial f_2(x_0)}{\partial x_2} & \cdots & \frac{\partial f_2(x_0)}{\partial x_n} \\ & \vdots & & \\ \frac{\partial f_n(x_0)}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_{n(x_0)}}{\partial x_n} \end{bmatrix},$$

$$u(t) = (u_1, u_2, \dots, u_n)^T$$

$$u_1 = (x_1 - x_1{}^0), u_2 = (x_2 - x_2{}^0), \dots, u_n = (x_n - x_n{}^0)$$

In order to analyze stability of the system, computation of eigenvalues of the corresponding system is the first step. For specific purpose below we will see how to compute determinant and eigenvalues.

## 1.4 Tridiagonal matrix

**Theorem** [1.4.1] (Jeffrey, 2010) (Laplace expansion theorem for determinant): Let $A = [a_{ij}]$ be an $n \times n$ matrix. The expansion of Laplace allows reducing the computation of $n \times n$ determinant to that of $(n-1) \times (n-1)$ determinants.

The formula, expanding with respect to the $i^{th}$ row is:

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in})$$

where $A_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by erasing the $i^{th}$ row and the $j^{th}$ column from $A$.

The formula, expanding with respect to the $j^{th}$ column is:

$$\det(A) = (-1)^{j+1} a_{1j} \det(A_{1j}) + \cdots + (-1)^{j+n} a_{nj} \det(A_{nj})$$

**Theorem** [1.4.2] (Slavik, 2013) (Gershgorin circle theorem): The eigenvalues of a tridiagonal matrix $A = [a_{ij}]$ are contained in the union of the intervals $[a_{ij} - r_i, a_{ij} + r_i]$, where

$$r_i = \sum_{j \epsilon \{1,\dots,n\} \setminus i} |a_{ij}|$$

for $1 \le i \le n$.

Given an $n \times n$ tridiagonal matrix $T_n(x)$ of the form

$$T_n(x) = \begin{bmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & x & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & x \end{bmatrix} \quad (1.1.4)$$

and its associated determinant $D_n(x) = det|T_n(x)|$ (Jeffrey, 2010).

**Theorem** [1.4.3] (Jeffrey, 2010): Let $T_n(x)$ be a tridiagonal matrix given in equation (1.1.4). Then

1) The eigenvalue of $T_n(x)$ is given by $\lambda_m = x - 2\cos\left(\frac{m\pi}{n+1}\right), m = 1,2,\dots,n$
2) The eigenvector of $T_n(x)$ is given by $u^m = \quad (u_1^m), (u_2^m), \dots, (u_n^m)$, for $m = 1,2,\dots,n$

   The first equation is

$$(x - \lambda_m)(u_1^m) + (u_2^m) = 0$$

   The $n - 2$ equations is

   $u_{i-1}^m + (x - \lambda_m)u_i^m + u_{i+1}^m = 0$, for $i = 2,3,\dots,n-1$

   while the last equation becomes

$$u_{n-1}^m + (x - \lambda_m)u_n^m = 0$$

## 1.5 Stability

Stability means, roughly speaking, that a small change (a small disturbance) of a physical system at some instant changes the behavior of the system only slightly at all future times. In general, perturbing the initial state in some direction results in the trajectory asymptotically approaching the given one and in other directions to the trajectory getting away from it (Edwards and Penney, 2008)

Stability plays an important role in system engineering and science fields. Stability of dynamic system with or without control input and external disturbances plays a key role to achieve success in the realistic world.

**Definition** 1.5.1 (Boyce, 2001): we say that a system (1.1.3) is said to be

a) Asymptotically stable if all trajectories of its solutions converge to the fixed point as $t \to \infty$. In other words, it is asymptotically stable if all of the eigenvalues of its coefficient matrix have negative real part.
b) Unstable if all trajectories (or all but a few, in the case of saddle point) start out at the fixed point as $t \to -\infty$, and then move away to infinitely distant out as $t \to \infty$. In other

words, it is unstable if at least one of the eigenvalues of its coefficient matrix has positive real part.

c) Stable (or neutrally stable) if each trajectory move about the fixed point within a finite range of distance. It never moves out to infinitely distance, nor (unlike in the case of asymptotically stable) does it ever go to the fixed point. In other words it is stable all of the eigenvalues of its coefficient matrix are purely imaginary.

# CHAPTER TWO

# SOME MATHEMATICAL MODELS INVOLVING FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

There are many applications of first order ordinary differential equations to real world problems. In this chapter we will discuss some of the applications particularly in describing some biological applications and mixing problems.

## 2.1. Mathematical models in Biological Processes

In this section we will discuss some of the biological processes that can be governed or described by first order ordinary differential equations in describing population growth and decay and prey predator interaction.

### 2.1.1. Population Growth or Decay Model

The use of differential equations with regard to population problems can be shown as a mathematical model to describe the population dynamics of a certain species.

If $p(t)$ denotes the size of population of a country at time $t$, then by Balance law for population (Zill, 2013) we have

$$\frac{dp}{dt} = B(p,t) - D(p,t) + M(p,t) \qquad (2.1.1)$$

where,

$B(p,t)$ represents inputs (birth rates)

$D(p,t)$ represents outputs (death rates)

$M(p,t)$ represents net migration

One of the simplest cases is that assuming a model (2.1.1) for birth and death rates are proportional to the population and no migrants. Thus

$$B(p,t) = bp(t), \quad D(p,t) = dp(t), \quad M(p,t) = 0$$

Hence equation (2.1.1) reduce to

$$\frac{dp}{dt} = (b-d)p = \gamma p \qquad (2.1.2)$$

where $b - d = \gamma$ is proportionality constant which indicates population growth for $\gamma > 0$ and population decay for $\gamma < 0$. Since equation (2.1.2) is a linear differential equation, we can get a solution of the form:

$$p(t) = p_0 e^{\gamma t}$$

where $p(t_0) = p_0$ is the initial population and $\gamma$ is called the growth or the decay constant.

