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By

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Fuzzy Algebraic Structures on JU-Algebra

A Dissertation Submitted to the Department of Mathematics in Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

By

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Co-supervisor: Mihret Alamneh Taye (PhD, Associate Prof.)

Declaration of Authorship

This is to certify that the dissertation entitled "**Fuzzy Algebraic Structures on JU-Algebra**", submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, Department of Mathematics, College of Science, Bahir Dar University, represents original research conducted by me. The work has not been submitted to this or any other institution for the award of any other degree, diploma, or certificate. All sources of assistance and contributions received during this research have been duly acknowledged.

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Approval of Dissertation for Oral Defense

We hereby certify that we have supervised, read, and evaluated this dissertation entitled "**Fuzzy Algebraic Structures on JU-Algebra**" by **Selamawit Hunie Gelaw** prepared under our guidance. We recommend the dissertation to be submitted for oral defense (mock-via and via voce).

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As members of the board of examiners, we examined this dissertation entitled "**Fuzzy Algebraic Structures on JU-Algebra**" by **Selamawit Hunie Gelaw**. We hereby certify that the dissertation is accepted for fulfilling the requirements for the award of the degree of doctor of philosophy in mathematics.

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Dedication

I dedicate my dissertation work to my beloved family, whose unwavering support has been my greatest strength. A special honor goes to my loving partner, Habtamu Dessie, whose words of encouragement and belief in me have been a constant source of inspiration. His wisdom and love continue to guide me every step of the way.

Publications

This dissertation is based on the following works.

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Abstract

Fuzzy set theory provides a mathematical framework for handling uncertainty and partial truth, extending classical set theory by allowing elements to have degrees of membership. This study aimed to introduce a new class of algebraic structures in fuzzy sets, including Q-fuzzy sets, T-fuzzy sets, and interval-valued fuzzy sets. The concepts of fuzzy JU-subalgebras and fuzzy JU-ideals have been introduced for JU-algebras, and their properties have been duly characterized. Also, we establish the relation between fuzzy JU-subalgebra and fuzzy JU-ideal on JU-algebra. This study begins with foundational concepts of fuzzy sets, Q-fuzzy sets, doubt Q-fuzzy sets, T-fuzzy sets, interval-valued fuzzy sets, and anti-fuzzy sets. It also introduces key algebraic systems such as BCK-algebras, BCI-algebras, UP-algebras, PS-algebras, B-algebras, BG-algebras, and KU-algebras to establish the necessary theoretical background for the development of JU-algebraic theory. In this dissertation, we investigate fuzzy subalgebras and fuzzy ideals in JU-algebras, with emphasis on prime fuzzy sets and p-multiplicative relations. Their behavior under homomorphisms and Cartesian products is rigorously analyzed. The theory is extended to Q-fuzzy and doubt Q-fuzzy structures, incorporating normality and doubt relations, and exploring their level subsets. Further, we studied T-fuzzy structures, characterized by chain conditions and Cartesian product. Interval-valued fuzzy structures, including $(\tilde{\mathrm{E}}, \tilde{\mathrm{F}})$-based models, are introduced to represent uncertainty more precisely, with analysis of their properties under mappings, level set, and product operations. The study also explores anti-fuzzy structures, presenting complementary notions through anti-fuzzy subalgebras and ideals, and investigating their relational dynamics. Overall, this dissertation contributes to the advancement of fuzzy algebraic theory by extending fuzzy set concepts to JU-algebras and providing a unified framework for analyzing their structural behavior.

Table of Contents

List of Symbols and Abbreviations

X	JU-Algebra
$I - V$	Interval-Valued
$IVFSs$	Interval-Valued Fuzzy Sets
$\vartheta, \beta, \lambda$	Fuzzy sets
η	Q-Fuzzy Sets
ϖ	T-Fuzzy sets
ϕ	Interval-Valued Fuzzy Sets
$\vartheta_1 \times \vartheta_2$	Cartesian product of ϑ_1 and ϑ_2
$\vartheta(x)$	Degree of membership of x in fuzzy set ϑ
$\Delta, \tilde{E}, \tilde{F}, \tilde{s}, p, q$	Parameters
ϕ^L	Lower bound of an interval-valued fuzzy set ϕ
ϕ^U	Upper bound of an interval-valued fuzzy set ϕ
ϑ^c	Complement of ϑ
χ_S	Fuzzy characteristic function of S
$\cup$	Union
$\cap$	Intersection
$\subseteq$	Subset
$\in$	Belongs to
$\notin$	Does not belong to
$\forall$	For all
$\exists$	There exists
Ω, Λ	Family of fuzzy sets
$U(\vartheta, t)$	Upper t-level set
$L(\vartheta, s)$	Lower s-level set
$f(\vartheta)$	Homomorphic image of ϑ

$f^{-1}(\vartheta)$	Homomorphic inverse image of ϑ
min	Minimum
max	Maximum
inf	Infimum
sup	Supremum
iff	If and only if

Introduction

Cantor started the study of classical set theory in 1874 [12]. A Cantor set (crisp set) is described using a membership function that assigns a value in binary terms to each element of the universe. Their development not only shaped the mathematical landscape but also prompted significant philosophical discussions regarding the nature of mathematical objects and the implications of categorization in various fields. Most of our conventional tools for explicit reasoning, modeling, and computing are precise, deterministic, and crisp; that is, they provide yes-or-no characterizations instead of more-or-less types. The transition of an element between membership and nonmembership in a given set in the universe is abrupt and well-defined. This phenomenon signifies that the system is unequivocal, and it involves no obscurity. The crisp set can be defined by the so-called characteristics function. This milestone development opened the eyes of researchers, inspiring advancements that enable more refined concepts and improved measurement of real-world phenomena. On the other hand, real situations are generally uncertain or ambiguous in various ways. In the present world, there exists information that is incomplete, unclear, uncertain, inexact, equivocal, or ambiguous. In the framework of crisp set theory, it is exceptionally hard to answer different questions since they do not have absolutely true answers.

To address this challenge, Zadeh [69] introduced the theory of fuzzy sets as a means of modeling uncertainty, imprecise, or vagueness within the unit interval $[0,1]$. Unlike classical sets, fuzzy sets allow elements to have degrees of membership ranging from 0 to 1. Fuzzy set theory provides a powerful framework for dealing with linguistic variables (e.g., ”tall”, ”satisfaction”, ”attitude”, ”disease condition”, ”hot”, ”fast”, etc.) and has been widely applied in fields such as artificial intelligence, control systems, decision making, and pattern recognition.

Building on Zadeh’s foundational work, in 1971, Rosenfeld [52] made a significant contribution by extending fuzzy set theory into the realm of algebraic structures. Specifically, Rosenfeld introduced the concept of fuzzy subgroups, initiating the application of fuzzy set theory to group theory and beyond. His work simplified the process of interpreting algebraic operations under fuzziness and laid the groundwork for the development of fuzzy algebraic systems. This integration allowed for the exploration of various fuzzy analogues of classical algebraic notions such as fuzzy subgroups, fuzzy ideals, and fuzzy subrings, which accommodate uncertainty and partial truth within algebraic frameworks. Rosenfeld’s development thus opened a new research direction, making fuzzy set theory not only a tool for uncertainty modeling but also a foundational concept in algebraic studies involving fuzziness. doubt fuzzy subsets[4, 7, 42] also studied to address situations involving hesitation or conflict in information, modeling uncertainty with an added dimension of doubt in membership assignment.

To further enhance the expressive power of fuzzy sets, especially in contexts involving parameterized uncertainty, researchers introduced the notion of Q-fuzzy subsets[5]. A Q-fuzzy subset incorporates an additional parameter set Q into the fuzzy framework, resulting in a membership function of the form $\eta : X \times Q \to [0,1]$. This allows for greater flexibility and granularity in describing the membership behavior of elements across varying conditions or criteria defined by Q. Several refinements of Q-fuzzy subsets have been developed to handle more complex forms of uncertainty.The concept of Normal is entertained in Q-fuzzy subsets, which ensure that for each parameter in Q, there exists at least one element in X with full membership (i.e., degree 1), aligning with the concept of a "core" or "center" of certainty within the fuzzy set. T-fuzzy subsets[27–29, 57] also incorporate triangular norms (T-norms) to define generalized intersection operations, supporting more structured and algebraically consistent fuzzy logic operations. These developments have enriched the theoretical landscape of fuzzy algebra, enabling more expressive and context-sensitive modeling of algebraic structures. They also play a critical role in applications across decision making, knowledge representation, and intelligent systems, where uncertainty is both inherent and unavoidable.

Despite the richness of these frameworks, standard fuzzy sets, whether classical or Q-fuzzy or T-fuzzy, may still fall short in fully capturing uncertainty due to vagueness in the membership degree itself. To address this limitation, the concept of interval valued fuzzy sets was introduced. In this extension, the degree of membership is no longer a precise value in $[0,1]$, but an interval $[\phi^L(x)$, $\phi^U(x)] \subseteq [0,1]$, where $0 \leqslant \phi^L(x) \leqslant \phi^U(x) \leqslant 1$. This reflects both the minimum and maximum confidence in the membership of an element, offering a richer representation of uncertainty, particularly when precise evaluation is infeasible. Interval valued fuzzy sets have been successfully integrated into various algebraic structures, leading to the development of interval valued fuzzy subalgebras, ideals, and other algebraic constructs. These advanced models enhance the degree of flexibility and realism in applications that involve uncertainty, incomplete information, or subjective assessment.

BCK and BCI-algebras are foundational structures of logical algebras, which enhance the flexibility in modeling logical systems and non-classical reasoning. The notion of BCK-algebra was introduced by Iseki and Tanaka [22]. In 1980, Iseki [23] introduced the notion of BCI-algebra as a generalization of a BCK-algebra and investigated some of its properties. In 2002, Neggers and Kim [38] introduced a notion called B-algebra, as a generalization of BCI/BCK-algebras and studied its properties through normalization. In 2007, Walendziak[65] introduced a new notion called BF-algebra, a generalization of BCI/BCK-algebras. In the same year, Walendziak[65] introduced a notion called BG-algebra, as a generalization of B/BF/BCI/BCK-algebras. In 2009, Prabayak and Leerawat[41] introduced a notion called KU-algebra, as a generalization of BCI/BCK–algebras and studied its properties through ideals. Mostafa in 2011[33] introduced the notion of fuzzy KU-ideal of KU-algebra, and then they investigated several basic properties which are related to fuzzy KU-subalgebra and fuzzy KU-ideal. They described how to deal with the homomorphic image and inverse image of fuzzy KU-ideals. They also proved that the Cartesian product of fuzzy KU-ideals in the Cartesian product of fuzzy KU-algebras was a fuzzy KU-ideal. Indeed, Prabpayak in 2009,[41] introduced the notion of homomorphisms in fuzzy KU-algebras. Salvem [54] also worked on the notion of fuzzy PMS-algebra under Homomorphism.

In 2014, Smarandache[45, 46] introduced and studied fuzzy translations and homomorphism image of PS-algebra. In 2016, Somjanta [60] introduced and studied fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras. He introduced a new algebraic structure, called UP-algebra, and a concept of UP-ideals, UP-subalgebras, UP-prime, alpha-translation, Q-fuzzy set, level sets, and UP-homomorphisms in UP-algebras, and investigated some related properties of them, and showed that the notion of UP-algebras is a generalization of BCK/BCI/KU-algebras. In 2015, Mostefa et al. [35] introduced PU-algebra and Iampan in 2017[21] introduced prime fuzzy translation. In 2020, Ansari et al. [3] introduced the JU-algebra as a generalization of weak KU-algebras. His work relaxes certain constraints of KU/BCK-algebras. In 2022, Romano [51] explored ideals and subalgebras within JU-algebras. In 2023, Ral [48] further applied these concepts to binary coding theory.

The concept of Q-fuzzy sets was introduced by many researchers and was extensively investigated in many algebraic structures, such that Solairaju and Nagarajan [59] developed a new algebraic structure called a Q-fuzzy group for implementing the Q-fuzzy set in group theories and investigated some of their related properties. In 2010, Muthuraj [37] introduced the notion of a Q-fuzzy subset of a BG-algebra as a Q-fuzzy BG-ideal.In 2018, [62] explored Q-fuzzy sets and their level subsets in UP-algebras. Furthermore, Barbhuiya in 2014, [7] extended the concept of doubt fuzzy ideals in BF-algebras, while [42] discussed doubt Fuzzy Ideals of B-Algebras. In 2018, [43] further contributed by introducing the normalization of fuzzy B-ideals in B-algebras.

In 2013, Selvam [55] studied the notion of anti Q-fuzzy KU-subalgebras and homomorphism of KU-algebras, and in 2014, Selvam studied anti Q-fuzzy R-closed KU-ideals in KU-algebras and their lower level cuts. In 2014, Priya[47] introduced the concept of anti Q-fuzzy R-closed PS-ideals of PS-algebras, lower level cuts of a fuzzy set, R-closed PS-ideals, and proved some results. In 2018, Tanamoon [62] introduced the notions of Q-fuzzy UP-ideals and Q-fuzzy UP-subalgebras of UP-algebras, and their properties were investigated.

T-Fuzzy relations on algebraic structures have emerged as a significant advancement in algebraic fuzzy logic. The foundation for these structures was laid by Kiyoshi Iséki (1966)[22] with crisp BCK-algebras, which model non-classical logics. Consequently, numerous research studies have been done by combining the notions of fuzzy sets and T-norm to different algebraic structures such as BCI [2, 50], KU [56, 58], BG [57], TM [63], and so on. Rosenfeld (1971) pioneered fuzzy groups, which inspired the extension to fuzzy BCK/BCI-algebras by Jun (2000) and others, where T-norms like the minimum, product, and Łukasiewicz operators define fuzzy partial orders and ideals. These T-norm-based fuzzy algebras address limitations in classical binary logic by enabling graded truth valuations, but they also introduce challenges, such as the dependence of algebraic properties on T-norm continuity (e.g., only continuous T-norms yield well-behaved residuation).

On the other hand, we notice that the interval valued (I-V) fuzzy sets introduced by Zadeh (1975) [70] and later formalized by Takeuti & Titani (1984)[61] and Gorzałczany (1987)[16]. These sets address the limitation of traditional fuzzy sets in handling uncertainty by representing membership degrees as intervals $[\phi^L(x), \phi^U(x)] \subseteq [0,1]$, where $\phi^L(x)$ and $\phi^U(x)$ denote the lower bound(minimum certainty) and upper bound(maximum plausibility) of membership, respectively. Interval valued fuzzy sets(IVFSs) are primarily used to model situations where the exact membership grade is imprecise or unknown, but can be bounded within a specific range. This makes IVFSs particularly useful in decision-making, medical diagnosis, and control systems, where expert opinions or measurements often come with inherent uncertainty. Compared to classical fuzzy sets, IVFSs provide a richer framework for capturing and propagating uncertainty, but they also introduce higher computational complexity due to interval-based operations. In 1994, Biswas [10] defined interval valued fuzzy subgroups and investigated some elementary properties. Likewise, in 2006, Saeid [53] introduced the concept of interval valued fuzzy set in BG-algebra. In 2008, Ma [31] introduced the concept of interval valued fuzzy set in BCI-algebra, and in 2011, Mostafa [34] KU-algebra.

In 2012, Khan [25, 40] developed the concept of interval valued (α, δ) fuzzy sets as a refined version of interval valued fuzzy sets, designed to offer greater control over uncertainty granularity. These sets incorporate two thresholds: α (a lower bound cutoff) and δ (an upper bound tolerance), which filter and adjust the membership intervals dynamically. This innovation brings to enhance flexibility in applications where certain levels of uncertainty need to be ignored (below α) or capped (above δ), such as in pattern recognition or risk analysis. Additionally, since the flexible membership condition is so important by introduced predefined interval [36] introduced the concept of $\tilde{\alpha}$-interval-valued fuzzy new-ideal of PU-algebra.

Despite the applications growing more complex, the need arose to also examine systems where elements do not belong to a set, leading to the study of anti-fuzzy sets and their algebraic counterparts. The concept of anti-fuzzy sets was first proposed by Biswas (1990) [9]as a dual counterpart to traditional fuzzy sets. The anti-fuzzy structures formalize the concept of explicit non-membership, offering a dual perspective that complements fuzzy systems in areas like conflict resolution and multi-criteria optimization. The anti-fuzzy concept further extends to BCK-algebras [20], BG-algebras [26], KU-algebras [55], and TM-algebras [49], enriching their ability to model uncertainty, logical negation, and conflict-based reasoning.

To the best of our knowledge, no researcher has yet explored the fuzzy structures on JU-algebra. This gap in the literature motivated us to initiate a study on fuzzy algebraic structures within this framework. This work explores fuzzy algebraic structures on JU-algebras, with an emphasis on investigating fuzzy subalgebras, ideals, prime fuzzy sets with p-multiplicative relations, and analyzing their properties under homomorphism. We extend further the theory to Q-fuzzy structures on JU-subalgebras, Q-fuzzy structures on JU-ideals, Normal relations, doubt relations, and analyze their properties under homomorphism and Cartesian products. We also extend the theory to T-fuzzy structures on JU-subalgebras, T-fuzzy structures on JU-ideals, Interval valued fuzzy structures on JU-subalgebras, Interval valued fuzzy structures on JU-ideals, investigating interval valued $(\tilde{E},\tilde{F})$ fuzzy subalgebras, interval valued $(\tilde{E},\tilde{F})$ fuzzy structures on JU-ideals, and analyzing their properties under a set of levels, homomorphism, and Cartesian products. This work also explores anti-fuzzy algebraic structures on JU-algebras, investigating anti-fuzzy subalgebras, ideals, their lower-level subsets, relations, and analyzing their properties under homomorphism and Cartesian products.

This dissertation consists of five chapters, and each chapter also consists of a number of sections. The content and coverage of each chapter are summarized as follows.
In the first chapter, we provide a foundational overview of key concepts of fuzzy sets, Q-fuzzy sets, doubt fuzzy sets, T-fuzzy sets, Interval-valued fuzzy sets, and anti-fuzzy sets, which are essential for the development of this dissertation. Additionally, we review fundamental definitions and properties of related algebraic structures, including BCK-algebras, BCI-algebras, UP-algebras, TM-algebras, B-algebras, BG-algebras, and KU-algebras, to establish the necessary background for subsequent discussions.

In the second chapter, we have introduced the notions of JU-subalgebras and JU-ideals in JU-algebra and have proved some interesting results. Moreover, we have defined how to deal with the JU-algebra homomorphic image and inverse image of fuzzy JU-subalgebras and fuzzy JU-ideals in JU-algebra. Further, we have also proved that the Cartesian product of fuzzy JU-subalgebras and fuzzy JU-ideals are fuzzy JU-subalgebra and a fuzzy JU-ideal, respectively, in the JU-algebra. The concept of prime fuzzy sets and fuzzy p-multiplicative relations is also studied and important results have been obtained. Fuzzy level subsets on JU-subalgebras and JU-ideals of a JU-algebra are defined and characterized in terms of their level subsets. The notion of a complement(anti-fuzzy) JU-subalgebra and anti-fuzzy JU-ideal is also discussed.

In the third chapter, the concept of Q-fuzzy JU-subalgebras and ideals of Q-fuzzy JU-algebras has been studied, and several related properties have been investigated. The level subsets of a Q-fuzzy JU-subalgebra and Q-fuzzy JU-ideals of a JU-algebra are defined and characterized in terms of their level subsets. The concepts of doubt Q-fuzzy JU-subalgebras and doubt Q-fuzzy JU-ideals of a JU-algebra are introduced, and associated properties have been investigated. We have also discussed the concept of Normal in a Q-fuzzy JU-subalgebras and JU-ideals of a JU-algebra. Additionally, the concept of Q-fuzzy JU-subalgebra and Q-fuzzy JU-ideal of a JU-algebra has also been studied under homomorphism and Cartesian product, and associated properties have been addressed.

In the fourth chapter, the notion of T-fuzzy subalgebra and T-fuzzy JU-ideal of a JU-algebra is introduced using the concept of a T-fuzzy set to JU-subalgebras and JU-ideals, and several related results are investigated. The T-fuzzy subalgebras and T-fuzzy ideals of JU-algebra are characterized using their level sets. The homomorphism of a T-fuzzy JU-subalgebra and T-fuzzy JU-ideal of a JU-algebra is studied, and its homomorphic image and inverse image are explored. The concept of the ascending and descending chain of a T-fuzzy JU-subalgebra and a T-fuzzy JU-ideal of a JU-algebra is also studied. The Cartesian product of any two T-fuzzy JU-subalgebras and T-fuzzy ideals is discussed, and some related results are investigated.

In the fifth chapter, the concept of an interval valued fuzzy JU-subalgebra and interval valued fuzzy JU-ideal of a JU-algebra is introduced and some of their related properties are discussed. The concept of the $\tilde{S}$-interval valued fuzzy JU-subalgebras and $\tilde{S}$-interval valued fuzzy JU-ideals of JU-algebras is studied. The concept of interval valued fuzzy JU-subalgebra of a JU-algebra is also considered under homomorphism and Cartesian product and some related results are investigated. Additionally, the concept of a more realistic interval valued ($\tilde{E}$,$\tilde{F}$)-fuzzy JU-subalgebra and interval valued ($\tilde{E}$,$\tilde{F}$)-fuzzy JU-ideal of a JU-algebra are introduced and some of their related properties are discussed. The concept of interval valued ($\tilde{E}$,$\tilde{F}$)-fuzzy JU-subalgebra and interval valued ($\tilde{E}$,$\tilde{F}$)-fuzzy JU-ideal of a JU-algebra are also studied under homomorphism and Cartesian product, and associated properties are addressed.

In the last part, we have presented the conclusion, which contains a brief overview of the obtained results as well as a discussion of future work that can be conducted in the study of the fuzzy structure of the JU-algebra. Finally, we have concluded the dissertation with a complete bibliography, and the published papers are appended.

Chapter 1
Preliminaries

In this chapter, we review the essential definitions and foundational results necessary for our study. The chapter is organized into six sections. The first section covers basic concepts and the basic definitions of fuzzy set theory. The second section covers basic definitions and concepts of Q-fuzzy sets. The third section outlines the core principles of T-fuzzy sets, followed by the fourth section, which discusses interval-valued fuzzy sets. The fifth section recalls the basic definitions and results of BCK/BCI, B, UP, PS, and KU-algebras, while the sixth section summarizes some basic definitions and results in JU-algebras.

1.1. Basic Concepts of Fuzzy Set Theory

In this section, we review fundamental definitions and key results from the theory of fuzzy sets.

Definition 1.1.1. *[69] Let X be a nonempty set. A mapping $\vartheta : X \to [0,1]$ is called a fuzzy subset of X, where $\vartheta(x)$ assigns a membership degree to each element $x \in X$, indicating how much x belongs to ϑ.*

Definition 1.1.2. *[69] Let ϑ and β be fuzzy subsets of a set X with membership functions $\vartheta, \beta : X \to [0,1]$. Then*

1. *ϑ is said to be contained in β, denoted by $\vartheta \subseteq \beta$, if and only if*
$$\vartheta(x) \leqslant \beta(x) \quad \forall x \in X.$$
2. *ϑ is said to be strictly contained in β, denoted by $\vartheta \subset \beta$, if and only if*
$$\vartheta(x) < \beta(x) \quad \textit{and} \quad \vartheta(x) \neq \beta(x) \quad \textit{for at least one} \quad x \in X$$
3. *ϑ is said to be equal to β, denoted by $\vartheta = \beta$, if and only if*
$$\vartheta(x) = \beta(x) \quad \forall x \in X.$$

4. *The union of ϑ and β, denoted by $\vartheta \cup \beta$, is given by*

$$(\vartheta \cup \beta)(x) = \max\{\vartheta(x), \beta(x)\}, \quad \exists x \in X.$$

5. *The intersection of ϑ and β, denoted by $\vartheta \cap \beta$, is given by*

$$(\vartheta \cap \beta)(x) = \min\{\vartheta(x), \beta(x)\}, \quad \forall x \in X.$$

6. *The complement of ϑ, denoted by ϑ^c, is given by*

$$\vartheta^c(x) = 1 - \vartheta(x), \quad \forall x \in X.$$

Definition 1.1.3. *[69] Let $\{\vartheta_i \mid i \in \Omega\}$ be a family of fuzzy subsets of a set X. The intersection and union of these fuzzy subsets, denoted $\bigcap_{i\in\Omega} \vartheta_i$ and $\bigcup_{i\in\Omega} \vartheta_i$ respectively, are defined point-wise as follows:*

$$\left(\bigcap_{i\in\Omega} \vartheta_i\right)(x) = \inf_{i\in\Omega} \vartheta_i(x), \quad \forall x \in X,$$

$$\left(\bigcup_{i\in\Omega} \vartheta_i\right)(x) = \sup_{i\in\Omega} \vartheta_i(x), \quad \forall x \in X.$$

Lemma 1.1.1. *[32] Let ϑ be a fuzzy set defined on the set X. Then, for any elements $x, y \in X$, the following properties hold:*

$$1 - \min\{\vartheta(x), \vartheta(y)\} = \max\{1 - \vartheta(x), 1 - \vartheta(y)\}.$$

$$1 - \max\{\vartheta(x), \vartheta(y)\} = \min\{1 - \vartheta(x), 1 - \vartheta(y)\}$$

Definition 1.1.4. *[21, 60] A fuzzy set ϑ in a set X is called a prime fuzzy set in X, if $\vartheta(x * y) \leqslant$* $\max\{\vartheta(x), \vartheta(y)\}$, $\forall x, y \in X$.

Definition 1.1.5. *[4] Let ϑ be a fuzzy set in a set X. The degree of doubt at an element $x \in X$, denoted by $\theta_D(x)$, is defined as: $\vartheta_D(x) = 1 - \vartheta(x) - \bar{\vartheta}(x)$ with $\vartheta(x) + \bar{\vartheta}(x) < 1$, where $\vartheta(x)$ is the degree of membership and $\bar{\vartheta}(x)$ is the degree of non-membership.*

Example 1.1.1. *Consider the problem of diagnosing a particular disease, denoted as Disease X, based on the presence of three clinical symptoms: fever, cough, and fatigue. Where each element is associated with its degree of diagnostic doubt (d_i)*
Let the doubt mapping for Disease X be defined as follows:

$$\vartheta_D(x) = \begin{cases} 0.1, & \textit{if } \ x \text{ is } \textit{fever}, \\ 0.3, & \textit{if } \ x \text{ is } \textit{cough}, \\ 0.2, & \textit{if } \ x \text{ is } \textit{fatigue}. \end{cases}$$

See [64], $\vartheta_D(x)$ is a fuzzy set application of doubt in medical diagnosis.
This fuzzy set of doubts captures the uncertainty in the symptom-based reasoning for disease X. To be clearer, a higher (d_i) means more diagnostic doubt or variability in how confidently the symptom supports the diagnosis.

Definition 1.1.6. *[14] Let ϑ be any fuzzy subset of a set X and let $t \in [0,1]$. The set $U(\vartheta,t) = \{x \in X \mid \vartheta(x) \geqslant t\}$, is called the* upper t-level subset *of ϑ. Clearly, $U(\vartheta,t) \subseteq U(\vartheta,s)$ whenever $t > s$. And, the set $L(\vartheta,t) = \{x \in X \mid \vartheta(x) \leqslant t\}$, is called the* lower t-level subset *of ϑ. Clearly, $L(\vartheta,t) \subseteq L(\vartheta,s)$ whenever $t < s$.*

Definition 1.1.7. *[52] A fuzzy subset ϑ defined on a set X is said to satisfy the* sup-property *if, for every nonempty subset $S \subseteq X$, there exists an element $a_0 \in S$ such that:*

$$\vartheta(a_0) = \sup_{a \in S} \vartheta(a)$$

Definition 1.1.8. *[52] Let f be a function from a set X to a set Y; ϑ be any fuzzy subset of X, and β be any fuzzy subset of Y. The image of ϑ under f, denoted by $f(\vartheta)$, is a fuzzy subset of Y defined by*

$$f(\vartheta)(y) = \begin{cases} \sup\{\vartheta(x) | x \in f^{-1}(y), \text{ if } \quad f^{-1}(y) \neq \emptyset\} \\ 0, \text{ otherwise} \end{cases}$$

where $y \in Y$. The inverse image of β under f, symbolized by $f^{-1}(\beta)$, is a fuzzy subset of X defined by $f^{-1}(\beta)(x) = \beta(f(x)), \forall x \in X$.

Definition 1.1.9. *[32, 70] Let X, Y, Z be any sets. Then the functions $f : X \to Y$ and $g : Y \to Z$ are homomorphisms. Let ϑ be any fuzzy subset of X and β be any fuzzy subset of Z. Then the image of the composition function ϑ under $g \circ f$, denoted by $(g \circ f)(\vartheta)$, is a fuzzy set in Z defined by*

$$(g \circ f)(\vartheta)(z) = \begin{cases} \sup\{\vartheta(x) | x \in (g \circ f)^{-1}(z)\}, \text{if} \quad (g \circ f)^{-1}(z) \neq \emptyset \\ 0, \text{otherwise} \end{cases}$$

The inverse image of β under $g \circ f$, denoted $(g \circ f)^{-1}(\beta)$, is a fuzzy set in X defined by
$(g \circ f)^{-1}(\beta)(x) = \beta(g \circ f)(x) = \beta(g(f(x)))$.

Definition 1.1.10. *[45, 46] Let $f : X \to X$ be an endomorphism and ϑ be a fuzzy set in X. We define a new fuzzy set in X by ϑ_f in X as $\vartheta_f(x) = \vartheta(f(x)), \forall x \in X$.*

Definition 1.1.11. *[52] Let f be a function from a set X to a set Y, and ϑ be fuzzy subsets of X. Then*

1. $f(f^{-1}(\vartheta)) = \vartheta$.
2. $\vartheta \subseteq f^{-1}(f(\vartheta))$.

Definition 1.1.12. *[8] A fuzzy relation on a nonempty set* X *is a fuzzy set with a membership function* $\vartheta : X \times X \to [0,1]$.

Definition 1.1.13. *[36] Let* ϑ_1 *and* ϑ_2 *be two fuzzy subsets in a sets X and Y, respectively. The Cartesian product of* $\vartheta_1 \times \vartheta_2 : X \times Y \to [0,1]$, *is given by* $(\vartheta_1 \times \vartheta_2)(x,y) = \min\{\vartheta_1(x), \vartheta_2(y)\}$, $\forall (x,y) \in X \times Y$.

Definition 1.1.14. *[8] Let* ϑ *be a fuzzy subset in a set* X, *then the* strongest fuzzy relation *on* X *is a fuzzy relation* β *on* ϑ *whose membership function* $\beta : X \times X \to [0,1]$ *is given by*

$$\beta(x,y) = \min\{\vartheta(x), \vartheta(y)\}, \quad \forall x,y \in X.$$

1.2. Basic Concepts of Q-Fuzzy Subset

Definition 1.2.1. *[62] A Q-fuzzy set is a nonempty set X is an arbitrary function* $\eta : X \times Q \to [0,1]$, *where Q is a nonempty set and [0, 1] is the unit segment of the real line.*

Beyond theoretical development, these concepts have many applications in different areas, including medical sciences, for better decision making by allowing different parameters to play.

Example 1.2.1. *[1] A doctor is assessing a patient's blood sugar levels to diagnose diabetes. Blood glucose can vary throughout the day depending on food intake, activity, and metabolic factors.*
Let

$$X = \{140\ mg/dL,\ 160\ mg/dL,\ 200\ mg/dL\}, \quad Q = \{Fasting,\ Post\text{-}meal\}.$$

According to the definition of Q-fuzzy set, we define the mapping

$$\eta : X \times Q \to [0,1]$$

, *where* $\eta(x,q)$ *represents the membership degree of glucose level* x *at time* q *with respect to the concept "diabetic condition."*
Let

$$q_1 = Fasting, \qquad q_2 = Post\text{-}meal.$$

Define the mapping $\eta(x,q)$ *for blood glucose levels as follows:*

$$\eta(x,q_1) = \begin{cases} 0.2, & \text{if } x \text{ is glucose level } 140\ mg/dL, \\ 0.5, & \text{if } x \text{ is glucose level } 160\ mg/dL, \\ 1.0, & \text{if } x \text{ is glucose level } 200\ mg/dL. \end{cases}, \quad \eta(x,q_2) = \begin{cases} 0.3, & \text{if } x \text{ is glucose level } 140\ mg/dL, \\ 0.6, & \text{if } x \text{ is glucose level } 160\ mg/dL, \\ 1.0, & \text{if } x \text{ is glucose level } 200\ mg/dL. \end{cases}$$

Thus, for instance:

$$\eta(\text{Blood Glucose 160 mg/dL}, Fasting) = 0.5 \quad \text{and} \quad \eta(\text{Blood Glucose 200 mg/dL}, Post\text{-}meal) = 1.0.$$

This means that a fasting blood sugar level of 160 mg/dL corresponds to a moderate risk of diabetes with membership degree 0.5, while a post-meal level of 200 mg/dL indicates a definite diabetic condition with membership degree 1.0.
Hence, the Q-fuzzy set η provides a flexible and quantitative representation of diabetic conditions under varying time contexts, helping medical practitioners make more accurate and graded diagnostic decisions.

Definition 1.2.2. *[62] Let η be a Q-fuzzy set in X. For $t, s \in [0,1]$, the sets*

$$U(\eta, t) = \{x \in X \mid \eta(x,q) \geqslant t,\, q \in Q\} \quad \textit{and} \quad L(\eta, s) = \{x \in X \mid \eta(x,q) \leqslant s,\, q \in Q\}$$

are called the upper t-level set and the lower s-level set of η, respectively, and can be used in the characterization of Q-fuzzy sets.

Definition 1.2.3. *[37] If η is a Q-fuzzy set in a set X, then the complement, denoted by η^c, is the Q-fuzzy subset of X given by $\eta^c(x,q) = 1 - \eta(x,q), \forall x \in X$ and $q \in Q$.*

Definition 1.2.4. *[37] The union of two Q-fuzzy subsets η_1 and η_2 of a set X is defined by:*

$$(\eta_1 \cup \eta_2)(x,q) = \max\{\eta_1(x,q), \eta_2(x,q)\}, \quad \textit{for all } x \in X \textit{ and } q \in Q.$$

Definition 1.2.5. *[37] The intersection of two Q-fuzzy subsets η_1 and η_2 of a set X is defined by:*

$$(\eta_1 \cap \eta_2)(x,q) = \min\{\eta_1(x,q), \eta_2(x,q)\}, \quad \textit{for all } x \in X \textit{ and } q \in Q.$$

Definition 1.2.6. *[43] A fuzzy set ϑ of a set X is said to be normal if there exists $x \in X$ such that $\vartheta(x) = 1$.*

1.3. Basic Concepts of T-Fuzzy Subset

Definition 1.3.1. *[63] A triangular norm(T-norm) is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:*

(a) $T(x,1) = x$
(b) $T(x,y) = T(y,x)$
(c) $T(x, T(y,z)) = T(T(x,y), z)$
(d) $T(x,y) \leqslant T(x,z)$ *whenever* $y \leqslant z$, $\forall x,y,z \in [0,1]$

Examples of T-norm are:
(a) Lukasiewicz T-norm $T_{Luk}(x,y) = max\{x+y-1, 0\}, \ \forall x,y \in [0,1]$
(b) Minimum T-norm $T_{min}(x,y) = min(x,y), \ \forall x,y \in [0,1]$
(c) Product T-norm $T_p(x,y) = x.y, \forall x,y \in [0,1]$

(d) Drastic T-norm $T_D(x,y) = \begin{cases} y, \text{ if} x=1 \\ x, \text{ if } y=1 , \forall x,y \in [0,1] \\ 0, \text{ otherwise} \end{cases}$

Definition 1.3.2. *[50] The union of two T-fuzzy subsets ϖ_1 and ϖ_2 of a set X is defined by:*

$$(\varpi_1 \cup \varpi_2)(x) = \max\{\varpi_1(x), \varpi_2(x)\}, \quad \textit{for some } x \in X.$$

Definition 1.3.3. *[50] The intersection of two T-fuzzy subsets ϖ_1 and ϖ_2 of a set X is defined by:*

$$(\varpi_1 \cap \varpi_2)(x) = \min\{\varpi_1(x), \varpi_2(x)\}, \quad \forall x \in X.$$

Definition 1.3.4. *[50] Let ϖ_1 and ϖ_2 be two T-fuzzy subsets of a set X and Y, respectively. The Cartesian product of ϖ_1 and ϖ_2 with respect to T-norm T denoted by $[\varpi_1 \times \varpi_2]_T$ is defined by $[\varpi_1 \times \varpi_2]_T(x,y) = T(\varpi_1(x), \varpi_2(y))$, $\forall (x,y) \in X \times Y$.*

Lemma 1.3.1. *[17, 58] Let T and T' be T-norms. Then $T'(T(p,q), T(r,s)) = T(T'(p,r), T'(q,s))$*

Definition 1.3.5. *[58] Let ϖ_1 and ϖ_2 be fuzzy subset in a set X and T be a T- norm. Then the T-fuzzy product of ϖ_1 and ϖ_2 denoted by $[\varpi_1 . \varpi_2]_T$, is defined by $[\varpi_1 . \varpi_2]_T(x) = T(\varpi_1(x), \varpi_2(x)), \forall x \in X$.*

1.4. Basic Concepts of Interval Valued Fuzzy Subset

Definition 1.4.1. *[11, 40, 53] Let X be a nonempty set. An interval valued fuzzy set (abbreviated as I-V fuzzy set) $\tilde{\phi}$ in X is given by $\tilde{\phi} = \{(x, [\phi^L(x), \phi^U(x)]), x \in X\}$, where ϕ^L and ϕ^U are the lower and upper bound of the membership degree respectively, such that $0 \leqslant \phi^L \leqslant \phi^U \leqslant 1$.*

Remark 1.4.1. *a) Let $\tilde{\phi} = [\phi^L, \phi^U]$, and let $C[0,1]$ denote the family of all closed sub-intervals of $[0,1]$. It is clear that if $\phi^L(x) = \phi^U(x) = a$, where $0 \leqslant a \leqslant 1$, then $\tilde{\phi}(x) = [a,a]$ in $C[0,1]$. Thus $\tilde{\phi}(x)$ in $[0,1]$, $\forall x \in X$. Therefore, the I-V fuzzy set ϕ is given by $\phi = \{(x, \tilde{\phi}(x)), x \in X\}$, where $\tilde{\phi} : X \to C[0,1]$.*

b) We define the refined minimum (abbreviated as $r\min$) and order "$\leqslant$" on elements $C_1 = [a_1, b_1]$ and $C_2 = [a_2, b_2]$ of $C[0,1]$ as follows:

$r\min(C_1, C_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$, $C_1 \leqslant C_2 \Leftrightarrow a_1 \leqslant a_2$ and $b_1 \leqslant b_2$. The same way to define $\geqslant$ and $=$. Also, if $C_j = [a_j, b_j]$, $j \in I$, then we define $r\sup(C_j) = [\sup(a_j), \sup(b_j)]$ and $r\inf(C_j) = [\inf(a_j), \inf(b_j)]$

Definition 1.4.2. *[36] Let $\tilde{\phi}$ be I-V fuzzy subset of a set X and let $t \in [0,1]$. The set $U(\tilde{\phi}, t) = \{x \in X | \tilde{\phi}(x) \geqslant t\}$, is called the upper t-level subset of $\tilde{\phi}$. Clearly, $U(\tilde{\phi}, t) \subseteq U(\tilde{\phi}, s)$ whenever $t > s$. And, the set $L(\tilde{\phi}, t) = \{x \in X | \tilde{\phi}(x) \leqslant t\}$, is called the lower t-level subset of $\tilde{\phi}$. Clearly, $L(\tilde{\phi}, t) \subseteq L(\tilde{\phi}, s)$ whenever $t < s$.*

Definition 1.4.3. *[36] If $\tilde{\phi}$ is an I-V fuzzy set in a set X, then the complement, denoted by $\tilde{\phi}^c$, is the I-V fuzzy subset of X given by $\tilde{\phi}^c(x) = 1 - \tilde{\phi}(x), \forall x \in X$*

Definition 1.4.4. *[36] The union of two I-V fuzzy subsets $\tilde{\phi}_1$ and $\tilde{\phi}_2$ of a set X is defined by:*

$$(\tilde{\phi}_1 \cup \tilde{\phi}_2)(x) = r\max\{\tilde{\phi}_1(x), \tilde{\phi}_2(x)\}, \quad \textit{for all } x \in X.$$

Definition 1.4.5. *[36] The intersection of two I-V fuzzy subsets $\tilde{\phi}_1$ and $\tilde{\phi}_2$ of a set X is defined by:*

$$(\tilde{\phi}_1 \cap \tilde{\phi}_2)(x) = r\min\{\tilde{\phi}_1(x), \tilde{\phi}_2(x)\}, \quad \textit{for all } x \in X.$$

Definition 1.4.6. *[36] Let ϕ_1 and ϕ_2 be two I-V fuzzy subsets in a set X. The Cartesian product of $\phi_1 \times \phi_2 : X \times X \to C[0,1]$, is given by $(\tilde{\phi_1} \times \tilde{\phi_2})(x,y) = r\min\{\tilde{\phi_1}(x), \tilde{\phi_2}(y)\}$, $\forall (x,y) \in X \times X$.*

Lemma 1.4.1. *[10] Let g be a mapping from a set X into a set Y. Let*

$$\phi = \{(x, [\phi^L(x), \phi^U(x)]), x \in X\} \quad \textit{and} \quad \psi = \{(y, [\psi^L(y), \psi^U(y)]), y \in Y\}$$

be I-V fuzzy sets in X and Y, respectively. Then

a) $g^{-1}(\tilde{\phi}) = \{(x, [g^{-1}(\phi^L)(x), g^{-1}(\phi^U)(x)]), x \in X\}$,

b) $g(\tilde{\psi}) = \{(y, [g(\psi^L)(y), g(\psi^U)(y)]), y \in Y\}$.

