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Autonomous Dynamical Systems and Stability Analysis

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A Project Report

On

Autonomous Dynamical Systems and Stability Analysis

By

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September, 2024

Bahir Dar, Ethiopia

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College of Science

Department of Mathematics

Autonomous Dynamical Systems and Stability Analysis

A project report submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of Masters of Science in Mathematics.

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Approval of the project for oral defense

I hereby certify that I have supervised, read and sumerized this project report entitled "Autonomous dynamical systems and stability analysis'' and prepered by Tadesse Akele under my guidance. I recommend that the project be submitted for oral defense.

Abdu Mohammed Seid (PhD)	____________________	____________________
Advisor name	signature	Date

Bahir Dar University

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We hereby certify that we have examined this project report entitled **“**Autonomous dynamical systems and stability analysis**”** by Tadesse Akele. We recommend that this project be approved for the degree of “Master of Science in Mathematics”.

Board of examiners

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External examiner’s name signature Date

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Internal examiner’s name signature Date

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Chairperson’s name signature Date

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Abstract

This project report is to explore the behavior and stability of autonomous dynamical systems, focusing on their theoretical and practical applications. The primary objective of this study is to analyze the stability properties and to develop methods for assessing their robustness aganst perturbations.The project involves the fundamental concepts in dynamical systems theory, including equillibrium points, Jacobean matrix, Lyapunov’s direct method, linearization, and bifurcation analysis.The results indicate that the proposed methods provide effective tools for predicting systm behavior and ensuring stability and some real-life applications including electric circuts, damped pendules and Lotka-Volterra or predator-prey equations, were covered thoroughly.

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CHAPTER ONE

1. Introduction

1.1. Background of the study

In our everyday lives we encounter various systems, known as the physical or dynamical systems. In dealing with the physical systems, we mainly focus upon those that behave in some desired fashion, i.e; whose behaviors we wish to be under control. These are known as the control systems. Generally, a set of differential or difference equations is used to formulate the mathematical model of a dynamical systems.

The theory of autonomous dynamical system is now well established after being studied intensively over the past years. In the system of differential equations,

$$\frac{dx_i}{dt} = F_i,\ i = 1,2,3, \ldots$$

If the function F_i depends on the variable t, then the system is called non - autonomous, but if it does not depend on t, the equation of the system is called an autonomous system of differential equations. Many physics facts, where time is typically believed to be the independent variable, are described as autonomous systems because it is thought that the laws of nature that govern how things work are the same for any point in the future. In mathematics, stability theory addresses the stability of solutions of differential equations and trajectories of dynamical systems under small perturbations of initial conditions.

Stability analysis is one of the basic problems in the field of systems, control, and signal processing.The goal of stability analysis of a time delay system is to determine the region in the delay parameter space at which the system is still stable. The stability analysis by the matrix method is a by product of research on the incremental nonlinear analysis of structure.Simo, J.C.,& T.S.R. (1978).

The Ruth-Hurwitz method is a mathematical test for analyzing the stability of a linear time invariant system. Autonomous systems operates in complex open-ended environments with a high level of independence, self-independence, and self-determination. We believe such systems have the potential to deeply impact society. This area of study will integrate faculty expertise from mechanical and aerospace engineering, electrical and computer engineering, and computer science. The problem of determining the stability region or region of attraction of a stable equilibrium point for a nonlinear autonomous dynamical system is an important one in many applications, such as electric power systems, economics, ecology, etc.

Stability theory plays a central role in system theory and engineering. For systems represented by state models, Stability is characterized by studying the asymptotic behavior of the state variables near a steady state. The analysis of whether the given system can reach steady state passing through the transits successfully is called stability analysis of the system.

The Jacobi stability of a dynamical system can be regarded as the robustness of the system to small perturbations of the whole trajectory[23].This is a very conveyable way of regarding the resistance of limit cycles to small perturbations of trajectories.

1.2. Statement of the problem

The study of autonomous dynamical systems and stability analysis of the corresponding governing equations is fundamental in applications. Related to this, detailed discussions on the issue aren't seen in applications locally. In this project, we aim to discuss autonomous dynamical system and stability analysis with some real life dynamical system applications.

1.3. Objectives

1.3.1. General Objective

The general objective of this project report is to explore and analyze the behavior of autonomous dynamical systems with a focus on understanding their stability characteristics. This involves studying the mathematical and computational frameworks that describe these systems and applying various stability analysis techniques to evaluate their behavior over time.

1.3.2. Specific Objectives

The specific objectives of this project

- State critical points and investigate classifications of critical points.
- Apply various stability analysis methods such as Lyapunov's direct method, linearization, and the fundamental eigenvalues matrix, to determine the stability of equilibrium points and trajectories in the systems.
- Apply the theoretical and computational techniques to real-world examples or to demonstrate the practical applications of stability analysis in autonomous dynamical systems.

1.4. Significance of the project

The significance of the study includes the following:

- To ensure the properties and stability analysis of an autonomous dynamical systems.
- To check autonomous dynamical systems and stability are important in various real-world life applications like population dynamics, electric circuits, pendulums, etc.

1.5. Scope of the project

This work focuses on discussing autonomous dynamical systems and stability analysis of the corresponding governing equations. The study is limited to the autonomous dynamical system only, and it does not consider non-autonomous dynamical systems.

CHAPTER TWO

2. Preliminary Studies

2.1. Stability of linear autonomous system

Definition2.1: The autonomous dynamical systems are mathematical models that describe the evolution of a system over time without external influences. Mathematically, autonomous dynamical systems can be represented by a set of ordinary differential equations of the form,

$$\frac{dx_i}{dt} = F_i(x_{1,}x_{2,} \_\_\_\_\_\_\_, x_n), \;\; i = 1,2,\dots,n \qquad 2.1$$

Where, x is a vector representing the state of the system, t is time and F_i is smooth function. Any systems of oridinary diferential equations that are autonomous or non-autonomous can be reduced by simple substitution to a dynamical systems.This implies, any systems of ODEs, wheather autonomous or non-autonomous can be handled with in the frame work of dynamical systems. Autonomous systems are alerady in the form of dynamical system, and no further reduction or substitution is required. On the other hand, non-autonomous systems can be transformed in to autonomous systems by using different methods like linerization.Thus, a two dimensional non-autonomous system

$$\frac{dx_i}{dt} = F_i\left(x_{1,}x_2, \dots t\right)$$

can be transformed into a three-dimension autonomous system simply by defining a new dependent variable $x_3 = t$. Any solution $x(t)$ of equation (2.1) is called trajectory or an integral curve of the system. If $x(t)$ is any trajectory and t_0 is any constant, then the curve $x(t - t_0)$ is also a trajectory of the same system because, by translating the t-axis, $dx/dt = d/d(t - t_0)$ is the trajectory $x = x(t - t_0)$ by considering to represent the same geometrical trajectory as $x = x(t)$.

2.1.1. Equilibrium point of a dynamical system

In a dynamical system, an equilibrium point (steady state) is a condition where the system remains constant over time, meaning that the state variables do not change.

Defnition2.2: (Critical points): Consider the autonomous system

$$y'(t) = f(y(t))$$

If we say that c is a critical point or equilibrium point of the system and constant solution of the system then, $f(c) = 0$.

For instance, the critical point of the linear system $y'(t) = Ay(t)$ are the constant vector c that satisfies $c = 0$. When A is invertible, the only critical point is thus the origin $c = 0$.

As another example, consider 2×2 non-linear system

$$x'(t) = 1 - y(t), \quad y'(t) = 4 - x(t)^2$$

Then the critical points are the point $(x_{0,}\, y_o)$ that satisfy $1 - y_o = 0, 4 - {x_0}^2 = 0$

Thus, it easily follows that the only critical points are $(\pm 2,1)$.

Definition2.3: Consider a system of second order differential equations:

$$\frac{d^2 x^i}{dt^2} = G(x,y), where\ i = 1,2,3, \dots, n$$

Here, $x^i(t)$ are the components of a vector function $x(t)$ in n-dimensional space, and $G(x,y)$ are smooth functions that depend on the variables x and y. If the real part of all eigenvalues of $J(x,y)$ are strictly negative, the system is considerd Jacobi stable. This means the small perturbations will decay over time and the system will return to its original tragectory. Conversly, if any eigenvalues has a non-negative real part, the system is Jacobi unstable. This means that small perturbations can grow leading to diverging trajectories. Jacobi stability captures how perturbations evolve in time. This definition provides a criteria for stability based on the behavior of small perturbations around the trajectory, analyzed through the eigenvalues.

Consider thes ystem of ODE

$$\begin{cases} \frac{du}{dt} = f(u,v) \\ \frac{dv}{dt} = g(u,v) \end{cases} \qquad 2.2$$

such that the point $(0,0)$ is a fixed point, $[i.e. f(u,v) = g(u,v) = 0]$. In general, even though the fixed point is $(u_{0\,,}v_0)$, by a change of variables $\bar{U} = u - u_0$, $\bar{V} = v - v_0$ one can obtain in most of the cases $(0,0)$ as a fixed point. We denote by J the Jacobean matrix

$J(u,v) = \begin{pmatrix} f_u\, f_v \\ g_u\, g_v \end{pmatrix}$, where the subscripts indicate partial derivatives with respect to u and v.