As a result the population grows and continues to expand to infinity if $\boldsymbol{\gamma > 0}$, while the population will shrink and tend to zero if $\boldsymbol{\gamma < 0}$.

However, populations cannot grow without bound there can be competition for food, resources or space. Suppose an environment is capable of sustaining no more than a fixed number $k$ of individuals in its population. The quantity $k$ is called the carrying capacity of the environment. Thus for other models equation (2.1.2) can be expected to decrease as the population $p$ increases in size.

The assumption that the rate at which a population grows (or decreases) is dependent only on the number $p(t)$ present and not on any time-dependent mechanisms such as seasonal phenomena can be stated as (Zill, 2013):

$$\frac{dp}{dt} = pf(p) \tag{2.1.3}$$

Now, assume that $f(p)$ is linear

$$f(p) = \alpha p + \beta$$

with conditions

$$\lim_{p(t)\to 0} fp(t) = \gamma, \qquad f(k) = 0$$

so $f$ takes on the form $f(p) = \gamma - \left(\gamma/_k\right)p$

Equation (2.1.3) becomes

$$\frac{dp}{dt} = p\left(\gamma - \frac{\gamma}{k}p\right) \tag{2.1.4}$$

This is called the logistic population model with growth rate $\gamma$ and carrying capacity $k$.

The exponential growth population model can be adjusted for a limited environment and limited resources by taking into account the assumptions:

a) For small population, the rate of growth of the population is proportional to its size.
b) For too large population which is supported by its environment and resources, the rate of growth of the population will decrease.

Clearly, when assuming $p(t)$ is small compared to $k$, then the equation reduces to the exponential one which is nonlinear and separable. The constant solutions $p = 0$ and $p = k$ are known as equilibrium solutions. We can also derive the qualitative information about the solutions to the differential equation from knowledge of where $\frac{dp}{dt}$ is zero, positive, and negative.

The long-term behavior of the population is very different for $p \neq 0$ and $p \neq k$. Equation (2.1.4) can be rewritten as

$$\frac{dp}{p\left(1-\frac{p}{k}\right)} = \gamma dt$$

Applying integrations with the technique of partial fraction gives

$$p(t) = \frac{kce^{\gamma t}}{k + ce^{\gamma t}}$$

where $c$ is a constant

If we consider the initial condition $p(0) = p_0$ (assuming that $p_0$ is not equal to both 0 or $k$), we get

$$p(t) = \frac{kp_0}{p_0 + (k - p_0)e^{-\gamma t}} \tag{2.1.5}$$

Observing its graphical response in Appendix B, One can easily tell what happens to the population if there will be variation in the initial population as $t \to \infty$ as follows.

| Value | Long term behavior of population | |
|---|---|---|
| $p_0 = 0$ | | ➢ no population |
| $0 < p_0 < k$ | | ➢ population grows towards the balance population $p = k$ |
| $p_0 = k$ | $\lim_{t\to\infty} p(t) = k$ | ➢ population level or perfect balance with its surroundings |
| $p_0 > k$ | | ➢ population decreases towards the balance population $p = k$ |

### 2.1.2 Prey predator model

In this study we completely characterize the qualitative behavior of a linear three species food chain. Suppose that three different species of animals interact within the same environment or ecosystem. The ecosystem that we wish to model is a linear three species food chain, where the lowest-level prey species $x$ is preyed up on by a mid-level species $y$, which, in turn, is preyed up on by a top-level predator species $z$. Examples of such three species ecosystems include: mouse-snake-owl and worm-robin- falcon (Paullet et al., 2002).

The model of predator and prey association includes only natural growth or decay and the predator-prey interaction itself. We assume all other relationships (factors) to be negligible. And the prey population grows according to a first order rate law in the absence of predators, while the predator population declines according to a first order rate law if the prey population is extinct.

If there were no predators in the ecosystem, then the prey's species would, with an added assumption of unlimited food supply, grow at a rate that is proportional to the number of prey species present at time $t$ (first order rate law) *:*

$$\frac{dx}{dt} = ax, \qquad a > 0. \tag{2.1.6}$$

But when predator species are present, the prey species population is decreased by $bxy,\ b > 0$, that is, decreased by the rate at which the preys population are eaten during their encounters with the predator species: adding this rate to equation (2.1.6) gives the model for the prey species population:

$$\frac{dx}{dt} = ax - bxy. \tag{2.1.7}$$

If there were no prey species in the ecosystem, then one might expect that the mid- level species, lacking an adequate food supply, would decline in number according to:

$$\frac{dy}{dt} = -cy \qquad c > 0. \tag{2.1.8}$$

When prey species are present in the environment, however, it seems reasonable that the number of encounters or interactions between these two species per unit time is jointly proportional to their populations (the product $xy$). Thus when prey species are present, there is a supply of food, so mid-level species are added to the system at a rate $exy,\ e > 0$. But when top-level predator species are present, the mid-level species population is decreased by $gyz, g > 0$, decreased by the rate at which the mid-level species population are eaten during their encounters with the top predator species: Adding this rate to equation (2.1.8) gives a model for the mid-level species population:

$$\frac{dy}{dt} = -cy + exy - gyz. \tag{2.1.9}$$

Similarly, if there were no mid-level species in the ecosystem, then one might expect that the top- level species, lacking an adequate food supply, would decline in number according to:

$$\frac{dz}{dt} = -hz. \tag{2.1.10}$$

When mid-level species are present in the environment, however, it seems reasonable that the number of encounters or interactions between these two species per unit time is jointly proportional to their populations (the product $yz$). Thus, when prey species are present, there is a supply of food, so top-level species are added to the system at a rate $lyz,\ l > 0$. Adding this rate to equation (2.1.10) gives a model for the top-level species population:

$$\frac{dz}{dt} = -hz + lyz \tag{2.1.11}$$

Equations (2.1.7), (2.1.9) and (2.1.11) constitute a system of nonlinear ordinary differential equations. Then the model we proposed to study becomes

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy \\ \dfrac{dy}{dt} = -cy + exy - gyz \\ \dfrac{dz}{dt} = -hz + lyz \end{cases} \tag{2.1.12}$$

where $x, y$ and $z$ take nonnegative values because populations are nonnegative and

- $a$ - natural growth rate of the prey in the absence of mid-level predator y.
- $b$ - effect of predation on the prey x.
- $c$ - natural death rate of the mid-level predator y in the absence of prey x.
- $e$ - efficiency and propagation rate of the mid-level predator y in the presence of prey x.
- $g$ - effect of predation on the species y by species z.
- $h$ - natural death rate of predator z in the absence of prey y .
- $l$ - efficiency and propagation of rate of predator z in the presence of prey.