Definition 1.4.7. *[36] Let $\tilde{\phi}$ be an interval valued fuzzy set in set X and let $\tilde{S} \in C[0,1]$. Then the interval valued fuzzy set $\tilde{\phi}^{\tilde{S}}$ of X is called the $\tilde{S}$-interval valued fuzzy set of X (w.r.t. interval valued fuzzy set $\tilde{\phi}$) and is defined by $\tilde{\phi}^{\tilde{S}}(x) = r\min\{\tilde{\phi}(x), \tilde{S}\}$, $\forall x \in X$.*

Remark 1.4.2. *Clearly, $\tilde{\phi}^{\tilde{1}} = \tilde{\phi}$ and $\tilde{\phi}^{\tilde{0}} = \tilde{0}$, where $\tilde{1} = [1,1]$ and $\tilde{0} = [0,0]$*

1.5. Some Types of Algebras

In this section, we recall the definitions of some algebras used in the study of the dissertation.

Definition 1.5.1. *[23] A BCK-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:*

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = 0$ *and* $y * x = 0$ *implies* $x = y$

5. $0 * x = 0$, *for all* $x, y, z \in X$.

Definition 1.5.2. *[24] A BCI-algebra* $(X, *, 0)$ *is a nonempty set* X *with a constant* 0 *and a binary operation* $*$ *satisfying the following axioms:*

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = 0$ *and* $y * x = 0$ *implies* $x = y$, *for all* $x, y, z \in X$.

Definition 1.5.3. *[38] A B-algebra is a nonempty set* X *with a constant* 0 *and a binary operation* $*$ *satisfying the following axioms:*

1. $x * x = 0$,
2. $x * 0 = x$,
3. $(x * y) * z = x * \big(z * (0 * y)\big)$, *for all* $x, y, z \in X$.

Definition 1.5.4. *[21] A UP-algebra is a nonempty set* A *together with a binary operation* $* : A \times A \to A$ *satisfying the following axioms for all* $x, y, z \in A$*:*

1. $x * x = x$
2. $x * y = y * x$
3. $(x * y) * z = (x * z) * (y * z)$
4. $x * y = x \Rightarrow x = y$

Definition 1.5.5. *[46] A nonempty set* X *with a constant* 0 *and a binary operation* $*$ *is called a PS-algebra if it satisfies the following axioms:*

1. $x * x = 0$
2. $x * 0 = 0$
3. $x * y = 0$ *and* $y * x = 0 \Rightarrow x = y$, *for all* $x, y \in X$.

Definition 1.5.6. *[33] A KU-algebra* $(X, *, 0)$ *is a nonempty set* X *with a constant 0 and a binary operation * satisfying the following conditions:*

1. $(x * y) * ((y * z) * (x * z)) = 0$
2. $x * 0 = 0$
3. $0 * x = x$

4. $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$, *for all* $x, y, z \in X$.

Definition 1.5.7. *[33] A subset* S *of a KU-algebra* X *is called a subalgebra of* X *if* $x * y \in S$, *whenever* $x, y \in S$.

Definition 1.5.8. *[33] A nonempty subset* A *of a KU-algebra* X *is called a KU-ideal of* X *if it satisfies the following conditions:*

1. $0 \in A$,

2. $x * (y * z) \in A$ *and* $y \in A$ *implies* $x * z \in A$, *for all* $x, y, z \in X$.

1.6. Basic Concepts of JU-Algebra

Definition 1.6.1. *[3] A JU-algebra is a structure* $(X; *, 1)$, *where: "$*$" is binary operations on* X *and "1" is an element of* X, $\forall x, y, z \in X$ *verifying the following axioms:*

(a) $(x * y) * [(y * z) * (x * z)] = 1$,

(b) $1 * x = x$,

(c) $x * y = 1$ *and* $y * x = 1 \Rightarrow x = y$, $\forall x, y \in X$.

In a JU-algebra X, we can define a partial ordering "$\leqslant$" by putting $x \leqslant y$ *if and only if* $y * x = 1$.

This feature enabling the algebra to handle uncertainty naturally than the traditional algebras and it makes it distinct and novel.

In what follows, let $(X, *, 1)$ denote a JU-algebra unless otherwise specified. For brevity, we also call X a JU-algebra.

Example 1.6.1. *Let X =* $\{1, 2, 3, 4\}$ *in which* $*$ *is defined by the following Cayley table:*

$*$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

See [3], $(X, *, 1)$ is a JU-algebra

Lemma 1.6.1. *[3, 48] If X is a JU-algebra, then the following properties satisfied for any* $x, y, z \in X$:

1. $x \leqslant y$ *implies* $y * z \leqslant x * z$
2. $x \leqslant y$ *and* $y \leqslant x \Rightarrow x = y$
3. $x * x = 1$
4. $z * (y * x) = y * (z * x)$
5. $(x * [(x * y) * y]) = 1$

Definition 1.6.2. *[51] A nonempty subset S of a JU-algebra X is called a subalgebra of X if* $x * y \in S, \forall x, y \in S$.

Definition 1.6.3. *[51] A nonempty subset J of a JU-algebra X is said to be a JU-ideal of X if it satisfies:*

(i). $1 \in J$

(ii). $x, x * y \in J \Rightarrow y \in J$, $\forall x, y \in X$.

Example 1.6.2. *Let X =* $\{1,2,3,4,5,6\}$ *in which* $*$ *is defined by the following Cayley table:*

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	6
6	1	1	2	1	1	1

See [3], $(X, *, 1)$ *is a JU-algebra. It is easy to show that* $J_1 = \{1,2\}$ *and* $J_2 = \{1,2,3,4,5\}$ *are JU-ideals of* X.

Definition 1.6.4. *[3, 41, 46] Let* X *and* Y *be any two JU-algebras. Then a mapping* $g \colon X \to Y$ *is said to be a homomorphism of JU-algebras if*

$$g(x * y) = g(x) * g(y) \quad \textit{for all } x, y \in X.$$

Definition 1.6.5. *[3, 41, 46] Let* $g \colon X \to Y$ *be a homomorphism of JU-algebras. Then* g *is called*

(i). a monomorphism if it is one-to-one,

(ii). an epimorphism if it is onto, and

(iii). an endomorphism if it is a mapping from a JU-algebra X *to itself.*

Note: *If* g *is a homomorphism of JU-algebras, then* $g(1) = 1$.

Definition 1.6.6. *[3, 41, 46] If* g *is a one-to-one and onto homomorphism from a JU-algebra* X *into itself, then* g *is called an automorphism of JU-algebras. The set of all automorphisms of a JU-algebra* X *is denoted by* $\mathrm{Aut}(X)$.

Theorem 1.6.2. *[3, 51] Let* $g \colon X \to Y$ *be a homomorphism of JU-algebras. Then*

(i). If S *is a JU-subalgebra of* X*, then* $g(S)$ *is a JU-subalgebra of* Y.

(ii). If S *is a JU-subalgebra of* Y*, then* $g^{-1}(S)$ *is a JU-subalgebra of* X.

(iii). If J *is a JU-ideal of* X*, then* $g(J)$ *is a JU-ideal of* Y.

(iv). If J *is a JU-ideal of* Y*, then* $g^{-1}(J)$ *is a JU-ideal of* X.

Chapter 2
Fuzzy Structures on JU-Algebra

In this chapter, we explore the fuzzy extension of JU-subalgebra and JU-ideal of JU-algebra and obtain some results. A fuzzy JU-subalgebra and a fuzzy JU-ideal of a JU-algebra are characterized in terms of their level subsets. We also investigate the concepts of prime fuzzy sets and fuzzy p-translation and p-multiplicative sets in the context of JU-algebras. Moreover, we discuss the cartesian product and homomorphism of fuzzy JU-subalgebra and fuzzy JU-ideal of JU-algebras.

2.1. Fuzzy Subalgebras of JU-Algebra

In this section, we discuss and investigate a new notion of fuzzy JU-subalgebras in JU-algebras and study their related properties.

Definition 2.1.1. *Let X be a JU-algebra, a fuzzy set* ϑ *in X is called a fuzzy JU-subalgebra of X if for all* $x, y \in X$, $\vartheta(x * y) \geqslant \min\{\vartheta(x), \vartheta(y)\}$, *where* $*$ *is the binary operation in* X.

Example 2.1.1. *Consider X* = $\{1,2,3,4,5\}$, *we construct the following table.*

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

See [3] $(X;*,1)$ is a JU-algebra. Define a fuzzy set $\vartheta : X \to [0,1]$ by

$$\vartheta(x) = \begin{cases} 0.7, \text{ if } x = 1 \\ 0.5, \text{ if } x = 2 \\ 0.3, \text{ otherwise} \end{cases}$$

To verify ϑ is a fuzzy JU-subalgebra, let we take
for $x = 1, y = 2$, then $\vartheta(1 * 2) = \vartheta(2) = 0.5 \cdots (*)$ and
$\min\{\vartheta(1), \vartheta(2)\} = \min\{0.7, 0.5\} = 0.5 \cdots (**)$,

From (*) and (**) we get $0.5 \geqslant 0.5$ which is true.
for $x = 4, y = 1$, then $\vartheta(4 * 1) = \vartheta(1) = 0.7 \cdots (***)$ and
$\min\{\vartheta(4), \vartheta(1)\} = \min\{0.3, 0.7\} = 0.3 \cdots (****)$,
From (***) and (****) we get $0.7 \geqslant 0.3$ which is true.
Similarly, the result holds for all values of x and y in X.
Therefore, we conclude that ϑ is a fuzzy JU-subalgebra of $(X, *, 1)$.

Lemma 2.1.1. *If ϑ is a fuzzy JU-subalgebra of a JU-algebra X. Then $\vartheta(1) \geqslant \vartheta(x)$, $\forall x \in X$.*

Proof. Suppose ϑ be a fuzzy JU-subalgebra of a JU-algebra X.
From Lemma 1.6.1 we get $\vartheta(1) = \vartheta(x * x) \geqslant \min\{\vartheta(x), \vartheta(x)\} = \vartheta(x)$ □

Theorem 2.1.2. *If ϑ and λ are fuzzy JU-subalgebras of JU-algebra X, then $\vartheta \cap \lambda$ is also a fuzzy JU-subalgebra of X.*

Proof. Suppose ϑ and λ are fuzzy JU-subalgebras of a JU-algebra X. Let $x, y \in X$. Then

$$\begin{aligned}
(\vartheta \cap \lambda)(x * y) &= \min\{\vartheta(x * y), \lambda(x * y)\} \\
&\geqslant \min\{\min\{\vartheta(x), \vartheta(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\
&= \min\{\min\{\vartheta(x), \lambda(x)\}, \min\{\vartheta(y), \lambda(y)\}\} \\
&= \min\{(\vartheta \cap \lambda)(x), (\vartheta \cap \lambda)(y)\}
\end{aligned}$$

□

Corollary 2.1.3. *Let $\{\vartheta_i | i \in \Omega\}$ be a family of fuzzy JU-subalgebras of a JU-algebra X. Then $\bigcap_{i \in \Omega} \vartheta_i$ is also a fuzzy JU-subalgebra of X. where $\bigcap_{i \in \Omega} \vartheta_i(x) = \inf_{i \in \Omega}\{\vartheta_i(x)\}$, $\forall x \in X$.*

Proof. Let $\{\vartheta_i | i \in \Omega\}$ be a family of JU-subalgebras of X, then for any $x, y \in X$.

$$\begin{aligned}
(\cap_{i \in \Omega} \vartheta_i)(x * y) &= \inf_{i \in \Omega}\{\vartheta_i(x * y)\} \\
&\geqslant \inf_{i \in \Omega}\{\min\{\vartheta_i(x), \vartheta_i(y)\}\} \\
&= \min\{\inf_{i \in \Omega}\{\vartheta_i(x)\}, \inf_{i \in \Omega}\{\vartheta_i(y)\}\} \\
&= \min\{(\cap_{i \in \Omega} \vartheta_i)(x), (\cap_{i \in \Omega} \vartheta_i)(y)\}
\end{aligned}$$

Therefore, $(\cap_{i \in \Omega} \vartheta_i)(x * y) \geqslant \min\{(\cap_{i \in \Omega} \vartheta_i)(x), (\cap_{i \in \Omega} \vartheta_i)(y)\}$, hence $\bigcap_{i \in \Omega} \vartheta_i$ is a fuzzy JU-subalgebra of X. □

Remark 2.1.1. *The union of any two fuzzy JU-subalgebras of a JU-algebra X may not be a fuzzy JU-subalgebra of X.*

Example 2.1.2. *Let X = $\{1, 2, 3, 4\}$ in which $*$ is defined by the following table.*

See [3] $(X; *, 1)$ is a JU-algebra of X. We define $\vartheta : X \to [0, 1]$ as follows:

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

$$\vartheta(x) = \begin{cases} 0.8, if\ x \in \{1,3\} \\ 0.6,\ otherwise \end{cases}$$

And define $\lambda : X \to [0,1]$ as follows:

$$\lambda(x) = \begin{cases} 0.9, if\ x = 1 \\ 0.7, if\ x = 2 \\ 0.5,\ otherwise \end{cases}$$

Then,

$$\begin{aligned}(\vartheta \cup \lambda)(2*3) &= \max\{\vartheta(2*3), \lambda(2*3)\} \\ &= \max\{\vartheta(4), \lambda(4)\} \\ &= max\{0.6, 0.5\} = 0.6 \cdots (*)\end{aligned}$$

And,

$$\begin{aligned}\max\{\vartheta(2*3), \lambda(2*3)\} &\geqslant \max\{min\{\vartheta(2), \vartheta(3)\}, min\{\lambda(2), \lambda(3)\}\} \\ &= \min\{max\{\vartheta(2), \lambda(2)\}, max\{\vartheta(3), \lambda(3)\}\} \\ &= \min\{max\{0.6, 0.7\}, max\{0.8, 0.5\}\} \\ &= \min\{0.7, 0.8\} = 0.7 \cdots (**)\end{aligned}$$

From $(*)$ and $(**)$ we get $0.6 \geqslant 0.7$ which is false.
Therefore, we conclude that the union of any two fuzzy JU-subalgebras of a JU-algebra X may not be a fuzzy JU-subalgebra of X.

Definition 2.1.2. *Let ϑ be a fuzzy set in a JU-algebra X. Then ϑ is called anti-fuzzy JU-subalgebra of X if for all $x, y \in X$, $\vartheta(x*y) \leqslant \max\{\vartheta(x), \vartheta(y)\}$.*

Theorem 2.1.4. *A fuzzy set ϑ is a fuzzy JU-subalgebra of X if and only if its complement ϑ^c is an anti-fuzzy JU-subalgebra of X.*

Proof. Suppose ϑ fuzzy JU-subalgebra of X. We claim that ϑ^c is anti-fuzzy JU-subalgebra of X. Let $x, y \in X$, then

$\vartheta^c(x*y) = 1 - \vartheta(x*y)$

$\leqslant 1-\min\{\vartheta(x),\vartheta(y)\}$
$= \max\{1-\vartheta(x),1-\vartheta(y)\}$ (By Lemma 1.1.1)
$= \max\{\vartheta^c(x),\vartheta^c(y)\}$

Thus, ϑ^c is anti- fuzzy JU-subalgebra of X.

Conversely, Suppose ϑ^c is anti-fuzzy JU-subalgebra of X. We claim that ϑ is fuzzy JU-subalgebra of X. Let $x,y \in X$, then

$1-\vartheta(x*y) = \vartheta^c(x*y)$
$\leqslant \max\{\vartheta^c(x),\vartheta^c(y)\}$
$= \max\{1-\vartheta(x),1-\vartheta(y)\}$
$= 1-\min\{\vartheta(x),\vartheta(y)\}$ (By Lemma 1.1.1)

Thus, $\vartheta(x*y) \geqslant \min\{\vartheta(x),\vartheta(y)\}$ □

2.1.1 Fuzzy p-Translation and Fuzzy p-Multiplication of JU-Subalgebras

Let X be a JU-algebra. For any fuzzy set ϑ of X, we define $\Delta = \inf\{1-\vartheta(x) | x \in X\}$, unless otherwise we specified.

Definition 2.1.1.1. *Let ϑ be a fuzzy subset of X and $p \in [0,\Delta]$. A mapping $\vartheta_p^T : X \to [0,1]$ is said to be a fuzzy p-translation of ϑ if it satisfies $\vartheta_p^T(x) = \vartheta(x) + p$, $\forall x \in X$.*

Definition 2.1.1.2. *Let ϑ be a fuzzy subset of X and $p \in [0,\Delta]$. A mapping $\vartheta_p^M : X \to [0,1]$ is said to be a fuzzy p-multiplication of ϑ if it satisfies $\vartheta_p^M(x) = (p)(\vartheta)(x)$, $\forall x \in X$.*

Remark 2.1.1.1. *In Definition 2.1.1.2, we use $p \in [0,\Delta]$, but for every $p \in [0,1]$, $\vartheta_p^M(x) = (p)(\vartheta)(x)$ is also a fuzzy subset of X.*

Example 2.1.1.1. *Let X = {1,2,3,4} in which $*$ is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

See [3], $(X,*,1)$ is a JU-algebra. Define a mapping $\vartheta : X \to [0,1]$ as follows $\vartheta(1) = 0.8, \vartheta(2) = 0.7, \vartheta(3) = 0.6, \vartheta(4) = 0.4$. Then $\Delta = \inf\{1-0.8, 1-0.7, 1-0.6, 1-0.4\} = 0.2$. Let $p = 0.1 \in [0,0.2]$, then a mapping $\vartheta_p^T : X \to [0,1]$ defined by $\vartheta_p^T(1) = 0.9, \vartheta_p^T(2) = 0.8, \vartheta_p^T(3) = 0.7$, $\vartheta_p^T(4) = 0.5$ And a mapping $\vartheta_p^M : X \to [0,1]$ defined by $\vartheta_p^M(1) = 0.08, \vartheta_p^M(2) = 0.07, \vartheta_p^M(3) = 0.06$, $\vartheta_p^M(4) = 0.04$

Therefore, for p= $0.1 \in [0,0.2]$, both the fuzzy translation ϑ_p^T and fuzzy multiplication ϑ_p^M of ϑ satisfy the fuzzy set membership conditions in X.

Theorem 2.1.1.1. *Let ϑ be a fuzzy subset of a JU-algebra X and $p \in [0,\Delta]$. Then, ϑ is a fuzzy JU-subalgebra of X if and only if the fuzzy p-translation $\vartheta_p^T(x)$ of ϑ is also a fuzzy JU-subalgebra of X.*

Proof. Suppose that ϑ is a fuzzy JU-subalgebra of X and $p \in [0, \Delta]$. We claim that ϑ_p^T is a fuzzy JU-subalgebra of X. For any $x, y \in X$, then

$$\begin{aligned}\vartheta_p^T(x*y) &= \vartheta(x*y) + p \\ &\geqslant \min\{\vartheta(x), \vartheta(y)\} + p \\ &= \min\{\vartheta(x) + p, \vartheta(y) + p\} \\ &= \min\{\vartheta_p^T(x), \vartheta_p^T(y)\}\end{aligned}$$

Thus, $\vartheta_p^T(x*y) \geqslant \min\{\vartheta_p^T(x), \vartheta_p^T(y)\}$, hence ϑ_p^T is a fuzzy JU-subalgebra of X.
Conversely, suppose ϑ_p^T is a fuzzy JU-subalgebra of X and for all $p \in [0, \Delta]$. We claim that ϑ is a fuzzy JU-subalgebra of X. For any $x, y \in X$, then

$$\begin{aligned}\vartheta(x*y) + p &= \vartheta_p^T(x*y) \\ &\geqslant \min\{\vartheta_p^T(x), \vartheta_p^T(y)\} \\ &= \min\{\vartheta(x) + p, \vartheta(y) + p\} \\ &= \min\{\vartheta(x), \vartheta(y)\} + p\end{aligned}$$

By subtracting p both sides, we get $\vartheta(x*y) \geqslant \min\{\vartheta(x), \vartheta(y)\}$. Hence, ϑ is a fuzzy JU-subalgebra of X. □

Theorem 2.1.1.2. *Let ϑ be a fuzzy JU-subalgebra of a JU-algebra X, and for every $p \in [0, \Delta]$. Then the fuzzy p-multiplicative ϑ_p^M of ϑ is a fuzzy JU-subalgebra of X.*

Proof. Suppose ϑ is a fuzzy subalgebra of a JU-algebra X and for any $p \in [0, \Delta]$. Then $\forall x, y \in X$,

$$\begin{aligned}\vartheta_p^M(x*y) &= (p)(\vartheta(x*y)) \\ &\geqslant (p)(\min\{\vartheta(x), \vartheta(y)\}) \\ &= \min\{(p)(\vartheta(x)), (p)(\vartheta(y))\} \\ &= \min\{\vartheta_p^M(x), \vartheta_p^M(y)\}.\end{aligned}$$

□

Definition 2.1.1.3. *A fuzzy JU-subalgebra ϑ of a JU- algebra X is called prime fuzzy JU-subalgebra of X if, ϑ is prime fuzzy set in X.*

Theorem 2.1.1.3. *Let ϑ be a fuzzy subset of a JU-algebra X and $p \in [0, \Delta]$. Then, ϑ is a prime fuzzy JU-subalgebra of X if and only if the prime fuzzy p-translation $\vartheta_p^T(x)$ of ϑ is also a prime fuzzy JU-subalgebra of X.*

Proof. Suppose that ϑ is a prime fuzzy JU-subalgebra of X and $p \in [0, \Delta]$. We claim that ϑ_p^T is a prime fuzzy JU-subalgebra of X. For any $x, y \in X$, then

$$\vartheta_p^T(x*y) = \vartheta(x*y) + p$$

$$
\begin{aligned}
&\leqslant \max\{\vartheta(x),\vartheta(y)\}+p\\
&=\max\{\vartheta(x)+p,\vartheta(y)+p\}\\
&=\max\{\vartheta_p^T(x),\vartheta_p^T(y)\}
\end{aligned}
$$

Thus, $\vartheta_p^T(x*y)\leqslant \max\{\vartheta_p^T(x),\vartheta_p^T(y)\}$, hence ϑ_p^T is a prime fuzzy JU-subalgebra of X.
Conversely, suppose ϑ_p^T is a prime fuzzy JU-subalgebra of X and for $p\in[0,\Delta]$. We claim that ϑ is a prime fuzzy JU-subalgebra of X. For any $x,y\in X$, then

$$
\begin{aligned}
\vartheta(x*y)+p&=\vartheta_p^T(x*y)\\
&\leqslant \max\{\vartheta_p^T(x),\vartheta_p^T(y)\}\\
&=\max\{\vartheta(x)+p,\vartheta(y)+p\}\\
&=\max\{\vartheta(x),\vartheta(y)\}+p
\end{aligned}
$$

By subtracting p both sides, we get $\vartheta(x*y)\leqslant \max\{\vartheta(x),\vartheta(y)\}$. Hence, ϑ is a prime fuzzy JU-subalgebra of X. □

Theorem 2.1.1.4. *If ϑ is a prime fuzzy JU-subalgebra of a JU-algebra X, then the fuzzy p-multiplicative ϑ_p^M of ϑ is a prime fuzzy JU-subalgebra of X for all $p\in[0,\Delta]$.*

Proof. Let ϑ be a prime fuzzy JU-subalgebra of a JU-algebra X and $p\in[0,\Delta]$.Then

$$
\begin{aligned}
\vartheta_p^M(x*y)&=(p)(\vartheta(x*y))\\
&\leqslant (p)(\max\{\vartheta(x),\vartheta(y)\})\\
&=\max\{(p)(\vartheta(x)),(p)(\vartheta(y))\}\\
&=\max\{\vartheta_p^M(x),\vartheta_p^M(y)\}
\end{aligned}
$$

□

Theorem 2.1.1.5. *Let $p\in(0,\Delta)$ such that the fuzzy p-multiplicative ϑ_p^M of ϑ is a prime fuzzy JU-subalgebra of a JU-algebra X, then ϑ is a prime fuzzy JU-subalgebra of X.*

Proof. Assume that ϑ_p^M of ϑ is a prime fuzzy JU-subalgebra of a JU-algebra X, for some $p\in(0,\Delta)$ and for any $x,y\in X$. Then

$$
\begin{aligned}
(p)(\vartheta(x*y))&=\vartheta_p^M(x*y)\\
&\leqslant \max\{\vartheta_p^M(x),\vartheta_p^M(y)\}\\
&=\max\{(p)(\vartheta(x)),(p)(\vartheta(y))\}\\
&=(p)(\max\{\vartheta(x),\vartheta(y)\})
\end{aligned}
$$

Dividing both sides by p we get, $\vartheta(x*y)\leqslant \max\{\vartheta(x),\vartheta(y)\}$. Hence, ϑ is a prime fuzzy JU-subalgebra of X. □

Definition 2.1.1.4. *Let ϑ be a fuzzy subset of* X, $p \in [0,\Delta]$ *and* $q \in [0,1]$. *A mapping* $\vartheta^{MT}_{(p,q)} : X \to [0,1]$ *is said to be a fuzzy magnified-*pq*-translation of* ϑ *if it satisfies:*

$$\vartheta^{MT}_{(p,q)}(x) = (q)(\vartheta(x)) + p.$$

Example 2.1.1.2. *Let X =* $\{1,2,3,4\}$ *in which* $*$ *is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

See [3], $(X,*,1)$ *is a JU-algebra. Define a mapping* $\vartheta : X \to [0,1]$ *as follows* $\vartheta(1) = 0.8, \vartheta(2) = 0.7, \vartheta(3) = 0.6, \vartheta(4) = 0.4$. *Then,* $\Delta = \inf\{1-0.8, 1-0.7, 1-0.6, 1-0.4\} = 0.2$. *Let* $p = 0.1 \in [0,0.2]$, *then a mapping* $\vartheta^T_p : X \to [0,1]$ *defined by* $\vartheta^T_p(1) = 0.9, \vartheta^T_p(2) = 0.8, \vartheta^T_p(3) = 0.7$, $\vartheta^T_p(4) = 0.5$ *and* $q = 0.6 \in [0,1]$. *Then, the mapping* $\vartheta^{MT}_{(0.1,0.6)} : X \to [0,1]$ *is defined by* $\vartheta^{MT}_{(0.1,0.6)}(1) = (0.6)(0.8) + 0.1 = 0.48 + 0.1 = 0.58$. $\vartheta^{MT}_{(0.1,0.6)}(2) = (0.6)(0.7) + 0.1 = 0.42 + 0.1 = 0.52$. $\vartheta^{MT}_{(0.1,0.6)}(3) = (0.6)(0.6) + 0.1 = 0.36 + 0.1 = 0.46$. $\vartheta^{MT}_{(0.1,0.6)}(4) = (0.6)(0.4) + 0.1 = 0.24 + 0.1 = 0.34$, *which satisfies* $\vartheta^{MT}_{(0.1,0.6)}(x) = (q)(\vartheta)(x) + p$ *for all* $x \in X$ *and is a fuzzy magnified-*$(0.1)(0.6)$*-translation.*

Theorem 2.1.1.6. *Let* ϑ *be a fuzzy subset of* X, $p \in [0,\Delta]$ *and* $q \in [0,1]$. *Then the fuzzy magnified-*pq*-translation* $\vartheta^{MT}_{(p,q)}$ *of* ϑ *is a fuzzy JU-subalgebra of* X.

Proof. Suppose that ϑ be a fuzzy JU-subalgebra of X, $p \in [0,\Delta]$ and $q \in [0,1]$. We claim that $\vartheta^{MT}_{(p,q)}$ is a fuzzy JU-subalgebra of X. For all $x, y \in X$, then

$$\begin{aligned}
\vartheta^{MT}_{(p,q)}(x * y) &= (q)(\vartheta(x * y)) + p \\
&\geqslant (q)(\min\{\vartheta(x), \vartheta(y)\}) + p \\
&= \min\{(q)(\vartheta(x)) + p, (q)(\vartheta(y)) + p\} \\
&= \min\{\vartheta^{MT}_{(p,q)}(x), \vartheta^{MT}_{(p,q)}(y)\}.
\end{aligned}$$

Hence, $\vartheta^{MT}_{(p,q)}$ is a fuzzy JU-subalgebra of X. □

Theorem 2.1.1.7. *If there exist* $q \in (0,1]$ *and for all* $p \in [0,\Delta]$ *such that the fuzzy magnified-*pq*-translation* $\vartheta^{MT}_{(p,q)}$ *of* ϑ *is a fuzzy JU-subalgebra of* X, *then* ϑ *is a fuzzy JU-subalgebra of X.*

Proof. Suppose that $\vartheta^{MT}_{(p,q)}$ is a fuzzy JU-subalgebra of X, for all $p \in [0,\Delta]$ and for some $q \in (0,1]$. We claim that ϑ is a fuzzy JU-subalgebra of X. For all $x, y \in X$, then

$$\begin{aligned}
(q)(\vartheta(x * y)) + p &= \vartheta^{MT}_{(p,q)}(x * y) \\
&\geqslant \min\{\vartheta^{MT}_{(p,q)}(x), \vartheta^{MT}_{(p,q)}(y)\} \\
&= \min\{(q)(\vartheta(x)) + p, (q)(\vartheta(y)) + p\}
\end{aligned}$$

$$= (q)(\min\{\vartheta(x), \vartheta(y)\} + p$$

Subtracting p from both sides and dividing by q gives

$$\vartheta(x * y) \geqslant \min\{\vartheta(x), \vartheta(y)\}$$

Therefore ϑ is a fuzzy JU-subalgebra of X. □

2.1.2 Level Subsets of Fuzzy JU-Subalgebras

In this section, we introduce the concept of level subsets for fuzzy JU-subalgebras within JU-algebras.

Theorem 2.1.2.1. *If ϑ is a fuzzy JU-subalgebra of X, then the set $X_\vartheta = \{x \in X | \vartheta(x) = \vartheta(1)\}$ is a JU-subalgebra of X.*

Proof. Since X is a JU-algebra, $1 \in X$ and clearly $1 \in X_\vartheta$. This shows that X_ϑ is nonempty. Now let $x, y \in X_\vartheta$. Then $\vartheta(x) = \vartheta(1) = \vartheta(y)$, and so

$$\begin{aligned} \vartheta(x * y) &\geqslant \min\{\vartheta(x), \vartheta(y)\} = \min\{\vartheta(1), \vartheta(1)\}. \\ &\Rightarrow \vartheta(x * y) \geqslant \vartheta(1) \end{aligned}$$

But,

$$\begin{aligned} \vartheta(x * y) &\leqslant \vartheta(1), \forall x, y \in X \quad \text{(by Lemma 2.1.1)} \\ &\Rightarrow \vartheta(x * y) = \vartheta(1) \\ &\Rightarrow x * y \in X_\vartheta. \end{aligned}$$

Hence, X_ϑ is a JU-subalgebra of X. □

Theorem 2.1.2.2. *Let S be a subset of a JU-algebra X. Then the characteristics function χ_S is a fuzzy JU-subalgebra of X if and only if S is a JU-subalgebra of X.*

Proof. Let $\chi_S : X \to \{0, 1\}$ be a fuzzy JU-subalgebra of a JU-algebra X and let $x, y \in S$ implies that $\chi_S(x) = \chi_S(y) = 1$. Then, $\chi_S(x * y) \geqslant \min\{\chi_S(x), \chi_S(y)\} = \min\{1, 1\}$, implies that $\chi_S(x * y) \geqslant 1$ but, $\chi_S(x * y) \leqslant 1$, $\chi_S(x * y) = 1$
Therefore $x * y \in S$, implies that S is a JU-subalgebra of X.
Conversely, let S be a JU-subalgebra of X and $x, y \in X$. Consider the following cases:

Case 1. if $x, y \in S$, then $x * y \in S$, we get that

$$\chi_S(x * y) = 1 = \min\{\chi_S(x), \chi_S(y)\}$$

Case 2. if $x, y \notin S$, then $\chi_S(x) = 0 = \chi_S(y)$

Thus, $\chi_S(x * y) = 0 = \min\{\chi_S(x), \chi_S(y)\}$

Case 3. if $x \in S$ and $y \notin S$, then $\chi_S(x) = 1$ and $\chi_S(y) = 0$

Then, $\chi_S(x * y) = 0 = \min\{1,0\} = \min\{\chi_S(x), \chi_S(y)\}$

Or, if $x \notin S$ and $y \in S$, then $\chi_S(x) = 0$ and $\chi_S(y) = 1$

Then, $\chi_S(x * y) = 0 = \min\{0,1\} = \min\{\chi_S(x), \chi_S(y)\}$

Hence, χ_S is a fuzzy JU-subalgebra of X. □

Theorem 2.1.2.3. *Let* S *be a subset of* X *and let* ϑ *be a fuzzy set on* X*, given by*

$$\vartheta(x) = \begin{cases} \alpha & \text{if } x \in S, \\ \beta & \text{otherwise} \end{cases}$$

for all $\alpha, \beta \in [0,1]$ *with* $\alpha \geqslant \beta$. *Then* ϑ *is a fuzzy JU-subalgebra of* X *if and only if* S *is a JU-subalgebra of* X. *Moreover, in this case,* $X_\vartheta = S$.

Proof. Suppose ϑ is a fuzzy JU-subalgebra of X. For any $x, y \in S$, we have $\vartheta(x) = \alpha = \vartheta(y)$. Then:

$$\vartheta(x * y) \geqslant \min\{\vartheta(x), \vartheta(y)\} = \min\{\alpha, \alpha\} = \alpha,$$
$$\Rightarrow \vartheta(x * y) \geqslant \alpha.$$

But from $\vartheta(x)$ definition, the maximum membership value is α, so:

$$\vartheta(x * y) \leqslant \alpha \quad \Rightarrow \quad \vartheta(x * y) = \alpha.$$

Hence, $x * y \in S$. Therefore, S is a JU-subalgebra of X.
Conversely, suppose that S is a JU-subalgebra of X, let $x, y \in X$. Let us consider the following cases:

Case 1. if $x, y \in S$ then $x * y \in S$, thus $\vartheta(x * y) \geqslant \alpha = \min\{\vartheta(x), \vartheta(y)\}$

Case 2. if $x \notin S$ or $y \notin S$, then $\vartheta(x * y) \geqslant \beta = \min\{\vartheta(x), \vartheta(y)\}$

Case 3. if $x \in S$ or $y \notin S$, then $\vartheta(x * y) \geqslant \beta = \min\{\alpha, \beta\} = \min\{\vartheta(x), \vartheta(y)\}$

Or, if $x \notin S$ or $y \in S$, then $\vartheta(x * y) \geqslant \beta = \min\{\beta, \alpha\} = \min\{\vartheta(x), \vartheta(y)\}$

Thus ϑ is a fuzzy JU-subalgebra of X.
Moreover, we have $X_\vartheta = \{x \in X \mid \vartheta(x) = \vartheta(1)\} = \{x \in X | \vartheta(x) = \alpha\} = S$ □

Theorem 2.1.2.4. *A fuzzy set* ϑ *of a JU-algebra X is a fuzzy JU-subalgebra of X if and only if for every* $t \in [0,1]$, $U(\vartheta, t) = \{x \in X \mid \vartheta(x) \geqslant t\}$ *is either empty set or a JU-subalgebra of X.*

Proof. Suppose that ϑ is a fuzzy JU-subalgebra of a JU-algebra X such that $U(\vartheta,t) \neq \emptyset$. We need to show that $U(\vartheta,t)$ is a JU-subalgebra of X. Now, for any $x,y \in U(\vartheta,t)$, we have $\vartheta(x) \geqslant t$ and $\vartheta(y) \geqslant t$

Then $\vartheta(x*y) \geqslant \min\{\vartheta(x),\vartheta(y)\} \geqslant \min\{t,t\} = t$

Therefore, $x*y \in U(\vartheta,t)$. Hence, $U(\vartheta,t)$ is a JU-subalgebra of X.

