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The stone weirestrass theorem

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THE STONE-WEIRESTRASS THEOREM

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Abstract

The goal of this report is to prove the Stone-Weirestrass theorem and gives a precise information about approximate by function in an algebra of continuous functions.

Suppose that we have a collection of continuous functions on a compact Hausdorff Space which is closed under addition, multiplication and scalar multiplication, and also that for any two distinct points in the space, there is a continuous function in the collection which takes distinct values at these points. Then the theorem says that this collection approximates any continuous function arbitrarily closely.

Notations

C (0, 1)   Space continuous function on (0, 1)

$||X||$ Norm of X

inf   infimum(greatest lower bound)

$\bar{A}$ The closure of A

R^n Euclidian Space

$x \wedge y$ min(x, y)

$x \vee y$ Max(x, y)

CHAPTER ONE

INTRODUCION AND PRELIMINARIES

1.1.Introduction

In analysis, the Weirstrass approximation theorem states that every continuous function defined on an interval [a, b] can be uniformly approximated as closely as desired by a polynomial function. Because polynomials are the simplest functions and computers can directly evaluate polynomials. This theorem has both practical and theoretical relevance especially in interpolation.

The original version of this result was established by Karl Weirstrass in 1855. Marshal H.stone considerably generalized the theorem in 1937, and simplified the proof in 1948. His result is known as the Stone- Weierstrass theorem generalizes the Weierstrass approximation theorem in two directions. Instead of the real interval [a, b] an arbitrarily compact Hausdorff Space X is considered and instead of the algebra of polynomial function, approximation with elements from more general lattice of (X, R) is investigated.

The Stone-Weirestrass theorem is a vital result in the study of the algebra of continuous functions on a compact Hausdorff Space.

1.2 PRELIMINARIES

1.2.1 Topological Space

Definition 1.1: Let X be a non-empty set. A collection τ of subset of X is said to be a topology on X if

i) X and $\emptyset$,belongs to τ
ii) The union of any number of sets in τ belongs to τ and
iii) The intersection of any finite sets in τ belongs to τ

The pair (X, τ) is called a topological space.

Example 1.1: Let X= {a, b, c, d, e, f} and

$$\tau_1 = \{X,\ \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$$

Then τ_1 is a topology on X as it satisfies conditions (i), (ii), (iii) of the above definitions.

Example 1.2: Let X = {a, b, c, d, e} and

$$\tau_2 = \{\ X\ , \emptyset, \{a\}, \{c, d\}, \{a, c, e\}, \{b, c, d,\}\}$$

Then τ_2 *is* not a topology on X as the union {c, d}∪{a, c, e}={a, c, d, e} of two members of τ_2 does not belongs to τ_2 ;that is, τ_2 does not satisfies condition (ii) of the above definition.

1.2.1.1 Closed Sets: Let X be a topological space. A subset A of X is a closed set if and only if its compliment A^c is an open set.

For instance, the closed interval [a, b] is closed in R since its compliment $(-\infty, a) \cup (b, \infty)$ the union of two infinite intervals is open.

Definition 1.2: A set O of real numbers is called open if for each $x \in O$ there is $\delta > 0$ such that for each y with $|x - y| < \delta$ belongs to O.

Example 1.3: The class $\tau = \{\ X\ , \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ defined a topology on $X = \{$ a, b, c, d, e$\}$. The closed subset of X are $\emptyset$, X, {b, c, d, e}, {a, b, e}, {b, e}, {a} that is, the compliments of the open subset of X. Note that there are subsets of X, such as {b, c, d, e}, which are both open and closed, and there are sub set of X, such as {a, b}, which are both neither open nor closed.

1.2.1.2 Limit Point

Definition 1.3: Let X be a topological space. A point p$\in$ X is an accumulation point or limit point (also derived point) as a subset A of X if and only if every open set G containing p contains a point different from p that is G open p$\in G$ implies (G/ {p})$\cap$A$\neq \emptyset$.the set of accumulation point of A, denoted by A' ,is called the derived set of A.

This concept profitably generalizes the notion of a limit and is underpinning of concepts such as closed set and the topological closure. Indeed a set is closed if and only if it contains all of its limit points.

Example1.4: The class τ ={ X , $\emptyset$, {a}, {c, d}, {a, c, d}, {b, c, d, e}} defines a topology on $X =\{$ a, b, c, d, e$\}$.Consider the subset $A =\{$ a, b, c$\}$ of X. Observe that b$\in X$ is a limit point of A since the open sets containing b are {b, c, d, e} and X, and each contains a point of A different from b, i.e. c. On the other hand, the point a$\in X$ is not a limit point of A since the open set {a} which contains a, does not contain a point of A different from a. Similarly, the points d and e are limit points of A and the point c is not a limit point of A.