By using theorem (1.2.1) each coordinate plane is invariant with respect to the system (2.1.12). The property of invariant coordinate planes matches biological considerations; if some species is extinct it will not reappear.

Next we solve each of the three corresponding planer (two variables) system in the respective coordinate planes. That is we solve the system (2.1.12) by taking situations when the absence of species at each level will be occurred.

1. In the absence of top predator ($z = 0$), the model reduces to:

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = -cy + exy \end{cases}$$

Clearly $dx/dt \leq ax$, then we have $x(t) \to \infty$ and this in turn will cause $y(t) \to \infty$ as $t \to \infty$. This behavior fits with what we would expect biologically in the absence of top predator. Thus, species $x$ and $y$ are free from predation.

Combining the two equations gives a separable differential equation

$$\frac{dy}{dx} = \frac{y(-c + ex)}{x(a - by)}.$$

whose solution in the $xy$ plane is of the form:

$$y^a x^c = Ee^{ex+by},$$

where $E$ is integration constant.

2. In the absence of midlevel species ($y = 0$), the model reduces to:

$$\begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dz}{dt} = -hz. \end{cases}$$

The equation $dz/dt = -hz$ implies that $z(t)$ decays exponentially while $dx/dt = ax$ implies that $x(t)$ grows exponentially as $t \to \infty$. This behavior fits with what we would expect biologically in the absence of midlevel species. Thus, species $x$ is free from predation and $z$ is without a source of food.

The solution in the $xz$ plane can be directly computed from the separable equation

$$\frac{dz}{dx} = \frac{-hz}{ax}$$

It follows that

$$z = Mx^{\frac{-h}{a}},$$

where $M$ is integration constant.

3. Finally, in the absence of bottom-level prey species ($x = 0$), the model reduces to:

$$\begin{cases} \dfrac{dy}{dt} = -cy - gyz \\ \dfrac{dz}{dt} = -hz + lyz. \end{cases}$$

Analogous to (1), we have $dy/dt \le -cy$ and $y(t) \to 0$ as $t \to \infty$. This in turn will cause $z(t) \to 0$ as $t \to \infty$. This behavior fits with what we would expect biologically in the absence of bottom-level species $x$. Thus, all species eventually become extinct in the absence of bottom level species$x$. The system reduce to a separable differential equation

$$\frac{dz}{dy} = \frac{z(-h + ly)}{-y(c + gz)}.$$

And has solution in the $yz$ plane of the form

$$z^c y^{-h} = Ne^{-gz-ly},$$

where $N$ is integration constant.

In the analysis of systems of differential equation, it is often useful to consider solutions that do not change with time, that is, for which point(s) the system $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$ and $\frac{dz}{dt} = 0$. Solution of Such system are called equilibria, steady states or fixed points of system (2.1.12).

The equilibrium points of (2.1.12) are solution of the algebraic system

$$\begin{cases} (a - by)x = 0 \\ (-c + ex - gz)y = 0 \\ (-h + ly)z = 0 \end{cases}$$

As a result, we have three equilibrium points $(x_0, y_0, z_0)$ located at $(0,0,0)$, $(c/e\,, a/b\,, 0)$ and $(0, h/l\,, -c/g)$.

Since the right hand sides of system (2.1.12) have continuous partial derivatives in $x, y$ and $z$ the system (2.1.12) can be linearized at one of the equilibrium $(x_0, y_0, z_0)$ and the associated linearized system takes the form

$$\begin{cases} \frac{dx}{dt} \approx (a - by_0)(x - x_0) - bx_0\,(y - y_0) \\ \frac{dy}{dt} \approx ey_0(x - x_0) + (-c + ex_0 - gz_0)(y - y_0) - gy_0(z - z_0) \\ \frac{dz}{dt} \approx lz_0(y - y_0) + (-h + ly_0)(z - z_0) \end{cases}$$

whose Jacobian matrix is given by

$$J(x_0, y_0, z_0) = \begin{bmatrix} a - by_0 & -bx_0 & 0 \\ ey_0 & -c + ex_0 - gz_0 & -gy_0 \\ 0 & lz_0 & -h + ly_0 \end{bmatrix}. \qquad (2.1.13)$$

The behavior of the linearized system at $(x_0, y_0, z_0)$ is determined by the eigenvalues of the Jacobian matrix. These eigenvalues give us information about the dynamics near the equilibrium point of the original system.

a. The Jacobian matrix (2.1.13) at the equilibrium point $(0, 0, 0)$ takes the form

$$J(0,0,0) = \begin{bmatrix} a & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -h \end{bmatrix}$$

with eigenvalues $\lambda_1 = a$, $\lambda_2 = -c$ and $\lambda_3 = -h$ and corresponding eigenvectors $\langle 1,0,0 \rangle$, $\langle 0,1,0 \rangle$ and $\langle 0,0,1 \rangle$ respectively. Since there is an eigenvalue with positive real part, namely $a$, the equilibrium point $(0,0,0)$ is unstable.

The general solution is written in the form of the linear combination of eigenvalue and its corresponding eigenvector. That is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{at} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-ct} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-ht}$$

where $c_1, c_2, c_3$ are arbitrary constants.

