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PROPERTIES OF STONE ALMOST DISTRIBUTIVE LATTICE

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Wisdom at the source of the Blue Nile

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Abstract

The purpose of this project is to discuss the concept of "Properties of Stone Almost Distributive lattice" in terms of its dense element and its center. we define different set of axioms which characterize the class of "Properties of Stone Almost Distributive Lattice" and proved that, this axioms independent, in addition to this, we gave a few more necessary and sufficient condition for ADL to be a Stone ADL in terms of principal ideals generated by the semi-complement of an element of a lattice.

CHAPTER ONE

1. INTRODUCTION AND PRELIMINARY

1.1 Introduction

In 1962 G.Grather [2] proved a representation theorem for stone lattices, that is" Every stone lattice is isomorphic to a sub lattice of the lattice of all ideals of a complete and atomic Boolean algebra." For proving the above theorem, G. Grazer[2] in his paper listed some important properties of stone lattices such as $a^* = a^{**}$, a is of the form b^* if and only if $a = a^{**}$, an element a has a complement if and only if there exists an element b such that $a = b^*$ and so on for a, b elements of a lattice L. The concept of almost distributive lattice (ADL) was introduced by Swamy U.M and G. C Rao[3] as a common abstraction of existing lattice theoretic and ring theoretic generalization of Boolean algebra.

The class of ADL with pseado-complementaion was introduced in [4] and it was observed that an ADL can have more than one pseado-complementation .

In 2000 Swamy Rao and Nanajiin[4] define the concept of a stone ADL L is a stone ADL with one pseado-complementation, Then L becomes a stone ADL with pseado-complementation on it.

In this paper we make an extensive study of properties of stone ADL in terms of its dense element and its center we define the concept of semi complement of an element a in an ADL and derive the necessary and sufficient for an ADL to be a stone ADL in terms of principal ideals generated by the semi compliments of L.

1.2. Preliminaries

In this paper we give some important definitions and results, that are more frequently used

Definition 1.2.1 An algebra (L, $\wedge$, $\vee$) of type (2, 2, 0) is called **Lattice,** if it satisfies the following axioms: for all $x, y, z \in L$.

. (1) $x \vee y = y \vee x$. and $x \wedge y = y \wedge x$..........................commutative

(2) $(x \vee y) \vee z = x \vee (y \vee z)$. and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.............. Assocativity

(3) $x \vee x = x$. and $x \wedge x = x$.................................Idempotent

(4) $(x \vee y) \wedge y = y$. and $(x \wedge y) \vee y = y$ .... Absorbation

NOTE In any lattice (L, $\vee$, $\wedge$) the following are equivalent.

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 1.2.2 A **Distributive lattice** is a lattice which satisfies either of the distributive law.

D1: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

D2: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

If (L, $\vee$, $\wedge$) is a Lattice, then an element 0 of L is called zero element or least element of L, thus $0 \wedge a = 0$ for all $a \in L$ and an element 1 of L is called one element or greater element we have $a \vee 1 = 1$ for all $a \in L$.

If L has 0 and 1, then L is a **bounded Lattice.**

A bounded Lattice L is saide to be **complemented**, if for all $a \in L$ there exists $b \in L$ such that

$a \vee b = 1$ and $a \wedge b = 0$.

A Lattice L is Said to be **relatively complement**, if for any $x, y \in L$ such that $x \leq y$, the bounded lattice. $[x, y] = \{z \in L;\ x \leq z \leq y\}$ is a complemented lattice.

Definition 1.2.3 A **stone lattice** (A; $e_0, \ldots, e_{n-1}$) of oreder n is an L algebra **A** in which there exists a chain $0 = e_0 \leq e_1 \leq \ldots \leq e_{n-1} = 1$ such that e_{i+1} is the smallest dense element of $[e_{i+1}, 1]$.

Definition 1.2.4 An algebra (L, ∨, ∧, 0) of type (2, 2, 0) is called **an almost distributive lattice (ADL)** if the following conditions hold:

(1) a ∧ (b ∨ c) = (a ∧ b) ∨ (a ∧ c).
(2) (a ∨ b) ∧ c = (a ∧ c) ∨ (b ∧ c).
(3) (a ∨ b) ∧ a = a.
(4) (a ∨ b) ∧ b = b.
(5) a ∨ (a ∧ b) = a.
(6) 0 ∧ a = 0.

Lemma 1.2.5 Let (L, ∨, ∧) be an ADL with 0. For any a, b, c ∈ L we have the following;

(1) a ∨ b = a ⟺ a ∧ b = b.

(2) a ∨ b = b ⟺ a ∧ b = a.

(3) ∧ is associative.

(4) a ∧ b ∧ c = b ∧ a ∧ c.

(5) (a ∨ b) ∧ c = (b ∨ a) ∧ c.

(6) a ∧ b = 0 ⟺ b ∧ a = 0.

(7) a ∨ (b ∧ c) = (a ∨ b) ∧ (a ∨ c).

