DSpace Institution

DSpace Repository http://dspace.org

Mathematics Thesis and Dissertations

2025-07

A study on the strong minimum/maximum principle and Liouville-type theorems for partial trace equations with nonlinear gradient terms

Bukayaw, Kindu

http://ir.bdu.edu.et/handle/123456789/16982

Downloaded from DSpace Repository, DSpace Institution's institutional repository

Bahir Dar University

College of Science
Department of Mathematics

A study on the strong minimum/maximum principle and Liouville-type theorems for partial trace equations with nonlinear gradient terms

By

Bukayaw Kindu

July, 2025
Bahir Dar, Ethiopia

Bahir Dar University
College of Science
Department of Mathematics

A study on the strong minimum/maximum principle and Liouville-type theorems for partial trace equations with nonlinear gradient terms

A Dissertation Submitted to the Department of Mathematics, College of Science, Bahir Dar University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

By
Bukayaw Kindu

Main supervisor: Ahmed Mohammed (Prof.)
(Ball State University)
Co-supervisor: Birilew Belayneh (PhD, Assoc. Prof.)
(Bahir Dar University)

Copyright ©2025 by Bukayaw Kindu

Declaration

This is to certify that the dissertation entitled "**A study on the strong minimum/maximum principle and Liouville-type theorems for partial trace equations with nonlinear gradient terms**", submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, Department of Mathematics, Bahir Dar University, is a record of original work carried out by me and has never been submitted to this or any other institution to get any other degree or certificates. The help and assistance I received during the course of this study have been acknowledged.

Bukayaw Kindu Ameshe		July 24, 2025 G.C
Name of PhD student	Signature	Date

Bahir Dar University

College of Science

Department of Mathematics

Approval of dissertation for oral defense

We hereby certify that we have supervised, read, and evaluated this dissertation entitled "**A study on the strong minimum/maximum principle and Liouville-type theorems for partial trace equations with nonlinear gradient terms**" by **Bukayaw Kindu** prepared under our guidance. We recommend that the dissertation be submitted for oral defense (mock-viva and viva voce).

Hunegnaw Dessie (Assoc. Prof.)		July 24, 2025 G.C
Department Head's Name	Signature	Date
Birilew Belayneh (Assoc. Prof.)		July 24, 2025 G.C
Co-Supervisor's Name	Signature	Date
Ahmed Mohammed (Professor)	Ahmed Mohammed	July 24, 2025 G.C
Main Supervisor's Name	Signature	Date

Bahir Dar University

College of science

Department of mathematics

Approval of dissertation for defense result

As members of the board of examiners, we examined this dissertation entitled "**A study on the strong minimum/maximum principle and Liouville-type theorems for partial trace equations with nonlinear gradient terms**" by **Bukayaw Kindu**. We hereby certify that the dissertation is accepted for the award of the degree of doctor of philosophy in mathematics.

Board of Examiners

Shiferaw Birhanu (Prof.)	S. Berhanu	July 24, 2025 G.C
External Examiner's Name 1	Signature	Date
Tadele Mengesha (Prof.)		July 24, 2025 G.C
External Examiner's Name 2	Signature	Date
Tsegaye Gedif (Assoc. Prof.)		July 24, 2025 G.C
Internal Examiner's Name	Signature	Date
Hunegnaw Dessie (Assoc. Prof.)		July 24, 2025 G.C
Chairperson's name	Signature	Date

Dedication

This dissertation is dedicated to the memory of my beloved mother, **Hymanotie Gashu** and my younger sister, **Sindu Kindu**.

Publications and Conference Presentations

Published papers

1. **B. Kindu**, A. Mohammed, and B. Tsegaw. A Vázquez-type strong minimum/maximum principle for partial trace operators. *Discrete and Continuous Dynamical Systems-S*, 2023, 16(11): 3146–3162. https://doi.org/10.3934/dcdss.2023115.

2. **B. Kindu**, A. Mohammed, and B. Tsegaw. Liouville-type theorems for partial trace equations with nonlinear gradient terms, *Journal of Mathematical Analysis and Applications, 543(2):129010, 2025.* https://doi.org/10.1016/j.jmaa.2024.129010

Under review paper

- On the existence of positive radial solutions to weighted partial trace equations with nonlinear gradient terms.

Conference presentations

- **B. Kindu**; A Vázquez-type strong minimum/maximum principle for partial trace operators. National Conference on Recent Trends in Scientific Research (NCRTSR–2024), which was held from April 12–13, 2024, at Bahir Dar University, Bahir Dar, Ethiopia.

- **B. Kindu**; Liouville-type theorems for partial trace equations with nonlinear gradient terms. The second workshop of "Ethio-Italy Colloquium on Applied Mathematics (ETICAM-2)", which was held in Addis Ababa, Ethiopia, was hosted by the Department of Mathematics at Addis Ababa University from 10$^{\text{th}}$ to 13$^{\text{th}}$ in February 2025.

Seminar presentation

- A study of Liouville-type theorems for partial trace equations with nonlinear gradient terms. Friday seminar: presented on 11 February 2024, Department of Mathematics, Bair Dar University, Bahir Dar, Ethiopia.

Acknowledgment

First and foremost, I would like to express my heartfelt gratitude to God, the son of the Virgin Mary, for His grace, faithfulness, and unwavering love throughout my academic journey, leading up to the development of my doctoral studies.

I express my sincere appreciation to **Prof. Ahmed Mohammed** and **Dr. Birilew Belayneh** for their insightful recommendations, invaluable comments, and continued professional support. Their enthusiastic guidance, persistent dedication, and genuine care have greatly contributed to my studies. I am deeply grateful for their willingness to generously invest their time.

Furthermore, I am deeply grateful to the Department of Mathematics at Bahir Dar University for fostering a warm and friendly atmosphere, as well as dedicated services throughout my studies. I would like to thank the faculty members in the Department of Mathematics, particularly Prof. Birhanu Assaye, Prof. Tsegaye Kassa, Dr. Mollalign Haile, Dr. Mulat Gebeyehu, Dr. Abdu Mohammed, Dr. Assaye Walelgn, Prof. Awoke Andargie, Dr. Eshetu Haile, Dr. Gurju Awugchew, Dr. Bekalu Tarekegn, Dr. Endalew Getnet, Dr. Getachew Mehabie, and Dr. Hunegnaw Dessie for their moral support.

I also want to thank my colleagues Habtamu Getachew, Gedefaw Mekuriaw, Tamiru Melese, Wubale Demis, Dagnachew Jenber, Yibekal Walle, Yassin Tesfaw, Maschal Yeshitla, Alachew Amaneh, Zenaw Asnake, Dagnaw Tantie, and Demissache Adem, who provided me with sincere friendship and delightful moments.

Special thanks go to my wife **Hana Wondafrash**, for her love, understanding, and sacrifice for our family. I am also deeply indebted to my father, **Ato Kindu Ameshe**, for his sacrifices and support, which is a constant source of strength in my life. I pray for God's blessings on him for a long and healthy life.

Abstract

The purpose of this dissertation is to study the strong minimum/maximum principle and to investigate various Liouville-type theorems for partial trace equations with nonlinear gradient terms. To investigate the problems and obtain the results presented in this dissertation, we will use various principles (minimum and maximum principles, a comparison principle for viscosity sub/supersolutions, boundary principles, and compact support principles), concepts of duality, and different mathematical estimates and inequalities.

In this study, we examine a strong minimum principle of Vázquez type for partial trace operators with gradient terms. More explicitly, given a n-tuple $\mathbf{a} = (a_1, \cdots, a_n)$ of non-negative real numbers with $a_n > 0$, we give sufficient conditions on a continuous function h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ in order for non-negative viscosity supersolutions of

$$\mathcal{P}_{\mathbf{a}}(D^2u) = h(u, |Du|) \tag{$\bigstar$}$$

in connected open subsets of $\mathbb{R}^n$ that vanish at some point to Ω vanish identically in Ω. When h depends only on the gradient, the condition is also necessary. Here $\mathcal{P}_{\mathbf{a}}$ belongs to a class of fully nonlinear degenerate elliptic operators that includes the Min-Max operator which is defined as the sum of the minimum and the maximum eigenvalues of the Hessian matrix. Under suitable conditions on h and $\mathbf{a} = (a_1, \cdots, a_n)$, both the Strong Maximum Principle and the Compact Support Principle for subsolutions are also investigated. The work covers a new class of degenerate operators and a wide class of Hamiltonians not investigated in the literature and some of the results are new even when $\mathcal{P}_{\mathbf{a}}$ reduces to the standard Laplacian. Illustrative examples are presented for such equations.

We investigate various Liouville-type theorems for partial trace equations with nonlinear gradient terms. Specifically, we establish sufficient conditions for which

their viscosity subsolutions vanish identically in $\mathbb{R}^n$. For a prototype of such equations, that is, when $h(u,|Du|) = f(u) + g(u)|Du|^q$ for $0 < q < 2$ in ($\bigstar$), we give necessary and sufficient conditions for viscosity subsolutions identically zero in $\mathbb{R}^n$.

Our results are important in the field of mathematical analysis. They serve as a fundamental tool for analyzing the existence and uniqueness, regularity, and blow-up estimate behavior of solutions for practical problems.

Keywords and Phrases. Fully nonlinear degenerate elliptic operators, Minimum/Maximum principle, Weighted partial trace operator, Comparison principle, Viscosity solutions, Liouville theorems, Keller-Osserman conditions.

Contents

Acronyms and Notations

Acronyms

Abbreviations	**Description**
PDEs	Partial differential equations
LSC(Ω)	Set of all lower semicontinuous functions in Ω
USC(Ω)	Set of all upper semicontinuous functions in Ω
SMP	Strong maximum principle
SmP	Strong minimum principle
CSP	Compact support principle
WMP	Weak maximum principle
WmP	Weak minimum principle

Symbols and Notations

$\mathbb{R}$	Entire real line
$\mathbb{R}^+$	$(0, \infty)$
$\mathbb{R}^+_0$	$[0, \infty)$
$\mathbb{R}^n$	Euclidean space of n dimensions
o	Origin in $\mathbb{R}^n$
$o(\|x - x_0\|^2)$	higher-order terms that decay faster than $\|x - x_0\|^2$ as $x \to x_0$.
(x', x_n)	Ordered n-tuples $x = (x_1, \cdots, x_{n-1}, x_n)$ in $\mathbb{R}^n$
$\mathbb{S}^n$	Set of all $n \times n$ real symmetric matrices
$\langle \cdot, \cdot \rangle$	Euclidean inner product
$x^\perp$	$\{y \in \mathbb{R}^n : \langle x, y \rangle = 0\}$ is the orthogonal complement of $x \in \mathbb{R}^n$
Ω	Open subset of $\mathbb{R}^n$

$\overline{\Omega}$	Topological closure of Ω
$\partial\Omega = \overline{\Omega} \setminus \Omega$	Topological boundary of Ω
$\operatorname{dist}(x, \partial\Omega)$	Distance of $x \in \Omega$ to the boundary $\partial\Omega$ of Ω
$C(\Omega)$	Space of all continuous real-valued functions in Ω
$C^k(\Omega)$	Space of k times continuously differentiable functions in Ω
$C^\infty(\Omega)$	Space of infinitely continuously differentiable functions in Ω
I_n	$n \times n$ identity matrix
$\mathbf{0}$	$n \times n$ zero matrix
D_i	$\dfrac{\partial}{\partial x_i}, \quad i = 1, 2, \cdots, n$
$Du = \nabla u$	Gradient of u
$\lvert \cdot \rvert$	Euclidean norm
D_{ij}	$\dfrac{\partial^2}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \cdots, n$
Δ	Laplacian operator
Δ_p	p-Laplacian operator
$D^2 u$	Hessian matrix of u
$\lambda_j(X)$	Eigenvalues of $X \in \mathbb{S}^n$, with $\lambda_i(X) \leq \lambda_{i+1}(X)$ for all $i = 1, 2, \cdots, n-1$
$X \leq Y$	$Y - X$ is a positive semidefinite matrix
A^T	Transpose of an $m \times n$ real matrix A
$\mathcal{P}_{\mathbf{a}}(X)$	$\displaystyle\sum_{j=1}^{n} a_j \lambda_j(X)$, where $\mathbf{a} := (a_1, \cdots, a_n) \in [0, \infty)^n$ with $\lvert\mathbf{a}\rvert := a_1 + \cdots + a_n$
$\mathcal{A}_j$	$\{\mathcal{P}_{\mathbf{a}} : a_j > 0\}$ for $1 \leq j \leq n$
$\bigotimes$	Tensor product or kronecker product
$B_r(x_0)$	Open ball with radius r and centre x_0
$\overline{B}_r(x_0)$	Topological closure of $B_r(x_0)$

Chapter 1

General Introduction

1.1. Background of the study

Historically, partial differential equations (PDEs) originated from the study of surfaces in geometry and various problems in mechanics. The use of PDEs began in the 18^{th} century with the work of Euler, D'Alembert, Lagrange, and Laplace as a central tool in the description of physical sciences [12]. Partial differential equations are used to model a wide range of phenomena, in particular in physics, engineering, chemistry, biology, and finance. For instance, they are fundamental in the modern understanding of sound, fluid dynamics, elasticity, general relativity, and quantum mechanics. They also play an important role in "pure mathematics", in particular, in geometry and analysis [28]. Among the most important equations in the application of PDEs are the second-order partial differential equations. In particular, nonlinear partial differential equations play an important role in science and engineering.

The maximum principle is one of the most useful and well-known tools employed in the study of PDEs. To investigate phenomena in both natural and social sciences, it is important to analyze solutions to PDEs derived from certain maximization-minimization principles. It enables us to obtain information about solutions (uniqueness, approximation, boundedness and symmetry, necessary conditions of solvability, etc. [21]) to differential equations and inequalities without explicit knowledge of the solutions.

The maximum principle of Eberhard Hopf is the classical and bedrock result of the theory of second-order elliptic PDEs and has an enormous number of important applications. It goes back to the maximum principle for harmonic functions, which was already known to Gauss in 1839. On the other hand, it carries forward the maximum principles of Gilbarg, Trudinger, and Serrin and the maximum principles initiated particularly by Vázquez and Díaz in the 1980's [66].

A well-known fact in the theory of harmonic function states that any nonnegative superharmonic function in a connected open set must either vanish identically or be strictly positive everywhere. This property is referred to as the Strong Minimum Principle, hereafter shortened as SmP.

The most well-known notion of solutions for PDEs of divergence type is weak solutions, which are weaker than classical solutions in the distribution sense. The failure of weak solutions to describe physical situations everywhere in a global domain and the need for the application of nonlinear PDEs in science, economics, and engineering to have global solutions have become interesting issues for many mathematicians and engineers over the past many years. As a consequence of this, the notion of a "viscosity solution" has been introduced.

The name of this theory originated from the "vanishing viscosity method", developed first for first-order fully nonlinear PDE (Hamilton-Jacobi PDE). The formal notions were introduced by P. L. Lions and M. G. Crandall in the early 1980s for first-order PDE, following the preceding contributions of L. C. Evans. The extension to the case of second-order PDEs came around the 1990s, as noted by H. Ishii and P. L. Lions [48]. This theory of viscosity solutions is developed for fully nonlinear degenerate second order PDEs (for reasons such as existence, uniqueness, stability, regularity, and so on). For second-order elliptic PDEs of nondivergence type, the notion of viscosity solutions is the most appropriate one for several reasons [52]. The primary advantages of this theory are:

- It allows merely continuous functions to be solutions of fully nonlinear second-order PDEs
- It provides general existence and uniqueness theorems,
- It yields precise formulations of general boundary conditions.

The notion of a viscosity solution has a wide range of applications, including common classes of PDE such as evolutionary problems and boundary value problems. It arises in optimal control theory (the Hamilton-Jacobi-Bellman equation), in differential games (the Isaacs equation), in stochastic optimal control (stochastic differential games), in geometric equations (mean curvature and Monge–Ampére equations), and so on [52].

Liouville's theorem is named after a French mathematician and engineer, Joseph Liouville (1809–1882), who noted that the only bounded holomorphic functions in the complex plane are constants. This was generalized to solutions of elliptic equations of various forms, and the results take different forms. Liouville-type theorems are powerful tools in elliptic partial differential equations in establishing a priori estimates for non-negative solutions. These theorems help us to understand the behavior of solutions and investigate the existence of solutions to different problems, for instance, periodic problems, optimal control problems, etc. [1, 40].

1.2. Statement of the problem

While the strong minimum/maximum principle and Liouville-type theorems are established for many PDEs, not enough attention is given to SmP (or SMP) and Liouville-type theorems for partial trace equations with nonlinear gradient terms. In this dissertation, we work to fill this gap and study a fully nonlinear degenerate elliptic equation

$$\mathcal{P}_{\mathbf{a}}(D^2u) = h(u, |Du|), \quad \text{in } \mathbb{R}^n \tag{1.2.1}$$

where the operator $\mathcal{P}_{\mathbf{a}}(\cdot)$ is the weighted partial trace operator, which will be defined in the next chapter. The function h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is continuous and satisfies some conditions. This study will attempt to

1. prove the SmP for (1.2.1) with suitable assumptions on h and the n-tuple $\mathbf{a}$ for $\mathcal{P}_{\mathbf{a}}$.

   Problem 1: Suppose h satisfies the following conditions for some constant $\nu > 0$:

   (M): $s \mapsto s + h(s, t_0)$ and $t \mapsto t + h(s_0, t)$ are non-decreasing in $(0, \nu)$ for each s_0, $t_0 \in (0, \nu)$,

   (P): $t + h(t, t)$ is positive in $(0, \nu)$, and

$$\int_0^{\nu} \frac{dt}{t + h(t, t)} = \infty.$$

Under the assumptions **(M)**, **(P)**, and $a_n > 0$, the behavior of nonnegative viscosity supersolutions u of (1.2.1) can be characterized in an open connected subset $\Omega \subseteq \mathbb{R}^n$. Specifically, if u is any non-negative viscosity supersolution of (1.2.1), it must be identically zero or strictly positive throughout Ω.

2 prove the SMP for (1.2.1) with suitable assumptions on h and the n-tuple $\mathbf{a}$ for $\mathcal{P}_{\mathbf{a}}$.

Problem 2: The following additional conditions are required to validate the SMP, in addition to **(M)** and **(P)**.

($\boldsymbol{\mathcal{D}_1}$): $t \mapsto h(s,t)$ is non-decreasing in $\mathbb{R}_0^+$ for each $s \in \mathbb{R}$.

($\boldsymbol{\mathcal{D}_2}$): $h(-s,0) + h(s,0) \geq 0$ for all $s \in \mathbb{R}$.

Assume **(M)**, **(P)**, ($\boldsymbol{\mathcal{D}_1}$), ($\boldsymbol{\mathcal{D}_2}$), and $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_1$ hold. If $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of (1.2.1) that satisfies $u(x) \leq u(x_0)$ and $u(x_0) \geq 0$ for some $x_0 \in \Omega$, it can be shown that $u \equiv u(x_0)$ in Ω. We obtain a SMP for the subsolutions of (1.2.1) from the SmP by applying duality to $\mathcal{P}_{\mathbf{a}}$.

3 show the Strong Boundary Minimum Principle for (1.2.1) with conditions **(M)**, **(P)** hold and $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_n$ in an open subset Ω of $\mathbb{R}^n$.

Problem 3: Suppose $x_0 \in \partial\Omega$ and $u \in \mathrm{LSC}(\Omega \cup \{x_0\})$ is a viscosity supersolution of (1.2.1) that satisfies

(i) $u(x) \geq u(x_0) = 0$ for all $x \in \Omega$,

(ii) there is a ball $B(z,r) \subset \Omega$ with $B(z,r) \cap \partial\Omega = \{x_0\}$.

Then

$$\liminf_{\substack{x \to x_0 \\ x \in \mathcal{C}_{x_0}}} \frac{u(x) - u(x_0)}{|x - x_0|} > 0,$$

where $\mathcal{C}_{x_0}$ is any circular cone with vertex x_0 and opening less than π with axis along the normal direction at x_0.

4 show that the Strong Boundary Maximum Principle for (1.2.1) within an open connected subset $\Omega \subset \mathbb{R}^n$ and h satisfies conditions ($\boldsymbol{\mathcal{D}_1}$) and ($\boldsymbol{\mathcal{D}_2}$). Furthermore, we assume conditions **(M)**, **(P)** hold and $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_1$.

Problem 4: Let $x_0 \in \partial\Omega$ and $u \in \mathrm{USC}(\Omega \cup \{x_0\})$ be a viscosity subsolution of (1.2.1) such that

(i) $u(x) \le u(x_0) = 0$ for all $x \in \Omega$,

(ii) there is a ball $B(z,r) \subset \Omega$ with $B(z,r) \cap \partial\Omega = \{x_0\}$.

Then

$$\liminf_{\substack{x \to x_0 \\ x \in \mathcal{C}_{x_0}}} \frac{u(x_0) - u(x)}{|x - x_0|} > 0,$$

where $\mathcal{C}_{x_0}$ is as defined in problem 3.

5 show that the Compact Support Principle (CSP) holds for subsolutions of (1.2.1) in $\Omega = \mathbb{R}^n \setminus \overline{B}(0, r_0)$, where $r_0 > 0$.

Problem 5: For any non-negative viscosity subsolution u of (1.2.1) in $\Omega = \mathbb{R}^n \setminus \overline{B}(0, r_0)$ such that $u(x) \to 0$ as $|x| \to \infty$, we prove that $u \equiv 0$ in $\mathbb{R}^n \setminus B(0, R)$ for some $R > r_0$, provided that the following assumptions on h hold.

There exists a constant $\nu > 0$ such that

(H-1): $h(0,t) > 0$ for all $t > 0$, the map $s \mapsto h(s,t)$ is strictly increasing in $[0, \nu)$ for all $t \in \mathbb{R}_0^+$, and

$$\int_0^{\nu} \frac{dt}{h(0,t)} < \infty.$$

(H-2): $h(s,0) \ge 0$ for all $s \ge 0$, the map $t \mapsto h(s,t)$ is strictly increasing in $\mathbb{R}_0^+$ for all $s \in (0, \nu)$, and

$$\int_0^{\nu} \frac{dt}{\sqrt{\mathcal{H}(t)}} < \infty,$$

where

$$\mathcal{H}(t) := \int_0^t h(s,0)\, ds, \quad s > 0. \tag{1.2.2}$$

6 prove various Liouville-type theorems for (1.2.1) with suitable assumptions on the n-tuple $\mathbf{a}$ and h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ that satisfy

(H-1): $s \mapsto h(s,t)$ is increasing on $\mathbb{R}_0^+$ for each $t \geq 0$.

(H-2): $\displaystyle\int_0^1 \frac{dt}{h(\varrho, t)} < \infty$ for sufficiently small $\varrho > 0$.

(H-2)$^\circ$: $\displaystyle\int_1^\infty \frac{dt}{h(t,t)} = \infty$.

(H-2)*: $\displaystyle\int_1^\infty \frac{ds}{\sqrt{\mathcal{H}(s)}} < \infty$, where $\mathcal{H}$ is defined in (1.2.2).

(H-3): $h(s,0) > 0$ for all $s > 0$ and $h(0,0) = 0$.

(H-3)*: $h(c,0) = 0$ for exactly one $c \in \mathbb{R}$.

Problem 6: Under suitable assumptions on the n-tuple $\mathbf{a} := (a_1, \cdots, a_n)$, we prove the following Liouville-type theorems:

(a) Suppose that **(H-1)**, **(H-2)**, and **(H-3)** hold, and let $\mathbf{a} = (a_1, \cdots, a_n)$ be a n-tuple of non-negative real numbers with $a_1 a_n > 0$. If u is a non-negative viscosity subsolution of (1.2.1), then $u \equiv 0$ in $\mathbb{R}^n$.

(b) If **(H-1)**, **(H-2)**^{0} and **(H-3)** hold, then (1.2.1), with $a_n > 0$ admits a positive radial viscosity solution.

(c) Suppose that **(H-1)**, **(H-2)*** and **(H-3)** hold. If u is a non-negative subsolution of (1.2.1) in $\mathbb{R}^n$, then $u \equiv 0$ in $\mathbb{R}^n$.

(d) Suppose that u is a viscosity solution of (1.2.1) and $u(x) = o(|x|^2)$ as $|x| \to \infty$. If there is exactly one $c \in \mathbb{R}$ such that $h(c,0) = 0$, then $u \equiv c$ in $\mathbb{R}^n$.

1.3. Objective of the study

1.3.1. General Objective

The main objective of this study is to examine a Vázquez-type strong minimum/maximum principle and formulate various Liouville-type theorems for a partial trace equation

$$\mathcal{P}_{\mathbf{a}}(D^2 u) = h(u, |Du|), \tag{1.3.1}$$

where h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function in a connected open set $\Omega \subseteq \mathbb{R}^n$.

1.3.2. Specific Objectives

The specific objectives of this study are summarized as follows.

1. To extend the Strong Minimum Principle (SmP) for non-negative supersolutions of the weighted partial trace equations involving nonlinear gradient terms.

2. To examine the Strong Maximum Principle (SMP) and the Compact Support Principle (CSP) for non-negative subsolutions of the weighted partial trace equations involving nonlinear gradient terms.

3. To examine the comparison principle and the strong boundary minimum/-maximum principle for weighted partial trace equations.

4. To investigate Liouville-type properties for viscosity subsolutions of weighted partial trace equations.

5. To provide the necessary and sufficient conditions for prototype weighted partial trace equations whose non-negative subsolutions to be identically zero in $\mathbb{R}^n$.

1.4. Significance of the study

The strong minimum/maximum principle and the Liouville-type theorems are crucial tools in both theoretical and practical PDE problems as well as other mathematical fields. Its significance is

1. to provide a good understanding of applications of the minimum/maximum principle and Liouville-type theorems for solving practical problems in various fields of mathematics.

2. to contribute for further advancement of the theory of more general weighted partial trace operators.

3. to provide background information for further research in the broader context of fully nonlinear partial differential equations.

In summary, studying the strong minimum/maximum principle and the Liouville-type theorems for weighted partial trace equations with nonlinear gradient terms provides fundamental results that can have far-reaching consequences for the theory and applications of different classes of PDEs.

1.5. Organization of the Dissertation

Following this introductory chapter, the remaining part of the dissertation is organized as follows.

1. In **Chapter 2**, we present a brief introduction to the basic concepts and definitions of the weighted partial trace operator, viscosity solution, and minimum/maximum principle that can be used in the next chapters.

2. In **Chapter 3**, we present a summary of previous research findings and identify gaps in the research problems under consideration.

3. In **Chapter 4**, we investigate the comparison principle, strong minimum/-maximum principles, strong boundary minimum/maximum principles, and compact support principle for weighted partial trace equations involving nonlinear gradient terms. The results obtained as propositions, lemmas, and theorems are stated along with their respective proofs and illustrative examples.

4. In **Chapter 5**, we examine various Liouville-type theorems for weighted partial trace equations with nonlinear gradient terms.

   Finally, we present our conclusions and future work.

1.6. Research Methodology

To investigate the problems and obtain the results presented in this dissertation, we apply various principles (minimum and maximum principles, a comparison principle for viscosity sub/supersolutions, boundary principles, and compact support principles), concepts of duality, and different mathematical estimates and inequalities.