So $A' = \{b, d, e\}$ is the limit point of A.

1.2.1.3 Closure of a Set

Definition 1.4: Let A be a subset of a topological space X. The closure of A, denoted by $\bar{A}$ is the intersection of all closed supersets of A. i.e. If $\{U_i, i \in I\}$ is the class of all closed subsets of X containing A, then $\bar{A}=\cap_i \cup_i$.

If X is a metric space, a point x$\in X$ is called a point of closure of the set A if for every $\delta > 0$ there is a point y$\in A$ such that p(x, y)$< \delta$. In other words, a point x$\in X$ is the point of closure of A if every open set containing x contains points of A. From the definition, we observe that $\bar{A}$ is closed since it is the intersection of closed sets. Furthermore, $\bar{A}$ is the smallest closed set containing A, that is if B is a closed set containing A, then A$\subseteq \bar{A} \subseteq B$.

Accordingly, a set A is closed if and only if A=$\bar{A}$

1.2.1.4 Hausdorff Space

Definition 1.5: A topological space X is said to be a Hausdorff Space if for each distinct pairs x, y$\in$ X there exist two disjoint open set U and V such that x$\in$U and y$\in$ V.

Example1.5: Let X= {a, b, c,} and

$\tau_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$ is a Hausedorff Space.

1.2.1.5 Open Cover of Set

Definition1.6: We say that a collection $\mathfrak{C}$ of sets cover a set F if $F \subset \cup \{O: O \in \mathfrak{C}\}$. In this case the collection $\mathfrak{C}$ is called a covering of F. If each $O \in \mathfrak{C}$ is open, we call $\mathfrak{C}$ is an open covering of F.

We now state a theorem which is fundamental in basic topology and will be necessary for what it follow.

Theorem1.1: If X is a compact Hausedorff Space and $\{O_x\}_{x\in X}$ is an open covering of X, then there exists a finite collection $\{O_{x_i}\}_{i=1}^{n}$ which also covers X. That is, for each x in X there exist an open set in $\{O_{x_i}\}_{i=1}^{n}$ such that x is in O_{x_i}.

1.2.1.6 Compact Set

The concept of compactness is no doubt motivated by the properties of closed and bounded intervals as stated in the classical Hein-borel theorem.

Definition 1.7: A subset A of a topological space X is compact if every open cover of A has a finite sub cover.

In other words, if A is compact and A$\subseteq \cup_i U_i$, where the U_i are open sets, then one can select a finite number of the open sets say $U_1, U_2, \dots, U_m$ so that A$\subseteq U_1 \cup U_2 \cup, \dots, \cup U_{i_m}$.

Example 1.6: By the Hein-borel, every closed and bounded interval [a, b] is reducible to a finite cover

1.2.2 Metric Space

Definition1.8: Let X be a non empty set. A real valued function ρ defined on $\rho: X \times X \rightarrow R$, i.e. ordered pairs of elements in X, is called a metric on X. If it satisfies for a, b, c ∈X the following axioms

1. ρ(a, b)≥ 0 and ρ(a, a)=0
2. ρ(a, b)= ρ(b, a)
3. ρ(a, c)≤ $\rho(a, b) + \rho(b, c)$

A space X is said to be metrizable if there exists a metric ρ on X

Example 1.7: Let X=C [0, 1] the set of all continuous function from [0, 1] to R. Then show that the function p defined by $\rho(f,g) = \underset{x\in[0,1]}{Sup}|f(x) - g(x)|$ where f, g, h $\in X$ is a metric on X.

Solution:

Let f, g, h ∈X. Then

1. $\rho(f,g) = \underset{x\in[0,1]}{Sup}|f(x) - g(x)| \geq 0$ and $\rho(f,f) = \underset{x\in[0,1]}{Sup}|f(x) - f(x)| = 0$

2. $\rho(f,g) = \underset{x\in[0,1]}{Sup}|f(x) - g(x)| = \underset{x\in[0,1]}{Sup}|g(x) - f(x)| = \rho(g,f)$

3. $\rho(f,g) = \underset{x\in[0,1]}{Sup}|f(x) - g(x)| = \underset{x\in[0,1]}{Sup}|f(x) - h(x) + h(x) - g(x)|$

$$\leq \underset{x\in[0,1]}{Sup}|f(x) - g(x)| + \underset{x\in[0'1]}{Sup}|h(x) - g(x)|$$

$$\leq \rho(f,g) + \rho(h,g)$$

From (1), (2), (3), we observe that p satisfies all the conditions in the definitions of metric space.

Therefore, p is a metric on X.