(8) a ∧ (a ∨ b) = a, (a ∧ b) ∨ b = b and a ∨ (b ∧ a) = a.

(9) a ∧ a = a and a ∨ a = a.

(10) a ∨ 0 = a = 0 ∨ a and a ∧ 0 = 0.

Definition 1.2.6 Let (L, ∨, ∧) be an ADL with 0. For any a, b ∈ L, define a ≤ b if and only if a ∧ b = a or equivalently, a ∨ b = b. Then "≤" is **a partial ordering** on L .

Lemma 1.2.7 Let L be an ADL with 0, and m ∈ L. Then the following are equivalent:

(1) m is maximal with respect to the partial ordering ≤.
(2) m ∨ a = m, for all a ∈ L .
(3) m ∧ a = a, for every a ∈ L.

Definition 1.2.8 Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then a unary operation $a \to a^*$ on L is called **a pseudo-complementation** on L if,

(1) $a \wedge a^* = 0$;

(2) $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$;

(3) $(a \vee b)^* = a^* \wedge b^*$ for all a, b $\in L$.

Here, the unary operation * is called a pseudo- complementation on L and a* is called a pseudo-compliment of a in L. Now, we list some results of pseudo-complementation.

Lemma 1.2.9 Let L be an ADL with 0 and * be a pseudo-complementation on L. Then for a, b $\in L$, the following conditions hold:

(1) 0^* is maximal.

(2) If a is maximal, then $a^* = 0$.

(3) $0^{**} = 0$.

(4) $a^{**} \wedge a = a$.

(5) $a^* = 0 \Leftrightarrow a^{**}$ is maximal.

(6) $a^* \leq 0^*$.

(7) $a^* \wedge b^* = b^* \wedge a^*$.

(8) $a \leq b \Rightarrow b^* \leq a^*$

(9) $a^* \leq (a \wedge b)^*$ and $b^* \leq (a \wedge b)^*$.

(10) $a^* \leq b* \Leftrightarrow b^{**} \leq a^{**}$

(11) $a = 0 \Leftrightarrow a^{**} = 0$.

Definition 1.2.10 A **homomorphism** between ADLs L_1 and L_2 is a mapping $f : L_1 \to L_2$ satisfying the following conditions:

1. $f(a \vee b) = f(a) \vee f(b)$;
2. $f(a \wedge b) = f(a) \wedge f(b)$;
3. $f(0) = 0$.

Definition 1.2.11 Let $(L, \vee, \wedge, 0)$ be an ADL with 0. A non empty subset I of L is an **ideal** of L if $x \vee y \in I$ and $x \wedge a \in I$ whenever $x, y \in I$ and a $\in L$.

CHAPTER TWO

PROPERTIES OF STONE ALMOST DISTRIBUTIVE LATTICE.

In[2], Grazer proved a representation theorem for stone Lattices, and for this he listed some important properties of stone lattices and, Grazer and Schmidt give some definitions of Stone lattices and proved related results. In these section we discuss these properties to Stone almost distributive lattice(Stone ADLs). We also characterize a Stone ADL L in terms of the semi-complements of an element of L. We give the following definition of Stone ADLs taken from[4]

Definition 2.1. Let $(L, \vee, \wedge, *, 0)$ be an ADL with 0 and * be a pseudo-complementation on L. Then L is called **a stone ADL** if, for any $a \in L$, $a^* \vee a^{**} = 0^*$.

Lemma 2.2. Let $(L, \vee, \wedge, *, 0)$ be a stone ADL and $a, b \in L$. Then the following conditions hold:

(1) $0^* \wedge a = a$ and $0^* \vee a = 0^*$.

(2) $a^{***} = a^*$ and $0^{**} = 0$.

(3) $(a \wedge b)^* = a^* \vee b^*$.

(4) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

(5) An element $a \in [0, 0^*]$ is a complimented if and only if $a = b^*$ for some $b \in L$.

Proof. Let L be an Stone ADL and $a, b \in L$.

(1) Since $0 \wedge a = 0$ by the definition 1.2.8. we get $0^* \wedge a = a$ and hence $0^* \vee a = 0^*$.
(2) From lemma 1.2.9(4) we have $a^{**} \wedge a = a$, so that $a^{**} \vee a = a^{**}$ and hence,
$a^{***} = (a^{**} \vee a)^* = a^{***} \wedge a^* = a^*$.
(3) Let $x = (a \wedge b)^*$. Then
$\Rightarrow a \wedge b \wedge x = a \wedge b \wedge (a \wedge b)^* = 0$
$\Rightarrow a^* \wedge b \wedge x = b \wedge x.$
$\Rightarrow a^{**} \wedge b \wedge x = a^{**} \wedge a^* \wedge b \wedge x = 0.$
$\Rightarrow b \wedge a^{**} \wedge x = 0.$
$\Rightarrow b^* \wedge a^{**} \wedge x = a^{**} \wedge x.$
$\Rightarrow b^* \vee (a^{**} \wedge x) = b^*.$