Chapter 2

Preliminary Concepts

The purpose of this chapter is to provide a brief overview of the main concepts, definitions, and notations that we will use later to establish the various results presented in the study. It covers several fundamental ideas, including weighted partial trace operators, viscosity solutions, minimum/maximum principles, Liouville-type theorems, and properties of non-decreasing functions. The goal is to introduce the reader to the key theoretical foundations of these preliminary concepts, which we will use in the subsequent chapters.

For a more comprehensive understanding, the chapter suggests referring to the works of L. A. Caffarelli and X. Cabré [16], Pucci et al. [65], Galise et al. [38], and the references therein.

2.1. Basic Definitions

Here and elsewhere in the following chapters, we use the following notation to denote partial derivatives.

- The partial derivatives of u in x are defined as

$$\frac{\partial u(x)}{\partial x_i} := \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} \quad \text{(if the limit exists)},$$

  for $i = 1, \cdots, n$, where e_i denotes the standard basis vector i of $\mathbb{R}^n$. Commonly used notations are $\dfrac{\partial u}{\partial x_i} = \partial_{x_i} u = u_{x_i}$.

- The second-order partial derivatives are defined as

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i}\left(\frac{\partial u(x)}{\partial x_j}\right) \quad \text{(if they exist)},$$

for $i, j = 1, \cdots, n$. Commonly used notations are $\dfrac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j} = \partial^2_{x_i x_j} u$.

- Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ be a multiindex. Its order is defined as:

$$|\alpha| := \sum_{i=1}^{n} \alpha_i,$$

and the corresponding $|\alpha|^{\text{th}}$ order partial derivatives of u are

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x) \quad \text{(if they exist)}.$$

Moreover, for $k \in \mathbb{N}$, we denote the collection of all k-th order partial derivatives of u in x by

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}.$$

As usual, we write $D^1 u(x)$ as a column vector

$$D^1 u(x) = Du(x) = \begin{bmatrix} \frac{\partial u(x)}{\partial x_1} \\ \vdots \\ \frac{\partial u(x)}{\partial x_n} \end{bmatrix} \quad \text{(gradient)},$$

and $D^2 u(x)$ as a matrix,

$$D^2 u(x) = \begin{bmatrix} \frac{\partial^2 u(x)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 u(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u(x)}{\partial x_n \partial x_n} \end{bmatrix} \quad \text{(Hessian matrix)}.$$

Note that $D^2 u(x)$ is a symmetric $n \times n$ matrix.

Definition 2.1.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open subset, $n \geq 2$ and $k \in \mathbb{N}$. An equation of the form

$$F(x, u(x), Du(x), \cdots, D^{k-1}u(x), D^k u(x)) = 0, \quad x \in \Omega, \qquad (2.1.1)$$

is called a k-th order PDE, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is a given function and u: $\Omega \to \mathbb{R}$ is the unknown function.

A classical solution of the PDE is a k-time continuously differentiable function $u : \Omega \to \mathbb{R}$ that satisfies (2.1.1).

Definition 2.1.2. Depending on the structure of the function F in (2.1.1), we classify PDEs as follows:

- The PDE (2.1.1) is **linear** if the function F is linear in u and its derivatives, that is if it is of the form

  $$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) + f(x) = 0,$$

  for given functions a_α and f. Moreover, if $f \equiv 0$, the PDE is called homogeneous and inhomogeneous otherwise.

- The PDE (2.1.1) is **semilinear** if F is linear in the highest-order derivatives, that is, if it is of the form

  $$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + a_0(D^{k-1}u(x), \ldots, u(x), x) = 0,$$

  for given functions a_α and a_0.

- The PDE (2.1.1) is **quasi-linear** if it is of the form

  $$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u(x), \cdots, u(x), x) D^\alpha u(x) + a_0(D^{k-1}u(x), \cdots, u(x), x) = 0,$$

  for given functions a_α and a_0.

- The PDE (2.1.1) is **fully nonlinear** if F is a nonlinear function of the highest order derivatives $D^k u$.

Definition 2.1.3. Let $u \in C^2(\Omega)$ be a real-valued function. Let us now consider a general second-order partial differential equation (PDE) of the form

$$\mathcal{F}[u] := F(x, u, Du, D^2u) = 0, \quad x \in \Omega \subseteq \mathbb{R}^n, \tag{2.1.2}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is a continuous function and $\mathbb{S}^n$ denotes the set of real symmetric $n \times n$ matrices. The set $\mathbb{S}^n$ is equipped with the standard partial ordering: for $X, Y \in \mathbb{S}^n$, we write $X \geq Y$ if $\langle (X - Y)\xi, \xi \rangle \geq 0$ for all $\xi \in \mathbb{R}^n$.

If the highest-order coefficients of (2.1.2) depend only on $(x, u, Du) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, then the equation is called a **quasilinear** equation, and it takes the form

$$Qu(x) := \sum_{i,j=1}^{n} a_{ij}(x, u(x), Du(x))D_{ij}u(x) + b(x, u(x), Du(x)). \tag{2.1.3}$$

A quasilinear equation (2.1.3) is called **semilinear** if its highest-order term is linear in $D^2u(x)$, that is,

$$Pu(x) := \sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) + b(x, u(x), Du(x)).$$

If all coefficients depend only on $x \in \Omega$, the equation is referred to as **linear**:

$$Lu(x) := \sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b_i(x)D_iu(x) + c(x)u(x),$$

where $D_iu = \partial u/\partial x_i$, $D_{ij}u = \partial^2 u/\partial x_i \partial x_j$, and $a_{ij}(x), b_i(x), c(x)$ are given functions defined on the open set $\Omega \subseteq \mathbb{R}^n$. Without loss of generality, we assume that the coefficient matrix of (2.1.3) satisfies symmetry: $a_{ij}(x) = a_{ji}(x)$ for all i, j.

Note that the coefficient matrix $A = (a_{ij})$ can depend on x, $u(x)$, and $Du(x)$. So, the eigenvalues λ and Λ also depend on x.

By definition, equation (2.1.3) is called **elliptic** if $\lambda(x) > 0$ for all $x \in \Omega$. **Strictly elliptic** if there exists ϵ such that $\inf_{x\in\Omega} \lambda(x) \geq \epsilon > 0$. This is equivalent to

$$\inf_{x\in\Omega} \langle A(x)\xi, \xi \rangle \geq \epsilon |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Uniformly elliptic if $\sup_{x\in\Omega} \Lambda(x) < \infty$.

A **fully nonlinear** equation of the form (2.1.2) is said to be **elliptic** if the matrix $\left(F_{ij}\right)_{i,j=1}^{n} := \left(\dfrac{\partial F}{\partial u_{x_i x_j}}\right)$ is positive definite.

Consider the following examples.

1. The differential equation with the Laplace operator

$$\Delta u = \operatorname{div}(Du) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2},$$

is linear and uniformly elliptic.

2. The differential equation with mean curvature operator

$$\mathcal{H}(u) = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\Delta u(1+|Du|^2) - |Du|^2}{\left(1+|Du|^2\right)^{3/2}}$$

is a second order quasi-linear elliptic PDE, which is nonlinear as $\mathcal{H}(\alpha u + \beta v) \neq \alpha\mathcal{H}(u) + \beta\mathcal{H}(v)$ for some functions u, v: $\Omega \to \mathbb{R}$.

3. The differential equation with the Monge-Ampère operator

$$\mathcal{K}(u) = \det(D^2 u),$$

is fully nonlinear.

4. The p-Laplace equation: This equation generalizes the Laplace equation $\Delta u = 0$ by introducing a power $p > 1$ in the equation. It is given by:

$$-\operatorname{div}(|Du|^{p-2} Du) = f(x, u, Du)$$

where Du represents the gradient of u and f is a nonlinear function.

5. The Allen-Cahn equation: This equation arises in the study of phase transitions and pattern formation. It is a semi-linear elliptic equation given by:

$$-\Delta u + u - u^3 = 0$$

where Δ represents the Laplacian operator.

6. The Hamilton-Jacobi equation: This equation appears in optimal control theory and Hamiltonian mechanics. It is a quasi-linear elliptic equation

given by:

$$h(x, Du) - \mathrm{tr}(\mathrm{A}(x)D^2u) = 0$$

where h represents a Hamiltonian function, Du denotes the gradient of u, $\mathrm{A}(x)$ is a symmetric matrix, D^2u is the Hessian matrix of second derivatives, and $\mathrm{tr}(\mathrm{A})$ denotes the trace of matrix A.

Definition 2.1.4. A second-order differential operator $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is said to be **degenerate elliptic**, if

$$X \leq Y \text{ in } \mathbb{S}^n \implies \mathcal{F}(x, r, p, X) \leq \mathcal{F}(x, r, p, Y) \quad \forall (x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

For example, $\Delta u = 0$ (where Δ denotes the Laplacian) is a degenerate elliptic differential equation, since

$$\mathrm{tr}(X) = \mathcal{F}(x, u, p, X) \leq \mathcal{F}(x, u, p, Y) = \mathrm{tr}(Y) \quad \text{for } X \leq Y.$$

Definition 2.1.5. We say that a differential operator $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is **uniformly elliptic** if there exist constants $0 < \lambda \leq \Lambda$ such that for any positive definite matrix A and any $X \in \mathbb{S}^n$, it holds that

$$\lambda \,\mathrm{tr}(\mathrm{A}) \leq \mathcal{F}(x, r, p, X + \mathrm{A}) - \mathcal{F}(x, r, p, X) \leq \Lambda \,\mathrm{tr}(\mathrm{A}). \tag{2.1.4}$$

Example 2.1.1. Note that $\mathcal{F}[u] = \Delta u$, which means $\mathcal{F}(x, r, p, X) = \mathrm{tr}(X)$. Since $\mathrm{tr}(X + N) = \mathrm{tr}(X) + \mathrm{tr}(N)$ for all positive semidefinite matrices N, we see that $\mathcal{F}$ is uniformly elliptic with $\lambda = 1 = \Lambda$.

When the operator $\mathcal{F}$ is differentiable, the inequalities (2.1.4) are equivalent to

$$\lambda I \leq \frac{\partial \mathcal{F}}{\partial X_{ij}} \leq \Lambda I.$$

Uniform ellipticity is the key concept in developing general regularity results for elliptic PDEs.

In linear algebra, it is known that if A and B are two $n \times n$ real symmetric matrices with $\mathrm{A} \leq \mathrm{B}$, then the corresponding eigenvalues $\lambda_j(\mathrm{A}) \leq \lambda_j(\mathrm{B})$ for each $j = 1, 2, \cdots, n$. A symmetric matrix has the following properties.

– Its eigenvalues are real,

– Its eigenvectors corresponding to eigenvalues are orthogonal,

– It is diagonalizable.

Definition 2.1.6. A $n \times n$ matrix A is said to be similar to another $n \times n$ matrix B if and only if there exists an invertible $n \times n$ matrix P such that

$$\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP}.$$

A symmetric matrix is similar to its diagonalization, whose entries are the eigenvalues. Two similar matrices have the same rank, trace, determinant, and eigenvalues.

Lemma 2.1.1. Every symmetric matrix can be uniquely decomposed as a difference of two non-negative symmetric matrices whose product is null. If $M \in \mathbb{S}^n$, then $M = M^+ - M^-$, where $M^\pm \in \mathbb{S}^n$, $M^\pm \geq \mathbf{0}$ and $M^+M^- = \mathbf{0}$. The matrices M^+ and M^- are called the positive and negative parts of M, respectively.

Proof. Let us first assume that M is a diagonal matrix: $M = D$ with $D_{ij} = 0$ for $i \neq j$. The diagonal matrices E and F with diagonal entries defined for $i = 1, \cdots, n$ by

$$E_{ii} = (D_{ii})^+ = \max\{D_{ii}, 0\}, \quad F_{ii} = (D_{ii})^- = -\min\{D_{ii}, 0\}$$

give the required decomposition. To prove the uniqueness of this decomposition, assume $D = \overline{E} - \overline{F}$ with $\overline{E}$, $\overline{F} \geq \mathbf{0}$, $\overline{EF} = \mathbf{0}$ and prove that $E = \overline{E}$, $F = \overline{F}$. In fact, for $i = 1, \cdots, n$

$$\begin{aligned}
0 &= \sum_{j=1}^{n} \overline{E}_{ij}\overline{F}_{ij} = \sum_{j=1}^{n} \left(D_{ij} + \overline{F}_{ij}\right)\overline{F}_{ij} \\
&= D_{ii}\overline{F}_{ii} + \overline{F}_{ii}^{\,2} + \sum_{i \neq j} \overline{F}_{ij}^2 \qquad (\because \quad D_{ij} = 0 \text{ for } i \neq j) \\
&= \left(D_{ii} + \overline{F}_{ii}\right)\overline{F}_{ii} + \sum_{j \neq i} \overline{F}_{ij}^2 = \overline{E}_{ii}\overline{F}_{ii} + \sum_{i \neq j} \overline{F}_{ij}^2 \\
&= \sum_{i \neq j} \overline{F}_{ij}^2
\end{aligned}$$

which implies that

$$\overline{F}_{ij} = 0, \quad \forall i \neq j.$$

Then $\overline{E}$ and $\overline{F}$ are diagonal matrices and $D_{ii} = \overline{E}_{ii} - \overline{F}_{ii}$. Taking into account $\overline{EF} = \mathbf{0}$, if $D_{ii} > 0$ it follows $\overline{E}_{ii} > \overline{F}_{ii}$, $\overline{F}_{ii} = 0$ and $\overline{E}_{ii} = D_{ii}$. Similarly, if $D_{ii} < 0$, then $\overline{E}_{ii} < \overline{F}_{ii}$, $\overline{E}_{ii} = 0$ and $D_{ii} = -\overline{F}_{ii}$. Thus, $\overline{E}_{ii} = \overline{F}_{ii}$ if $D_{ii} = 0$. This proves the assertion for the diagonal case. The positive and negative parts of D are

$$D^+ = (D_{ii})^+ \quad \text{and} \quad D^- = (D_{ii})^-.$$

Now, let us prove the general case. Since M is symmetric and diagonalizable, there exists an orthogonal matrix O such that $M = ODO^T$, where D is a diagonal matrix whose entries D_{ii} are the eigenvalues of M. Then $M = OD^+O^T - OD^-O^T$. Indeed, both OD^+O^T and OD^-O^T satisfy the hypothesis.

Finally, let us prove the uniqueness. Assume that $M = P - Q$ with P, $Q \geq 0$ and $PQ = 0$, then we have $D = O^TMO = O^TPO - O^TQO$ and from the uniqueness for the diagonal case $D^+ = O^TPO$, $D^- = O^TQO$. Hence, we conclude $P = OD^+O^T$ and $Q = OD^-O^T$. $\square$

Let $X \in \mathbb{S}^n$ have eigenvalues $e_1, e_2, \cdots, e_n$. The Pucci extremal operators $\mathcal{P}^+_{\lambda,\Lambda}$ and $\mathcal{P}^-_{\lambda,\Lambda}$ with ellipticity constants $0 < \lambda \leq \Lambda$ are defined by

$$\begin{aligned}\mathcal{P}^+_{\lambda,\Lambda}(X) &:= \sup_{\lambda I \leq \mathrm{A} \leq \Lambda I} \operatorname{tr}(\mathrm{A}X) = \lambda \sum_{e_i \geq 0} e_i + \Lambda \sum_{e_i < 0} e_i,\\ \mathcal{P}^-_{\lambda,\Lambda}(X) &:= \inf_{\lambda I \leq \mathrm{A} \leq \Lambda I} \operatorname{tr}(\mathrm{A}X) = \Lambda \sum_{e_i \geq 0} e_i + \lambda \sum_{e_i < 0} e_i.\end{aligned}$$

where A and X are $n \times n$ real symmetric matrices, and I is the identity matrix. When $\lambda = \Lambda = 1$, $\mathcal{P}^{\pm}_{\lambda,\Lambda}$ coincide with the Laplace operator. Another useful property, see [16], of the Pucci extremal operators:

$$\begin{aligned}\mathcal{P}^+_{\lambda,\Lambda}(-X) &= -\mathcal{P}^-_{\lambda,\Lambda}(X),\\ \mathcal{P}^-_{\lambda,\Lambda}(X) + \mathcal{P}^-_{\lambda,\Lambda}(Y) &\leq \mathcal{P}^-_{\lambda,\Lambda}(X+Y),\\ \mathcal{P}^-_{\lambda,\Lambda}(X+Y) &\leq \mathcal{P}^-_{\lambda,\Lambda}(X) + \mathcal{P}^+_{\lambda,\Lambda}(Y),\end{aligned}$$

and

$$\mathcal{P}^+_{\lambda,\Lambda}(X+Y) \leq \mathcal{P}^+_{\lambda,\Lambda}(X) + \mathcal{P}^+_{\lambda,\Lambda}(Y).$$

For more properties of the Pucci operators, see, e.g., L. A. Caffarelli and X.

Cabré [16] or I. C. Dolcetta and A. Vitolo [27]. It is convenient to give an equivalent definition for a fully nonlinear uniformly elliptic equation in terms of Pucci extremal operators. The operator $\mathcal{F}(x, u, Du, D^2u)$ is uniformly elliptic with the ellipticity constant $0 < \lambda \leq \Lambda < \infty$ if and only if

$$\mathcal{P}^-_{\lambda,\Lambda}(M - N) \leq \mathcal{F}(x, t, p, M) - \mathcal{F}(x, t, p, N) \leq \mathcal{P}^+_{\lambda,\Lambda}(M - N).$$

consequently, if $\mathcal{F}$ is uniformly elliptic, then

$$\mathcal{P}^-_{\lambda,\Lambda}(X) \leq \mathcal{F}(x, r, p, X) \leq \mathcal{P}^+_{\lambda,\Lambda}(X),$$

and $\mathcal{F}(x, r, p, \mathbf{0}) = 0$.

Since any $X \in \mathbb{S}^n$ can be decomposed uniquely as $X = X^+ - X^-$ with X^+, $X^- \geq \mathbf{0}$, and $X^+X^- = \mathbf{0}$, it follows that

$$\mathcal{P}^+_{\lambda,\Lambda}(X) = \Lambda \mathrm{tr}(X^+) - \lambda \mathrm{tr}(X^-),$$

and

$$\mathcal{P}^-_{\lambda,\Lambda}(X) = -\mathcal{P}^+_{\lambda,\Lambda}(-X) = \lambda \mathrm{tr}(X^+) - \Lambda \mathrm{tr}(X^-).$$

We refer the reader to [16, 20, 68] and references therein for proofs and the monograph [38] for details on the extremal Pucci operators.

2.2. The weighted partial trace operator

In this section, we recall the notation and some facts about the operators that we will use in the sequel.

Given two vectors ξ, $\eta \in \mathbb{R}^n$, we denote their scalar product by

$$\langle \xi, \eta \rangle = \begin{bmatrix} \xi_1 & \dots & \xi_n \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} = \sum_{i=1}^{n} \xi_i \eta_i$$

and their tensor product as the matrix

$$\xi \otimes \eta = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \begin{bmatrix} \eta_1 & \dots & \eta_n \end{bmatrix} = \begin{bmatrix} \xi_i \eta_j \end{bmatrix}_{1 \leq i,j \leq n}.$$

For a symmetric matrix A, the eigenvectors corresponding to distinct eigenvalues are orthogonal. Indeed, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvectors of A corresponding to different eigenvalues λ and μ, then

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle \mathrm{A}x, y\rangle = \langle x, \mathrm{A}^T y\rangle = \langle x, \mathrm{A}y\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle.$$

Therefore, $(\lambda - \mu)\langle x, y\rangle = 0$. Since $\lambda - \mu \neq 0$, then $\langle x, y\rangle = 0$, that is, $\boldsymbol{x} \perp \boldsymbol{y}$.

If X is a diagonal matrix with diagonal entries λ_j, $j = 1, 2, \dots, n$, then by rotational invariance we have

$$\langle X\xi, \xi\rangle = \sum_{j=1}^{n} \lambda_j \xi_j^2.$$

For a given n-tuple $\mathbf{a} := (a_1, \cdots, a_n)$ of non-negative real numbers such that $a_j > 0$ for some $1 \leq j \leq n$. Then, the weighted partial trace operator is defined as (see for details [33, 50, 51, 76]).

$$\mathcal{P}_{\mathbf{a}}(X) := \sum_{j=1}^{n} a_j \lambda_j(X), \quad X \in \mathbb{S}^n, \tag{2.2.1}$$

where $\lambda_1(X) \leq \lambda_2(X) \leq \cdots \leq \lambda_n(X)$ are the real eigenvalues of $X \in \mathbb{S}^n$. The Laplace operator is a special type of this operator, as the truncated Laplace operators $\mathcal{P}_k^-(X)$ and $\mathcal{P}_k^+(X)$ are equal, where

$$\mathcal{P}_k^-(X) = \sum_{j=1}^{k} \lambda_j(X), \qquad \mathcal{P}_k^+(X) = \sum_{j=n-k+1}^{n} \lambda_j(X),$$

for a positive integer $1 \leq k \leq n$.

Remark 2.2.1. Let $\mathbf{a} = e_1 + \cdots + e_n$, where $\{e_1, \cdots, e_n\}$ are the standard basis of $\mathbb{R}^n$. Then

1. If $k = n$, then $\mathcal{P}_k^{\pm}(X)$ coincides with the Laplace operator.

2 If $e_i = 0$ for all $1 < i < n$, then we obtain a min-max operator denoted by

$$\mathcal{M}(X) = \lambda_1(X) + \lambda_n(X).$$

Except for the Laplace operator and the Min-Max operator for $n = 2$, the operator $\mathcal{P}_{\mathbf{a}}$ has nonlinearity, nonconcavity, and nonconvexity properties, see [33].

For example, consider the matrices

$$X_1 = e_1 \otimes e_1 - e_3 \otimes e_3, \quad X_2 = -e_1 \otimes e_1 + e_2 \otimes e_2,$$

and

$$X_3 = e_1 \otimes e_1 - e_2 \otimes e_2 - e_3 \otimes e_3.$$

Then $\lambda_1(X_i) = -1$ and $\lambda_3(X_i) = 1$, giving $\mathcal{M}(X_i) = 0$ for all $i = 1, 2, 3$.

1 The operator $\mathcal{M}$ is not linear for $n \geq 3$. In ffact,

$$X_1 - X_2 = 2e_1 \otimes e_1 - e_2 \otimes e_2 - e_3 \otimes e_3,$$

and

$$\lambda_1(X_1 - X_2) = -1, \quad \lambda_3(X_1 - X_2) = 2.$$

Hence,

$$\mathcal{M}(X_1) - \mathcal{M}(X_2) = 0 \neq 1 = \mathcal{M}(X_1 - X_2).$$

2 The operator $\mathcal{M}$ is neither convex nor concave. In fact, for every $t \in [0, 1]$, we have

$$\begin{aligned} tX_1 + (1-t)X_2 &= t\,(e_1 \otimes e_1 - e_3 \otimes e_3) + (1-t)\,(-e_1 \otimes e_1 + e_2 \otimes e_2) \\ &= (2t-1)e_1 \otimes e_1 + (1-t)e_2 \otimes e_2 - te_3 \otimes e_3. \end{aligned}$$

Thus, the three eigenvalues are $2t - 1$, $1 - t$, and $-t$. So

$$\begin{aligned} \mathcal{M}(tX_1 + (1-t)X_2) &= \lambda_1\left(tX_1 + (1-t)X_2\right) + \lambda_3\left(tX_1 + (1-t)X_2\right) \\ &= \min\{2t-1, -t\} + \max\{2t-1, 1-t\}. \end{aligned}$$

Therefore,

$$\mathcal{M}\left(tX_1 + (1-t)X_2\right) = 1 - 2t \quad \text{for} \quad t \in \left(\frac{1}{3}, \frac{2}{3}\right).$$

From this, we have

$$\mathcal{M}\left(tX_1 + (1-t)X_2\right) > 0 \quad \text{for} \quad t \in \left(\frac{1}{3}, \frac{1}{2}\right),$$

and

$$\mathcal{M}\left(tX_1 + (1-t)X_2\right) < 0 \quad \text{for} \quad t \in \left(\frac{1}{2}, \frac{2}{3}\right),$$

Moreover, for every $t \in [0,1]$, we have

$$t\mathcal{M}(X_1) + (1-t)\mathcal{M}(X_2) = 0.$$

Thus, $\mathcal{M}$ is neither convex nor concave in $\mathbb{S}^3$.

Proposition 2.2.2. The weighted partial trace operator $\mathcal{P}_{\mathbf{a}}$ is degenerate and uniformly elliptic with ellipticity constants $\lambda = \underline{a} := \min\{a_j : 1 \le j \le n\}$ and $\Lambda = \overline{a} := \max\{a_j : 1 \le j \le n\}$ for $\underline{a} > 0$.

Proof. Suppose $X \le Y$. Then

$$\begin{aligned}\mathcal{P}_{\mathbf{a}}(Y) - \mathcal{P}_{\mathbf{a}}(X) &= \sum_{j=1}^{n} a_j\lambda_j(Y) - \sum_{j=1}^{n} a_j\lambda_j(X) = \sum_{j=1}^{n} a_j\left(\lambda_j(Y) - \lambda_j(X)\right)\\ &\ge \underline{a}\sum_{j=1}^{n}\left(\lambda_j(Y) - \lambda_j(X)\right) = \underline{a}\,\mathrm{tr}(Y-X) \ge 0,\end{aligned} \tag{2.2.2}$$

where we used the fact that $\lambda_j(X) \le \lambda_j(X)$ for $X \le Y$ for all $j = 1, 2, \ldots, n$.

From (2.2.2), $\mathcal{P}_{\mathbf{a}}(Y) \ge \mathcal{P}_{\mathbf{a}}(X)$, and hence the weighted partial trace operator $\mathcal{P}_{\mathbf{a}}$ is a **degenerate elliptic**.

Similarly, for $X \leq Y$, we have

$$\begin{aligned}\mathcal{P}_{\mathbf{a}}(Y) - \mathcal{P}_{\mathbf{a}}(X) &= \sum_{i=1}^{n} a_i \big(\lambda_i(Y) - \lambda_i(X)\big) \\ &\leq \overline{a} \sum_{i=1}^{n} \big(\lambda_i(Y) - \lambda_i(X)\big) = \overline{a}\mathrm{tr}(Y - X).\end{aligned} \tag{2.2.3}$$

Thus, from (2.2.2) and (2.2.3), we have

$$\underline{a}\mathrm{tr}(Y - X) \leq \mathcal{P}_{\mathbf{a}}(Y) - \mathcal{P}_{\mathbf{a}}(X) \leq \overline{a}\mathrm{tr}(Y - X).$$

Therefore, the operator $\mathcal{P}_{\mathbf{a}}$ is **uniformly elliptic** with ellipticity constants $\lambda = \underline{a} := \min\{a_j : 1 \leq j \leq n\}$ and $\Lambda = \overline{a} := \max\{a_j : 1 \leq j \leq n\}$.