Conversely, assume that $U(\vartheta,t)$ is a JU-subalgebra of X and $U(\vartheta,t) \neq \emptyset$. We need to show that ϑ is a fuzzy JU-subalgebra of X.

Let $x,y \in X$ and there exist $t_1,t_2 \in [0,1]$ such that $\vartheta(x) = t_1$ and $\vartheta(y) = t_2$. Take, $t = \min\{t_1,t_2\}$.

Then $x \in U(\vartheta,t_1)$ and $y \in U(\vartheta,t_2)$.

Choose $t = \min\{t_1,t_2\}$, then $t \leqslant t_1$ and $t \leqslant t_2$.

$\Rightarrow U(\vartheta,t_1) \subseteq U(\vartheta,t)$ and $U(\vartheta,t_2) \subseteq U(\vartheta,t)$

$\Rightarrow x,y \in U(\vartheta,t)$

By assumption, $U(\vartheta,t)$ is a JU-subalgebra of X, which implies that $x*y \in U(\vartheta,t)$. Then $\vartheta(x*y) \geqslant t = \min\{t_1,t_2\} = \min\{\vartheta(x),\vartheta(y)\}$.

Therefore, ϑ is a fuzzy JU-subalgebra of X. □

Theorem 2.1.2.5. *Let ϑ be a fuzzy JU-subalgebra of a JU-algebra X. Two upper-level JU-subalgebras $U(\vartheta,t_1)$ and $U(\vartheta,t_2)$ (with $t_1 < t_2$) of a fuzzy JU-subalgbera ϑ are equal if and only if there is no $x \in X$ such that $t_1 \leqslant \vartheta(x) < t_2$.*

Proof. Let ϑ be a fuzzy JU-subalgebra of a JU-algebra X. Suppose $U(\vartheta,t_1) = U(\vartheta,t_2)$ (with $t_1 < t_2$). Then we claim that there is no $x \in X$ such that $t_1 \leqslant \vartheta(x) < t_2$.

Assume that if there exist $x \in X$ such that $t_1 \leqslant \vartheta(x) < t_2$

$\Rightarrow x \in U(\vartheta,t_1)$, but $x \notin U(\vartheta,t_2)$

$\Rightarrow U(\vartheta,t_1)$ is a proper subset of $U(\vartheta,t_2)$, which is contradict to our assumption that is $U(\vartheta,t_1) = U(\vartheta,t_2)$.

Conversely, assume that there is no $x \in X$ such that $t_1 \leqslant \vartheta(x) < t_2$. We went to show that $U(\vartheta,t_1) = U(\vartheta,t_2)$.

Since $t_1 < t_2$, we get $U(\vartheta,t_2) \subseteq U(\vartheta,t_1) \cdots (*)$

Now, $x \in U(\vartheta,t_1) \Rightarrow \vartheta(x) \geqslant t_1 \Rightarrow \vartheta(x) > t_2$, (Since, $\vartheta(x)$ does not lie between t_1 and t_2)

$\Rightarrow x \in U(\vartheta,t_2)$

Hence, $U(\vartheta,t_1) \subseteq U(\vartheta,t_2) \cdots (**)$

From (*) and (**) we get $U(\vartheta,t_1) = U(\vartheta,t_2)$ □

Remark 2.1.2.1. *Following Theorem 2.1.2.5, the level JU-subalgebras of a fuzzy JU-subalgebra ϑ in a finite JU-algebra X form a chain:*

$U(\vartheta,t_0) \subset U(\vartheta,t_1) \subset U(\vartheta,t_2) \subset \cdots \subset U(\vartheta,t_n) = X$, *where* $t_0 > t_1 > t_2 > \cdots > t_n$.

Corollary 2.1.2.6. *If ϑ is a fuzzy JU-subalgebra of finite JU-algebra X with $\mathrm{Im}(\vartheta) = \{t_1,\ldots,t_n\}$, then the family of JU-subalgebras $\{U(\vartheta,t_i) \mid 1 \leqslant i \leqslant n\}$ constitutes all upper level JU-subalgebras of ϑ in X.*

Proof. Let ϑ be a fuzzy JU-subalgebra of a JU-algebra X such that $\mathrm{Im}(\vartheta) = \{t_1, t_2, \ldots, t_n\}$ with $t_1 < \cdots < t_n$. Let $t \in [0,1]$ and $t \notin \mathrm{Im}(\vartheta)$, then consider the following cases.

Case 1. If $t \leqslant t_1$, then $U(\vartheta, t_1) = X = U(\vartheta, t)$.

Case 2. If $t > t_n$, then $U(\vartheta, t) = \{x \in X \mid \vartheta(x) \geqslant t\} = \{x \in X \mid \vartheta(x) \geqslant t_n\} = \emptyset$.

Case 3. If $t_{i-1} < t < t_i$, then $U(\vartheta, t) = U(\vartheta, t_i)$ by Theorem 2.1.2.5, since there is no $x \in X$ such that $t \leqslant \vartheta(x) < t_i$.

Thus for any $t \in [0,1]$, the level JU-subalgebra is one of $\{U(\mu, t_i) \mid i = 1, 2, \ldots, n\}$. □

Theorem 2.1.2.7. *Let ϑ be a fuzzy JU-subalgebra of a JU-algebra X. Then ϑ^c is an anti-fuzzy JU-subalgebra of X if and only if, for every $s \in [0,1]$, the set $L(\vartheta^c, s) = \{x \in X \mid \vartheta^c(x) \leqslant s\}$ is either empty or a JU-subalgebra of X.*

Proof. Suppose that ϑ^c is an anti-fuzzy JU-subalgebra of X and $L(\vartheta^c, s)$ is nonempty. We need to show that $L(\vartheta^c, s)$ is a JU-subalgebra of X. Now, for all $x, y \in L(\vartheta^c, s)$, then by Definition 1.1.6 we have $\vartheta^c(x) \leqslant s$ and $\vartheta^c(y) \leqslant s$.
Since ϑ^c is anti-fuzzy, $\vartheta^c(x * y) \leqslant \max\{\vartheta^c(x), \vartheta^c(y)\} \leqslant \max\{s, s\} = s$.
Thus, $x * y \in L(\vartheta^c, s)$. Hence, $L(\vartheta^c, s)$ is a JU-subalgebra.
Conversely, assume that $L(\vartheta^c, s)$ is a JU-subalgebra of JU-algebra X and $L(\vartheta^c, s) \neq \emptyset$. We need to show that ϑ^c is anti-fuzzy JU-subalgebra of X.
Let $x, y \in X$ with $s_1 = \vartheta^c(x)$ and $s_2 = \vartheta^c(y)$. Take $s = \max\{s_1, s_2\}$, for $s_1, s_2 \in [0,1]$. Then $x \in L(\vartheta^c, s_1)$ and $y \in L(\vartheta^c, s_2)$.
Choose $s = \max\{s_1, s_2\}$, then $s_1 \leqslant s$ and $s_2 \leqslant s$.

$\Rightarrow L(\vartheta^c, s_1) \subseteq L(\vartheta^c, s)$ and $L(\vartheta^c s_2) \subseteq L(\vartheta^c, s)$

$\Rightarrow x, y \in L(\vartheta^c, s)$.

Then $x, y \in L(\vartheta^c, s)$, so by assumption $L(\vartheta^c, s)$ is a JU-subalgebra of X, implies that
$x * y \in L(\vartheta^c, s)$ $\vartheta^c(x * y) \leqslant s = \max\{\vartheta^c(x), \vartheta^c(y)\}$.
Therefore, ϑ^c is an anti-fuzzy JU-subalgebra. □

Theorem 2.1.2.8. *Let ϑ^c be an anti-fuzzy JU-subalgebra of a JU-algebra X. Two Lower-level JU-subalgebras $L(\vartheta^c, s_1)$ and $L(\vartheta^c, s_2)$ (with $s_1 > s_2$) of anti-fuzzy JU-subalgbera ϑ^c are equal if and only if there is no $x \in X$ such that $s_1 \geqslant \vartheta^c(x) > s_2$.*

Proof. Let ϑ^c be anti-fuzzy JU-subalgebra of a JU-algebra X.
Suppose $L(\vartheta^c, s_1) = L(\vartheta^c, s_2)$ (with $s_1 > s_2$). Then we claim that there is no $x \in X$ such that $s_1 \geqslant \vartheta^c(x) > s_2$.
Assume that if there exist $x \in X$ such that $s_1 \geqslant \vartheta^c(x) > s_2$

$\Rightarrow x \in L(\vartheta^c, s_1)$, but $x \notin L(\vartheta^c, s_2)$

$\Rightarrow L(\vartheta^c, s_2)$ is a proper subset of $L(\vartheta^c, s_1)$, which is contradict to our assumption that is $L(\vartheta^c, s_1) = L(\vartheta^c, s_2)$.

Conversely, assume that there is no $x \in X$ such that $S_1 \geqslant \vartheta^c(x) > s_2$. We went to show that $L(\vartheta^c, s_1) = L(\vartheta^c, s_2)$. Since $s_1 > s_2$, we get $L(\vartheta^c, s_2) \subseteq L(\vartheta^c, s_1) \cdots (*)$
Now, $x \in L(\vartheta^c, s_1) \Rightarrow \vartheta^c(x) \leqslant s_1 \Rightarrow \vartheta^c(x) < s_2$ (Since $\vartheta^c(x)$ does not lie between s_1 and s_2)
$\Rightarrow x \in L(\vartheta^c, s_2$, hence $L(\vartheta^c, s_1) \subseteq L(\vartheta^c, s_2) \cdots (**)$
From (*) and (**) we get $L(\vartheta^c, s_1) = L(\vartheta^c, s_2)$ □

Theorem 2.1.2.9. *Every JU-subalgebra of X can be realized as a level JU-subalgebra of some fuzzy JU-subalgebra of X.*

Proof. Let S be a JU-subalgebra of X and let ϑ be a fuzzy set in X defined by

$$\vartheta(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{other wise} \end{cases}$$

,where $t \in [0,1]$.
We show that ϑ is a fuzzy JU-subalgebra of X. Let $x, y \in X$. Then consider the following cases:

Case 1. If $x, y \in S$, then $x * y \in S$, and so

$\vartheta(x * y) \geqslant t = \min\{\vartheta(x), \vartheta(y)\}$.

Case 2. If $x, y \notin S$, then $\vartheta(x) = 0 = \vartheta(y)$ and thus

$\vartheta(x * y) \geqslant 0 = \min\{\vartheta(x), \vartheta(y)\}$.

Case 3. If $x \in S$ and $y \notin S$, then

$\vartheta(x * y) \geqslant 0 = \min\{1, 0\} = \min\{\vartheta(x), \vartheta(y)\}$.

Or if $x \notin S$ and $y \in S$ we get $\vartheta(x * y) \geqslant 0 = \min\{0, 1\} = \min\{\vartheta(x), \vartheta(y)\}$.

Therefore, ϑ is a fuzzy JU-subalgebra of X. □

Theorem 2.1.2.10. *Let* $\{S_i\}$ *be any family of JU-subalgebras of JU-algebra* X *such that* $S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n = X$. *Then there exist fuzzy JU-subalgebras which are exactly the JU-subalgebras of this chain* $\{S_i\}$.

Proof. Consider a set of numbers $t_0 > t_1 > t_3 > \cdots > t_n$ where each $t_i \in [0,1]$. Let $\vartheta : X \to [0,1]$ be a fuzzy set defined by

$$\vartheta(x) = \begin{cases} t_0, & \text{if } x \in S_0 \\ t_i, & \text{if } x \in S_i - S_{i-1},\ 0 < i \leqslant n. \end{cases}$$

Now, we claim that ϑ is a fuzzy JU-subalgebra of X and $U(\vartheta, t_i) = S_i$ for $0 < i \leqslant n$. Let $x, y \in X$, then we consider the following cases:

Case 1. If $x, y \in S_i - S_{i-1}$ for some i, then $x, y \in S_i$. Therefore, by definition of ϑ, we have $\vartheta(x) = t_i = \vartheta(y)$. Since S_i is a JU-subalgebra of X, it follows that $x * y \in S_i$ and so either $x * y, \in S_i - S_{i-1}$ or $x * y, \in S_{i-1}$. Implies that $\vartheta(x) \geqslant t_i$ or $\vartheta(x) \geqslant t_i - 1 > t_i$. In any case we get $\vartheta(x * y) \geqslant t_i = \min\{\vartheta(x), \vartheta(y)\}$

Case 2. For $i > j$, $t_j > t_i$ and $S_j \subset S_i$. Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$ implies $\vartheta(x) = t_i$, $\vartheta(y) = t_j \geqslant t_i$. Since S_i is a JU-subalgebra of X and $S_j \subset S_i$, then $x * y \in S_i$. Now, $\vartheta(x * y) \geqslant t_i = \min\{\vartheta(x), \vartheta(y)\}$. Thus ϑ is a fuzzy JU-subalgebra of X. Also, from the definition of ϑ, it follows that $\mathrm{Im}(\vartheta) = \{t_0, t_1, t_2, \ldots, t_n\}$. Hence, the upper level JU-subalgebras of ϑ are given by the chain of JU-subalgebras:

$$U(\vartheta, t_0) \subset U(\vartheta, t_1) \subset \cdots \subset U(\vartheta, t_n) = X, \quad \text{for } 0 < i \leqslant n.$$

Now, $U(\vartheta, t_0) = \{x \in X : \mu(x) \geqslant t_0\} = S_0$. Finally, we prove that $U(\vartheta, t_i) = S_i$, for $0 < i \leqslant n$. Now let $x \in S_i$, then $\vartheta(x) \geqslant t_i$. This implies $x \in U(\vartheta, t_i)$. Hence, $S_i \subseteq U(\vartheta, t_i)$. If $x \in U(\vartheta, t_i)$, then $\vartheta(x) \geqslant t_i$. Which implies that $x \notin S_j$ for $j > i$. Otherwise, if $x \in S_j$, then $\vartheta(x) \geqslant t_j$ which implies $t_i \geqslant \vartheta(x) \geqslant t_j$. This contradicts the assumption that $x \in U(\vartheta, t_i)$. Hence $\vartheta(x) \in \{t_0, t_1, t_2, \ldots, t_n\}$. So, $x \in S_k$ for some $k \leqslant i$. As $S_k \subseteq S_i$, it follows that $x \in S_i$. Hence $U(\vartheta, t_i) \subseteq S_i$. Therefore, $U(\vartheta, t_i) = S_i$, for $0 < i \leqslant n$.

□

2.1.3 Homomorphism of Fuzzy JU-Subalgebras

In this section, we examine the notion of homomorphisms for fuzzy JU-subalgebras in the context of JU-algebras.

Theorem 2.1.3.1. *Let $(X, *, 1)$ and $(Y, *', 1')$ be JU-algebras and a map $f : X \to Y$ be an onto homomorphism, and ϑ be a fuzzy JU-subalgebra of a JU-algebra X. Then the image $f(\vartheta)$ is a fuzzy JU-subalgebra of Y with the sup property.*

Proof. Let $f : X \to Y$ be an onto homomorphism and ϑ be a fuzzy JU-subalgebra of X. By Definition 1.1.7 and let $m, n \in Y$ with $p \in f^{-1}(m)$ and $q \in f^{-1}(n)$ such that $\vartheta(p) = \sup\{\vartheta(r) : r \in f^{-1}(m)\}$ and $\vartheta(q) = \sup\{\vartheta(r) : r \in f^{-1}(n)\}$

$$\begin{aligned} \text{Then, } f(\vartheta)(m * n) &= \sup\{\vartheta(r) : r \in f^{-1}(m * n)\} \\ &\geqslant \vartheta(p * q) \\ &\geqslant \min\{\vartheta(p), \vartheta(q)\} \\ &= \min\{\sup\{\vartheta(r) : r \in f^{-1}(m)\}, \sup\{\vartheta(r) : r \in f^{-1}(n)\}\} \\ &= \min\{f(\vartheta)(m), f(\vartheta)(n)\} \end{aligned}$$

Hence,$f(\vartheta)$ is a fuzzy JU-subalgebra of X. □

Theorem 2.1.3.2. *Let $(X, *, 1)$ and $(Y, *', 1')$ be JU-algebras and a map $f : X \to Y$ be homomorphism and λ be a fuzzy JU-subalgebra of Y. Then the Per-image $f^{-1}(\lambda)$, is a fuzzy JU-subalgebra of X.*

Proof. Let λ be a fuzzy JU-subalgebra of Y and for all $x, y \in X$, then

$$\begin{aligned} f^{-1}(\lambda)(x*y) &= \lambda(f(x*y)) \\ &= \lambda(f(x)*f(y)) \\ &\geqslant \min\{\lambda(f(x)), \lambda(f(y))\} \\ &= \min\{f^{-1}(\lambda)(x), f^{-1}(\lambda)(y)\} \end{aligned}$$

□

Theorem 2.1.3.3. *Let* X, Y, Z *be JU-algebras. Then the function* $f : X \to Y$ *and* $g : Y \to Z$ *be JU-homomorphism then the following holds:*

(a) If f, g *are surjective, then* $(g \circ f)(\vartheta)$ *is a fuzzy JU-subalgebra Z, for any fuzzy JU-subalgebra of* ϑ *of X, provided that the sup property holds.*

(b) $(g \circ f)^{-1}(\beta)$ *is a fuzzy JU-subalgebra of X for any fuzzy JU-subalgebra* λ *of Z.*

Proof. (a) Let ϑ be a fuzzy JU-subalgebra of X, for $m, n \in Z$, there exist $p, q \in X$ such that, $(g \circ f)(p) = m$ and $(g \circ f)(q) = n$. Then $\vartheta(s) = \sup\{\vartheta(r) | r \in (g \circ f)^{-1}(m)\}$ and $\vartheta(t) = \sup\{\vartheta(r) | r \in (g \circ f)^{-1}(n)\}$ for some $s, t \in X$. Since f, g are surjective, so is $g \circ f$, and hence $s * t \in (g \circ f)^{-1}(m * n)$. Now,

$$\begin{aligned} (g \circ f)(\vartheta)(m*n) &\geqslant \vartheta(s*t) \\ &\geqslant \min\{\vartheta(s), \vartheta(t)\} \\ &= \min\{\sup\{\vartheta(r) | r \in (g \circ f)^{-1}(m), \sup\{\vartheta(r) | r \in (g \circ f)^{-1}(n)\}\} \\ &= \min\{(g \circ f)(\vartheta)(m), (g \circ f)(\vartheta)(n)\}. \end{aligned}$$

Therefore,$(g \circ f)(\vartheta)$ is a fuzzy JU-subalgebra of Z.

(b) Let β be a fuzzy JU-subalgebra of Z, and let $x, y \in X$. Then,

$$\begin{aligned} (g \circ f)^{-1}(\beta)(m*n) &= \beta\big((g \circ f)(m*n)\big) \\ &= \beta\big((g \circ f)(m) * (g \circ f)(n)\big) \\ &\geqslant \min\big\{\beta((g \circ f)(m)), \beta((g \circ f)(n))\big\} \\ &= \min\big\{(g \circ f)^{-1}(\beta)(m), (g \circ f)^{-1}(\beta)(n)\big\} \end{aligned}$$

□

Theorem 2.1.3.4. *Let* $f : X \to Y$ *be an epimorphism of JU-algebra. If* ϑ_f *is a fuzzy JU-subalgebra of X, then* ϑ *is a fuzzy JU-subalgebra of Y.*

Proof. Let ϑ_f be a fuzzy JU-subalgebra of X and let $y \in Y$. Then there exist $x \in X$ such that $f(x) = y$. Let $y_1, y_2 \in Y$, then

$$\begin{aligned} \vartheta(y_1 * y_2) = \vartheta(f(x_1 * x_2)) &= \vartheta_f(x_1 * x_2) \\ &\geqslant \min\{\vartheta_f(x_1), \vartheta_f(x_2)\} \\ &= \min\{\vartheta(f(x_1), \vartheta(f(x_2)\} \\ &= \min\{\vartheta(y_1), \vartheta(y_2\} \end{aligned}$$

Therefore, $\vartheta(y_1 * y_2) \geqslant \min\{\vartheta(y_1), \vartheta(y_2\}$, hence ϑ is a fuzzy JU-subalgebra of Y. □

Theorem 2.1.3.5. *Let $f : X \to Y$ be a homomorphism of a JU-algebra. If ϑ is a fuzzy JU-subalgebra of Y, then ϑ_f is a fuzzy JU-subalgebra of X.*

Proof. Let ϑ be a fuzzy JU-subalgebra of Y and let $x, y \in X$, then

$$\begin{aligned} \vartheta_f(x * y) &= \vartheta(f(x * y)) \\ &= \vartheta(f(x) * f(y)) \\ &\geqslant \min\{\vartheta(f(x)), \vartheta(f(y))\} \end{aligned}$$

Therefore, $\vartheta_f(x * y) \geqslant \min\{\vartheta_f(x), \vartheta_f(y)\}$, hence ϑ_f is a fuzzy JU-subalgebra of X. □

Corollary 2.1.3.6. *Let f be an endomorphism of a JU-algebra X. If ϑ is a fuzzy JU-subalgebra of X, then so is ϑ_f.*

Proof. Suppose f is an endomorphism of the JU-algebra X. Let $x, y \in X$, then

$$\begin{aligned} \vartheta_f(x * y) &= \vartheta(f(x * y)) \\ &= \vartheta(f(x) * f(y)) \\ &\geqslant \min\{\vartheta(f(x)), \vartheta(f(y))\} \\ &= \min\{\vartheta_f(x), \vartheta_f(y)\} \end{aligned}$$

Therefore, $\vartheta_f(x * y) \geqslant \min\{\vartheta_f(x), \vartheta_f(y)\}$, $\forall x, y \in X$, hence ϑ_f is a fuzzy JU-subalgebra of X. □

2.1.4 Cartesian Product of Fuzzy JU-Subalgebras

In this section, the Cartesian product of fuzzy subalgebras in JU-algebra is discussed, and some of their properties are investigated.

Lemma 2.1.4.1. *Let ϑ_1 and ϑ_2 be two fuzzy JU-subalgebras of X and Y, respectively. Then $(\vartheta_1 \times \vartheta_2)(1,1) \geqslant (\vartheta_1 \times \vartheta_2)(x,y)$, for all $(x,y) \in X \times Y$.*

Proof. Suppose ϑ_1 is a fuzzy JU-subalgebra of X and ϑ_2 is a fuzzy JU-subalgebra of Y. For any $(x,y) \in X \times Y$ then,

$$\begin{aligned}(\vartheta_1 \times \vartheta_2)(1,1) &= \min\{\vartheta_1(1), \vartheta_2(1)\} \\ &\geqslant \min\{\vartheta_1(x), \vartheta_2(y)\} \text{(By Lemma 2.1.1)} \\ &= (\vartheta_1 \times \vartheta_2)(x,y)\end{aligned}$$

□

Theorem 2.1.4.2. *Let ϑ_1, ϑ_2 be fuzzy JU-subalgebras of a JU-algebra X, and β_1, β_2 be fuzzy JU-subalgebras of a JU-algebra Y. If $\vartheta_1 \subseteq \vartheta_2$ and $\beta_1 \subseteq \beta_2$, then $\vartheta_1 \times \beta_1 \subseteq \vartheta_2 \times \beta_2$.*

Proof. Suppose ϑ_1, ϑ_2 be fuzzy JU-subalgebras of a JU-algebra X, and β_1, β_2 be fuzzy JU-subalgebras of a JU-algebra Y and $\vartheta_1 \subseteq \vartheta_2$ and $\beta_1 \subseteq \beta_2$. We claim that $\vartheta_1 \times \beta_1 \subseteq \vartheta_2 \times \beta_2$. For any $(x,y) \in X \times Y$, then by Definition 1.1.13 we have

$(\vartheta_1 \times \beta_1)(x,y) = \min\{\vartheta_1(x), \beta_1(y)\}$,

$(\vartheta_2 \times \beta_2)(x,y) = \min\{\vartheta_2(x), \beta_2(y)\}$.

And from assumption, we have $\vartheta_1(x) \leqslant \vartheta_2(x)$ and $\beta_1(y) \leqslant \beta_2(y)$). It follows that,

$\min\{\vartheta_1(x), \beta_1(y)\} \leqslant \min\{\vartheta_2(x), \beta_2(y)\}$

$\Rightarrow (\vartheta_1 \times \beta_1)(x,y) \leqslant (\vartheta_2 \times \beta_2)(x,y), \quad \forall (x,y) \in X \times Y.$

Hence, $\vartheta_1 \times \beta_1 \subseteq \vartheta_2 \times \beta_2$. □

Theorem 2.1.4.3. *Let ϑ_1 and ϑ_2 be fuzzy subsets of JU-algebras X and Y, respectively such that $\vartheta_1(x) \leqslant \vartheta_2(1')$, for all $x \in X$, when $1'$ is a constant element in Y. If $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-subalgebra of $X \times Y$, then ϑ_1 is a fuzzy JU-subalgebra of X.*

Proof. Let $(x,y) \in X \times Y$. Then $(x,1'),(y,1') \in X \times Y$. Since $\vartheta_1(x) \leqslant \vartheta_2(1'), \vartheta_1(y) \leqslant \vartheta_2(1')$, for all $x,y \in X$. We get

$$\begin{aligned}\vartheta_1(x*y) = \min\{\vartheta_1(x*y), \vartheta_2(1'*1')\} &= (\vartheta_1 \times \vartheta_2)(x*y, 1'*1') \\ &= (\vartheta_1 \times \vartheta_2)((x,1')*(y,1')) \\ &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(x,1'), (\vartheta_1 \times \vartheta_2)(y,1')\} \\ &= \min\{\min\{\vartheta_1(x), \vartheta_2(1')\}, \min\{\vartheta_1(y), \vartheta_2(1')\}\} \\ &= \min\{\vartheta_1(x), \vartheta_1(y)\}.\end{aligned}$$

Hence, ϑ_1 is a fuzzy JU-subalgebra of X. □

Theorem 2.1.4.4. *Let ϑ_1 and ϑ_2 be fuzzy subsets of JU- algebras X and Y, respectively such that $\vartheta_2(y) \leqslant \vartheta_1(1)$, for all $y \in Y$, when 1 is a constant element in X. If $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-subalgebra of $X \times Y$, then ϑ_2 is a fuzzy JU-subalgebra of Y.*

Proof. Let $x, y \in Y$. Then $(1,x),(1,y) \in X \times Y$. Since $\vartheta_2(x) \leqslant \vartheta_1(1), \vartheta_2(y) \leqslant \vartheta_1(1)$, for all $x, y \in Y$. We get

$$\begin{aligned}\vartheta_2(x*y) = \min\{\vartheta_1(1*1), \vartheta_2(x*y)\} &= (\vartheta_1 \times \vartheta_2)(1*1, x*y) \\ &= (\vartheta_1 \times \vartheta_2)((1,x)*(1,y)) \\ &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(1,x), (\vartheta_1 \times \vartheta_2)(1,y)\} \\ &= \min\{\min\{\vartheta_1(1), \vartheta_2(x)\}, \min\{\vartheta_1(1), \vartheta_2(y)\}\} \\ &= \min\{\vartheta_2(x), \vartheta_2(y)\}.\end{aligned}$$

Hence, ϑ_2 is a fuzzy JU-subalgebra of Y. □

From Theorems 2.1.4.3 and 2.1.4.4 we have the following theorem.

Theorem 2.1.4.5. *Let ϑ_1 and ϑ_2 be fuzzy subsets of JU-algebras X and Y respectively. If $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-subalgebra of $X \times Y$, then either ϑ_1 is a fuzzy JU-subalgebra of X or ϑ_2 is a fuzzy JU-subalgebra of Y.*

Proof. Suppose $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-subalgebra of $X \times Y$. We need to show that either ϑ_1 is a fuzzy JU-subalgebra of X or ϑ_2 is a fuzzy JU-subalgebra of Y. Since $\vartheta_1 \times \vartheta_2$ is fuzzy JU-subalgebra of $X \times Y$, then

$$(\vartheta_1 \times \vartheta_2)((x_1,y_1)*(x_2,y_2)) \geqslant \min\{(\vartheta_1 \times \vartheta_2)(x_1,y_1), (\vartheta_1 \times \vartheta_2)(x_2,y_2)\} \cdots (*)$$

If we put $x_1 = 1 = x_2$ in (*), we get

$$\begin{aligned}(\vartheta_1 \times \vartheta_2)((1,y_1)*(1,y_2)) &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(1,y_1), (\vartheta_1 \times \vartheta_2)(1,y_2)\} \\ \Rightarrow (\vartheta_1 \times \vartheta_2)(1*1, y_1*y_2) &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(1,y_1), (\vartheta_1 \times \vartheta_2)(1,y_2)\} \\ \Rightarrow (\vartheta_1 \times \vartheta_2)(1, y_1*y_2) &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(1,y_1), (\vartheta_1 \times \vartheta_2)(1,y_2)\} \\ \Rightarrow \min\{\vartheta_1(1), \vartheta_2(y_1*y_2)\} &\geqslant \min\{\min\{\vartheta_1(1), \vartheta_2(y_1)\}, \min\{\vartheta_1(1), \vartheta_2(y_2)\}\} \\ \Rightarrow \vartheta_2(y_1*y_2) &\geqslant \min\{\vartheta_2(y_1), \vartheta_2(y_2)\}.\end{aligned}$$

Hence, ϑ_2 is a fuzzy JU-subalgebra of Y.
We prove that ϑ_1 is a fuzzy JU-subalgebra of X by putting $y_1 = 1 = y_2$ in (*).
If we put $y_1 = 1 = y_2$ in (*), we get

$$\begin{aligned}(\vartheta_1 \times \vartheta_2)((x_1,1)*(x_2,1)) &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(x_1,1), (\vartheta_1 \times \vartheta_2)(x_2,1)\} \\ \Rightarrow (\vartheta_1 \times \vartheta_2)(x_1*x_2, 1*1) &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(x_1,1), (\vartheta_1 \times \vartheta_2)(x_2,1)\} \\ \Rightarrow (\vartheta_1 \times \vartheta_2)(x_1*x_2, 1) &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(x_1,1), (\vartheta_1 \times \vartheta_2)(x_2,1)\} \\ \Rightarrow \min\{\vartheta_1(x_1*x_2), \vartheta_2(1)\} &\geqslant \min\{\min\{\vartheta_1(x_1), \vartheta_2(1)\}, \min\{\vartheta_1(x_2), \vartheta_2(1)\}\} \\ \Rightarrow \vartheta_1(x_1*x_2) &\geqslant \min\{\vartheta_1(x_1), \vartheta_1(x_2)\}.\end{aligned}$$

Hence, ϑ_1 is a fuzzy JU-subalgebra of X. $\square$

Theorem 2.1.4.6. *Let ϑ be a fuzzy subset of a JU-algebra X and let β be a strong fuzzy relation on X. If β is a fuzzy JU-subalgebra of $X \times X$, then $\vartheta(1) \geqslant \vartheta(x)$, for all $x \in X$.*

Proof. Since β is a fuzzy JU-subalgebra of $X \times X$, it follows that $\beta(x,y) = \min\{\vartheta(x),\vartheta(y)\}$. By Lemma 2.1.4.1, $\beta(1,1) \geqslant \beta(x,x)$. Then we have

$$\vartheta(1) = \min\{\vartheta(1),\vartheta(1)\} = \beta(1,1) \geqslant \beta(x,x) = \min\{\vartheta(x),\vartheta(x)\} = \vartheta(x)$$

Hence, $\vartheta(1) \geqslant \vartheta(x)$, for all $x \in X$. $\square$

Theorem 2.1.4.7. *Let ϑ be a fuzzy subset of a JU-algebra X and let β be the strongest fuzzy relation on a JU-algebra X. Then ϑ is a fuzzy JU-subalgebra of X if and only if β is a fuzzy JU-subalgebra of $X \times X$.*

Proof. Suppose that ϑ is a fuzzy JU-subalgebra of X. Let $(x_1,x_2),(y_1,y_2) \in X \times X$. Then we have

$$\begin{aligned}
\beta((x_1,x_2)*(y_1,y_2)) &= \beta(x_1*y_1,x_2*y_2) \\
&= \min\{\vartheta(x_1*y_1),\vartheta(x_2*y_2)\} \\
&\geqslant \min\{\min\{\vartheta(x_1),\vartheta(y_1)\},\min\{\vartheta(x_2),\vartheta(y_2)\}\} \\
&= \min\{\min\{\vartheta(x_1),\vartheta(x_2)\},\min\{\vartheta(y_1),\vartheta(y_2)\}\} \\
&= \min\{\beta(x_1,x_2),\beta(y_1,y_2)\}.
\end{aligned}$$

Thus $\beta((x_1,x_2)*(y_1,y_2)) \geqslant \min\{\beta(x_1,x_2),\beta(y_1,y_2)\}$.
Hence, β is a fuzzy JU-subalgebra of $X \times X$.
Conversely, assume that β is a fuzzy JU-subalgebra of $X \times X$. We need to show that ϑ is a fuzzy JU-subalgebra of X. Let $(x_1,x_2),(y_1,y_2) \in X \times X$. Then

$$\begin{aligned}
\min\{\vartheta(x_1*y_1),\vartheta(x_2*y_2)\} &= \beta(x_1*y_1,x_2*y_2) \\
&= \beta((x_1,x_2)*(y_1,y_2)) \\
&\geqslant \min\{\beta(x_1,x_2),\beta(y_1,y_2)\} \\
&= \min\{\min\{\vartheta(x_1),\vartheta(x_2)\},\min\{\vartheta(y_1),\vartheta(y_2)\}\}
\end{aligned}$$

In particular, if we take $x_2 = 1 = y_2$, then we get $\vartheta(x_1*y_1) \geqslant \min\{\vartheta(x_1),\vartheta(y_1)\}$, for all $x_1,y_1 \in X$.
Hence, ϑ is a fuzzy JU-subalgebra of X. $\square$

2.2. Fuzzy JU-Ideals of JU-Algebras

In this section, we discuss and investigate a new notion of fuzzy JU-ideals in JU-algebra and study their related properties.

Definition 2.2.1. *Let X be a JU-algebra, a fuzzy set* ϑ *in X is called a fuzzy JU-ideal of X if it satisfies the following conditions:*

(a) $\vartheta(1) \geqslant \vartheta(x)$

(b) $\vartheta(y) \geqslant \min\{\vartheta(x), \vartheta(x * y)\}, \forall x, y \in X.$

Example 2.2.1. *Let X =* $\{1,2,3,4,5,6\}$ *in which* $*$ *is defined by the following Cayley table:*

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	6
6	1	1	2	1	1	1

See [3] $(X;*,1)$ *is a JU-algebra. Define a fuzzy set* $\vartheta : X \to [0,1]$ *by*

$$\vartheta(x) = \begin{cases} 0.6, \text{ if } x = 1 \\ 0.3, \text{ if } x \in \{2,3,4\} \\ 0.2, \text{ otherwise} \end{cases}$$

Let $x, y \in X$. *By the definition of a fuzzy JU-ideal, we verify that*

$$\vartheta(y) \geqslant \min\{\vartheta(x), \vartheta(x * y)\}.$$

Now take, $x = 1$, $y = 2$, *then* $\vartheta(2) = 0.3 \cdots (*)$ *and*
$\min\{\vartheta(1), \vartheta(1 * 2)\} = \min\{\vartheta(1), \vartheta(2)\} = \min\{0.6, 0.3\} = 0.3 \cdots (**)$.
From (*) and (**) we get $0.3 \geqslant 0.3$ *which is true.*
For $x = 3, y = 1$, *then* $\vartheta(1) = 0.6 \cdots (***)$
And, $\min\{\vartheta(3), \vartheta(3 * 1)\} = \min\{\vartheta(3), \vartheta(1)\} = \min\{0.3, 0.6\} = 0.3 \cdots (****)$,
From (***) and (****) we get $0.6 \geqslant 0.3$ *which is true.*
Similarly, the result holds for all values of x and y in X.
Therefore, we conclude that ϑ *is a fuzzy JU-ideal of* $(X;*,1)$.

Theorem 2.2.1. *Every fuzzy JU-ideal of a JU-algebra X is order reversing.*

Proof. Let ϑ be a fuzzy JU-ideal of a JU-algebra X, and for any $x, y \in X$ be such that $x \leqslant y$, then $y * x = 1$

$$\begin{aligned} \text{Then } \vartheta(x) &\geqslant \min\{\vartheta(y * x), \vartheta(y)\} \\ &= \min\{\vartheta(1), \vartheta(y)\} \\ &= \vartheta(y) \end{aligned}$$ □

Theorem 2.2.2. *If* ϑ *is a fuzzy JU-ideal of X, then* $\vartheta((x * y) * y) \geqslant \vartheta(x)$.

Proof. Since $(x * [(x * y) * y]) = 1$ (By Lemma 1.6.1)
Now, $\vartheta((x * y) * y) \geqslant \min\{\vartheta(x), \vartheta(x * [(x * y) * y])\}$

$$= \min\{\vartheta(x), \vartheta(1)\}$$
$$= \vartheta(x)$$ □

Theorem 2.2.3. *Every fuzzy JU-ideal of a JU-algebra* X *is a fuzzy JU-subalgebra of X.*

Proof. Suppose ϑ is a fuzzy JU-ideal of X. That is:

a) $\vartheta(1) \geqslant \vartheta(x)$,

b) $\vartheta(y) \geqslant \min\{\vartheta(x), \vartheta(x * y)\}$, for all $x, y \in X$.

We claim that ϑ is a fuzzy JU-subalgebra. That is, for all $x, y \in X$
$\vartheta(x * y) \geqslant \min\{\vartheta(x), \vartheta(y)\}$
Assume that without loss of generality:
$\vartheta(x) \leqslant \vartheta(y) \Rightarrow \min\{\vartheta(x), \vartheta(y)\} = \vartheta(x)$
Now we consider two cases:

Case 1. If $\vartheta(x * y) \geqslant \vartheta(x)$, then

$\vartheta(x * y) \geqslant \vartheta(x) = \min\{\vartheta(x), \vartheta(y)\}$

$\Rightarrow \vartheta(x * y) \geqslant \min\{\vartheta(x), \vartheta(y)\}$

The inequality is satisfied directly.