The long term behavior of this general solution is

$$\text{As} \quad t \to \infty, \quad \begin{cases} x(t) \to \infty \\ y(t) \to 0 \\ z(t) \to 0 \end{cases}$$

The physical interpretation of this solution is that of the mid-level and top-level predators were eradicated, the prey population would grow in this simple model.

b. The Jacobian matrix (2.1.13) at the equilibrium point $(c/e\,, a/b\,, 0)$ is of the form

$$J(c/e\,, a/b\,, 0) = \begin{bmatrix} 0 & \frac{-cb}{e} & 0 \\ \frac{ae}{b} & 0 & \frac{-ag}{b} \\ 0 & 0 & \frac{-hb+al}{b} \end{bmatrix}$$

with eigenvalues

$$\lambda_1 = \frac{la-hb}{b}, \qquad \lambda_2 = -i\sqrt{ac}, \qquad \lambda_3 = i\sqrt{ac}$$

and corresponding eigenvectors respectively:

$$\langle 1, \frac{(hb-al)e}{b^2c}, \frac{ab^2ce + b^2eh^2 - 2abehl + a^2el^2}{ab^2cg} \rangle \;, \langle 1, \frac{ie\sqrt{ac}}{bc}, 0 \rangle, \;\; \langle 1, \frac{-ie\sqrt{ac}}{bc}, 0 \rangle$$

Thus, the equilibrium point is stable if $la - hb < 0$ and unstable if $la - hb > 0$.

For $la - hb = 0$, the jacobian matrix evaluated at this equilibrium point have three eigenvalues with zero real part. Thus, each such equilibrium point is stable.

The stability of this equilibrium point is of significance. Since it is stable, non-zero populations might be attracted towards the equilibrium point and as such the dynamics of the system might lead towards the extinction of all the three species for many cases of initial population levels.

In this case, the general solution becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{(hb-al)e}{b^2c} \\ \frac{ab^2ce + b^2eh^2 - 2abehl + a^2el}{ab^2cg} \end{pmatrix} e^{\left(\frac{la-hb}{b}\right)t}$$
$$+ c_2 \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos\sqrt{ac}t - \begin{pmatrix} 0 \\ \frac{e\sqrt{ac}}{bc} \\ 0 \end{pmatrix} \sin\sqrt{ac}t \right]$$
$$+ c_3 \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} sin\sqrt{ac}t + \begin{pmatrix} 0 \\ \frac{e\sqrt{ac}}{bc} \\ 0 \end{pmatrix} \cos\sqrt{ac}t \right]$$

where $c_{1,}c_{2,}c_3$ are arbitrary positive constants.

The long term behavior of the general solution is illustrated in the table bellow

For $la - hb > 0$

| | |
|---|---|
| As $t \to \infty$ | $x(t) \to \infty$ |
| | $y(t) \to -\infty$ |
| | $\begin{cases} z(t) \to \infty, \text{if } ab^2ce + b^2eh^2 - 2abehl + a^2el > 0 \\ z(t) - \infty, \text{if } ab^2ce + b^2eh^2 - 2abehl + a^2el < 0 \end{cases}$ |

This solution can be interpreted physically, when the predator population decreases in time the prey population grows. The decreasing of the mid-level predator is due to they are eaten by the top-predator as a result the top-level predator population increases, on the other hand as time increases the top-level predator faced shortage of food due to this their population decreases again.

For $la - hb < 0$

As time increases the prey population and the mid-level predator remains bounded. That is there is no total extinction and unbounded growth, but the top level predator extinct totally due to shortage of food initially.

Starting from a state in which all the mid-level predator, top-level predators and prey populations are relatively small, the prey first increase because there is little predation. Then the mid-level predators are with abundant food, as a result increase in population, but the top level predator disappears due to luck of food supply as it takes long time to get food from the mid-level predator. This causes heavier predation and the preys tend to decrease. Finally, with a diminished food supply, both the predator population also decreases, and the system returns to the original state.

For $la - hb = 0$

As $t \to \infty, x(t), y(t), z(t)$ neither converges to the critical point nor move to infinite-distant away, rather remains bounded. This solution can be interpreted physically, the prey, mid-level and top-level species are neither grow without bound nor decay, rather their population remains bounded.

c. For the analysis of the equilibrium point $(0, h/l, -c/g)$ of the jacobian matrix (2.1.13), one can follow the same procedure as in case b) and analogously interpret the result.

## 2.2 Mathematical models in Mixing Problem

In this section we will discuss the application of first order ordinary differential equation in mixing problems.

### 2.2.1 Single Tank Mixture Problem Model

A large tank of some sort contains a starting volume of a liquid solution of some substance. The liquid that is being added to the tank is poured in at a specified rate and may have a concentration of the substance of interest which is different from the concentration currently in the tank. The liquid is being drained from the tank at a rate which may be the same or different from the rate liquid is being added; the concentration of the substance in the draining liquid is related in some way to the concentration presently contained in the tank. Such process can involve any sort of fluid – liquid or gas – being added to and drawn from a container. (Alan, 2013)

Suppose that we have two chemical substances where one is solvable in the other, such as salt and water. Suppose that we have a tank containing a mixture of these substances, and the mixture of them is poured in and the resulting “well-mixed” solution pours out through a valve at the bottom.

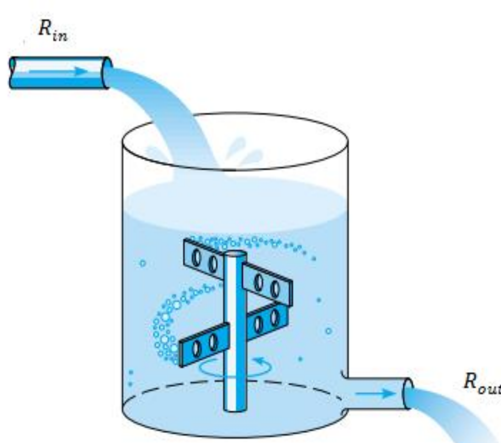


Figure 2: mixing Solutions in the tank

Let

- $c_{in}$ − Concentration of salt in the solution being poured into the tank.
- $c_{out}$ − Concentration of salt in the solution being poured out of the tank.
- $R_{in}$ = rate at which the salt is being poured into the tank
- $R_{out}$= rate at which the salt is being poured out of the tank

There are two types of flow rates for which we can set up differential equations in connection with the mixing problem. Each of these is used to describe what is happening within the tank of liquid. These are volume flow rate and mass flow rate.