1.2.3 Maximum and Minimum

Definition1.9: Let a *and* b are real numbers. Then

i. The maximum of a and b is

$$\text{Max (a, b)} = a \vee b = \begin{cases} a, if\ a \geq b \\ b, if\ b \geq a \end{cases}$$

ii. The minimum of a and b is

$$\text{Min (a, b)} = a \wedge b = \begin{cases} a, if\ a \leq b \\ b, if\ b \leq a \end{cases}$$

Definition1.10: Let f and g are two real valued functions. Then

i. The maximum of f and g is

$$\text{Max (f, g) (t)} = (f \vee g)(t) = \begin{cases} f(t), when\ ever\ f(t) \geq g(t) \\ g(t), when\ ever\ g(t) \geq f(t) \end{cases}$$

ii. The minimum of f and g is

$$\text{Min (f, g) (t)} = (f \wedge g)(t) = \begin{cases} f(t), whenever\ f(t) \leq g(t) \\ g(t), when\ ever\ g(t) \leq f(t) \end{cases}$$

Example 1.8: Let f (t) = t and g (t) $=t^2$. Then

$$(f \vee g)(t) = \begin{cases} t^2, if\ -\infty < t \leq 0 \\ t, if\ 0 \leq t \leq 1 \\ t^2\ if\ 1 \leq t < \infty \end{cases}$$

$$(f \wedge g)(t) = \begin{cases} t, if\ -\infty < t \leq 0 \\ t^2,\ if\ 0 \leq t \leq 1 \\ t\ if, 1 \leq t < \infty \end{cases}$$

1.2.4 Linear Space

Definition1.11: A space X of a real valued function is called a linear space (or vector space) if it has a property that $\alpha f + \beta g$ belongs to X for each pair f and g belongs to X for each constants α and β.

Example 1.9: The set C (X, R) is a linear space since any constant multiple of continuous real valued function is continuous.

1.2.4.1 Normed Linear Space

Definition1.12: A non negative real valued function ||. || defined on a linear space X is said to be a norm if for all x, y$\in X$

i. $\|x\| \geq 0$

ii. $\|x\| = 0 \ iff \ x = 0$

iii. $\|x + y\| \leq \|x\| + \|y\|$

iv. $\|kx\| = |k|\|x\|, k \in R$

A linear space X together with a norm is celled normed linear space or simply normed space. The real number is called the norm of the vector x.

Example 1.10: Let X=C [0, 1] the set of all continuous function on [0, 1]. Suppose that for each f$\in X$, define $\|f\| = \underset{x\in[0,1]}{Sup}|f(x)|$. Then show that X is a normed space.

Solution:

Let f, g$\in X$ and let $\alpha, \beta \in R$. Then $(\alpha f + \beta g)(x)$= $\alpha f(x) + \beta g(x)$ for all x$\in$ $[0, 1]$. Thus, X is a linear space. Now, let f, g$\in X$, then

i. $\|f\| = \underset{x\in[0,1]}{Sup}|f(x)| \geq 0$

ii. $f = 0 \Leftrightarrow f(x) = 0$ for all $x \in [0,1] \Leftrightarrow \underset{x\in[0,1]}{Sup}|f(x)| = 0 \Leftrightarrow \|f\| = 0$

iii. $\|f + g\| = \underset{x\in[0,1]}{Sup}|(f + g)(x)| \leq \underset{x\in[0,1]}{Sup}|f(x)| + \underset{x\in[0,1]}{Sup}|g(x)|$

$$= \|f\| + \|g\|$$

iv. $\|kf\| = \underset{x\in[0,1]}{Sup}|kf| = |k|\underset{x\in[0,1]}{Sup}|f(x)| = |k|\|f\|$

From (i), (ii), (iii), and (iv), X is a normed space. Since X is a linear space, X is normed linear space.

Therefore, X is a normed linear space.

Theorem1.2. In every normed space X the norm $\|.\|: X \to R$ is uniformly continuous.

Proof:

Let $x, y \in X$. Then given $\varepsilon > 0$ there exist $\delta > 0$ such that $|\|x\| - \|y\|| < \delta$ whenever

$\|x - y\| < \delta$.

Now $|\|x\| - \|y\|| \leq \|x - y\| < \delta$. Chose$\delta = \varepsilon$. Hence $\|.\|$ is continuous.

1.2.4.2 Boundedness of Function

Definition1.13: A function f on a normed linear space X is said to be bounded if $\|f(x)\| \leq \|f\| \|x\|$ for all x∈X

Example 1.11 Let f is a bounded linear function on a normed space X. Then show that X is continuous.

Solution:

Let f be a bounded linear function on a normed space X. Given$\varepsilon > 0$, there is $a\ \delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$, whenever $\|x - y\| < \delta$.

Now, $\|f(x) - f(y)\| = \|f(x - y)\| \leq \|f\| \|x - y\| < \|f\|\delta$. So we chose $\delta = \frac{\varepsilon}{\|f\|}$, f≠ 0

If f=0, then the zero function is continuous. Therefore f is continuous.