Case 2. If $\vartheta(x * y) < \vartheta(x)$, then from the assumption that ϑ is fuzzy JU-ideal, apply condition (b):

$\vartheta(y) \geqslant \min\{\vartheta(x), \vartheta(x * y)\} = \vartheta(x * y)$

$\Rightarrow \vartheta(y) \geqslant \vartheta(x * y)\}$

Since we also know, $\vartheta(x * y) < \vartheta(x)$ and $\vartheta(x) \leqslant \vartheta(y)$, it follows that

$\vartheta(x * y) < \vartheta(x) \leqslant \vartheta(y) \Rightarrow \vartheta(x * y) < \min\{\vartheta(x), \vartheta(y)\}$.

Which is contradict the inequality $\vartheta(x * y) \geqslant \min\{\vartheta(x), \vartheta(y)\}$

Thus, Case 1 must always hold, which implies:

$\vartheta(x * y) \geqslant \vartheta(x) = \min\{\vartheta(x), \vartheta(y)\}$

Hence, ϑ is a fuzzy JU-subalgebra of X.

□

Remark 2.2.1. *The converse of the above theorem may not be true.*

Example 2.2.2. *Let X =* $\{1,2,3,4\}$ *in which* $*$ *is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $(X;*,1)$ *is a JU-algebra. Define* $\vartheta(x)$ *as follows:*

$$\vartheta(x) = \begin{cases} 0.9, & \text{if } x = 1 \\ 0.7 & \text{if } x \in \{2,4\} \\ 0.4 & \text{if } x = 3 \end{cases}$$

Since ϑ *is a fuzzy JU-subalgebra of X. The fuzzy JU-ideal condition requires* $\vartheta(y) \geqslant \min\{\vartheta(x), \vartheta(x*y)\}$*, for all* $x, y \in X$. *Let* $x = 2$, $y = 3 \Rightarrow x*y = 4$ *and* $\vartheta(2) = 0.7, \quad \vartheta(3) = 0.4, \quad \vartheta(4) = 0.7$. *Then,* $\vartheta(3) = 0.4 \cdots (*)$

And, $\quad \min\{\vartheta(2), \vartheta(4)\} = \min\{0.7, 0.7\} = 0.7 \cdots (**)$

From $(*)$ *and* $(**)$ *we get* $0.4 \geqslant 0.7$ *which is false. This shows that every fuzzy JU-subalgebra of a JU-algebra X may not a fuzzy JU-ideal of X.*

Theorem 2.2.4. *The intersection of any two fuzzy JU-ideals is also a fuzzy JU-ideal of X.*

Proof. Suppose ϑ and λ are fuzzy JU-ideals of a JU-algebra X, for any $x \in X$. Then

$(\vartheta \cap \lambda)(1) = \min\{\vartheta(1), \lambda(1)\} \geqslant \min\{\vartheta(x), \vartheta(x)\} = (\vartheta \cap \lambda)(x)$

And, for any $x, y \in X$, then

$$\begin{aligned} (\vartheta \cap \lambda)(y) &= \min\{\vartheta(y), \lambda(y)\} \\ &\geqslant \min\{\min\{\vartheta(x), \vartheta(x*y)\}, \min\{\lambda(x), \lambda(x*y)\}\} \\ &= \min\{\min\{\vartheta(x), \lambda(x)\}, \min\{\vartheta(x*y), \lambda(x*y)\}\} \\ &= \min\{(\vartheta \cap \lambda)(x), (\vartheta \cap \lambda)(x*y)\} \end{aligned}$$

□

Corollary 2.2.5. *Let* $\{\vartheta_i | i \in \Omega\}$ *be a family of fuzzy JU-ideals of a JU-algebra X. Then* $\bigcap_{i \in \Omega} \vartheta_i$ *is also a fuzzy JU-ideal of X.*

Proof. Let $\{\vartheta_i | i \in \Omega\}$ be a family of JU-ideals of X, then for any $x \in X$.

$(\bigcap_{i \in \Omega} \vartheta_i)(1) = \inf_{i \in \Omega}\{\vartheta_i(1)\} \geqslant \inf_{i \in \Omega}\{\vartheta_i(x)\}$

And, for any $x, y \in X$ then

$$\begin{aligned} (\textstyle\bigcap_{i \in \Omega} \vartheta_i)(y) &= \inf_{i \in \Omega}\{\vartheta_i(y)\} \\ &\geqslant \inf_{i \in \Omega}\{\vartheta_i(x), \vartheta_i(x*y)\} \\ &= \min\{\inf_{i \in \Omega}\{\vartheta_i(x)\}, \inf_{i \in \Omega}\{\vartheta_i(x*y)\}\} \\ &= \min\{(\textstyle\bigcap_{i \in \Omega} \vartheta_i)(x), (\bigcap_{i \in \Omega} \vartheta_i)(x*y)\} \end{aligned}$$

□

Remark 2.2.2. *The union of any two fuzzy JU-ideals of a JU-algebra X may not be a fuzzy JU-ideal of X.*

Example 2.2.3. *Let X =* $\{1,2,3,4,5,6\}$ *be a JU-algebra. (See the table from the above example 2.2.1). Let us define* $\vartheta_1(x)$ *and* $\vartheta_2(x)$ *by:*

$$\vartheta_1(x) = \begin{cases} 0.8 & \text{if } x \in \{1,2\} \\ 0.6 & \text{if } x = 3 \\ 0.4 & \text{otherwise} \end{cases}$$

and

$$\vartheta_2(x) = \begin{cases} 0.9 & \textit{if } x \in \{1,2\} \\ 0.5 & \textit{if } x \in \{3,4\} \\ 0.2 & \textit{otherwise} \end{cases}$$

Then, $\vartheta_1 \cup \vartheta_2(4) = \max\{\vartheta_1(4), \vartheta_2(4)\} = max\{0.4, 0.5\} = 0.5 \cdots (*)$
And,

$$\begin{aligned} \max\{\vartheta_1(4), \vartheta_2(4)\} &\geqslant \max\{\min\{\vartheta_1(3), \vartheta_1(3*4)\}, \min\{\vartheta_2(3), \vartheta_2(3*4)\}\} \\ &= \min\{\max\{\vartheta_1(3), \vartheta_2(3)\}, \max\{\vartheta_1(3*4), \vartheta_2(3*4)\}\} \\ &= \min\{\max\{\vartheta_1(3), \vartheta_2(3)\}, \max\{\vartheta_1(2), \vartheta_2(2)\}\} \\ &= \min\{\max\{0.6, 0.5\}, \max\{0.8, 0.9\}\} \\ &= \min\{0.6, 0.9\} = 0.6 \cdots (**) \end{aligned}$$

From $(*)$ *and* $(**)$ *we get* $0.5 \geqslant 0.6$ *which is false.This shows that the union of any two fuzzy JU-ideals of a JU-algebra X may not be a fuzzy JU-ideal of X.*

2.2.1 Fuzzy p-Translation and Fuzzy p-Multiplication of JU-Ideals

Theorem 2.2.1.1. *Let* ϑ *be a fuzzy subset of a JU-algebra* X *and* $p \in [0, \Delta]$. *Then,* ϑ *is a fuzzy JU-ideal of* X *if and only if the fuzzy* p*-translation* $\vartheta_p^T(x)$ *of* ϑ *is also a fuzzy JU-ideal of* X.

Proof. Suppose that ϑ is a fuzzy JU-ideal of X and $p \in [0, \Delta]$. We claim that ϑ_p^T is a fuzzy JU-ideal of X. For any $x, y \in X$, then
$\vartheta_p^T(1) = \vartheta(1) + p \geqslant \vartheta(x) + p = \vartheta_p^T(x)$. And,

$$\begin{aligned} \vartheta_p^T(y) = \vartheta(y) + p &\geqslant \min\{\vartheta(x), \vartheta(x*y)\} + p \\ &= \min\{\vartheta(x) + p, \vartheta(x*y) + p\} \\ &= \min\{\vartheta_p^T(x), \vartheta_p^T(x*y)\} \end{aligned}$$

Thus, $\vartheta_p^T(y) \geqslant \min\{\vartheta_p^T(x), \vartheta_p^T(x*y)\}$, hence ϑ_p^T is a fuzzy JU-ideal of X.
Conversely, suppose ϑ_p^T is a fuzzy JU-ideal of X and for all $p \in [0, \Delta]$. We claim that ϑ is a fuzzy JU-ideal of X. For any $x, y \in X$, then
$\vartheta(1) + p = \vartheta_p^T(1) \geqslant \vartheta_p^T(x) = \vartheta(x) + p$
By subtracting p both sides, we get $\vartheta(1) \geqslant \vartheta(x)$. And,

$$\begin{aligned} \vartheta(y) + p = \vartheta_p^T(y) &\geqslant \min\{\vartheta_p^T(x), \vartheta_p^T(x*y)\} \\ &= \min\{\vartheta(x) + p, \vartheta(x*y) + p\} \\ &= \min\{\vartheta(x), \vartheta(x*y)\} + p \end{aligned}$$

By subtracting p both sides, we get $\vartheta(y) \geqslant \min\{\vartheta(x), \vartheta(x * y)\}$. Hence, ϑ is a fuzzy JU-ideal of X. □

Theorem 2.2.1.2. *Let ϑ be a fuzzy JU-ideal of a JU-algebra X, then the fuzzy* p*-multiplicative* ϑ_p^M *of ϑ is a fuzzy JU-ideal of X for every* $p \in [0, \Delta]$.

Proof. Let ϑ be a fuzzy JU-ideal of a JU-algebra X and $\forall p \in [0, \Delta]$. Then $\forall x, y \in X$.
$\vartheta_p^M(1) = (p)(\vartheta(1)) \geqslant (p)(\vartheta(x)) = \vartheta_p^M(x)$. And,

$$\begin{aligned}
\vartheta_p^M(y) = (p)(\vartheta(y)) &\geqslant (p)(\min\{\vartheta(x), \vartheta(x * y)\}) \\
&= \min\{(p)(\vartheta(x)), (P)(\vartheta(x * y))\} \\
&= \min\{\vartheta_p^M(x), \vartheta_p^M(x * y)\}
\end{aligned}$$

□

Theorem 2.2.1.3. *If there exists* $p \in (0, \Delta)$ *such that the fuzzy* p*-multiplicative* ϑ_p^M *of ϑ is a fuzzy JU-ideal of a JU-algebra X, then ϑ is a fuzzy JU-ideal of X.*

Proof. Suppose the fuzzy p-multiplicative ϑ_p^M of ϑ is a fuzzy JU-ideal of a JU-algebra X, and for some $p \in (0, \Delta)$. Then $\forall x, y \in X$.
$(p)(\vartheta(1)) = \vartheta_p^M(1) \geqslant \vartheta_p^M(x) = (p)(\vartheta(x))$, hence $\vartheta(1) \geqslant \vartheta(x)$. And,

$$\begin{aligned}
(p)(\vartheta(x)) = \vartheta_p^M(y) &\geqslant \min\{\vartheta_p^M(x), \vartheta_p^M(x * y)\} \\
&= \min\{(p)(\vartheta(x)), (p)(\vartheta(x * y))\} \\
&= (p)(\min\{\vartheta(x), \vartheta(x * y)\})
\end{aligned}$$

By canceling p both sides we get, $\vartheta(y) \geqslant \min\{\vartheta(x), \vartheta(x * y)\}$. Thus, ϑ is a fuzzy JU-ideal of X. □

Definition 2.2.1.1. *A fuzzy JU-ideal ϑ of a JU- algebra X is called prime fuzzy JU-ideal of X if ϑ is a prime fuzzy set in X.*

Theorem 2.2.1.4. *If ϑ is a prime fuzzy JU-ideal of a JU-algebra X, then the fuzzy* p*-multiplicative* ϑ_p^M *of ϑ is a prime fuzzy JU-ideal of X for all* $p \in [0, \Delta]$.

Proof. Let ϑ be a prime fuzzy JU-ideal of a JU-algebra X and $p \in [0, \Delta]$. Then

$$\begin{aligned}
\vartheta_p^M(x * y) = (p)(\vartheta(x * y)) &\leqslant (p)(\max\{\vartheta(x), \vartheta(y)\}) \\
&= \max\{(p)(\vartheta(x)), (p)(\vartheta(y))\} \\
&= \max\{\vartheta_p^M(x), \vartheta_p^M(y)\}
\end{aligned}$$

□

Theorem 2.2.1.5. *If there exists* $p \in (0,\Delta)$ *such that the fuzzy* p*-multiplicative* ϑ_p^M *of* ϑ *is a prime fuzzy JU-ideal of a JU-algebra X, then* ϑ *is a prime fuzzy JU-ideal of X.*

Proof. Assume that the fuzzy p-multiplicative ϑ_p^M of ϑ is a prime fuzzy ideal of a JU-algebra X, for some $p \in (0,\Delta)$ and for any $x,y \in X$. Then

$$\begin{aligned}(p)(\vartheta(x*y)) = \vartheta_p^M(x*y) &\leqslant \max\{\vartheta_p^\Delta(x),\vartheta_p^M(y)\}\\ &= \max\{(p)(\vartheta(x)),(p)(\vartheta(y))\}\\ &= (p)(\max\{\vartheta(x),\vartheta(y)\})\end{aligned}$$

By canceling p both sides we get, $\vartheta(x*y) \leqslant \max\{\vartheta(x),\vartheta(y)\}$. Thus, ϑ is a prime fuzzy JU-ideal of X. □

Theorem 2.2.1.6. *Let* ϑ *be a fuzzy subset of* X, $p \in [0,\Delta]$ *and* $q \in [0,1]$. *Then the fuzzy magnified-*pq*-translation* $\vartheta^{MT}_{(p,q)}$ *of* ϑ *is a fuzzy JU-ideal of* X.

Proof. Suppose that ϑ is a fuzzy JU-ideal of X, $p \in [0,\Delta]$ and $q \in [0,1]$. We claim that $\vartheta^{MT}_{(p,q)}$ is a fuzzy JU-ideal of X. For all $x,y \in X$, we have

$$\vartheta^{MT}_{p,q}(1) = (q)(\vartheta(1)) + p \geqslant (q)(\vartheta(x)) + p = \vartheta^{MT}_{p,q}(x)$$

And,

$$\begin{aligned}\vartheta^{MT}_{(p,q)}(y) &= (q)(\vartheta(y)) + p\\ &\geqslant (q)(\min\{\vartheta(x),\vartheta(x*y)\}) + p\\ &= \min\{(q)(\vartheta(x)) + p, (q)(\vartheta(x*y)) + p\}\\ &= \min\{\vartheta^{MT}_{(p,q)}(x), \vartheta^{MT}_{(p,q)}(x*y)\}.\end{aligned}$$

Hence, $\vartheta^{MT}_{(p,q)}$ is a fuzzy JU-ideal of X. □

Theorem 2.2.1.7. *If there exists* $q \in (0,1]$ *such that for all* $p \in [0,\Delta]$, *the fuzzy magnified-*(p,q)*-translation* $\vartheta^{MT}_{(p,q)}$ *of* ϑ *is a fuzzy JU-ideal of* X, *then* ϑ *is a fuzzy JU-ideal of* X.

Proof. Suppose that $\vartheta^{MT}_{(p,q)}$ is a fuzzy JU-ideal of X for all $p \in [0,\Delta]$ and for some $q \in (0,1]$. We claim that ϑ is a fuzzy JU-ideal of X. For all $x,y \in X$, we have
$(q)(\vartheta(1)) + p = \vartheta^{MT}_{p,q}(1) \geqslant \vartheta^{MT}_{p,q}(x) = (q)(\vartheta(x)) + p$, hence $\vartheta(1) \geqslant \vartheta(x)$. And,

$$\begin{aligned}(q)(\vartheta(y)) + p = \vartheta^{MT}_{(p,q)}(y) &\geqslant \min\{\vartheta^{MT}_{(p,q)}(x), \vartheta^{MT}_{(p,q)}(x*y)\}\\ &= \min\{(q)(\vartheta(x)) + p, (q)(\vartheta(x*y)) + p\}\\ &= (q)(\min\{\vartheta(x),\vartheta(x*y)\}) + p\end{aligned}$$

Subtracting p from both sides and dividing by q, we obtain, $\vartheta(y) \geqslant \min\{\vartheta(x),\vartheta(x*y)\}$. Therefore, ϑ is a fuzzy JU-ideal of X. □

2.2.2 Level Subsets of Fuzzy JU-Ideals

Theorem 2.2.2.1. *If ϑ is a fuzzy JU-ideal of X, then the set $X_\vartheta = \{x \in X | \vartheta(x) = \vartheta(1)\}$ is a JU-ideal of X.*

Proof. Assume that ϑ is a fuzzy JU-ideal of X. We claim that X_ϑ is a JU-ideal of X. Clearly, $1 \in X_\vartheta$. Now let $x, x * y \in X_\vartheta$. Then $\vartheta(x) = \vartheta(1) = \vartheta(x * y)$, and so

$$\vartheta(y) \geqslant \min\{\vartheta(x), \vartheta(x * y)\} = \min\{\vartheta(1), \vartheta(1)\} = \vartheta(1).$$
$$\Rightarrow \vartheta(y) \geqslant \vartheta(1)$$

But $\vartheta(y) \leqslant \vartheta(1), \forall y \in X$ (by Definition 2.2.1)
$\Rightarrow \vartheta(y) = \vartheta(1) \Rightarrow y \in X_\vartheta$. Hence X_ϑ is a JU-ideal of X. □

Theorem 2.2.2.2. *Let J be a subset of a JU-algebra X. Then the characteristics function χ_J is a fuzzy JU-ideal of X if and only if J is a JU-ideal of X.*

Proof. Let $\chi_J : X \to \{0, 1\}$ be a fuzzy JU- ideal of X. Let $x \in J$, then $\chi_J(x) = 1$. Since χ_J is a fuzzy JU-ideal of X, then $\chi_J(1) \geqslant \chi_J(x), \forall x \in X$. Clearly, $1 \in J$. And,
Let $x, x * y \in J$ implies that $\chi_J(x) = \chi_J(x * y) = 1$
Thus, $\chi_J(y) \geqslant \min\{\chi_J(x), \chi_J(x * y)\} = \min\{1, 1\} = 1$
$\Rightarrow \chi_J(y) \geqslant 1$ but $\chi_J(y) \leqslant 1$
$\Rightarrow \chi_J(y) = 1$, implies that $y \in J$.
Hence, J is an ideal of X.
Conversely, let J be an ideal of X. Since $1 \in J$. Then
$\chi_J(1) = \chi_J(x * x)$ (By Lemma 1.6.1)
$\geqslant \min\{\chi_J(x), \chi_J(x)\}$
$= \chi_J(x)$
And consider the following cases:

Case 1. if $x, x * y \in J$, then $y \in J$, we get that

$$\chi_J(y) = 1 = \min\{\chi_J(x), \chi_J(x * y)\}$$

Case 2. if x, $x * y \notin J$, then $\chi_J(x) = 0 = \chi_J(x * y)$

Thus, $\chi_J(y) = 0 = \min\{\chi_J(x), \chi_J(x * y)\}$

Case 3. if x or $x * y$ not in J, then $\min\{\chi_J(x), \chi_J(x * y)\} = 0$.

Then, $\chi_J(y) = 0 = \min\{\chi_J(x), \chi_J(x * y)\}$

Hence χ_J is a fuzzy ideal of X.

□

Theorem 2.2.2.3. *A fuzzy set ϑ of a JU-algebra X is a fuzzy JU-ideal of X if and only if for every $t \in [0, 1]$, $U(\vartheta, t) = \{x \in X \mid \vartheta(x) \geqslant t\}$ is either the empty set or a JU-ideal of X.*

Proof. Suppose ϑ is a fuzzy JU-ideal in a JU-algebra X and $U(\vartheta,t) \neq \emptyset$
For any $x \in U(\vartheta,t)$ such that $\vartheta(x) \geqslant t$. Since ϑ is a fuzzy JU-ideal of X, we have $\vartheta(1) \geqslant \vartheta(x)$, for all $x \in X$. Thus $\vartheta(1) \geqslant \vartheta(x) \geqslant t$. Therefore $1 \in U(\vartheta,t)$.
And, let $x, x*y \in U(\vartheta,t)$ implies that $\vartheta(x) \geqslant t$ and $\vartheta(x*y) \geqslant t$
Then $\vartheta(y) \geqslant \min\{\vartheta(x),\vartheta(x*y)\} \geqslant \min\{t,t\} = t$

$\Rightarrow \vartheta(y) \geqslant t$, implies that $y \in U(\vartheta,t)$ and hence $U(\vartheta,t)$ is a JU-ideal of X.

Conversely, assume that the level subset $U(\vartheta,t)$ is a JU-ideal of X. Further, let we assume that there exist some $x_0 \in X$ such that $\vartheta(1) < \vartheta(x_0)$
Take $\beta = \frac{1}{2}[\vartheta(1) + \vartheta(x_0)]$

$\Rightarrow \vartheta(1) < \beta < \vartheta(x_0)$

$x_0 \in \vartheta_\beta$ and $1 \notin U(\vartheta,\beta)$, which is contradict to our assumption that $U(\vartheta,\beta)$ is a JU-ideal of X.

Therefore, $\vartheta(1) \geqslant \vartheta(x)$, $\forall x \in X$.
And, assume that $x_0, y_0 \in X$ such that

$\vartheta(y_0) < \min\{\vartheta(x_0),\vartheta(x_0*y_0)\}$

Take $\beta = \frac{1}{2}[\vartheta(y_0) + \min\{\vartheta(x_0),\vartheta(x_0*y_0)\}]$

$\Rightarrow \vartheta(y_0) < \beta < \min\{\vartheta(x_0),\vartheta(x_0*y_0)\}$

$\Rightarrow \beta > \vartheta(y_0)$ and $\beta < \min\{\vartheta(x_0),\vartheta(x_0*y_0)\}$

$\Rightarrow \beta > \vartheta(y_0)$, $\beta < \vartheta(x_0)$, and $\beta < \vartheta(x_0*y_0)$

$y_0 \notin U(\vartheta,\beta)$, a contradiction, since $U(\vartheta,\beta)$ is a JU-ideal of X.

Therefore, $\vartheta(y_0) \geqslant \min\{\vartheta(x_0),\vartheta(x_0*y_0)\}$ for any $x_0, y_0 \in X$. □

Theorem 2.2.2.4. *Let ϑ be a fuzzy JU-ideal of a JU-algebra X. Two upper-level JU-ideals $U(\vartheta,t_1)$ and $U(\vartheta,t_2)$ (with $t_1 < t_2$) of a fuzzy JU-ideal ϑ are equal if and only if there is no $x \in X$ such that $t_1 \leqslant \vartheta(x) < t_2$.*

Proof. Let ϑ be a fuzzy JU-ideal of a JU-algebra X. Suppose $U(\vartheta,t_1) = U(\vartheta,t_2)$ (with $t_1 < t_2$). Then we claim that there is no $x \in X$ such that $t_1 \leqslant \vartheta(x) < t_2$.
Assume that if there exist $x \in X$ such that $t_1 \leqslant \vartheta(x) < t_2$

$\Rightarrow x \in U(\vartheta,t_1)$, but $x \notin U(\vartheta,t_2)$

$\Rightarrow U(\vartheta,t_1)$ is a proper subset of $U(\vartheta,t_2)$, which is contradict to our assumption that is $U(\vartheta,t_1) = U(\vartheta,t_2)$.

Conversely, assume that there is no $x \in X$ such that $t_1 \leqslant \vartheta(x) < t_2$. We went to show that $U(\vartheta,t_2) = U(\vartheta,t_1)$. Since $t_1 < t_2$, we get $U(\vartheta,t_2) \subseteq U(\vartheta,t_1) \cdots (*)$
Now, $x \in U(\vartheta,t_1) \Rightarrow \vartheta(x) \geqslant t_1 \Rightarrow \vartheta(x) > t_2$, (Since, $\vartheta(x)$ does not lie between t_1 and t_2)
$\Rightarrow x \in U(\vartheta,t_2)$
Hence, $U(\vartheta,t_1) \subseteq U(\vartheta,t_2) \cdots (**)$
From (*) and (**) we get $U(\vartheta,t_1) = U(\vartheta,t_2)$ □

Remark 2.2.2.1. *Following Theorem 2.2.2.4, the level JU-ideals of a fuzzy JU-ideal ϑ in a finite JU-algebra X form a chain:*

$U(\vartheta,t_0) \subset U(\vartheta,t_1) \subset U(\vartheta,t_2) \subset \cdots \subset U(\vartheta,t_n) = X$, *where* $t_0 > t_1 > t_2 > \cdots > t_n$.

Corollary 2.2.2.5. *If ϑ is a fuzzy JU-ideal of finite JU-algebra X with $\mathrm{Im}(\vartheta) = \{t_1, \ldots, t_n\}$, then the family of JU-ideals $\{U(\vartheta, t_i) \mid 1 \leqslant i \leqslant n\}$ constitutes all upper level JU-ideals of ϑ in X.*

Proof. Let ϑ be a fuzzy JU-ideal of a JU-algebra X such that $\mathrm{Im}(\vartheta) = \{t_1, t_2, \ldots, t_n\}$ with $t_1 < \cdots < t_n$. Let $t \in [0,1]$ and $t \notin \mathrm{Im}(\vartheta)$, then consider the following cases.

Case 1. If $t \leqslant t_1$, then $U(\vartheta, t_1) = X = U(\vartheta, t)$.

Case 2. If $t > t_n$, then $U(\vartheta, t) = \{x \in X \mid \vartheta(x) \geqslant t\} = \{x \in X \mid \vartheta(x) \geqslant t_n\} = \emptyset$.

Case 3. If $t_{i-1} < t < t_i$, then $U(\vartheta, t) = U(\vartheta, t_i)$ by Theorem 2.2.2.4, since there is no $x \in X$ such that $t \leqslant \vartheta(x) < t_i$.

Thus for any $t \in [0,1]$, the level JU-ideal is one of $\{U(\vartheta, t_i) \mid i = 1, 2, \ldots, n\}$. □

Theorem 2.2.2.6. *Let ϑ be a fuzzy JU-ideal of a JU-algebra X. Then ϑ^c is an anti-fuzzy JU-ideal of X if and only if for every $s \in [0,1]$, the lower s-level set $L(\vartheta^c, s) = \{x \in X \mid \vartheta^c(x) \leqslant s\}$ is either empty or a JU-ideal of X.*

Proof. Suppose that ϑ^c is an anti-fuzzy JU-ideal of X and $L(\vartheta^c, s) \neq \emptyset$. We claim that $L(\vartheta^c, s)$ is a JU-ideal of X.

For any $x \in L(\vartheta^c, s)$, by definition we have $\vartheta^c(x) \leqslant s$. Since $L(\vartheta^c, s)$ is an anti-fuzzy JU-ideal of X, it follows that $\vartheta^c(1) \leqslant \vartheta^c(x)$ for all $x \in X$. Thus, $\vartheta^c(1) \leqslant \vartheta^c(x) \leqslant s$, and hence $1 \in L(\vartheta^c, s)$.

And, let $x, x * y \in L(\vartheta^c, s)$ implies that $\vartheta^c(x) \leqslant s$ and $\vartheta^c(x * y) \leqslant s$

Then $\vartheta^c(y) \leqslant \min\{\vartheta^c(x), \vartheta^c(x * y)\} = \min\{s, s\} = s$

$\Rightarrow \vartheta^c(y) \leqslant s$, implies that $y \in L(\vartheta^c, s)$

This implies that $y \in L(\vartheta^c, s)$ and hence $L(\vartheta^c, s)$ is a JU-ideal of X.

Conversely, assume that the level subset $L(\vartheta^c, s)$ is a JU-ideal of X. Further, let we assume that there exist some $x_0 \in X$ such that $\vartheta^c(1) > \vartheta^c(x_0)$

Take $\beta = \frac{1}{2}[\vartheta^c(1) + \vartheta^c(x_0)]$

$\Rightarrow \vartheta^c(1) > \beta > \vartheta^c(x_0)$

$x_0 \in L(\vartheta^c, \beta)$ and $1 \notin L(\vartheta^c, \beta)$, which is contradict to our assumption that $L(\vartheta^c, \beta)$ is a JU-ideal of X.

Therefore, $\vartheta^c(1) \leqslant \vartheta^c(x), \forall x \in X$.

And, assume that $x_0, y_0 \in X$ such that

$\vartheta^c(y_0) > \min\{\vartheta^c(x_0), \vartheta^c(x_0 * y_0)\}$

Take $\beta = \frac{1}{2}[\vartheta^c(y_0) + \min\{\vartheta^c(x_0), \vartheta^c(x_0 * y_0)\}]$

$\Rightarrow \vartheta^c(y_0) > \beta > \min\{\vartheta^c(x_0), \vartheta^c(x_0 * y_0)\}$

$\Rightarrow \beta < \vartheta^c(y_0)$ and $\beta > \min\{\vartheta^c(x_0), \vartheta^c(x_0 * y_0)\}$

$\Rightarrow \beta < \vartheta^c(y_0)$, $\beta > \vartheta^c(x_0)$, and $\beta > \vartheta^c(x_0 * y_0)$

$y_0 \notin L(\vartheta^c, \beta)$, a contradiction, since $L(\vartheta^c, \beta)$ is a JU-ideal of X.

Therefore, $\vartheta^c(y_0) \leqslant \min\{\vartheta^c(x_0), \vartheta^c(x_0 * y_0)\}$ for any $x_0, y_0 \in X$. □

Theorem 2.2.2.7. *Let ϑ^c be anti-fuzzy JU-ideal of a JU-algebra X. Two Lower-level JU-ideals $L(\vartheta^c, s_1)$ and $L(\vartheta^c, s_2)$ (with $s_1 > s_2$) of anti-fuzzy JU-ideal ϑ^c are equal if and only if there is no $x \in X$ such that $s_1 \geqslant \vartheta^c(x) > s_2$.*

Proof. Let ϑ^c be anti-fuzzy JU-ideal of a JU-algebra X. Suppose $L(\vartheta^c, s_1) = L(\vartheta^c, s_2)$ (with $s_1 > s_2$). Then we claim that there is no $x \in X$ such that $s_1 \geqslant \vartheta^c(x) > s_2$.
Assume that if there exist $x \in X$ such that $s_1 \geqslant \vartheta^c(x) > s_2$

$\Rightarrow x \in L(\vartheta^c, S_1)$, but $x \notin L(\vartheta^c, S_2)$

$\Rightarrow L(\vartheta^c, S_2)$ is a proper subset of $L(\vartheta^c, S_1)$, which is contradict to our assumption that is $L(\vartheta^c, S_1) = L(\vartheta^c, S_2)$.
Conversely, assume that there is no $x \in X$ such that $S_1 \geqslant \vartheta^c(x) > s_2$. We went to show that $L(\vartheta^c, S_1) = L(\vartheta^c, S_2)$. Since $s_1 > s_2$, we get $L(\vartheta^c, S_2) \subseteq L(\vartheta^c, S_1) \cdots (*)$
Now, $x \in L(\vartheta^c, s_1) \Rightarrow \vartheta^c(x) \geqslant s_1 \Rightarrow \vartheta^c(x) > s_2$, (Since, $\vartheta^c(x)$ does not lie between s_1 and s_2)
$\Rightarrow x \in L(\vartheta^c, s_2)$
Hence, $L(\vartheta^c, s_1) \subseteq L(\vartheta^c, S_2) \cdots (**)$
From (*) and (**) we get $L(\vartheta^c, s_1) = L(\vartheta^c, s_2)$ □

Theorem 2.2.2.8. *Every JU-ideal of X can be realized as level JU-ideal of a fuzzy JU-ideal of X.*

Proof. Let J be a JU-ideal of X and let ϑ be a fuzzy set in X defined by

$$\vartheta(x) = \begin{cases} t & \text{if } x \in J \\ 0 & \text{other wise} \end{cases}$$

,where $t \in [0,1]$.
We went to show that ϑ is a fuzzy JU-ideal of X. Since J is a JU-ideal of X, then $1 \in J$ and $\vartheta(1) = t$. Therefore $\vartheta(1) \geqslant \vartheta(x)$, $\forall x \in X$. Let $x, y \in X$. And also, consider the following cases:

Case 1. If $x, x * y \in J$, then $y \in J$, and so $\vartheta(y) \geqslant t = \min\{\vartheta(x), \vartheta(x * y)\}$.

Case 2. If $x, x * y \notin J$, then $\vartheta(x) = 0 = \vartheta(x * y)$ and thus $\vartheta(y) \geqslant 0 = \min\{\vartheta(x), \vartheta(x * y)\}$.

Case 3. If $x \in J$ and $x * y \notin J$, then $\vartheta(y) \geqslant 0 = \min\{1, 0\} = \min\{\vartheta(x), \vartheta(x * y)\}$.

Or if $x \notin J$ and $x * y \in J$ we get $\vartheta(y) \geqslant 0 = \min\{0, 1\} = \min\{\vartheta(x), \vartheta(x * y)\}$.

Therefore ϑ is a fuzzy JU-ideal of X. □

Theorem 2.2.2.9. *Let $\{J_i\}$ be any family of JU-ideals of JU-algebra X such that $J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = X$. Then there exist fuzzy JU-ideals which are exactly the JU-ideals of this chain $\{J_i\}$.*

Proof. Consider a set of numbers $t_0 > t_1 > t_3 > \cdots > t_n$ where each $t_i \in [0,1]$. Let $\vartheta : X \to [0,1]$ be a fuzzy set defined by

$$\vartheta(x) = \begin{cases} t_0, & \text{if } x \in J_0 \\ t_i, & \text{if } x \in J_i - J_{i-1},\ 0 < i \leqslant n. \end{cases}$$

Now, we claim that ϑ is a fuzzy JU-ideal of X and $U(\vartheta,t_i) = J_i$ for $0 < i \leqslant n$. Since J_i is JU-ideal of X, then $1 \in J_i$ for $i = 1,2,...,n$. If $x \in J_0$ then $\vartheta(1) = t_0 = \vartheta(x)$ and if $x \in J_i - J_{i-1}$ then $\vartheta(1) = t_i = \vartheta(x)$. In all cases $\vartheta(1) \geqslant \vartheta(x)$

And, for any $x,y \in X$, then we consider the following cases:

Case 1. Let $x, x*y \in J_i - J_{i-1}$. Therefore, by definition of ϑ, we have $\vartheta(x) = t_i = \vartheta(x*y)$. Since J_i is a JU-ideal of X, it follows that $y \in J_i$ and so either $y \in J_i$ or $y \in J_{i-1}$. In any cases we get $\vartheta(y) \geqslant t_i = \min\{\vartheta(x), \vartheta(x*y)\}$.

Case 2. For $i > j$, $t_j > t_i$ and $J_j \subset J_i$. Let $x \in J_i - J_{i-1}$ and $x*y \in J_j - J_{j-1}$ implies $\vartheta(x) = t_i$, $\vartheta(x*y) = t_j \geqslant t_i$. Then $y \in J_i$, since J_i is a JU-ideal of X and $J_j \subset J_i$. Hence, $\vartheta(y) \geqslant t_i = \min\{\vartheta(x), \vartheta(x*y)\}$. Consequently, ϑ is a fuzzy JU-ideal of X. Also, from the definition of ϑ, it follows that $\mathrm{Im}(\vartheta) = \{t_0, t_1, t_2, \dots, t_n\}$. Hence the upper level JU-ideals of ϑ are given by the chain of JU-ideals:

$$U(\vartheta,t_0) \subset U(\vartheta,t_1) \subset \cdots \subset U(\vartheta,t_n) = X, \quad \text{for } 0 < i \leqslant n.$$

Now, $U(\vartheta,t_0) = \{x \in X : \mu(x) \geqslant t_0\} = J_0$.

Finally, we prove that $U(\vartheta,t_i) = J_i$, for $0 < i \leqslant n$. Now let $x \in J_i$, then $\vartheta(x) \geqslant t_i$. This implies $x \in U(\vartheta,t_i)$. Hence, $J_i \subseteq U(\vartheta,t_i)$. If $x \in U(\vartheta,t_i)$, then $\vartheta(x) \geqslant t_i$. Which implies that $x \notin J_j$ for $j > i$. Otherwise, if $x \in J_j$, then $\vartheta(x) \geqslant t_j$ which implies $t_i \geqslant \vartheta(x) \geqslant t_j$. This contradicts the assumption that $x \in U(\vartheta,t_i)$. Hence $\vartheta(x) \in \{t_0, t_1, t_2, \dots, t_n\}$. So, $x \in J_k$ for some $k \leqslant i$. As $J_k \subseteq J_i$, it follows that $x \in J_i$. Hence $U(\vartheta,t_i) \subseteq J_i$. Therefore, $U(\vartheta,t_i) = J_i$, for $0 < i \leqslant n$. □

2.2.3 Homomorphism of Fuzzy JU-Ideals

Theorem 2.2.3.1. *Let* $(X,*,1)$ *and* $(Y,*',1')$ *be JU-algebras and a map* $g : X \to Y$ *be an onto homomorphism, and* ϑ *be a fuzzy JU-ideal of a JU-algebra X. Then the image* $g(\vartheta)$ *is a fuzzy JU-ideal of Y with the sup property.*

Proof. Let $g : X \to Y$ be an onto homomorphism and ϑ be a fuzzy JU-ideal of X. We want to show that $g(\vartheta)$ is a fuzzy JU-ideal in Y. Let $y \in Y$. Since g is onto, there exist $x \in X$ such that $g(x) = y$ Also g is a homomorphism, $g(1) = 1'$. Thus:

$$\begin{aligned} g(\vartheta)(1') &= \sup_{1 \in g^{-1}(1')}\{\vartheta(1)\} \\ &\geqslant \sup_{x \in g^{-1}(y)} \vartheta(x)\}. \\ &= g(\vartheta)(y) \end{aligned}$$

And, let $y_1, y_2 \in Y$. Since g is onto, there exist $x_1, x_2 \in X$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$ Also g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2)$. Now

$$\begin{aligned} g(\vartheta)(y_2) &= \sup_{x_2 \in g^{-1}(y_2)}\{\vartheta(x_2)\} \\ &\geqslant \min\{\sup_{x_1 \in g^{-1}(y_1)}\{\vartheta(x_1)\}, \sup_{x_1 * x_2} \in g^{-1}(y_1 *' y_2)\{\vartheta(x_1 * x_2)\}\} \\ &= \min\{g(\vartheta)(y_1), g(\vartheta)(y_1 *' y_2)\} \end{aligned}$$

Hence $g(\vartheta)$ is a fuzzy ideal of y. □

Theorem 2.2.3.2. *Let $(X,*,1)$ and $(Y,*^{'},1^{'})$ be JU-algebras and a map $g: X \to Y$ be homomorphism. Let ϑ be a fuzzy JU-ideal of Y. Then the Per-image $g^{-1}(\vartheta)$, is a fuzzy JU-ideal of X.*

Proof. Let ϑ be a fuzzy JU-ideal of Y. For any $x \in X$ and $y \in Y$ such that $g(x) = y$. Since g is a homomorphism, $g(1) = 1'$. Then,

$g^{-1}(\vartheta)(1) = \vartheta(g(1)) = \vartheta(1') \geqslant \vartheta(y) = \vartheta(g(x)) = g^{-1}(\vartheta)(x))$

And, for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$

Since g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Then,

$$\begin{aligned} g^{-1}(\vartheta)(x_2) = \vartheta(g(x_2)) &= \vartheta(y_2) \\ &\geqslant \min\{\vartheta(y_1), \vartheta(y_1 *' y_2)\} \\ &= \min\{\vartheta(g(x_1)), \vartheta(g(x_1) *' g(x_2))\} \\ &= \min\{\vartheta(g(x_1)), \vartheta(g(x_1 * x_2))\} \\ &= \min\{g^{-1}(\vartheta)(x_1), g^{-1}(\vartheta)(x_1 * x_2)\} \end{aligned}$$

□

Theorem 2.2.3.3. *Let X, Y, Z be JU-algebras. Then the function $f: X \to Y$ and $g: Y \to Z$ be JU-homomorphism then the following holds:*

(a) If f, g are surjective then $(g \circ f)(\vartheta)$ a fuzzy JU-ideal Z, for any fuzzy JU-ideal of ϑ of X, provided that sup property holds.