Now,

$$a^* \vee b^* = a^* \vee [b^* \vee (a^{**} \wedge x)] = a^* \vee [(b^* \vee a^{**}) \wedge (b^* \vee x)].$$
$$= [a^* \vee (b^* \vee a^{**})] \wedge [a^* \vee (b^* \vee x)] = 0^* \wedge [a^* \vee (b^* \vee x)].$$
$$= a^* \vee (b^* \vee x).$$

Thus, $(a^* \vee b^*) \wedge x = [a^* \vee (b^* \vee x)] \wedge x = x.$

Now, $(a \wedge b)^* = (a^* \vee b^*) \wedge (a \wedge b)^* = [a^* \wedge (a \wedge b)^*] \vee [b^* \wedge (a \wedge b)^*]$.
$= [a \vee (a \wedge b)]^* \vee [b \vee (a \wedge b)]^* = a^* \vee b^*$.
Hence, $(a \wedge b)^* = a^* \vee b^*$.

4 . $(a \wedge b)^{**} = (a^{**} \wedge b^{**})$. *by* (3)above.

$= a^{**} \wedge b^{**} = (a^* \vee b^*)^*$.

$= a^{**} \wedge b^{**}$. *by* (3) above.

5. Suppose $a \in [0, 0^*]$ is a complemented element of L. Then , there exists $b \in L$.

such that $a \wedge b = 0$ and $a \vee b = 0^*$. Since $b \wedge a = 0$,we get $b^* \wedge a = a$.

Now $b^* = (b \vee 0)^* = b^* \wedge 0^* = b^* \wedge (a \vee b) = b^* \wedge a = a$.

Conversely, assume that $a = b^*$ for some $b \in L$. Then, $a \wedge b^{**} = a \wedge a^* = 0$, and $a \vee b^{**} = b^* \vee b^{**} = 0^*$. There fore, b^{**} is the complement of a in $[0, 0^*]$.

Definition 2.3. If L is an ADL with a maximal element, then the element $a \in L$ is called **complemented element,** if there exists an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is maximal element of L . Here b is called **the complement** of a.

Unlike in the distributive lattice, a complemented element in an ADL L not have a unique complement. If a is a complemented element in L, then we denote the set of all complements of a by C(a), and it can be easily observed that C(a) is a sub-ADL of L.

Definition 2.4. Let L be an ADL with 0 and a maximal element. The **center** of L is defined as the set of all complemented elements of L and it is denoted by B(L) or simply, B.

Lemma 2.5. If $(L, \vee, \wedge, 0)$ is an ADL with a maximal element and a is a complemented element in L then B(a) is a sub ADL of L.

Proof. Let $(L, \vee, \wedge, 0)$ be an ADL with a maximal element and $c, d \in B(a)$. Then $a \wedge c = 0$, $a \wedge d = 0$, $a \vee c$ and $a \vee d$ are maximal element of L.

Now $a \wedge c \wedge d = 0 \wedge d = 0$ and $a \vee (c \wedge d) = (a \vee c) \wedge (a \vee d) = a \vee d$ wich is maximal .thus

$c \wedge d$ is a complement of a and hence $c \wedge d \in B(a)$.

Again, $a \wedge (c \vee d) = (a \wedge c) \vee (a \wedge d) = 0$.

Now , We show that $a \vee (c \vee d)$ is maximal.

Let $x \in L$, then $[a \vee (c \vee d)] \wedge x = [(a \vee c) \vee d] \wedge x$

$= [(a \vee c) \wedge x] \vee (d \wedge x)$.

$= x \vee (d \wedge x)$, since $a \vee c$ is maximal.

$= x$.

Thus $c \vee d$ is a complement of a and hence $c \vee d \in B(a)$.

There fore B(a) is a sub ADL of L.

Lemma 2.6. Let L be a pseudo-complemented ADL with center B and $a \in L$. *Then* $a \in B$ if and only if $a \vee a^*$ is maximal.

***Proof*.** Suppose $a \in B$. then there exists an element $b \in L$ such that $a \wedge b = 0$ *and* $a \vee b$ is maximal.

Since $a \wedge b = 0$ we get $a^* \wedge b = b$ and hence $(a \vee a^*) \wedge (a \vee b) = a \vee (a^* \wedge b) = a \vee b$, which is maximal. Thus $a \vee a^*$ is maximal.

Conversely suppose $a \vee a^*$ is maximal. But $a \wedge a^* = 0$.

Hence $a \in B$.

Now We prove the following results.

Theorem 2.7. Let L be a stone ADL. Then the center B(L) coincides with the center of the distributive lattice[0, 0*] and hence $(B(L), \vee, \wedge)$ *is* Boolean algebra.

Proof. Let B(L) = B. Suppose $a \in B$. then there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is maximal element of L .

Now, $a \wedge b = 0 \Rightarrow a^* \wedge b = b$.

$\Rightarrow a^{**} \wedge b = a^{**} \wedge a^* \wedge b = 0 \Rightarrow b \wedge a^{**} = 0$.