On the other hand, for $Y \geq 0$ (that is, Y is a positive semidefinite matrix), we have

$$\begin{aligned}\mathcal{P}_{\mathbf{a}}(X + Y) &= \underline{a}\mathcal{P}_{\mathbf{a}}(X + Y) + \sum_{i=1}^{n} (a_i - \underline{a})\, \lambda_i(X + Y) \\ &\geq \underline{a}\mathcal{P}_{\mathbf{a}}(X + Y) + \sum_{i=1}^{n} (a_i - \underline{a})\, \lambda_i(X) \\ &= \underline{a}\mathcal{P}_{\mathbf{a}}(Y) + \mathcal{P}_{\mathbf{a}}(X).\end{aligned} \tag{2.2.4}$$

Analogously,

$$\begin{aligned}\mathcal{P}_{\mathbf{a}}(X + Y) &= \overline{a}\mathcal{P}_{\mathbf{a}}(X + Y) - \sum_{i=1}^{n} (\overline{a} - a_i)\, \lambda_i(X + Y) \\ &\leq \overline{a}\mathcal{P}_{\mathbf{a}}(X + Y) - \sum_{i=1}^{n} (\overline{a} - a_i)\, \lambda_i(X) \\ &= \overline{a}\mathcal{P}_{\mathbf{a}}(Y) + \mathcal{P}_{\mathbf{a}}(X).\end{aligned} \tag{2.2.5}$$

Thus, combining (2.2.4) and (2.2.5), gives

$$\underline{a}\mathcal{P}_{\mathbf{a}}(Y) \leq \mathcal{P}_{\mathbf{a}}(X + Y) - \mathcal{P}_{\mathbf{a}}(X) \leq \overline{a}\mathcal{P}_{\mathbf{a}}(Y).$$

Definition 2.2.1. Let $\mathcal{F}$ be a fully nonlinear degenerate elliptic operator. The dual operator for $\mathcal{F}$ is defined as

$$\widetilde{\mathcal{F}}(X) = -\mathcal{F}(-X).$$

Let $\mathbf{a} = (a_1, \cdots, a_n)$, and denote $\mathbf{a}' = (a_n, \cdots, a_1)$. The operator $\mathcal{P}_{\mathbf{a}'}$ defined by

$$\mathcal{P}_{\mathbf{a}'}(X) = -\mathcal{P}_{\mathbf{a}}(-X)$$

is the dual of the operator $\mathcal{P}_{\mathbf{a}}$. For any $X \in \mathbb{S}^n$, the dual of the weighted partial trace operator

$$\mathcal{P}_{\mathbf{a}}(X) = \sum_{j=1}^{n} a_j \lambda_j(X)$$

is again a weighted partial trace operator given by

$$\mathcal{P}_{\mathbf{a}'}(X) = \sum_{j=1}^{n} a_{n-j+1} \lambda_j(X).$$

We will focus on the two classes of weighted partial trace operators:

$$\mathcal{A}_1 := \{\mathcal{P}_{\mathbf{a}} : a_1 > 0\}, \quad \text{and} \quad \mathcal{A}_n := \{\mathcal{P}_{\mathbf{a}} : a_n > 0\}.$$

It is clear that if $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_n$, then the dual operator $\mathcal{P}_{\mathbf{a}'} \in \mathcal{A}_1$ and vice versa.

In particular, the Laplace operator $\mathcal{P}_n^-(D^2u) = \mathcal{P}_n^+(D^2u) = \Delta u$ and the Min-Max operator $\mathcal{M}(X) = \lambda_1(X) + \lambda_n(X)$ are self-dual operators. This means that

$$\mathcal{P}_{\mathbf{a}}(X) = \mathcal{P}_{\mathbf{a}'}(X).$$

Specifically, $\mathcal{P}_k^+(X) = -\mathcal{P}_k^-(-X)$. We state the results for $\mathcal{P}_k^-(X)$, with obvious equivalents when considering the operator $\mathcal{P}_k^+(X)$. Such operators are positively homogeneous of degree one and degenerate elliptic.

2.3. Viscosity Solutions

In this section, we introduce the notion of viscosity solutions for second-order elliptic partial differential equations [19]. A classical solution of the equation $\mathcal{F}[u] = 0$ is a C^2 function u such that for every point $x \in \Omega$, it satisfies $\mathcal{F}(x, u(x), Du(x), D^2u(x)) = 0$.

A viscosity solution is a generalized notion of a solution in which we only require a priori that our function be continuous. In some cases, PDEs do have solutions in a generalized sense. The generalized solutions are unknown functions that do not possess the required regularity, and their derivatives may not exist. The notion of viscosity solutions allows us to make sense of how a non-smooth continuous function may solve an elliptic PDE.

For a scalar degenerate elliptic equation, one can define a weak solution type called a **viscosity solution**. Under the concept of viscosity solution, u does not need to be differentiable everywhere. There may be points where Du or D^2u does not exist and yet u satisfies the equation in the generalized sense. The definition allows only certain kinds of singularity, so existence, uniqueness, and stability under uniform limits hold for a large class of equations. The function u may not be C^2, but u solves the equation in a weak sense.

Let us begin by recalling the following definitions. We say that a function $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is upper (resp. lower) semicontinuous at $x \in \Omega$ provided that

$$\limsup_{\Omega \ni y \to x} u(y) \leq u(x) \quad (\text{resp. } \liminf_{\Omega \ni y \to x} u(y) \geq u(x)).$$

where

$$\limsup_{\Omega \ni y \to x} u(y) := \inf_{r>0} \sup\{u(y) : y \in \Omega \cap B(x,r)\},$$
$$\liminf_{\Omega \ni y \to x} u(y) := \sup_{r>0} \inf\{u(y) : y \in \Omega \cap B(x,r)\}.$$

Let $\mathrm{USC}(\Omega)$ (respectively, $\mathrm{LSC}(\Omega)$) denote the collection of functions that are upper (respectively, lower) semicontinuous at all points in Ω.

Definition 2.3.1. We say that a function $u\colon \Omega \to \mathbb{R}$ satisfies the inequality $\mathcal{F}(x, u, Du, D^2u) \geq 0$ in Ω in the viscosity sense if u is upper semicontinuous in Ω and there exist a ball $B_r(x_0) \subset \Omega$ and a C^2 function $\psi\colon B_r(x_0) \to \mathbb{R}$ such that $\psi(x_0) = u(x_0)$ and $\psi \geq u$ in $B_r(x_0)$. That is, u is a viscosity subsolution if

$$\mathcal{F}(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0.$$

Reversing all inequalities in **Definition 2.3.1**, we call u is a viscosity supersolution.

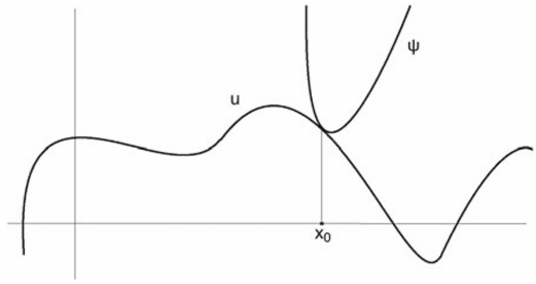

Figure 2.1: A 2D illlustration of the upper test function

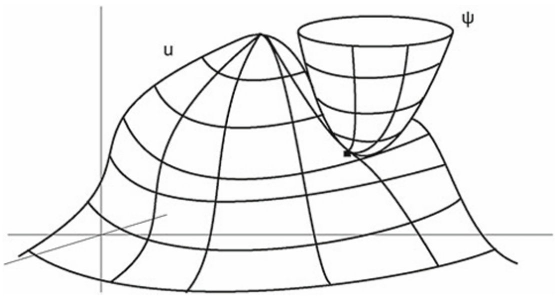

Figure 2.2: A 3D illlustration of the upper test function

Note that u is a supersolution (that is, $\mathcal{F}[u] \leq 0$) if and only if $-u$ is a subsolution of $\widetilde{\mathcal{F}}(x, u, p, A) = -\mathcal{F}(x, -u, -p, -A) \geq 0$. In particular, supersolutions are lower semicontinuous.

A viscosity solution of $\mathcal{F}(\cdot, u, Du, D^2u) = 0$ is a continuous function u, which is at the same time a viscosity subsolution and a viscosity supersolution.

If $u \in C^2$ is a classical subsolution, then it is a subsolution in the viscosity sense, which is equivalent to $\mathcal{F}(x, u, Du, D^2u) \geq 0$ in the classical sense. In fact, if u is a classical C^2 subsolution and there exists a test function φ as in **Definition 2.3.1**, we would have

$$\varphi(x) = u(x), \quad D\varphi(x) = Du(x), \quad D^2\varphi(x) \geq D^2u(x).$$

By virtue of the ellipticity assumption in the function $\mathcal{F}$, this would imply that

$$\mathcal{F}(x, \varphi(x), D\varphi(x), D^2\varphi(x)) \geq \mathcal{F}(x, u(x), Du(x), D^2u(x)) \geq 0.$$

Conversely, if u is a C^2 viscosity subsolution, then using itself as a test function (that is, $\varphi = u$), we observe that $\mathcal{F}[u] \geq 0$ classically.

The following definition is equivalent to **Definition 2.3.1**, which is convenient for proofs of theorems.

Definition 2.3.2. Let $\Omega \subseteq \mathbb{R}^n$, $u \in C(\Omega)$ and consider the degenerate elliptic PDE (see **Definition 2.1.4**). An upper semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a viscosity subsolution of (2.1.2), if

$$\mathcal{F}(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \tag{2.3.1}$$

for any $x_0 \in \Omega$ and any $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a maximum at x_0.

Similarly, a lower semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a viscosity supersolution of (2.1.2), if

$$\mathcal{F}(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0 \tag{2.3.2}$$

for any $\varphi \in C^2(\Omega)$ and any $x_0 \in \Omega$ such that $u - \varphi$ has a minimum at x_0.

A continuous function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Proposition 2.3.1. Let u be a classical solution of equation (2.1.2) and $\varphi\colon \Omega \to \mathbb{R}$ be any C^2 function. Then if $u - \varphi$ has a local maximum (local minimum) at a point $x_0 \in \Omega$, then

$$\mathcal{F}(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq \ (\leq)\ 0.$$

Proof. We prove the subsolution part and leave the supersolution part as it is similar. The proof is a simple consequence of the maximum principle. If $u - \varphi$ has a local maximum at x_0, then $Du(x_0) = D\varphi(x_0)$ and $D^2u(x_0) - D^2\varphi(x_0) \leq 0$. Now, using the degenerate ellipticity, we conclude. $\square$

Remark 2.3.2. Let $k \geq 2$ be an integer. Then

1. For PDEs of order k, we can give the same definition, requiring a C^k-regularity for the test functions.

2. To check the condition for viscosity subsolution (respectively, supersolution), it is enough to require the function to be upper (respectively, lower) semicontinuous.

3. Equations $\mathcal{F}(x, u, Du, D^2u) = 0$ and $-\mathcal{F}(x, u, Du, D^2u) = 0$ are not equivalent in the viscosity sense.

4. A classical solution is a viscosity solution, and a continuously differentiable viscosity solution is a classical solution.

Let us now recall the definition of viscosity solution to

$$\mathcal{P}_{\mathbf{a}}(D^2u) = h(u, |Du|). \tag{2.3.3}$$

A function u in $\mathrm{USC}(\Omega)$ (respectively, $\mathrm{LSC}(\Omega)$) is said to be a viscosity subsolution (respectively, supersolution) of (2.3.3) in an open set $\Omega \subseteq \mathbb{R}^n$ if and only if for each $x_0 \in \Omega$ and for every $\psi \in C^2(\Omega)$ such that $u - \psi$ has a local maximum (respectively, minimum) at x_0, then

$$\mathcal{P}_{\mathbf{a}}(D^2\psi(x_0)) \geq (\text{respectively, } \leq)\ h(u(x_0), |D\psi(x_0)|).$$

We say $u \in C(\Omega)$ is a viscosity solution of (2.3.3) provided that u is both a viscosity subsolution and a viscosity supersolution of (2.3.3) in Ω. See [16] for more on viscosity solutions and their properties.

Remark 2.3.3. Let $\psi \in C^2(\Omega)$. Then

1. The condition "ψ touches the graph of u from above at x_0" means "$u-\psi$ has a local maximum at x_0". Similarly, "ψ touches the graph of u from below at x_0" means "$u - \psi$ has a local minimum at x_0".

2. A continuous function u is a viscosity solution of (2.3.3) if it is both a viscosity subsolution and a supersolution of (2.3.3).

3. Any viscosity solution is continuous.

One of the main mathematical tools for asserting the existence and uniqueness of viscosity solutions is the comparison principle.

Definition 2.3.3 (Comparison Principle)**.** We say that (2.3.3) satisfies a comparison principle if $w^* \in \mathrm{USC}(\overline{\Omega})$ and $w_* \in \mathrm{LSC}(\overline{\Omega})$ are subsolution and supersolution, respectively, such that

$$w^* \leq w_* \quad \text{on } \partial\Omega,$$

implies

$$w^* \leq w_* \quad \text{in } \overline{\Omega}.$$

Remark 2.3.4. The uniqueness immediately follows from the comparison principle.

2.4. Weak and strong minimum/maximum principles

In this section, we shall look at a brief description of the weak and strong minimum/maximum principles of elliptic operators.

If u is a harmonic function (that is, $\Delta u = 0$) in a domain D, and if u is non-constant, then u cannot attain a maximum or minimum at any point in the interior of D. This principle implies that the maximum and minimum values of a non-constant harmonic function on a closed and bounded domain must occur on the boundary of the domain, not in the interior.

Given an unknown function u on $\Omega \subseteq \mathbb{R}^n$. For any connected open subset Ω of the domain of u, the weak minimum/maximum principle states that the maximum of u on the closure of Ω is the same as that on the boundary $\partial\Omega$, that is,

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

Similarly, the minimum of u on the closure of Ω is the same as that on the boundary $\partial\Omega$, that is,

$$\min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x).$$

The strong minimum/maximum principle says that, unless u is a constant function, the minimum/maximum cannot be attains anywhere on Ω itself. It tells us that, for a solution of an elliptic equation, extrema can be attained in the interior if and only if the function is constant.

Definition 2.4.1. The boundary $\partial B_r(x_0)$ of a ball $B_r(x_0) \subseteq \mathbb{R}^n$ is called a sphere, the (outward) unit normal at a point $x \in \partial B_r(x_0)$ is the unit vector:

$$\vec{n} = \frac{x - x_0}{|x - x_0|}.$$

The normal derivative of a function $u \in C^1(\overline{B}_r(x_0))$ at a point x is the directional derivative

$$\frac{\partial u(x)}{\partial \vec{n}} = \vec{n} \cdot \nabla u(x).$$

One of the most important refinements, known as the Hopf maximum principle, asserts that at a maximum point on the boundary, the outward normal derivative is positive (unless the function is identically constant).

Lemma 2.4.1 (Hopf boundary point lemma, [33])**.** Let $u \in \mathrm{USC}(\overline{B})$ be a viscosity subsolution of equation $\mathcal{P}_{\mathbf{a}}(D^2u) = 0$ in a ball B, with $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_1$. Let $x_0 \in \partial B$. If $u(x_0) > u(x)$ for all $x \in B$, then the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality

$$\frac{\partial u(x_0)}{\partial \vec{n}} > 0.$$

On the other hand, let $u \in \mathrm{LSC}(\overline{B})$ be a viscosity supersolution of equation $\mathcal{P}_{\mathbf{a}}(D^2u) = 0$ in a ball B, with $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_n$. If $u(x_0) < u(x)$ for all $x \in B$, then the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality

$$\frac{\partial u(x_0)}{\partial \vec{n}} < 0.$$

Let $\Omega \subseteq \mathbb{R}^n$ be a domain (not necessarily bounded) and

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}u + \sum_{i=1}^{n} b_i(x) D_i u + c(x)u,$$

be a uniformly elliptic operator in Ω with bounded functions $a_{ij}(x)$, $b_i(x)$, and $c(x)$. Suppose $Lu \geq 0$ in Ω for $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then

1. if $c \leq 0$, u cannot attain a non-negative maximum in Ω unless it is constant.
2. if $c \equiv 0$, u cannot attain a maximum in Ω unless it is constant.
3. if x_0 is an interior minimum/maximum point such that $u(x_0) = 0$, then $u \equiv 0$ irrespective of the sign of c.

Definition 2.4.2. Consider $\mathcal{F}$ to be a second-order differential operator as discussed in the previous section. Then:

1. $\mathcal{F}$ satisfies the weak maximum principle (or WMP) in Ω if

$$\mathcal{F}[u] \geq 0 \text{ in } \Omega, \quad \limsup_{x \to \partial\Omega} u(x) \leq 0 \quad \text{on } \partial\Omega,$$

   implies $u(x) \leq 0$ in Ω.

2. $\mathcal{F}$ satisfies the strong maximum principle (or SMP) if

$$\mathcal{F}[u] \geq 0 \text{ in } \Omega, \quad u(x) \leq 0 \quad \text{in } \Omega,$$

implies $u(x) \equiv 0$ in Ω or $u(x) < 0$ in Ω.

3. $\mathcal{F}$ satisfies the weak minimum principle (or WmP) in Ω if

$$\mathcal{F}[u] \leq 0 \text{ in } \Omega, \quad \liminf_{x \to \partial\Omega} u(x) \geq 0 \quad \text{on } \partial\Omega,$$

implies $u(x) \geq 0$ in Ω.

4. $\mathcal{F}$ satisfies the strong minimum principle (or SmP) if

$$\mathcal{F}[u] \leq 0 \text{ in } \Omega, \quad u(x) \geq 0 \quad \text{in } \Omega,$$

implies $u(x) \equiv 0$ in Ω or $u(x) > 0$ in Ω.

Remark 2.4.1. We recall that

1. If $v \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of (2.1.2) in Ω, then $u = -v$ is a viscosity subsolution of

$$-\mathcal{F}(x, -u(x), -Du(x), -D^2u(x)) = 0.$$

2. The conditions under which the SmP holds for $\mathcal{F}$ can be easily deduced from the assumptions of SMP for

$$G(x, r, p, X) := -\mathcal{F}(x, -r, -p, -X).$$

Of course, if $\mathcal{F}$ is an odd function with respect to r, p and X, then the maximum and minimum principles are equivalent.

The classical versions of the WMP (or WmP) and SMP (or SmP) can be summarized as follows.

1. A solution u to a partial differential equation is said to satisfy the WMP (or WmP) if it has the property that u achieves the maximum/minimum value on the boundary of its domain.

2. A solution u to a partial differential equation is said to satisfy SmP (or SMP) if it has the property that u achieves its minimum/maximum in the interior of its domain. This is the case where u is constant.

A function has **compact support** if it is zero outside of **compact set**. Alternatively, one can say that a function has compact support if its support is a compact set.

2.5. Liouville-type Theorems

Liouville-type theorems are important in the field of differential equations because they provide conditions under which a solution must be trivial.

The statement of Liouville's theorem has several versions.

Theorem 2.5.1 (Louville's Theorem in complex analysis)**.** A bounded entire (i.e. every where differentiable) function is constant.

The proof of this theorem is found in [57]. This profound result leads to arguably the most natural proof of the Fundamental Theorem of Algebra [37].

This classical theorem generalizes to real harmonic functions on $\mathbb{R}^n$ which are bounded only on one side. That is, if u is a real harmonic function on $\mathbb{R}^n$ for $n \geq 2$ and bounded above or below, then u must be constant.

Theorem 2.5.2 (Liouville's Theorem for Harmonic Function)**.** A bounded harmonic function on $\mathbb{R}^n$ is constant.

The proof of this theorem is found in [1].

A harmonic function on a compact set is determined by its restriction to the boundary. However, the harmonic function of a closed half-space is not determined by its restriction to the boundary (this follows from the maximum principle).

Recall that we identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$, writing a typical point $z \in \mathbb{R}^n$ as $z = (x, y)$, where $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. The upper half-space $\mathbf{H} = \mathbf{H}_n$ is the set

$$\mathbf{H} = \{(x, y) \in \mathbb{R}^n : y > 0\}.$$

We identify $\mathbb{R}^{n-1}$ with $\mathbb{R}^{n-1} \times \{0\}$; with this convention, we then have

$$\partial \mathbf{H} = \mathbb{R}^{n-1}.$$

For u a function on $\mathbf{H}$ and $y > 0$, we let u_y denote the function on $\mathbb{R}^{n-1}$ defined by

$$u_y(x) = u(x, y).$$

The functions u_y play the same role on the upper half-space that the dilations play on the ball.

The next result shows that this behavior cannot occur if we consider only harmonic functions that are bounded.

Corollary 2.5.3. Suppose u is a continuous bounded function on $\overline{\Omega}_n$ that is harmonic on Ω_n. If $u = 0$ on $\partial\Omega_n$, then $u \equiv 0$ on $\overline{\Omega}_n$.

Proof. For $x \in \mathbb{R}^{n-1}$ and $y < 0$, define $u(x, y) = -u(x, -y)$, thereby extending u to a bounded continuous function defined in all $\mathbb{R}^n$. Clearly u satisfies the local mean value property, so u is harmonic on $\mathbb{R}^n$. **Theorem 2.5.2** now shows that u is constant on $\mathbb{R}^n$. See the proof in [1]. □

Liouville-type theorems in the context of PDE theory state that if a function f belongs to a specific function space (or class), then f is either constant or identically zero.

Now, recall the following Liouville-type theorems.

Theorem 2.5.4. Let u be a positive bounded classical solution of

$$-\Delta u = u^q |Du|^p \quad \text{in} \quad \mathbb{R}^n,$$

where $p > 2$, $q > 0$ and $n \geq 1$. Then u is constant.

The proof of this theorem is found in [36].

Theorem 2.5.5. Let $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_1 \cap \mathcal{A}_n$. If u is an entire viscosity solution of an equation $\mathcal{P}_{\mathbf{a}}(D^2 u) = 0$ that is bounded above or below, then u is constant.

The proof of this theorem is found in [33].

2.6. Some results on non-decreasing functions

Here we recall some results from calculus and analysis that play a great role and are useful throughout the study. To meet specific analytical needs, we adopt the following lemmas from [6, 23, 32, 60]. For detailed proofs, the reader is referred to these references.

Lemma 2.6.1. Let g: $\mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous and non-decreasing function on $\mathbb{R}_0^+$, and $g(s) > 0$ for $s > 0$. Let $G(t) = \displaystyle\int_0^t g(s)\,ds, \quad t \geq 0$. Then, the following properties hold:

1. For any $a > 0$ and $s > 0$, we have
$$aG\left(\frac{s}{a}\right) = a\int_0^{\frac{s}{a}} g(\tau)\,d\tau = \int_0^s g\left(\frac{\tau}{a}\right)\,d\tau. \tag{2.6.1}$$

2. For a fixed s, we have
$$\lim_{a\to\infty} aG\left(\frac{s}{a}\right) = sg(0).$$

3. For $t > 0$, we have
$$\frac{1}{2}G(t) \geq G\left(\frac{t}{2}\right).$$

Proof. Let $a > 0$ and $s > 0$.

1. By the definition of the integral, we have
$$aG\left(\frac{s}{a}\right) = a\int_0^{s/a} g(\tau)\,d\tau = \int_0^s g\left(\frac{\tau}{a}\right)\,d\tau,$$
where the last step follows from the substitution $\tau = \frac{s}{a}$.

2. Taking the limit as $a \to \infty$ in (2.6.1), we get
$$\lim_{a\to\infty} aG\left(\frac{s}{a}\right) = \lim_{a\to\infty}\int_0^s g\left(\frac{\tau}{a}\right)\,d\tau = sg(0),$$
where the last step follows from the fact that g is continuous at 0.

3. For $t > 0$, we have:
$$G\left(\frac{t}{2}\right) = \int_0^{t/2} g(\tau)\,d\tau = \frac{1}{2}\int_0^t g\left(\frac{s}{2}\right)\,ds \leq \frac{1}{2}\int_0^t g(s)\,ds = \frac{1}{2}G(t),$$
where the inequality follows from the fact that g is non-decreasing. This completes the proof of **Lemma 2.6.1**. $\square$

Lemma 2.6.2. If $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a non-decreasing function such that

$$\int_0^1 \frac{dt}{g(t)} < \infty,$$

then

$$\lim_{t\to 0^+} \frac{t}{g(t)} = 0.$$

Proof. Let $t > 0$ be fixed. Since g is non-decreasing, we have $g(s) \leq g(t)$ for all $0 \leq s \leq t$. This implies that

$$\frac{1}{g(t)} \leq \frac{1}{g(s)}, \quad \forall s \in [0, t].$$

Hence,

$$\int_0^t \frac{ds}{g(t)} \leq \int_0^t \frac{ds}{g(s)}. \tag{2.6.2}$$

From the left-hand side of (2.6.2), we get

$$\int_0^t \frac{ds}{g(t)} = \frac{1}{g(t)} \int_0^t ds = \frac{t}{g(t)}. \tag{2.6.3}$$

By combining (2.6.2) and (2.6.3), we obtain

$$\frac{t}{g(t)} \leq \int_0^t \frac{ds}{g(s)}, \quad t > 0$$

Since $\dfrac{1}{g(s)} \in L^1(0,1)$, by absolute continuity, the integral $\displaystyle\int_0^t \frac{ds}{g(s)} \to 0$ as $t \to 0^+$. $\square$

Lemma 2.6.3. Let $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous function. Assume that for some $k > 0$, the map $t \mapsto kt + g(t)$ is non-decreasing. Then

$$\int_0^1 \frac{dt}{g(t)} = \infty \quad \text{if and only if} \quad \int_0^1 \frac{dt}{kt + g(t)} = \infty.$$

Proof. The backward implication is obvious from the comparison theorem for improper integrals, since $kt > 0$ for $t \in (0,1)$. To prove the forward implication, we use the proof by contradiction. That is to say, suppose that

$$\int_0^1 \frac{dt}{kt+g(t)} < \infty. \tag{2.6.4}$$

Then, since the map $t \mapsto kt + g(t)$ is non-decreasing, by **Lemma 2.6.2**, we have

$$\lim_{t\to 0^+} \frac{t}{kt+g(t)} = 0,$$

which implies that

$$\frac{t}{kt+g(t)} \leq \frac{1}{k+1}, \quad t \in (0,\delta)$$

for some $\delta > 0$.

Consequently, we have

$$\frac{t}{g(t)} \leq 1, \quad t \in (0,\delta). \tag{2.6.5}$$

Therefore, from (2.6.4) and (2.6.5), we conclude that

$$\int_0^\delta \frac{dt}{g(t)} = \int_0^\delta \left(1 + \frac{kt}{g(t)}\right) \frac{dt}{kt+g(t)} \leq (k+1) \int_0^1 \frac{dt}{kt+g(t)} < \infty$$

contradicts the conclusion. This completes the proof of **Lemma 2.6.3**. □

Chapter 3

Review of Related Literature

3.1. Minimum/Maximum Principle

Over the years, SmP (or SMP) has been extended to various elliptic differential operators [6, 41, 60, 63–67, 72] and the references therein. For our investigation, we recall some results that are of particular interest to this study. The geometric minimum (or maximum) principle is a consequence of the classical Aleksandrov-Bakelman-Pucci [17] SMP (or SmP) for C^2-subsolution and supersolution to non-linear uniformly elliptic equations, which is in turn based on Hopf's (1927) minimum/maximum principle for linear elliptic equations [47]. Hopf studied a strictly elliptic operator

$$Lu := -\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u + \sum_{i=1}^{n} b_i(x)D_i u + c(x)u, \quad x \in \Omega \subseteq \mathbb{R}^n.$$

According to Hopf's minimum/maximum principle, if $c = 0$ and $Lu \leq 0$ (respectively, $Lu \geq 0$), then u is a constant function if it achieves its minimum (respectively, maximum) in Ω.