(b) $(g \circ f)^{-1}(\lambda)$ is a fuzzy JU-ideal of X for any fuzzy JU-ideal λ of Z.

Proof. (a) Let ϑ be a fuzzy JU-ideal of X, for $z \in Z$,there exist $x \in X$ such that, $(g \circ f)(x) = z$. Then

$(g \circ f)(\vartheta)(1_z) = \sup_{1_x \in (g \circ f)^{-1}(1_z)}\{\vartheta(1_x)\} \geqslant \sup_{1_x \in (g \circ f)^{-1}(z)}\{\vartheta(x)\} = (g \circ f)(\vartheta)(z)$

And, for any $z_1, z_2 \in Z$,there exist $x_1, x_2 \in X$ such that, $(g \circ f)(x_1) = z_1$, $(g \circ f)(x_2) = z_2$. Taking $\vartheta(s) = \sup_{r \in (g \circ f)^{-1}(z_1)}\{\vartheta(r)\}$ and $\vartheta(t) = \sup_{r \in (g \circ f)^{-1}(z_2)}\{\vartheta(r)\}$ for some $s, t \in X$. Also f, g are homomorphism, so is $g \circ f$, and hence $s * t \in (g \circ f)^{-1}(z_1 * z_2)$. Now,

$$\begin{aligned} (g \circ f)(\vartheta)(z_2) \geqslant \vartheta(t) &= \sup_{r \in (g \circ f)^{-1}(z_2)}\{\vartheta(r)\} \\ &\geqslant \min\{\sup_{r \in (g \circ f)^{-1}(z_1)}\{\vartheta(r)\}, \sup_{r \in (g \circ f)^{-1}(z_1 * z_2)}\{\vartheta(r)\}\} \\ &= \min\{\vartheta(s), \vartheta(s * t)\} \\ &= \min\{(g \circ f)(\vartheta)(z_1), (g \circ f)(\vartheta)(z_1 * z_2)\} \end{aligned}$$

Therefore,$(g \circ f)(\vartheta)$ is a fuzzy JU-ideal of Z.

(b). Let β be a fuzzy JU-ideal of Z, and let $x \in X$, $z \in Z$ such that $(g \circ f)(1_X) = 1_Z$. Then,

$(g \circ f)^{-1}(\beta)(1_X) = \beta((g \circ f)(1_X)) \geqslant \beta((g \circ f)(x)) = g \circ f)^{-1}(\beta)(x)$.

Also, for any $x, y \in X$, we have:

$$\begin{aligned}(g\circ f)^{-1}(\beta)(y) &= \beta((g\circ f)(y))\\ &\geqslant \min\left\{\beta((g\circ f)(x)),\, \beta((g\circ f)(x*y))\right\}\\ &= \min\left\{(g\circ f)^{-1}(\beta)(x),\, (g\circ f)^{-1}(\beta)(x*y)\right\}.\end{aligned}$$

□

Theorem 2.2.3.4. *Let f be endomorphism of a JU-algebra X. If ϑ is a fuzzy JU-ideal of X, then so is ϑ_f.*

Proof. Let ϑ be a fuzzy JU-ideal of X.
Now, $\vartheta_f(x) = \vartheta(f(x)) \leqslant \vartheta(f(1)) = \vartheta_f(1), \forall x \in X$.
Therefore, $\vartheta_f(1) \geqslant \vartheta_f(x)$
Let $x, y \in X$, then

$$\begin{aligned}\vartheta_f(y) = \vartheta(f(y)) &\geqslant \min\{\vartheta(f(x)), \vartheta(f(x)*f(y))\}\\ &= \min\{\vartheta(f(x)), \vartheta(f(x*y))\\ &= \min\{\vartheta_f(x), \vartheta_f(x*y)\}\end{aligned}$$

Therefore, $\vartheta_f(y) \geqslant \min\{\vartheta_f(x), \vartheta_f(x*y)\}$, hence ϑ_f is a fuzzy JU-ideal of X. □

Theorem 2.2.3.5. *Let $f : X \to Y$ be an epimorphism of JU-algebra. If ϑ_f is a fuzzy JU-ideal of X, then ϑ is a fuzzy JU-ideal of Y.*

Proof. Let ϑ_f be a fuzzy JU- ideal of X and let $y \in Y$. Then there exist $x \in X$ such that $f(x) = y$

$$\vartheta(1) = \vartheta(f(1)) = \vartheta_f(1) \geqslant \vartheta_f(x) = \vartheta(f(x)) = \vartheta(y)$$

Hence $\vartheta(1) \geqslant \vartheta(y)$.
Let $y_1, y_2 \in Y$, then

$$\begin{aligned}\vartheta(y_1) = \vartheta(f(x_1)) &= \vartheta_f(x_1)\\ &\geqslant \min\{\vartheta_f(x_2*x_1), \vartheta_f(x_2)\}\\ &= \min\{\vartheta(f(x_2*x_1), \vartheta(f(x_2)\}\\ &= \min\{\vartheta(f(x_2)*f(x_1)), \vartheta(f(x_2)\}\\ &= \min\{\vartheta(y_2*y_1), \vartheta(y_2\}\end{aligned}$$

Thus $\vartheta(y_1) \geqslant \min\{\vartheta(y_2*y_1), \vartheta(y_2\}$
Therefore, ϑ is a fuzzy JU-ideal of Y. □

Theorem 2.2.3.6. *Let* $f : X \to Y$ *be a homomorphism of a JU-algebra. If* ϑ *is a fuzzy JU-ideal of Y, then* ϑ_f *is a fuzzy JU-ideal of X.*

Proof. Let ϑ be a fuzzy JU- ideal of Y and let $x, y \in X$, then
$\vartheta_f(1) = \vartheta(f(1)) \geqslant \vartheta(f(x)) = \vartheta_f(x)$, hence $\vartheta_f(1) \geqslant \vartheta_f(x)$

$$\begin{aligned} \text{Also} \quad \vartheta_f(x) = \vartheta(f(x)) &\geqslant \min\{\vartheta(f(y) * \vartheta(f(x)), \vartheta(f(y))\} \\ &= \min\{\vartheta(f(y * x)), \vartheta(f(y))\} \\ &= \min\{\vartheta_f(y * x), \vartheta_f(y)\} \end{aligned}$$

Therefore, $\vartheta_f(x) \geqslant \min\{\vartheta_f(y * x), \vartheta_f(y)\}$, hence ϑ_f is a fuzzy JU-ideal of X. □

2.2.4 Cartesian Product of Fuzzy JU-Ideals

In this section, the Cartesian product interval-valued-fuzzy ideals of X are discussed and some of their properties are investigated.

Theorem 2.2.4.1. *Let* ϑ_1 *and* ϑ_2 *be two fuzzy JU-ideals of X and Y, respectively. Then the Cartesian product of* $\vartheta_1 \times \vartheta_2$ *is also a fuzzy JU-ideal of* $X \times Y$.

Proof. Let ϑ_1 be a fuzzy JU-ideal of X and ϑ_2 be a fuzzy JU-ideal of Y. We went to show that $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-ideal of $X \times Y$. Let $(x, y) \in X \times Y$. Then

$(\vartheta_1 \times \vartheta_2)(1,1) = \min\{\vartheta_1(1), \vartheta_2(1)\} \geqslant \min\{\vartheta_1(x), \vartheta_2(y)\} = (\vartheta_1 \times \vartheta_2)(x, y)$, for all $(x, y) \in X \times Y$.

And, let $(x_1, x_2), (y_1, y_2) \in X \times Y$, then

$$\begin{aligned} (\vartheta_1 \times \vartheta_2)(y_1, y_2) &= \min\{\vartheta_1(y_1), \vartheta_2(y_2)\} \\ &\geqslant \min\{\min\{\vartheta_1(x_1), \vartheta_1(x_1 * y_1\}, \min\{\vartheta_2(x_2), \vartheta_2(x_2 * y_2\}\} \\ &= \min\{\min\{\vartheta_1(x_1), \vartheta_2(x_2\}, \min\{\vartheta_1(x_1 * y_1), \vartheta_2(x_2 * y_2\}\} \\ &= \min\{(\vartheta_1 \times \vartheta_2)((x_1, x_2), (\vartheta_1 \times \vartheta_2)((x_1 * y_1, x_2 * y_2)\} \\ &= \min\{(\vartheta_1 \times \vartheta_2)((x_1, x_2), (\vartheta_1 \times \vartheta_2)((x_1, x_2) * (y_1, y_2))\} \end{aligned}$$

□

Theorem 2.2.4.2. *Let* ϑ_1 *and* ϑ_2 *be two fuzzy subsets of* X *and* Y, *respectively such that* $\vartheta_1 \times \vartheta_2$ *is a fuzzy JU-ideal of* $X \times Y$. *Then either* ϑ_1 *is a fuzzy JU-ideal of* X *or* ϑ_2 *is a fuzzy JU-ideal of* Y.

Proof. Let ϑ_1 and ϑ_2 be fuzzy subsets of X and Y respectively. Suppose that $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-ideal of $X \times Y$. We need to show that either ϑ_1 is a fuzzy JU-ideal of X or ϑ_2 is a fuzzy JU-ideal of Y. Then $(\vartheta_1 \times \vartheta_2)(1,1) \geqslant (\vartheta_1 \times \vartheta_2)(x, y)$, for all $(x, y) \in X \times Y$. Suppose $\vartheta_1(x) \geqslant \vartheta_1(1)$ and $\vartheta_2(y) \geqslant \vartheta_2(1)$, for some $(x, y) \in X \times Y$. Then $(\vartheta_1 \times \vartheta_2)(x, y) = \min\{\vartheta_1(x), \vartheta_2(y)\} \geqslant \min\{\vartheta_1(1), \vartheta_2(1)\} =$

$(\vartheta_1 \times \vartheta_2)(1,1)$, this leads to a contradiction. Thus $\vartheta_1(1) \geqslant \vartheta_1(x)$ and $\vartheta_2(y) \geqslant \vartheta_2(1)$, for $(x,y) \in X \times Y$.
Take $(x_1,x_2),(y_1,y_2) \in X \times Y$, and let $x = (x_1,x_2)$, $y = (y_1,y_2)$ Then

$$\begin{aligned}
(\vartheta_1 \times \vartheta_2)(y) &\geqslant \min\{(\vartheta_1 \times \vartheta_2)(x),(\vartheta_1 \times \vartheta_2)(x * y)\} \\
&= \min\{(\vartheta_1 \times \vartheta_2)(x_1,x_2),(\vartheta_1 \times \vartheta_2)((x_1,x_2) * (y_1,y_2))\} \\
&= \min\{(\vartheta_1 \times \vartheta_2)(x_1,x_2),(\vartheta_1 \times \vartheta_2)((x_1 * y_1),(x_2 * y_2))\} \\
&= \min\{\min\{\vartheta_1(x_1),\vartheta_2(x_2)\},\min\{\vartheta_1(x_1 * y_1),\vartheta_2(x_2 * y_2)\}\}.
\end{aligned}$$

As $(\vartheta_1 \times \vartheta_2)(y) = (\vartheta_1 \times \vartheta_2)(y_1,y_2)$, we have

$$\begin{aligned}
&(\vartheta_1 \times \vartheta_2)(y_1,y_2) \geqslant \min\{\min\{\vartheta_1(x_1),\vartheta_2(x_2)\},\min\{\vartheta_1(x_1 * y_1),\vartheta_2(x_2 * y_2)\}\}. \\
\Rightarrow\ &\min\{\vartheta_1(y_1),\vartheta_2(y_2)\} \geqslant \min\{\min\{\vartheta_1(x_1),\vartheta_2(x_2)\},\min\{\vartheta_1(x_1 * y_1),\vartheta_2(x_2 * y_2)\}\} \\
&\Rightarrow \text{either } \vartheta_1(y_1) \geqslant \min\{\vartheta_1(x_1),\vartheta_1(x_1 * y_1)\} \\
&\qquad \text{or } \vartheta_2(y_2) \geqslant \min\{\vartheta_2(x_2),\vartheta_2(x_2 * y_2)\}.
\end{aligned}$$

Hence, either ϑ_1 is a fuzzy JU-ideal of X or ϑ_2 is a fuzzy JU-ideal of Y. □

Theorem 2.2.4.3. *Let ϑ_1 and ϑ_2 be fuzzy subsets of X and Y, respectively such that $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-ideal of $X \times Y$. Then*

(i) either $\vartheta_1(1) \geqslant \vartheta_1(x)$ or $\vartheta_2(1) \geqslant \vartheta_2(y)$, for all $(x,y) \in X \times Y$.

(ii) If $\vartheta_1(1) \geqslant \vartheta_1(x)$, for all $x \in X$, then either $\vartheta_2(1) \geqslant \vartheta_2(y)$ or $\vartheta_2(1) \geqslant \vartheta_1(x)$, for all $(x,y) \in X \times Y$.

(iii) If $\vartheta_2(1) \geqslant \vartheta_2(y)$, for all $y \in Y$, then either $\vartheta_1(1) \geqslant \vartheta_1(x)$ or $\vartheta_1(1) \geqslant \vartheta_2(y)$, for all $(x,y) \in X \times Y$.

Proof. Let $\vartheta_1 \times \vartheta_2$ be a fuzzy JU-ideal of $X \times Y$. Then

(i) Suppose that $\vartheta_1(1) < \vartheta_1(x)$, for some $x \in X$ and $\vartheta_2(1) < \vartheta_2(y)$, for some $y \in Y$. Then $(\vartheta_1 \times \vartheta_2)(x,y) = \min\{\vartheta_1(x),\vartheta_2(y)\} > \min\{\vartheta_1(1),\vartheta_2(1)\} = (\vartheta_1 \times \vartheta_2)(1,1)$. This is a contradiction to the fact that $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-ideal of $X \times Y$.

   Therefore, either $\vartheta_1(1) \geqslant \vartheta_1(x)$ or $\vartheta_2(1) \geqslant \vartheta_2(y)$, for all $(x,y) \in X \times Y$.

(ii) Suppose that $\vartheta_1(1) \geqslant \vartheta_1(x)$, for all $x \in X$. Assume that there exist $x \in X$ and $y \in Y$ such that $\vartheta_2(1) < \vartheta_1(x)$ and $\vartheta_2(1) < \vartheta_2(y)$. Then $\vartheta_2(1) < \vartheta_1(x) \leqslant \vartheta_1(1)$. Thus $(\vartheta_1 \times \vartheta_2)(x,y) = \min\{\vartheta_1(x),\vartheta_2(y)\} > \min\{\vartheta_2(1),\vartheta_2(1)\} = \vartheta_2(1) = \min\{\vartheta_1(1),\vartheta_2(1)\} = (\vartheta_1 \times \vartheta_2)(1,1)$. This is a contradiction since $\vartheta_1 \times \vartheta_2$ is a fuzzy JU-ideal of $X \times Y$.

   Hence, either $\vartheta_2(1) \geqslant \vartheta_2(y)$, for all $y \in Y$ or $\vartheta_2(1) \geqslant \vartheta_1(x)$, for all $x \in X$.

(iii) Assume that $\vartheta_2(1) \geqslant \vartheta_2(y)$, for all $y \in Y$. Assume that there exist $x \in X$ and $y \in Y$ such that $\vartheta_1(1) < \vartheta_1(x)$ and $\vartheta_1(1) < \vartheta_2(y)$. Then $\vartheta_1(1) < \vartheta_2(y) \leqslant \vartheta_2(y)$. Thus $(\vartheta_1 \times \vartheta_2)(x,y) = \min\{\vartheta_1(x), \vartheta_2(y)\} > \min\{\vartheta_1(1), \vartheta_1(1)\} = \vartheta_1(1) = \min\{\vartheta_1(1), \vartheta_2(1)\} = (\vartheta_1 \times \vartheta_2)(1,1)$ which is a contradiction. Hence, either $\vartheta_1(1) \geqslant \vartheta_1(x)$, for all $x \in X$ or $\vartheta_1(1) \geqslant \vartheta_2(y)$, for all $y \in Y$.

$\square$

Chapter 3
Q-Fuzzy Structures on JU-Algebra

In this chapter, we discuss the notions of Q-fuzzy JU-sub algebras and Q-fuzzy JU-ideals of JU-algebras, along with their lower and upper level subsets. We also discuss the doubt and Normal Q-fuzzy JU-sub algebras and Q-fuzzy JU-ideals of a JU-algebra, and investigate some of their properties.

3.1. Q-Fuzzy JU-Subalgebras of JU-Algebra

Definition 3.1.1. *Let X be a JU-algebra, a Q-fuzzy set* η *in X is called Q-fuzzy JU-subalgebra of X if for any* $x, y \in X$

$$\eta(x * y, q) \geqslant \min\{\eta(x,q), \eta(y,q)\}, \forall q \in Q$$

Example 3.1.1. *Let X =* $\{1,2,3,4\}$ *in which* $*$ *is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3], $(X, *, 1)$ *is a JU-algebra. Let* $Q = \{q_1, q_2\}$*, given a Q-fuzzy set* $\eta : Q \times X \to [0,1]$ *by*

$$\eta(x,q_1) = \begin{cases} 0.9, & x = 1, \\ 0.6, & x = 2, \\ 0.4, & x = 3, \\ 0.5, & x = 4. \end{cases}, \quad \eta(x,q_2) = \begin{cases} 0.7, & x = 1, \\ 0.5, & x = 2, \\ 0.3, & x = 3, \\ 0.4, & x = 4. \end{cases}$$

To verify η is a Q-fuzzy JU-subalgebra of X,
let we take, $x = 1, y = 2$, then $\eta(1 * 2, q_1) = \eta(2, q_1) = 0.6 \cdots (*)$
And, $\min\{\eta(1,q_1), \eta(2,q_1)\} = \min\{0.9, 0.6\} = 0.6 \cdots (**)$.
From (*) and (**) we get $0.6 \geqslant 0.6$ which is true.
For $x = 2, y = 3$, then $\eta(2 * 3, q_1) = \eta(4, q_1) = 0.5 \cdots (***)$
And, $\min\{\eta(2,q_1), \eta(3,q_1)\} = \min\{0.6, 0.4\} = 0.4 \cdots (****)$.

From (***) and (****) we get $0.5 \geqslant 0.4$ which is true.
Similarly, the result holds for all values of $x, y \in X$ and $q_1, q_2 \in Q$.
Therefore, we conclude that η is a Q-fuzzy JU-subalgebra of X.

Lemma 3.1.1. *If η is a Q-fuzzy JU-subalgebra of a JU-algebra X. Then $\eta(1,q) \geqslant \eta(x,q)$, $\forall x \in X$ and $\forall q \in Q$.*

Proof. Suppose η a be a Q-fuzzy JU-subalgebra of a JU-algebra X.
Since, $\eta(1,q) = \eta(x * x, q)$ (By Lemma 1.6.1)

$$\begin{aligned} &\geqslant \min\{\eta(x,q), \eta(x,q)\} \\ &= \eta(x,q) \end{aligned}$$

□

Theorem 3.1.2. *Let η be a Q-fuzzy JU-subalgebra of a JU-algebra X such that $\eta(1) \neq 0$. Let $\eta^+ : X \times Q \to [0,1]$ be a Q-fuzzy set defined by $\eta^+(x,q) = \frac{\eta(x,q)}{\eta(1,q)}$, for all $x \in X$ and $q \in Q$. Then η^+ is a Q-fuzzy JU-subalgebra of X.*

Proof. Suppose that η is a Q-fuzzy JU-subalgebra of X, with $\eta(1,q) \neq 0$. Let $x, y \in X$ and $q \in Q$. Then:

$$\begin{aligned} \eta^+(x * y, q) = \frac{\eta(x * y, q)}{\eta(1,q)} &= \frac{1}{\eta(1,q)} \eta(x * y, q) \\ &\geqslant \frac{1}{\eta(1,q)} \min\{\eta(x,q), \eta(y)\} \\ &= \min\{\frac{\eta(x,q)}{\eta(1,q)}, \frac{\eta(y,q)}{\eta(1,q)}\} \\ &= \min\{\eta^+(x,q), \eta^+(y,q)\} \end{aligned}$$

Therefore, η^+ is a Q-fuzzy JU-subalgebra of X. □

Theorem 3.1.3. *Let η_1 and η_2 be any two Q-fuzzy JU-subalgebras of X. Then $\eta_1 \cap \eta_2$ is a Q-fuzzy JU-subalgebra of X.*

Proof. Suppose η_1 and η_2 are Q-fuzzy JU-subalgebras of X, for all $x, y \in X$, $q \in Q$. Then

$$\begin{aligned} (\eta_1 \cap \eta_2)(x * y, q) &= \min\{\eta_1(x * y, q), \eta_2(x * y, q)\} \\ &\geqslant \min\{\min\{\eta_1(x,q), \eta_1(y,q)\}, \min\{\eta_2(x,q), \eta_2(y,q)\}\} \\ &= \min\{\min\{\eta_1(x,q), \eta_2(x,q)\}, \min\{\eta_1(y,q), \eta_2(y,q)\}\} \\ &= \min\{\eta_1 \cap \eta_2(x,q), \eta_1 \cap \eta_2(y,q)\} \end{aligned}$$

$\Rightarrow (\eta_1 \cap \eta_2)(x * y, q) \geqslant \min\{(\eta_1 \cap \eta_2)(x,q), (\eta_1 \cap \eta_2)(y,q)\}$, hence $\eta_1 \cap \eta_2$ is a Q-fuzzy JU-subalgebra of X. □

Corollary 3.1.4. *Let $\{\eta_j : j \in \Omega\}$ be a family of Q-fuzzy JU-subalgebras of X. Then $\bigcap_{j \in \Omega} \eta_j$ is also a Q-fuzzy JU-subalgebra of X.*

Proof. Define a Q-fuzzy set $\eta = \bigcap_{j \in \Omega} \eta_j$ pointwise by

$$\eta(x,q) = \inf_{j \in \Omega} \eta_j(x,q), \quad \forall x \in X, q \in Q.$$

We show that η satisfies the defining closure conditions of a Q-fuzzy JU-subalgebra. For each $j \in \Omega$ and all $x, y \in X$, $q \in Q$, we have

$$\eta_j(x * y, q) \geqslant \min\{\eta_j(x,q), \eta_j(y,q)\}.$$

Then

$$\begin{aligned}\eta(x * y, q) = \inf_{j \in \Omega} \eta_j(x * y, q) &\geqslant \inf_{j \in \Omega} \min\{\eta_j(x,q), \eta_j(y,q)\} \\ &= \min\left\{ \inf_{j \in \Omega} \eta_j(x,q), \inf_{j \in \Omega} \eta_j(y,q) \right\} \\ &= \min\{\eta(x,q), \eta(y,q)\}.\end{aligned}$$

□

Remark 3.1.1. *The union of two Q-fuzzy JU-subalgebras may not be a Q-fuzzy JU-subalgebra.*

Example 3.1.2. *Let X = $\{1,2,3,4\}$ in which $*$ is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $(X; *, 1)$ *is a JU-algebra. And we define* $\eta_1 : X \times Q \to [0,1]$ *as follows:*

$$\eta_1(1,q) = 0.7 = \eta_1(3,q), \eta_1(2,q) = 0.5 = \eta_1(4,q).$$

and $\eta_2 : X \times Q \to [0,1]$ *as follows:*

$$\eta_2(1,q) = 0.8, \eta_2(2,q) = 0.6, \eta_2(3,q) = 0.4 = \eta_1(4,q).$$

Then:

$$\begin{aligned}(\eta_1 \cup \eta_2)(2 * 3, q) &= \max\{\eta_1(2 * 3, q), \eta_2(2 * 3, q)\} \\ &= \max\{\eta_1(4,q), \eta_2(4,q)\} = \max\{0.5, 0.4\} = 0.5\end{aligned}$$

Hence $(\mu \cup \eta)(2 * 3, q) = 0.5 \cdots (*)$. *And,*

$$\begin{aligned}\max\{\eta_1(2 * 3, q), \eta_2(2 * 3, q)\} &\geqslant \max\{\min\{\eta_1(2,q), \eta_1(3,q)\}, \min\{\eta_2(2,q), \eta_2(3,q)\}\} \\ &= \min\{\max\{\eta_1(2,q), \eta_2(2,q)\}, \max\{\eta_1(3,q), \eta_2(3,q)\}\} \\ &= \min\{\max\{0.5, 0.6\}, \max\{0.7, 0.4\}\}\end{aligned}$$

$$= \min\{0.6, 0.7\} = 0.6 \cdots (**).$$

From $(*)$ *and* $(**)$ *we get* $0.5 \geqslant 0.6$, *which is false. This shows that the union of two Q-fuzzy JU-subalgebras of a JU-algebra may not be a Q-fuzzy JU-subalgebra.*

Theorem 3.1.5. *If* η *is a Q-fuzzy JU-subalgebra of X, then the set* $X_\eta = \{x \in X, q \in Q | \eta(x,q) = \eta(1,q)\}$ *is a JU-subalgebra of X.*

Proof. Let $x, y \in X_\eta$. Then $\eta(x,q) = \eta(1,q) = \eta(y,q)$, and so

$\eta(x*y,q) \geqslant \min\{\eta(x,q), \eta(y,q)\} = \min\{\eta(1,q), \eta(1,q)\}$.

$\Rightarrow \eta(x*y,q) \geqslant \eta(1,q)$ but $\eta(x*y,q) \leqslant \eta(1,q)$, $\forall x,y \in X$ (by Lemma 3.1.1)

$\Rightarrow \eta(x*y,q) = \eta(1,q)$

$\Rightarrow x*y \in X_\eta$.

Hence X_η is a JU-subalgebra of X. □

Lemma 3.1.6. *Let* η_1 *and* η_2 *be two Q-fuzzy JU-subalgebras of a JU-algebra* X. *If* $\eta_1 \subseteq \eta_2$ *and* $\eta_1(1,q) = \eta_2(1,q)$, *for* $q \in Q$ *then* $X_{\eta_1} \subseteq X_{\eta_2}$.

Proof. Assume that η_1 and η_2 be two Q-fuzzy JU-subalgebras of a JU-algebra X and $\eta_1 \subseteq \eta_2$. Then from definition of $' \subseteq '$ we have $\eta_1(x,q) \leqslant \eta_2(x,q)$ for all $x \in X$ and $q \in Q$. Then, for any $x \in X_{\eta_1}$, we get $\eta_2(x,q) \geqslant \eta_1(x,q) = \eta_1(1,q) = \eta_2(1,q)$
$\Rightarrow \eta_2(x,q) \geqslant \eta_2(1,q)$, but $\eta_2(x,q) \leqslant \eta_2(1,q)$ (by Lemma 3.1.1)
$\Rightarrow \eta_2(x,q) = \eta_2(1,q)$.
Thus, $x \in X_{\eta_2}$, hence, $X_{\eta_1} \subseteq X_{\eta_2}$. □

Theorem 3.1.7. *Let S be a subset of a JU-algebra X. Then the characteristics function* χ_S *is a Q-fuzzy JU-subalgebra of X if and only if S is a subalgebra of X.*

Proof. Let $\chi_S : X \times Q \to \{0,1\}$ be a Q-fuzzy JU-subalgebra of a JU-algebra X and let $x, y \in S$ and implies that $\chi_S(x,q) = \chi_S(y,q) = 1$. Then,

$\chi_S(x*y,q) \geqslant \min\{\chi_S(x,q), \chi_S(y,q)\} = \min\{1,1\} = 1$

$\Rightarrow \chi_S(x*y,q) \geqslant 1$, but $\chi_S(x*y,q) \leqslant 1$

$\Rightarrow \chi_S(x*y,q) = 1$

$\Rightarrow x*y \in S$, implies that S is a JU-subalgebra of X.

Conversely, let S be JU-subalgebra of X and $x, y \in X$ and $q \in Q$. Consider the following cases:

Case 1. if $x, y \in S$, then $x*y \in S$, we get that

$$\chi_S(x*y,q) = 1 = \min\{\chi_S(x,q), \chi_S(y,q)\}$$

Case 2. if $x, y \notin S$, then $\chi_S(x,q) = 0 = \chi_S(x,q)$

Thus, $\chi_S(x*y,q) = 0 = \min\{\chi_S(x), q, \chi_S(y,q)\}$

Case 3. if x or y not in S, then $\min\{\chi_S(x,q), \chi_S(y,q)\} = 0$.

Then, $\chi_S(x*y,q) = 0 = \min\{\chi_S(x,q), \chi_S(y,q)\}$

Hence, χ_S is a Q-fuzzy JU-subalgebra of X. $\square$

Theorem 3.1.8. *Suppose* η *is a* Q*-fuzzy JU-subalgebra of* X*. Then for* $t \in [0,1]$ *such that* $t \leqslant \eta(x,q)$*, the level subset* $U(\eta;t,q)$ *of* η *is a JU-subalgebra of* X*.*

Proof. Suppose that η is a Q-fuzzy JU-subalgebra of X. Then for all $x,y \in U(\eta;t,q)$ and $q \in Q$, we have $\eta(x,q) \geqslant t$ and $\eta(y,q) \geqslant t$. Then,

$$\eta(x*y,q) \geqslant \min\{\eta(x,q),\eta(y,q)\} \geqslant \min\{t,t\} = t$$

$\Rightarrow \eta(x*y,q) \geqslant t$, and hence $x*y \in U(\eta;t,q)$.
Therefore, $U(\eta;t,q)$ is a JU-subalgebra of X. $\square$

Theorem 3.1.9. *A Q-fuzzy set* η *of a JU-algebra X is a Q-fuzzy JU-subalgebra of X if and only if for every* $t \in [0,1]$ *and* $q \in Q$, $U(\eta;t,q) = \{x \in X \mid \eta(x,q) \geqslant t\}$ *is either the empty set or a JU-subalgebra of X.*

Proof. Suppose that η is a Q-fuzzy JU-subalgebra of a JU-algebra X. We need to show that $U(\eta;t,q)$ is a JU-subalgebra of X. If the condition is not satisfied by any element, $U(\eta;t,q)$ is empty. Now assume that U(μA; t) is nonempty. Then for all $x,y \in U(\eta;t,q)$ and $q \in Q$, we have $\eta(x,q) \geqslant t$ and $\eta(y,q) \geqslant t$. Then,

$$\eta(x*y,q) \geqslant \min\{\eta(x,q),\eta(y,q)\} \geqslant \min\{t,t\} = t$$

which implies $\eta(x*y,q) \geqslant t$, and hence $x*y \in U(\eta;t,q)$.
Therefore, $U(\eta;t,q)$ is a JU-subalgebra of X.
Conversely, assume that $U(\eta;t,q)$ is a JU-subalgebra of X and $U(\eta;t,q)$ nonempty set. We need η is a Q-fuzzy JU-subalgebra of X.
Let $x,y \in X$ and $q \in Q$, there exist $t_1,t_2 \in [0,1]$ such that $\eta(x,q) = t_1$ and $\eta(y,q) = t_2$. Take, $t = \min\{t_1,t_2\}$. Then $x \in U(\eta;t_1,q)$ and $y \in U(\eta;t_2,q)$.
Choose $t = \min\{t_1,t_2\}$, then $t \leqslant t_1$ and $t \leqslant t_2$.
$\Rightarrow U(\eta;t_1,q) \subseteq U(\eta;t,q)$ and $U(\eta;t_2,q) \subseteq U(\eta;t,q)$
$\Rightarrow x,y \in U(\eta;t,q)$
By assumption $U(\eta;t,q)$ is a JU-subalgebra of X, implies that $x*y \in U(\eta;t,q)$. Then,
$\eta(x*y,q) \geqslant t = \min\{t_1,t_2\} = \min\{\eta(x,q),\eta(y,q)\}$.
Therefore η is a Q-fuzzy JU-subalgebra of X. $\square$

Theorem 3.1.10. *Let* η *be a* Q*-fuzzy set in a JU-algebra X. Then* η^c *is a* Q*-fuzzy JU-subalgebra of* X *if and only if for* $s \in [0,1]$ *and* $q \in Q$, $L(\eta;s,q)$ *is either empty or a JU-subalgebra of* X*.*

Proof. Suppose that η^c is a Q-fuzzy JU-subalgebra of X. Let $q \in Q$ and $s \in [0,1]$ be such that $L(\eta;s,q) \neq \emptyset$ and let $x,y \in L(\eta;s,q)$. Then $\eta(x,q) \leqslant s$ and $\eta(y,q) \leqslant s$. Now,

$$\eta^c(x*y,q) \geqslant \min\{\eta^c(x,q),\eta^c(y,q)\}$$

$$
\begin{aligned}
&= \min\{1-\eta(x,q), 1-\eta(y,q)\} \\
&= 1-\max\{\eta(x,q), \eta(y,q)\} \quad \text{(By Lemma 1.1.1)} \\
\Longrightarrow\ & \eta(x*y,q) \leqslant \max\{\eta(x,q), \eta(y,q)\} \leqslant s \\
\Longrightarrow\ & x*y \in L(\eta; s, q).
\end{aligned}
$$

Hence, $L(\eta; s, q)$ is a JU-subalgebra of X.
Conversely, suppose that $L(\eta; s, q)$ is a nonempty JU-subalgebra of X. Let $x, y \in X$, $q \in Q$, and let $s = \max\{\eta(x,q), \eta(y,q)\}$. Thus $\eta(x,q) \leqslant s$ and $\eta(y,q) \leqslant s$, so $x, y \in L(\eta; s, q)$. By assumption, $L(\eta; s, q)$ is a JU-subalgebra of X. It follows that $x*y \in L(\eta; s, q)$. Thus,

$$
\begin{aligned}
&\eta(x*y,q) \leqslant s = \max\{\eta(x,q), \eta(y,q)\} \\
&1-\eta(x*y,q) \geqslant 1-\max\{\eta(x,q), \eta(y,q)\} \\
&\qquad = \min\{1-\eta(x,q), 1-\eta(y,q)\}, \quad \text{(By Lemma 1.1.1)} \\
\Longrightarrow\ & \eta^c(x*y,q) \geqslant \min\{\eta^c(x,q), \eta^c(y,q)\}.
\end{aligned}
$$

□

Theorem 3.1.11. *Every Q-fuzzy JU-subalgebra of X can be realized as level JU-subalgebra of some Q-fuzzy JU-subalgebra of X.*

Proof. Let S be a JU-subalgebra of X and let η be a Q-fuzzy set in X defined by

$$
\eta(x,q) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{other wise} \end{cases}
$$

, for all $x \in X$ and $q \in Q$, where $t \in [0,1]$.
We show that η is a Q-fuzzy JU-subalgebra of X. Let $x, y \in X$. Then consider the following cases:

Case 1. If $x, y \in S$, then $x*y \in S$, and so

$$\eta(x*y,q) \geqslant t = \min\{\eta(x,q), \eta(y,q)\}.$$

Case 2. If $x, y \notin S$, then $\eta(x,q) = 0 = \eta(y,q)$ and thus

$$\eta(x*y,q) \geqslant 0 = \min\{\eta(x,q), \eta(y,q)\}.$$

Case 3. If $x \in S$ and $y \notin S$, then

$$\eta(x*y,q) \geqslant 0 = \min\{1,0\} = \min\{\eta(x,q), \eta(y,q)\}.$$

Or if $x \notin S$ and $y \in S$ we get $\eta(x*y,q) \geqslant 0 = \min\{0,1\} = \min\{\eta(x,q), \eta(y,q)\}$.