Lemma 1 $\sqrt{1-t}$, has a power series representation $\sqrt{1-t}$ = 1-$\sum_{n=1}^{\infty} a_n t^n$, a_n=$\frac{(2n-2)!}{2^{2n-1}(n-1)!n!}$, for n∈ N which is uniformly convergent on the interval -1≤ $t \leq 1$.

Proof: Observe that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$=$\lim_{n\to\infty} \frac{2(n-1)}{2n+1}$=1

Hence, by the ratio test the series converges point wise for t ∈(-1, 1), we will show the series does indeed converge to $\sqrt{1-t}$.

Define $\varphi(t)$= 1-$\sum_{n=1}^{\infty} n a_n t^n$. So, $\varphi(t)$ converges for t∈(-1, 1). Similarly we obtain

$\frac{d\varphi(t)}{dt}$=-$\sum_{n=1}^{\infty} n a_n t^n$ multiplying both sides by -2(1-t) we obtain

$$\frac{d\varphi(t)}{dt}(-2(1-t)) = -2(1-t)\left(-\sum_{n=1}^{\infty} na_n t^n\right)$$

$$= \sum_{n=1}^{\infty} 2na_n t^{n-1} - \sum_{n=1}^{\infty} 2na_n t^n$$

$$= 2a_1 - 2a_1 t + 4a_2 t^2 + 6a_3 t^2 - 6a_3 t^3 + \cdots$$

$$= 2a_1 - (2a_1 - 4a_2)\,t - (4a_2 - 6a_3)t^2 - \ldots$$

$$= 1 - \frac{1}{2}t - \frac{1}{8}t^2 - \ldots$$

$$= 1 - \sum_{n=1}^{\infty} a_n t^n$$

$$= \varphi(t)$$

Therefore, $\frac{d\varphi(t)}{dt}(-2(1-t)) = \varphi(t)$.

Then, $-\frac{1}{2}\int\frac{dt}{1-t} = \int\frac{d\varphi}{\varphi} \Rightarrow \varphi(t) = c\sqrt{1-t}$ for some $c \in R$. Evaluating both sides at t=0 gives c=1, Hence, $\varphi(t) = c\sqrt{1-t}$ point wise for t$\in [0,1)$. Now we must show that $\varphi(1) = 0$.

Stirling's inequality states

$$e^{\frac{7}{8}-n}n^{n+\frac{1}{2}} < n! < e^{1-n}n^{n-\frac{1}{2}}$$

Hence, for n≥ 2

$$a_n < 2^{1-2n}\frac{(2n-2)^{2n-\frac{3}{2}}}{e^{\frac{7}{4}}n^{n+\frac{1}{2}}(n-1)^{n-\frac{1}{2}}} < \frac{1}{\sqrt{2}e^{\frac{7}{4}}}\frac{(n-1)^{2n-\frac{3}{2}}}{(n-1)^{2n}}$$

$$= \frac{1}{\sqrt{2}e^{\frac{7}{4}}}\frac{1}{(n-1)^{\frac{3}{2}}}$$

So, $\sum_{n=1}^{n} a_n = \frac{1}{2} + \frac{1}{\sqrt{2}e^{\frac{7}{4}}}\sum_{n=2}^{\infty}\frac{1}{(n-2)^{\frac{3}{2}}} < \infty$

Where comparison test was used in the first inequality and p-series test in the second. Since $(\sum_{n=1}^{k} a_n)_{k\geq 1}$ is monotonically increasing and bounded above, $\varphi(t)$ exists. Then by Abel's theorem,

$$\varphi(1) = 1 - \sum_{n=1}^{\infty} a_n = \lim_{n\to 1^-}\sum_{n=1}^{\infty} a_n t^n = \lim_{n\to 1^-}\sqrt{1-t} = 0$$

Hence again using Abel's theorem

$\varphi(t) = \sqrt{1-t}$ uniform for t$\in$ [0, 1].

CHAPTER TWO

THE STONE WEIERSTRASS THEOREM

Under this chapter, we concentrate on certain aspects of Lattice, Separation of points, Algebra of function, the Ston-Weirestrass theorem and its application.

2.1 Lattice of functions

Definition2.1: A collection $\mathcal{L}$ of real valued functions on a set X is called a lattice if it is closed under maximum and minimum. That is

i. $f, g \in \mathcal{L} \Rightarrow f \vee g \in \mathcal{L}$.
ii. $f, g \in \mathcal{L} \Rightarrow f \wedge g \in \mathcal{L}$.