Since L is a stone ADL, we have $a^{**} \vee a^* = 0^*$. Now, $a^{**} = (a \vee b) \wedge a^{**} = (a \wedge a^{**}) \vee (b \wedge a^{**})$

$= a \vee 0 = a$. There fore, $a \vee a^* = a^* \vee a = a^* \vee a^{**} = 0^*$. Hence the result follows.

The following result follows directly from the above theorem and previous results.

Corollary 2.8. Let L be a stone ADL with center B. Then B $= \{a * : a \in L\} = \{a \in L : a = a * *\}$.

Proof. Let L be a stone ADL with center B and $C = \{a \in L : a = a **\}$.

Let $a \in B$. Then from the proof of the above lemma 2.7 , $a = a^{**}$.Hence $a \in C$,So that $B \subseteq C$.

Let $a \in C$.Then $a = b^*$ for some $b \in L$. Now, $a \wedge a^* = 0$ and $a \vee a^* = b^* \vee b^{**} = 0^*$and henc $a \in B$,

So that $C \subseteq B$. There fore $B = C$. Since $a^{***} = a^*$, we get $C = \{a \in L: a = a^{**}\}$.

Corollary 2.9. If L is a stone ADL and $a \in L$ is complemented, then $B(a) = \{a^*\}$.

Proof. Let a be a complemented element of L. Then $a \in B$ and hence $a = b^*$ for some $b \in L$.

Now, $a \wedge a^* = 0$ and $a \vee a^* = b^* \vee b^{**} = 0^*$.hence $a^* \in B(a)$.

Let $c \in B(a)$.then $a^*, c \in$ so that $a^* \wedge c = 0$ and $a \vee c$ is maximal.

$$\text{Hence, } c = 0^* \wedge c = (a \vee a^*) \wedge c = (a \wedge c) \vee (a^* \wedge c) = a^* \wedge c = c \wedge a^*$$

$$= (a \wedge a^*) \vee (c \wedge a^*) = (a \vee c) \wedge a^* = a^*.$$

There fore $B(a) = a^*$.

Definition 2.10.. An ADL $(L, \vee, \wedge)$ is siad to be **relative complement** if the interval $[a, b]$ is a complemented lattice for any a, b in L with $a \leq b$.

Theorem 2.11. Let L be a complete relatively complemented ADL with a maximal element m. Then the following conditions hold.

(1) The set of all Ideals $I(L)$ of L is a stone lattice.
(2) The center of $I(L) = PI(L)$.

Proof. (1) Let $I(L)$ be the set of all ideals of L. Clearly $I(L)$ is a complete lattice. for $I \in I(L)$ define $I^* = \{a \in L: a \wedge x = 0, for\ every\ x \in I\}$. Then clearly $I \cap I^* = (0]$. Let $J \in I(L)$ such That $I \cap J = (0]$. Now Let $y \in J$. Then for every $x \in I, x \wedge y \in I \cap J$ and Hence $x \wedge y = 0$. Thus $J \subseteq I^*$. Hence $I(L)$ is pseudo-complemented lattice. Finally, let $I \in I(L)$. Since L is a complete ADL, $PI(L)$ is a complete lattice. Hence $\vee_{y \in I^*} (y] = (t]$, for some $t \in L$.Let s be a complement of t in $[0, t \vee 0^*]$. Then $t \wedge s = 0$ and $t \vee s = t \vee 0^*$. If $y \in I^*$, then $y \in y(t]$ and Hence $t \wedge y = y$. Now, $y \wedge s = t \wedge y \wedge s = 0$. There fore , $s \in I^{**}$. Again since $t \in (t] = \vee_{y \in I^*} (y]$. there exists $y_1, y_2, \dots, y_n \in I^*$such that $t \in \vee_{i=1}^{n} (y_i] = (\vee_{i=1}^{n} y_i] \Rightarrow (\vee_{i=1}^{n} y_i) \wedge t = t$. Now, for any $x \in I, (\vee_{i=1}^{n} y_i) \wedge x = (\vee_{i=1}^{n} y_i \wedge x) = 0$, and hence $x \wedge t = x \wedge (\vee_{i=1}^{n} y_i) \wedge t = 0$. Thus $t \in I^*$. Thus,

$t \vee 0^* = t \vee s \in I^* \vee I^{**}$. Hence , $I^* \vee I^{**} = L$. There fore $I(L)$ is a stone lattice.

(2) Let I be complemented ideal of L. Then there exists an ideal J such That $I \wedge J = (0]$ and $I \vee J = L$. Since $m \in L$ we get $m = x \vee y$, for some $x \in I$ and $y \in J$. Now for any $t \in I$. $t = m \wedge t = (x \vee y) \wedge t = x \wedge t \in (x]$, we get $I = (x]$. Conversely suppose $x \in L$. Let y be the

complement of x in $[0, x \vee m]$. Then $x \wedge y = 0$ and $x \vee y = x \vee m$. Now $x \wedge y = 0 \Rightarrow (x] \cap (y] = (x \wedge y] = (0]$. More over , $x \vee m = x \vee y \in (x] \vee (y]$ and $x \vee m$ is maximal. Therefore $(x] \vee (y] = L$ and hence $I = (x]$ is a complemented ideal of L.