The characteristic equation is $\lambda^2 - \lambda\{tr\}(A) + detA = 0$. where $\{tr\}(A)$ the trace and detA is the determinant of the matrix. From the characteristic equation, one can determine the eigenvalues and investigate the stabilityof the system around the fixed point or the equilibrium.

Theorem2.1: Through each given point $X^0 \in R^n$, there passes a unique geometrical trajectory of dynamical system given in equation (2.2). Consequently, no two distinct geometrical trajectories can intersect at any point except the point where,

$$F_1 = F_2 = \cdots = F_n = 0$$

Proof: We prove the two dimensional case. The theorem implies that distinct orbits cannot cross each other. Suppose we have two orbits for the system above that start from distinct points but latter cross at a point, $(u_{0,}v_0)$. If they cross each other, then the tangent T_1 to the first curve does not have the same slope at $(u_{0,}v_0)$ as the tangent T_2, to the second curve. Then if we denote by $m(T_k)$, the slope of T_k, k= 1, 2, at the point $(u_{0,}v_0)$ where the curves intersect, then;

$$m(T_1) = \frac{dv}{du}]_{C_1} = \frac{V'(t)}{U'(t)}|_{C_1} = \frac{F(U_o,V_0)}{G(U_o,V_0)} = \frac{V'(t)}{U'(t)}|_{C_2} = \frac{dv}{du}]_{C_2} = m(T_2).$$

But this says that since both curves are orbits, their slopes are determined exclusively by the values of $F(u_{0,}v_0)$ and $G(u_{0,}v_0)$. But the slopes at $(u_{0,}v_0)$ cannot be different, which is to say, the curves cannot intersect.

Defnition2.4: The system of the form,

$$\begin{cases} \frac{dx}{dt} = ax + by + p = U(x,y) \\ \frac{dy}{dt} = cx + dy + q = V(x,y) \end{cases} \tag{2.3}$$

where a, b, c, d, p and q are constants.

Note that: The point (x_0, y_0) where $U(x_0, y_0) = V(x_0, y_0) = 0$ is called critical or stationary or equilibrium point (which is of course constant solution of the system), and the curves $U(x,y) = 0$ and $V(x,y) = 0$ are called isoclines.When the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ vanishes, system (2.3) has a unique equilibrium point(x_0, y_0) where,

$$\begin{cases} ax_0 + by_0 = -p \\ cx_0 + dy_0 = -q \end{cases}$$

By the translation of axes, i.e. $x' = x - x_0$ And $y' = y - y_0$ we can place the equilibrium point at the origin and obtain the following homogeneous linear system

$$\begin{cases} \frac{dx}{dt} = ax + by = F(x,y) \\ \frac{dy}{dt} = cx + dy = G(x,y) \end{cases} \tag{2.4}$$

To discuss the behavior of the trajectories of the linear system [eqn. (2.4)] for the case when it has only one equilibrium point (0,0) let us determine the solutions $x(t)$ & $y(t)$ of the system.

Suppose $X(t) = \partial e^{\lambda t}$ and $Y(t) = \beta e^{\lambda t}$ are solutions of (2.4), then we have,

$$X' = \partial\lambda e^{\lambda t} = a\partial e^{\lambda t} + b\beta e^{\lambda t}$$

$$Y' = \beta\lambda e^{\lambda t} = c\partial e^{\lambda t} + d\beta e^{\lambda t}$$

Dividing both sides by $e^{\lambda t}$ us

$$\begin{cases} (\lambda - a)\partial - b\beta = 0 \\ -c\partial + (\lambda - d)\beta = 0 \end{cases}$$

This implies that;

$$\begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix}\begin{pmatrix} \partial \\ \beta \end{pmatrix} = 0$$

If neither $\partial \neq 0$ nor $\beta \neq 0$ then;

$$\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = 0$$

where, ∂ and β have to be found. If λ_1 and λ_2 are the roots of the equation;

$$Det(A - \lambda I) = \begin{vmatrix} a - \lambda & -b \\ -c & d - \lambda \end{vmatrix} = \lambda^2 - (a + d)\lambda + ad - bc = 0$$

Thus, the solutions are;

$$\begin{cases} X_1(t) = \partial_1 e^{\lambda_1 t} \\ X_2(t) = \partial_2 e^{\lambda_2 t} \end{cases} \text{And} \begin{cases} Y_1(t) = \beta_1 e^{\lambda_1 t} \\ Y_2(t) = \beta_2 e^{\lambda_2 t} \end{cases}$$

■ If $x_1(t)$ and $x_2(t)$ are linearly independent solutions of the system, then the combination $x_1(t)$ and $x_2(t)$ are also a solution. The general solution is given by as follows:

Case1: If λ_1 and $\lambda_2 \epsilon R$, then

$$X(t) = \partial_1 e^{\lambda_1 t} + \partial_2 e^{\lambda_2 t} \text{ and } Y(t) = \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t}$$

Case2: If $\lambda_1 = \lambda_2 \epsilon R$ then;

$$X(t) = \partial_1 e^{\lambda_1 t} + (\partial_2 + \partial_3 \mathrm{t})\, e^{\lambda_1 t} \text{ and } Y(t) = \beta_1 e^{\lambda_1 t} + (\beta_2 + \beta_3 \mathrm{t})\, e^{\lambda_1 t}$$

Case3: If λ_1 and $\lambda_2 \epsilon \mathbb{C}$, and $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ then;

$$X(t) = C_1 e^{at}(A_1 \cos bt - A_2 \sin bt) + C_2 e^{at}(A_1 \cos bt + A_2 \sin bt)$$

$$Y(t) = C_1 e^{at}(B_1 \cos bt - B_2 \sin bt) + C_2 e^{at}(B_1 \cos bt + B_2 \sin bt)$$

Note that: The path of the trajectories of the system is the solution of the differential equation:

$$\frac{dy}{dx} = \frac{G\,(x,y)}{F\,(x,y)}$$

And the phase plane is the xy-plane. The vector field $[F(x,y), G(x,y)]$ indicates which direction along the trajectory to travel in order to "go with the flow" described by the system.

Defnition2.5: If $\forall \varepsilon > 0, \exists \delta > 0$, and $|x_0 - x^*| < \delta \Rightarrow |x(t) - x^*| < \varepsilon, \forall t \geq t_0$, then the point x^* is called stable point. Otherwise it is unstable.If it is stable and $x(t) \rightarrow x^*$ as t$\rightarrow \infty$, then it is called asymptotically stable. Now let us see some examples of dynamical systems by starting where A is not singular $[i.e. |A| \neq 0]$.

Example: Consider the system

$$\begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dy}{dt} = by \end{cases} \tag{2.5}$$

Since the matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $det(A - \lambda I) = \begin{vmatrix} a-\lambda & 0 \\ 0 & b-\lambda \end{vmatrix} = (\mathrm{a} - \lambda)\,(b - \lambda) = 0$

Which implies that $\lambda_1 = a$ and $\lambda_2 = b$ are the eigenvalues. Then the general solutions are $x(t) = c_1 e^{at}$ and $y(t) = c_2 e^{bt}$

The equation of the path of the trajectories of the system (2.5) is $y = cx^{b/a}$, where c is constant related to c_1 and c_2 and $[0,0]$ is the equilibrium point. The trajectories are defined for all t; they move forward infinitely as $t \to \infty$. the equilibrium point $[0,0]$ is called source node. In case the direction along the trajectories is reversed and tends to the origin and hence the origin is called sink node.

2.1.2. Classifications of critical points using eigenvalues

Equilibrium points or critical points can be classified by considering the signs of eigenvalues of the linear system. There are six types of critical points which are described below.

- **Improper node (stable node):**
  - Eigenvalues: All real and negative.
  - Behavior: The system returns to equilibrium after any small disturbance.
  - Stability: Asymptotically stable.

Example: Let us consider the autonomous system $X' = AX$ where, $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$

- Eigenvalues: $\lambda_1 = -2$ and $\lambda_2 = -4$ (both real and negative).
- Corresponding eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- Solution: $x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$
- Interpretation: The origin is an improper node (stable node) and asymptotically stable.

- **Proper node (unstable node):**
  - Eigenvalues: All real and positive.
  - Behavior: The system moves away from equilibrium after a small disturbance.

- Stability: unstable.

Example: Let us consider the system $x' = Ax$, where, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Eigenvalues: $\lambda_1 \ = 1$ and $\lambda_2 = 1$ (both real and positive)
- Corresponding eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Solution: $x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$
- Interpretation: The origin is a proper node (unstable node).

➢ **Saddle point:**

- Eigenvalues: All real, with one positive and one negative.
- Behavior: The system exhibits instability in the direction of the positive eigenvalues and stability in the direction of the negative eigenvalues.
- Stability: always unstable

Example: Let us consider the system $X' = Ax$

Where, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- Eigenvalues: $\lambda_1 \ = 1$ and $\lambda_2 = -1$ (one negative, one positive).
- Corresponding eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Solution: $x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$
- Interpretation: The origin is a saddle point (always unstable).