The volume flow rate equation tells us how the amount of liquid in the tank is changing. The net rate of change of the volume in the tank is given by

$$\frac{dv}{dt} = \left(\frac{dv}{dt}\right)_{in} - \left(\frac{dv}{dt}\right)_{out} \tag{2.2.1}$$

where $v(t)$ is the volume of liquid present in the tank at time $t$ measured from some chosen moment.

By the law of conservation of salt, the two rates in the difference represent the constant rate at which liquid is being added to (input flow rate) and at which it is being drained from (output flow rate) the tank.

The mass flow rate equation describes the net rate of change of the mass of dissolved substance in the tank.

$$\frac{dm}{dt} = \left(\frac{dm}{dt}\right)_{in} - \left(\frac{dm}{dt}\right)_{out} \tag{2.2.2}$$

By definition of concentration, the mass of the substance in a particular portion of liquid is equal to the volume of that liquid times the concentration of the substance in it. Thus, the solution found in the tank at a given time $t$ has a mass of the substance given by the formula

$$m(t) = v(t).c(t)$$

Now, one can relate the mass flow of the substance to the volume flow of the liquid. For the solution being added to the tank, assuming that the concentration is the same everywhere (constant) in the tank because of instantaneous mixing, this means that the liquid being drained out has the same concentration of the substance as is found in the tank the inflow and outflow of substance mass is therefore

$$\left(\frac{dm}{dt}\right)_{in} = \left(\frac{dv}{dt}\right)_{in}.c_{in} \ , \qquad \left(\frac{dm}{dt}\right)_{out} = \left(\frac{dv}{dt}\right)_{out} c_{out} = \left(\frac{dv}{dt}\right)_{out}.\ c(t)$$

Thus, for any of the mixing problems, the mass flow rate equation is given by

$$\frac{d}{dt}m(t) = \frac{d}{dt}[v(t).c(t)] = \left[\left(\frac{dv}{dt}\right)_{in}.c_{in}\right] - \left[\left(\frac{dv}{dt}\right)_{out}.c(t)\right] \qquad (2.2.3)$$

First, consider the situation in which the rate at which liquid is added is the same as the rate at which liquid is drained out of the tank. That is

$$\left(\frac{dv}{dt}\right)_{in} = \left(\frac{dv}{dt}\right)_{out} = \left(\frac{dv}{dt}\right)_{flow}$$

Then it follows that

$$\frac{d}{dt}v(t) = \left(\frac{dv}{dt}\right)_{flow} - \left(\frac{dv}{dt}\right)_{flow} = 0$$

which implies that $v(t)$ is *a* constant. Since the volume will thus always remain the same as it was at time $t = 0$, the volume in the tank is given by

$$v(t) = v(0) = v_0$$

The condition of having equal rates of filling and draining permits us to rewrite the mass flow rate equation (2.2.3) as

$$\frac{d}{dt}m(t) = \left[\left(\frac{dv}{dt}\right)_{flow}.c_{in}\right] - \left[\left(\frac{dv}{dt}\right)_{flow}.\frac{m(t)}{v(t)}\right] = \left(\frac{dv}{dt}\right)_{flow}.\left[c_{in} - \frac{m(t)}{v_0}\right]$$

Then, it follows that

$$\frac{dm}{c_{in} - \frac{m}{v_0}} = \left(\frac{dv}{dt}\right)_{flow} dt, \quad \text{for } c_{in} \neq \frac{m}{v_0}$$

with solution

$$-v_0 \ln\left|c_{in} - \frac{m}{v_0}\right| = \left(\frac{dv}{dt}\right)_{flow} t + C$$

Assuming the initial condition $m_0 := m(0) = v(0).c(0) = v_0.c_0$ gives

$$\ln\left|\frac{v_0c_{in} - m_0}{v_0c_{in} - m}\right| = kt$$

where

$$k = \frac{\left(\frac{dv}{dt}\right)_{flow}}{v_0}$$

Next, we have the following case

| case | Substance mass | $m(t=0)$ | $\lim_{t\to\infty} m(t)$ |
|---|---|---|---|
| $c_{in} = \frac{m}{v_0}$ | $m(t) = m_0 = v_0 c_{in}$ | $m_0$ | $v_0 c_{in}$ |
| $c_{in} > \frac{m}{v_0}$ | $m(t) = v_0 c_{in} - (v_0 c_{in} - m_0)e^{-kt}$ | $m_0$ | $v_0 c_{in}$ |
| $c_{in} < \frac{m}{v_0}$ | $m(t) = v_0 c_{in} + (m_0 - v_0 c_{in}).e^{-kt}$ | $m_0$ | $v_0 c_{in}$ |

In other words, the initial mass of substance in the tank is either $m_0 < v_0 c_{in}$ or $m_0 > v_0 c_{in}$ both asymptotically approach the constant solution function $m(t) = v_0 c_{in}$

Analogously, equation (2.2.3) for the rate of change of concentration in the tank can be rewritten as

$$\frac{dc}{dt} = \frac{\left(\frac{dv}{dt}\right)_{flow}}{v_0}.(c_{in} - c)$$

whose solution

$$\frac{c_{in} - c}{|c_{in} - c_0|} = e^{-kt}, \quad \text{for } c_{in} \neq c_0$$