After Hopf, the classical minimum/maximum principle was studied in various works, such as the well-known books by Protter and Weinberger [62], Gilbarg and Trudinger [41], and the monograph by P. Pucci and J. Serrin [66].

In [72], J. L. Vázquez proved a SmP for supersolutions of

$$\Delta u = f(u),$$

where $f\colon \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous and non-decreasing function with $f(0) = 0$ such that

$$\int_0^1 \frac{dt}{\sqrt{F(t)}} = \infty, \tag{3.1.1}$$

where, F is the antiderivative of f that vanishes at zero. In [29], it is shown that the condition (3.1.1) is also necessary for SmP to hold for the equation.

The work [72] was extended to a wide class of quasilinear differential inequalities by Pucci and Serrin in the papers [64, 66], and by P. Pucci, J. Serrin and H. Zou in [67]. We also direct the reader to the excellent monograph [65] that includes a chapter on SmP for many divergence structure elliptic inequalities.

Another interesting study, Felmer et al. [31] studied the solvability of coercive quasi-linear and fully nonlinear elliptic equation

$$\Delta u = f(u) + g(|Du|), \quad \text{in } \mathbb{R}^n$$

with f and g continuous non-decreasing functions such that $f(0) = 0 = g(0)$, under the integral condition

$$\int_0^1 \frac{dt}{at + f(t) + g(t)} = \infty, \quad \forall a > 0, \tag{3.1.2}$$

which is equivalent to

$$\int_0^1 \frac{dt}{f(t) + g(t)} = \infty,$$

by **Lemma 2.6.3** above.

The authors also studied the strong maximum principle of the more general equation

$$\Delta_p u = f(u) + g(|Du|),$$

where Δ_p is the p-Laplace operator and f and g are modified nonlinearities that satisfy (3.1.2).

A. Mohammed and A. Vitolo [60, Theorem 1.1] proved the validity of SmP (or

SMP) to equations of the form

$$\Delta_p u = h(u, |Du|), \tag{3.1.3}$$

where h satisfies certain conditions.

The authors provided sufficient conditions on h to ensure that the SmP is maintained. In other words, if u is a non-negative viscosity supersolution of (3.1.3) in a connected open set $\Omega \subseteq \mathbb{R}^n$, then $u \equiv 0$ in Ω or $u > 0$ in Ω.

T. Bhattacharya and A. Mohammed [6] investigated the SmP (or SMP) for non-negative viscosity subsolution or supersolution of a class of nonlinear possibly degenerate subelliptic equations of the type

$$\mathcal{F}(x, D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) = h(u, |D_{\mathcal{X}}u|) \quad \text{in } \Omega,$$

where $\mathcal{F}$ and h satisfy certain conditions. The equation involves a family of smooth vector fields $\mathcal{X} = \{X_1, X_2, \cdots, X_m\}$ in Ω that satisfy the Hörmander rank condition at every point of Ω. For a basis σ^j in Euclidean space, we write $X_j = \sigma^j \cdot D$, with $\sigma^j\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma = \sigma(x) = [\sigma^1(x), \cdots, \sigma^m(x)] \in \mathbb{R}^{n\times m}$. Then

$$D_{\mathcal{X}}u = \sigma^T Du = (\sigma^1 \cdot Du, \cdots, \sigma^m \cdot Du),$$

and

$$X_i(X_j u) = (\sigma^T D^2 u \sigma)_{ij} + (D\sigma^j \sigma^i) \cdot Du.$$

Birindelli et al. [8] investigated the strong maximum principle for degenerate elliptic operators whose higher-order term involves the truncated Laplacian, namely

$$\mathcal{P}_k^-(D^2u) = \sum_{j=1}^{k} \lambda_j(D^2u) \quad \text{and} \quad \mathcal{P}_k^+(D^2u) = \sum_{j=n-k+1}^{n} \lambda_j(D^2u),$$

where $\lambda_1(X) \leq \cdots \leq \lambda_n(X)$ are eigenvalues of the Hessian matrix, $X = D^2u$. Equations involving the operators $\mathcal{P}_k^{\pm}$ have been studied in [7, 10, 11, 14, 15, 25, 26, 43–46, 50, 51, 76].

Weighted partial trace operators have received increased attention motivated by geometric problems of mean curvature [70, 71, 78]. In the context of PDE, refer to L. A. Caffarelli and X. Cabré [16], and Harvey and Lawson [46]. The existence and uniqueness of solutions, regularity results, Harnack inequalities, and Aleksandrov-Bakelman-Pucci estimates for weighted partial trace operators have been extensively studied in [10, 11, 33, 74, 76, 77].

In a recent paper [33], F. Ferrari and A. Vitolo maintain the Aleksandrov-Bakelman-Pucci estimate, the weak Harnack inequality, the global Hölder estimate, the strong minimum/maximum principle, and a Liouville theorem for solutions to $\mathcal{P}_{\mathbf{a}}(D^2u) = 0$. In the case of the strong minimum/maximum principle, for a given n -tuple $\mathbf{a} = (a_1, \cdots, a_n)$ of non-negative real numbers with $a_n > 0$ and any non-negative viscosity solution u of $\mathcal{P}_{\mathbf{a}}(D^2u) \leq 0$ in a connected open set $\Omega \subseteq \mathbb{R}^n$, then either $u \equiv 0$ in Ω or $u > 0$ in Ω. The main tool used to prove this is the weak Harnack inequality (see [33, Theorem 5.2]). The authors also examined various characteristics of the weighted partial trace operator (2.2.1) and showed that it shares many properties with the standard Laplacian.

As far as our knowledge is concerned, a Vázquez-type SmP (or SMP) for partial trace equations with nonlinear gradient terms is new, even when $\mathcal{P}_{\mathbf{a}}$ is reduced to the standard Laplacian. So, our interest in this study is mainly focused on the validity of a Vázquez-type SmP (or SMP) for (1.2.1).

3.2. Liouville-type theorems

The study of Liouville-type theorems has attracted the interest of many researchers since the work of J. Liouville and A. Cauchy, where it was noted that the only bounded holomorphic functions in the complex plane are constants. This was generalized to solutions of elliptic equations of various forms, and the results take different forms. We mention the pioneering work of A. Farina [30] and Mitidieri and Pokhozaev [58] for their investigations of Liouville-type results for general semilinear elliptic equations and quasilinear inequalities with reaction and convection terms, respectively. We should also highlight the influential works [13, 59], in which Liouville-type results are obtained for quasilinear and semilinear elliptic equations based on the maximum principle of P-functions, as well as the classical Bernstein method to obtain global gradient bounds. We also recall the paper [34] for interesting results on Liouville-type properties for quasilinear elliptic equations, where the author investigates both coercive and non-coercive cases. We also refer to the paper [35], which studied non-coercive elliptic systems of divergence type. The equations studied in [35] include the p-Laplacian and mean curvature operators, involving gradient nonlinearity.

Over the years, researchers have employed various methods to derive Liouville-type properties. Among the most common and powerful techniques is the blow-up method, which involves re-scaling solutions around a point and then passing to a limit. This often reduces the equation to an easier form. Maximum principles (both weak and strong) and comparison principles are also utilized to compare solutions with subsolutions or supersolutions to force non-existence results. These methods are often effective for coercive equations, where there is good control over the growth of solutions and their regularity. They can also be suitably adapted to handle degenerate elliptic equations. Other methods are better suited to investigate the more delicate case of Liouville-type properties of non-coercive equations.

For degenerate elliptic equations, De Giorgi-Nash-Moser iteration techniques have been employed to derive Harnack inequalities, from which Liouville theorems are deduced (see [69]). An adaptation of the Hadamard three-sphere theorem has also been used effectively to study the Liouville-type properties of highly degenerate equations, such as subelliptic equations [9, 20, 33, 42, 53].

Gidas and Spruck [40] were the first to investigate Liouville-type theorems for semilinear equations of the form

$$\Delta u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^n$$

where $1 < p < \dfrac{n+2}{n-2}$.

In [24], Dolbeault and Monneau established a Liouville-type theorem for a more general free boundary of Serrin-type fully nonlinear elliptic equations

$$\Delta u + f(u) = 0.$$

Lu-Wang-Zhu [54] established Liouville-type theorem and the decay estimate for a classical non-negative solution to a higher-order elliptic equation

$$(-\Delta)^m u = u^q \quad \text{in } \mathbb{R}^n.$$

Berestycki et al. [4] also obtained Liouville-type results for semilinear elliptic equations of the form

$$\sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{n} b_i(x) D_i u(x) + f(x, u(x)) = 0, \quad x \in \mathbb{R}^n.$$

Cutrì et al. [20] proved Liouville type results for viscosity subsolution and supersolution of

$$\mathcal{F}(x, D^2 u) + u^p = 0,$$

with $\mathcal{F}$ uniformly elliptic, $\mathcal{F}(x, \mathbf{0}) = 0$ and p is in a suitable range.

In [3], Bardi and Goffi studied the Liouville properties for viscosity subsolution or supersolution of fully nonlinear degenerate elliptic equations of type

$$\mathcal{F}(x, u, Du, D^2 u) = 0 \quad \text{in } \mathbb{R}^n,$$

where $\mathcal{F}$: $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ such that $\mathcal{F}(x, r, p, X)$ is at least continuous and proper. That is, non-increasing (respectively, non-decreasing) with respect to r and X.

In 1957, J. B. Keller [49] and R. Osserman [61] independently and simultaneously proved that the semilinear differential inequality

$$\Delta u \geq f(u)$$

has an entire solution u: $\mathbb{R}^n \to \mathbb{R}$ if and only if

$$\int_0^{\infty} \frac{dt}{\sqrt{F(t)}} = \infty.$$

Here, $F(t)$ denotes an antiderivative of a positive, continuous and non-decreasing function f.

Felmer et al. [32] studied Liouville-type theorems for viscosity subsolutions of

$$\mathcal{P}^{+}_{\lambda,\Lambda}(D^2 u) = f(u) + g(|Du|) \quad \text{in } \mathbb{R}^n, \tag{3.2.1}$$

where f and g are strictly increasing continuous functions on $\mathbb{R}^{+}_{0} = [0, \infty)$ with

$f(0) = 0 = g(0)$, and $\mathcal{P}^+_{\lambda,\Lambda}$ is Pucci's maximal operator

$$\mathcal{P}^+_{\lambda,\Lambda}(X) := \Lambda \sum_{\lambda_j > 0} \lambda_j + \lambda \sum_{\lambda_j < 0} \lambda_j,$$

with ellipticity constants $0 < \lambda \leq \Lambda$ and $\lambda_j := \lambda_j(X)$, $j = 1, \cdots, n$ denote the eigenvalues of the real $n \times n$ symmetric matrix X.

In [32, Theorem 1], the authors showed that any non-negative viscosity subsolution of (3.2.1) vanishes identically in $\mathbb{R}^n$ if and only if the growth conditions on f and g satisfy either of the following:

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{or} \quad \int_1^\infty \frac{dt}{g(t)} < \infty. \tag{3.2.2}$$

where $F(t) = \displaystyle\int_0^t f(\tau)\, d\tau$.

Dolcetta et al. [25] established a necessary and sufficient condition for the existence of a continuous viscosity subsolution to

$$\mathcal{P}^+_k(D^2 u) = f(u) \quad \text{in } \mathbb{R}^n, \tag{3.2.3}$$

for a positive, continuous and non-decreasing function f, where $\mathcal{P}^+_k$ is a truncated Laplacian operator defined by

$$\mathcal{P}^+_k(X) := \sum_{j=k+1}^{n} \lambda_j(X).$$

In [25, Theorem 1.1], it is shown that (3.2.3) has a continuous viscosity subsolution if and only if the function f satisfies the Keller-Osserman condition

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty,$$

for $F(t)$ as defined in (3.2.2). The author obtained a necessary and sufficient condition for the existence of viscosity subsolutions to (3.2.3).

Liouville-type theorems, regularity results, singular solutions, and other related results for the truncated Laplacian operator P^+_k have been studied in [9, 14, 15, 33, 75, 76].

Later in [26], the authors extended their investigation of viscosity subsolutions to

$$\mathcal{F}(x, D^2u) = f(u) + g(u)|Du|^q.$$

Here, f and g are assumed to be positive, continuous and monotone increasing functions, $q \in (0, 2]$, and $\mathcal{F}$ is a second-order degenerate elliptic operator. They established a general condition similar to the condition (3.2.2).

C. Doerr and A. Mohammed [23] proved several Liouville-type theorems for non-negative viscosity subsolutions of

$$S_k(D^2u) = h(u, |Du|) \quad \text{in } \mathbb{R}^n,$$

where $S_k(D^2u)$ is the k-Hessian operator acting on $u \in C^2(\Omega)$ defined by

$$S_k(D^2u) := \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} \lambda_{i_j}(D^2u)$$

for fixed positive integers n and k such that $1 \le k \le n$. They established a necessary and sufficient condition for the existence of non-trivial non-negative entire solutions to

$$S_k(D^2u) = f(u) + g(u)|Du|^q \quad \text{in } \Omega,$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set and $0 \le q < k + 1$ is an integer.

Several Liouville-type theorems have been extended for nonlinear elliptic PDEs in [5, 9, 20, 22, 36, 39, 73] and the references therein.

We refer to the interesting paper [36] and the references therein for many results on Liouville-type theorems related to other classes of degenerate elliptic equations. We also mention recent works [2, 18] on Liouville theorems related to the p-Laplacian with nonlinear gradient terms.

F. Ferrari and A. Vitolo [33, Theorem 7.1] proved that if $a_1 a_n > 0$ and u is an entire viscosity solution of $\mathcal{P}_{\mathbf{a}}(D^2u) = 0$ in a connected open set $\Omega \subseteq \mathbb{R}^n$, then $u \equiv 0$ in Ω. In addition, the authors showed that the condition $a_1 a_n > 0$ is a necessary condition for a Liouville-type theorem to hold.

Our work is motivated by the recent paper [33] in which it was shown that if $a_1 a_n > 0$, then any viscosity solution of $\mathcal{P}_{\mathbf{a}}(D^2u) = 0$ in $\mathbb{R}^n$ that is bounded from above or from below must be a constant. The authors established the result from the Harnack inequality. Our investigation in this study is more aligned with work [25, 26], which considers fully nonlinear, uniformly elliptic equations and uses

Keller-Osserman-type conditions on the lower-order terms to derive Liouville-type theorems. See also [22, 55, 56].

Liouville-type theorems for partial trace equations with nonlinear gradient terms have yet to be investigated. In this study, our main interest is to extend various Liouville-type theorems for partial trace equations that include nonlinear gradient terms.

Chapter 4

Vázquez-type Strong Minimum/Maximum Principle for partial trace operator

4.1. Introduction

A well-known fact in harmonic function theory states that any non-negative superharmonic function in a connected open set must be either identically vanish or strictly positive everywhere (see [41]). This property is referred to as the Strong Minimum Principle.

We are motivated by the recent work [33], in which F. Ferrari and A. Vitolo studied several properties of partial trace operators:

$$\mathcal{P}_{\mathbf{a}}(X) := \sum_{j=1}^{n} a_j \lambda_j(X). \tag{4.1.1}$$

Here, $\mathbf{a} = (a_1, \cdots, a_n)$ is an n-tuple of non-negative real numbers, at least one of which is positive, and $\lambda_1(X) \leq \cdots \leq \lambda_n(X)$ are eigenvalues of an $n \times n$ real symmetric matrix X. A good example of (4.1.1) is the min-max operator

$$\mathcal{M}(x) := \lambda_1(X) + \lambda_n(X).$$

For $n > 2$, this is a fully nonlinear, nonconvex and degenerate elliptic operator.

In this study, we examine the Strong Minimum Principle for viscosity supersolutions of

$$\mathcal{P}_{\mathbf{a}}(D^2u) = h(u, |Du|) \quad \text{in } \Omega, \tag{4.1.2}$$

where Ω is an open connected subset of $\mathbb{R}^n$, and h is required to satisfy some suitable assumptions. We will also consider the Strong Maximum Principle (SMP) for subsolutions of (4.1.2). Finally, we investigate the compact support principle (CSP) for non-negative subsolutions of (4.1.2).

4.2. Main results

We begin by introducing the conditions we use on the Hamiltonian h in equation (4.1.2) for the validity of the Strong Minimum/Maximum Principles. Let h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ be a continuous function. Assume the following conditions for some constant $\nu > 0$:

(M): $s \mapsto s + h(s, t_0)$ and $t \mapsto t + h(s_0, t)$ are non-decreasing in $(0, \nu)$ for each s_0, $t_0 \in (0, \nu)$,

(P): $t + h(t, t)$ is positive in $(0, \nu)$, and $\displaystyle\int_0^\nu \frac{dt}{t + h(t,t)} = \infty$.

Our first main result concerns the Strong Minimum Principle for viscosity supersolutions of (4.1.2).

Theorem 4.2.1 (Strong Minimum Principle – SmP)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open connected set and assume that conditions **(M)** and **(P)** hold. Furthermore, suppose that $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_n$. If $u \in \mathrm{LSC}(\Omega)$ is a non-negative supersolution of the problem (4.1.2) in Ω, then either $u \equiv 0$ in Ω or $u > 0$ in Ω.

A Strong Maximum Principle (SMP) on the subsolutions of problem (4.1.2) is obtained from the Strong Minimum Principle by duality on $\mathcal{P}_{\mathbf{a}}$. Some additional conditions are needed on h as follows:

($\boldsymbol{\mathcal{D}_1}$): $s \mapsto h(s, t)$ is non-decreasing in $\mathbb{R}$ for each $t \in \mathbb{R}_0^+$.

($\boldsymbol{\mathcal{D}_2}$): $h(-s, t) + h(s, t) \geq 0$ for all $(s, t) \in \mathbb{R} \times \mathbb{R}_0^+$.

Theorem 4.2.2 (Strong Maximum Principle – SMP)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open connected subset, and let h satisfy conditions ($\boldsymbol{\mathcal{D}_1}$) and ($\boldsymbol{\mathcal{D}_2}$). We further suppose that h satisfies **(M)**, **(P)** and that $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_1$. If $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of (4.1.2) such that $u(x) \leq u(x_0)$ on Ω for some $x_0 \in \Omega$ and $u(x_0) \geq 0$, then $u \equiv u(x_0)$ in Ω.

The next main result shows that condition **(P)** is actually the necessary and sufficient condition for (4.1.2) to have the Strong Minimum Principle (SmP) provided that, in addition to **(M)**, we assume the following condition on h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$.

(Q): $h(t,t)$ is positive and $s \mapsto h(s,t)$ is non-increasing in $(0,\nu)$ for each $t \in \mathbb{R}^+$.

Theorem 4.2.3. Let $\Omega \subseteq \mathbb{R}^n$ be an open connected subset and consider $\mathcal{P}_{\mathbf{a}}$ with $a_n > 0$. Suppose h satisfies **(M)** and **(Q)**. The Strong Minimum Principle holds for the problem (4.1.2) if and only if h satisfies **(P)**.

Now, let us recall the definition of the compact support principle [65] as follows.

A Compact Support Principle (CSP) is said to hold for (4.1.2) in $\mathbb{R}^n \backslash \overline{B}(o, r_0)$, where $r_0 > 0$, if for any non-negative subsolution u of (4.1.2) in $\mathbb{R}^n \backslash \overline{B}(o, r_0)$ such that $u(x) \to 0$ as $|x| \to \infty$, it follows that $u \equiv 0$ in $\mathbb{R}^n \backslash B(o, R)$ for some $R \geq r_0$. To state the next result suppose there exists a constant $\nu > 0$ such that the following conditions on h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ hold.

(H-0): $s \mapsto h(s,t)$ is increasing in $(0,\nu)$, $\forall t \in \mathbb{R}_0^+$.

(H-1): $h(0,t) > 0$ for $0 < t < \nu$, and

$$\int_0^\nu \frac{dt}{h(0,t)} < \infty.$$

(H-2): $h(s,0) > 0$, $t \mapsto h(s,t)$ is non-decreasing in $\mathbb{R}_0^+$, $\forall s \in (0,\nu)$, and

$$\int_0^\nu \frac{dt}{\sqrt{\mathcal{H}(t)}} < \infty,$$

where

$$\mathcal{H}(t) := \int_0^t h(s,0)\, ds.$$

The following result shows that **(H-0)**, together with either **(H-1)** or **(H-2)**, leads to the compact support principle (CSP) for non-negative subsolutions in exterior domains. We direct the reader to [31,67] for important results on CSP for a wide class of singular elliptic equations and quasi-linear elliptic equations.

Theorem 4.2.4. Suppose that either **(H-0)** and **(H-1)** or **(H-0)** and **(H-2)** hold, and that $|\mathbf{a}| > 0$ for the partial trace operator $\mathcal{P}_{\mathbf{a}}$. If, for some $r_0 > 0$, u is a non-negative subsolution of (4.1.2) on $\mathcal{O} := \mathbb{R}^n \backslash \overline{B}(o, r_0)$ with $u(x) \to 0$ as $|x| \to \infty$, then u has a compact support in $\mathcal{O}$.

4.3. Preliminary Results

In the following proposition, [60, Lemma 2.1] has been slightly modified to meet our specific needs in this study.

Proposition 4.3.1 (A Comparison Principle)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\mathcal{F}\colon \mathbb{S}^n \to \mathbb{R}$ an elliptic operator in Ω and for some constant $\mu > 0$, consider a function $\Phi\colon \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ which is non-decreasing in $(0, \mu)$ in the first variable for each fixed second variable in $\mathbb{R}_0^+$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $v \in \mathrm{LSC}(\Omega)$ such that

$$\mathcal{F}(D^2 u) \geq \Phi(u, |Du|) \quad \text{in } \Omega \quad \text{and} \quad \mathcal{F}(D^2 v) \leq \Phi(v, |Dv|) \quad \text{in } \Omega \tag{4.3.1}$$

in the viscosity sense. If $u(x) < \mu$ and $v(x) > 0$ for all $x \in \Omega$, one of the differential inequalities in (4.3.1) is strict and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. Suppose that $u > v$ at some point in Ω and assume that $u - v$ has a maximum at $x_0 \in \Omega$. Note that $0 < v(x_0) < u(x_0) < \mu$. First, suppose that $\mathcal{F}(D^2 u) > \Phi(u, |Du|)$ in Ω. Since v is a viscosity supersolution, $v - u$ has a minimum at x_0, and recalling the assumption that $u \in C^2(\Omega)$, we find

$$\mathcal{F}(D^2 u(x_0)) \leq \Phi(v(x_0), |Du(x_0)|) \leq \Phi(u(x_0), |Du(x_0)|).$$

This is an obvious contradiction. Suppose $\mathcal{F}(D^2 v) < \Phi(v, |Dv|)$ in Ω. Since $v - u$ has a minimum at x_0 and v is a strict viscosity supersolution, we have

$$\mathcal{F}(D^2 u(x_0)) < \Phi(v(x_0), |Du(x_0)|) \leq \Phi(u(x_0), |Du(x_0)|).$$

This contradicts the first inequality in (4.3.1). $\square$

Remark 4.3.2. Some remarks on conditions **(M)** and **(P)**.

(a) It follows from **(M)** that $t \mapsto 2t + h(t,t)$ is non-decreasing on $(0,\nu)$.

(b) Conditions **(M)** and **(P)** imply

$$\int_0^\nu \frac{dt}{2t + h(t,t)} = \infty. \tag{4.3.2}$$

To see this, let $g(t) := t + h(t,t)$, which is positive in $(0,\nu)$, by **(P)**. Then $g(t) + t = 2t + h(t,t)$ is non-decreasing in $(0,\nu)$, by **(M)**. Hence, **Lemma 2.6.3** and **(P)** show that (4.3.2) is true.

(c) Suppose that $h(t,t)$ is non-negative in $(0,\nu)$, and condition **(M)** holds. It follows from **Lemma 2.6.3** that the integral condition in **(P)** is equivalent to

$$\int_0^\nu \frac{dt}{h(t,t)} = \infty.$$

We refer to [60] for examples of Hamiltonians satisfying conditions **(M)** and **(P)**.

For a non-negative function u, a typical example is given by $h(u,|Du|) = au^\alpha + b|Du|^\beta$, with $\min(\alpha,\beta) \geq 1$. The same applies to the multiplicative case $h(u,|Du|) = au^\alpha|Du|^\beta$, where α, $\beta \geq 0$ and $\alpha + \beta \geq 1$. In this case, we have $h(t,t) = at^{\alpha+\beta}$, which ensures that both conditions **(M)** and **(P)** are satisfied.

Let us now consider the following boundary-value problem for given constants $\rho > 0$, $\varepsilon > 0$ and $\kappa > 0$.

$$\begin{cases} \varphi'' \geq \kappa\varphi' + \kappa\varphi + h(\kappa\varphi, \kappa\varphi') & \text{in } (0,\rho) \\ \varphi(0) = 0, \quad \varphi(\rho) \leq \varepsilon. \end{cases} \tag{BVP}$$

Lemma 4.3.1. Suppose h satisfies **(M)** and **(P)** for some $\nu > 0$. Let $0 < \rho \leq 1$, $\varepsilon > 0$, and $\kappa > 0$ be given. Then for each $0 < \gamma < \min\{\varepsilon/\rho, \nu/\kappa\}$ there exists a convex solution $\varphi \in C^2([0,\rho])$ of the problem (BVP) such that $0 \leq \varphi \leq \varepsilon$ and $0 < \varphi' \leq \gamma$ on $(0,\rho)$.