Example 2.1: The following are lattice of functions contained in C ([0, 1], R):

i. C([0, 1], R) itself
ii. The collection of non negative continuous function on [0, 1]

Proposition1. Let $\mathcal{L}$ is a lattice of continuous real-valued functions on X where X is compact. Suppose the function h defined by $h(x) = \inf_{x \in \mathcal{L}} f(x)$ is continuous and finite, then given any $\varepsilon > 0$, there is $g \in \mathcal{L}$ such that $0 \le g(x) - h(x) \le \varepsilon$ for all $x \in X$.

Proof: Let $\varepsilon > 0$ and let $x \in X$. By definition of h, there is $f_x \in \mathcal{L}$ such that

$0 \le f_x(x) - h(x) < \frac{\varepsilon}{3}$. But both f_x and h are continuous. So there is an open set O_x containing x such that if $y \in O_x$, then $|f_x(x) - f_x(y)| < \frac{\varepsilon}{3}$ and $|h(x) - h(y)| < \frac{\varepsilon}{3}$. Hence for each $y \in O_x$, we have

$$0 \le f_x(y) - h(y) = |f_x(y) - h(y)| \le |f_x(x) - f_x(y)| + |f_x(x) - h(x)| + |h(x) - h(y)| < \varepsilon.$$

We have for each $x \in X$ the associated f_x and O_x. So $\{O_x\}_{x \in X}$ is an open covering of X. By compactness, there is a finite sub covering $\{O_{x_1}, O_{x_2}, \dots, O_{x_N}\}$ of X. Let $g = f_{x_1} \wedge f_{x_2} \dots \wedge f_{x_N}$. So $g \in \mathcal{L}$ and we know that for any $y \in X$, there is O_x containing y such that

$0 \le g(y) - h(y) \le f_x(x) - h(y) < \varepsilon$. As defined.

Remark

1. In the proof of **proposition2** we will first have continuous function h and then show that h can be acquired by the above means (h is the point infimum of a sub lattice) so as desired approximation property.

2.2. Separation of Points

Definition 2.2: A collection F of functions on a set X is said to be separate points of X, if whenever x and y are disjoint elements of X, there is an f∈ F such that f(x) $\neq f(y)$.

Examples 2.2:

i. If F contains a one-to-one function, then it separate points.
ii. The collection of function $\{\sin x, \cos x\}$ separate points of [0, 2π), but it does not separate points of [0, 2π].

Proposition2. Let X is a compact Haousdorff Space and $\mathcal{L}$ a lattice of continuous real-valued functions on X such that it has the following two properties;

i. $\mathcal{L}$ separate points
ii. For any $c \in R$ and $f \in \mathcal{L}$ we have $c + f \in \mathcal{L}$ and $cf \in \mathcal{L}$. Then for any $h \in C(X, R)$ and any $\varepsilon > 0$, there is $g \in \mathcal{L}$ such that for all $x \in X$, $0 \leq g(x) - h(x) < \varepsilon$ and thus $\bar{\mathcal{L}} = C(X, R)$.

To prove **propostion2** we prove the following two lemmas first.

Lemma2. Let $\mathcal{L}$ be a lattice of continuous real-valued functions on X. If f has property (i) and (ii), then given any $a, b \in R$ and distinct $x, y \in R$, there is $f \in \mathcal{L}$ such that $f(x) = a$ and $f(y) = b$.

Proof: Since $\mathcal{L}$ separate points, there is $g \in \mathcal{L}$ such that $g(x) - g(y) \neq 0$ so f is defined by

$$f(t) = \frac{a(g(t)-g(y))+b(g(x)-g(t))}{g(x)-g(y)}$$

Is well-defined and $f \in \mathcal{L}$ by property (i) and (ii). Notice that $f(x) = a$ and $f(y) = b$

Lemma3. Let $\mathcal{L}$ be a lattice of continuous real-valued functions on X where X is compact and $\mathcal{L}$ satisfies (i) and (ii). Then given $a, b \in R$ with $a \leq b$, F a closed subset of X and $p \notin F$, there is a function $f \in \mathcal{L}$ such that $f \geq a$, $f(p) = a$ and $f(x) > b$ for all $x \in F$.

Proof: For any $x \in F$, $x \neq p$. So by **Lemma2**, there is $f_x \in \mathcal{L}$ such that $f_x(p) = a$ and

$f_x(x) = b + 1$. Notice that since f_x is continuous, there is an open set O_x containing x such that for all $y \in O_x$, $f_x(y) - f_x(x) \geq -\frac{1}{2}$ and hence $f_x(y) \geq f_x(x) - \frac{1}{2} = b + \frac{1}{2} > b$.For each $x \in F$, let O_x and f_x be the associated open set and function satisfying the previous condition. Then $\{O_x\}_{x \in F}$ cover F. But F is compact being a closed subset of a compact set. So there is a finite sub covering $\{O_{x_1}, O_{x_2}, \dots, O_{x_N}\}$ of F. Let $g = f_{x_1} \vee f_{x_2} \vee \dots \vee f_{x_N}$. So $g(p) = a$ and for any $y \in F$, there is O_x containing y so that $g(y) \geq f_x(y) > b$. Finally, let $f = a \vee b$ and this is the desired function ($f \geq a, f(p) = a, f > b$ on F.)