Upon examining the solution of the equation for concentration rate of change, we have the following properties

| case | Concentration | $c(t=0)$ | $\lim_{t\to\infty} c(t)$ |
|---|---|---|---|
| $c_{in} = c$ | $c(t) = c_0$ | $c_0$ | $c_{in}$ |
| $c_{in} > c$ | $c(t) = c_{in} - (c_{in} - c_o)e^{-kt}$ | $c_0$ | $c_{in}$ |
| $c_{in} < c$ | $c(t) = c_{in} + (c_0 - c_{in})e^{-kt}$ | $c_0$ | $c_{in}$ |

Similarly, the concentration functions satisfy the initial condition in the tank is either $c_0 < c_{in}$

or $c_0 > c_{in}$ both approach the constant solution function asymptotically

Next, consider the case in which the rate at which liquid is added is not the same as the rate at which liquid is drained out of the tank. That is

$$\left(\frac{dv}{dt}\right)_{in} \neq \left(\frac{dv}{dt}\right)_{out}$$

Here there are two cases.

a) If the rate at which liquid added is faster than the rate at which liquid is drained out of the tank.

$$\left(\frac{dv}{dt}\right)_{in} > \left(\frac{dv}{dt}\right)_{out}$$

Then it follows that

$$\frac{d}{dt}v(t) = \left(\frac{dv}{dt}\right)_{in} - \left(\frac{dv}{dt}\right)_{out} > 0$$

By assuming the input and output rates are constant, the net flow rate will also be a constant, say $\left(\frac{dv}{dt}\right)_{net} = \left(\frac{dv}{dt}\right)_{in} - \left(\frac{dv}{dt}\right)_{out} = \Delta v$

which implies that $v(t) = \Delta v . t + C$. At time $t = 0$ the volume in the tank is $v(0) = v_{0,}$ so the volume in the tank is given by

$$v(t) = \Delta v . t + v_0$$

Because $\Delta v$ is positive, this means that the volume of solution would grow without limit However, in a practical problem, we would need to specify the actual capacity of the tank as a limit at which we would cut off adding liquid to the tank.

The mass flow rate equation for this case becomes

$$\frac{dm}{dt} = \left[\left(\frac{dv}{dt}\right)_{in} . c_{in}\right] - \left[\left(\frac{dv}{dt}\right)_{out} . \frac{m(t)}{v_0 + \Delta v . t}\right] \qquad (2.2.4)$$

Then, it follows that

$$\frac{dm}{dt} + \left[\frac{\left(\frac{dv}{dt}\right)_{out}}{v_0 + \Delta v . t}\right] . m(t) = \left(\frac{dv}{dt}\right)_{in} . c_{in}$$

with solution

$$m(t) = \frac{C}{(v_0 + \Delta v . t)^{\Omega}} + c_{in} . (v_0 + \Delta v . t)$$

where C is integration constant,

$$\Omega = \frac{\left(\frac{dv}{dt}\right)_{out}}{\Delta v}$$

Applying the initial condition $m(0) = m_0$ gives

$$m(t) = (m_0 - v_0.c_{in}).\left(\frac{v_0}{v_0 + \Delta v.t}\right)^{\Omega} + c_{in}.(v_0 + \Delta v.t) \qquad (2.2.5)$$

With the input flow rate being larger than the output flow rate, that it is generally impossible for $\frac{dm}{dt}$ to approach zero. This also makes sense physically speaking: the volume of liquid being added to the tank keeps increasing forever and, if it carries a nonzero concentration of dissolved substance, the mass of dissolved substance in the tank can only increase.

Analogously, equation (2.2.3) for the rate of change of concentration in the tank can be rewritten as

$$\frac{dc}{dt} + \left[\frac{\left(\frac{dv}{dt}\right)_{in}}{v_0 + \Delta v.t}\right] c(t) = \frac{\left(\frac{dv}{dt}\right)_{in}.c_{in}}{v_0 + \Delta v.t}$$

With solution

$$c(t) = \frac{1}{(v_0 + \Delta v.t)^{\Phi}}(C + c_{in}.(v_0 + \Delta v.t)^{\Phi})$$

where $C$ is integration constant, $\Phi = \frac{\left(\frac{dv}{dt}\right)_{in}}{\Delta v}$ . Applying initial conditions $c(0) = c_0$, gives

$$c(t) = (c_0 - c_{in}).\left(\frac{v_0}{v_0 + \Delta v.t}\right)^{\Phi} + c_{in}$$

As $t \to \infty, c(t) = c_{in}$. Its physical interpretation is in the "long run", the concentration of dissolved substance in the tank would become essentially the same as that in the liquid being added. What is interesting about the case of adding liquid faster than it is drained is that the differential equation for mass does not have a "steady-state solution", but the differential equation for concentration does.

b) If the rate at which liquid is drained out of the tank is faster than the rate at which liquid is added

$$\left(\frac{dv}{dt}\right)_{in} < \left(\frac{dv}{dt}\right)_{out}$$

In this case we have

$$0 < \left(\frac{dv}{dt}\right)_{out} - \left(\frac{dv}{dt}\right)_{in} = -\Delta v$$

with

$$v(t) = C - \Delta v.t$$

At time $t = 0$ the volume in the tank is $v(0) = v_{0,}$ so the volume in the tank is given by

$$v(t) = v_0 - \Delta v.t$$

Applying similar methods equation (2.2.3) becomes

$$\frac{dm}{dt} + \left[\frac{\left(\frac{dv}{dt}\right)_{out}}{v_0 - \Delta v.t}\right].m(t) = \left(\frac{dv}{dt}\right)_{in}.c_{in}$$

Applying the initial conditions, the solution of mass function is given by

$$m(t) = (m_0 - v_0.c_{in}).\left(\frac{v_0 - \Delta v.t}{v_0}\right)^{\Omega} + c_{in}.(v_0 - \Delta v.t)$$

As $t \to \infty, m(t) \to -\infty$. Since the output flow rate being larger than the input flow rate, This also makes sense physically speaking: the volume of liquid being flow out of the tank keeps increasing forever and, if it carries a zero concentration of dissolved substance, the mass of dissolved substance in the tank can only decrease.