Proof. Let $0 < \rho \leq 1$, $\varepsilon > 0$, and $\kappa > 0$ be given. Let us take any

$$0 < \gamma < \min\{\varepsilon/\rho, \nu/\kappa\}. \tag{4.3.3}$$

According to (4.3.2), we now fix $\gamma_0 > 0$, depending on γ, κ and ρ, such that

$$\int_{\gamma_0}^{\gamma} \frac{dt}{2\kappa t + h(\kappa t, \kappa t)} = \rho. \tag{4.3.4}$$

Set

$$\Psi(r) := \int_{\gamma_0}^{r} \frac{dt}{2\kappa t + h(\kappa t, \kappa t)}, \quad \gamma_0 \le r \le \gamma.$$

Clearly, Ψ is a C^1 function such that $\Psi'(r) > 0$ for $r \in [\gamma_0, \gamma]$ and $\Psi(\gamma_0) = 0$, $\Psi(\gamma) = \rho$. Let ψ: $[0, \rho] \to [\gamma_0, \gamma]$ be the inverse of Ψ so that

$$\int_{\gamma_0}^{\psi(r)} \frac{dt}{2\kappa t + h(\kappa t, \kappa t)} = r, \quad 0 \le r \le \rho. \tag{4.3.5}$$

We note that $\psi \in C^1([0, \rho])$ is increasing and from the identity (4.3.5) we find that for $0 \le r \le \rho$,

$$\psi'(r) = 2\kappa\psi(r) + h(\kappa\psi(r), \kappa\psi(r)); \quad \psi(0) = \gamma_0, \quad \psi(\rho) = \gamma. \tag{4.3.6}$$

Let us now define

$$\varphi(r) := \int_0^r \psi(s)\, ds, \quad 0 \le r \le \rho.$$

Then $\varphi' = \psi > 0$ on $[0, \rho]$ and $\varphi \in C^2\left([0, \rho]\right)$. Note that $\varphi'(0) = \gamma_0 > 0$ and $0 < \varphi' = \psi \le \gamma$ on $[0, \rho]$. Moreover, for $0 \le r \le \rho$, we have

$$\varphi(r) = \int_0^r \psi(s)\, ds \le r\psi(r) \le \rho\psi(\rho) = \rho\gamma < \varepsilon. \tag{4.3.7}$$

Here, we have used (4.3.3) and (4.3.6). Thus, $\varphi(0) = 0$ and $\varphi(r) \le \varepsilon$ for $r \in (0, \rho)$. Recalling $0 < \rho \le 1$, we should also note from (4.3.7) that

$$\varphi(r) \le \psi(r), \quad r \in [0, \rho]. \tag{4.3.8}$$

Note that $\kappa\varphi \leq \kappa\psi < \nu$ on $(0,\rho)$. Differentiating the equation $\varphi' = \psi$ on $(0,\rho)$ and using (4.3.6), we find that the following holds on $(0,\rho)$:

$$\begin{aligned}\varphi'' = \psi' &= 2\kappa\psi + h(\kappa\psi,\kappa\psi)\\ &= \kappa\varphi' + \kappa\psi + h(\kappa\psi,\kappa\varphi')\\ &\geq \kappa\varphi' + \kappa\varphi + h(\kappa\varphi,\kappa\varphi'), \quad \text{by (4.3.8) and condition } \mathbf{(M)}.\end{aligned}$$

This completes the proof of **Lemma 4.3.1**. □

4.4. Proofs of main theorems

Proof of Theorem 4.2.1. Let $\Omega^+ := \{x \in \Omega : u(x) > 0\}$ and $\Omega^0 := \{x \in \Omega : u(x) = 0\}$. If Ω^+ or Ω^0 is empty, then there is nothing to show. Suppose both are non-empty. Since $u \in \mathrm{LSC}(\Omega)$, it is clear that Ω^+ is an open set. Let $z \in \Omega^+$ be such that $\mathrm{dist}(z,\partial\Omega^+) < \mathrm{dist}(z,\partial\Omega)$, and let $r_0 := \mathrm{dist}(z,\partial\Omega^+)$. Note that $r_0 := |z - x_0|$ for some $x_0 \in \partial\Omega^+ \cap \Omega^0$ and that $B(z,r_0) \subseteq \Omega^+$. We fix $\delta > 0$ small enough so that $3\delta < r_0$ and $r_0 + \delta < \mathrm{dist}(z,\partial\Omega)$. Let $R := r_0 + \delta$ and $\varrho := r_0 - \delta$ be so that $R/2 < \varrho < R$.

Recalling that $a_n > 0$, we set

$$\alpha := \frac{2|\mathbf{a}|}{a_n R^2}, \quad \mu := \frac{R\sqrt{a_n}}{2|\mathbf{a}|}\exp\left(\frac{2|\mathbf{a}|}{a_n}\right), \tag{4.4.1}$$

and

$$\kappa := \max\left\{1, \frac{2}{\sqrt{a_n}}\exp\left(\frac{3}{2}\frac{|\mathbf{a}|}{a_n}\right)\right\}. \tag{4.4.2}$$

We further choose δ sufficiently small that[1]

$$\rho := \mu\left(e^{-\alpha\varrho^2} - e^{-\alpha R^2}\right) \leq 1.$$

[1]Note that

$$\rho := \frac{(r_0+\delta)\sqrt{a_n}}{2|\mathbf{a}|}\exp\left(\frac{2|\mathbf{a}|}{a_n}\right)\left\{\exp\left[-\frac{2|\mathbf{a}|}{a_n}\cdot\left(\frac{r_0-\delta}{r_0+\delta}\right)^2\right] - \exp\left(-\frac{2|\mathbf{a}|}{a_n}\right)\right\}$$

which tends to zero as δ tends to zero.

We consider the following function

$$w(x) := e^{-\alpha|x-z|^2} - e^{-\alpha R^2}, \quad x \in \mathcal{A}_\varrho^R(z),$$

where $\mathcal{A}_\varrho^R(z)$ denotes the annulus $\{x \in \Omega : \varrho < |x-z| < R\}$.

Recalling that $\overline{B}(z,\varrho) \subseteq \Omega^+$ and that $u \in \mathrm{LSC}(\Omega)$, let us fix $\varepsilon > 0$, such that

$$0 < \varepsilon < \min\{u(x) : x \in \overline{B}(z,\varrho)\}, \quad \text{and} \quad \varepsilon\kappa < \nu. \tag{4.4.3}$$

Taking into account the constants $0 < \rho \leq 1$, $\varepsilon > 0$, and κ, we fix γ as in (4.3.3). Next, we invoke **Lemma 4.3.1** to find $\varphi \in C^2\left([0,\rho]\right)$ that satisfies the boundary-value problem (BVP). We set

$$v(x) := \varphi\left(\mu w(x)\right), \quad x \in \mathcal{A}_\varrho^R(z),$$

where μ is as in (4.4.1). According to **Lemma 4.3.1**, we have $0 \leq v(x) \leq \varepsilon$ in $\mathcal{A}_\varrho^R(z)$.

To find the second partial derivative of $v(x) := \varphi\left(\mu w(x)\right)$, we need to differentiate twice with respect to each variable x_i and x_j, where $x = (x_1, x_2, \cdots, x_n)$, respectively.

Let us define $r = |x-z|$. Then, the function $w(x)$ can be expressed as:

$$w(x) = e^{-\alpha r^2} - e^{-\alpha R^2}.$$

First, we compute the first partial derivative of $w(x)$ with respect to x_j:

$$\frac{\partial w}{\partial x_j} = -2\alpha(x_j - z_j)e^{-\alpha r^2}.$$

Consequently, we find

$$Dw \otimes Dw = Dw(Dw)^T = 4\alpha^2 e^{-2\alpha|x-z|^2}(x-z)(x-z)^T.$$

In this expression, $(x-z)(x-z)^T$ represents the outer product of the vector $(x-z)$ with itself. And to compute the second partial derivative of $w(x)$ with respect to

x_i and x_j, we proceed as follows:

$$\begin{aligned}
\frac{\partial^2 w}{\partial x_i \partial x_j} &= -2\alpha \frac{\partial}{\partial x_i}\left[\left(x_j - z_j\right) e^{-\alpha r^2}\right] \\
&= -2\alpha\left(-2\alpha(x_j - z_j)(x_i - z_i)e^{-\alpha r^2} + \frac{\partial}{\partial x_i}(x_j - z_j)e^{-\alpha r^2}\right) \\
&= 4\alpha^2(x_j - z_j)(x_i - z_i)e^{-\alpha r^2} - 2\alpha\delta_{ij}e^{-\alpha r^2} \\
&= 2\alpha e^{-\alpha r^2}\left(2\alpha(x_j - z_j)(x_i - z_i) - \delta_{ij}\right).
\end{aligned}$$

Here, δ_{ij} represents the Kronecker delta. Thus, the Hessian matrix $D^2w(x)$ of the function $w(x)$ resulting in a matrix is given by

$$D^2 w = 2\alpha e^{-\alpha|x-z|^2}\left\{2\alpha(x-z)(x-z)^T - I_n\right\},$$

where I_n is the $n \times n$ identity matrix.

Therefore, the first derivative of v can be expressed as:

$$\frac{\partial v}{\partial x_i} = \phi'(\mu w(x)) \cdot \mu \frac{\partial w}{\partial x_i}$$

and we have $Dv = \mu\varphi'(\mu w)Dw$.

The second derivative $\dfrac{\partial^2 v}{\partial x_i \partial x_j}$ requires differentiating the first derivative:

$$\frac{\partial^2 v}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\phi'(\mu w(x)) \cdot \mu \frac{\partial w}{\partial x_i}\right)$$

As a consequence, we have

$$\begin{aligned}
D^2 v &= \mu^2\varphi''(\mu w)Dw \otimes Dw + \mu\varphi'(\mu w)D^2 w \\
&= 4\alpha^2\mu^2 e^{-2\alpha|x-z|^2}\varphi''(\mu w)(x-z)(x-z)^T \\
&\quad + 2\alpha\mu e^{-\alpha|x-z|^2}\varphi'(\mu w)\left\{2\alpha(x-z)(x-z)^T - I_n\right\}.
\end{aligned}$$

In what follows, we suppose that $x \neq z$. We observe that $D^2v(x)$ has one eigenvalue

$$4\alpha^2\mu^2 e^{-2\alpha|x-z|^2}|x-z|^2\varphi''(\mu w) + 2\alpha\mu e^{-\alpha|x-z|^2}\varphi'(\mu w)\left\{2\alpha|x-z|^2 - 1\right\}$$

corresponding to the eigenvector $x-z$ and $n-1$ eigenvalues

$$-2\alpha\mu e^{-\alpha|x-z|^2}\varphi'(\mu w)$$

corresponding to each of the $n-1$ basis elements of the orthogonal complement $(x-z)^\perp$ of $x-z$. Therefore, with $r=|x-z|$, we have

$$\begin{aligned}\lambda_1(D^2v)=\cdots=\lambda_{n-1}(D^2v)&=-2\alpha\mu e^{-\alpha|x-z|^2}\varphi'(\mu w)\\&=-2\alpha\mu e^{-\alpha r^2}\varphi'(\mu w),\end{aligned}$$

and[2]

$$\begin{aligned}\lambda_n(D^2v)&=4\alpha^2\mu^2e^{-2\alpha|x-z|^2}|x-z|^2\varphi''(\mu w)+2\alpha\mu e^{-\alpha|x-z|^2}\varphi'(\mu w)\left\{2\alpha|x-z|^2-1\right\}\\&=4\alpha^2\mu^2e^{-2\alpha r^2}r^2\varphi''(\mu w)+2\alpha\mu e^{-\alpha r^2}\varphi'(\mu w)\left\{2\alpha r^2-1\right\}.\end{aligned}$$

Therefore, for $x\neq z$, we have

$$\begin{aligned}&\mathcal{P}_{\mathbf{a}}(D^2v)=\sum_{j=1}^{n}a_j\lambda_j(D^2v)\\&=-2\alpha\mu e^{-\alpha r^2}\varphi'(\mu w)\sum_{j=1}^{n-1}a_j+a_n\Big[4\alpha^2\mu^2e^{-2\alpha r^2}r^2\varphi''(\mu w)+2\alpha\mu e^{-\alpha r^2}\varphi'(\mu w)\big\{2\alpha r^2-1\big\}\Big]\\&=4a_n\alpha^2\mu^2e^{-2\alpha r^2}r^2\varphi''(\mu w)+2\alpha\mu e^{-\alpha r^2}\varphi'(\mu w)\Big[2a_n\alpha r^2-\sum_{j=1}^{n}a_j\Big].\end{aligned}\tag{4.4.4}$$

Recalling that $R/2\le r\le R$ and using (4.4.1) in (4.4.4) we find that

$$\mathcal{P}_{\mathbf{a}}(D^2v)\ge\varphi''(\mu w)\quad\text{in }\mathcal{A}_\varrho^R(z).$$

Note that for $R/2\le r\le R$, we have

$$2\alpha\mu r e^{-\alpha r^2}\le\frac{2}{\sqrt{a_n}}\exp\left(\frac{3|\mathbf{a}|}{2a_n}\right).$$

[2]Recall that $\lambda_n(D^2v)\ge\lambda_j(D^2v)$ for all $j=1,\cdots,n-1$.

Therefore, in $\mathcal{A}_\varrho^R(z)$, we get

$$|Dv| = \mu\varphi'(\mu w)|Dw| = 2\alpha\mu r e^{-\alpha r^2}\varphi'(\mu w) \le \kappa\varphi'(\mu w) \le \kappa\gamma. \tag{4.4.5}$$

The choice of γ in **Lemma 4.3.1** shows that $\kappa\varphi'(\mu w) \le \kappa\gamma \le \nu$ in $\mathcal{A}_\varrho^R(z)$, with $\nu > 0$ as in conditions **(M)** and **(P)**. Furthermore, recalling our choice of ε in (4.4.3), we see that $0 < \kappa\varphi \le \kappa\varepsilon \le \nu$. Then, by **Lemma 4.3.1**, we have the following on $\mathcal{A}_\varrho^R(z)$.

$$\begin{aligned}
\mathcal{P}_{\mathbf{a}}(D^2 v) &\ge \varphi''(\mu w) \\
&\ge \kappa\varphi'(\mu w) + \kappa\varphi(\mu w) + h(\kappa\varphi(\mu w), \kappa\varphi'(\mu w)) \\
&\ge |Dv| + \kappa\varphi(\mu w) + h(\kappa\varphi(\mu w), |Dv|), && \text{by condition } \textbf{(M)} \text{ and } (4.4.5) \\
&> \kappa v + h(\kappa v, |Dv|), && \text{since } |Dv| > 0 \text{ on } \mathcal{A}_\varrho^R(z) \\
&\ge v + h(v, |Dv|), && \text{by condition } \textbf{(M)}\text{, since } \kappa \ge 1.
\end{aligned}$$

Thus, we have

$$\mathcal{P}_{\mathbf{a}}(D^2 v) > v + h(v, |Dv|) \quad \text{in } \mathcal{A}_\varrho^R(z). \tag{4.4.6}$$

On the other hand, since $u \ge 0$ is a supersolution of (4.1.2), we have

$$\mathcal{P}_{\mathbf{a}}(D^2 u) \le h(u, |Du|) \le u + h(u, |Du|) \quad \text{in } \mathcal{A}_\varrho^R(z). \tag{4.4.7}$$

Observe that $v \in C^2(\mathcal{A}_\varrho^R(z))$ and $0 < v \le \nu$ (see (4.4.3)). Moreover, we have

$$v(x) \le \varepsilon \le u(x) \quad \text{for } |x - z| = \varrho$$

and

$$v(x) = 0 \le u(x) \quad \text{for } |x - z| = R.$$

Therefore, from (4.4.6) and (4.4.7), by the comparison principle in **Proposition 4.3.1**, we find that $v \le u$ in $\mathcal{A}_\varrho^R(z)$. In particular, $0 < v(x_0) \le u(x_0)$, which is not possible since $x_0 \in \Omega^0$ (that is, $u(x_0) = 0$). □

Remark 4.4.1. Let us make some remarks on **Theorem 4.2.1**.

(a) If $h(t,t) \equiv 0$ in $(0, \nu)$, then h satisfies **(M)** and **(P)**. In this case, **Theorem 4.2.1** reduces to [33, Theorem 6.2].

(b) Assume that conditions **(M)** and **(P)** hold. Let $u \in \mathrm{LSC}(\Omega)$ be a supersolution of (4.1.2), and let $m := \inf_\Omega u$. If, in addition

   (i) $s \mapsto h(s, t_0)$ is non-decreasing in $\mathbb{R}$ for each $t_0 \geq 0$ and $m \leq 0$, or

   (ii) $s \mapsto h(s, t_0)$ is non-increasing in $\mathbb{R}$ for each $t_0 \geq 0$ and $m > 0$,

   then **Theorem 4.2.1** remains true.

In this case, note that $z = u - m \geq 0$ is a solution of

$$\begin{aligned}\mathcal{P}_{\mathbf{a}}(D^2 z) = \mathcal{P}_{\mathbf{a}}(D^2 u) &= h(u, |Du|)\\ &\leq h(u-m, |D(u-m)|) = h(z, |Dz|).\end{aligned}$$

Therefore, by **Theorem 4.2.1**, we have $z \equiv 0$ on Ω or $z > 0$ on Ω. That is $u \equiv m$ or $u > m$ in Ω.

However, note that the assumption that $m \leq 0$ cannot be removed when $s \mapsto h(s, t_0)$ is non-decreasing. To see this, note that

$$u(x) = \frac{1}{2n}|x|^2 + 1,$$

satisfies

$$\Delta u = 1 \leq \frac{1}{2n}|x|^2 + 1 = u,$$

and $\inf_{B(o,1)} u = 1$. Here, $h(s,t) = s$ and note that both conditions **(M)** and **(P)** hold.

Here, we give illustrative examples to show that some of the conditions needed for our main results cannot be removed.

(a) Let $u(x) = {x_j}^2$ for some $1 \leq j \leq n$. Then $\lambda_k(D^2 u) = 0$ for $k \neq n$, and $\lambda_n(D^2 u) = 2$. Therefore,

$$\mathcal{P}_{\mathbf{a}}(D^2 u) = 2a_n.$$

   If $a_n = 0$, then $\mathcal{P}_{\mathbf{a}}(D^2 u) = 0 \leq u = h(u, |Du|)$. But $u(0) = 0$ and yet $u \not\equiv 0$ in $\Omega := B(o, 1)$.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be connected. Given constants $p > 1$, $q > 1$ and α, $\beta \in \mathbb{R}$, we note that any non-negative viscosity solution $u \in C(\Omega)$ of $\mathcal{P}_{\mathbf{a}}(D^2 u) \leq \alpha u^p + \beta |Du|^q$ in Ω such that $u(x_0) = 0$ for some $x_0 \in \Omega$ vanishes identically in Ω. To see this, let $h(s,t) = \alpha \left(s^+\right)^p + \beta t^q$, where $s^+ := \max\{s, 0\}$ and $t \geq 0$. As the case $\alpha = \beta = 0$ is obvious, we assume that $|\alpha| + |\beta| > 0$. Then it is easily seen that h satisfies **(M)** and **(P)** with

$$\nu := \min \left\{ 1, \left(\frac{1}{p|\alpha| + q|\beta|} \right)^{\dfrac{1}{\min\{p-1, q-1\}}} \right\}.$$

Therefore, the desired conclusion follows from **Theorem 4.2.1**.

Proof of Theorem 4.2.2. Let $w = u(x_0) - u$. Then, by duality, we have

$$\begin{aligned}
\mathcal{P}_{\mathbf{a}'}(D^2 w) &= \mathcal{P}_{\mathbf{a}'}(-D^2 u) \\
&= -\mathcal{P}_{\mathbf{a}}(D^2 u) \leq -h(u, |Du|) \\
&\leq h(-u, |Du|), && \text{by } (\boldsymbol{\mathcal{D}_2}) \\
&\leq h(u(x_0) - u, |Du|), && \text{by } (\boldsymbol{\mathcal{D}_1}) \\
&= h(w, |Dw|).
\end{aligned}$$

We note that $\mathcal{P}_{\mathbf{a}'} \in \mathcal{A}_n$ and that w is a non-negative supersolution of (4.1.2). By Theorem 4.2.1, we see that $w \equiv 0$ in Ω or $w > 0$ in Ω. Since $w(x_0) = 0$, we see that $u \equiv u(x_0)$ in Ω. □

Proof of Theorem 4.2.3. The backward implication follows from Theorem 4.2.1. So we focus on the forward implication. Assume that conditions **(M)** and **(Q)** hold for h, but that h does not satisfy condition **(P)**. Since **(M)** holds, **Lemma 5.3.1** shows that

$$\int_0^\nu \frac{ds}{h(s,s)} < \infty.$$

We follow the ideas of J. I. Díaz in [29] to show that SmP fails for problem (4.1.2). Let ψ be the inverse of

$$\Psi(r) := |\mathbf{a}| \int_0^r \frac{ds}{h(s,s)}, \quad 0 \leq r \leq \nu.$$

Then ψ: $(0,\rho) \to (0,\nu)$, where $\rho := \Psi(\nu)$, satisfies (recall that $|\mathbf{a}| > 0$)

$$\int_0^{\psi(r)} \frac{ds}{h(s,s)} = \frac{r}{|\mathbf{a}|}.$$

Without loss of generality, we can suppose that ν is small enough for $\rho \leq 1$. Note that ψ is a C^1 function in $(0,\rho)$ such that

$$\psi'(r) = \frac{1}{|\mathbf{a}|} h(\psi(r), \psi(r)).$$

Next, we define

$$\varphi(r) := \int_0^r \psi(t)dt, \quad 0 \leq r < \rho.$$

Note that $\varphi \leq \psi$ on $[0,\rho]$. Then $\varphi'(r) = \psi(r)$, $\varphi(0) = \varphi'(0) = 0$ and

$$\begin{aligned}\varphi''(r) = \psi'(r) &= \frac{1}{|\mathbf{a}|} h(\psi(r), \varphi'(r)) \\ &\leq \frac{1}{|\mathbf{a}|} h(\varphi(r), \varphi'(r)), \quad \text{by } \textbf{(Q)} \text{ and the fact that } \varphi \leq \psi \text{ on } [0,\rho).\end{aligned}$$

In particular, we see $\varphi''(0) = |\mathbf{a}|^{-1} h(0,0)$. We define u on $\Omega = \mathbb{R}^{n-1} \times (-\rho,\rho)$ by

$$u(x', x_n) := \begin{cases} 0, & \text{if } -\rho < x_n \leq 0 \\ \varphi(x_n), & \text{if } 0 \leq x_n < \rho. \end{cases}$$

Notice that $u \in C^1(\Omega)$. Obviously $D^2 u \equiv 0$ in $\Omega^- := \{x : -\rho < x_n < 0\}$ and therefore $\mathcal{P}_{\mathbf{a}}(D^2 u) \leq h(u, |Du|)$ in Ω^-. On $\Omega^+ := \{x : 0 < x_n < \rho\}$, we see that $Du(x) = \varphi'(x_n) e_n$. For $x \in \Omega^+$, we see that $D^2 u(x)$ has eigenvalues 0 and $\varphi''(x_n) > 0$ of multiplicities $n-1$ and 1, respectively. Therefore (note that $0 < \varphi'(x_n) < \nu$ for $0 < x_n < \rho$)

$$\begin{aligned}\mathcal{P}_{\mathbf{a}}(D^2 u(x)) = a_n \varphi''(x_n) &\leq \frac{a_n}{|\mathbf{a}|} h(\varphi(r), \varphi'(r)) \\ &\leq h(u(x), |Du(x)|), \quad x \in \Omega^+.\end{aligned}$$

Now, we consider $x_0 = (x_0', 0) \in \Omega$ and $\eta \in C^2(\Omega)$ so that $u - \eta$ has a minimum at x_0 and $u(x_0) = \eta(x_0)$. Then

$$
\begin{aligned}
&o(|x - x_0|^2) + \frac{1}{2}\langle D^2\eta(x_0)(x - x_0), x - x_0\rangle + \langle D\eta(x_0), x - x_0\rangle + \eta(x_0) \\
&\leq u(x', x_n) \quad \text{as } x \to x_0.
\end{aligned}
$$

Note that $D\eta(x_0) = Du(x_0) = o$. Given a unit vector $\mathbf{b}$ and $0 < t < \rho$, let

$$
x := x_0 + t\mathbf{b} = (x_0' + t\mathbf{b}', tb_n).
$$

Then, as $t \to 0^+$, we have

$$
\begin{aligned}
o(t^2) + \frac{t^2}{2}\langle D^2\eta(x_0)\mathbf{b}, \mathbf{b}\rangle &\leq u(x_0' + tb', tb_n) - u(x_0', 0) \\
&\leq \varphi(t|b_n|) - \varphi(0).
\end{aligned}
$$

Dividing by t^2 and then taking the limit as $t \to 0^+$, we obtain

$$
\begin{aligned}
\frac{1}{2}\langle D^2\eta(x_0)\mathbf{b}, \mathbf{b}\rangle &\leq \lim_{t\to 0^+} \frac{\varphi(t|b_n|) - \varphi(0)}{t^2} \\
&= \frac{1}{2}b_n^2\varphi''(0) \leq \frac{1}{2}\varphi''(0).
\end{aligned}
$$

Consequently, we find

$$
\lambda_n(D^2\eta(x_0)) \leq \varphi''(0).
$$

Therefore,

$$
\begin{aligned}
\mathcal{P}_{\mathbf{a}}(D^2\eta(x_0)) &= \sum_{j=1}^{n} a_j\lambda_j(D^2\eta(x_0)) \leq \sum_{j=1}^{n} a_j\lambda_n(D^2\eta(x_0)) \\
&= \lambda_n(D^2\eta(x_0))\sum_{j=1}^{n} a_j = |\mathbf{a}|\lambda_n(D^2\eta(x_0)) \\
&\leq |\mathbf{a}|\varphi''(0) = h(0,0) = h(u(x_0), |D\eta(x_0)|).
\end{aligned}
$$

In summary, we have shown that u is a non-negative viscosity supersolution of (4.1.2). Yet, u vanishes for (x', x_n) with $-\rho < x_n \leq 0$, but it is positive when $0 < x_n < \rho$. Therefore, Equation (4.1.2) does not satisfy the SmP. $\square$

It is interesting to note that the condition

$$\int_0^{\nu} \frac{ds}{\sqrt{G(s)}} = \infty$$

is not sufficient to hold the SmP for the equation (4.1.2).

Remark 4.4.2. Suppose g is continuous non-decreasing in $(0, \nu)$, and set

$$G(t) = \int_0^t g(s)\, ds.$$

For $0 < t < \nu$, we have

$$\begin{aligned} \frac{\sqrt{G(t)}}{g(t)} \leq \frac{\sqrt{tg(t)}}{g(t)} = \sqrt{\frac{t}{g(t)}} = \frac{t}{\sqrt{tg(t)}} &\leq \int_0^t \frac{ds}{\sqrt{sg(s)}} \\ \leq \int_0^{\nu} \frac{ds}{\sqrt{sg(s)}} \leq \int_0^{\nu} \frac{ds}{\sqrt{G(s)}} &:= \mathrm{C}. \end{aligned}$$

Therefore

$$\frac{\sqrt{G(t)}}{g(t)} \leq \mathrm{C}, \quad 0 < t < \nu.$$

Hence, we have

$$\int_0^{\nu} \frac{ds}{g(s)} \leq \mathrm{C} \int_0^{\nu} \frac{ds}{\sqrt{G(s)}}.$$

We note that the left-hand side integral can be finite, while the right-hand side integral is infinite.

In fact, consider the function

$$g(t) = t\log^2\left(1 + \frac{1}{t}\right), \quad t > 0.$$

Note that there is $\sigma > 0$ such that g is increasing on $(0, \sigma)$ and $\lim_{t \to 0^+} g(t) = 0$. One can see that

$$\int_0^{\sigma} \frac{ds}{g(s)} < \infty \quad \text{and} \quad \int_0^{\sigma} \frac{ds}{\sqrt{G(s)}} = \infty.$$

As we saw in the proof of **Theorem 4.2.3**, the equation

$$\mathcal{P}_{\mathbf{a}}(D^2u) = |Du|\log^2\left(1 + \frac{1}{|Du|}\right)$$

admits a non-trivial supersolution that vanishes in a portion of the set $\Omega := \mathbb{R}^{n-1} \times (-\rho, \rho)$ for some $\rho > 0$.