➢ **Spiral point (stable focus):**

- Eigenvalues: Complex conjugate pairs with negative real parts.
- Behavior: Trajectories spiral towards the equilibrium point.
- Stability: Asymptotically stable.

Example: Let us consider the system $X' = Ax$ where, $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$

- Eigenvalues: $\lambda_{1,2} = -1 \pm i$ (complex conjugates with negative real part)
- Corresponding eigenvectors: $v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$
- Solution: $x(t) = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1+i)t} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1-i)t}$

- Interpretation: The origin is a spiral point (stable focus). The negative real part indicates that trajectories will spiral inward towards the equilibrium point, indicating stability.

➢ **Spiral point (unstable focus):**

- Eigenvalues: Complex conjugate pairs with positive real part.
- Behavior: Trajectories spiral away from the equilibrium point.
- Stability: unstable.

Example: Let us consider the system $X' = Ax$ Where, $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

- Eigenvalues: $\lambda_{1,2} = 1 \pm i$ (complex conjugates with positive real part).
- Corresponding eigenvectors: $v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$
- Solution: $x(t) = c_1 e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix}$
- Interpretation: unstable (spiral point or unstable focus). The positive real part indicates that the trajectories of the system will spiral outwared from the equilibrium point indicating instability.

➢ **Center:**

- Eigenvalues: Puerly imaginary ($i.e \ \pm i\beta$).
- Behavior: Trajectories form closed orbits arround the equilibrium point.
- Stability: Orbitally stable (neither asymptotically stable nor unstable).

Example: Let us consider the system $X' = Ax$ Where, $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

- Eigenvalues: $\lambda_{1,2} = \pm 2i$ (puerly imaginary).
- Corresponding eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$
- Solution: $x(t) = c_1 e^{2it} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + c_2 e^{-2it} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$
- Interpretation: The origin is center (stable oscilatory).

Table2.1: The characteristics of equilibrium points for a given eigenvalues

Eigenvalues	Characteristics of equilibrium points
$\lambda_1 > \lambda_2 > 0$	proper node
$\lambda_1 < \lambda_2 < 0$	improper node
$\lambda_1 < 0 < \lambda_2$	Saddle point
$\lambda_1 = \lambda_2 > 0$	Source node
$\lambda_1 = \lambda_2 < 0$	Sink node
$\lambda_1, \lambda_2 = \alpha \pm i\beta, \alpha > 0$	spiral/ outward focus
$\lambda_1, \lambda_2 = \alpha \pm i\beta, \alpha < 0$	spiral/ inward focus
$\lambda_1, \lambda_2 = \alpha \pm i\beta, \alpha = 0$	Center

Note: for linear autonomous system if we know the coefficient matrix A, then we can easily determine the nature of stability of the system by finding the eigenvalues of A.

Important remarks:

- Different equilibrium points have different stability properties.
- Only very small disturbances are allowable in stability analysis.
- Asymptotical stability is more desirable for practical purpose.

2.2. Stability of non-linear autonomous system

2.2.1. Non-linear dynamical system

Most non-linear systems of differential equations are impossible to solve analytically. One reason for this is the unfortunate fact that we simply do not have enough functions with specific names that we can use to write down explicit solutions of these systems. Hence we are forced to resort to different means in order to understand these systems. There are techniques that arise in the field of dynamical systems, Wiggins,S. (2003). So we totally focus on the qualitative theory of the dynamical systems around the equilibrium point because of the difficulty in determine the quantitative solutions and the importance of interpreting on a life application.

Consider the system $\begin{cases} X' = F(x,y) \\ Y' = G(x,y) \end{cases}$, (2.6)

where F&G are smooth and non-linear functions. Unlike in the case of a linear system of differential equations, it is very difficult to discuss directly about the type and character of the equilibrium points of the system (2.6). For this reason, we use the concepts of linerization, Hartman-Grobman theorem, and Lyapunov functions.

2.2.2. Concept of linearization

So far, we have discussed the stability properties of linear systems. However, most of the practical systems are non-linear in nature, which cannot be studied easily like linear systems. A non-linear system differs from the usual linear system in a number of ways, like

- Principle of the superposition is not applicable to non-linear systems.
- Altering the size of the input does not change the shape of the response of a linear system, but this is not in the case of a non-linear system.
- In a non-linear system, stability is a characteristic of the system, independent of the magnitude of the system input or the initial conditions. But in a linear system, stability may depend on the magnitude of the input as well as the initial conditions.

Suppose that (x_0, y_0) be an equilibrium point of equation (2.6), then the linearization of $F(x,y)$ and $G(x,y)$ about (x_0, y_0) in the approximated linear functions about the point(x_0, y_0),

$$F(x,y) = F(x_0,y_0) + \partial_x F(x_0,y_0)(x - x_0) + \partial_y F(x_0,y_0)(y - y_0) + h_1(x,y)$$

$$G(x,y) = G(x_0,y_0) + \partial_x G(x_0,y_0)(x - x_0) + \partial_y G(x_0,y_0)(y - y_0) + h_2(x,y)$$

Where h_1 and h_2 are functions of x and y with higher order terms.

Since, $F(x_0,y_0) = G(x_0,y_0) = 0$ then we obtain;

$$F(x,y) = \partial_x F(x_0,y_0)(x - x_0) + \partial_y F(x_0,y_0)(y - y_0) + h_1(x,y)$$

$$G(x,y) = \partial_x G(x_0,y_0)(x - x_0) + \partial_y G(x_0,y_0)(y - y_0) + h_2(x,y)$$

Then, we say (eqn. (2.6)) is linearized as,

$$\frac{d}{dt}\begin{bmatrix} \mathrm{x} - x_0 \\ \mathrm{y} - y_0 \end{bmatrix} = \begin{bmatrix} \partial_x \text{ F } (x_0, y_0) & \partial_y \text{ F } (x_0, y_0) \\ \partial_x \text{ G } (x_0, y_0) & \partial_y \text{ G } (x_0, y_0) \end{bmatrix} \begin{bmatrix} \mathrm{x} - x_0 \\ \mathrm{y} - y_0 \end{bmatrix}$$

Then, the matrix;

$$J(x_0, y_0) = \begin{bmatrix} \partial_x \text{ F } (x_0, y_0) & \partial_y \text{ F } (x_0, y_0) \\ \partial_x \text{ G } (x_0, y_0) & \partial_y \text{ G } (x_0, y_0) \end{bmatrix}$$ is constant matrix which is called the

Jacobean matrix.

Example1: Consider the critical point of the 2×2 non-linear system.

$$X'(t) = x^2 - y, \; Y'(t) = x - y$$

To find them explicitly, one needs to solve the equation

$$x^2 - y = x - y = 0 \Longrightarrow x = y, x^2 - y = 0$$

It easily follows that the critical points are $P(0,0)$& $Q(1,1)$.

To study their stability properties, we look at the Jacobean matrix,

$$A = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{bmatrix} 2x & -1 \\ 1 & -1 \end{bmatrix}$$

and we compute its eigenvalues. Since its eigenvalues depend on x, one needs to examine each of the critical points separately.When it comes to the critical point $P(0,0)$,the Jacobean matrix is $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ and its eigenvalues are the roots of the characteristic polynomial,

$$\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = \frac{-1 \pm i\sqrt{3}}{2}.$$

Thus, their real part is negative and P is asymptotically stable or spiral sink. When it comes to a critical point $Q(1,1)$, the Jacobean matrix is $A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$ and its eigenvalues are the roots of the characteristics polynomial,

$$\lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1 \pm i\sqrt{3}}{2}$$

Hence, the eigenvalues are complex with a posetive real part. So, Q is spiral source or unstable.

Example2: We study the critical point of the 2×2 non-linear system

$$x'(t) = x(y - 1)$$

$$y'(t) = x - y - 1$$

To find them explicitly, one needs to solve the equations $x(y - 1) = 0, x = y + 1$ when $x = 0$, the second equation gives $y = -1$. When $y = 1$, the second equation gives $x = 2$. This means that the system has two critical points, namely, R (0, -1) and S (2, 1). To study their stability properties, we look at the Jacobean matrix

$$A = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{bmatrix} y - 1 & x \\ 1 & -1 \end{bmatrix}$$

and we compute the eigenvalues of A for each of the critical points.

When it comes the critical point $(0, -1)$, the Jacobean matrix is

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}.$$

This is lower triangular, so, its eigenvalues are the diagonal entries, as those are both negative. We conclude that R is asymptotically stable.

When it comes to the critical point $S(2,1)$, the Jacobean matrix is

$$A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}.$$

And the eigenvalues are the roots of the characteristic polynomial is

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = \frac{-1 \pm 3}{2} = -2,1$$

In particular, one of the eigenvalues is positive and S is unstable.

Definition2.6: (Hyperbolic fixed point): A fixed point (x_0, y_0) of a dynamical system is hyperbolic if and only if all the eigenvalues of the linearization of the system about (x_0, y_0) have non-zero real part. Our objective is to investigate the behavior of trajectories of this system near a critical point $\left(x_{0,}y_0\right)$.