Similarly, equation (2.2.3) becomes

$$\frac{dc}{dt} + \left[\frac{\left(\frac{dv}{dt}\right)_{in}}{v_0 - \Delta v.t}\right].c(t) = \left[\frac{\left(\frac{dv}{dt}\right)_{out}.c_{in}}{v_0 - \Delta v.t}\right]$$

Applying the initial conditions, the solution of concentration becomes

$$c(t) = (c_0 - c_{in}).\left(\frac{v_0 - \Delta v.t}{v_0}\right)^{\Phi} + c_{in}$$

As $t \to \infty, c(t) \to -\infty$. But physically from the long term behavior we can interpret as the concentration of dissolved substance in the tank would become decrease. Since the out flow rate is greater than the inflow rate.

### 2.2.2. Mixing problem with many tanks

Consider $n$ tanks filled with brine, which are connected by a pair of pipes. One pipe brings brine from the $i^{th}$ tank to the $(i+1)^{th}$ tank at a given rate, while the second pipe carries brine in the opposite direction (to the $i^{th}$ tank) at the same rate, for $i = 1, 2, \ldots, n-1$. Assuming that the initial concentrations in all tanks are known and that we have a perfect mixing, finding the concentrations in all tanks after a given period of time leads to a system of linear ordinary differential equations (Alan, 2013).

Consider the case where $n$ tanks (with $n > 1$) of the same volume $v$ are arranged in linear shape. We have a row of $n$ tanks $T_1, T_2, \dots, T_n$ with neighboring tanks connected by a pair of pipes.

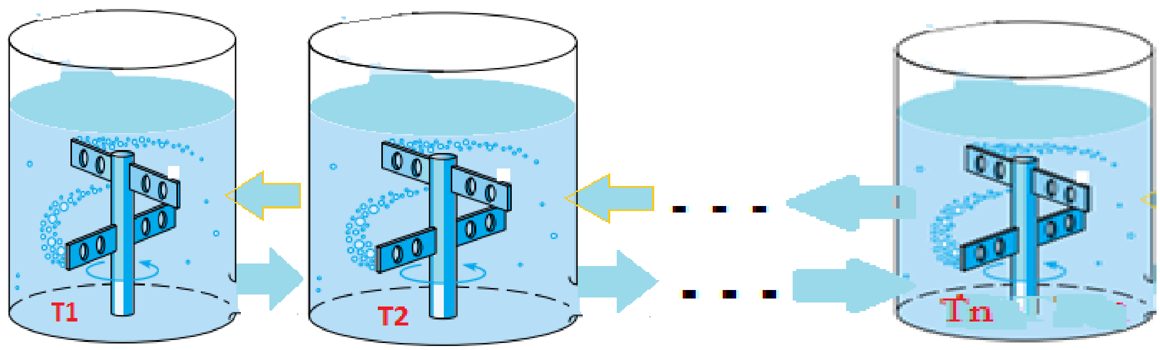


Figure 3: A linear arrangements of n tanks

Assume that the flow through each pipe is $f$ gallons per unit of time. Consequently, the volume $v$ in each tank remains constant. Let $x_i(t)$ be the amount of salt in tank $T_i$ at time $t$. This yields the differential equations

$$x'_1(t) = -f\frac{x_1(t)}{v} + f\frac{x_2(t)}{v}$$

$$x'_i(t) = f\frac{x_{i-1}(t)}{v} - 2f\frac{x_i(t)}{v}) + f\frac{x_{i+1}(t)}{v}, \quad \text{for } 2 \le i \le n-1$$

$$x'_n(t) = f\frac{x_{n-1}(t)}{v} - f\frac{x_n(t)}{v}$$

Without loss of generality, we assume that $f = v$ and switch to the vector form

Then, the system in matrix form becomes

$$x' = Ax(t),$$

where

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}$$

$$x(t) = (x_1, x_1, \dots, x_n)^T$$

Clearly A is tridiagonal and symmetric. Since $A$ is a symmetric matrix, all eigenvalues must be real. According to the Gershgorin circle theorem (In this case $a_{11} = a_{nn} = -1, r_1 = r_n = 1$, while $a_{ii} = -2, r_i = 2$), the eigenvalues of $A$ are contained in the intervals of [-4, 0].

Applying Laplace expansion theorem (theorem 1.4.1) and linear recurrence relation (Elayedi, 2005)

$$\det(A - \lambda I) = 2\cot\left(\frac{\gamma}{2}\right)\sin(n\gamma), \gamma\epsilon(0, \pi]$$

when $\gamma = \frac{k\pi}{n}$, for $k\epsilon\{1,2, \dots, n\}$.

By using theorem (1.4.3), the eigenvalues of the coefficient matrix $A$ is given by

$$\lambda_k = -2cos\frac{k\pi}{n} - 2, \text{ for } k\epsilon\{1,2, \dots, n\}$$

It follows that the components of an eigenvector $V = (v_1, v_2, \dots, v_n)$ corresponding to the eigenvalues $\lambda_k$ are given by the relations

$$v_2 = -\left(1 + 2cos\frac{k\pi}{n}\right)v_1$$

where $v_1$ can be an arbitrary nonzero number.

In general, while

$$v_i = -2\left(cos\frac{k\pi}{n}\right)v_{i-1} - v_{i-2}, \text{ for } i\epsilon\{3,4, \dots, n\}$$

Then, the general solution is given by

$$x(t) = \sum_{i=1}^{n} c_i v_i\, e^{\lambda_i t}$$

Since, $\lambda_n = 0$, and the corresponding eigenvectors have all components identical and the remaining eigenvalues are negative, we conclude that the long term behavior of the general solution always approaches the state with all tanks containing the same amount of salt.

## SUMMARY

This project attempted to discuss the application of first order ordinary differential equation in modeling phenomena of real world problems. Some of the models included are Population growth and decay, Prey-predator interaction, mixing problems in a single tank and multiple tank systems.