Proof of propostion3

We want to show that $\bar{\mathcal{L}} = C(X, R)$. Given any $h \in C(X, R)$, let $\mathcal{L} = \{f \in \mathcal{L}: f \geq h\}$. If we can show for any $p \in X$, $h = \inf\{m, h(p) + \varepsilon\} \geq h(p) + \epsilon$. Let $F = \{x \in X: h(x) \geq h(p) + \varepsilon\}$.

So $p \notin F$ and F is a closed because h is continues. Thus, by **Lemma3**, there is $f \in \mathcal{L}$ such that $f(p) = h(p) + \varepsilon$, $f \geq h(p) + \varepsilon$ and $f > m$ and thus $f > h$ on F. Notice that on F^c,

$h < h(p) + \varepsilon \leq f$. Consequently, $f > h$. Hence $f \in \mathcal{L}$. Also because $0 \leq f - h = \varepsilon \leq \varepsilon$ and ε is arbitrarily, $h(p) = \inf_{f \in \mathcal{L}} f(p)$, as desired.

2.3 Algebra of Functions

Central to the Stone-Werstrass theorem is the concept of an algebra which have already briefly encountered. We define it as follow.

A collection A of a real valued function or complex functions on a set X is called an algebra, if it is closed under addition, scalar multiplication, and multiplication. That is, if $f, g \in A$ and α is a scalar, then

i. $f + g \epsilon A$.
ii. $\alpha f \epsilon A$.
iii. $fg \epsilon A$.

Example 2.3.Let X is a topological space. Then the collection C (X, R) of real-valued continuous function on X is an algebra of functions.

Example 2.4 Let X is a topological space. Then the collection C (X) of complex-valued continuous function on X is an algebra of functions

Let A be an algebra of bounded function on [0, 1].we can consider the collection of all functions f: [0, 1] which are uniform limits of sequences $\{f_i\}$, where f_i ∈f .Of course each element f of A is a uniform limit of a sequence of function in A –just take the constant sequence f_i=f for all i. We aspire to get more however; recall that what we want to show is that every continuous function arises as a uniform limit of sequence of function in p [0, 1], so why not consider more generally the set $\bar{A}$ of all uniform limits of functions in our algebra A? The following result gives some indication that might be a sensible to do.

Proposition3 For any algebra A of bounded functions on [0, 1], $\bar{A}$ is also an algebra of bounded functions on [0, 1].

Proof: Let $a, b \in R$, $f, g \in \bar{A}$ and f_n, g_n in A such that $f_n \to f$ and $g_n \to g$ (uniformly).

Then $af_n + bg_n \to af + bg$. Hence, $af + bg \in \bar{A}$ which shows $\bar{A}$ is linear space. Furthermore, since $f_n g_n \to fg$.

$\therefore$ There for $\bar{A}$ is an algebra.

We call $\bar{A}$ the uniform closure of A. Thus we can restate the Weierstrass approximation theorem as: show that the uniform closure of the algebra of polynomial functions on [0, 1] is the algebra of all continues function on [0, 1].

2.3.1 Sub Algebra

Definition2.3: A set S is sub algebra of the algebra A if

i. S is a linear sub space of A
ii. S is closed under the operation$\times$.

Now we also the following question: Given a subset T of C (X, R) we consider the set

(In C (X, R)) generated by taking sums, scalar products, products, uniform limits of elements of T. where is the class is this class of generated equals to C (X, R)?

The Stone–Weirstrass theorem provides answer to this question.

Theorem 2.1 Let S be a sub algebra of C(X, R) containing f$\equiv$ 1. Then if f and g are in $\bar{s}$, so are $|f|$, $f \vee g$, and f$\wedge$g.