Theorem2.2: $(\boldsymbol{Hartman - Grobman\ theorem})$

Let F: $R^n \to R^n$ be a smooth map of a dynamical with differential equations

$\frac{dx_i}{dt} = F_i(x_i)$, where i= 1, 2, 3, …, n.

$A = \left[\frac{\partial F_i}{\partial X_j}\right]$, where i, j =1, 2, 3, …, n of $F(x)$ at point x_0 has hyperbolic fixed point.

Then there exists a neighborhood N of x_0 and a homomorphism h: $N \to R^n$ such that $h(x_0) = 0$. $\frac{dx_i}{dt} = F_i(x_i)$, where i= 1, 2, 3, ……., n is topologically conjugate by the smooth map $X = h(x)$ to the flow of its linearization, $X' = Ax$; where A is the coefficient matrix of the linearized system. Let r_1 and r_2 be the eigenvalues of A. Then by Hartman-Grobman theorem the type and stability of the critical point of the linear system $X' = Ax$ and the non-linear system are $X' = F(x)$ as given the table below.

Table 2.2: type and stability of linear system and non-linear systems.

Eigenvalues	Linear system		Non-linear system	
	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	Node	Unstable	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable	Node	Asymptotically stable
$r_2 < 0 < r_1$	Saddle	Unstable	Saddle	Unstable
$r_2 = r_1 > 0$	Node	Unstable	Node/spiral	Unstable
$r_2 = r_1 < 0$	Node	Asymptotically stable	Node/spiral	Asymptotically stable
$r_1, r_2 = \lambda \pm i\beta; \lambda > 0$	Spiral	Unstable	Spiral	Unstable
$r_1, r_2 = \lambda \pm i\beta; \lambda < 0$	Spiral	Asymptotically stable	Spiral	Asymptotically stable
$r_1, r_2 = \lambda \pm i\beta; \lambda = 0$	Center	Stable	Center	Stable

2.2.3 Concept of stability in the sense of Lyapunov

One of the powerful tools for stability analysis of systems of differential equations including non-linear systems are lyapunov functions. It is used to determine the stability without solving the system. Alexander Mihailov Lyapunov (1857-1987) elaborated an extremely general method for investigating the solution of the system of differential equations for stability.

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n)\ \forall_i = 1,2, \dots, n$$

Definition2.7: Let D be open connected subset of R^n and

$V: D \to R$. Then, V is said to be

- Positive definite in D if $V(x_1, x_2, \dots, x_n) > 0\ \forall_x \in D$
- Positive semi-definite in D if $V(x_1, x_2, \dots, x_n) \geq 0\ \forall_x \in D$. Except at one or more points in the state space including the origin $x = 0$ where $V(x_1, x_2, \dots, x_n) = 0$
- Negative-definite in D if. $V(x_1, x_2, \dots, x_n) < 0\ \forall_x \in D$
- Negative- semi definite in D if $V(x_1, x_2, \dots, x_n) \leq 0\ \forall_x \in D$. Except at one or more points in the state space including the origin $x = 0$ where, $V(x_1, x_2, \dots, x_n) = 0$

There is no universal single concept of stability and it is defined in many different ways. But, we shall consider the following fundamental definition of stability due to the Russian mathematician Lyapunov.

Definition2.8: A function V: D→R is said to be Lyapunov function if

- V is differentiable
- V is positive function
- V is decreasing function along the solution system

Theorem2.3: Suppose that there exists a differentiable positive- definite function $: D \to R$. $\frac{dv}{dt}$ along trajectories of the system is negative semi-definite, then the equilibrium point is stable.That is,

$$\frac{dv}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(t, x_1, x_2, ---, x_n \le 0), \forall_x \in D \,/\, \{0\} \;\; V(x) = 0 \text{ only if } x = 0.$$

Proof: Let $\varepsilon > 0$ and $t_0 \in R$ be given. Assume without loss of generality, that $B(x_0, \varepsilon)$ is contained in D. Assume positive-definite function $W: D \to R$, such that $V(t, x) \ge W(x)$ for every $(t, x) \in R \times D$. Let $m = min\{V(t_0, x), |x| \le \varepsilon\}$. Since W is continuous and positive-definite, m is well-defined and positive. Given that $\delta > 0$ small enough that $\delta < \varepsilon$ and $max\{\{V(t_0, x), |x| \le \delta\} < m\}$.Since V is positive-definite and continuous functions, it is possible, if $x(t)$ solution of $x'(t) = f(x)$ and $|x(t_0)| < \delta$. Then $V(t_0)$, $X(t_0) < m$ and $\frac{d}{dt} v(t, x(t = v't(x, t)) \le 0$ For all t. so $(t, x(t)) < m$ for every $t \ge t_0$. Thus, $w(x) < m$ and $|x(t)| \ne \varepsilon$, for every $t \ge t_0$. Since $|x(t)| \le \varepsilon$, for every $t \ge t_0$. The solution of $x' = f(x)$ is stable.

Theorem2.4: Suppose that a dynamical system $x' = f(x)$ has a fixed point at the origin. If a Lyapunov function exists, as defined above, then the origin is Lyapunov stable. If in addition $v' < 0$ for $x \ne 0$ then V is a Strict Lyapunov function and the origin is asymptotically stable.

Proof: We want to show that for any sufficiently small neighborhood U of the origin, there is a neighborhood V such that if $x_0 \epsilon v$, then $\varphi t(x_0) \in U)$ for all positive t. (where $\varphi t(x_0)$ is the solution at time t of the system $x' = f(x)$ starting at X_0 when $t = 0$).We start by choosing δ sufficiently small such that $|x| < \delta \Rightarrow x \epsilon U$.

Let $\alpha = min\{V: |x| = \delta\}$. Clearly $\alpha > 0$ from the definition of V. Now consider the set of points $U_1 = \{x: v(x) < \delta\} and |x| < \delta$.Then since for $x_0 \in u_1$,$(V(x_0) < \alpha$ and v does not increase along trajectories, $|\emptyset t(x_0)|$ can never reach δ, as if this happened V$\ge \alpha$.Thus clearly

$x\epsilon U_1$ at $t = 0 \Rightarrow |x(t)| < \delta \Rightarrow x(t)\epsilon U\ \forall t \geq 0$, as required. Again, since x remains in the domain U. For any initial point $x_0 \neq 0$, $V(x_0) > 0)$ and $v' < 0$ along trajectories. Thus $V(\emptyset t(x_0))$ decreases monotonically and is bounded below by 0. Then we have $V \to \alpha \geq 0$; suppose $\alpha > 0$. Then $|x|$ is bounded away from zero (by continuity of V), and so $W \equiv v' < -b < 0$.

Thus, $V(\emptyset t(x_0)) < V(x_0) - bt$ and so is certainly negative after a finite time. Thus there is a contradiction, and so $\alpha = 0$. Hence $v \to 0$ as $t \to \infty$ and so $|x| \to 0$ (again by continuity). This proves asymptotic stability. To prove results about instability just reverse the sense of time. Sometimes we can demonstrate asymptotic stability even when V is not a strict Lyapunov function. But there is no cut-and-dried method of finding Lyapunov functions; it is usually a matter of creativity or trial and error in each case.

Theorm2.5: If a differentiable positive-definite function $V: D \to R$. Along $\frac{dv}{dt}$ the trajectories of the system is positive-definite, then the equilibrium point is unstable. i.e.

$$\frac{dv}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(t, x_1, x_2 - -- , x_n) > 0, \forall_x \in D/\{0\}$$

Example1: Check the stability of the system given by

$$X'(t) = y^2 - x^3 , y'(t) = -y - 2xy$$

Consider the function $V(x, y) = ax^2 + by^2$ for some $a, b > 0$. This is certainly continuous and non-negative. While $V(x, y) = 0$, if, $x = y = 0$.To show that V is a Lyapunov function, we compute,

$$\nabla V. f = \frac{\partial v}{\partial x} x'(t) + \frac{\partial v}{\partial y} y'(t)$$

$$= 2ax(y^2 - x^3) + 2by(-y - 2xy)$$

$$= (2a - 4b)xy^2 - 2ax^4 - 2by^2$$

We need to ensure that $\nabla V. f \leq 0.$ and this obviously true. If we let $a = 2b > 0$. In other words, $V(x, y) = 2bx^2 + by^2$ is a Lyapunov function for any constant $b > 0$, so the zero solution is stable.