Mathematically it is possible to represent the population variations of prey and predator relationship to a certain extent of accuracy by mathematical model which is described by systems of non-linear order ordinary differential equations. And also mathematical model have been used widely to estimate the population dynamics of animals as well as the human population. From this discussion we got the idea how first order ordinary differential equations are closely associated with physical applications and also how the law of nature in different fields of science is formulated in terms of ordinary differential equations.

The idea of the exponential model is the assumption that a model for the birth and death rates are proportional to the population. But logistic model remedy the weakness of exponential model. That is one problem with the exponential model is that it predicts either the population grows without bound or that it decays to extinction. Of course, population cannot grow without bound there can be competition for food, resources or space and this effect can be modeled by supposing that the growth rate depends on the population.

Investigating the behavior of a mixed solution of some substance is also another application of mathematical model, finding the concentration of the mixed solution after a given period of time leads to the resulting well mixed solution.

As a conclusion many fundamental problems in biological, physical sciences and engineering are described by first order ordinary differential equations. It is believed that many unsolved problems of future technologies will be solved using ordinary differential equations. On the other hand, physical problems motivate the development of applied mathematics, and this is especially true for differential equations that help to solve real world problems in the field.

## **APPENDIX A:** DEFINITION OF TERMS

**Migration:** movement of people from one country to another.

**Logistic model**: the rate of population increase may be limited, i.e., it may depend on population density.

**Non-logistic model**: a model which is not logistic.

**Predator**: is an organism that eats another organism.

**Prey**: is the organism which the predator eats.

**Predator-prey relationship**: An interaction between two organisms of unlike species in which one of them acts as predator that captures and feeds on the other organism that serves as the prey.

**Linear food chain**: It is a food chain system that the lowest-level prey species is preyed up on by a mid-level species , which, in turn, is preyed up on by a top-level predator species.

**Concentration**: refers to the amount of a substance per defined space. Or concentration is the ratio of solute in a solution to either solvent or total solution.

**Brine**: water containing salt.

**Well-mixed**: is used to indicate that the fluid being poured in is assumed to instantly dissolve into a homogeneous mixture the moment it goes into the tank.

**Instantaneous mixing**: this means that the liquid being drained out has the same concentration of the substance as is found in the tank.

## **APPENDIX B:** MATLAB CODE FOR LOGISTIC POPULATION GROWTH

We plot the solution to the logistic population growth equation for initial conditions $p_0$ =10, 15, 20, 25, 30, and 40. And the caring capacity $k = 25$.

```
p0=[10 15 20 25 30 40];
t=linspace(0,5);
for i=1:length(p0)
p=25*p0(i)./(p0(i)+(25-p0(i)).*exp(-t));
plot(t,p);hold on
end
xlabel('time(in year)'); ylabel('population(in million)'); title('Logistic population growth');
```

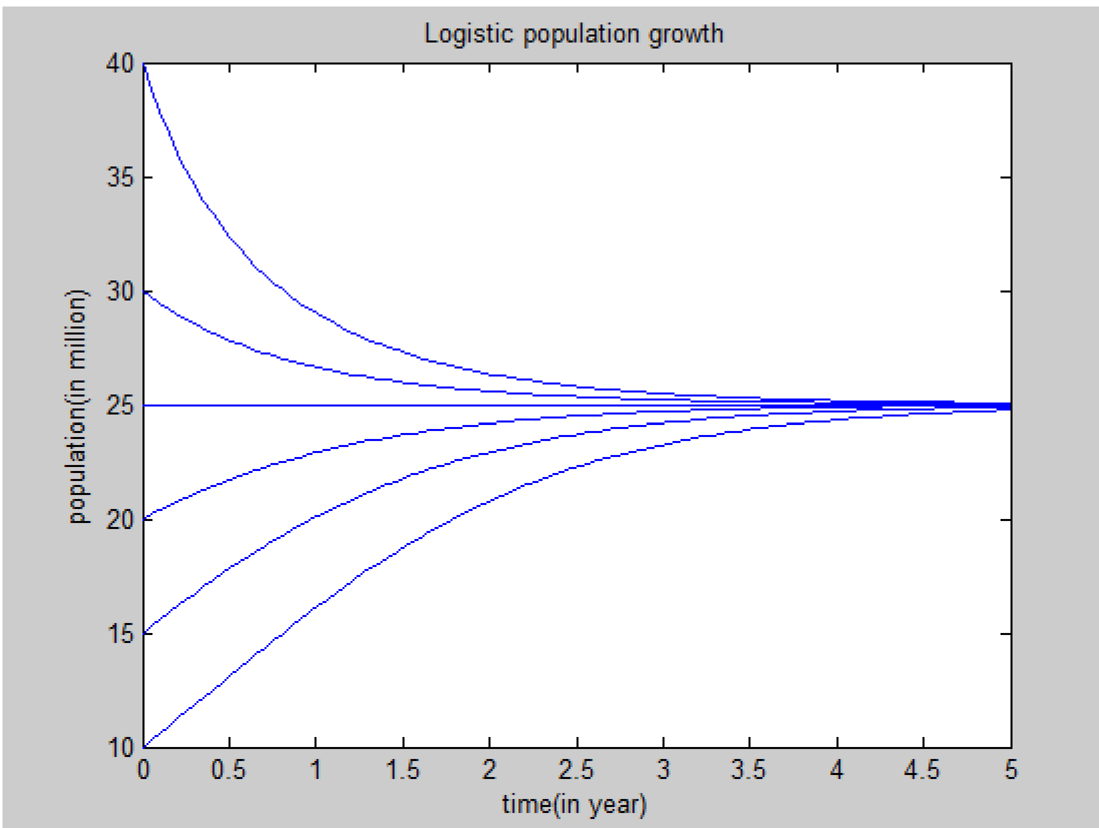


Figure 4: Logistic population growth:

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