Therefore, while $\displaystyle\int_0^\rho \frac{ds}{\sqrt{G(s)}} = \infty$ is necessary and sufficient for non-negative solutions of $\mathcal{P}_{\mathbf{a}}(D^2u) = g(u)$ to satisfy the SmP, as was noted in **Theorem 4.2.3**, it is $\displaystyle\int_0^\nu \frac{ds}{g(s)} = \infty$ that is necessary and sufficient condition for non-negative solutions of $\mathcal{P}_{\mathbf{a}}(D^2u) = g\left(|Du|\right)$ to satisfy the SmP.

Now, we give an illustrative example of h that satisfies conditions **(M)** and **(Q)**, but not **(P)**.

Let $h(s,t) := f(s) + \varpi(t)$, where $f \in C^1(\mathbb{R}, \mathbb{R})$ with $f(0) = f'(0) = 0$ is non-increasing in $(0, \delta)$ for some $\delta > 0$. Furthermore, let $\varpi \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ be non-decreasing on $(0, \delta)$ such that $f(t) + \varpi(t) > 0$ in $(0, \delta)$. Since $1 + f'(s) > 0$ for sufficiently small $s > 0$, we see that $s \mapsto s + h(s,t)$ is non-decreasing for sufficiently small $s > 0$. Therefore, we easily see that **(M)** and **(Q)** hold for a suitable ν with $\delta \geq \nu > 0$. We observe that h satisfies condition **(P)** if and only if

$$\int_0^\nu \frac{dt}{\varpi(t)} = \infty. \tag{4.4.8}$$

In fact, we note that

$$t + h(t,t) = t + f(t) + \varpi(t) \leq t + \varpi(t), \quad t \in (0, \nu).$$

Therefore, if (4.4.8) holds, then (it follows from **Lemma 5.3.1**) that

$$\int_0^\nu \frac{dt}{t + h(t,t)} = \infty.$$

On the other hand, since $t + f(t) \geq 0$ for sufficiently small $t > 0$, for such t we have

$$t + h(t,t) = t + f(t) + \varpi(t) \geq \varpi(t).$$

Therefore, we see that **(P)** fails to hold if (4.4.8) does not hold.

Proof of Theorem 4.2.4. Case (i). Let us first suppose that **(H-0)** and **(H-1)** hold. We use condition **(H-1)** to define

$$\Psi(r) := |\mathbf{a}| \int_0^r \frac{ds}{h(0,s)}, \quad 0 \leq r \leq \nu.$$

Let ψ be the inverse of Ψ, and $\rho := \Psi(\nu)$. Then ψ: $(0,\rho) \to (0,\nu)$ satisfies

$$\int_0^{\psi(r)} \frac{ds}{h(0,s)} = \frac{r}{|\mathbf{a}|}.$$

Note that ψ is a C^1 function in $(0,\rho)$ such that

$$\psi'(r) = \frac{1}{|\mathbf{a}|} h(0,\psi(r)). \tag{4.4.9}$$

Next, we define

$$\varphi(r) := \int_0^r \psi(t)\, dt, \quad 0 \le r \le \rho. \tag{4.4.10}$$

We fix $0 < \varrho < \rho$ such that $0 < \varphi(\varrho) < \nu$. From (4.4.10), we have $\varphi'(r) = \psi(r)$. This, together with (4.4.9), leads to

$$\varphi''(r) = \psi'(r) = \frac{1}{|\mathbf{a}|} h(0,\varphi'(r)). \tag{4.4.11}$$

Therefore, by **(H-0)** and (4.4.11) we find

$$|\mathbf{a}|\varphi''(r) \le h(\varphi(r),\varphi'(r)), \quad 0 < r < \rho. \tag{4.4.12}$$

Now, let u be a non-negative subsolution of (4.1.2) in $\mathbb{R}^n \backslash \overline{B}(o,r_0)$ such that $u(x) \to 0$ as $|x| \to \infty$. We choose $r_1 \ge r_0$ such that $0 \le u(x) \le \varphi(\varrho) < \nu$ for $|x| \ge r_1$. Let

$$v(x) := \begin{cases} \varphi(\varrho + r_1 - |x|), & \text{for } r_1 < |x| \le \varrho + r_1 \\ 0, & \text{for } |x| > \varrho + r_1. \end{cases} \tag{4.4.13}$$

We note that v is C^1 in $\mathbb{R}^n \backslash \overline{B}(o,r_1)$. We also recall that for $r_1 < |x| < \varrho + r_1$, the Hessian matrix $D^2 v(x)$ has eigenvalues $-|x|^{-1}\varphi'(\varrho + r_1 - |x|)$ of multiplicity $n-1$ and eigenvalue $\varphi''(\varrho + r_1 - |x|)$ of multiplicity 1. Therefore, for $r_1 < |x| < \varrho + r_1$, we have

$$
\begin{aligned}
\mathcal{P}_{\mathbf{a}}(D^2 v) = \sum_{j=1}^{n} a_j \lambda_j(D^2 v) &= -\frac{\varphi'(\varrho + r_1 - |x|)}{|x|} \sum_{j=1}^{n-1} a_j + a_n \varphi''(\varrho + r_1 - |x|) \\
&\leq |\mathbf{a}| \varphi''(\varrho + r_1 - |x|) \\
&\leq h(v, |Dv|), \quad \text{by (4.4.12)}. \qquad (4.4.14)
\end{aligned}
$$

Obviously, $\mathcal{P}_{\mathbf{a}}(D^2 v(x)) = 0 \leq h(v, |Dv|)$ for $|x| > \varrho + r_1$. Now, let us suppose $|x_0| = \varrho + r_1$. We take $\eta \in C^2(\Omega)$ such that $v - \eta$ has a minimum at x_0, and $v(x_0) = \eta(x_0) = 0$. Clearly $D\eta(x_0) = Dv(x_0) = o$. Therefore

$$
\begin{aligned}
o(|x - x_0|^2) + \frac{1}{2} \langle D^2 \eta(x_0)(x - x_0), x - x_0 \rangle + \langle D\eta(x_0), x - x_0 \rangle + \eta(x_0) \\
\leq v(x) \quad \text{as } x \to x_0.
\end{aligned}
$$

Let $x := x_0 - t\mathbf{b}$, where $\mathbf{b}$ is a unit vector in $\mathbb{R}^n$. Clearly, for $t > 0$, we have $|x_0| - |x_0 - t\mathbf{b}| \leq t$. But then $v(x) \leq \max\{\varphi(|x_0| - |x_0 - t\mathbf{b}|), 0\} \leq \varphi(t)$. Therefore

$$
o(t^2) + \frac{t^2}{2} \langle D^2 \eta(x_0)\mathbf{b}, \mathbf{b} \rangle \leq \varphi(t) - \varphi(0), \quad \text{as } t \to 0^+.
$$

Dividing by t^2 and then taking the limit as $t \to 0^+$, we find that

$$
\langle D^2 \eta(x_0)\mathbf{b}, \mathbf{b} \rangle \leq \varphi''(0).
$$

Hence

$$
\lambda_n(D^2 \eta(x_0)) \leq \varphi''(0). \qquad (4.4.15)
$$

Thus, from (4.4.12) and (4.4.15), we find

$$
\begin{aligned}
\mathcal{P}_{\mathbf{a}}(D^2 \eta(x_0)) = \sum_{j=1}^{n} a_j \lambda_j(D^2 \eta(x_0)) &\leq \sum_{j=1}^{n} a_j \lambda_n(D^2 \eta(x_0)) \\
&= \lambda_n(D^2 \eta(x_0)) \sum_{j=1}^{n} a_j = |\mathbf{a}| \lambda_n(D^2 \eta(x_0)) \\
&\leq |\mathbf{a}| \varphi''(0) \leq h(\varphi(0), |D\eta(x_0)|).
\end{aligned}
$$

Thus, v is a viscosity supersolution of (4.1.2) in $|x| > r_1$. Moreover, our choice of r_1 shows that $u \le v$ on $|x| = r_1$. Given $\varepsilon > 0$, we can choose $R > \varrho + r_1$ such that $u \le \varepsilon = v + \varepsilon$ on $|x| > R$. Let $z := v + \varepsilon$ such that

$$\mathcal{P}_{\mathbf{a}}(D^2 z) \le h(v, |Dv|) < h(z, |Dz|) \quad \text{in} \ \ r_1 < |x| < R' \text{ for any } R' > R. \tag{4.4.16}$$

Therefore, by the Comparison Principle in **Proposition 4.3.1**, we conclude that $u \le v + \varepsilon$ for $|x| > r_1$. Thus $u \le \varepsilon$ for all $|x| > \varrho + r_1$ and arbitrary $\varepsilon > 0$. Therefore, we conclude that $u \equiv 0$ for $|x| > \varrho + r_1$.

Case (ii). Suppose now that conditions **(H-0)** and **(H-2)** hold. In this case, we define

$$\Psi(r) := \sqrt{|\mathbf{a}|} \int_0^r \frac{ds}{\sqrt{2\mathcal{H}(s)}}, \quad 0 \le r \le \nu$$

and let ψ be the inverse of Ψ so that

$$\int_0^{\psi(r)} \frac{ds}{\sqrt{2\mathcal{H}(s)}} = \frac{r}{\sqrt{|\mathbf{a}|}}, \quad 0 \le r \le \rho := \Psi(\nu).$$

We fix $0 < \varrho < \rho$ such that $0 < \psi(\varrho) < \nu$. Clearly, we have

$$\psi'(t) = \frac{1}{\sqrt{|\mathbf{a}|}} \sqrt{2\mathcal{H}(\psi(t))}, \quad 0 < t < \rho.$$

Thus, we get

$$\begin{aligned}
\psi''(t) &= \frac{1}{\sqrt{2|\mathbf{a}|\mathcal{H}(\psi(t))}} h(\psi(t), 0)\psi'(t) \\
&= \frac{1}{|\mathbf{a}|} h(\psi(t), 0) \le \frac{1}{|\mathbf{a}|} h(\psi(t), \psi'(t)), \quad 0 < t < \rho.
\end{aligned}$$

As in Case (i), we fix $r_1 \ge r_0$ such that $0 \le u(x) \le \psi(\varrho) < \nu$ for all $|x| \ge r_1$. We define v for $|x| > r_1$ by

$$v(x) := \begin{cases} \psi(\varrho + r_1 - |x|), & \text{for } r_1 < |x| \le \varrho + r_1 \\ 0, & \text{for } |x| > \varrho + r_1. \end{cases} \tag{4.4.17}$$

As in the proof of Case (i), we can show that

$$\mathcal{P}_{\mathbf{a}}(D^2 v) \leq h(v, |Dv|) \quad \text{for} \quad |x| > r_1.$$

Given $\varepsilon > 0$, we can choose $R > \varrho + r_1$ such that $u \leq \varepsilon = v + \varepsilon$ on $|x| > R$. Then $z = v + \varepsilon$ is a viscosity solution of (4.4.16) in $r_1 < |x| < R'$ for any $R' > R$. We proceed as in the proof of Case (i) to conclude the desired result. □

Let $\varpi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous and non-decreasing function such that $\varpi(0) = 0$ and $\varpi(t) > 0$ for $t > 0$. Moreover, let f, $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions such that f is increasing on $(0, \nu)$ and g is non-decreasing on $(0, \nu)$ for some $\nu > 0$. Furthermore, suppose that $f(0) = 0$, $f(s) > 0$ for $s > 0$, and $g(s) > 0$ for $s \geq 0$. We define

$$h(s, t) = f(s) + g(s)\varpi(t).$$

We assume that

$$\int_0^{\nu} \frac{ds}{\varpi(s)} < \infty, \quad \text{or} \quad \int_0^{\nu} \frac{ds}{\sqrt{F(s)}} < \infty,$$

where $F(s) = \displaystyle\int_0^s f(\tau)\, d\tau$. Then, if $|\mathbf{a}| > 0$, **Theorem 4.2.4** shows that non-negative subsolutions of

$$\mathcal{P}_{\mathbf{a}}(D^2 u) = f(u) + g(u)\varpi\left(|Du|\right)$$

satisfy the Compact Support Principle. □

Theorem 4.4.3 (Strong Boundary Minimum Principle)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and assume conditions **(M)** and **(P)** hold and that $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_n$. Suppose $x_0 \in \partial\Omega$ and $u \in \mathrm{LSC}(\Omega \cup \{x_0\})$ is a viscosity supersolution of (4.1.2) in Ω such that

(i) $u(x) > u(x_0) = 0$ for all $x \in \Omega$,

(ii) there is a ball $B := B(z, R) \subseteq \Omega$ with $\overline{B} \cap \partial\Omega = \{x_0\}$.

Then

$$\liminf_{\substack{x \to x_0 \\ x \in \mathcal{C}_{x_0}}} \frac{u(x) - u(x_0)}{|x - x_0|} > 0, \tag{4.4.18}$$

where $\mathcal{C}_{x_0}$ is any circular cone with vertex x_0, opening less than π, and with axis along the normal direction at x_0.

Proof. We will use the function v constructed in the proof of **Theorem 4.2.1** with some slight modifications. With R as in (ii), we take $R/2 < \varrho < R$ and define $v(x) := \varphi(\mu w(x))$ in $\mathcal{A}_\varrho^R(z)$ where

$$w(x) := e^{-\alpha|z-x|^2} - e^{-\alpha R^2}, \quad x \in \mathcal{A}_\varrho^R(z).$$

Here α, μ, and κ are defined as in the proof of **Theorem 4.2.1**. We also choose ϱ close enough to R so that

$$\rho := \mu\left(e^{-\alpha\varrho^2} - e^{-\alpha R^2}\right) \leq 1.$$

We choose $\varepsilon > 0$ as in (4.4.3). Proceeding as in the proof of **Theorem 4.2.1**, we can show that $0 < v \leq u$ in the annulus $\mathcal{A}_\varrho^R(z)$. Let $\boldsymbol{n}$ be the unit outer normal vector at $x_0 \in \partial\Omega$. Then for sufficiently small $t > 0$, we have

$$\frac{u(x_0 - t\boldsymbol{\eta}) - u(x_0)}{t} \geq \frac{v(x_0 - t\boldsymbol{\eta}) - v(x_0)}{t},$$

where $\boldsymbol{\eta}$ is any unit vector such that $\langle \boldsymbol{\eta}, \boldsymbol{n}\rangle = 0$. Hence

$$\begin{aligned}\liminf_{t\to 0^+} \frac{u(x_0 - t\boldsymbol{\eta}) - u(x_0)}{t} &\geq \lim_{t\to 0^+} \frac{v(x_0 - t\boldsymbol{\eta}) - v(x_0)}{t} \\ &= -\langle Dv(x_0), \boldsymbol{\eta}\rangle \\ &= 2\alpha\mu\varphi'(0)e^{-\alpha R^2}\langle x_0 - z, \eta\rangle. \end{aligned} \tag{4.4.19}$$

Given $x \in \mathcal{C}_{x_0}$, let $\theta(x)$ be the angle between $x_0 - z$ and $x - x_0$, and let us take $\boldsymbol{\eta} := |x - x_0|^{-1}(x_0 - x)$. Then, with $t := |x - x_0|$, we find that (4.4.19) implies

$$\begin{aligned}\liminf_{\substack{x\to x_0 \\ x\in\mathcal{C}_{x_0}}} \frac{u(x) - u(x_0)}{|x - x_0|} &\geq 2\alpha\mu\varphi'(0)e^{-\alpha R^2}\langle x_0 - z, \boldsymbol{\eta}\rangle \\ &= 2\alpha\mu\varphi'(0)e^{-\alpha R^2}\left\langle x_0 - z, \frac{x_0 - x}{|x - x_0|}\right\rangle \\ &= 2\alpha\mu\varphi'(0)e^{-\alpha R^2}|z - x_0|\cos\theta(x) \\ &> 0,\end{aligned}$$

as desired. $\square$

Corollary 4.4.4 (Strong Boundary Maximum Principle)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and h satisfy conditions $(\boldsymbol{\mathcal{D}_1})$, and $(\boldsymbol{\mathcal{D}_2})$. Furthermore, we assume conditions **(M)**, **(P)** hold and $\mathcal{P}_{\mathbf{a}} \in \mathcal{A}_1$. Let $x_0 \in \partial\Omega$ and $u \in \mathrm{USC}(\Omega \cup \{x_0\})$ be a viscosity subsolution of (4.1.2) in Ω such that $u\,(x_0) \geq 0$ and

(i) $u(x) < u\,(x_0)$ for all $x \in \Omega$,

(ii) there is a ball $B(z, R) \subseteq \Omega$ with $\overline{B} \cap \partial\Omega = \{x_0\}$.

Then

$$\liminf_{\substack{x \to x_0 \\ x \in \mathcal{C}_{x_0}}} \frac{u(x_0) - u(x)}{|x - x_0|} > 0, \tag{4.4.20}$$

where $\mathcal{C}_{x_0}$ is as defined in (4.4.18).

Proof. Let $\mathbf{a}' := (a_n, \cdots, a_1)$ and note that $\mathcal{P}_{\mathbf{a}'}(X) = -\mathcal{P}_{\mathbf{a}}(-X)$ and $\mathcal{P}_{\mathbf{a}'} \in \mathcal{A}_n$. Let $z(x) := u\,(x_0) - u(x)$. Then

$$\begin{aligned}
\mathcal{P}_{\mathbf{a}'}(D^2 z(x)) = \mathcal{P}_{\mathbf{a}'}(-D^2 u(x)) &= -\mathcal{P}_{\mathbf{a}}(D^2 u(x)) \\
&\leq -h(u, |Du|) \leq h(-u, |Du|) \\
&\leq h(u(x_0) - u(x), |Du|) = h(z(x), |Dz(x)|).
\end{aligned}$$

Therefore, $z(x) > z(x_0) = 0$ for all $x \in \Omega$ and satisfies all the hypotheses in **Proposition 4.4.3**. Moreover, we see that $\mathcal{P}_{\mathbf{a}'} \in \mathcal{A}_n$. Therefore, we can apply **Proposition 4.4.3** to conclude the inequality (4.4.20). $\square$

Chapter 5

Liouville-type theorems for partial trace equations with nonlinear gradient terms

5.1. Introduction

The primary goal of this chapter is to examine the Liouville-type properties pertaining to viscosity subsolutions of the partial trace equations:

$$\mathcal{P}_{\mathbf{a}}(D^2u) = h(u, |Du|), \quad \text{in } \mathbb{R}^n, \tag{5.1.1}$$

where h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function. A good prototype for the problem (5.1.1) considered is when $h(u, |Du|) = f(u) + g(u)|Du|^q$ for $0 < q < 2$. This will be discussed in more detail later. In (5.1.1), $\mathcal{P}_{\mathbf{a}}$ refers to the following degenerate elliptic operator

$$\mathcal{P}_{\mathbf{a}}(X) = \sum_{j=1}^{n} a_j \lambda_j(X), \tag{5.1.2}$$

where $\lambda_j(X)$ are the eigenvalues of a real $n \times n$ symmetric matrix X arranged in non-decreasing order $\lambda_1(X) \le \cdots \le \lambda_n(X)$, and $\mathbf{a} = (a_1, \cdots, a_n)$ is an n-tuple of non-negative real numbers, not all of which are zero. The operators (5.1.2) encompass those of the form

$$\mathcal{P}_k^-(X) := \sum_{j=1}^{k} \lambda_j(X), \quad \text{and} \quad \mathcal{P}_k^+(X) := \sum_{j=n-k+1}^{n} \lambda_j(X),$$

where $1 \le k \le n$ is a fixed but arbitrary integer. In particular, if $k = n$, the operators $\mathcal{P}_k^{\pm}$ reduce to the Laplace operator, while for $1 \le k < n$ these operators $\mathcal{P}_k^{\pm}$ are degenerate elliptic Bellman operators.

Another prototype of the partial trace operators (5.1.2) is the Min-Max operator

$$\mathcal{M}(X) = \lambda_1(X) + \lambda_n(X),$$

which is a degenerate elliptic for $n > 2$. The operator $\mathcal{M}$ can be thought of as a degenerate elliptic Isaac operator. In general, these are fully nonlinear degenerate elliptic equations that cover a large class of degenerate equations that have recently been the focus of many researchers. See [33] for a discussion of these operators and more.

In this chapter, we use Keller-Osserman growth conditions of nonlinearities and appropriate comparison principles to study Liouville-type theorems.

5.2. Main Results

Throughout this chapter, $h\colon \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ will represent a continuous function that is nondecreasing in each variable. Below we list some of the conditions in h, we will refer to statements of subsequent theorems.

(h-1): $s \mapsto h(s,t)$ is increasing on $\mathbb{R}_0^+$ for each $t \ge 0$.

(h-2): $\displaystyle\int_1^{\infty} \frac{ds}{h(\varrho, s)} < \infty$ for all sufficiently small $\varrho > 0$.

(h-3): $h(s,0) > 0$ for all $s > 0$ and $h(0,0) = 0$.

Theorem 5.2.1. Suppose h satisfies conditions **(h-1)**, **(h-2)** and **(h-3)**, and suppose $\mathbf{a} = (a_1, \cdots, a_n)$ is an n-tuple of non-negative real numbers with $a_1 a_n > 0$. If u is a non-negative viscosity subsolution of (5.1.1), then $u \equiv 0$ in $\mathbb{R}^n$.

Our next result shows the existence of a positive radial viscosity solution of (5.1.1) in $\mathbb{R}^n$ under conditions **(h-1)**, **(h-3)**, and the following additional condition that replaces condition **(h-2)**.

(h-2)*: $\displaystyle\int_1^\infty \frac{ds}{h(s,s)} = \infty.$

Theorem 5.2.2. Suppose conditions **(h-1)**, **(h-2)***, and **(h-3)** hold. Then problem (5.1.1), with $a_n > 0$, admits infinitely many positive radial viscosity solutions.

If, instead of **(h-3)**, we assume that

(h-3)*: $h(c, 0) = 0$ for exactly one $c \in \mathbb{R}$,

then the following Liouville-type result holds.

Theorem 5.2.3. Suppose that condition **(h-3)*** holds. If u is a viscosity solution of Problem (5.1.1) such that

$$u(x) = o(|x|^2) \quad \text{as } |x| \to \infty, \tag{5.2.1}$$

then $u \equiv c$ in $\mathbb{R}^n$.

Let us now consider the problem (5.1.1) with $h(s,t) := f(s) + g(s)\varpi(t)$, where $\varpi\colon \mathbb{R}_0^+ \to \mathbb{R}$ will be assumed to satisfy the following condition.

> **($\mathcal{W}$)**: ϖ is a non-decreasing continuous function such that $\varpi(0) = 0$ and $\varpi(t) > 0$ for $t > 0$.

In addition, we will need the following conditions on f and g.

> **(M)**: f, $g\colon \mathbb{R} \to \mathbb{R}_0^+$ are continuous, non-decreasing functions, such that f is increasing on $\mathbb{R}_0^+$, and $f(0) = 0$.

With $h(s,t) := f(s) + g(s)\varpi(t)$, our problem (5.1.1) takes the form

$$\mathcal{P}_{\mathbf{a}}(D^2u) = f(u) + g(u)\varpi(|Du|) \quad \text{in } \mathbb{R}^n. \tag{5.2.2}$$

It should be noted that if f and g satisfy condition **(M)**, and ϖ satisfies condition **($\mathcal{W}$)**, then $h(s,t) = f(s) + g(s)\varpi(t)$ satisfies both conditions **(h-1)** and **(h-3)**. If, in addition,

$$\int_1^\infty \frac{ds}{\varpi(s)} < \infty, \tag{5.2.3}$$

and we also assume that $g(s) > 0$ for $s > 0$, then condition **(h-2)** holds as well. As a result of **Theorem 5.2.1** shows that any non-negative viscosity subsolution of (5.2.2) vanishes identically on $\mathbb{R}^n$. Therefore, in what follows, we investigate the problem (5.2.2) when (5.2.3) does not hold. In fact, we will study the problem (5.2.2) when ϖ satisfies the following conditions:

$$\textbf{(}\mathcal{W}\textbf{-1)}\text{: } \textbf{(i)} \quad \int_0^1 \frac{s}{\varpi(s)}\, ds < \infty, \quad \textbf{(ii)} \quad \int_1^\infty \frac{ds}{\varpi(s)} = \infty.$$

Under the assumption that **(**$\mathcal{W}$**-1) (i)** holds, we define an increasing function Φ: $\mathbb{R}_0^+ \to \mathbb{R}_0^+$ by

$$\Phi(t) := \int_0^t \frac{s}{\varpi(s)}\, ds, \quad \text{for } t > 0.$$

We should note that under condition **(**$\mathcal{W}$**-1)**, we have

$$\begin{aligned}
\Phi(\infty) &= \lim_{t\to\infty} \int_0^t \frac{s}{\varpi(s)}\, ds = \int_0^1 \frac{s}{\varpi(s)}\, ds + \lim_{t\to\infty} \int_1^t \frac{s}{\varpi(s)}\, ds \\
&\geq \int_0^1 \frac{s}{\varpi(s)}\, ds + \lim_{t\to\infty} \int_1^t \frac{ds}{\varpi(s)} && (\because \quad s \geq 1) \\
&= \int_0^1 \frac{s}{\varpi(s)}\, ds + \int_1^\infty \frac{ds}{\varpi(s)} = \infty.
\end{aligned}$$

Inspired by the classical Keller-Osserman conditions [49, 61], we use the inverse function Φ^{-1} to introduce the following Keller-Osserman type condition:

$$(\mathbf{KO})_\varpi\text{: } \int_1^\infty \frac{dt}{\sqrt{F(t) + \Phi^{-1}(G(t))}} < \infty.$$

In $(\mathbf{KO})_\varpi$, F and G are the anti-derivatives of f and g, respectively, that vanish at 0.

We can now state the following Liouville-type result.

Theorem 5.2.4. Suppose that f, g: $\mathbb{R} \to \mathbb{R}_0^+$ satisfy condition **(M)**, with $g(s) > 0$ for $s > 0$, and ϖ: $\mathbb{R}_0^+ \to \mathbb{R}$ satisfies **(**$\mathcal{W}$**)**. Furthermore, assume that conditions **(**$\mathcal{W}$**-1)**, and $(\mathbf{KO})_\varpi$ hold. If u is a non-negative viscosity subsolution of (5.2.2), with $a_n > 0$, then u is identically zero in $\mathbb{R}^n$.

The next set of conditions will serve as part of sufficient conditions for the existence of positive radial solutions to the problem (5.2.2):

$$\textbf{(}\mathcal{W}\textbf{-2)}\text{: } \textbf{(i)} \quad \int_0^1 \frac{dt}{\Phi^{-1}\big(F(t) + G(t)\big)} = \infty, \quad \textbf{(ii)} \quad \lim_{s\to\infty} \frac{s^2}{\varpi(s)} = \infty.$$

We can now state our existence results as follows.