Lemma 2.12. Let L be an ADL and $a \in L$. *Then* $L_a = \{a \wedge x : x \in L\}$ is a sub-ADL of L.

Proof. Let L be an ADL and for $a \in L$ *let* $L_a = \{a \wedge x : x \in L\}$

Suppose $t, s \in L_a$ then $t = a \wedge x$ *and* $s = a \wedge y$ *for some* $x, y \in L$.

Now , $t \wedge s = a \wedge x \wedge a \wedge y = a \wedge x \wedge y \in L_a$ and

$t \vee s = (a \wedge x) \vee (a \wedge y) = a \wedge (x \vee y) \in L_a$.

thus $(L_a, \vee, \wedge)$ is a sub-ADL of L.

Theorem 2.13. Let L be astone ADL and $a \in L$, then the map $\psi : L \to L_a \times L_{a^*}$ defined by $\psi(x) = (a \wedge x,\ a^* \wedge x)$ for all $x \in L$ is an isomorphism if and only if a is the center of L .

Proof. Suppose the map ψ is an isomorphism. Now,

$\psi(a) = (a \wedge a,\ a^* \wedge a) = (a, 0)$ and $\psi(a^{**}) = (a \wedge a^{**},\ a^* \wedge a^{**}) = (a, 0)$.

Thus , $\psi(a) = \psi(a^{**})$ implies $a = a^{**}$ and Hence a is in the center $B(L)$.

Conversely, assume that a is in B(L). We want to show that ψ is an isomorphism.

Now, $\psi(x \wedge y) = (a \wedge x \wedge y,\ a^* \wedge x \wedge y)$.

$= (a \wedge x,\ a^* \wedge x) \wedge (a \wedge y,\ a^* \wedge y)$.

$= \psi(x) \wedge \psi(y)$.

similarly $\psi(x \vee y) = (a \wedge (x \vee y),\ a^* \wedge (x \vee y))$

$= (a \wedge x,\ a^* \wedge x) \vee (a \wedge y,\ a^* \wedge y)$

$= \psi(x) \vee \psi(y)$.

Hence ψ is a homomorphism.

Let $x, y \in L$ *and* $\psi(x) = \psi(y)$. Then, $x = 0^* \wedge x = (a \vee a^*) \wedge x = (a \wedge x) \vee (a^* \wedge x)$

$= (a \wedge y) \vee (a^* \wedge y) = (a \vee a^*) \wedge y = 0^* \wedge y = y$.

Hence ψ is one- to- one

Finally suppose $(a \wedge x, a^* \wedge y) \in L_a \times L_{a^*}$, for some $x, y \in L$.

Write $z = (a \wedge x) \vee (a^* \wedge y)$.

Then, $\psi(z) = (a \wedge z, a^* \wedge z)$.

$= (a \wedge [(a \wedge x) \vee (a^* \wedge y)], a^* \wedge [(a \wedge x) \vee (a^* \wedge y)])$.

$= ((a \wedge x) \vee 0, 0 \vee (a^* \wedge y))$.

$= (a \wedge x, a^* \wedge y)$.

There fore , ψ is an Isomorphism.

Definition 2.14. Let L be an ADL with 0. An element b *in* L is saide to be **a semi-complement** of the element a in L if $a \wedge b = 0$. We denote the set of all semi-complement of a by S(a).

The following result can be verified easily.

Lemma 2.15. Let L be an ADL and $a \in L$.Then $S(a)$ is a $S(a)n$ *ideal of* L and $S(a) = (a^*]$.

Proof. Let L be an ADL whith pseudo-Complementation and let S(a) be the set of all semi complements of $a \in L$. Let $t \in S(a)$. Then $a \wedge t = 0$ and hence $a^* \wedge t = t$, so that $t \in (a^*]$. On the other hand, if $t \in (a^*]$, then $a^* \wedge t = t$. Now $a \wedge t = a \wedge a^* \wedge t = 0$. Thus $t \in S(a)$. Hence $S(a) = (a^*]$. So $S(a)$ is an Ideal of L.

Let us recall that an ideal I in an ADL L is called a direct factor if there exists an ideal J *of* L such that $I \cap J = \{0\}$ and $I \vee J = L$. Now We prove the following theorem.

Theorem 2.16. Let L be pseudo complimented ADL. Then L is a stone ADL if and only if, for any $a \in L$, The ideal $S(a)$ is a direct factor of L.

Proof. Suppose L is a Stone ADL and $a \in L$. *Then* by the above lemma $S(a) = (a^*]$. Since L is a stone ADL, we have $a^* \wedge a^{**} = 0$ and and $a^* \vee a^{**} = 0^*$ Hence $(a^*] \wedge (a^{**}] = 0$ and $(a^*] \vee (a^{**}] = L$.There fore $(a^*]$ is a direct factor of L.