Example2: Check the stability of the system given by

$$\begin{cases} x_1{}' = x_2 \\ x_2{}' = -x_1 - x_2 \end{cases}$$

Define $V(x_1, x_2) = \frac{1}{2}[(x_1 + x_2)^2 + 2x_1{}^2 + x_2{}^2]$ is positive- definite

$\Rightarrow$ V $(x_1, x_2) > 0, \forall\ (x_1, x_2)\ \in R^2\ /\ \{0, 0\}$ and

$$\frac{dv}{dt} = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}\right)\begin{pmatrix} x_2 \\ -x_1 - x_2 \end{pmatrix}$$

$$= \frac{1}{2}(6x_1 + 2x_2)(x_2) + 2x_1 + 4x_2(-x_1 - x_2)\ .$$

$$= \frac{1}{2}(6x_1 * x_2 + 2{x_2}^2 - 2{x_1}^2 - 2x_1 * x_2 - 4x_1 * x_2 - 4{x_2}^2)$$

$$= ({x_1}^2 + {x_2}^2) < 0, (\forall x_1, x_2 \in R^2).$$

Since, $\frac{dv}{dt} < 0$, it is negative-definite. Therefore, $(x_1, x_2) \equiv$ (0,0)) is asymptotically stable

Example3: Consider the system;

$$\begin{cases} X' = xy^2 + x^2y + x^3 \\ Y' = y^3 - x^3 \end{cases}$$

Assume that $V(x, y) = x^2 + y^2$ then V is positive- definite and its derivative with respect to t is given by;

$$v'(x, y) = 2x(xy^2 + x^2y + x^3) + 2y(y^3 - x^3) = 2(x^2y^2 + x^4 + y^4)$$

which is positive definite. Hence the origin is unstable. Actually, finding a Lyapunov function for a particular system is very difficult.

CHAPTER THREE

3. Applications of Linear & Non-linear Dynamical Systems

3.1. Applications of Linear Dynamical System

Linear autonomous dynamical systems are foundational models in various fields of science and engineering due to their simplicity and the powerful insights they provide into complex phenomena. These systems are characterized by their linearity and autonomy. Linearity means the system's behavior can be described using linear equations. This property allows for analytical solutions and makes it possible to leverage tools from linear algebra and calculus. Autonomos indicates that the system's behavior depends only on its current state and not explicitly on time or external inputs.

Generally, linear autonomous dynamical systems are versatile tools that apply to a wide range of fields. Like in RC circutes, signal processing, motion of spring, control systems, etc.

Example1: (Electric circuit)

Consider a closed electric circuit which has a resistance R, inductance L, capacitance C, the charge on the capacitor Q, electromotive force E and current I. It is known from physics that the voltage drops across the resistor, inductor and capacitor $\frac{R}{L}=0$ rspectively are;

RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$. Kirchhoff's voltage law says that the sum of this voltage drops is equal to the supplied voltage; $\frac{dI}{dt}+RI+\frac{Q}{C}=E$. Differentiating this equation with respect to t and substituting

$\frac{dQ}{dt}=I$ we get; $I''\frac{R}{L}I'-\frac{1}{LC}I$ and hence the system;

$$\begin{cases} I' = y \\ Y' = -\frac{R}{L}Y - \frac{1}{LC}I \end{cases}$$

Is a system of first order differential equation with variable I &Y

The determinant of the coefficient matrix is given by

$$Det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{-1}{\lambda C} & \frac{-R}{L}-\lambda \end{vmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

Hence the Eigenvalues are; $\lambda = \frac{-R\pm\sqrt{R^2-\frac{4L}{C}}}{2}$. We have the following three cases about the above eigenvalues.

If $R^2-\frac{4L}{C}>0$, then λ_1 and $\lambda_2$are real distinct & negative the origin is stable node.

If $R^2 - \frac{4L}{C} = 0$, then $\lambda_1 = \lambda_2 = \frac{-R}{2L}$, so that the zero solution is again stable node.

If $R^2 - \frac{4L}{C} < 0$, then λ_1 and λ_2 are complex conjugates with negative real parts, so the origin is stable focus. In all three cases the equilibrium point $(0,0)$ is asymptotically stable and the transit current tends to zero as t→ ∞.

Generally, RC circuts are used in filters and signal processing applications. For example, in audio electronics, RC circuits can filter out unwanted frequencies from audio signals.

Example 2: Motion of spring

We consider the motion of an object with mass m at the end of a spring that is either vertical or horizontal on a level surface. by Hooke's Law, which says that if the spring is stretched or (compressed) units from its natural length, then it exerts a force that is proportional to x (restoring force = - kx), where k is a positive constant (called the spring constant). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have;

$$m\frac{d^2x}{dt^2} = -kx \text{ or } m\frac{d^2x}{dt^2} + kx = 0$$

But we are interested in the motion of a spring that is subject to a frictional force. An example is the damping force supplied by a shock absorber in a car or a bicycle. We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. (This has been confirmed, at least approximately, by some physical experiments).Thus damping force is equal to $-c(\frac{d_x}{d_t})$, where c is a positive constant, called the damping constant. Thus, in this case, Newton's Second Law gives:

$$\frac{d^2x}{dt^2} + \frac{kx}{m} + \frac{cdx}{mdt} = 0; c, k > 0$$

If we substitute $y = x$, then we have

$$\begin{cases} x' = y \\ y' = -\dfrac{k}{m}x - \dfrac{c}{m}y \end{cases} \tag{2.7}$$

$Clearly,$ From (2.7) the critical point is $(0,0)$ and the eigenvalues of the coefficient matrix of the system are;

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2}$$

Case1:If $c^2 - 4mk > 0$ (over damping): In this case $\lambda_1$and λ_2 are distinct real roots, thus $x = c_1e^{\lambda_1 t} + c_1e^{\lambda_2 t}$ since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < 0$. so, the eigenvalues $\lambda_1$and λ_2 must both be negative. This shows that $(0,0)$ is stable node. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means that there is a strong damping force (high friction) compared with a week spring or small mass.

Case 2: If $c^2 - 4mk = 0$ ($c$ritical damping): This case corresponds to equal roots,

$$\lambda_1 = \lambda_2 = \frac{-c}{2m} \text{ thus } X(t) = \left(c_1 + (c_2 t) e^{\left(\frac{-c}{2m}\right)t}\right)$$

Thus $(0,0)$ is a stable node. In this case the damping is just sufficient to defeat vibrations.

Case3: If $c^2 - 4mk < 0$ (under damping): Here the roots are complex:

$$\lambda = \frac{-c}{2m} \pm \omega i \text{ where } \omega = \frac{\sqrt{4mk - c^2}}{2}$$

Thus the solution is; $X = e^{\left(\frac{-c}{2m}\right)t}(c_1 \cos \omega t \, c_2 \sin \omega t)$

Therefore, $(0,0)$ is stable focus.

3.2. Applications of non-Linear Dynamical Systems

Non-limear dynamical systems are a fundamental area of study in applied mathematics and engineering characterized by their complex behavior and rich range of applications across various scientific and practical domains. Unlike linear systems, where superposition principles apply and solutions are straightforward to analyze, non-linear systems exhibit intricate dynamics due to their non-linearty. Non-linear autonomous dynamical systems are described by different equations that capture the system's internal dynamics.

Nonlinear dynamical systems can exhibit different types of behavioral patterns depending on the values of the physical parametres and intrinsic features of the respective systems.These behavioral futures are characterized by complex dynamic phenomena such as multiple steady state behavior, limit cycles, bifurcations, complex attractors and chaos.

The study of non-linear autonomous dynamical system is essential for advancing our understanding of complex phenomena acros various fields.These system's ability to exhibit a wide range of behaviors, from predictable cycles to chaotic dynamics, makes them a powerful tool for modeling, analyzing real-world problems.

Exmple1: Damped Pendulum System

Consider a mathematical model for a frictionless pendulum consisting of a mass, m, on a rigid weightless rod of length L. If the angular deflection from the rest position is denoted by θ, then the component of the gravitational force in the direction normal to the rod is equal to $-\text{mg}\sin\theta$.

Then conservation of angular momentum states that; $-mL\theta''(t) = -mg \sin\theta\,(t)$.

To see when friction is added, the momentum equation is modified as; $-mL\theta''(t) = -mg\sin\theta\,(t) - c\theta'(t); c > 0$

The term $-c\theta'(t)$ represents a force which is in opposition to the momentum and is proportional to the speed of the pendulum, e.g., a force created by friction with the air. Here $-c\theta'(t)$waste energy. To see this let $u = \theta$ and $v = \theta'$ and write;

$$U' = v, v' = -\omega^2 \sin u - cv; c > 0$$

Then we have;

$$vv'(t) + \omega^2 \sin(u)\, U'(t) = \frac{d}{dt}\left[\frac{1}{2}v^2 + \omega^2(1 - \cos(u(t))\right] = -cv^2 < 0.$$

Which implies that $\frac{d}{dt}E(U,V) = -cv^2 < 0$

So, energy decreases along the orbits of this system. Note that the new system has the same critical points as the frictionless system. Then (letting $\omega = 1$, for simplicity) we compute;

$$J(U,V) = \begin{bmatrix} 0 & 1 \\ \cos u & -c \end{bmatrix}$$

Therefore, $J(\pm\pi, 0) = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$ at critical points $(\pm\pi, 0)$. Assuming $C^2 < 4$, the eigenvalues are real and opposite signs. So, the critical point is a saddle point. The eigenvalues can be found to be;

$$\lambda_1 = \frac{-1}{2}c - \frac{1}{2}\sqrt{c^2 + 4} < 0 \text{ and } \lambda_2 = \frac{-1}{2}c + \frac{1}{2}\sqrt{c^2 + 4} > 0.$$

At $(0,0)$; $(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix}$ and the eigenvalues are;

$$\lambda_1 = \frac{-1}{2}c - \frac{1}{2}\sqrt{c^2 - 4} \text{ and } \lambda_2 = \frac{-1}{2}c + \frac{1}{2}\sqrt{c^2 - 4}$$

That is; the eigenvalues are a complex conjugate pair with negative real part so the critical point is stable focus. This shows that the closed orbits around the origin have become, in the presence of friction, spirals tending toward the origin (i.e.,the rest position.) We interpret this to mean that as the pendulum loses energy to friction, the amplitude of the motion decreases to zero. The unbounded orbits in the upper phase plane have become spirals that tend toward the next critical point located at x = π, while the some of the unbounded orbits in the lower phase plane are attracted to the critical point at the origin.