Theorem 5.2.5. Suppose that f, g: $\mathbb{R} \to \mathbb{R}_0^+$ satisfy condition **(M)**, and that ϖ: $\mathbb{R}_0^+ \to \mathbb{R}$ satisfies $(\mathcal{W})$. Furthermore, assume that conditions ($\mathcal{W}$-**1**), and ($\mathcal{W}$-**2**) hold. Then problem (5.2.2), with $a_n > 0$, admits many positive radial viscosity solutions in $\mathbb{R}^n$.

If, in problem (5.2.2), we take $\varpi(t) = |t|^q$ and assume condition **(M)**, then $(\mathbf{KO})_\varpi$ gives a necessary and sufficient condition for the identical vanishing of any non-negative viscosity solutions to problem (5.2.2). Toward this end, let us rewrite condition $(\mathbf{KO})_\varpi$ in the following explicit form, when $0 < q < 2$.

$$(\mathbf{KO})_q\colon \quad \int_0^1 \frac{ds}{\sqrt{F(s)} + G(s)^{\frac{1}{2-q}}} < \infty.$$

Note that if $1 < q < 2$ and $g(s) > 0$ for some $s > 0$, then it is easy to see that $(\mathbf{KO})_q$ holds.

We now state our last main result which shows that $(\mathbf{KO})_q$ is a necessary and sufficient condition for the identical vanishing of any non-negative viscosity solution of

$$\mathcal{P}_{\mathbf{a}}(D^2u) = f(u) + g(u)|Du|^q \quad \text{in } \mathbb{R}^n. \tag{5.2.4}$$

Theorem 5.2.6. Suppose that f, g: $\mathbb{R} \to \mathbb{R}_0^+$ satisfy condition **(M)**, and suppose $a_n > 0$. If $(\mathbf{KO})_q$ holds, then any non-negative viscosity subsolution of (5.2.4) vanishes identically in $\mathbb{R}^n$. Conversely, if condition $(\mathbf{KO})_q$ fails, then problem (5.2.4) admits infinitely many positive radial viscosity solutions in $\mathbb{R}^n$.

5.3. Preliminary Results

In our subsequent discussions, h: $\mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is continuous, and except for **Theorem 5.2.3** and its proof, we will assume that **(h-3)** holds and that $a_n > 0$.

Given $\varepsilon > 0$ and $c > 0$, we consider the following initial-value problem:

$$\begin{cases} r^{1-c}(r^{c-1}\phi')' = {a_n}^{-1}h(\phi(r), \phi'(r)), \\ \phi(0) = \varepsilon, \quad \phi'(0) = 0. \end{cases} \tag{IVP(ε)}$$

This problem is known to admit a solution $\phi \in C^2([0, R))$ in some maximal interval of existence. Moreover, $\phi > 0$, $\phi' > 0$, and $\phi'' > 0$ in $(0, R)$. We refer the reader to [26] for a detailed discussion of these results. Furthermore, proceeding as in [26], we integrate the equation in the initial-value problem (IVP(ε)) on $(0, r)$ for any $0 < r < R$ to get

$$\frac{\phi'(r)}{r} \leq \frac{1}{ca_n} h(\phi(r), \phi'(r)). \tag{5.3.1}$$

Next, we expand the left-hand side of the equation in the initial-value problem (IVP(ε)). We find

$$\frac{c-1}{r}\phi'(r) + \phi''(r) = \frac{1}{a_n} h(\phi(r), \phi'(r)). \tag{5.3.2}$$

In particular, by taking the limit as $r \to 0^+$ in (5.3.2) and using the definition $\phi''(0) = \lim_{r \to 0^+} \dfrac{\phi'(r)}{r}$, we find that

$$\phi''(0) = \frac{1}{ca_n} h(\phi(0), \phi'(0)). \tag{5.3.3}$$

In the following, we will always assume that $c \geq 1$. Using (5.3.1) in (5.3.2) and rearranging, we get

$$\phi''(r) \geq \frac{1}{ca_n} h(\phi(r), \phi'(r)) \tag{5.3.4}$$

From (5.3.1) and (5.3.4), we find that

$$\phi''(r) \geq \frac{\phi'(r)}{r} \tag{5.3.5}$$

Now, let us take

$$c := \frac{|\mathbf{a}|}{a_n}.$$

Letting $v(x) := \phi(|x - z|)$. We see that for $x \neq z$, we have

$$D^2 v = \phi''(|x-z|) \frac{(x-z) \otimes (x-z)}{|x-z|^2} + \phi'(|x-z|) \left[\frac{I_n}{|x-z|} - \frac{(x-z) \otimes (x-z)}{|x-z|^3} \right].$$

Therefore, the eigenvalues of $D^2 v(x)$ for $x \neq z$ are

$$\phi''(|x-z|) \text{ of multiplicity } 1 \text{ and } \frac{\phi'(|x-z|)}{|x-z|} \text{ of multiplicity } n-1. \tag{5.3.6}$$

Recalling (5.3.6) and using (5.3.5), we conclude that for $x \neq z$

$$\mathcal{P}_{\mathbf{a}}(D^2 v) = \frac{\phi'(|x-z|)}{|x-z|}\sum_{j=1}^{n-1} a_j + a_n \phi''(|x-z|) \tag{5.3.7}$$

$$= a_n\phi''(r) + \frac{|\mathbf{a}| - a_n}{r}\phi'(r), \quad r := |x-z|,$$

$$= h(\phi(r), \phi'(r)). \tag{5.3.8}$$

In (5.3.8), we used our choice of c and (5.3.2). Therefore, v is a classical solution of $\mathcal{P}_{\mathbf{a}}(D^2 v) = h(v, |Dv|)$ in $B(z,R)\backslash\{z\}$. Let us now show that, in fact, $v(x) = \phi(|x-z|)$ is a viscosity solution of (5.1.1) in the ball $B := B(z,R)$, and for this, it suffices to verify the definition of a viscosity solution at the centre z of the ball B. First, we notice that $|Dv(x)| = \phi'(|x-z|)$ for all $x \in B(z,R)\backslash\{z\}$. Therefore, since $|Dv(z)| = \phi'(0) = 0$, it follows that $Dv(z) = o$.

Now, let $\varphi \in C^2(B)$ be such that $v - \varphi$ has a maximum at z. Then, for $0 < |t| < R$ and any $e \in \mathbb{R}^n$ such that $\|e\| = 1$, we have (recall that $D\varphi(z) = Dv(z) = o$)

$$\begin{aligned}\phi(t) - \phi(0) = v(z+te) - v(z) &\leq \varphi(z+te) - \varphi(z)\\ &= \frac{t^2}{2}\langle D^2\varphi(z)e, e\rangle + o(t^2) \quad \text{as } t \to 0^+.\end{aligned}$$

Thus

$$\frac{\phi(t)-\phi(0)}{t^2} \leq \frac{1}{2}\langle D^2\varphi(z)e, e\rangle + \frac{o(t^2)}{t^2} \quad \text{as } t \to 0^+.$$

Taking the limit as $t \to 0^+$, we find that

$$\phi''(0) \leq \langle D^2\varphi(z)e, e\rangle$$

for all unit vectors e. Thus, $\phi''(0) \le \lambda_1(D^2\varphi(z))$. But we notice from (5.3.3) that

$$\begin{aligned}\mathcal{P}_{\mathbf{a}}(D^2\varphi(z)) &= \sum_{j=1}^{n} a_j\lambda_j(D^2\varphi(z)) \ge \sum_{j=1}^{n} a_j\lambda_1(D^2\varphi(z)) \\ &= \lambda_1(D^2\varphi(z))\sum_{j=1}^{n} a_j = |\mathbf{a}|\lambda_1(D^2\varphi(z)) \\ &\ge \phi''(0)|\mathbf{a}| = h(\phi(0), \phi'(0)) = h(v(z), |D\varphi(z)|).\end{aligned}$$

A similar argument shows that $\mathcal{P}_{\mathbf{a}}(D^2\psi(z)) \le h(v(z), |D\psi(z)|)$ for any function $\psi \in C^2(B)$ such that $v - \psi$ has a minimum at z, thus completing the verification that $v = \phi(|x - z|)$ is a viscosity solution of $\mathcal{P}_{\mathbf{a}}(D^2 v) = h(v, |Dv|)$ in $B(z, R)$.

Next, we establish a comparison principle, which is a key tool in our study of the main results. For this, we first reformulated the definition of viscosity subsoluton and supersolution in the framework of subjets and superjets. See [19] for more details. We recall these concepts as follows:

We use $\mathbb{S}^n$ to denote the collection of $n \times n$ symmetric matrices with real entries. Let u: $\Omega \to \mathbb{R}$, and $x_0 \in \Omega$. The second order superjet $J^{2,+}_{\Omega}u(x_0)$ of u at x_0 and the second order subjet $J^{2,-}_{\Omega}u(x_0)$ of u at x_0 are, respectively, the sets

$$\begin{aligned}J^{2,+}_{\Omega}u(x_0) := \big\{(p, X) \in \mathbb{R}^n \times \mathbb{S}^n : u(x_0 + \eta) &\le u(x_0) + \langle p, \eta\rangle \\ &+ \frac{1}{2}\langle X\eta, \eta\rangle + o(|\eta|^2) \quad \text{as } |\eta| \to 0\big\},\end{aligned}$$

and

$$\begin{aligned}J^{2,-}_{\Omega}u(x_0) := \big\{(p, X) \in \mathbb{R}^n \times \mathbb{S}^n : u(x_0 + \eta) &\ge u(x_0) + \langle p, \eta\rangle \\ &+ \frac{1}{2}\langle X\eta, \eta\rangle + o(|\eta|^2) \quad \text{as } |\eta| \to 0\big\}.\end{aligned}$$

Equivalent definitions of subsolution and supersolution in terms of subjets and superjets can be given as follows.

A function u in USC(Ω) (respectively, LSC(Ω)) is a viscosity subsolution (respectively, viscosity supersolution) of (5.1.1) if and only if for each x_0 in Ω and each (p, X) in $J^{2,+}_{\Omega}u(x_0)$ (respectively, $J^{2,-}_{\Omega}u(x_0)$), we have

$$\mathcal{P}_{\mathbf{a}}(X) \ge \text{(respectively, } \le) \ h(u(x_0), |p|).$$

We are now in a position to state our comparison principle as follows.

Proposition 5.3.1 (A Comparison Principle)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset, and let $h\colon \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ be a continuous function. Suppose $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ such that

$$\mathcal{P}_{\mathbf{a}}(D^2u) \geq h(u, |Du|) \quad \text{in } \Omega, \quad \text{and} \quad \mathcal{P}_{\mathbf{a}}(D^2v) \leq h(v, |Dv|) \quad \text{in } \Omega.$$

We assume that $s \mapsto h(s,t)$ is increasing on the interval $[\min_{\overline{\Omega}} v, \infty)$ for each $t \geq 0$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. The method of proof is standard; see [19], and we have chosen to include it for the reader's convenience. By way of contradiction, suppose that $u - v$ achieves a positive maximum at some point $x_0 \in \Omega$. We apply [19, Theorem 3.2] with $u_1 := u$, $u_2 := -v$, and $\varphi := \varphi_\alpha(x,y) = \alpha|x-y|^2/2$ for each $\alpha > 0$. According to [19, Theorem 3.2], there exist x_α, $y_\alpha \in \Omega$ and matrices X_α, $Y_\alpha \in \mathbb{S}^n$ such that

$$\big(\alpha(x_\alpha - y_\alpha), X_\alpha\big) \in J^{2,+}u(x_\alpha), \quad \big(\alpha(x_\alpha - y_\alpha), Y_\alpha\big) \in J^{2,-}v(y_\alpha), \tag{5.3.9}$$

and

$$-3\alpha \begin{pmatrix} I_n & \mathbf{0} \\ \mathbf{0} & I_n \end{pmatrix} \leq \begin{pmatrix} X_\alpha & \mathbf{0} \\ \mathbf{0} & -Y_\alpha \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}. \tag{5.3.10}$$

Note that

$$u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \geq u(x_0) - v(x_0) > 0. \tag{5.3.11}$$

Therefore, using (5.3.9) and the fact that $X_\alpha \leq Y_\alpha$ (which follows from (5.3.10)), we conclude that

$$\begin{aligned} h(u(x_\alpha), \alpha|x_\alpha - y_\alpha|) &\leq \mathcal{P}_{\mathbf{a}}(X_\alpha) \\ &\leq \mathcal{P}_{\mathbf{a}}(Y_\alpha) \\ &\leq h(v(y_\alpha), \alpha|x_\alpha - y_\alpha|). \end{aligned} \tag{5.3.12}$$

Recalling that $s \mapsto h(s,t)$ is increasing on $[\min_{\overline{\Omega}} v, \infty)$ for each $t \geq 0$ and that $u(x_\alpha) > v(y_\alpha)$ (see (5.3.11)), we note that (5.3.12) is not possible. This concludes the proof of the proposition. $\square$

The next lemma will be used in the proof of **Theorem 5.2.1**.

Lemma 5.3.1. Let u be a non-negative subsolution of (5.1.1), and $x_0 \in \mathbb{R}^n$. Suppose that there is a constant $R > 0$ and $\phi \in C^2([0,R)) \cap C([0,R])$ with $\phi'(r) \to \infty$ as $r \to R^-$ such that for some constant $\kappa \geq 0$, the following hold.

(i) $u(z) = \phi(R) + \kappa$ for some $z \in \partial B(x_0, R)$, and

(ii) $u(x) \leq \phi(|x - x_0|) + \kappa$ for all $x \in B(x_0, R)$.

Given $\rho > R$ and $\alpha > 0$, there is y with $R \leq |y - x_0| < \rho$ such that

$$u(z) + \alpha(|y - x_0| - R) < u(y).$$

Proof. Suppose the conclusion is not true. Then there is $\rho > R$ and $\alpha > 0$ such that

$$u(z) + \alpha(|x - x_0| - R) \geq u(x) \quad \text{for all } x \text{ with } R \leq |x - x_0| < \rho. \tag{5.3.13}$$

Let $\vartheta(x) := u(z) + \alpha(|x - x_0| - R)$ and take $\beta > 0$ such that $R + \dfrac{\alpha}{\beta} < \rho$, and set

$$\begin{aligned}\theta(x) :&= u(z) + 2\alpha(|x - x_0| - R) - \beta(|x - x_0| - R)^2 \\ &= \vartheta(x) + \alpha(|x - x_0| - R) - \beta(|x - x_0| - R)^2.\end{aligned}$$

Our choice of β and (5.3.13) allows us to estimate

$$\theta(x) \geq \vartheta(x) \geq u(x) \quad \text{for } R \leq |x - x_0| \leq R + \frac{\alpha}{\beta}. \tag{5.3.14}$$

Since $\phi'(r) \to \infty$ as $r \to R^-$, we can find $\delta := \delta(\alpha, \beta)$ such that[1]

$$\theta(x) - \phi(|x - x_0|) \geq u(z) - \phi(R)$$

[1]Note that $\theta(x) - u(z) \geq \phi(r) - \phi(R)$ if and only if $2\alpha(r - R) - \beta(r - R)^2 \geq \phi(r) - \phi(R)$, where $r := |x - z|$. The inequality follows since

$$\lim_{r \to R}\left[\frac{\phi(r) - \phi(R)}{r - R} + \beta(r - R)\right] = \infty.$$

for all $R-\delta < |x-x_0| < R$. Observe that, by the assumption in **(i)**, we have $u(z)-\phi(R)=\kappa$.

Therefore, we see that $\theta(x) \geq \phi(|x-x_0|)+\kappa$ for all $R-\delta < |x-x_0| \leq R$. This, together with (5.3.14) and the assumption **(ii)**, shows that

$$\theta(x) \geq u(x) \quad \text{for } R < |x-x_0| < R+\frac{\alpha}{\beta}$$

Since $\theta(z)=u(z)$, we see that $u-\theta$ has a maximum at z. Since u is a viscosity subsolution of (5.1.1), we find that

$$\mathcal{P}_{\mathbf{a}}(D^2\theta(z)) \geq h(u(z), |D\theta(z)|) \geq 0. \tag{5.3.15}$$

For $x \neq x_0$, let us note that

$$\begin{aligned} D^2\theta(x) = {} & \frac{2\alpha}{|x-x_0|}\left(I_n - \frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2}\right) \\ & - 2\beta\frac{(|x-x_0|-R)}{|x-x_0|}\left(I_n - \frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2}\right) - 2\beta\frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2} \end{aligned}$$

Therefore, the eigenvalues of $D^2\theta(x)$ are

$$-2\beta \text{ of multiplicity } 1 \text{ and } -2\beta+\frac{2\alpha+2\beta R}{|x-x_0|} \text{ of multiplicity } n-1.$$

Hence

$$\begin{aligned} \mathcal{P}_{\mathbf{a}}(D^2\theta(x)) &= -2\beta a_1 + \left(\frac{2\alpha+2\beta R}{|x-x_0|} - 2\beta\right)\sum_{j=2}^{n} a_j \\ &= -2\beta\left[a_1 + \left(1-\frac{R}{|x-x_0|}\right)(|\mathbf{a}|-a_1)\right] + \frac{2\alpha}{|x-x_0|}(|\mathbf{a}|-a_1) \end{aligned}$$

Since $z \in \partial B(x_0,R)$, we see that

$$\mathcal{P}_{\mathbf{a}}(D^2\theta(z)) = -2\beta a_1 + \frac{2\alpha}{R}(|\mathbf{a}|-a_1).$$

Since $a_1>0$, we can choose $\beta>0$ large enough such that $\mathcal{P}_{\mathbf{a}}(D^2\theta(z))<0$, which contradicts (5.3.15). This contradiction justifies the conclusion of the lemma. $\square$

5.4. Proofs of Main Theorems

Proof of Theorem 5.2.1. Suppose, contrary to the conclusion we wish to establish, that $u(x_0) > 0$ for some $x_0 \in \Omega$. We fix $0 < \varepsilon < u(x_0)$, sufficiently small that the condition **(h-2)** holds with $\varrho = \varepsilon$. We take a solution ϕ to (IVP(ε)) on a maximal interval of existence $[0, R)$. Rearranging the inequality in (5.3.4), we find

$$\frac{1}{|\mathbf{a}|} \le \frac{\phi''(s)}{h(\phi(s), \phi'(s))} \le \frac{\phi''(s)}{h(\phi(0), \phi'(s))}, \quad 0 < s < R.$$

Fixing $0 < r_0 < R$ and integrating the above inequality on (r_0, r) for any $r_0 < r < R$, we find

$$\frac{1}{|\mathbf{a}|}(r - r_0) \le \int_{\phi'(r_0)}^{\phi'(r)} \frac{dt}{h(\phi(0), t)} \le \int_{\phi'(r_0)}^{\infty} \frac{dt}{h(\varepsilon, t)}, \quad r_0 < r < R.$$

Invoking the assumption **(h-2)** shows that $R < \infty$.

Recall that $a_n > 0$. Let $v(x) := \phi(|x - x_0|)$, for $x \in B := B(x_0, R)$, so that v is a viscosity solution of

$$\mathcal{P}_{\mathbf{a}}(D^2 v) = h(v, |Dv|) \quad \text{in } B(x_0, R). \tag{5.4.1}$$

If $u \le v$ on the boundary ∂B, then the comparison principle would show that $u \le v$. In particular, we would have $u(x_0) \le v(x_0) = \phi(0) = \varepsilon$. But this contradicts the choice of ε. As a consequence, we must have $v < u$ at some point in ∂B. This implies that $\phi(R) < \infty$, and therefore we have $v \in C(\overline{B})$. Since $[0, R)$ is a maximal interval of existence for ϕ, we conclude that $\phi'(r) \to \infty$ as $r \to R^-$.

Now, let $z \in \partial B$ be a point of maximum of $u - v$ on ∂B. Note that $u(z) > v(z)$.

$$w(x) := v(x) + u(z) - v(z), \quad x \in \overline{B}.$$

By **(h-1)** and (5.4.1), we see that

$$\mathcal{P}_{\mathbf{a}}(D^2 w) = \mathcal{P}_{\mathbf{a}}(D^2 v) \le h(w, |Dw|) \quad \text{in } B, \quad \text{and } u \le w \text{ on } \partial B.$$

An application of the comparison principle shows that

$$u \leq w \quad \text{in } B. \tag{5.4.2}$$

Given a constant $\alpha > 0$, we introduce the function

$$\vartheta_\alpha(x) := w(z) + \alpha(|x - x_0| - R), \quad x \in \mathbb{R}^n.$$

Note that $\vartheta_\alpha(x) = w(z) = u(z)$ on ∂B.

For $|x - x_0| \geq R$, we have

$$\mathcal{P}_{\mathbf{a}}(D^2\vartheta_\alpha(x)) = \frac{\alpha}{|x - x_0|}\left(|\mathbf{a}| - a_1\right) \leq \frac{\alpha}{R}\left(|\mathbf{a}| - a_1\right). \tag{5.4.3}$$

From condition **(h-2)** and **Lemma 2.6.2**, we recall that

$$\lim_{t\to\infty} \frac{t}{h(\varepsilon, t)} = 0.$$

Therefore, we see that for $\alpha_0 > 0$ large enough

$$h(\varepsilon, \alpha) \geq \left(\frac{\alpha}{R}\right)\left(|\mathbf{a}| - a_1\right) \quad \alpha \geq \alpha_0. \tag{5.4.4}$$

Note that on $\{x : |x - x_0| \geq R\}$, we have

$$h(\vartheta_\alpha, |D\vartheta_\alpha|) \geq h(w(z), \alpha) \geq h(\varepsilon, \alpha).$$

For $\alpha \geq \alpha_0$, from (5.4.3) and (5.4.4), we can derive

$$\mathcal{P}_{\mathbf{a}}(D^2\vartheta_\alpha) \leq h(\vartheta_\alpha, |D\vartheta_\alpha|), \quad |x - x_0| \geq R.$$

We now fix $\alpha \geq \alpha_0$ and $\rho > R$ such that

$$\vartheta_\alpha \geq u, \quad \text{for } |x - x_0| = \rho.$$

On $|x-x_0| = R$, we recall that $\vartheta_\alpha(x) = w(z) = u(z) \geq v(z)+u(x)-v(x) = u(x)$ (since $v(z) = v(x)$ when $|x-x_0| = R$). By the comparison principle, **Proposition**

5.3.1, we have

$$u(x) \le \vartheta_\alpha(x) \quad \text{for } R \le |x - x_0| < \rho. \tag{5.4.5}$$

Let us now recall that $\phi'(r) \to \infty$ as $r \to R^-$. If we take $\kappa := u(z) - v(z)$ so that $u(z) = \phi(R) + \kappa$, we see from (5.4.2) that $u(x) \le v(x) + \kappa = \phi(x - z) + \kappa$ in $B(z, R)$. Therefore, all the hypotheses of **Lemma 5.3.1** hold. But (5.4.5) contradicts the conclusion of **Lemma 5.3.1**. This completes the proof of the theorem. $\square$

Proof of Theorem 5.2.2. Let $\phi \in C^2([0, R))$ be a solution of problem (IVP(ε)) for some $0 < \varepsilon < 1/2$, where we assume that $[0, R)$ is the maximal interval of existence and $0 < R \le \infty$. We claim that $R = \infty$, and this would show that $v := \phi(|x|)$ is a positive radial solution of the problem (5.1.1) in $\mathbb{R}^n$. Suppose that $R < \infty$. Since $[0, R)$ is a maximal interval of existence and

$$\phi(r) = \phi(0) + \int_0^r \phi'(t)\, dt, \quad r \in (0, R), \tag{5.4.6}$$

we must have $\phi'(r) \to \infty$ as $r \to R^-$. Since $h(s, t)$ is non-decreasing in each variable, we see from (5.3.2) and (5.4.6) that (with $\rho := \max\{1, R\}$)

$$\begin{aligned} \phi''(r) &\le \frac{1}{a_n} h(\phi(r), \phi'(r)) \\ &\le \frac{1}{a_n} h(\phi(0) + R\phi'(r), \phi'(r)) \\ &\le \frac{1}{a_n} h(\phi(0) + \rho\phi'(r), \phi(0) + \rho\phi'(r)), \quad 0 < r < R. \end{aligned} \tag{5.4.7}$$

Next, we fix $0 < r_0 < R$ such that $\phi'(r_0) > 1$, and we use (5.4.7) to compute

$$\int_{r_0}^R \frac{\phi''(t)}{h(\phi(0) + \rho\phi'(r), \phi(0) + \rho\phi'(r))}\, dt \le \frac{1}{a_n} (R - r_0)\,.$$

In other words, we find

$$\int_{\phi(0)+\rho\phi'(r_0)}^{\infty} \frac{ds}{h(s,s)} \le \frac{\rho}{a_n} (R - r_0)\,.$$

Since we assume that $R < \infty$, this last inequality contradicts the assumption that **(h-2)*** holds. Therefore, $R = \infty$, and hence $u(x) = \phi(|x|)$ is a positive radial viscosity solution of (5.1.1). $\square$

Proof of Theorem 5.2.3. Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ be arbitrary, and consider

$$w(x) := u(x) - u(x_0) + 2\varepsilon - \frac{\varepsilon}{2}|x - x_0|^2, \quad x \in \mathbb{R}^n. \tag{5.4.8}$$

By the assumption that $u(x) = o(|x|^2)$ as $|x| \to \infty$, we see that

$$\lim_{|x|\to\infty} w(x) = -\infty.$$

Since $w(x_0) > 0$, we find that w attains a positive maximum in $\mathbb{R}^n$ at some point $y_0 \in \mathbb{R}^n$. Considering u as a viscosity subsolution to (5.1.1), and using

$$\varphi(x) := u(x_0) - 2\varepsilon + \frac{\varepsilon}{2}|x - x_0|^2$$

as a test function, we see that

$$\varepsilon|\mathbf{a}| = \mathcal{P}_{\mathbf{a}}(D^2\varphi(y_0)) \geq h(u(y_0), |D\varphi(y_0)|) = h(u(y_0), \varepsilon|y_0 - x_0|). \tag{5.4.9}$$

From here $w(y_0) > 0$, we note that

$$u(y_0) > u(x_0) - 2\varepsilon.$$

Therefore, this last inequality, together with (5.4.9), show that

$$h(u(x_0) - 2\varepsilon, \varepsilon|y_0 - x_0|) \leq h(u(y_0), \varepsilon|y_0 - x_0|) \leq \varepsilon|\mathbf{a}|.$$

Taking the limit as $\varepsilon \to 0$, we conclude that $h(u(x_0), 0) \leq 0$. Using the auxiliary function

$$v(x) := u(x) - u(x_0) - 2\varepsilon + \frac{\varepsilon}{2}|x - x_0|^2, \quad x \in \mathbb{R}^n,$$

in place of (5.4.8), and arguing in a similar manner where now u is considered as a viscosity supersolution of (5.1.1), one can also show that $h(u(x_0), 0) \geq 0$. Since we picked $x_0 \in \mathbb{R}^n$ arbitrarily, we conclude that $h(u(x), 0) \equiv 0$ in $\mathbb{R}^n$. We now use **(h-3)*** to conclude that $u(x) \equiv c$ in $\mathbb{R}^n$, where c is the constant in condition

(h-3)*. We point out that in the proof we used the non-decreasing monotonicity of h in the first variable only (cf. beginning of Section 2). $\square$

In the following, we use the notation $\mathcal{F}(s,t) := F(s) - F(t)$, and $\mathcal{G}(s,t) := G(s) - G(t)$.