Therefore, η is a Q-fuzzy JU-subalgebra of X. □

Theorem 3.1.12. *Let $f: X \to Y$ be an epimorphism of JU-algebras and Q be any nonempty set. If η is a Q-fuzzy JU-subalgebra of X, then $f(\eta)$ is a Q-fuzzy JU-subalgebra ofY.*

Proof. Let η be a Q-fuzzy JU-subalgebra of X and $f : X \to Y$ be an epimorphism of JU-algebras. Then $f(x*y) = f(x)*f(y)$ and $f(x), f(y) \in Y, \forall x, y \in X, q \in Q$.
We claim that $f(\eta)$ is a Q-fuzzy JU-subalgebra of Y. Then for any $x, y \in X$ and $q \in Q$. Then

$$\begin{aligned} f(\eta)(f(x)*f(y),q) = f(\eta)(f(x*y),q) &\geqslant \eta(x*y,q) \quad \text{(by Definition 1.1.11)} \\ &\geqslant \min\{\eta(x,q),\eta(y,q)\} \\ &= \min\{f(\eta)(f(x),q), f(\eta)(f(y),q)\} \\ \Rightarrow f(\eta)(f(x)*f(y),q) &\geqslant \min\{f(\eta)(f(x),q), f(\eta)(f(y),q)\}. \end{aligned}$$

Therefore, $f(\eta)$ is a Q-fuzzy JU-subalgebra of Y. ☐

Theorem 3.1.13. *Let $f : X \to Y$ be an epimorphism of JU-algebras and Q be any non empty set. Then η is a Q-fuzzy JU-subalgebra of Y if and only if $f^{-1}(\eta)$ is a Q-fuzzy JU-subalgebra of X.*

Proof. Assume that η is a Q-fuzzy JU-subalgebra of Y. We need to show that $f^{-1}(\eta)$ is a Q-fuzzy JU-subalgebra of X. Let $x, y \in X$, then $f(x), f(y) \in Y$. Now we have

$$\begin{aligned} f^{-1}(\eta)(x*y,q) &= \eta(f(x*y),q) = \eta(f(x)*f(y),q) \\ &\geqslant \min\{\eta(f(x),q),\eta(f(y),q)\} \\ &= \min\{f^{-1}(\eta)(f(x),q), f^{-1}(\eta)(f(y),q)\} \\ \Rightarrow f^{-1}(\eta)(x*y,q) &\geqslant \min\{f^{-1}(\eta)(f(x),q), f^{-1}(\eta)(f(y),q)\}. \end{aligned}$$

Conversely, assume that $f^{-1}(\eta)$ is a Q-fuzzy JU-subalgebra of X. We need to show that η is a Q-fuzzy JU-subalgebra of Y. Let $a, b \in Y$ and $q \in Q$. Since f is epimorphism of JU-algebras, there exist $x, y \in X$ such that $f(x) = a, f(y) = b$. Now

$$\begin{aligned} \eta(a*b,q) = \eta(f(x)*f(y),q) &= \eta(f(x*y),q) \\ &= f^{-1}(\eta)(x*y,q) \\ &\geqslant \min\{f^{-1}(\eta)(x,q), f^{-1}(\eta)(y,q)\} \\ &= \min\{\eta(f(x),q),\eta(f(y),q)\} \\ &= \min\{\eta(a,q),\eta(b,q)\} \\ \Rightarrow \eta(a*b,q) &\geqslant \min\{\eta(a,q),\eta(b,q)\}. \end{aligned}$$

Hence, η is a Q-fuzzy JU-subalgebra of Y. ☐

Lemma 3.1.14. *Let η_1 and η_2 be any two Q-fuzzy JU-subalgebras of X and Y respectively, then $(\eta_1 \times \eta_2)((1,1),q) \geqslant (\eta_1 \times \eta_2)((x,y),q)$, for $(x,y) \in X \times Y$, $q \in Q$.*

Proof. Suppose that η_1 and η_2 are any two Q-fuzzy JU-subalgebras of X and Y respectively. For any $(x,y) \in X \times Y$, then

$$(\eta_1 \times \eta_2)((1,1),q) = \min\{\eta_1(1,q),\eta_2(1,q)\}$$

$$\geqslant \min\{\eta_1(x,q),\eta_2(y,q)\}$$
$$= (\eta_1 \times \eta_2)((x,y),q).$$

Thus, $(\eta_1 \times \eta_2)((1,1),q) \geqslant (\eta_1 \times \eta_2)((x,y),q)$. □

Theorem 3.1.15. *Let η_1 and η_2 be Q-fuzzy JU-subalgebras of X and Y respectively. Then $\eta_1 \times \eta_2$ is a Q-fuzzy JU-subalgebra of $X \times Y$.*

Proof. Let η_1 and η_2 be any two Q-fuzzy JU-subalgebras of X and Y respectively. Let $(x_1,y_1),(x_2,y_2) \in X \times Y$ and $q \in Q$, then

$$\begin{aligned}
\eta_1 \times \eta_2[(x_1,y_1) * (x_2,y_2),q] &= \eta_1 \times \eta_2((x_1 * x_2, y_1 * y_2),q) \\
&= \min\{\eta_1(x_1 * x_2,q),\eta_2(y_1 * y_2,q)\} \\
&\geqslant \min\{\min\{\eta_1(x_1,q),\eta_1(x_2,q)\},\min\{\eta_2(y_1,q),\eta_2(y_2,q)\}\} \\
&= \min\{\min\{\eta_1(x_1,q),\eta_2(y_1,q)\},\min\{\eta_1(x_2,q),\eta_2(y_2,q)\}\} \\
&= \min\{\eta_1 \times \eta_2((x_1,y_1),q),\eta_1 \times \eta_2((x_2,y_2),q)\}
\end{aligned}$$

□

Theorem 3.1.16. *Let η_1 and η_2 be Q-fuzzy subset of X and Y respectively. If $\eta_1 \times \eta_2$ is Q-fuzzy JU-subalgebra $X \times Y$, then either (i) or (ii) holds:*

i) $\eta_2(1,q) \geqslant \eta_1(x,q)$

ii) $\eta_1(1,q) \geqslant \eta_2(y,q)$

for all $x \in X$, $y \in Y$, *and* $q \in Q$.

Proof. Suppose $\eta_1 \times \eta_2$ is Q-fuzzy JU-subalgebra of $X \times Y$. Assume, for contradiction, that neither (i) nor (ii) holds. Then there exist $x \in X$ and $y \in Y$ such that:

(i) $\eta_1(x,q) > \eta_2(1,q)$

(ii) $\eta_2(y,q) > \eta_1(1,q)$

, for $q \in Q$.
Then we have

$$\eta_1 \times \eta_2((x,y),q) = \min\{\eta_1(x,q),\eta_2(y,q)\} > \min\{\eta_1(1,q),\eta_2(1,q)\} = \eta_1 \times \eta_2((1,1),q)$$

Thus,
$$\eta_1 \times \eta_2((x,y),q) > \eta_1 \times \eta_2((1,1),q)$$

which contradicts Lemma 3.1.14
Hence, either (i) or (ii) holds for all $x \in X$, $y \in Y$, $q \in Q$. □

Theorem 3.1.17. *Let η_1 and η_2 be Q-fuzzy subsets of X and Y, respectively, such that $\eta_1 \times \eta_2$ is a Q-fuzzy JU-subalgebra of $X \times Y$. Then the following statements hold true:*

(i) If $\eta_1(x,q) \leqslant \eta_2(1,q)$, for all $x \in X$, then η_1 is a Q-fuzzy JU-subalgebra of X.

(ii) If $\eta_2(x,q) \leqslant \eta_1(1,q)$, for all $x \in Y$, then η_2 is a Q-fuzzy JU-subalgebra of Y.

(iii) Either η_1 is a Q-fuzzy JU-subalgebra of X or η_2 is a Q-fuzzy JU-subalgebra of Y.

Proof. Let $\eta_1 \times \eta_2$ be a Q-fuzzy JU-subalgebra of $X \times Y$.
(i) Let $x, y \in X$. Then $(x,1),(y,1) \in X \times Y$. Using the property $\eta_1(x,q) \leqslant \eta_2(1,q)$, for all $x \in X$ and $q \in Q$, we obtain

$$\begin{aligned}\eta_1(x*y,q) &= \min\{\eta_1(x*y,q), \eta_2(1*1,q)\}\\ &= \eta_1 \times \eta_2((x*y, 1*1), q)\\ &= \eta_1 \times \eta_2((x,1)*(y,1), q)\\ &\geqslant \min\{\eta_1 \times \eta_2((x,1),q), \eta_1 \times \eta_2((y,1),q)\} \quad (\text{since } \eta_1 \times \eta_2\end{aligned}$$

is a Q-fuzzy JU-Subalgebra of $X \times Y$.)

$$\begin{aligned}&= \min\{\min\{\eta_1(x,q), \eta_2(1,q)\}, \min\{\eta_1(y,q), \eta_2(1,q)\}\}\\ &= \min\{\eta_1(x,q), \eta_1(y,q)\}.\end{aligned}$$

Hence,

$$\eta_1(x*y,q) \geqslant \min\{\eta_1(x,q), \eta_1(y,q)\}, \quad \forall x, y \in X, q \in Q.$$

(ii) Let $x, y \in Y$. Then $(1,x),(1,y) \in X \times Y$. Using the property $\eta_2(x,q) \leqslant \eta_1(1,q)$, for all $x \in Y$ and $q \in Q$, we get

$$\begin{aligned}\eta_2(x*y,q) &= \min\{\eta_1(1*1,q), \eta_2(x*y,q)\}\\ &= \eta_1 \times \eta_2((1*1, x*y), q)\\ &= \eta_1 \times \eta_2((1,x)*(1,y), q)\\ &\geqslant \min\{\eta_1 \times \eta_2((1,x),q), \eta_1 \times \eta_2((1,y),q)\}\\ &= \min\{\min\{\eta_1(1,q), \eta_2(x,q)\}, \min\{\eta_1(1,q), \eta_2(y,q)\}\}\\ &= \min\{\eta_2(x,q), \eta_2(y,q)\}.\end{aligned}$$

Hence,

$$\eta_2(x*y,q) \geqslant \min\{\eta_2(x,q), \eta_2(y,q)\}, \quad \forall x, y \in Y, q \in Q.$$

(iii) Similarly, assume $\eta_2(1,q) \geqslant \eta_2(y,q)$ for all $y \in Y$. Then

$$\min\{\eta_1(1,q), \eta_2(1,q)\} \geqslant \min\{\eta_1(x,q), \eta_2(y,q)\}$$

it gives,

$$\min\{\eta_1(1,q),\eta_2(y,q)\} \geqslant \min\{\eta_1(x,q),\eta_2(y,q)\}.$$

This inequality holds if either

$$\eta_1(1,q) \geqslant \eta_1(x,q) \quad \text{or} \quad \eta_1(1,q) \geqslant \eta_2(y,q),$$

which proves (iii).

Hence, all three statements holds the Cartesian product of Q-fuzzy JU-subalgebras. □

3.1.1 Normal Q-Fuzzy JU-Subalgebras

Definition 3.1.1.1. *A Q-fuzzy JU-subalgebra* η *of X is said to be normal if there exist* $x \in X$ *such that* $\eta(x,q) = 1$

Example 3.1.1.1. *Consider X* = $\{1,2,3,4,5\}$, *we construct the following table.*

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

See [3], $(X,*,1)$ *is a JU-algebra.*
Let $Q = \{q_1,q_2\}$, *given a Q-fuzzy set* $\eta : Q \times X \to [0,1]$ *by*

$$\eta(1,q_1) = 1, \eta(2,q_1) = 0.8, \eta(3,q_1) = 0.7, \eta(4,q_1) = 0.5, \eta(5,q_1) = 0.3$$

and

$$\eta(1,q_2) = 1, \eta(2,q_2) = 0.9, \eta(3,q_2) = 0.6, \eta(4,q_2) = 0.4, \eta(5,q_2) = 0.1$$

Based on the given, we have $\eta(1,q_1) = 1$ and $\eta(1,q_2) = 1$, so η is normal.
Now again, a Q-fuzzy set η is a Q-fuzzy JU-subalgebra if, for all $x,y \in X$ and $q \in Q$,

$$\eta(x*y,q) \geqslant \min\{\eta(x,q),\eta(y,q)\}.$$

For $x = 2$, $y = 3$, q_1, then $\eta(x*y,q_1) = \eta(3,q_1) = 0.7, \cdot(*)$ and
$\min(\eta(2,q_1),\eta(3,q_1)) = \min(0.8,0.7) = 0.7 \cdot (**)$
Now, $\eta(x*y,q) \geqslant \min\{\eta(x,q),\eta(y,q)\} = 0.7 \geqslant 0.7$, which is true.
For $x = 4$, $y = 5$, q_2, then $\eta(4*5,q_2) = \eta(3,q_2) = 0.6, \cdot(***)$ and
$\min(\eta(4,q_2),\eta(5,q_2)) = \min(0.4,0.1) = 0.1, \cdot(****)$
$\eta(x*y,q) \geqslant \min\{\eta(x,q),\eta(y,q)\} = 0.6 \geqslant 0.1$, which is true.
Similarly, the result holds for all values of $x,y \in X$ and $q_1,q_2 \in Q$.

Hence, η satisfies a Q-fuzzy JU-subalgebra and the normality condition, we conclude that η is a normal Q-fuzzy JU-subalgebra of X.

Lemma 3.1.1.1. *If η is a normal Q-fuzzy JU-subalgebra of a JU-algebra X, then $\eta(1,q) = 1$*

Proof. Suppose η a be normal Q-fuzzy JU-subalgebra of a JU-algebra X.
Since, $\eta(1,q) \geqslant \eta(x,q) = 1$ (By Lemma 3.1.1)
$\Rightarrow \eta(1,q) \geqslant 1$ but, $\eta(1,q) \leqslant 1$
Therefore, $\eta(1,q) = 1$. □

Lemma 3.1.1.2. *Define a set $X_\eta = \{x \in X, q \in Q \mid \eta(x,q) = \eta(1,q)\}$ and let η and β be normal Q-fuzzy JU-subalgebra of X. If $\eta \subseteq \beta$, then $X_\eta \subseteq X_\beta$.*

Proof. Let $x \in X_\eta$, and $q \in Q$ Then: $\beta(x,q) \geqslant \eta(x,q)$ (since $\eta \subseteq \beta$)
$= \eta(1,q) = 1$ (by definition of X_η)
$= 1$ (since η is normal)
$= \beta(1,q)$ (since β is normal)
$\Rightarrow x \in X_\beta$
$\Rightarrow X_\eta \subseteq X_\beta$ □

Theorem 3.1.1.3. *For a Q-fuzzy JU-subalgebra of η of X, we can generate a normal fuzzy subalgebra of X which contains η.*

Proof. Let η be a Q-fuzzy JU-subalgebra of X. Define a Q-fuzzy subset η^* of X as $\eta^*(x,q) = \eta(x,q) + \eta^c(1,q), x \in X$. Let $x, y \in X$, then

$$\begin{aligned}
\eta^*(x*y,q) &= \eta(x*y,q) + \eta^c(1,q) \\
&\geqslant \min\{\eta(x,q), \eta(y,q)\} + \eta^c(1,q) \\
&= \min\{\eta(x,q) + \eta^c(1,q), \eta(y,q) + \eta^c(1,q)\} \\
&= \min\{\eta^*(x,q), \eta^*(y,q)\} \\
&\quad \text{And} \quad \eta^*(1,q) = \eta(1,q) + \eta^c(1,q) \\
&= \eta(1,q) + 1 - \eta(1,q) = 1
\end{aligned}$$

Therefore η^* is normal Q-fuzzy JU-algebra of X. Clearly $\eta \subseteq \eta^*$ thus η^* is a normal JU-subalgebra of X which contains η. □

Corollary 3.1.1.4. *i. If η itself is normal then $\eta = \eta^*$.*
ii. If η is a Q-fuzzy JU-subalgebra of X then $(\eta^*)^* = \eta^*$

Proof. i. Suppose η is normal. Then $\eta(1,q) = 1$ for all $q \in Q$. Thus,

$$\eta^*(x,q) = \eta(x,q) + \eta^c(1,q) (\text{By Theorem } 3.1.1.3) = \eta(x,q) + (1 - \eta(1,q)) = \eta(x,q) + (1-1) = \eta(x,q).$$

Hence, $\eta = \eta^*$.

ii. For any η, we have

$$(\eta^*)^*(x,q) = \eta^*(x,q) + (\eta^*)^c(1,q).(\text{By Theorem } 3.1.1.3)$$

Since $\eta^*(1,q) = \eta(1,q) + \eta^c(1,q) = 1$, it follows that

$$(\eta^*)^c(1,q) = 1 - \eta^*(1,q) = 0.$$

Therefore,

$$(\eta^*)^*(x,q) = \eta^*(x,q) + 0 = \eta^*(x,q),$$

which implies $(\eta^*)^* = \eta^*$. □

Theorem 3.1.1.5. *Let η be a Q-fuzzy JU-subalgebra of X. If η contains a normal Q-fuzzy JU-subalgebra of X generated by any other Q-fuzzy JU-subalgebra of X, then η is normal.*

Proof. Let λ be a Q-fuzzy JU- subalgebra of X. by theorem 3.1.1.3 λ^* is a normal Q-fuzzy JU-subalgebra of X. That is, $\lambda^* = 1$.
Let η is a Q-fuzzy JU-subalgebra of X such that $\lambda^* \subseteq \eta$
$\eta(x,q) \geqslant \lambda^*(x,q), \forall x \in X$
Put $x = 1 \Rightarrow \eta(1,q) \geqslant \lambda^* = 1$
$\Rightarrow \eta(1,q) \geqslant 1$. Hence η is normal. □

Theorem 3.1.1.6. *Let η be a Q-fuzzy JU-subalgebra of X and let $f : [1,\eta(1,q)] \to [0,1]$ be an increasing function. Define a Q-fuzzy set $\eta^f : X \to [0,1]$ by $\eta^f = f(\eta(x,q)), \forall x \in X$ then*

i. η^f is a Q-fuzzy JU-subalgebra of X.

ii. If $f(\eta(1,q)) = 1$ then η^f is normal

iii. If $f(t) \geqslant t$, $\forall t \in [1,\eta(1,q)]$ then $\eta \subseteq \eta^f$

Proof. i. $\eta^f(x*y,q) = f(\eta(x*y,q))$
$\geqslant f(\min\{\eta(x,q),\eta(y,q)\})$
$= \min\{f(\eta(x,q)), f(\eta(y,q))\}$
$= \min\{\eta^f(x,q),\eta^f(y,q)\}$
$\Rightarrow \eta^f$ is a Q-fuzzy subalgebra of X.
ii. If $f(\eta(1,q)) = 1 \Rightarrow \eta^f(1,q) = 1$
$\Rightarrow \eta^f$ is normal
iii. Let $f(t) \geqslant t, \forall t \in [1,\eta(1,q)]$
Then $\eta^f(x,q) = f(\eta(x,q)) \geqslant \eta(x,q), \forall x \in X$
Hence $\eta \subseteq \eta^f$ □

3.1.2 Doubt Q-Fuzzy JU-Subalgebras

Definition 3.1.2.1. *A Q-fuzzy set* η *of a JU-algebra X is called a doubt Q-fuzzy JU-subalgebra of X if* $\forall x, y \in X$ *and* $q \in Q$, $\eta_D(x * y, q) \leqslant \max\{\eta_D(x, q), \eta_D(y, q)\}$.

Example 3.1.2.1. *Let X =* $\{1, 2, 3, 4\}$ *in which* $*$ *is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

See [3], $(X; *, 1)$ *is a JU-algebra. Let* $Q = \{q_1, q_2\}$ *and we define a Q-fuzzy set* $\eta_D : Q \times X \to [0, 1]$ *by:*

$$\eta(x, q_1) = \begin{cases} 0.3, & x = 1, \\ 0.5, & x = 2, \\ 0.8, & x = 3, \\ 0.9, & x = 4. \end{cases}, \quad \eta(x, q_2) = \begin{cases} 0.1, & x = 1, \\ 0.6, & x = 2, \\ 0.7, & x = 3, \\ 0.8, & x = 4. \end{cases}$$

Recall: A Q*-fuzzy set* η_D *of a JU-algebra* X *is called a* doubt Q-fuzzy JU-subalgebra *of* X *if for all* $x, y \in X$ *and for all* $q \in Q$,

$$\eta_D(x * y, q) \leqslant \max\{\eta_D(x, q), \eta_D(y, q)\}.$$

Let we take, $x = 1, y = 2$, *we have* $\eta_D(1 * 2, q_1) = \eta_D(2, q_1) = 0.5, \cdots (*)$. *and*
$\max\{\eta_D(1, q_1), \eta_D(2, q_1)\} = \max\{0.3, 0.5\} = 0.5 \cdots (*)$.
Hence, from (*)and (**) $0.5 \leqslant 0.5$, *which is true.*
Similarly, for $x = 3, y = 4$, *we have* $\eta_D(3 * 4, q_1) = \eta_D(3, q_1) = 0.8, \cdots (* * *)$. *and*
$\max\{\eta_D(3, q_1), \eta_D(4, q_1)\} = \max\{0.8, 0.9\} = 0.9 \cdots (* * **)$.
Hence, from (***)and (****) $0.8 \leqslant 0.9$, *which is true.*
Similarly, the result holds for all values of $x, y \in X$ *and* $q_1, q_2 \in Q$
Therefore, we conclude that η_D *is a doubt Q-fuzzy JU-subalgebra of X.*

Lemma 3.1.2.1. *If* η_D *is a doubt Q-fuzzy JU-subalgebra of a JU-algebra X. Then* $\eta_D(1, q) \leqslant \eta_D(x, q)$, $\forall x \in X$ *and* $\forall q \in Q$.

Proof. Suppose η_D is a doubt Q-fuzzy JU-subalgebra of a JU-algebra X.
Since, $\eta_D(1, q) = \eta_D(x * x, q)$ (By Lemma 1.6.1)

$$\leqslant \max\{\eta_D(x, q), \eta_D(x, q)\}$$
$$= \eta_D(x, q)$$ □

Theorem 3.1.2.2. *If* η_D *is a doubt Q-fuzzy JU-subalgebra of X, then the set* $X_{\eta_D} = \{x \in X, q \in Q | \eta_D(x, q) = \eta_D(1, q)\}$ *is a JU-subalgebra of X.*

Proof. Let $x,y \in X_{\eta_D}$. Then $\eta_D(x,q) = \eta_D(1,q) = \eta_D(y,q)$, and so

$\eta_D(x*y,q) \leqslant \max\{\eta_D(x,q),\eta_D(y,q)\} = \max\{\eta_D(1,q),\eta_D(1,q)\}$.

$\Rightarrow \eta_D(x*y,q) \leqslant \eta_D(1,q)$ but $\eta_D(x*y,q) \geqslant \eta_D(1,q)$, $\forall x,y \in X$ (By Lemma 3.1.2.1)

$\Rightarrow \eta_D(x*y,q) = \eta_D(1,q)$

$\Rightarrow x*y \in X_{\eta_D}$.

Hence X_{η_D} is a JU-subalgebra of X. □

Theorem 3.1.2.3. *Let* η *be a doubt* Q*-fuzzy JU-subalgebra of* X *and let* $g : [1,\eta(1,q)] \to [0,1]$ *be an increasing function. Define a* Q*-fuzzy set* $\eta^g : X \to [0,1]$ *by* $\eta^g(x,q) = g(\eta(x,q))$, $\forall x \in X$ *then:*

i. η^g *is a doubt* Q*-fuzzy JU-subalgebra of* X.

ii. If $g(\eta(1,q)) = 1$ *then* η^g *is normal.*

iii. If $g(s) \leqslant s$, $\forall s \in [1,\eta(1,q)]$ *then* $\eta^g \subseteq \eta$.

Proof. i. Suppose η is a doubt Q-fuzzy JU-subalgebra of X. Then:

$$\begin{aligned}
\eta^g(x*y,q) &= g(\eta(x*y,q)) \\
&\leqslant g(\max\{\eta(x,q),\eta(y,q)\}) \quad \text{(since } g \text{ is increasing)} \\
&= \max\{g(\eta(x,q)),g(\eta(y,q))\} \\
&= \max\{\eta^g(x,q),\eta^g(y,q)\}
\end{aligned}$$

Therefore, η^g is a doubt Q-fuzzy JU-subalgebra of X.

ii. If $g(\eta(1,q)) = 1$ then $\eta^g(1,q) = 1$, which means η^g is normal.

iii. Let $g(s) \leqslant s$, $\forall s \in [1,\eta(1,q)]$. Then:

$\eta^g(x,q) = g(\eta(x,q)) \leqslant \eta(x,q)$, $\forall x \in X$. Hence $\eta^g \subseteq \eta$.

□

Theorem 3.1.2.4. *Let* $f : X \to Y$ *be an epimorphism of JU-algebras and* Q *be any nonempty set. If* η_D *is a doubt Q-fuzzy JU-subalgebra of* X*, then* $f(\eta_D)$ *is a doubt Q-fuzzy JU-subalgebra of* Y.

Proof. Let η_D be a doubt Q-fuzzy JU-subalgebra of X and $f : X \to Y$ be an epimorphism of JU-algebras. Then $f(x*y) = f(x)*f(y)$, $f(x), f(y) \in Y, \forall x,y \in X, q \in Q$. Then we have to prove that $f(\eta_D)$ is a doubt Q-fuzzy JU-subalgebra of Y. By Definition 1.1.11, we have

$$\begin{aligned}
f(\eta_D)(f(x)*f(y),q) = f(\eta_D)(f(x*y),q) &\leqslant \eta_D(x*y,q) \\
&\leqslant \max\{\eta_D(x,q),\eta_D(y,q)\} \\
&= \max\{f(\eta_D)(f(x),q), f(\eta_D)(f(y),q)\} \\
\Rightarrow f(\eta_D)(f(x)*f(y),q) &\leqslant \max\{f(\eta_D)(f(x),q), f(\eta_D)(f(y),q)\}.
\end{aligned}$$

Therefore, $f(\eta_D)$ is a doubt Q-fuzzy JU-subalgebra of Y. □

Theorem 3.1.2.5. *Let* $f : X \to Y$ *be an epimorphism of JU-algebras and* Q *be any nonempty set. Then* η_D *is a doubt Q-fuzzy JU-subalgebra of* Y *if and only if* $f^{-1}(\eta_D)$ *is a doubt Q-fuzzy JU-subalgebra of* X.

Proof. Assume that η_D is a doubt Q-fuzzy JU-subalgebra of Y. We need to show that $f^{-1}(\eta_D)$ is a doubt Q-fuzzy JU-subalgebra of X. Let $x, y \in X$, then $f(x), f(y) \in Y$. Now we have

$$\begin{aligned} f^{-1}(\eta_D)(x * y, q) = \eta_D(f(x * y), q) &= \eta_D(f(x) * f(y), q) \\ &\leqslant \max\{\eta_D(f(x), q), \eta_D(f(y), q)\} \\ &= \max\{f^{-1}(\eta_D)(f(x), q), f^{-1}(\eta_D)(f(y), q)\} \\ \Rightarrow f^{-1}(\eta_D)(x * y, q) &\leqslant \max\{f^{-1}(\eta_D)(f(x), q), f^{-1}(\eta_D)(f(y), q)\}. \end{aligned}$$

Conversely, assume that $f^{-1}(\eta_D)$ is a doubt Q-fuzzy JU-subalgebra of X. We need to show that η_D is a doubt Q-fuzzy JU-subalgebra of Y. Let $a, b \in Y$ and $q \in Q$. Since f is epimorphism of J algebras, there exist $x, y \in X$ such that $f(x) = a, f(y) = b$. Now

$$\begin{aligned} \eta_D(a * b, q) = \eta_D(f(x) * f(y), q) &= \eta_D(f(x * y), q) \\ &= f^{-1}(\eta_D)(x * y, q) \\ &\leqslant \max\{f^{-1}(\eta_D)(x, q), f^{-1}(\eta_D)(y, q)\} \\ &= \max\{\eta_D(f(x), q), \eta_D(f(y), q)\} \\ &= \max\{\eta_D(a, q), \eta_D(b, q)\} \\ \Rightarrow \eta_D(a * b, q) &\leqslant \max\{\eta_D(a, q), \eta_D(b, q)\}. \end{aligned}$$

Therefore, η_D is a doubt Q-fuzzy JU-subalgebra of Y. □

Lemma 3.1.2.6. *Let* η_{D_1} *and* η_{D_2} *be any two doubt* Q*-fuzzy JU-subalgebras of* X *and* Y *respectively, then* $(\eta_{D_1} \times \eta_{D_2})((1,1), q) \leqslant (\eta_{D_1} \times \eta_{D_2})((x,y), q)$, $\forall (x,y) \in X \times Y$, $q \in Q$.

Proof. Suppose that η_{D_1} and η_{D_2} are any two doubt Q-fuzzy JU-subalgebras of X and Y respectively. For any $(x, y) \in X \times Y$, then

$$\begin{aligned} (\eta_{D_1} \times \eta_{D_2})((1,1), q) &= \max\{\eta_{D_1}(1, q), \eta_{D_2}(1, q)\} \\ &\leqslant \max\{\eta_{D_1}(x, q), \eta_{D_2}(y, q)\} \\ &= (\eta_{D_1} \times \eta_{D_2})((x, y), q). \end{aligned}$$

Thus, $(\eta_{D_1} \times \eta_{D_2})((1,1), q) \leqslant (\eta_{D_1} \times \eta_{D_2})((x,y), q)$. □

Theorem 3.1.2.7. *Let* η_{D_1} *and* η_{D_2} *be doubt* Q*-fuzzy JU-subalgebras of* X *and* Y *respectively. Then* $\eta_{D_1} \times \eta_{D_2}$ *is a doubt* Q*-fuzzy JU-subalgebra of* $X \times Y$.

Proof. Let η_{D_1} and η_{D_2} be any two doubt Q-fuzzy JU-subalgebras of X and Y respectively. Let

$(x_1,y_1),(x_2,y_2) \in X \times Y$ and $q \in Q$, then

$$\begin{aligned}
\eta_{D_1} \times \eta_{D_2}[(x_1,y_1)*(x_2,y_2),q] &= \eta_{D_1} \times \eta_{D_2}((x_1*x_2,y_1*y_2),q) \\
&= \max\{\eta_{D_1}(x_1*x_2,q),\eta_{D_2}(y_1*y_2,q)\} \\
&\leqslant \max\{\max\{\eta_{D_1}(x_1,q),\eta_{D_1}(x_2,q)\},\max\{\eta_{D_2}(y_1,q),\eta_{D_2}(y_2,q)\}\} \\
&= \max\{\max\{\eta_{D_1}(x_1,q),\eta_{D_2}(y_1,q)\},\max\{\eta_{D_1}(x_2,q),\eta_{D_2}(y_2,q)\}\} \\
&= \max\{\eta_{D_1} \times \eta_{D_2}((x_1,y_1),q),\eta_{D_1} \times \eta_{D_2}((x_2,y_2),q)\}
\end{aligned}$$

□

3.2. Q-Fuzzy JU-Ideals of a JU-Algebra

In this section, we discuss and investigate a new notion of Q-fuzzy JU-ideals in JU-algebras and study their related properties.

Definition 3.2.1. *Let X be a JU-algebra, a Q-fuzzy set η in X is called Q-fuzzy JU-ideal of X if it satisfies the following conditions:*
(a) $\eta(1,q) \geqslant \eta(x,q)$
(b) $\eta(y,q) \geqslant \min\{\eta(x,q),\eta(x*y,q)\}, \forall x,y \in X$ *and* $\forall q \in Q$.

Example 3.2.1. *Consider X* = {1,2,3,4,5}, *we construct the following table.*

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

See [3], $(X,*,1)$ *is a JU-algebra. Let* $Q = \{q_1,q_2\}$, *given a Q-fuzzy set* $\eta : Q \times X \to [0,1]$ *by*

$$\eta(x,q_1) = \begin{cases} 0.6, & x=1, \\ 0.3, & x=2, \\ 0.2, & \text{otherwise.} \end{cases} \quad , \quad \eta(x,q_2) = \begin{cases} 0.8, & x=1, \\ 0.5, & x=2, \\ 0.4, & x=3, \\ 0.1, & \text{otherwise.} \end{cases}$$

To verify η *is a Q-fuzzy JU-ideal of X,*
(a) $\eta(1,q_1) = 0.6 \cdots (*)$ *and*
let $x=2$, *then* $\eta(2,q_1) = 0.3 \cdots (**)$,
From (*) and (**) we get $0.6 \geqslant 0.3$ *which is true.*
Thus, $\eta(1,q) \geqslant \eta(x,q), \forall x \in X$ *and* $q \in Q$.

Likewise,
(b) Let $x = 3, y = 4$*, then* $\eta(4, q_1) = 0.2 \cdots (***)$ *and*
$\min\{\eta(3, q_1), \eta(3 * 4, q_1)\} = \min\{\eta(3, q_1), \eta(4, q_1)\} = \min\{0.2, 0.2\} = 0.2 \cdots (****)$,
From (***) and (****) we get $0.2 \geqslant 0.2$ *which is true.*
Similarly, the result holds for all values of $x, y \in X$ *and* $q_1, q_2 \in Q$.
Therefore, we conclude that η *is a Q-fuzzy JU-ideal of X.*

Theorem 3.2.1. *Every Q-fuzzy JU-ideal of a JU-algebra X is order reversing.*

Proof. Let η be a Q-fuzzy JU-ideals of a JU-algebra X and for any $x, y \in X$ be such that $x \leqslant y$, then $y * x = 1$

$$\begin{aligned}\eta(x, q) &\geqslant \min\{\eta(y * x, q), \eta(y, q)\}, \forall q \in Q\\ &= \min\{\eta(1, q), \eta(y, q)\}\\ &= \eta(y, q)\end{aligned}$$

□

Theorem 3.2.2. *If* η *is a Q-fuzzy JU-ideals of X, then* $\eta((x * y) * y, q) \geqslant \eta(x, q)$.

Proof. Since $(x * [(x * y) * y]) = 1$ (By Lemma 1.6.1)
Now, $\eta((x * y) * y, q) \geqslant \min\{\eta(x, q), \eta(x * [(x * y) * y]), q\}, \forall q \in Q$
$= \min\{\eta(x, q), \eta(1, q)\}$
$= \eta(x, q)$ □

Theorem 3.2.3. *Every Q-fuzzy JU-ideal of a JU-algebra* X *is a Q-fuzzy JU-subalgebra of* X.

Proof. Suppose η is a Q-fuzzy JU-ideal of X. That is:

a) $\eta(1, q) \geqslant \eta(x, q)$,

b) $\eta(y, q) \geqslant \min\{\eta(x, q), \eta(x * y, q)\}$, for all $x, y \in X$ and $q \in Q$.

We claim that η is a Q-fuzzy JU-subalgebra. That is, for all $x, y \in X$ and $q \in Q$:
$\eta(x * y, q) \geqslant \min\{\eta(x, q), \eta(y, q)\}$
Assume without loss of generality:
$\eta(x, q) \leqslant \eta(y, q) \Rightarrow \min\{\eta(x, q), \eta(y, q)\} = \eta(x, q)$
Now we consider two cases:

Case 1. If $\eta(x * y, q) \geqslant \eta(x, q)$, then

$\eta(x * y, q) \geqslant \eta(x, q) = \min\{\eta(x, q), \eta(y, q)\}$

$\Rightarrow \eta(x * y, q) \geqslant \min\{\eta(x, q), \eta(y, q)\}$

The inequality is satisfied directly.

Case 2. If $\eta(x*y,q) < \eta(x,q)$, then from the assumption that η is Q-fuzzy JU-ideal, apply condition (b):

$\eta(y,q) \geqslant \min\{\eta(x,q), \eta(x*y,q)\} = \eta(x*y,q)$

$\Rightarrow \eta(y,q) \geqslant \eta(x*y,q)$

Since we also know $\eta(x*y,q) < \eta(x,q)$ and $\eta(x,q) \leqslant \eta(y,q)$, it follows that

$\eta(x*y,q) < \eta(x,q) \leqslant \eta(y,q) \Rightarrow \eta(x*y,q) < \min\{\eta(x,q), \eta(y,q)\}$

which contradicts the inequality $\eta(x*y,q) \geqslant \min\{\eta(x,q), \eta(y,q)\}$.

Thus, Case 1 must always hold, which implies:

$\eta(x*y,q) \geqslant \eta(x,q) = \min\{\eta(x,q), \eta(y,q)\}$

Hence, η is a Q-fuzzy JU-subalgebra of X. □

Remark 3.2.1. *The converse of the above theorem may not be true.*

Example 3.2.2. *Let X = {1,2,3,4} in which $*$ is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $(X;*,1)$ *is a JU-algebra of X And Define* $\eta(x,q)$ *as follows:*

$$\eta(x,q) = \begin{cases} 0.9, & \textit{if } x = 1 \\ 0.7, & \textit{if } x \in \{2,4\} \\ 0.4, & \textit{if } x = 3 \end{cases}$$

Since η *is a Q-fuzzy JU-subalgebra of* X.
The Q-fuzzy JU-ideal condition requires:

$$\eta(y,q) \geqslant \min\{\eta(x,q), \eta(x*y,q)\}, \textit{ for all } x,y \in X \textit{ and } q \in Q$$

Let $x = 2,\ y = 3 \Rightarrow x*y = 4$ *and*

$$\eta(2,q) = 0.7, \eta(3,q) = 0.4, \eta(4,q) = 0.7$$

Then,

$$\eta(3,q) = 0.4 \quad \cdots(*)$$

And,

$$\min\{\eta(2,q), \eta(4,q)\} = \min\{0.7, 0.7\} = 0.7 \quad \cdots(**)$$

From (*) and (**) we get $0.4 \geqslant 0.7$ *which is false.*
Therefore, we showed that every Q-fuzzy JU-subalgebra of a JU-algebra X is not Q-fuzzy JU-ideal of X

Theorem 3.2.4. *Let* η *be a Q-fuzzy JU-ideal of a JU-algebra* X *such that* $\eta(1,q) \neq 0$. *Let* $\eta^+ : X \times Q \to [0,1]$ *be a Q-fuzzy set defined by* $\eta^+(x,q) = \frac{\eta(x,q)}{\eta(1,q)}$, *for all* $x \in X$ *and* $q \in Q$. *Then* η^+ *is a Q-fuzzy JU-ideal of* X.

Proof. Suppose that η is a Q-fuzzy JU-ideal of X, with $\eta(1,q) \neq 0$. Let $x,y \in X$ and $q \in Q$. Then,
$\eta^+(1,q) = \frac{\eta(1,q)}{\eta(1,q)} = \frac{1}{\eta(1,q)}\eta(1,q) \geqslant \frac{1}{\eta(1,q)}\eta(x,q) = \eta^+(x,q)$. And,

$$\begin{aligned}
\eta^+(y,q) = \frac{\eta(y,q)}{\eta(1,q)} &= \frac{1}{\eta(1,q)}\eta(y,q) \\
&\geqslant \frac{1}{\eta(1,q)}\min\{\eta(x,q),\eta(x*y)\} \\
&= \min\{\frac{\eta(x,q)}{\eta(1,q)}, \frac{\eta(x*y,q)}{\eta(1,q)} \\
&= \min\{\eta^+(x,q),\eta^+(x*y,q)\}
\end{aligned}$$

Therefore, η^+ is a Q-fuzzy JU-ideal of X. □

Theorem 3.2.5. *Let* η_1 *and* η_2 *be any two Q-fuzzy JU-ideals of* X. *Then* $\eta_1 \cap \eta_2$ *is a Q-fuzzy JU-ideal of* X.