Conversely, assume that S(a) is a direct factor of L, for all $a \in L$. Then there exists an ideal J *in* L such that $S(a) \cap J = \{0\}$ and $S(a) \vee J = L$. Write $0^* = b \vee (a^* \wedge x)$ for some $x \in L, b \in J$ and hence $b \vee a^*$ *is maximal*. Also since $(a^* \wedge b) \in S(a) \cap J$. I get $a^* \wedge b = 0$. So that $a^{**} \wedge b = b$ and $a^{**} = a^{**} \vee b$.

Now , $(a^* \vee a^{**}) \wedge 0^* = (a^* \vee a^{**} \vee b) \wedge 0^*$.

$= (a^{**} \vee a^* \vee b) \wedge 0^*$

$= (a^{**} \wedge 0^*) \vee [(a^* \vee b) \wedge 0^*]$

$= (a^{**} \wedge 0^*) \vee [(b \vee a^*) \wedge 0^*] = 0^*.$

Hence we get, $0^* = (a^* \wedge 0^*) \vee (a^{**} \wedge 0^*)$

$= (a \vee 0)^* \vee (a^* \vee 0)^*$

$= a^* \vee a^{**}.$

There fore L is a stone ADL.

Definition 2.17. An ADL L is saide to be a **dense ADL** if $a, b \in L$ *and* $a \wedge b = 0$ implies either $a = 0$ or $b = 0$.

The following result relates Stone ADL to dense ADL.

Lemma 2.18. If L is a stone ADL, then it is either a dense ADL or there exists an element $a \in L$ with $0 < a < 0^*$ and $a^* \neq 0$.

Proof. Let L be a *Stone* ADL, and assume that L is not dense. Then there exists $a, b \in L - \{0\}$ such that $a \wedge b = 0$. Hence $b \in (a^*]$ and $b \neq 0$, so that $a^* \neq 0$.

Lemma 2.19. Let L be a stone ADL . Then every principal ideal of L is a stone lattice.

Proof. Let L be a Stone ADL and $a \in L$. If $I = (a]$, then clearly $(I, \vee, \wedge)$ is a bounded distributive lattice. For each $x \in I$, define $x^{\perp} = a \wedge x^*$ Then we have the following.

(i) $x \wedge x^{\perp} = x \wedge a \wedge x^* = 0.$

(ii) Suppose $y \in I$ *and* $x \wedge y = 0$. Then, $x^* \wedge y = y$ and hence $x^{\perp} \wedge y = a \wedge x^* \wedge y = a \wedge y = y.$

(iii) $(x \vee y)^{\perp} = a \wedge (x \vee y)^*$

$= a \wedge x^* \wedge y^*$

$= a \wedge x^* \wedge a \wedge y^*$

$= x^{\perp} \wedge y^{\perp}.$

for every $x, y \in I$. There fore, $\perp$ is pseudo-complementation on $I = (a]$. more over, $x^{\perp\perp} = (x^* \wedge a)^{\perp} = (x^* \wedge a)^* \wedge a = (x^{**} \vee a^*) \wedge a = x^{**} \wedge a$. Hence $x^{\perp} \vee x^{\perp\perp} = (x^* \wedge a) \vee (x^{**} \wedge a) = (x^* \vee x^{**}) \wedge a = 0^* \wedge a = a$, where a is a greatest element in I. There fore $I = (a]$ is a stone lattice.

Theorem 2.20. Every dense ADL with a maximal element m is a stone ADL.

Proof. Let L be dese ADL with a maximal element m. For $a \in L$, $a^* = 0$ if $a \neq 0$ and $a^* = m$ if $a = 0$.

(i) Let $a \in L$. Then by the definition, either a = 0 or a* = 0 if a≠ 0 and a* = m if a = 0.

Hence, a ∧ a* = 0.

(ii) Let a, b $\in L$ and a ∧ b = 0. Then either a = 0 or b=0.

Suppose a = 0. then a*∧ b = m $\wedge b = b$, and If a≠ 0 *then* $b = 0$ and hence a* $\wedge b$ = 0 = b.

There for, in any case, a*$\wedge b = b$.

(iii) Let a ,b $\in L$,and assume that a ∨ b = 0. Then a = 0 and b = 0,Thes (a ∨ b)* = m = m ∧ m = a*∧ b*. If a ∨ b ≠ 0.Then (a ∨ b)* = 0 now a ∨ b ≠ 0, implies a $\neq 0$ *or* $b \neq 0$ and hence a* = 0 or b* = 0. in either cases a* ∧ b* = $(a \vee b)^*$ = 0.

(iv) Finally, let $a \in L$. Suppose a≠ 0. Then a* = 0, a** = m and hence a* ∨ a** = 0 ∨ m = m.

assume that a = 0. then a* = m and a** = 0. Hence a* ∨ a** = m ∨ 0 = m.