Proof of Theorem 5.2.4. Let $u \in \mathrm{USC}(\mathbb{R}^n)$ be a non-negative viscosity subsolution of (5.2.2), and by contradiction suppose that $u(x_0) > 0$ for some $x_0 \in \mathbb{R}^n$. Fix $0 < \varepsilon < u(x_0)$, and let $\phi \in C^2([0,R))$ be a solution of the initial value problem (IVP(ε)) with the maximal interval of existence $[0,R)$. Note that $\phi > 0$, $\phi' > 0$, and $\phi'' > 0$ in $(0,R)$. We claim $R < \infty$. Multiplying both sides of the inequality (5.3.4) by ϕ', we find

$$\frac{1}{|\mathbf{a}|}\Big[f\left(\phi(r)\right)\phi'(r) + g\left(\phi(r)\right)\phi'(r)\varpi\left(\phi'(r)\right)\Big] \le \phi''(r)\phi'(r). \tag{5.4.10}$$

From (5.4.10) we find the following two inequalities

$$\begin{aligned}&\frac{1}{|\mathbf{a}|}f\left(\phi(r)\right)\phi'(r) \le \phi''(r)\phi'(r), \quad \text{and}\\ &\frac{1}{|\mathbf{a}|}g\left(\phi(r)\right)\phi'(r)\varpi\left(\phi'(r)\right) \le \phi''(r)\phi'(r).\end{aligned}$$

Integrating each of the above inequalities on $(0,r)$ for any $0 < r < R$, we get, respectively,

$$\begin{aligned}&\sqrt{\varrho\mathcal{F}\left(\phi(r),\varepsilon\right)} \le \phi'(r), \quad \text{and}\\ &\Phi^{-1}\Big(\varrho\Big(\mathcal{G}\left(\phi(r),\varepsilon\right)\Big)\Big) \le \phi'(r), \quad \text{where} \quad \varrho := \frac{1}{|\mathbf{a}|}.\end{aligned}$$

Adding the above inequalities and dividing them, we find that

$$\frac{1}{2} \le \frac{\phi'(r)}{\sqrt{\varrho\mathcal{F}\left(s,\varepsilon\right)} + \Phi^{-1}\Big(\varrho\left(\mathcal{G}\left(s,\varepsilon\right)\right)\Big)}, \quad 0 < r < R.$$

By integrating the last inequality on $(0,r)$ for some fixed $r_0 < r < R$, we get

$$\frac{1}{2}\left(r - r_0\right) \le \int_{\varepsilon}^{\phi(r)} \frac{ds}{\sqrt{\varrho\mathcal{F}\left(\phi(r),\varepsilon\right)} + \Phi^{-1}\Big(\varrho\Big(\mathcal{G}\left(\phi(r),\varepsilon\right)\Big)\Big)}. \tag{5.4.11}$$

It is easily shown that

$$\varrho\mathcal{F}\left(s,\varepsilon\right) \geq \mathcal{F}\left(\varrho^* s, \varrho^*\varepsilon\right), \quad \text{and} \quad \varrho\mathcal{G}\left(s,\varepsilon\right) \geq \mathcal{G}\left(\varrho^* s, \varrho^*\varepsilon\right),$$

where $\varrho^* := \min\{1, \varrho\}$.

Using this observation in (5.4.11), we see that

$$\frac{1}{2}\left(r - r_0\right) \leq \frac{1}{\varrho^*}\int_{\varrho^*\varepsilon}^{\infty} \frac{ds}{\sqrt{\varrho\mathcal{F}\left(s,\varrho^*\varepsilon\right) + \Phi^{-1}\left(\varrho\left(\mathcal{G}\left(s,\varrho^*\varepsilon\right)\right)\right)}}, \quad r_0 < r < R.$$

Therefore, condition $(\mathbf{KO})_{\varpi}$ shows that $R < \infty$, (see the Appendix of [5]), as claimed. Since $[0, R)$ is a maximal interval of existence, it follows from this and (5.4.6) that $\phi'(r) \to \infty$ as $r \to R^-$. Next, we show that $\phi(r) \to \infty$ as $r \to R^-$. By way of contradiction, assume that $\phi(R^-) := \lim_{r \to R^-} \phi(r) < \infty$. From (5.3.2), we see that

$$\phi''(r) \leq \frac{1}{a_n}\Big(f\left(\phi(r)\right) + g\left(\phi(r)\right)\varpi\left(\phi'(r)\right)\Big). \tag{5.4.12}$$

Rewriting this inequality, we see that

$$\frac{\phi''(r)}{\varpi\left(\phi'(r)\right)} \leq \frac{1}{a_n}\left(\frac{f(\phi(R^-))}{\varpi\left(\phi'(r_0)\right)} + g(\phi(R^-))\right), \quad 0 < r < R.$$

Integrating this last inequality in (r_0, r) for $r_0 < r < R$, we find that

$$\int_{\phi'(r_0)}^{\phi'(r)} \frac{ds}{\varpi(s)} \leq \frac{R}{a_n}\left(\frac{f(\phi(R^-))}{\varpi\left(\phi'(r_0)\right)} + g(\phi(R^-))\right).$$

This, however, contradicts the assumption $(\mathcal{W}$**-1) (ii)**. Therefore, we must have $\phi(r) \to \infty$ as $r \to R^-$, as claimed.

We now define $v(x) := \phi(|x - x_0|)$, so that v is a viscosity solution of (5.2.2) with $v = \infty$ on the boundary of $B(x_0, R)$. Recalling that u is a viscosity subsolution, by the comparison principle we find that $u \leq v$ in the ball $B(x_0, R)$. In particular, $u(x_0) \leq v(x_0) = \varepsilon$, which contradicts our choice of ε. This shows that u cannot be positive anywhere in $\mathbb{R}^n$. $\square$

Proof of Theorem 5.2.5. Let $\phi \in C^2([0,R))$ be a solution of the initial value problem (IVP(ε)) for some $\varepsilon > 0$. We will show that $R = \infty$. We multiply (5.4.12) by ϕ', then divide both sides of the resulting inequality by $\varpi(\phi')$. For some fixed $0 < r_0 < R$, we integrate the derived inequality on (r_0, r) to find

$$\int_{r_0}^{r} \frac{\phi''(s)\phi'(s)}{\varpi(\phi'(s))} ds \leq \frac{1}{a_n}\left(\int_{r_0}^{r} \frac{f\left(\phi(s)\right)\phi'(s)}{\varpi\left(\phi'(s)\right)} ds + \int_{r_0}^{r} g(\phi(s))\phi'(s) ds\right).$$

Using condition **($\mathcal{W}$)**, we see that

$$\int_{\phi'(r_0)}^{\phi'(r)} \frac{s}{\varpi(s)} ds \leq \frac{1}{a_n} \max\left\{1, \frac{1}{\varpi(\phi'(r_0))}\right\}\left(F(\phi(r)) + G(\phi(r)\right), \quad r_0 < r < R.$$

In other words, we find that

$$\Phi(\phi'(r)) - \Phi(\phi'(r_0)) \leq \varrho\left(F(\phi(r)) + G(\phi(r))\right), \quad r_0 < r < R, \tag{5.4.13}$$

where

$$\varrho := \frac{1}{a_n} \max\left\{1, \frac{1}{\varpi(\phi'(r_0))}\right\}.$$

We observe that (5.4.13) shows that $\phi(r) \to \infty$ as $r \to R^-$. Otherwise, we would have both ϕ and ϕ' bounded on $[0,R)$, which contradicts the maximality of the interval $[0,R)$. This and (5.4.6) show that $\phi'(r) \to \infty$ as $r \to R^-$.

Note that

$$\begin{aligned}
\Phi(t) &= \int_0^t \frac{s}{\varpi(s)}\, ds \\
&\geq \frac{1}{\varpi(t)} \int_0^t s\, ds, \quad \text{since } \varpi \text{ is non-decreasing on } \mathbb{R}^+ \\
&= \frac{t^2}{2\varpi(t)}.
\end{aligned}$$

Thus,

$$\Phi(\phi'(r)) \geq \frac{(\phi'(r))^2}{2\varpi(\phi'(r))}, \quad r > 0. \tag{5.4.14}$$

Since $\phi'(r) \to \infty$ as $r \to R^-$ we conclude from (5.4.14) and condition **($\mathcal{W}$-2) (ii)** that

$$\lim_{r\to R^-}\Phi(\phi'(r))=\Phi\left(\lim_{r\to R^-}\phi'(r)\right)=\infty.$$

In particular, there is $r_0\le r<R$ such that

$$\Phi(\phi'(r))\ge 2\Phi(\phi'(r_0)),\quad r_0<r<R.$$

Substituting this into (5.4.13), we find that for $r_0\le r<R$,

$$\frac{1}{2}\Phi(\phi'(r))\le\Phi(\phi'(r))-\Phi(\phi'(r_0))\le\varrho\left(F(\phi(r))+G(\phi(r))\right).$$

Thus, we get

$$\phi'(r)\le\Phi^{-1}\Big(2\varrho\left(F(\phi(r))+G(\phi(r))\right)\Big),\quad r_0<r<R. \tag{5.4.15}$$

Integrating (5.4.15) on (r_0,r) for any $r_0<r<R$, we find

$$\int_{\phi'(r_0)}^{\phi'(r)}\frac{ds}{\Phi^{-1}\Big(2\varrho\left(F(s)+G(s)\right)\Big)}\le r-r_0. \tag{5.4.16}$$

Taking the limit $r\to R^-$ in (5.4.16), we find that condition **($\mathcal{W}$-2) (i)** is contradictory. Thus, we must have $R=\infty$. As a consequence, we see that $v(x):=\phi(|x|)$ is a positive radial viscosity solution of (5.2.2) in $\mathbb{R}^n$. □

Given $0<q<2$, $c\ge 1$, and $\varepsilon>0$, let $\phi\in C^2([0,R))$ be a solution of (IVP(ε)) in a maximal interval of existence $[0,R)$. As shown in [26, Lemma 2.3], we see that for $0<r<R$

$$d_1\int_\varepsilon^{\phi(r)}\frac{ds}{\sqrt{\mathcal{F}(s,\varepsilon)+\mathcal{G}(s,\varepsilon)^{\frac{1}{2-q}}}}\le r\le d_2\int_\varepsilon^{\phi(r)}\frac{ds}{\sqrt{\mathcal{F}(s,\varepsilon)+\mathcal{G}(s,\varepsilon)^{\frac{1}{2-q}}}}, \tag{5.4.17}$$

where d_1 and d_2 are positive constants such that d_1 depends on q and d_2 depends on q and c.

Remark 5.4.1. It is known (see [5]) that

$$\int_\varepsilon^\infty\frac{ds}{\sqrt{\mathcal{F}(s,\varepsilon)+\mathcal{G}(s,\varepsilon)^{\frac{1}{2-q}}}}<\infty$$

for all $\varepsilon>0$, if and only if f and g satisfy the condition $(\mathbf{KO})_q$.

Proof of Theorem 5.2.6. The first statement follows from **Theorem 5.2.4** when talking to $\varpi(t) = |t|^q$. Recalling that $(\mathbf{KO})_\varpi$ is just $(\mathbf{KO})_q$, the conclusion follows.

Now, we prove the second statement. For this, we assume that $(\mathbf{KO})_q$ does not hold. In this case, we again invoke [5, Theorem 5.2] or [23, Lemma 3.5] to find a non-decreasing and convex solution $\phi \in C^2([0,\infty))$ for the problem (IVP(ε)) (any fixed $\varepsilon > 0$). Then $u(x) = \phi(|x|)$ is a positive radial viscosity solution of problem (5.2.4) in $\mathbb{R}^n$. $\square$

Conclusion and Future Works

General Conclusion

This dissertation investigated the Compact Support Principle for viscosity supersolution (subsolution) of equation $\mathcal{P}_{\mathbf{a}}(D^2u) = h(u, |Du|)$, validated the Strong Minimum/Maximum Principle, and explored the Strong Boundary Minimum/-Maximum Principle. Our results give a necessary condition, namely $a_n > 0$ on a n-tuple $\mathbf{a} = (a_1, \cdots, a_n)$, and the Hamiltonian h depends only on the gradient to satisfy those principles. Notably, our work covers a new class of degenerate operators and a wide class of Hamiltonians not previously investigated in the literature. Furthermore, some of our findings are novel even when $\mathcal{P}_{\mathbf{a}}$ is reduced to the standard Laplacian.

On the other hand, various Liouville-type theorems for partial trace equations with nonlinear gradient terms were studied. Specifically, we have provided sufficient conditions for non-negative viscosity subsolutions of these equations in $\mathbb{R}^n$ to vanish identically. Furthermore, for a prototype of such equations, we have given necessary and sufficient conditions for non-negative subsolutions in $\mathbb{R}^n$ to be identically zero.

A variety of techniques and tools were used to analyze the problems in our proofs, such as duality, various principles (comparison principle, boundary principles, compact support principle, minimum/maximum principles), as well as different mathematical estimates and inequalities.

The study improves our understanding of the qualitative behaviour and applications of the well-posedness, uniqueness, and regularity of solutions to partial differential equations, paving the way for future research to explore their applications in various fields.

Future Work

Based on the findings of this dissertation and previous research, in our future work we plan to investigate the strong minimum/maximum principle for k-Hessian equations with nonlinear gradient terms. We will also study Liouville-type theorems for p-Laplacian operators and the existence of radial solutions to partial trace operators that include nonlinear gradient terms.

Bibliography

[1] S. Axler, P. Bourdon, and W. Ramey. *Harmonic Function Theory*, volume 137. Springer Science & Business Media, 2013.

[2] L. Baldelli and R. Filippucci. Existence results for elliptic problems with gradient terms via a priori estimates. *Nonlinear Analysis*, 198:111894, 2020.

[3] M. Bardi and A. Goffi. Liouville Results for Fully Nonlinear Equations Modeled on Hörmander Vector Fields: I. The Heisenberg Group. *Mathematische Annalen*, 379:611–650, 2021.

[4] H. Berestycki, F. Hamel, and L. Rossi. Liouville-type results for semilinear elliptic equations in unbounded domains. *Annali Di Matematica Pura Ed Applicata*, 186(3):469–507, 2007.

[5] T. Bhattacharya and A. Mohammed. Maximum principles for k-Hessian equations with lower order terms on unbounded domains. *The Journal of Geometric Analysis*, 31:3820–3862, 2021.

[6] T. Bhattacharya and A. Mohammed. On a strong maximum principle for fully nonlinear subelliptic equations with Hörmander condition. *Calculus of Variations and Partial Differential Equations*, 60(1):1–20, 2021.

[7] I. Birindelli and F. Demengel. Liouville theorems for p-laplace operators with gradient terms. *Journal of Mathematical Analysis and Applications*, 455(2):1152–1180, 2017.

[8] I. Birindelli, G. Galise, and H. Ishii. A family of degenerate elliptic operators: Maximum principle and its consequences. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 35(2):417–441, 2018.

[9] I. Birindelli, G. Galise, and F. Leoni. Liouville theorems for a family of very degenerate elliptic nonlinear operators. *Nonlinear Analysis*, 161:198–211, 2017.

[10] P. Blanc, C. Esteve, and J. D. Rossi. The evolution problem associated with eigenvalues of the Hessian. *Journal of the London Mathematical Society*, 102(3):1293–1317, 2020.

[11] P. Blanc and J. D. Rossi. Games for eigenvalues of the Hessian and concave/-convex envelopes. *Journal de Mathematiques Pures et Appliquees*, 127:192–215, 2019.

[12] H. Brezis and F. Browder. Partial differential equations in the 20th century. *Advances in mathematics*, 135(1):76–144, 1998.

[13] L. Caffarelli, N. Garofalo, and F. Segala. A gradient bound for entire solutions of quasi-linear equations and its consequences. *Communications on Pure and Applied Mathematics*, 47(11):1457–1473, 1994.

[14] L. Caffarelli, Y. Li, and L. Nirenberg. Some remarks on singular solutions of nonlinear elliptic equations III: Viscosity solutions including parabolic operators. *Communications on Pure and Applied Mathematics*, 66(1):109–143, 2013.

[15] L. Caffarelli, Y. Y. Li, and L. Nirenberg. Some remarks on singular solutions of nonlinear elliptic equations. I. *Journal of Fixed Point Theory and Applications*, 5:353–395, 2009.

[16] L. A. Caffarelli and X. Cabré. *Fully Nonlinear Elliptic Equations*, volume 43 of *Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

[17] I. Capuzzo-Dolcetta, F. Leoni, and A. V. and. The Alexandrov-Bakelman-Pucci Weak Maximum Principle for Fully Nonlinear Equations in Unbounded Domains. *Communications in Partial Differential Equations*, 30(12):1863–1881, 2005.

[18] C. Chang, B. Hu, and Z. Zhang. Liouville-type theorems and existence of solutions for quasilinear elliptic equations with nonlinear gradient terms. *Nonlinear Analysis*, 220:112873, 2022.

[19] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American mathematical society*, 27(1):1–67, 1992.

[20] A. Cutrı and F. Leoni. On the Liouville property for fully nonlinear equations. *Annales de l'Institut Henri Poincaré C*, 17(2):219–245, 2000.

[21] C.-P. Danet. The classical maximum principle. some of its extensions and applications. *Annals of the Academy of Romanian Scientists Series on Mathematics and its Applications*, 3:273–299, 2011.

[22] G. Díaz. A note on the Liouville method applied to elliptic eventually degenerate fully nonlinear equations governed by the Pucci operators and the Keller–Osserman condition. *Mathematische Annalen*, 353(1):145–159, 2012.

[23] C. Doerr and A. Mohammed. On Liouville-type theorems for k-Hessian equations with gradient terms. *Proceedings of the American Mathematical Society*, 152(06):2479–2495, 2024.

[24] J. Dolbeault and R. Monneau. On a Liouville type theorem for isotropic homogeneous fully nonlinear elliptic equations in dimension two. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 2(1):181–197, 2003.

[25] C. Dolcetta. Entire subsolutions of fully nonlinear degenerate elliptic equations. *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)*, 9(2):147, 2014.

[26] I. C. Dolcetta, F. Leoni, and A. Vitolo. On the inequality $\mathcal{F}(x, D^2u) \geq f(u) + g(u)|Du|^q$. *Mathematica Annalen*, 365:423–448, 2016.

[27] I. C. Dolcetta and A. Vitolo. On the maximum principle for viscosity solutions of fully nonlinear elliptic equations in general domain. *Le Matematiche*, 62(2):69–91, 2007.

[28] S. Doma and A. El-Sharif. Second order partial differential equations and their applications in physics and engineering. *Verlag/Publisher: Noor Publishing, Germany*, 2016.

[29] J. I. Díaz. *Nonlinear Partial Differential Equations and Free Boundaries*, volume 106 of *Research Notes in Mathematics*. Pitman Advanced Publishing Program, Boston, MA, 1985.

[30] A. Farina. Liouville-type results for solutions of $-\Delta u = |u|^{p-1}u$ on unbounded domains of $\mathbb{R}^N$. *Comptes Rendus Mathematique*, 341(7):415–418, 2005.

[31] P. Felmer, M. Montenegro, and A. Quaas. A note on the strong maximum principle and the compact support principle. *Journal of Differential Equations*, 246(1):39–49, 2009.

[32] P. Felmer, A. Quaas, and B. Sirakov. Solvability of nonlinear elliptic equations with gradient terms. *Journal of Differential Equations*, 254(11):4327–4346, 2013.

[33] F. Ferrari and A. Vitolo. Regularity properties for a class of non-uniformly elliptic isaacs operators. *Advanced Nonlinear Studies*, 20(1):213–241, 2020.

[34] R. Filippucci. Nonexistence of positive weak solutions of elliptic inequalities. *Nonlinear Analysis: Theory, Methods & Applications*, 70(8):2903–2916, 2009.

[35] R. Filippucci. Nonexistence of nonnegative solutions of elliptic systems of divergence type. *Journal of Differential Equations*, 250(1):572–595, 2011.

[36] R. Filippucci, P. Pucci, and P. Souplet. A Liouville-type theorem for an elliptic equation with superquadratic growth in the gradient. *Advanced Nonlinear Studies*, 20(2):245–251, 2020.

[37] B. Fine and G. Rosenberger. *The fundamental theorem of algebra*. Springer Science & Business Media, 1997.

[38] G. Galise. *Maximum principles, entire solutions and removable singularities of fully nonlinear second-order equations*. PhD thesis, Università di Salerno, 2012.

[39] G. Galise, S. Koike, O. Ley, and A. Vitolo. Entire solutions of fully nonlinear elliptic equations with a superlinear gradient term. *Journal of Mathematical Analysis and Applications*, 441(1):194–210, 2016.

[40] B. Gidas and J. Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Communications in Partial Differential Equations*, 6(8):883–901, 1981.

[41] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag, Berlin, Heidelberg, 2nd edition, 1977.

[42] A. Goffi. Some New Liouville-Type Results for Fully Nonlinear PDEs on the Heisenberg Group. *Nonlinear Analysis*, 200:112013, 2020.

[43] F. R. Harvey and H. B. L. Jr. Dirichlet duality and the nonlinear dirichlet problem. *Communications on Pure and Applied Mathematics*, 62(3):396–443, 2009.

[44] F. R. Harvey and H. B. L. Jr. Removable singularities for nonlinear subequations. *Indiana University Mathematics Journal*, 63(5):1525–1552, 2014.

[45] F. R. Harvey and H. B. Lawson. Existence, uniqueness and removable singularities for nonlinear partial differential equations in geometry. *Surveys in Differential Geometry*, 18(1):103–156, 2013.

[46] F. R. Harvey and H. B. Lawson Jr. Dirichlet Duality and the Nonlinear Dirichlet Problem on Riemanninan Manifolds. *Journal of Differential Geometry*, 88(3):395–482, 2011.

[47] E. Hopf. *Elementary remarks on the solutions of second-order partial differential equations of the elliptic type*. American Mathematical Society, 1927.

[48] N. Katzourakis. *An introduction to viscosity solutions for fully nonlinear PDE with applications to calculus of variations in L^∞*. Springer, 2014.

[49] J. B. Keller. On solutions of $\Delta u = f(u)$. *Communications on Pure and Applied Mathematics*, 10:503–510, 1957.

[50] B. Kindu, A. Mohammed, and B. Tsegaw. A Vázquez-type strong minimum/-maximum principle for partial trace operators. *Discrete and Continuous Dynamical Systems-S*, 16(11):3146–3162, 2023.

[51] B. Kindu, A. Mohammed, and B. Tsegaw. Liouville-type theorems for partial trace equations with nonlinear gradient terms. *Journal of Mathematical Analysis and Applications*, 543(2):129010, 2025.

[52] S. Koike and T. Kosugi. Remarks on the comparison principle for quasilinear pde with no zeroth order terms. *Communications on Pure & Applied Analysis*, 14(1), 2015.

[53] F. Leoni. Explicit subsolutions and a Liouville theorem for fully nonlinear uniformly elliptic inequalities in halfspaces. *Journal de mathématiques pures et appliquées*, 98(5):574–590, 2012.

[54] G. Lu, P. Wang, and J. Zhu. Liouville-type theorems and decay estimates for solutions to higher order elliptic equations. *Annales de l'Institut Henri Poincaré C*, 29(5):653–665, 2012.

[55] M. Magliaro, L. Mari, P. Mastrolia, and M. Rigoli. Keller–Osserman type conditions for differential inequalities with gradient terms on the heisenberg group. *Journal of Differential Equations*, 250(6):2643–2670, 2011.

[56] L. Mari, M. Rigoli, and A. G. Setti. Keller–Osserman conditions for diffusion-type operators on Riemannian manifolds. *Journal of Functional Analysis*, 258(2):665–712, 2010.

[57] D. E. Marshall. *Complex analysis*. Cambridge University Press, 2019.

[58] E. Mitidieri. Absence of positive solutions for nonvariational elliptic inequalities. *Differential and Integral Equations*, 10(3):579–588, 1998.

[59] L. Modica. A gradient bound and a liouville theorem for nonlinear poisson equations. *Communications on pure and applied mathematics*, 38(5):679–684, 1985.

[60] A. Mohammed and A. Vitolo. On the strong maximum principle. *Complex Variables and Elliptic Equations*, 65(8):1299–1314, 2020.

[61] R. Osserman. On the inequality $\Delta u \geq f(u)$. *Pacific J. Maths*, 7:1641–1647, 1957.

[62] M. H. Protter and H. F. Weinberger. *Maximum Principles in Differential Equations*. Springer-Verlag, New York, 1984.

[63] P. Pucci and V. Radulescu. The maximum principle with lack of monotonicity. *Electronic Journal of Qualitative Theory of Differential Equations*, 2018(58):1–11, 2018.

[64] P. Pucci and J. Serrin. A note on the strong maximum principle for elliptic differential inequalities. *Journal de Mathématiques pures et appliquées*, 79(1):57–71, 2000.

[65] P. Pucci and J. Serrin. *The Maximum Principle*. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Basel, 2007.

[66] P. Pucci and J. Serrin. The strong maximum principle revisited. *Journal of Differential Equations*, 229(2):377–396, 2007.

[67] P. Pucci, J. Serrin, and H. Zou. A strong maximum principle and a compact support principle for singular elliptic inequalities. *Journal de mathématiques pures et appliquées*, 78(8):769–789, 1999.

[68] A. Quaas and B. Sirakov. Existence results for nonproper elliptic equations involving the pucci operator. *Communications in Partial Differential Equations*, 31(7):987–1003, 2006.

[69] J. Serrin and H. Zou. Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities. *Acta Math*, 189(1):79–142, 2002.

[70] J.-P. Sha. p-convex Riemannian manifolds. *Inventiones mathematicae*, 83(3):437–447, 1986.

[71] J.-P. Sha. Handlebodies and p-convexity. *Journal of Differential Geometry*, 25(3):353–361, 1987.

[72] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Applied Mathematics and Optimization*, 12(1):191–202, 1984.

[73] A. Vitolo. Existence of positive entire solutions of fully nonlinear elliptic equations. *Journal of Elliptic and Parabolic Equations*, 4(2):293–304, 2018.

[74] A. Vitolo. Maximum principles for viscosity solutions of weakly elliptic equations. *Bruno Pini Mathematical Analysis Seminar*, 10:1–17, 2019.

[75] A. Vitolo. Singular elliptic equations with directional diffusion. *Math. Eng*, 3(3):1–16, 2021.

[76] A. Vitolo. Lipschitz estimates for partial trace operators with extremal Hessian eigenvalues. *Advances in Nonlinear Analysis*, 11(1):1182–1200, 2022.

[77] A. Vitolo. The Alexandroff–Bakelman–Pucci estimate via positive drift. *Pure and Applied Analysis*, 5(2):261–283, 2023.

[78] H. Wu. Manifolds of partially positive curvature. *Indiana University Mathematics Journal*, 36(3):525–548, 1987.