Example2: Lotka-Volterra or Predator-prey equation

The most common simple applications of non-linear systems are the so-called predator-prey or Lotka-Volterra systems. It was developed by Alferd Lotka and Vvito One Voltrrian in the early 20^{th} century. It is a relations of the interaction between two species of animals that live in the same ecosystem. Where by the quantity of each group of the species depends on the birth or dath rate and the successful meetings with the individuals of the other species. For example, in biology, this systems of equations the natural periodic varations of population of different species in nature.We take two eamples of hares & foxes in a forest, it makes easy to understand.

X = hares (prey)

Y = foxes (predator)

When there are a lot of hares, there is a lot of food for the foxes, so the fox population grows. However, when the fox population grows, the foxe seat more haires, so when there are lots of foxes, the hares population go down, and vice versa.

The Lotka-Volttera model proposes that this behavior described by the systems of equations.

$$\begin{cases} X' = x(a - by) \\ Y' = y(-c + dx) \end{cases}$$

where a, b, c, d are some parameters that describe the interactions of the foxes and hares.In this model, all parameteres are all positive.The model is slightly more complicated idea based on the exponential population model. That is, when $y = 0$ we have $x' = ax$ where $a > 0$. So in this case $x(t) = x_0 e^{at}$. When predators present, we assume that the prey population decreases at a rate proportional to the number of predator/prey encounters that is bxy where $b > 0$. So the differential equation for the prey population is

$$x' = ax - bxy.$$

For the predator population, we make more or less the opposite assumptions. In the absence of prey, the predator population declines at a rate proportional to the current population. So when $x = 0$ we have $y' = -cy$ with $c > 0$, which implies that $y(t) = y_o e^{-ct}$. we only consider $x, y \geq 0$. Now our first job is to locate the equilibrium points. These occur at the origin and at $(x, y) = \left({}^{c}/_{d} , {}^{a}/_{b} \right)$.

The Jacobean matrix is given by ;

$$J(x, y) = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}.$$

So, at the origin the system is (unstable) saddle point with eigenvalues a and −c. At the other equilibrium point (c /d, a/b), the eigenvalues are pure imaginary $\pm i\sqrt{ac}$ and so we cannot conclude anything at this stage about stability of this equilibrium point. Hence the solutions wind in the counter clockwise direction about the equilibrium point. From this, we cannot determine the precise behavior of solutions: They could possibly spiral in toward the equilibrium point, spiral toward a limit cycle, spiral out toward “infinity” and the coordinate axes, or else lie on closed orbits. To make this determination, we search for a Lyapunov function L. Employing the trick of separation of variables;we look for a function of the form $L(x,y)=F(x)+G(y)$. Recall that L' denotes the time derivative of L along solutions. We compute;

$$L'(x,y)=\frac{dF}{dx}x'+\frac{dG}{dy}y'=x\frac{dF}{dx}(a-by)+y\frac{dG}{dy}(-c+dx).$$

We obtain $L'=0$ provided that;

$$\frac{x\frac{dF}{dx}}{dx-c}=\frac{y\frac{dG}{dx}}{by-a}=constant$$

Setting the constant equal to one, we obtain; $\frac{dF}{dx}=d-\frac{c}{x}$ and $\frac{dG}{dy}=b-\frac{a}{y}$.

Integrating the above equation we get;

$$F(x)=dx-c\ln x \text{ and } G(y)=by-a\ln y.$$

Thus the function $L(x,y)=dx-c\ln x-by-a\ln y$ is constant $(L'=0)$ on solution curves of the system when $x,y>0$, by considering the signs of $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial y}$ it is easy to see that the critical point $Z=(\frac{c}{d}=\frac{a}{b})$ is an absolute minimum for L, it follows that $L-L(Z)$ is a Lyapunov function for the system. Therefore, Z is a stable equilibrium.

3.3. Bifurcation

Bifurcation in a dynamical system describe the qualitative change in behavior under a variation or change of some parameters of the system. Parameters are constants that are tunable. In linear systems, the bifurcations are related to changes in the stability of the system, as that is the main type of change that can happen in the dynamics of such systems and the point at which the stability changes for some parameter value is bifurcation point.

Bifurcation analysis is the analysis of a system of ordinary differential equations under parameter variation. Performing a local bifurcation analysis is often a powerful way to analyse the properties of such systems, since it pridicts what kind of behavior (system is in equilibrium, or there is cycling) occures where in parameter space. A bifurcation is a point in parameter space where equilibria appear, disappear, or change stability, and the bifurcation curve indicates the parameter values for which a change may occur. **Example:**Consider the following example to analyzing stabilities and bifurcate point;

$$\begin{cases}\dot{x}=\mu x+x^3\\ \dot{y}=-y\end{cases} \quad \text{And} \quad \begin{cases}\dot{x}=\mu x-x^3\\ \dot{y}=-y\end{cases}$$

If $\mu>0$, there are no equilibrium points, whereas, if $\mu<0$ there are three equilibrium points in the first system. And on the second system if $\mu<0$ there are no equilibrium points, whereas, if $\mu > 0$ there are three equilibrium points. Let as discuses critical point, eigenvalues and stability on the following table.

Table3.1: Types of critical point, eigenvalues and stability.

Systems of equation	$\begin{cases}\dot{x} = \mu x + x^3 \\ \dot{y} = -y\end{cases}$ and $J(x,y) = \begin{bmatrix}\mu + 3x^2 & 0 \\ 0 & -1\end{bmatrix}$			$\begin{cases}\dot{x} = \mu x - x^3 \\ \dot{y} = -y\end{cases}$ and $J(x,y) = \begin{bmatrix}\mu - 3x^2 & 0 \\ 0 & -1\end{bmatrix}$		
Critical point	For $\mu < 0$			for $\mu > 0$		
	(0,0)	$(\sqrt{-\mu}, 0)$	$(-\sqrt{-\mu}\ ,0)$	(0,0)	$(\sqrt{\mu}\,, 0)$	$(-\sqrt{\mu}\,,0)$
Linearization at the critical point	$\begin{bmatrix}\mu & 0 \\ 0 & -1\end{bmatrix}$	$\begin{bmatrix}-\mu & 0 \\ 0 & -1\end{bmatrix}$	$\begin{bmatrix}-\mu & 0 \\ 0 & -1\end{bmatrix}$	$\begin{bmatrix}\mu & 0 \\ 0 & -1\end{bmatrix}$	$\begin{bmatrix}-\mu & 0 \\ 0 & -1\end{bmatrix}$	$\begin{bmatrix}-\mu & 0 \\ 0 & -1\end{bmatrix}$
Eigenvalues	$\lambda_1 = u$ $\lambda_2 = -1$	$\lambda_1 = -\mu$ $\lambda_2 = -1$	$\lambda_1 = -\mu$ $\lambda_2 = -1$	$\lambda_{1=}\mu$ $\lambda_2 = -1$	$\lambda_1 = -\mu$ $\lambda_2 = -1$	$\lambda_1 = -\mu$ $\lambda_2 = -1$
Stability	Stable node	Unstable/ $saddlepoint$	Unstable/ $saddlepoint$	Unstable/ $saddlepoint$	Stable node	Stable node

3.3.1. Saddle- node bifurcation

For a tangent or tran scritical bifurcation to occur in a two- dimentional system one of the eigenvalues has to be equal to zero. Sadle-node bifurcation, tangential bifurcation or fold bifurcation is a local bifurcation in which two fixed points or equilibria of dynamical system collide and annihilate each other.

Example: Saddle-node bifurication in 2D

$$\dot{x} = \alpha - x^2$$
$$\dot{y} = -y$$

- Equilibra: since $f(x,y) = (\alpha - x^2, -y) = (0,0)$ if and only if $(x,y) = (\pm\sqrt{\alpha, 0})$.
- Stability: For the equilibrium point $(\sqrt{\alpha}, 0)$ the eigenvalues of the Jacobian matrix at equlibrium is $\lambda = -1$. Thus, the equilibrium $(\sqrt{\alpha}, 0)$ is stable if $\alpha > 0$
- For the equilibrium poit $(-\sqrt{\alpha}, 0)$, the eigenvalues of the Jacobian matix at equilibrium are $\lambda = 2\sqrt{\alpha}$ exists for $\alpha > 0$.

- Thus, two equilibrium emerges as α passes zero to the right and hence $\alpha = 0$ is saddle-node bifurcation.

3.3.2. Trans critical bifurcation

Trans critical bifurcation occures when there is an exchange of stablities between equilibras as the bifurcation parametr varies.