Therefore for any $a \in L$, a* ∨ a** = m = 0* and hence L is a stone ADL.

Definition 2.21. Let L be An ADL. An element $0 \neq a \in L$is called **an atom** in L if (a] is an atom in the lattice $PI(L)$.

Lemma 2.22. Let L be an ADL and $0 \neq a \in L$. Then a is an atom in L if and only if $0 \neq x \in L$ and a $\wedge x = x$ implies $x \wedge a = a$.

Proof. Let L be an ADL and $0 \neq a \in L$. Suppose a is an atom in L . Then (a] is an atom in $PI(L)$. Let $0 \neq x \in L$ and a $\wedge x = x$. Then (a] ∧ (x] = (x]. That is (x] ⊆ (a] Since (x] ≠ (0] we get (x] = (a]. Thus, a $\in (x]$ and we have $x \wedge$ a = a.

Conversely, suppose the condition holds to show that a is an atom , that is (a] an atom in $PI(L)$. Let $0 \neq x \in L$ such that (x] ⊆ (a]. Then $x \in (a]$. So that a $\wedge x = x$ and hence $x \wedge$ a = a.

There fore (a] ⊆ (x]. Thus (x] = (a] and hence (a] is an atom in $PI(L)$.

Definition 2.23. Let L be an ADL. The element a, b $\in L$ are said to be **equivalent** (Written as $a \sim b$) if $(a] = (b]$ or equivalently a ∧ b = b, and b ∧ a = a.

Theorem 2.24. A finite dense ADL has an atom which is unique up to equivalence .

Proof. Let L be a finite dense ADL and $L - \{0\} = \{a1, a2, a3, \ldots, an\}$. Write a = $a_1 \wedge a_2 \wedge \ldots \wedge a_n$ Then $a \neq 0$. Now let $x \in L$ such that $0 < x \leq a$. Since $x \leq a = a_1 \wedge a_2 \wedge \ldots \wedge a_i \wedge \ldots \wedge a_n$ We get $x = a_i$ for some $i\ (1 \leq i \leq n)$. Hence $ai = x \leq a \leq ai$. There fore $x = a$, so that a is an atom in L. If $a_1 \wedge a_2 \neq 0$. Then since $0 <$ a1 ∧ a2 ≤ a2 and a2 is an atom ,we get $a_1 \wedge a_2 = a_2$. Similarly, we get $a_2 \wedge a_1 = a_1$ There fore $(a_1] = (a_2]$, and hence $a_1 \sim a_2$.

Finally, We prove that the characterization Theorem finite stone ADL. But before that, we need the following result.

Lemma 2.25. If $L_1$and L_2 are two stone ADLs, then the direct product $L_1 \times L_2$ is also a stone ADL under point wise operations, defined by $(a_1,b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2), b_1 \wedge b_2)$, $(a_1,b_1) \vee (a_2,b_2) = (a_1 \vee a_2), b_1 \vee b_2)$ and $(a, b)^* = (a^*,b^*)$ for every $(a_1, b_1), (a_2, b_2) \in L_1 \times L_2$.

Proof. Let $L_1$and L_2 are two stone ADLs. We know that $L_1 \times L_2$ is an ADL under the point wise operation .

Let $(a, b) \in L_1 \times L_2$

(i) $(a, b) \wedge (a, b)^* = (a, b) \wedge (a, b)^*$.

$$= (a \wedge a^*, b \wedge b^*).$$
$$= (0, 0).$$

(ii) Suppose $(x,y) \in L_1 \times L_2$ and $(a, b) \wedge (x, y) = (0, 0)$.

Then ,$(a \wedge x) = 0$ and $(b \wedge y) = 0$ .Thus $a^* \wedge x = x$ and $b^* \wedge y = y$, so that

$$(a, b)^* \wedge (x, y) = (a^*, b^*) \wedge (x, y) = (a^* \wedge x, b^* \wedge y) = (x, y).$$

(iii) $[(a, b) \vee (c, d)]^* = (a \vee c, b \vee d)^* = ((a \vee c)^*, (b \vee d)^*) = (a^* \wedge c^*, b^* \wedge d^*)$

$$= (a^*, b^*) \wedge (c^*, d^*) = (a, b)^* \wedge (c, d)^*.$$

There for $L_1 \times L_2$ is pseudo-complemented.

(iv) $(a, b)^* \vee (a, b)^{**} = (a^*,b^*) \vee (a^{**},b^{**}) = (a^* \vee a^{**}), (b^* \vee b^{**}) = (0^*,0^*)$

Hence, $L_1 \times L_2$ is a stone ADL.

Lemma 2.26. Let L be a stone ADL and $a \in L$. Write $La = \{ a \wedge x : x \in L \}$. Define a unary operation * on L_a by $(a \wedge x)^* = a \wedge x^*$ for every $x \in L$. Then , $(L_a ; \vee, \wedge, *, 0, a \wedge m)$ is a stone ADL

Proof . Clearly $(L_a ; \vee, \wedge)$ Is an ADL .Let $x, y \in L$. Then , we have the following:

(i) $(a \wedge x) \wedge (a \wedge x)^* = a \wedge x \wedge a \wedge x^* = 0$.