Proof: Let f be in$\bar{s}$. Then since$\|f\| = \sup_{x \in X} |f(x)|$, $|f(x)| \leq \|x\|$, for all x$\in X$.Let us assume temporarily that $\|f\| \leq 1$ since this can always obtained by multiplying f by $\frac{1}{\|f\|}$, when f has a greater norm. Now, let ε>0. There exist N_ε such that

$$\left|\sqrt{1-t} - (1 - \textstyle\sum_1^\infty \frac{(2n-2)!t^n}{2^{2n-1}(n-1)!n!})\right| < \varepsilon \text{ When m} \geq N_\varepsilon, \text{ for all t in } -1 \leq t \leq 1.$$

Thus letting t= $(1 - f^2)$

$$\left|\sqrt{1-(1-f^2)} - (1 - \textstyle\sum_1^m \frac{(2n-2)!(1-f^2)^n}{2^{2n-1}(n-1)!n!})\right| < \varepsilon \text{ for all x inX.}$$

Since 1-$f^2 \leq$ 1, and $|f| = \sqrt{1-(1-f^2)}$

$$|f| = 1 - \textstyle\sum_1^\infty \frac{(2n-2)!(1-f^2)^n}{2^{2n-1}(n-1)!n!}, \text{ which is in } \bar{s}.$$

When $\|f\| >$ 1 we replace f by $\frac{f}{\|f\|}$ and then we multiply by $\|f\|$ to find

$$|f| = \|f\|[1 - \sum_1^\infty \frac{(2n-2)!(1-[\frac{f}{\|f\|}]^2)^n}{2^{2n-1}(n-1)!n!}]$$ Which is in $\bar{s}$. Since,

$$fVg = \frac{f+g+|f-g|}{2} \text{ And } f\Lambda g = \frac{f+g-|f-g|}{2}$$

Thus fVg and fΛg are also in $\bar{s}$.

Before we prove the Stone–Weirstrass theorem, we still have an important polynomial approximation lemma. It builds a connection from algebras to lattices.

Lemma4. Given ε>0, there is a polynomial p in one variable such that for all $t \in [-1, 1]$ we have $\big||t| - p(t)\big| < \varepsilon$.

Theorem 2.2: (Ston-Weirestrass theorem) Let X is a Compact Hausdorff Space. Suppose that $A \subset C(X, R)$ satisfies the following conditions:

i. A is an algebra.
ii. A separate point of X.
iii. 1$\in$A.

Then $\bar{A} = C(X, R)$

Proof: Since A is an algebra of continuous real valued function, then by **proposition 3** $\bar{A}$ is also an algebra of continuous real valued function on X. If we can prove that $\bar{A}$ is also a lattice, then the verification will be complete on **Propostion2**.

We first note that

$$fVg = \frac{f+g+|f-g|}{2} \text{ And } f\Lambda g = \frac{f+g-|f-g|}{2}$$. Thus to proof $\bar{A}$ is a lattice, it suffices to show that

$$f \in \bar{A} \Rightarrow |f| \in \bar{A}. \qquad (1)$$

Let $f \in \bar{A}$, if $f = 0$, then $|f| = 0 \in \bar{A}$, if not, $\|f\| \neq 0$, consider g=$\frac{f}{\|f\|}$ and observe that $g \in \bar{A}$. By **Lemma 4**, for any $\varepsilon > 0$ there is a polynomial P such that $\big||t| - p(t)\big| < \varepsilon$ for all t $\in$[-1, 1]. Because the range of g is contained in [-1, 1], it follow that

$$\||g| - pog\| < \varepsilon.$$

And because pog is a polynomial in power of g and $\bar{A}$ is an algebra containing the constant functions, we conclude that $pog \in \bar{A}$. Thus, $|g| \in \bar{\bar{A}}$=$\bar{A}$. Finally, because f is a scalar multiple of g and $\bar{A}$ is an algebra, it follows that$|f| \in \bar{A}$.

∴ Therefore by **Proposition2** $\bar{A} = C(X, R)$.

2.4 Complex version of the stone –weirestrass theorem

If C (X, R) is replaced by C (X), the hypotheses of the Stone-Weirestrss theorem must be augmented in order to obtain analogous result.

Theorem 2.3 Stone –Weierstrass Theorem (Complex Version)

Let X be a Compact Hausdorff Space. Suppose that $A \subset C(X)$ satisfies the following conditions:

i. A is an algebra.
ii. A separate point of X.
iii. 1∈A.
iv. f∈ A⟹ $\bar{f}$ ∈A

Then $\bar{A}$ = C (X, R)

Proof: Let f be in A, then $Ref = \frac{f+\bar{f}}{2}$, $Imf = \frac{f+\bar{f}}{2i}$ are also in A.

Let ReA denote the set of real parts of functions in A. Then ReA separate points of X, since if $x_1, x_2 \in X$, there exist $f \in A$ such that $f(x_1) \neq f(x_2)$.