Proof. Suppose η_1 and η_2 are Q-fuzzy JU-ideals of X, for any $x \in X$, $q \in Q$. Then
$(\eta_1 \cap \eta_2)(1,q) = \min\{\eta_1(1,q),\eta_2(1,q)\} \geqslant \min\{\eta_1(x,q),\eta_2(x,q)\} = (\eta_1 \cap \eta_2)(x,q)$
And for any $x,y \in X$, $q \in Q$. Then

$$\begin{aligned}
(\eta_1 \cap \eta_2)(y,q) &= \min\{\eta_1(y,q),\eta_2(y,q)\} \\
&\geqslant \min\{\min\{\eta_1(x,q),\eta_1(x*y,q)\},\min\{\eta_2(x,q),\eta_2(x*y,q)\}\} \\
&= \min\{\min\{\eta_1(x,q),\eta_2(x,q)\},\min\{\eta_1(x*y,q),\eta_2(x*y,q)\}\} \\
&= \min\{\eta_1 \cap \eta_2(x,q),\eta_1 \cap \eta_2(x*y,q)\}
\end{aligned}$$

$\Rightarrow (\eta_1 \cap \eta_2)(y,q) \geqslant \min\{\eta_1 \cap \eta_2(x,q),\eta_1 \cap \eta_2(x*y,q)\}$, hence $\eta_1 \cap \eta_2$ is a Q-fuzzy JU-ideal of X. □

Corollary 3.2.6. *Let* $\{\eta_j : j \in \Omega\}$ *be a family of Q-fuzzy JU-ideals of* X. *Then* $\bigcap_{j \in J}\eta_J$ *is also a Q-fuzzy JU-ideal of* X.

Proof. Define a Q-fuzzy set $\eta = \bigcap_{j\in\Omega}\eta_j$ by

$$\eta(x,q) = \inf_{j\in\Omega}\eta_j(x,q), \quad \forall x \in X, q \in Q.$$

We show that η satisfies the defining properties of a Q-fuzzy JU-ideal.
$\eta(1,q) = \inf_{j\in\Omega}\eta_j(1,q) \geqslant \inf_{j\in\Omega}\eta_j(x,q) = \eta(x,q)$.
And for each $j \in \Omega$ and all $x,y \in X$, $q \in Q$, a Q-fuzzy JU-ideal satisfies

$$\eta_j(y,q) \geqslant \min\{\eta_j(x,q),\eta_j(x*y,q)\}$$

Then, for η, we have

$$\begin{aligned}\eta(y,q) = \inf_{j\in\Omega}\eta_j(y,q) &\geqslant \inf_{j\in\Omega}\min\{\eta_j(x,q),\eta_j(x*y,q)\}\\ &= \min\left\{\inf_{j\in\Omega}\eta_j(x,q),\inf_{j\in\Omega}\eta_j(x*y,q)\right\}\\ &= \min\{\eta(x,q),\eta(x*y,q)\}.\end{aligned}$$

$\square$

Remark 3.2.2. *The union of two Q-fuzzy JU-ideals of a JU-algebra may not be a Q-fuzzy JU-ideal.*

Example 3.2.3. *Let $X = \{1,2,3,4,5\}$ in which $*$ is defined by the following table.*

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

See [3], $(X;*,1)$ is a JU-algebra. We define $\eta_1 : X \times Q \to [0,1]$ as follows $\eta_1(1,q) = 0.9, \eta_1(2,q) = 0.6, \eta_1(3,q) = 0.5, \eta_1(4,q) = 0.3, \eta_1(5,q) = 0.2$, and $\eta_2 : X \times Q \to [0,1]$ as follows $\eta_2(1,q) = 0.8, \eta_2(2,q) = 0.7, \eta_2(3,q) = 0.6, \eta_2(4,q) = 0.4, \eta_2(5,q) = 0.1$. Then

$$(\eta_1 \cup \eta_2)(5,q) = \max\{\eta_1(5,q),\eta_2(5,q)\} = \max\{0.2,0.1\} = 0.2\cdots(*)$$

And,

$$\begin{aligned}\max\{\eta_1(5,q),\eta_2(5,q)\} &\geqslant \max\{\min\{\eta_1(4,q),\eta_1(4*5,q)\},\min\{\eta_2(4,q),\eta_2(4*5,q)\}\}\\ &= \min\{\max\{\eta_1(4,q),\eta_2(4,q)\},\max\{\eta_1(4*5,q),\eta_2(4*5,q)\}\}\\ &= \min\{\max\{0.3,0.4\},\max\{0.5,0.6\}\}\\ &= \min\{0.4,0.6\} = 0.4\cdots(**)\end{aligned}$$

From (*) and (**) we get $0.2 \geqslant 0.4$ which is false.
Therefore, we verified that the union of two Q-fuzzy JU-ideals of a JU-algebra may not be a Q-fuzzy JU-ideal.

Theorem 3.2.7. *If* η *is a* Q*-fuzzy JU-ideal of* X*, then the set* $X_\eta := \{x \in X \mid \eta(x,q) = \eta(1,q), q \in Q\}$ *is a JU-ideal of* X*.*

Proof. Let $x, x*y \in X_\eta$. Then $\eta(x,q) = \eta(1,q) = \eta(x*y,q)$, and so

$$\eta(y,q) \geqslant \min\{\eta(x,q), \eta(x*y,q)\} = \min\{\eta(1,q), \eta(1,q)\} = \eta(1,q)$$

But by Definition 3.2.1, $\eta(y,q) \leqslant \eta(1,q)$ for all $y \in X$. Therefore: $\eta(y,q) = \eta(1,q)$ implies that $y \in X_\eta$. Hence X_η is a JU-ideal of X. □

Lemma 3.2.8. *Let* η_1 *and* η_2 *be two* Q*-fuzzy JU-ideals of a JU-algebra* X*. If* $\eta_1 \subseteq \eta_2$ *and* $\eta_1(1,q) = \eta_2(1,q)$ *for* $q \in Q$*, then* $X_{\eta_1} \subseteq X_{\eta_2}$*.*

Proof. Assume that η_1 and η_2 are two Q-fuzzy JU-ideals of X with $\eta_1 \subseteq \eta_2$. Then $\eta_1(x,q) \leqslant \eta_2(x,q)$ for all $x \in X$ and $q \in Q$.
For any $x \in X_{\eta_1}$ then $\eta_2(x,q) \geqslant \eta_1(x,q) = \eta_1(1,q) = \eta_2(1,q)$
$\Rightarrow \eta_2(x,q) \geqslant \eta_2(1,q)$. But $\eta_2(x,q) \leqslant \eta_2(1,q)$. Thus $\eta_2(x,q) = \eta_2(1,q)$, and $x \in X_{\eta_2}$. Thus, $\eta_1 \subseteq \eta_2$. □

Theorem 3.2.9. *Let* J *be a subset of a JU-algebra* X*. Then the characteristics function* χ_J *is a* Q*-fuzzy JU-ideal of* X *if and only if* J *is an ideal of* X*.*

Proof. Let $\chi_J : X \times Q \to \{0,1\}$ be a Q-fuzzy JU-ideal. Let $q \in Q$ and let $x, x*y \in J$. Then $\chi_J(x,q) = \chi_J(x*y,q) = 1$.
$\chi_J(y,q) \geqslant \min\{\chi_J(x,q), \chi_J(x*y,q)\} = \min\{1,1\} = 1$
$\Rightarrow \chi_J(y,q) = 1$ (since $\chi_J(y,q) \leqslant 1$)
$\Rightarrow y \in J$, hence J is a ideal of X.
Let J be a JU-ideal of X. Consider cases for $x, y \in X$:

Case 1. $x, x*y \in J$ implies that $y \in J$

$$\chi_J(y,q) = 1 = \min\{\chi_J(x,q), \chi_J(x*y,q)\}$$

Case 2. $x, x*y \notin J$ implies that$\chi_J(y,q) = 0 = \min\{\chi_J(x,q), \chi_J(x*y,q)\}$

Case 3. $x \in J, x*y \notin J$ implies that $\min\{\chi_J(x,q), \chi_J(x*y,q)\} = 0$

$$\chi_J(y,q) \geqslant 0 = \min\{\chi_J(x,q), \chi_J(x*y,q)\}$$

Hence χ_J is a Q-fuzzy JU-ideal of X. □

Theorem 3.2.10. *Let* η *be a* Q*-fuzzy subset of* X *such that the nonempty level sets* $U(\eta;t,q)$ *are JU-ideals of* X *for all* $t \in [0,1]$*. Then* η *is a* Q*-fuzzy JU-ideal of* X*.*

Proof. Suppose $U(\eta;t,q)$ is a JU-ideal of X for all $t \in [0,1]$. We need to show that η is a Q-fuzzy JU-ideal of X.

Assume that $\eta(1,q) < \mu(x,q)$ for some $x \in X$ and $q \in Q$. Take

$$t = \frac{1}{2}[\eta(1,q) + \eta(x,q)],$$

then $\eta(1,q) < t < \eta(x,q)$. This implies that $x \in U(\eta;t,q)$ and $1 \notin U(\eta;t,q)$, which is a contradiction since $U(\eta;t,q)$ is a JU-ideal of X. Therefore, $\eta(1,q) \geqslant \eta(x,q)$.
And let $x,y \in X$ and $q \in Q$. Assume that $\eta(y,q) \geqslant \min\{\eta(x,q),\eta(x*y,q)\}$ is not true. Then there exist $x_0,y_0 \in X$ such that

$$\eta(y_0,q) < \min\{\eta(x_0,q),\eta(x_0*y_0,q)\}.$$

Take

$$t = \frac{1}{2}[\eta(y_0,q) + \min\{\eta(x_0,q),\eta(x_0*y_0,q)\}].$$

Thus $t \in [0,1]$ and

$$\eta(y_0,q) < t < \min\{\eta(x_0,q),\eta(x_0*y_0,q)\}.$$

Hence, $\eta(y_0,q) < t$, $\eta(x_0,q) > t$, and $\eta(x_0*y_0,q) > t$, which means $y_0 \notin U(\eta;t,q)$ but $x_0, x_0*y_0 \in U(\eta;t,q)$. This contradicts the assumption that $U(\eta;t,q)$ is a JU-ideal of X. Therefore, $\eta(y,q) \geqslant \min\{\eta(x,q),\eta(x*y,q)\}$. Hence, η is a Q-fuzzy JU-ideal of X. □

Theorem 3.2.11. *If η is a Q-fuzzy JU-ideal of X such that $t \in Im(\eta)$ and for all $t \in [0,1]$. Then $U(\eta;t,q)$ is a JU-ideal of X.*

Proof. Assume that η is a Q-fuzzy JU-ideal of X such that $t \in Im(\eta)$ and for all $t \in [0,1]$. We need to show that $U(\eta;t,q)$ is a JU-ideal of X. Then there exists $x \in X$ such that $\eta(x,q) = t$. By Definition 3.2.1 $\eta(1,q) \geqslant \eta(x,q) = t$ So, $1 \in U(\eta;t,q)$.
Also let $x,y \in X$ and let x, $x*y \in U(\eta;t,q)$ such that $\eta(x,q) \geqslant t$ and $\eta(x*y,q) \geqslant t$. Since η is a Q-fuzzy JU-ideal of X, then $\eta(y,q) \geqslant \min\{\eta(x,q),\eta(x*y,q)\} \geqslant \min\{t,t\} = t$. Hence, $\eta(y,q) \geqslant t$. Implies that $y \in U(\eta;t,q)$. Therefore, $U(\eta;t,q)$ is a JU-ideal of X. □

Theorem 3.2.12. *Let η be a Q-fuzzy set in a JU-algebra X. Then η^c is a Q-fuzzy JU-ideal of X if and only if for $s \in [0,1]$ and $q \in Q$, $L(\eta;s,q)$ is either empty or a JU-ideal of X.*

Proof. Suppose that η^c is a Q-fuzzy JU-ideal of X. Let $q \in Q$ and $s \in [0,1]$ be such that $L(\eta;s,q) \neq \emptyset$ and let $x,y \in L(\eta;s,q)$. Then $\eta(x,q) \leqslant s$ and $\eta(y,q) \leqslant s$. Now,

$$\begin{aligned} \eta^c(y,q) &\geqslant \min\{\eta^c(x,q),\eta^c(x*y,q)\} \\ &= \min\{1-\eta(x,q),1-\eta(x*y,q)\} \\ &= 1-\max\{\eta(x,q),\eta(x*y,q)\} \quad \text{By Lemma 1.1.1} \\ \Longrightarrow \eta(y,q) &\leqslant \max\{\eta(x,q),\eta(x*y,q)\} \leqslant s \\ \Longrightarrow y &\in L(\eta;s,q). \end{aligned}$$

Hence, $L(\eta;s,q)$ is a JU-ideal of X.

Conversely, suppose that $L(\eta;s,q)$ is a nonempty JU-ideal of X. Let $x,y \in X$, $q \in Q$, and let $s = \max\{\eta(x,q),\eta(x*y,q)\}$. Thus $\eta(x,q) \leqslant s$ and $\eta(x*y,q) \leqslant s$, so $x, x*y \in L(\eta;s,q) \neq \emptyset$. By assumption, $L(\eta;s,q)$ is a JU-ideal of X. It follows that $y \in L(\eta;s,q)$. Thus,

$$\begin{aligned}
\eta(y,q) &\leqslant s = \max\{\eta(x,q),\eta(x*y,q)\}, \\
1-\eta(y,q) &\geqslant 1-\max\{\eta(x,q),\eta(x*y,q)\} \\
&= \min\{1-\eta(x,q),1-\eta(x*y,q)\} \quad \text{By Lemma 1.1.1} \\
\Longrightarrow\ \eta^c(y,q) &\geqslant \min\{\eta^c(x,q),\eta^c(x*y,q)\}.
\end{aligned}$$

□

Theorem 3.2.13. *Every Q-fuzzy JU-ideal of X can be realized as level JU-ideal of some Q-fuzzy JU-ideal of X.*

Proof. Let J be a JU-ideal of X and let η be a Q-fuzzy set in X defined by

$$\eta(x,q) = \begin{cases} t & \text{if } x \in J \\ 0 & \text{otherwise} \end{cases}$$

,for all $x \in X$ and $q \in Q$, where $t \in [0,1]$.
We show that η is a Q-fuzzy JU-ideal of X. Let $x, x*y \in X$. Then consider the following cases:

Case 1. If $x, x*y \in J$ then

$$\eta(y,q) \geqslant t = \min\{\eta(x,q),\eta(x*y,q)\}$$

Case 2. If $x, x*y \notin J$ then $\eta(x,q) = 0 = \eta(x*y,q)$ and thus

$$\eta(y,q) \geqslant 0 = \min\{\eta(x,q),\eta(x*y,q)\}$$

Case 3. If $x \in J$ and $x*y \notin J$ then

$$\eta(y,q) \geqslant 0 = \min\{1,0\} = \min\{\eta(x,q),\eta(x*y,q)\}$$

Or, $x \notin J$ and $x*y \in J$ then $\eta(y,q) \geqslant 0 = \min\{0,1\} = \min\{\eta(x,q),\eta(x*y,q)\}$

Therefore, η is a Q-fuzzy JU-ideal of X. □

Theorem 3.2.14. *Let $f: X \to Y$ be an epimorphism of JU-algebras and Q be any nonempty set. Then η is a Q-fuzzy JU-ideal of Y if and only if $f^{-1}(\eta)$ is a Q-fuzzy JU-ideal of X.*

Proof. Assume that η is a Q-fuzzy JU-ideal of Y. We need to show that $f^{-1}(\eta)$ is a Q-fuzzy JU-ideal of X. Let $x, x*y \in X$, then $f(x), f(x*y) \in Y$. Now we have

$$\begin{aligned}
f^{-1}(\eta)(y,q) &= \eta(f(y),q) \\
&\geqslant \min\{\eta(f(x),q),\eta(f(x*y),q)\} \\
&= \min\{f^{-1}(\eta)(x,q),f^{-1}(\eta)(x*y,q)\}
\end{aligned}$$

$\Rightarrow f^{-1}(\eta)(y,q) \geqslant \min\{f^{-1}(\eta)(x,q), f^{-1}(\eta)(x*y,q)\}$.
Conversely, assume that $f^{-1}(\eta)$ is a Q-fuzzy JU-ideal of X. We need to show that η is a Q-fuzzy JU-ideal of Y. Let $a,b \in Y$ and $q \in Q$. Since f is an epimorphism of JU-algebras, there exist $x, x*y \in X$ such that $f(x) = a$, $f(x*y) = b$. Now

$$\begin{aligned}
\eta(b,q) = \eta(f(y),q) &= f^{-1}(\eta)(y,q) \\
&\geqslant \min\{f^{-1}(\eta)(x,q), f^{-1}(\eta)(x*y,q)\} \\
&= \min\{\eta(f(x),q), \eta(f(x*y),q)\} \\
&= \min\{\eta(a,q), \eta(b,q)\} \\
\Rightarrow \eta(b,q) &\geqslant \min\{\eta(a,q), \eta(a*b,q)\}.
\end{aligned}$$

Therefore, η is a Q-fuzzy JU-ideal of Y. $\square$

Theorem 3.2.15. *Let* η_1 *and* η_2 *be Q-fuzzy JU-ideals of* X *and* Y *respectively. Then* $\eta_1 \times \eta_2$ *is a Q-fuzzy JU-ideal of* $X \times Y$.

Proof. Let η_1 and η_2 be any two Q-fuzzy JU-ideals of X and Y respectively.
Let $(x,y) \in X \times Y$, then

$$\begin{aligned}
(\eta_1 \times \eta_2)((1,1),q) &= \min\{\eta_1(1,q), \eta_2(1,q)\} \\
&> \min\{\eta_1(x,q), \eta_2(y,q)\} \\
&= (\eta_1 \times \eta_2)((x,y),q).
\end{aligned}$$

Hence $(\eta_1 \times \eta_2)((1,1),q) \geqslant (\eta_1 \times \eta_2)((x,y),q)$.
And let $(x_1,y_1),(x_2,y_2) \in X \times Y$ and $q \in Q$, then

$$\begin{aligned}
\eta_1 \times \eta_2[(y_1,y_2),q] &= \eta_1 \times \eta_2((y_1,y_2),q) \\
&= \min\{\eta_1(y_1,q), \eta_2(y_2,q)\} \\
&\geqslant \min\{\min\{\eta_1(x_1,q), \eta_1((x_1*y_1),q)\}, \min\{(x_2,q), \eta_1((x_2*y_2),q)\}\} \\
&= \min\{\min\{\eta_1(x_1,q), \eta_2(x_2,q)\}, \min\{\eta_1((x_1*y_1),q), \eta_2((x_2*y_2),q)\}\} \\
&= \min\{\eta_1 \times \eta_2((x_1,x_2),q), \eta_1 \times \eta_2(((x_1,x_2)*(y_1,y_2),q))\}.
\end{aligned}$$

Therefore, $\eta_1 \times \eta_2(y_1,y_2),q] \geqslant \min\{\eta_1 \times \eta_2((x_1,x_2),q), \eta_1 \times \eta_2(((x_1,x_2)*(y_1,y_2),q),q))\}$.
Hence, $\eta_1 \times \eta_2$ is a Q-fuzzy JU-ideal of $X \times Y$. $\square$

3.2.1 Normal Q-Fuzzy JU-Ideals

Definition 3.2.1.1. *A Q-fuzzy JU-ideal* η *of X is said to be normal if there exist* $x \in X$ *such that* $\eta(x,q) = 1$

Lemma 3.2.1.1. *If* η *be a normal Q-fuzzy JU-ideal of a JU-algebra X, then* $\eta(1,q) = 1$

Proof. Suppose η be normal Q-fuzzy JU-ideal of a JU-algebra X.
Since, $\eta(1,q) \geqslant \eta(x,q) = 1$
$\Rightarrow \eta(1,q) \geqslant 1$ but, $\eta(1,q) \leqslant 1$
Therefore, $\eta(1,q) = 1$. □

Lemma 3.2.1.2. *Define a set* $X_\eta = \{x \in X, q \in Q \mid \eta(x,q) = \eta(1,q)\}$ *and let* η *and* β *be normal Q-fuzzy JU-ideal of* X. *If* $\eta \subseteq \beta$, *then* $X_\eta \subseteq X_\beta$.

Proof. Let $x \in X_\eta$, and $q \in Q$ Then:

$$\begin{aligned}\beta(x,q) &\geqslant \eta(x,q) \quad (\text{since } \eta \subseteq \beta)\\ &= \eta(1,q) \quad (\text{by definition of } X_\eta)\\ &= 1 \quad (\text{since } \eta \text{ is normal})\\ &= \beta(1,q) \quad (\text{since } \beta \text{ is normal})\\ &\Rightarrow x \in X_\beta\\ &\Rightarrow X_\eta \subseteq X_\beta\end{aligned}$$

□

Theorem 3.2.1.3. *For a Q-fuzzy JU-ideal of* η *of X, we can generate a normal Q-fuzzy JU-ideal of X which contains* η.

Proof. Let η be a Q-fuzzy JU-ideal of X. Define a Q-fuzzy subset η^* of X as:
$\eta^*(x,q) = \eta(x,q) + \eta^c(1,q), \forall x \in X$.
Let $x, y \in X$, $\eta^*(1,q) = \eta(1,q) + \eta^c(1,q) \geqslant \eta(x,q) + \eta^c(1,q) = \eta^*(x,q)$
And,

$$\begin{aligned}\eta^*(y,q) &= \eta(y,q) + \eta^c(1,q)\\ &\geqslant \min\{\eta(x,q), \eta(x*y,q)\} + \eta^c(1,q)\\ &= \min\{\eta(x,q) + \eta^c(1,q), \eta(x*y,q) + \eta^c(1,q)\}\\ &= \min\{\eta^*(x,q), \eta^*(x*y,q)\}\end{aligned}$$

Also, $\eta^*(1,q) = \eta(1,q) + \eta^c(1,q) = \eta(1,q) + 1 - \eta(1,q) = 1$
$\Rightarrow \eta^*$ is normal Q-fuzzy JU-ideal of X. Clearly $\eta \subseteq \eta^*$ thus η^* is a normal Q-fuzzy JU- ideal of X which contains η. □

Corollary 3.2.1.4. *If* η *itself is normal then* $\eta = \eta^*$
If η *is a Q-fuzzy JU-ideal of X then* $(\eta^*)^* = \eta^*$

Proof. i. Suppose η is normal. Then $\eta(1,q) = 1$ for all $q \in Q$. Thus,

$$\eta^*(x,q) = \eta(x,q) + \eta^c(1,q) \quad (\text{By Theorem } 3.2.1.3)$$

$$= \eta(x,q) + (1 - \eta(1,q))$$
$$= \eta(x,q) + (1 - 1) \quad (\text{Since} \quad \eta \quad \text{is} \quad \text{normal})$$
$$= \eta(x,q).$$

Hence, $\eta = \eta^*$.
ii. For any η, we have

$$(\eta^*)^*(x,q) = \eta^*(x,q) + (\eta^*)^c(1,q)(\text{By Theorem } 3.2.1.3)$$

Since $\eta^*(1,q) = \eta(1,q) + \eta^c(1,q) = 1$, it follows that

$$(\eta^*)^c(1,q) = 1 - \eta^*(1,q) = 0.$$

Therefore,

$$(\eta^*)^*(x,q) = \eta^*(x,q) + 0 = \eta^*(x,q),$$

which implies $(\eta^*)^* = \eta^*$. □

Theorem 3.2.1.5. *Let η be a Q-fuzzy JU-ideal of X. If η contains a normal Q-fuzzy JU-ideal of X generated by any other Q-fuzzy JU-ideal of X, then η is normal.*

Proof. Let λ be a Q-fuzzy JU-ideal of X. By Lemma 3.2.1.1 λ^* is a normal Q-fuzzy JU-ideal of X.
i,e $\lambda^* = 1$
Let η is a Q-fuzzy JU-ideal of X such that $\lambda^* \subseteq \eta$
$\eta(x,q) \geqslant \lambda^*(x,q), \forall x \in X$
Put $x = 1 \Rightarrow \eta(1,q) \geqslant \lambda^* = 1$
$\Rightarrow \eta(1,q) \geqslant 1$. but, $\eta(1,q) \leqslant 1$
Therefore, $\eta(1,q) = 1$. □

Theorem 3.2.1.6. *Let η be a Q-fuzzy JU-ideal of X and let $f : [1,\eta(1,q)] \to [0,1]$ be an increasing function. Define a Q-fuzzy set $\eta^f : X \to [0,1]$ by $\eta^f = f(\eta(x,q)), \forall x \in X$ then*

i. η^f *is a Q-fuzzy JU-ideal of X.*

ii. *If* $f(\eta(1,q)) = 1$ *then* η^f *is normal*

iii. *If* $f(t) \geqslant t$, $\forall t \in [1,\eta(1,q)]$ *then* $\eta \subseteq \eta^f$

Proof. i, $\eta^f(1,q) = f(\eta(1,q)) \geqslant f(\eta(x,q)) = \eta^f(x,q)$
Also $\eta^f(y,q) = f(\eta(y,q))$
$$\geqslant f(\min\{\eta(x,q), \eta(x*y,q)\})$$
$$= \min\{f(\eta(x,q)), f(\eta(x*y,q))\}$$
$$= \min\{\eta^f(x,q), \eta^f(x*y,q)\}$$
$\Rightarrow \eta^f$ is a Q-fuzzy JU-ideal of X.
ii, If $f(\eta(1,q)) = 1 \Rightarrow \eta^f(1,q) = 1$

$\Rightarrow \eta^f$ is normal

iii, Let $f(t) \geqslant t, \forall t \in [1, \eta(1,q)]$

Then $\eta^f(x,q) = f(\eta(x,q)) \geqslant \eta(x,q), \forall x \in X$

Hence $\eta \subseteq \eta^f$ □

3.2.2 Doubt Q-Fuzzy JU-Ideals

Definition 3.2.2.1. *A Q-fuzzy set η_D of a JU-algebra X is called a doubt Q-fuzzy JU-ideal of X if it satisfies the following conditions:*

1. $\eta_D(1,q) \leqslant \eta_D(x,q)$

2. $\eta_D(y,q) \leqslant \max\{\eta_D(x,q), \eta_D(x*y,q)\}, \forall x,y \in X$ *and* $q \in Q$

Theorem 3.2.2.1. *If η_D is a doubt Q-fuzzy JU-ideal of a JU-algebra X. Then the set $X_{\eta_D} = \{x \in X | \eta_D(x,q) = \eta_D(1,q), \forall q \in Q\}$ is a JU-ideal of X.*

Proof. If η_D is a doubt Q-fuzzy JU-ideal of a JU-algebra X. Obviously, $1 \in X_{\eta_D}$. Let $x, x*y \in X_{\eta_D}$ implies that $\eta_D(x,q) = \eta_D(x*y,q) = \eta_D(1,q)$

Now, $\eta_D(y,q) \leqslant \max\{\eta_D(x,q), \eta_D(x*y,q)\}, \forall x,y \in X$ and $q \in Q$

$= \max\{\eta_D(1,q), \eta_D(1,q)\} = \eta_D(1,q)$

$\Rightarrow \eta_D(y,q) \leqslant \eta_D(1,q)$ but $\eta_D(y,q) \geqslant \eta_D(1,q)$ (by Definition 3.2.2.1)

$\Rightarrow \eta_D(y,q) = \eta_D(1,q)$, implies that $y \in X_{\eta_D}$. Hence X_{η_D} is a JU-ideal of X. □

Theorem 3.2.2.2. *Let J be a subset of a JU-algebra X. Then the characteristics function χ_J is a doubt Q-fuzzy JU-ideal of X if and only if J is a JU-ideal of X.*

Proof. Let $\chi_J : X \times Q \to \{0,1\}$ be a doubt Q-fuzzy JU-ideal. Let $q \in Q$ and let $x, x*y \in J$, then $\chi_J(x,q) = \chi_J(x*y,q) = 1$. Thus,

$\chi_J(y,q) \leqslant \max\{\chi_J(x,q), \chi_J(x*y,q)\} = \max\{1,1\} = 1$

$\Rightarrow \chi_J(y,q) = 1$ (since $\chi_J(y) \leqslant 1$)

$\Rightarrow y \in J$, hence J is a JU-ideal of X.

Conversely, Suppose J is a JU-ideal of X and for $x,y \in X$. Then consider the following cases:

Case 1. If $x, x*y \in J$ implies that $y \in J$

$$\chi_J(y,q) = 1 = \max\{\chi_J(x,q), \chi_J(x*y,q)\}$$

Case 2. If $x, x*y \notin J$ implies that$\chi_J(y,q) = 0 = \max\{\chi_J(x,q), \chi_J(x*y,q)\}$

Case 3. If either $x \in J$ or $x*y \notin J$ implies that $\max\{\chi_J(x,q), \chi_J(x*y,q)\} = 1$. Thus,

$$\chi_J(y,q) \leqslant 1 = \max\{1,0\} = \max\{0,1\} = \max\{\chi_J(x,q), \chi_J(x*y,q)\}$$

Hence χ_J is a doubt Q-fuzzy JU-ideal of X. □

Theorem 3.2.2.3. *Every doubt Q-fuzzy JU-ideal of X can be realized as level JU-ideal of some doubt Q-fuzzy JU-ideal of X.*

Proof. Let J be a JU-ideal of X and let η_D be a doubt Q-fuzzy set in X defined by

$$\eta_D(x,q) = \begin{cases} t & \text{if } x \in J \\ 0 & \text{otherwise} \end{cases}$$

,for all $x \in X$ and $q \in Q$, where $t \in [0,1]$.
We show that η_D is a doubt Q-fuzzy JU-ideal of X. Let $x, x*y \in X$. Then consider the following cases:

Case 1. If $x, x*y \in J$ then

$\eta_D(y,q) \leqslant t = \max\{\eta_D(x,q), \eta_D(x*y,q)\}$

Case 2. If $x, x*y \notin J$ then $\eta_D(x,q) = 0 = \eta_D(x*y,q)$ and thus

$\eta_D(y,q) \leqslant 0 = \max\{\eta_D(x,q), \eta_D(x*y,q)\}$

Case 3. If $x \in J$ and $x*y \notin J$ then

$\eta_D(y,q) \leqslant t = \max\{t,0\} = \max\{\eta_D(x,q), \eta_D(x*y,q)\}$

Or, $x \notin J$ and $x*y \in J$ then $\eta_D(y,q) \leqslant t = \max\{0,t\} = \max\{\eta_D(x,q), \eta_D(x*y,q)\}$

Therefore η_D is a doubt Q-fuzzy JU-ideal of X. □

Theorem 3.2.2.4. *Let η_D be a doubt Q-fuzzy JU-ideal of X and let $g : [1, \eta_D(1,q)] \to [0,1]$ be a increasing function. Define a Q-fuzzy set $\eta_D^g : X \to [0,1]$ by $\eta_D^g = g(\eta_D(x,q)), \forall x \in X$ then*

i. η_D^g *is a doubt Q-fuzzy JU-ideal of X.*

ii. If $g(\eta_D(1,q)) = 1$ *then* η_D^g *is normal*

iii. If $g(t) \leqslant t$, $\forall t \in [1, \eta_D(1,q)]$ *then* $\eta_D^g \subseteq \eta_D$

Proof. i. Suppose η_D is a doubt Q-fuzzy JU-ideal of X. Then
$\eta_D^g(1,q) = g(\eta_D(1,q)) \leqslant g(\eta_D(x,q)) = \eta_D^g(x,q)$. And,

$$\begin{aligned} \eta_D^g(y,q) = g(\eta_D(y,q)) &\leqslant g(max\{\eta_D(x,q), \eta_D(x*y,q)\}) \\ &= \max\{g(\eta_D(x,q)), g(\eta_D(x*y,q))\} \\ &= \max\{\eta_D^g(x,q), \eta_D^g(x*y,q)\} \end{aligned}$$

$\Rightarrow \eta_D^g$ is a doubt Q-fuzzy JU-ideal of X.
ii. If $g(\eta_D(1,q)) = 1 \Rightarrow \eta_D^g(1,q) = 1$. Thus, η_D^g is normal.
iii. Let $g(t) \leqslant t, \forall t \in [1, \eta_D(1,q)]$. Then,
$\eta_D^g(x,q) = g(\eta_D(x,q)) \leqslant \eta_D(x,q), \forall x \in X$. Hence, $\eta_D^g \subseteq \eta_D$ □

Theorem 3.2.2.5. *Let $f : X \to Y$ be an epimorphism of JU-algebras and Q be any nonempty set. Then η_D is a doubt Q-fuzzy JU-ideal of Y if and only if $f^{-1}(\eta_D)$ is a doubt Q-fuzzy JU-ideal of X.*

Proof. Assume that η_D is a doubt Q-fuzzy JU-ideal of Y. We need to show that $f^{-1}(\eta_D)$ is a doubt Q-fuzzy JU-ideal of X. Let $x, x*y \in X$, then $f(x), f(x*y) \in Y$. Now we have

$$\begin{aligned} f^{-1}(\eta_D)(y,q) &= \eta_D(f(y),q) \\ &\leqslant \max\{\eta_D(f(x),q), \eta_D(f(x*y),q)\} \\ &= \max\{f^{-1}(\eta_D)(x,q), f^{-1}(\eta_D)(x*y,q)\} \end{aligned}$$

$\Rightarrow f^{-1}(\eta_D)(y,q) \leqslant \max\{f^{-1}(\eta_D)(x,q), f^{-1}(\eta_D)(x*y,q)\}$.
Conversely, assume that $f^{-1}(\eta_D)$ is a doubt Q-fuzzy JU-ideal of X. We need to show that η_D is a doubt Q-fuzzy JU-ideal of Y. Let $a, b \in Y$ and $q \in Q$. Since f is an epimorphism of JU-algebras, there exist $x, x*y \in X$ such that $f(x) = a$, $f(y) = b$. Now

$$\begin{aligned} \eta_D(b,q) &= \eta_D(f(y),q) \\ &= f^{-1}(\eta_D)(y,q) \\ &\leqslant \max\{f^{-1}(\eta_D)(x,q), f^{-1}(\eta_D)(x*y,q)\} \\ &= \max\{\eta_D(f(x),q), \eta_D(f(x*y),q)\} \\ &= \max\{\eta_D(a,q), \eta_D(a*b,q)\} \\ \Rightarrow \eta_D(b,q) &\leqslant \max\{\eta_D(a,q), \eta_D(a*b,q)\}. \end{aligned}$$

Therefore, η_D is a doubt Q-fuzzy JU-ideal of Y. □

Theorem 3.2.2.6. *Let η_{D_1} and η_{D_2} be doubt Q-fuzzy JU-ideals of X and Y respectively. Then $\eta_{D_1} \times \eta_{D_2}$ is a doubt Q-fuzzy JU-ideal of $X \times Y$.*

Proof. Let η_{D_1} and η_{D_2} be any two doubt Q-fuzzy JU-ideals of X and Y respectively. Then

$$\begin{aligned} \mathrm{Let}(x,y) \in X \times Y, (\eta_{D_1} \times \eta_{D_2})((1,1),q) &= \max\{\eta_{D_1}(1,q), \eta_{D_2}(1,q)\} \\ &\leqslant \max\{\eta_{D_1}(x,q), \eta_{D_2}(y,q)\} \\ &= (\eta_{D_1} \times \eta_{D_2})((x,y),q). \end{aligned}$$

Hence $(\eta_{D_1} \times \eta_{D_2})((1,1),q) \leqslant (\eta_{D_1} \times \eta_{D_2})((x,y),q)$.
And let $(x_1,y_1),(x_2,y_2) \in X \times Y$ and $q \in Q$, then

$$\begin{aligned} \eta_{D_1} \times \eta_{D_2}[(y_1,y_2),q] &= \eta_{D_1} \times \eta_{D_2}((y_1,y_2),q) \\ &= \max\{\eta_{D_1}(y_1,q), \eta_{D_2}(y_2,q)\} \\ &\leqslant \max\{\max\{\eta_{D_1}(x_1,q), \eta_{D_1}((x_1*y_1),q)\}, \max\{(x_2,q), \eta_{D_1}((x_2*y_2),q)\}\} \\ &= \max\{\max\{\eta_{D_1}(x_1,q), \eta_{D_2}(x_2,q)\}, \max\{\eta_{D_1}((x_1*y_1),q), \eta_{D_2}((x_2*y_2),q)\}\} \\ &= \max\{\eta_{D_1} \times \eta_{D_2}((x_1,x_2),q), \eta_{D_1} \times \eta_{D_2}(((x_1,x_2)*(y_1,y_2),q))\}. \end{aligned}$$

Therefore, $\eta_{D_1} \times \eta_{D_2}(y_1,y_2),q] \leqslant \max\{\eta_{D_1} \times \eta_{D_2}((x_1,x_2),q), \eta_{D_1} \times \eta_{D_2}(((x_1,x_2)*(y_1,y_2),q),q))\}$.
Hence, $\eta_{D_1} \times \eta_{D_2}$ is a doubt Q-fuzzy JU-ideal of $X \times Y$. □

Chapter 4
T-Fuzzy Structures on JU-Algebra

4.1. T-Fuzzy JU-Subalgebras of JU-Algebra

In this section, we introduce the notion of T-fuzzy JU-subalgebra of a JU-algebra and discuss some of its properties.

Definition 4.1.1. *Let ϖ be a fuzzy set in X. Then the set ϖ is a T-fuzzy JU-subalgebra with the binary operation $*$ if* $\varpi(x*y) \geqslant T\{\varpi(x), \varpi(y)\}, \forall x,y \in X$.

Example 4.1.1. *Let X =* $\{1,2,3,4\}$ *in which $*$ is defined by the following Cayley table:*

$*$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

See [3] $(X;*,1)$ *is a JU-algebra. Let* $T : [0,1] \times [0,1] \to [0,1]$ *be a function. The given T-norm defined by* $T(x,y) = \max\{x+y-1, 0\}$, $\forall x,y \in [0,1]$. *Define a fuzzy set ϖ in X by* $\varpi(1) = 0.8$, $\varpi(2) = 0.6$, $\varpi(3) = 0.5$ *and* $\varpi(4) = 0.3$.
Now, take for $x = 2$*;* $y = 3$, *then* $\varpi(x*y)$=$\varpi(2*3)$=$\varpi(2) = 0.6, \cdots(*)$ *and*
$T\{\varpi(x), \varpi(y)\}$= $T\{\varpi(2), \varpi(3)\}$=$T\{0.6, 0.5\} = \max\{0.6+0.5-1, 0\} = \max\{0.1, 0\} = 0.1. \cdots(**)$
Hence, from (*) and (**) $0.6 \geqslant 0.1$, *which is true.*
Similarly, the result holds for all values of x and y in X.
Therefore, ϖ is a T-fuzzy JU-subalgebra of X.