(ii) Suppose $(a \wedge x) \wedge (a \wedge y) = 0$. Then, $x \wedge a \wedge y = 0$. So that $(a \wedge x)^* \wedge a \wedge y = a \wedge x^* \wedge a \wedge y = x^* \wedge a \wedge y = a \wedge y$.

(iii) $[(a \wedge x) \vee (a \wedge y)]^* = [a \wedge (x \vee y)]^* = a \wedge x^* \wedge y^* = (a \wedge x)^* \wedge (a \wedge y)^*$. Thus

L is pseudo-complementation.

(iv) $(a \wedge x)^* \vee (a \wedge x)^{**} = (a \wedge x)^* \vee (a \wedge x^*)^* = (a \wedge x^*) \vee (a \wedge x^{**})$

$$= a \wedge (x^* \vee x^{**}) = a \wedge 0^* = (a \wedge 0)^* = 0^*.$$

There fore, (L_a ; $\vee$, $\wedge$, *, 0, $a \wedge m$) is a stone ADL.

Lemma 2.27. Let L be a stone ADL and $a \in L$.Then , the map $f: L \longrightarrow L_{a^*} \times L_{a^{**}}$ defined by $f(x) = (a^* \wedge x, a^{**} \wedge x)$ is an isomorphism, where L_a and $L_{a^{**}}$ are defined as in the above lemma.

Proof. Let $x, y \in L$.

(i) $f(x \wedge y) = (a^* \wedge x \wedge y, a^{**} \wedge x \wedge y) = (a^* \wedge x , a^{**} \wedge x) \wedge (a^* \wedge y, a^{**} \wedge y)$

$$= f(x) \wedge f(y).$$

Similarly, $f(x \vee y) = f(x) \vee f(y)$, and hence f is a homomorphism.

(ii) Suppose $f(x) = f(y)$ Then, $(a^* \wedge x, a^{**} \wedge x) = (a^* \wedge y, a^{**} \wedge y)$ so that $a^* \wedge x = a^* \wedge y$ and $a^{**} \wedge x = a^{**} \wedge y$. Thus, $x = (a^* \vee a^{**}) \wedge x = (a^* \vee a^{**}) \wedge y = y$,

and hence f is one-to-one.

(iii) *Let* $(x, y) \in L_{a^*} \times L_{a^{**}}$ and $z = (a^* \wedge x) \vee (a^{**} \wedge y)$. Then

$a^* \wedge z = a^* \wedge [(a^* \wedge x) \vee (a^{**} \wedge y) = a^* \wedge x = x$ and $a^{**} \wedge z = a^{**} \wedge [(a^* \wedge x) \vee (a^{**} \wedge y)]$ $= a^{**} \wedge y = y$.

Hence, $f(z) = (a^* \wedge z , a^{**} \wedge z) = (x, y)$, so that f is on to. There fore, f is an Isomorphism.

Theorem 2.28. A finite ADL is a Stone ADL if and only if it is a direct product of dens ADLs .

Proof. Suppose L is a finite Stone ADL. Then by Lemma 2.15 L is either a dense ADL or there exists or an element $a \in L$ with $0 < a < 0^*$ such that $a^* \neq 0$. If L is a dense ADL, there is nothing to prove. Thus we assume the second condition. Since L is finite L, assume that $|L| = n$

,for some positive Integer n , and I prove by induction .If $|L| = 1$, then $L = \{0\}$ and hence L is adense ADL. Suppose the result is true for all ADLs L such that $|L| < n$. Let $L_{a^*} = \{a \wedge x : x \in L\}$ and $L_2 = \{a^{**} \wedge x : x \in L\}$ .Where a is defined as above Clearly ,a^{**} is not in L_1 and a^* is not L_2. Then $|L_1| < n$ and $|L_2| < n$, and hence by Induction hypothesis L_1 and L_2 are direct product of dens ADLs. Hence by lemma2.26 L is Isomorphic to $L_1 \times L_2$ so that L is a direct product of dens ADLs.

Conversely, assume that L is a direct product of dens ADLs, $L_1, L_2 \ldots L_n$. Then by Theorem 2.20 ,each dense ADL, $L_1, L_2 \ldots L_n$ is a Stone ADL. Thus, $L_1 \times L_2 \times \ldots \times L_n$ is a Stone ADL, by lemma 2.25 and Hence L is a Stone ADL.

Corollary 2.29. Every finite Stone ADL with more than one element is a direct product of finite stone ADLs, of which each has an atom and is unique up to equivalence.

Proof. Let L be a finite Stone ADLs L whith more than one element. Then by lemma 2.28 every finite dens ADL has an atom which is unique up to equivalence and hence the result follows Theorem 2.20.

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