Thus either $Ref(x_1) \neq Ref(x_2)$ or $Imf(x_1) \neq IMf(x_2)$ and these are both in ReA and ReA also contains 1. Hence

$$\overline{ReA} = C(X, R) \text{ and } \overline{ReA} + \imath\overline{ReA} = C(X, R) + iC(X, R) = C(X, R)$$

∴ There fore $\bar{A}$ = C (X, R)

Example2.5 In the Euclidian space R^2

Let $X = \{(x, y): x^2 + y^2 = 1\}$ be the unit circle and consider $C(X)$. It is evident that $C(X)$ is equivalent to the space of complex valued continuous function with period 2π, each f *in* $C(X)$ can be represented by

$$f = \{f(t): 0 \leq t \leq 2\pi\}$$

Now, consider the sub algebra S generated by $1, e^{it}, e^{-it}$. If $t_1 \neq t_2$, then $e^{it_1} \neq e^{it_2}$. So, S separates points. Hence any complex valued continuous function with period 2π can be uniformly approximated by a trigonometric polynomial $\sum_{n=-N}^{n} c_n e^{int}$

Corollary2.3. Every continuous function on a closed bounded set X in R^n can be uniformly approximated on X by a polynomial (in coordinates).

Proof: The set of all polynomials in the coordinates $x_1, x_2, \ldots, x_n$ for some n$\in N$ forms an algebra containing the constants. It separate points, since the given distinct points in R^n, one of the coordinate functions takes different values on these points. Hence **theorem 2.2** can be applied.

Remark

i. The case in which n=1 in the above corollary is the Weierstrass approximation theorem

ii. A result similar to the Weierstrass approximation theorem occurs in the theory of Fourier series, and was also first proved by Weierstrass. It states that a continuous 2π periodic function can be uniformly approximated on R by the trigonometric polynomials.

2.5 Trigonometric Polynomials

Definition2.4: The space of real valued trigonometric polynomials TP (R, R) is functions $\varphi: R \to R$ which are finite sums of the form

$$\varphi(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$$

Example 2.6 Let f be a continuous real valued function on R with period 2π; that is,

$f(x + 2\pi) = f(x)$. Show that given $\varepsilon > 0$, there is a finite Fourier series, φ given by

$\varphi(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$, such that $|\varphi(x) - f(x)| < \varepsilon$ for all x.

Solution: Let A be the set of finite Fourier series φ given by

$$\varphi(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx), n\in N.$$

Then A is a linear subspace of function in C (X) where x is taken to be the unit circle in the set of complex numbers. From the trigonometric identities

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$$

We see that A is a sub algebra of C (x). Furthermore A separate points of X and contains the constant functions. By **The Stone-Weirstrass** theorem given any continuous periodic real valued function f on R with period 2π and $\varepsilon > 0$ there is a finite Fourier series φ such that

$|\varphi(x) - f(x)| < \varepsilon$ for all x.

Example 2.7: Let X be algebra of continuous real valued functions on a compact space X, and assume that A separates the points of X. Then either $\bar{A}$= C (X) or there is a point p∈ X and

$\bar{A} = \{f: f \in C(\Omega), f(p) = 0\}$.

Solution: let X be algebra of continuous real valued functions on a compact space X, and assumes that A separates the points of X. If for each x ∈ X, there is an f_x ∈A with $f_x(x) \neq 0$, then by continuity, there is an open neighborhood O_x of x such that f_x(y) ≠ 0 for y ∈ O_x .The set $\{O_x\}$ cover X. So by compactness finitely many of them cover X say$\{O_{x_1}, O_{x_2}, \dots, O_{x_n}\}$.

Let g= $f_{x_1}^2 + f_{x_2}^2 + \cdots + f_{x_n}^2$ then g ∈A and g ≠ 0 everywhere. The closure of the range of g is a compact set k not containing 0. The function h given by h (t) = $\frac{1}{t}$ for t ∈ k and h (0) =0 is continuous on k∪{0}. So it can be uniformly approximated by a polynomial h_n so that

$h_n o g \to \frac{1}{g}$. Note that if h_n is uniformly within $\frac{\varepsilon}{2}$ of h, then$|h_n(0)| < \varepsilon$, but $h_n(0)$is the constant terms of h_n so substituting the constant term results in a polynomial still within in ε of h. Thus we may assume that the polynomials h_n have no constant term. Thus $\frac{1}{g} \in \bar{A}$, so 1∈ $\bar{A}$ and $\bar{A}$ contains the constant functions. Hence $\bar{A} = C$ (X).

Example: 2.8: Let F be a family of continuous real valued function on a compact Hausdorff Space X, and suppose that F separates the points of X. Then every continuous real valued function on X can be uniformly approximated by a polynomial in a finite number of functions of F.

Solution: let F be a family of continuous real valued function on a compact Hausdorff Space X, and supposes that F separates the points of X. Let A be the set of polynomials in a finite number of functions of F. Then A is a sub algebra of C (X) that separates points X and contains the constant functions. By the **Stone-Weirstrass** theorem, A is dense in C (X). Hence every real valued function on X can be uniformly approximated by a polynomial in a finite number of functions of F.

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