Example: Trans critical bifurcation in 2D

$$\dot{x} = \mathrm{a}\mathrm{x} - x^2$$
$$\dot{y} = -\mathrm{y}$$

- Equilibra: Since$f(x, y) = (\alpha x - x^2, -y) = (0,0)$ if and only if $(x, y) = (0,0)$ and $(x, y) = (\alpha, 0)$.
- Stability: For the equilibrium point (0,0), the eigenvalues of the Jacobian matrix at equilibrium are $\lambda = -1$ and $\lambda = \alpha$.Thus, the equilibrium (0,0) is stable if $\alpha < 0$ and unstable if $\alpha > 0$.
- For the equilibrium point $(\alpha, 0)$, the eigenvalues of the Jacobian matrix at equilibrium are $\lambda = -1$ and $\lambda = -\alpha$, thus, the equilibrium$(\alpha, 0)$ is stable if $\alpha > 0$ and unstable if $\alpha < 0$.

3.3.3. Hopf Bifurcation

Hopf Bifurcation is a bifurcation that occurs when the real part of a complex conjugate pair of eigenvalues, of the linearized flow at a fixed point, switchs from positive to negative or negative to positive that is when the eigenvalues become purely imaginary. Hopf bifurcation only occurs in systems of two or more dimensions. If an equilibrium point, showing an oscilatory behavior around itself, switches its stability, the resulting bifurcation is called a Hopf bifurcation. A Hopf bifurcation typically causes the appearance (or disapearance) of a limit cycle around the equilibrium point. One can check wheather the bifurcation is Hopf or not by looking at the imaginary components of the dominant eigenvalues whose real parts are at a critical value (zero); if there are non-zero imaginary components, it must be a Hopf bifurcation. A Hopf bifurcation is said to be supercritical Hopf bifurcation when the real part of the eigenvalues goes from negative to positive the fixed point goes from being a stable focus to an unstable focus. A Hopf bifurcation is said to be subcritical Hopf bifurcation when the real part of the eigenvalue goes from positive to negative the fixed point goes from being an unstable focus to a stable focus. Hopf bifurcation is a fundamental concept in a study of dynamical systems, particularly in the context of understanding how system behavior changes as a parameter is varied. A Hopf bifurcation is also known as Poincare-Andronov,Hopf bifurcation named after Heneri Poincare, AleksanderAndronov and Eberhard Hopf. Hopf bifurcation provides a crucial framework for understanding how dynamic systems transition from steady states to oscilatory behavior. By analyzing the conditions under which periodic solutions emerge or disappear, researchers or engineers can gain valuable insights into system stability and design, as well as, predict and control complex across a wide range of applications.

Example1: Van der pol oscilator

$$\dot{x} = x_2$$
$$\dot{x}_2 = -x_1 + r(\,1 - {x_1}^2)x_2$$

- Equilibra: Since $f(x_1, x_2) = (x_2, -x_1 + r(\,1 - {x_1}^2)x_2) = (0,0)$ if and only if $(x_1, x_2) = (0,0)$.
- Stability: Since eigenvalues of the Jacobian matrix at equilibrium are

  $\lambda_{1,2} = 0.5(r \pm \sqrt{r^2 - 4})$. The equilibrium is stable if $r < 0$ and unstable if $r > 0$.
- The critical condition for a Bifurcation to occur in this case is $Re(\lambda) = 0$.

$$Re(\lambda) = \begin{cases} 0.5(r \pm \sqrt{r^2 - 4}) \\ 0.5r, if\ r^2 - 4 < 0 \end{cases}$$

- The first case cant be zero, so the only critical condition for a bifurcation to occur is the second case, i.e. $r = 0$,when $Re(\lambda) = 0$ and $Im(\lambda) = \pm i$.
- This is is a Hopf bifurcation because the eigenvalues have non-zero imaginary parts when the stability change occures.
- When real part of the eigenvalues goes from negative to positive, it yields the equilibrium, (0,0), to change from stable equilibrium to unstable equilibrium and hence this bifurcation is supercritical Hopf bifurcation.

Example2: Consider the system

$$x' = \mu x - y - x(x^2 + y^2)$$
$$y' = x + \mu y - y(x^2 + y^2)$$

These is an equilibrium point at the origin and the linearized system is

$$X' = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} x$$

The eigenvalues at $\mu \pm i$ we expect a bifurcation when $\mu = 0$.

To see that happens as a passes through 0, we change to polar coordinates.

The system becomes; $r' = \mu r - r^3$

$\theta' = 1$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$

Note that: The origin is the only critical point for the system, since $\theta' \neq 1$. For $\mu < 0$ the origin is a sink since $\mu r - r^3 < 0 \; \forall r > 0$. Thus, all solutions tend to the origin in this case. When $\mu > 0$ then the equilibrium becomes a source. When $\mu > 0$ we have 0 if $r' = \sqrt{\mu}$. So the circle of radius $\sqrt{\mu}$ is a periodic solution with period 2π.We also have $r' > 0$ if $0 < r < \sqrt{\mu}$ while $r' > 0$ if $0 > \sqrt{\mu}$.

Thus, all non-zero solution spiral towards this circular solution at $t \to \infty$. This type of bifurcation is called Hopf bifurcation. Thus, at a Hopf bifurcation no new equilibrium arise. Instead a periodic solution is born at equilibrium point as passes through the bifurcation value.

Note that: From the above discussion we observe that the system does not undergo a bifurcation at μ_o if the eigenvalues of the coefficient matrix of its linearized system are;

- One eigenvalues is positive and the other is negative.
- Both have negative real part.
- Both have positive real part.

3.4. Limit Cycles and Bendixon-du Lac Criterion Theorem

A limit cycle is an oscillation unique to non-linear systems, discovered by Henri Poinccare.

It is an isolated closed trajectory (periodic orbit) in a phase plane. It is a closed trajectory in phase space having the property that at least one other trajectory spirals in to it as time approaches infinity in either direction. A limit cycle is a cyclic, closed trajectory in the phase space that is defined as an asymptotic limit of other oscilatory trajectories nearby. Limit cycles are a fundamental concept in the study of dynamical systems, representing a type of periodic solution that describes the behavior of a system over time. The techniques for studying limit cycles include the Poincare-Bendixon theorem, Lyapunove functions and bifurcation analysis. Limit cycles have broad applications across various fields in engineering, biology, physics, ecology, etc. Limit cycles represent a fundamental aspect of dynamical systems, providing valuable insights into periodic behavior and stability. By studying limit cycles, researchers and engineers can better understand and control oscilatory phenomena across a wide range of disciplines, from mechanical systems to biological rhythms and ecological dynamics.

Definition3.1: (1) an orbit (trajectory) $x(t)$ of a system of differential equations is said to be periodic if there exists a constant $T > 0$ such that $x(t+T) = x(t)$ for all t but $x(t+\tau) \neq x(t)$ for any $\tau \varepsilon (0, T)$. The period of the orbit is T.

(2) An orbit $x(t)$ of a system of differential equations is called a limit cycle if it is periodic and there are no nearby periodic orbits.

Example: Consider the nonlinear system;

$$\begin{cases} \dfrac{dx}{dt} = x + y - x(x^2 + y^2) \\ \dfrac{dy}{dt} = -x + y - y(x^2 + y^2) \end{cases}$$

It can be shown that (0, 0) is the only critical point and the corresponding linear system is:

$$\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = -x + y \end{cases}$$

which has the complex eigenvalues $\lambda = 1 \pm i$

It follows by table 3.1 that the nonlinear system looks like a spiral source near $(0,0)$. Now let's ask the question of what happens far away from $(0; 0)$. Being far away from (0; 0) just means that we are taking $x^2 + y^2$ to be large. If this term is sufficiently large, then the non-linear term in each equation dominates and the linear terms (e.g. $(-x(x^2 + y^2)$ dominates $x + y$ in the given system for x'). Scaling by $x^2 + y^2$ and recognizing it is positive, we have the following effective behaviors for large x and y; $\frac{dx}{dt}$ behaves like –x and $\frac{dy}{dt}$ bhaves like –y. We know that this corresponds to exponential decay toward $(0,0)$ so that trajectories may not become unbounded. This suggests something very strange is happening in the system. We have that the system is repelling locally at $(0; 0)$ but that the trajectories may not become unbounded in any direction. In other words, they must stay within a bounded region, but this bounded region excludes $(0; 0)$, which was the only fixed point of the system. So where exactly are the trajectories going? To apply Lyapunov's stability concept, consider the function $L(x, y) = x^2 + y^2$. This just keeps track of how far a point is away from (0,0) (it is the distance squared). Taking time derivative tells us how far the trajectories are travelling from (0,0) as time progress. We have:

$$\begin{aligned} \frac{dL}{dt} &= \left(\frac{dL}{dx}\right)\left(\frac{dx}{dt}\right) + \left(\frac{dL}{dt}\right)\left(\frac{dy}{dt}\right) \\ &= 2x\big(x + y - x(x^2 + y^2)\big) + 2y(-x + y - y(x^2 + y^2)) \\ &= 2(x^2 + xy - x^4 - x^2y^2 - xy + y^2 - x^2y^2 + y^4) \\ &= 2(x^2 + y^2 - x^4 - 2x^2y^2 + y^4) \\ &= 2[(x^2 + y^2) - (x^2 + y^2)^2] \end{aligned}$$

$$= 2(x^2 + y^2)(1 - (x^2 + y^2))$$

Now the quantity $x^2 + y^2$ just keeps path of how far we are away from $(0;0)$. The quantity itself is positive, so the only potential change in sign comes from the second term. Following our argument back to the start, we have:

- $\frac{dL}{dt} > 0 \ if \ x^2 + y^2 < 1$
- $\frac{dL}{dt} < 0 \ if \ x^2 + y^2 > 1$
- $\frac{dL}{dt} = 0 \ if \ x^2 + y^2 = 1$

In other words, the distance from $(0;0)$ is growing near $(0;0)$, shrinking far away, and staying exactly the same if the distance is exactly equal to one. It should not take much convincing that trajectories are in fact converging in on the unit circle. But what are they doing on that circle? There are no fixed points, so trajectories on the circle are always moving. Since, there are only two ways to move on the circle, it follows that there must be a trajectory restricted entirely on the unit circle which travels around and eventually repeats. In other words, there is a periodic orbit on the circle!

In general, a closed trajectory in the phase plane such that other non-closed trajectories spiral toward it, either from the inside or the outside, as $t \to \infty$, is called a limit cycle.

Theorem3.1: (Bendixon-du Lac Criterion): Consider the dynamical system;

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

In which functions F and G are assumed to be smooth in a given simply connected domain D (a domain with no holes). Let $B(x,y)$ be a smooth function in D such that;

$$\frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y}$$

does not change sign in D. Then the system has no closed orbits (trajectories) in D.

Proof: Suppose, to the contrary, that C is a closed trajectory in D. Then we have

by hypothesis.

$$I = \oint_C -BGdx + BFdy = \iint_{D_0} [\frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y}] \, dx \, dy \neq 0$$

Here D_0 represents the interior of C, and Green's theorm has been employed in the second equality. But, on the other hand, we have;

$$I = \int B(-G\frac{dx}{dt} + F\frac{dy}{dt})dt = \int -B(-GF + FG)dt = 0$$

This contradiction establishes the validity of the theorem.

Example: In case of a crowded environment where self-computation is there, the Lotka-Volterra equation is given by;

$$\begin{cases} \dfrac{dx}{dt} = x(a_1 - b_1 y - c_1 x) \\ \dfrac{dy}{dt} = y(-a_2 - b_2 y - c_2 x) \end{cases}$$

Where all the parameters are positive. Therefore, we show there is no periodic orbit in the first quadrant.

Consider the function $B(x, y) = (xy)^{-1}$. Then we have;

$$\frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y} = \frac{\partial}{\partial x}\left(y^{-1}(a_1 - b_1 y - c_1 x)\right) + \frac{\partial}{\partial y}(x^{-1}(-a_2 + b_2 y - c_2 x))$$

$$= -c_1 y^{-1} - c_2 x^{-1} < 0$$

So there is no periodic orbit in first quadrant. But in the case when there is no over crowding of species and self-computation $(i.e\, c_1 = c_2)$. We have seen that there are an infinite number of periodic orbits, which are trajectories of

$$L(x, y) = b_2 x - a_2 \log x + b_2 y - a_1 \log y.$$

Therom3.2: The system

$$\begin{cases} \frac{dx}{dt} = x(a_1 x + b_1 y + c_1) \\ \frac{dy}{dt} = y(a_2 x + b_2 y + c_2) \end{cases}$$

Has no limit cycle in first quadrant, provided that

$a_1 b_2 - a_2 b_1 \neq 0$ and $c_2 a_1 (b_1 - b_2) + c_1 b_2 (a_2 - a_1) \neq 0$

Proof: by applying Bendixon-duLac test with $B(x, y) = x^{\delta-1} y^{\gamma-1}$,

where $\delta = \frac{b_2(a_2 - a_1)}{a_1 b_2 - a_2 b_1}$ and

$$\gamma = \frac{a_2(b_1 - b_2)}{a_1 b_2 - a_2 b_1}$$

Now, $\frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y} = \mathrm{F}\frac{\partial B}{\partial X} + \mathrm{B}\frac{\partial G}{\partial Y} + \mathrm{G}\frac{\partial B}{\partial Y}$

$=(\delta - 1)\left(x^{\delta-2} y^{\gamma-1}\right)[x(a_1 x + b_1 y + c_1)] + x^{\delta-1} y^{\gamma-1}(2a_1 x + b_1 y + c_1) + x^{\delta-1} y^{\gamma-1}(a_2 x + 2b_2 y + c_2) + x^{\delta-1} y^{\gamma-2}(\gamma - 1)y((a_2 x + b_2 y + c_2)$

$$= (x^{\delta-1} y^{\gamma-1})\,[(\delta - 1)(a_1 x + b_1 y + c_1) + (2a_1 x + b_1 y + c_1) + (a_2 x + 2b_2 y + c_2) + (\gamma - 1)(a_2 x + b_2 y + c_2)\,]$$

$$= \left(x^{\delta-1} y^{\gamma-1}\right)(\delta a_1 + a_1 + \gamma a_2)x + (\delta a_1 + b_2 + \gamma b_2)y + \delta c_1 + \gamma c_2)$$

$$= \left(x^{\delta-1}y^{\gamma-1}\right)\left\{\left[a_1\left[\frac{b_2(a_2-a_1)}{a_1b_2-\ a_2b_1}\right]+a_1+a_2\left[\frac{a_1(b_1-b_2)}{a_1b_2-\ a_2b_1}\right]\right]x + \left[b_1\left[\frac{b_2(a_2-a_1)}{a_1b_2-\ a_2b_1}\right]+b_1+b_2\left[\frac{a_1(b_1-b_2)}{a_1b_2-\ a_2b_1}\right]\right]y + \left[c_1\left[\frac{b_2(a_2-a_1)}{a_1b_2-\ a_2b_1}\right]+c_1+c_2\left[\frac{a_1(b_1-b_2)}{a_1b_2-\ a_2b_1}\right]\right]\right\}$$

$$= (x^{\delta-1}y^{\gamma-1})\left\{\left[\frac{a_1b_2a_2-{a_1}^2b_2}{a_1b_2-a_2b_1}+\frac{{a_1}^2b_2-a_1b_2a_2}{a_1b_2-a_2b_1}+\frac{a_2a_1b_1-b_2a_2a_1}{a_1b_2-a_2b_1}\right]\mathrm{x} + \left[\frac{b_1b_2a_2-b_1b_2a_1}{a_1b_2-a_2b_1}+\frac{{b_2}^2a_1-b_1b_2a_2}{a_1b_2-a_2b_1}+\frac{b_2a_1b_1-b_2b_2a_1}{a_1b_2-a_2b_1}\right]\mathrm{y} + \left[\frac{c_1b_2(a_2-a_1)}{a_1b_2-\ a_2b_1}+\frac{c_2a_1(b_1-b_2)}{a_1b_2-\ a_2b_1}\right]\right\}$$

$$\frac{\partial(BF)}{\partial x}+\frac{\partial(BG)}{\partial y}=(x^{\delta-1}y^{\gamma-1})\left[\frac{c_1b_2(a_2-a_1)}{a_1b_2-\ a_2b_1}+\frac{c_2a_1(b_1-b_2)}{a_1b_2-\ a_2b_1}\right]$$

Since $K=\left[\frac{c_1b_2(a_2-a_1)}{a_1b_2-\ a_2b_1}+\frac{c_2a_1(b_1-b_2)}{a_1b_2-\ a_2b_1}\right]$ is a non-zero constant then;

- If, $K<0, then\ \frac{\partial(BF)}{\partial x}+\frac{\partial(BG)}{\partial y}<0$ in first quadrant.
- If $K>0\ then\ \frac{\partial(BF)}{\partial x}+\frac{\partial(BG)}{\partial y}>0$ in first quadrant.

In either case $\frac{\partial(BF)}{\partial x}+\frac{\partial(BG)}{\partial y}$ does not change sign in first quadrant. Hence there is no limit cycle for the given system in first quadrant.

In general; for the case of nonlinear dynamical system we cannot easily identify whether or not the system has limit cycle in the given region. Theorems like Hartman-Grobman the only secure for the property of the trajectories near in to the equilibrium point, but else, they (especially source trajectories emitted out from the equilibrium points) might proceed in that property or approach to a limit cycle after a while. Most results are only capable of guaranteeing limit cycles do not exist. They simply test if the system has no limit cycle; rather than whether it has limit cycle.

Conclusion

The main purpose of this project report is to examine the nature of stability in both linear and non-linear autonomous dynamical systems by using the eigenvalues of the fundamental matrix, the Jacobi linearization method and in the sense of Lyapunov. Moreover, autonomous dynamical systems apply to real-life applications in both linear and non-linear dynamical systems such as in electric circuits, motion of springs, damped pendulums and Lotka-Volterra or predator prey equations.

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