Theorem 4.1.1. *If ϖ is a T-fuzzy JU-subalgebra of X, then* $\varpi(1) \geqslant \varpi(x)$

Proof. Suppose ϖ is a T-fuzzy JU-subalgebra. Since $\varpi(1) = \varpi(x*x) \geqslant T\{\varpi(x), \varpi(x)\} = \varpi(x)$ □

Theorem 4.1.2. *The intersection of any two T-fuzzy JU-subalgebras of X is also a T-fuzzy JU-subalgebra of X.*

Proof. Suppose ϖ_1 and ϖ_2 are two T-fuzzy JU-subalgebras of X and $\forall x,y \in X$. Then,

$$\varpi_1 \cap \varpi_2(x*y) = \min\{\varpi_1(x*y), \varpi_2(x*y)\}$$

$$\geqslant \min\{T\{\varpi_1(x),\varpi_1(y)\},T\{\varpi_2(x),\varpi_2(y)\}\}$$
$$\geqslant T\{\min\{\varpi_1(x),\varpi_2(x)\},\min\{\varpi_1(y),\varpi_2(y)\}$$
$$= T\{\varpi_1\cap\varpi_2(x),\varpi_1\cap\varpi_2(y)\}$$

□

Corollary 4.1.3. *Let* $\{\varpi_j : j\in\Omega\}$ *be a family of T-fuzzy JU-subalgebras of X. Then* $\cap_{j\in\Omega}\varpi_j$ *is also T-fuzzy JU-subalgebra of X, where* $\cap_{j\in\Omega}\varpi_j = \{(x,\inf_{j\in\Omega}\varpi_j)|x\in X\}$

Proof. For each $j\in\Omega$, let ϖ_j be a T-fuzzy JU-subalgebra of X. Hence, for all $x,y\in X$,

$$\varpi_j(x*y)\geqslant T(\varpi_j(x),\varpi_j(y)).$$

Define a fuzzy set $\varpi : X\to[0,1]$ by

$$\varpi(x)=\inf_{j\in\Omega}\varpi_j(x),\quad \forall x\in X.$$

We will show that ϖ is a T-fuzzy JU-subalgebra of X. For any $x,y\in X$, we have

$$\begin{aligned}\varpi(x*y) &= \inf_{j\in\Omega}\varpi_j(x*y)\\ &\geqslant \inf_{j\in\Omega}T(\varpi_j(x),\varpi_j(y))\\ &\geqslant T\left(\inf_{j\in\Omega}\varpi_j(x),\inf_{j\in\Omega}\varpi_j(y)\right)\\ &= T(\varpi(x),\varpi(y))\end{aligned}$$

Therefore, $\bigcap_{j\in\Omega}\varpi_j$ is a T-fuzzy JU-subalgebra of X. □

Remark 4.1.1. *The union of any two T-fuzzy JU-subalgebras of a JU-algebra X may not be a T-fuzzy JU-subalgebra of X.*

Example 4.1.2. *Let X =* $\{1,2,3,4\}$ *in which* $*$ *is defined by the following Cayley table:*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $(X;*,1)$ *is a JU-algebra. Let* $T:[0,1]\times[0,1]\to[0,1]$ *be a function. The given T-norm defined by* $T(x,y)=\min\{x,y\}$, $\forall x,y\in[0,1]$. *Let us define the fuzzy set* ϖ_1 *and* ϖ_2 *in X by* $\varpi_1(1)=0.7$, $\varpi_1(2)=0.5$, $\varpi_1(3)=0.7,\varpi_1(4)=0.5$ *and* $\varpi_2(1)=0.8$, $\varpi_2(2)=0.6$, $\varpi_2(3)=0.4,\varpi_2(4)=0.4$. *Then,* $\varpi_1\cup\varpi_2(2*3)=\max\{\varpi_1(2*3),\varpi_2(2*3)\}=\max\{\varpi_1(4),\varpi_2(4)\}=\max\{0.5,0.4\}=0.5\cdots(*)$

And,

$$\begin{aligned}\max\{\varpi_1(2*3),\varpi_2(2*3)\} &\geqslant \max\{T(\varpi_1(2),\varpi_1(3)),T(\varpi_2(2),\varpi_2(3))\}\\ &= \max\{\min\{\varpi_1(2),\varpi_1(3)\},\min\{\varpi_2(2),\varpi_2(3)\}\}\\ &= \min\{\max\{\varpi_1(2),\varpi_2(2)\},\max\{\varpi_1(3),\varpi_2(3)\}\}\\ &= \min\{\max\{0.5,0.6\},\max\{0.7,0.4\}\}\\ &= \min\{0.6,0.7\} = 0.6\cdots(**)\end{aligned}$$

From $(*)$ *and* $(**)$ *we get* $0.5 \geqslant 0.6$ *which is false.*
Therefore, we showed that the union of any two T-fuzzy JU-subalgebras of a JU-algebra X may not be a T-fuzzy JU-subalgebra of X.

Theorem 4.1.4. *Let S be a nonempty subset of a JU-algebra X. Then the characteristics function* X_S *is a T-fuzzy JU-subalgebra of X if and only if S is a JU-subalgebra of X.*

Proof. Suppose X_S is a T-fuzzy JU-subalgebras of X and $S \neq \emptyset$. Let $x,y \in S$ implies that $X_S(x) = 1 = X_S(y)$. Now

$X_J(x*y) \geqslant T\{X_S(x),X_S(y)\} = T\{1,1\} = 1$
$\Rightarrow X_S(x*y) \geqslant 1$ but $X_S(x*y) \leqslant 1$
$\Rightarrow X_S(x*y) = 1$
$\Rightarrow x*y \in S$
$\Rightarrow$ S is a JU-subalgebra of X.

Conversely, suppose S is a JU-subalgebra of X. We need to show that X_S is a T-fuzzy JU-subalgebra of X. Now consider the following cases:

Case 1. if $x,y \in X$ where $x*y \in S$ then

$X_S(x*y) = 1 \geqslant T\{X_S(x),X_S(y)\}$

Case 2. if $x \in S$ and $y \notin S$(or $x \notin S$ and $y \in S$),then

$X_S(x) = 1, X_S(y) = 0$. Thus

$X_S(x*y) \geqslant 0 = T\{1,0\} = T\{0,1\} = T\{X_S(x),X_S(y)\}$

Case 3. Suppose $x,y \notin S$ then

$X_S(x) = 0 = X_S(y)$. Thus

$X_S(x*y) \geqslant 0 = T\{0,0\} = T\{X_S(x),X_S(y)\}$

□

Theorem 4.1.5. *Let* $U(\varpi : t)$ *be nonempty and* $\forall t \in [0,1]$. *A fuzzy subset* ϖ *of a JU-algebra X is a T-fuzzy JU-subalgebra of X, if and only if* $U(\varpi : t)$ *is a JU-subalgebra of X.*

Proof. Suppose that ϖ is a T-fuzzy JU-subalgebra of X such that $U(\varpi : t) \neq \emptyset$. Then for any $x, y \in X$ and let $x, y \in U(\varpi : t)$. Then, $\varpi(x) \geqslant t$ and $\varpi(y) \geqslant t$. We have

$\varpi(x * y) \geqslant T\{\varpi(x), \varpi(y)\} \geqslant t$

This implies that $x * y \in U(\varpi : t)$ and hence $U(\varpi : t)$ is a subalgebra of X.

Conversely, suppose that $U(\varpi : t)$ is a JU-subalgebra of X for any $t \in [0,1]$ and $U(\varpi : t) \neq \emptyset$. Assume that ϖ is not T-fuzzy JU-subalgebra of X. Then there exist some $z_0, z_1 \in X$ such that $\varpi(z_0 * z_1) < T\{\varpi(z_0), \varpi(z_1)\}$

Take $\beta = \frac{1}{2}[\varpi(z_0 * z_1) + T\{\varpi(z_0), \varpi(z_1)\}]$

$\Rightarrow \varpi(z_0 * z_1) < \beta < T\{\varpi(z_0), \varpi(z_1)\}$

$\Rightarrow z_0 * z_1 \notin U(\varpi : \beta)$, a contradiction, since $U(\varpi : \beta)$ is a JU-subalgebra of X. Therefore, $\varpi(z_0 * z_1) \geqslant T\{\varpi(z_0), \varpi(z_1)\}$ for any $z_0, z_1 \in X$. □

Theorem 4.1.6. *Let T be a T-norm and let ϖ be a fuzzy set in a JU-algebra of X with $Im(\varpi) = \{t_1, t_2, ..., t_n\}$ where $t_i < t_j$ whenever $i > j$. Assume that there exist an ascending chain of subalgebra $S_o \subseteq S_1 \subseteq S_2 \subseteq, ..., \subseteq S_n = X$ of X such that $\varpi(\tilde{S_m}) = t_m$, where $\tilde{S_m} = S_m / S_{m-1}$ for $m = 1,2,3,.....,n$ and $\tilde{S_o} = S_o$. Then ϖ is a T-fuzzy JU-subalgebra of X.*

Proof. Suppose that there exist an ascending chain of subalgebra $S_o \subseteq S_1 \subseteq S_2 \subseteq, ..., \subseteq S_n = X$ of X such that $\varpi(\tilde{S_m}) = t_m$, where $m = 1,2,3,...,n$. Let $x, y \in X$ and let $x, y \in \tilde{S_m}$ then $\varpi(x) = t_m = \varpi(y)$ and $x * y \in \tilde{S_m}$. Now

$\varpi(x * y) \geqslant t_m = \min\{\varpi(x), \varpi(y)\} \geqslant T\{\varpi(x), \varpi(y)\}$

Suppose that $x \in \tilde{S_p}$ and $y \in \tilde{S_q}$ for $p \neq q$ Without loss of generality, we may assume that $p > q$. Then $\varpi(x) = t_p < t_q = \varpi(y)$ and $x * y \in \tilde{S_p}$. Thus

$\varpi(x * y) \geqslant t_p = \min\{\varpi(x), \varpi(y)\} \geqslant T\{\varpi(x), \varpi(y)\}$

Hence ϖ is a T-fuzzy JU-subalgebra of X. □

Theorem 4.1.7. *let S be a JU-subalgebra of X and ϖ be a fuzzy set in X given by*

$$\varpi(x) = \begin{cases} p, \text{ if } x \in S \\ q, \text{ otherwise } \forall\, p, q \in [0,1] \end{cases}$$

With $p \geqslant q$. Then ϖ is a T_{Luk} fuzzy JU-subalgebra of X. Particularly if $p = 1$ and $q = 0$ then ϖ is T_{Luk} fuzzy subalgebra X. Additionally, $Im(\varpi) = S$

Proof. Suppose S is a JU-subalgebra of X and let $x, y \in X$. Then consider the following cases:

Case 1. If $x, y \in S$, then

$$T_{Luk}(\varpi(x), \varpi(y)) = T_{Luk}(p, p) = \max(2p - 1, 0)$$

$$= \begin{cases} 2p - 1, \text{ if } p \geqslant 1/2 \\ 0, \text{ Otherwise} \end{cases}$$

$$\leqslant p$$

$$= \varpi(x * y)$$

Case 2. If $x, y \notin S$,then

$$T_{Luk}(\varpi(x), \varpi(y)) = T_{Luk}(q, q) = max(2q - 1, 0)$$

$$= \begin{cases} 2q - 1, \text{ if } q \geqslant 1/2 \\ 0, \text{ Otherwise} \end{cases}$$

$$\leqslant q$$

$$= \varpi(x * y)$$

Case 3. If $x \in S$ and $y \notin S$ (or $x \notin S$ and $y \in S$), then

$$T_{Luk}(\varpi(x), \varpi(y)) = T_{Luk}(p, q) = max(p + q - 1, 0)$$

$$= \begin{cases} p + q - 1, \text{ if } p + q \geqslant 1 \\ 0, \text{ Otherwise} \end{cases}$$

$$\leqslant q$$

$$= \varpi(x * y)$$

Hence ϖ is a T-fuzzy JU-subalgebra of X.
Suppose that $p = 1$ and $q = 0$. Then $T_{Luk}(p, p) = max\{p + p - 1, 0\} = 1 = p$ and $T_{Luk}(q, q) = max\{q + q - 1, 0\} = 0 = q$ and $Im(\varpi) = S$ □

4.2. T-fuzzy JU-Ideals of JU-Algebras

In this section, we introduce the notion of T-fuzzy JU-ideals of JU-algebra and discuss some of their properties.

Definition 4.2.1. *Let ϖ be a fuzzy set in X. Then the set ϖ is a T-fuzzy JU-ideal with the binary operation $*$ if it satisfies the following axioms:*
a) $\varpi(1) \geqslant \varpi(x)$
b) $\varpi(y) \geqslant T\{\varpi(x), \varpi(x * y)\}, \forall x, y \in X$.

Example 4.2.1. *Let X = $\{1, 2, 3, 4, 5, 6\}$ in which $*$ is defined by the following Cayley table:*

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	1
6	1	1	2	1	1	1

See [3] $(X;*,1)$ *is a JU-algebra. Let* $T:[0,1]\times[0,1]\to[0,1]$ *be a function. The given T-norm defined by* $T(x,y)=x.y$, *for all* $x,y\in[0,1]$. *Define a fuzzy set* ϖ *in X by* $\varpi(1)=0.9$, $\varpi(2)=0.7$, $\varpi(3)=0.5=\varpi(4)$, $\varpi(5)=0.3=\varpi(6)$. *Now let we take* $x=1$ *and* $y=4$, *then* $\varpi(4)=0.5\cdots(*)$ *and* $T\{\varpi(1),\varpi(1*4)\}=T\{0.9,0.5\}=(0.9).(0.5)=0.45\cdots(**)$.
From (*) and (**) we get $0.5\geqslant 0.45$, *which is true.*
Similarly, the result holds for all values of x and y in X.
Therefore, ϖ *is a T-fuzzy JU-ideal of X.*

Theorem 4.2.1. *The intersection of any two T-fuzzy JU-ideals of X is also a T-fuzzy JU-ideal of X.*

Proof. Suppose ϖ_1 and ϖ_2 are two T-fuzzy JU-ideals of X and $\forall x,y\in X$. Then,

$\varpi_1\cap\varpi_2(1)=\min\{\varpi_1(1),\varpi_2(1)\}\geqslant\min\{\varpi_1(x),\varpi_2(x)\}=\varpi_1\cap\varpi_2(x)$

And,

$$\begin{aligned}\varpi_1\cap\varpi_2(y)&=\min\{\varpi_1(y),\varpi_2(y)\}\\&\geqslant\min\{T\{\varpi_1(x),\varpi_1(x*y)\},T\{\varpi_2(x),\varpi_2(x*y)\}\}\\&\geqslant T\{\min\{\varpi_1(x),\varpi_2(x)\},\min\{\varpi_1(x*y),\varpi_2(x*y)\}\}\\&=T\{\varpi_1\cap\varpi_2(x),\varpi_1\cap\varpi_2(x*y)\}\end{aligned}$$

□

Corollary 4.2.2. *Let* $\{\varpi_j:j\in\Omega\}$ *be a family of T-fuzzy JU-ideals of X. Then* $\cap_{j\in\Omega}\varpi_j$ *is also T-fuzzy JU-ideal of X, where* $\cap_{j\in\Omega}\varpi_j=\{(x,\inf_{j\in\Omega}\varpi_j)|x\in X\}$

Proof. For each $j\in\Omega$, let ϖ_j be a T-fuzzy JU-ideal of X. Hence, for all $x,y\in X$, the following properties hold:

(i) $\varpi_j(1)\geqslant\varpi_j(x)$,

(ii) $\varpi_j(x*y)\geqslant T(\varpi_j(x),\varpi_j(y))$.

Define a fuzzy set $\varpi:X\to[0,1]$ by

$$\varpi(x)=\inf_{j\in\Omega}\varpi_j(x),\quad\forall x\in X.$$

We will show that ϖ is a T-fuzzy JU-ideal of X.

(i) For any $x \in X$,

$$\varpi(1) = \inf_{j\in\Omega} \varpi_j(1) \geqslant \inf_{j\in\Omega} \varpi_j(x) = \varpi(x).$$

Hence, $\varpi(1) \geqslant \varpi(x)$.

(ii) And for any $x, y \in X$,

$$\begin{aligned}\varpi(y) &= \inf_{j\in\Omega} \varpi_j(y)\\ &\geqslant \inf_{j\in\Omega} T(\varpi_j(x), \varpi_j(x*y))\\ &\geqslant T\left(\inf_{j\in\Omega} \varpi_j(x), \inf_{j\in\Omega} \varpi_j(x*y)\right)\\ &= T(\varpi(x), \varpi(x*y)).\end{aligned}$$

Hence,

$$\varpi(y) \geqslant T(\varpi(x), \varpi(x*y)).$$

Therefore, $\bigcap_{j\in\Omega} \varpi_j$ is a T-fuzzy JU-ideal of X. □

Theorem 4.2.3. *Let J be a nonempty subset of a JU-algebra X. Then the characteristics function* X_J *is a T-fuzzy JU-ideal of X if and only if J is an ideal of X.*

Proof. Suppose X_J is a T-fuzzy JU-ideals of X and $J \neq \emptyset$.
Let $x \in J$ implies that $X_J(x) = 1$.
Hence, $X_J(1) = X_J(x*x) \geqslant T\{X_J(x), X_J(x)\} = T\{1,1\} = 1$
$\Rightarrow X_J(1) \geqslant 1$ but $X_J(1) \leqslant 1$ implies that $X_J(1) = 1$. Thus $1 \in J$
And let $x, x*y \in J$ implies that $X_J(x) = 1 = X_J(x*y)$. Now
$X_J(y) \geqslant T\{X_J(x), X_J(x*y)\} = T\{1,1\} = 1$
$\Rightarrow X_J(y) \geqslant 1$ but $X_J(y) \leqslant 1 \Rightarrow X_J(y) = 1$
$\Rightarrow y \in J$. Therefore, J is a JU-ideal of X.
Conversely, suppose J is a JU-ideal of X. We need to show that X_J is a T-fuzzy JU-ideal of X. Now consider the following cases:

Case 1. If $x, x*y \in J$ where $y \in J$ then

$$X_J(y) = 1 \geqslant T\{X_J(x), X_J(x*y)\}$$

Case 2. If $x \in J$ and $x*y \notin J$ (or $x \notin J$ and $x*y \in J$),then $X_J(x) = 1, X_J(x*y) = 0$. Thus

$$X_J(y) \geqslant 0 = T\{1,0\} = T\{0,1\} = T\{X_J(x), X_J(x*y)\}$$

Case 3. If $x, x*y \notin J$ then $X_J(x) = 0 = X_J(x*y)$. Thus

$$X_J(y) \geqslant 0 = T\{0,0\} = T\{X_J(x), X_J(x*y)\}$$

□

Theorem 4.2.4. *Let* $U(\varpi : t)$ *be nonempty for all* $t \in [0,1]$*. A fuzzy subset* ϖ *of a JU-algebra* X *is a T-fuzzy JU-ideal of* X *if and only if* $U(\varpi : t)$ *is a JU-ideal of* X*.*

Proof. Suppose ϖ is a T-fuzzy JU-ideal of X. Let $t \in [0,1]$ and assume $U(\varpi : t) \neq \emptyset$.
For any $x \in U(\varpi, t)$ such that $\varpi(x) \geqslant t$. Since ϖ is a T-fuzzy JU-ideal of X, we have $\varpi(1) \geqslant \varpi(x)$, for all $x \in X$. Thus $\varpi(1) \geqslant \varpi(x) \geqslant t$. Therefore $1 \in U(\varpi, t)$.
And let $x, x * y \in U(\varpi : t)$. Then $\varpi(x) \geqslant t$ and $\varpi(x * y) \geqslant t$. Hence

$$\varpi(y) \geqslant T\{\varpi(x), \varpi(x * y)\} \geqslant t.$$

$\Rightarrow y \in U(\varpi : t)$, hence $U(\varpi : t)$ is a JU-ideal of X.
Conversely, suppose that $U(\varpi : t)$ is a JU-ideal of X for every $t \in [0,1]$ with $U(\varpi : t) \neq \emptyset$. Assume, for contradiction, that ϖ is not a T-fuzzy JU-ideal of X.
Then there exists some $z_0 \in X$ such that $\varpi(1) < \varpi(z_0)$. Take

$$\beta = \tfrac{1}{2}\big(\varpi(1) + \varpi(z_0)\big).$$

Then $\varpi(1) < \beta < \varpi(z_0)$, which implies $z_0 \in U(\varpi : \beta)$ and $1 \notin U(\varpi : \beta)$ — a contradiction, since $U(\varpi : \beta)$ is assumed to be a JU-ideal of X. Therefore, we must have $\varpi(1) \geqslant \varpi(x)$ for all $x \in X$.
Next, assume that $z_0, z_1 \in X$ such that

$$\varpi(z_1) < T\{\varpi(z_0), \varpi(z_0 * z_1)\}.$$

Take

$$\beta = \tfrac{1}{2}\big(\varpi(z_1) + T\{\varpi(z_0), \varpi(z_0 * z_1)\}\big).$$

Then

$$\varpi(z_1) < \beta < T\{\varpi(z_0), \varpi(z_0 * z_1)\}.$$

This implies $\beta > \varpi(z_1)$ and $\beta < T\{\varpi(z_0), \varpi(z_0 * z_1)\}$. Hence $z_1 \notin U(\varpi : \beta)$, a contradiction since $U(\varpi : \beta)$ is a JU-ideal of X.
Therefore, $\varpi(z_1) \geqslant T\{\varpi(z_0), \varpi(z_0 * z_1)\}$ for all $z_0, z_1 \in X$. Thus, ϖ is a T-fuzzy JU-ideal of X. □

Theorem 4.2.5. *Let* ϖ *be a T-fuzzy JU-ideals of X. Then the set* $X_\varpi = \{x \in X | \varpi(x) = \varpi(1)\}$ *is an ideal of X.*

Proof. Let ϖ be a T-fuzzy JU-ideals of X. Obviously, $1 \in X_\varpi$. Let $x, x * y \in X_\varpi$ implies that $\varpi(x) = \varpi(1) = \varpi(x * y)$. Now

$\varpi(y) \geqslant T\{\varpi(x), \varpi(x * y)\} = T\{\varpi(1), \varpi(1)\} = \varpi(1)$
$\Rightarrow \varpi(y) \geqslant \varpi(1)$ but $\varpi(y) \leqslant \varpi(1)$ by Definition 4.2.1
$\Rightarrow \varpi(y) = \varpi(1)$
$\Rightarrow y \in X_\varpi$, hence X_ϖ is JU-ideals of X. □

Theorem 4.2.6. *If every T-fuzzy JU-ideal* ϖ *of X has a finite image, then every descending chain of JU-ideals of X converges at finite steps.*

Proof. Assume that there exists a strictly descending chain $J_1 \supsetneq J_2 \supsetneq J_3 ...$ of JU-ideal of X which does not converge at a finite step. Define a fuzzy set ϖ in X by

$$\varpi(x) = \begin{cases} \frac{n-1}{n}, \text{ if } x \in J_n \setminus J_{n+1} \\ 1, \ x \in \cap_{n=1}^{\infty} J_n \end{cases}$$

, where $n \in N$ and $J_1 = X$.
Since $1 \in J_n, \forall\, n$, $\varpi(1) = 1 \geqslant \varpi(x), \forall x \in X$.
And for any $x, y \in X$, then by the above assumption, consider the following cases.

Case 1. If $x \in J_n \setminus J_{n+1}$ and $x * y \in J_m \setminus J_{m+1}$ for $n = 1,2,3,...;\ m = 1,2,3,...$ Without loss of generality, we may assume that $n \leqslant m$.

Then x and $x * y \in J_n$, (since $J_m \subsetneq J_n$). Thus $y \in J_n$ since J_n is a JU-ideals of X. Hence, $\varpi(y) \geqslant \frac{n-1}{n} = T\{\varpi(x), \varpi(x * y)\}$.

Case 2. If $x, x * y \in \cap_{n=1}^{\infty} J_n$, then $y \in \cap_{n=1}^{\infty} J_n$.

Thus $\varpi(y) = 1 = T\{\varpi(x), \varpi(x * y)\}$.

Case 3. If $x \notin \cap_{n=0}^{\infty} J_n$ and $x * y \in \cap_{n=1}^{\infty} J_n$, then there exists a positive integer p such that $x \in J_p \setminus J_{p+1}$.

It follows that $y \in J_p$, then $\varpi(y) \geqslant \frac{p-1}{p} = T\{\varpi(x), \varpi(x * y)\}$.

Or, if $x \in \cap_{n=1}^{\infty} J_n$ and $x * y \notin \cap_{n=1}^{\infty} J_n$, then there exists a positive integer q such that $x * y \in J_q \setminus J_{q+1}$. Implies that $y \in J_q$, then $\varpi(y) \geqslant \frac{q-1}{q} = T\{\varpi(x), \varpi(x * y)\}$.

Thus, ϖ is a T-fuzzy JU-ideal with an infinite number of different values, which is a contradiction.

□

Theorem 4.2.7. *Every ascending chain of JU-ideals of X terminates at a finite step if and only if the set of values of any T-fuzzy JU-ideals is a well-ordered subset of* $[0, 1]$.

Proof. Let ϖ be a T-fuzzy JU-ideal of X. Assume that the set of values of ϖ is not a well-ordered subset of $[0, 1]$. Then there exist a strictly decreasing sequence $\{\gamma_n\}$ such that $\varpi(x_n) = \gamma_n$. This implies that $U(\varpi : \gamma_1) \subsetneq U(\varpi : \gamma_2) \subsetneq U(\varpi : \gamma_3) \subsetneq ...$ is a strictly ascending chain of JU-ideal of X which is not terminating. This is impossible.
Conversely, assume that there exists a strictly ascending chain

$J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq ....(*)$

,of JU-ideals of X which does not converge at a finite step. Remember that $J = \cup_{n \in N} J_n$ is the JU-ideals of X. Now let us define a fuzzy set ϖ in X by

$$\varpi(x) = \begin{cases} \frac{1}{t}, \text{ if } t = \min\{n \in N | x \in J_n\} \\ 0, \text{ if } x \notin J_n \end{cases}$$

We went to show that ϖ is a T-fuzzy JU-ideal of X. Since $1 \in J_n, \forall n$, $\varpi(1) \geqslant \frac{1}{n} = \varpi(x)$, $\forall x \in X$.
And for any $x, y \in X$, then by the above assumption consider the following cases.

Case 1. if $x, x*y \in J_n \setminus J_{n-1}$ for $n = 2, 3, ...$ then $y \in J_n \setminus J_{n-1}$ implies that

$\varpi(y) \geqslant \frac{1}{n} = T\{\varpi(x), \varpi(x*y)\}$

Case 2. If $x \in J_n$ and $x*y \in J_n \setminus J_m$ and $x*y \in J_n$ for any $m < n$. Since ϖ is a JU-ideal of X, then $y \in J_n$.Thus $\varpi(y) \geqslant \frac{1}{n} \geqslant \frac{1}{m+1} \geqslant \varpi(x*y)$.

Or, If $x \in J_n \setminus J_m$ and $x*y \in J_n$ and $x \in J_n$ for any $m < n$. Since ϖ is a JU-ideal of X, then $y \in J_n$.Thus $\varpi(y) \geqslant \frac{1}{n} \geqslant \frac{1}{m+1} \geqslant \varpi(x*y)$.

Therefore, ϖ is a T-fuzzy JU-ideal of X. Since the sequence (*) does not converge, ϖ has a strictly descending sequence of values. This contradicts the value set of any T-fuzzy JU-ideal is well-ordered. $\square$

4.3. Cartesian Product of T-Fuzzy JU-Algebras

In this section, the Cartesian product of T-fuzzy JU-subalgebras and T-fuzzy JU-ideals of X is discussed and some of their properties are investigated.

Theorem 4.3.1. *The Cartesian product of any two T-fuzzy JU-subalgebras of X is also a T-fuzzy JU-subalgebra of X.*

Proof. Suppose ϖ_1 and ϖ_2 be two T-fuzzy JU-subalgebras of a JU-algebra X. Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$\begin{aligned}
[\varpi_1 \times \varpi_2]_T((x_1, x_2) * (y_1, y_2)) &= [\varpi_1 \times \varpi_2]_T((x_1 * y_1), (x_2 * y_2) \\
&= T(\varpi_1(x_1 * y_1), \varpi_2(x_2 * y_2)) \quad \text{(By Definition 1.3.4)} \\
&\geqslant T(T(\varpi_1(x_1), \varpi_1(y_1)), T(\varpi_2(x_2), \varpi_2(y_2))) \\
&= T(T(\varpi_1(x_1), \varpi_2(x_2)), T(\varpi_1(y_1), \varpi_2(y_2))) \\
&= T([\varpi_1 \times \varpi_2]_T(x_1, x_2), [\varpi_1 \times \varpi_2]_T(y_1, y_2))
\end{aligned}$$

$\square$

Theorem 4.3.2. *Let $\{X_j\}_{j=1}^n$ be the finite family of JU-algebras and $X = \Pi_{j=1}^n X_j$ the Cartesian product of JU-algebras of $\{X_j\}$. Let ϖ_j be a T-fuzzy JU-subalgebra of X_j, where $1 \leqslant j \leqslant n$. Then $\varpi = \Pi_{j=1}^n \varpi_j$ is defined by $\varpi(x_1, x_2, ..., x_n) = (\Pi_{j=1}^n \varpi_j)_T(x_1, x_2, ..., x_n) = T_n(\varpi_1(x_1), \varpi_2(x_2), ..., \varpi_n(x_n))$ is also T-fuzzy JU-subalgebra of X.*

Proof. Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be any elements of $X = \Pi_{j=1}^n X_j$. Then

$$\begin{aligned}
\varpi(x*y) &= \varpi((x_1, x_2, ..., x_n) * (y_1, y_2, ..., y_n)) \\
&= \varpi(x_1 * y_1, x_2 * y_2, ..., x_n * y_n) \\
&= T_n(\varpi_1(x_1 * y_1), \varpi_2(x_2 * y_2), ..., \varpi_n(x_n * y_n)) \quad \text{(By Definition 1.3.4)}
\end{aligned}$$

$$
\begin{aligned}
&\geqslant T_n(T(\varpi_1(x_1),\varpi_1(y_1)),T(\varpi_2(x_2),\varpi_2(y_2)),...,T(\varpi_n(x_n),\varpi_n(y_n)))\\
&= T(T_n(\varpi_1(x_1),\varpi_2(x_2),...,\varpi_n(x_n)),T_n(\varpi_1(y_1),\varpi_2(y_2),...,\varpi_n(y_n)))\\
&= T(\varpi(x_1,x_2,...,x_n),\varpi(y_1,y_2,...,y_n))\\
&= T(\varpi(x),\varpi(y))
\end{aligned}
$$

□

Theorem 4.3.3. *Let ϖ_1 and ϖ_2 be T-fuzzy JU-subalgebras of X. If T' is a T-norm which dominates T, i.e $T'(T(p,q),T(r,s)) \geqslant T(T'(p,r),T'(q,s)), \forall p,q,r,s \in [0,1]$. Then the T' fuzzy product of ϖ_1 and ϖ_2, $[\varpi_1.\varpi_2]_{T'}$ is a T-fuzzy JU-subalgebra of X.*

Proof. Let $x,y \in X$, then

$$
\begin{aligned}
[\varpi_1.\varpi_2]_{T'}(x*y) &= T'(\varpi_1(x*y),\varpi_2(x*y)) \quad \text{(By Definition 1.3.5)}\\
&\geqslant T'(T(\varpi_1(x),\varpi_1(y)),T(\varpi_2(x),\varpi_2(y)))\\
&\geqslant T(T'(\varpi_1(x),\varpi_2(x)),T'(\varpi_1(y),\varpi_2(y)))\text{(By Lemma 1.3.1)}\\
&= T([\varpi_1.\varpi_2]_{T'}(x),[\varpi_1.\varpi_2]_{T'}(y))
\end{aligned}
$$

□

Theorem 4.3.4. *The Cartesian product of any two T-fuzzy JU-ideals of X is also a T-fuzzy JU-ideal of X.*

Proof. : Suppose $\varpi_1 = (X,\varpi_1)$ and $\varpi_2 = (Y,\varpi_2)$ be two T-fuzzy JU-ideals of a JU-algebra X. Let $x,y \in X \times Y$. Then $[\varpi_1 \times \varpi_2]_T(1,1) = T(\varpi_1(1),\varpi_2(1)) \geqslant T(\varpi_1(x),\varpi_2(y)) = [\varpi_1 \times \varpi_2]_T(x,y)$.
And, let $(x_1,x_2),(y_1,y_2) \in X \times Y$. Then

$$
\begin{aligned}
[\varpi_1 \times \varpi_2]_T(y_1,y_2) &= T(\varpi_1(y_1),\varpi_2(y_2)) \quad \text{(By Definition 1.3.4)}\\
&\geqslant T(T(\varpi_1(x_1),\varpi_1(x_1*y_1)),T(\varpi_2(x_2),\varpi_2(x_2*y_2)))\\
&= T(T(\varpi_1(x_1),\varpi_2(x_2)),T(\varpi_1(x_1*y_1),\varpi_2(x_2*y_2)))\\
&= T\{[\varpi_1 \times \varpi_2]_T((x_1,x_2),[\varpi_1 \times \varpi_2]_T((x_1*y_1,(x_2*y_2)\}\\
&= T\{[\varpi_1 \times \varpi_2]_T((x_1,x_2),[\varpi_1 \times \varpi_2]_T((x_1,x_2)*(y_1,y_2))\}
\end{aligned}
$$

□

Theorem 4.3.5. *Let ϖ_1 and ϖ_2 be T-fuzzy JU-ideals of X. If T' is a T-norm which dominates T, i.e $T'(T(p,q),T(r,s)) \geqslant T(T'(p,r),T'(q,s)), \forall p,q,r,s \in [0,1]$. Then the T' fuzzy product of ϖ_1 and ϖ_2, $[\varpi_1.\varpi_2]_{T'}$ is a T-fuzzy JU-ideal of X.*

Proof. Let $x,y \in X$, then

$$[\varpi_1.\varpi_2]_{T'}(1) = T'(\varpi_1(1), \varpi_2(1)) \geqslant T'(\varpi_1(x), \varpi_2(x)) = [\varpi_1.\varpi_2]_{T'}(x)$$

$$\begin{aligned}
[\varpi_1.\varpi_2]_{T'}(y) &= T'(\varpi_1(y), \varpi_2(y)) \quad \text{(By Definition 1.3.5)} \\
&\geqslant T'(T(\varpi_1(x), \varpi_1(x * y)), T(\varpi_2(x), \varpi_2(x * y))) \\
&\geqslant T(T'(\varpi_1(x), \varpi_2(x)), T'(\varpi_1(x * y), \varpi_2(x * y))) \quad \text{(By Lemma 1.3.1)} \\
&= T([\varpi_1.\varpi_2]_{T'}(x), [\varpi_1.\varpi_2]_{T'}(x * y))
\end{aligned}$$

□

Chapter 5
Interval valued Fuzzy Structures on JU-Algebra

In this study, we introduce the concept of interval valued fuzzy JU-subalgebras and JU-ideals (abbreviated as I-V fuzzy JU-subalgebras and JU-ideals), and we discuss level set, homomorphism and Cartesian product within the framework of JU-algebras.

5.1. Interval valued Fuzzy JU-Subalgebras of JU-Algebras

In this section, we introduce the notion of interval valued fuzzy JU-subalgebras of JU-algebras, and some of their properties are discussed.

Definition 5.1.1. *Let $\tilde{\phi}$ be an I-V fuzzy set in X. Then the set $\tilde{\phi}$ is an I-V fuzzy JU-subalgebra of X with the binary operation $*$ if*

$$\tilde{\phi}(x*y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}, \quad \forall x, y \in X.$$

Example 5.1.1. *Let X = $\{1,2,3,4\}$ in which $*$ is defined by the following Cayley table:*

$*$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

See [3] $(X;*,1)$ is a JU-algebra. Define $\tilde{\phi}(x)$ as follows:

$$\tilde{\phi}(x) = \begin{cases} [0.5, 0.8] & \textit{if } x \in \{1,2\} \\ [0.3, 0.4] & \textit{otherwise} \end{cases}$$

For $\tilde{\phi}$ to be an I-V fuzzy JU-subalgebra of X, the following condition must hold for all $x, y \in X$:
$\tilde{\phi}(x*y) \geqslant \min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$.
Now from the Cayley table: let we take,

for $x = 1$ *and* $y = 2$*, then* $\tilde{\phi}(1 * 2) = \tilde{\phi}(2) = [0.5, 0.8], \cdots (*)$ *and*
$\min\{\tilde{\phi}(1), \tilde{\phi}(2)\} = \min\{[0.5, 0.8], [0.5, 0.8]\} = [0.5, 0.8]. \cdots (**)$
From (*) and (**) we get $[0.5, 0.8] \geqslant [0.5, 0.8]$*, which is true.*
Likewise,
For $x = 2$ *and* $y = 3$*, then* $\tilde{\phi}(2 * 3) = \tilde{\phi}(2) = [0.5, 0.8], \cdots (***)$ *and*
$\min\{\tilde{\phi}(2), \tilde{\phi}(3)\} = \min\{[0.5, 0.8], [0.3, 0.4]\} = [0.3, 0.4]. \cdots (****)$
From (***) and (****) we get $[0.5, 0.8] \geqslant [0.3, 0.4]$*, which is true.*
Similarly, the result holds for all values of x and y in X.
Therefore, $\tilde{\phi}$ *is an I-V fuzzy JU-subalgebra of X.*

Theorem 5.1.1. *An I-V fuzzy set* $\tilde{\phi} = \{(x, [\phi^L(x), \phi^U(x)]), x \in X\}$ *is an I-V fuzzy JU-subalgebra of* X *if and only if* ϕ^L *and* ϕ^U *are fuzzy JU-subalgebra of* X.

Proof. Let ϕ^L and ϕ^U be fuzzy JU-subalgebras of X and $x, y \in X$. Then

$$\begin{aligned}
\tilde{\phi}(x * y) &= [\phi^L(x * y), \phi^U(x * y)] \\
&\geqslant [\min\{\phi^L(x), \phi^L(y)\}, \min\{\phi^U(x), \phi^U(y)\}] \\
&= r\min\{[\phi^L(x), \phi^U(x)], [\phi^L(y), \phi^U(y)]\} \\
&= r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}.
\end{aligned}$$

Conversely, suppose $\tilde{\phi}$ is an I-V fuzzy JU-subalgebra of X and $\forall x, y \in X$. Then

$$\begin{aligned}
[\phi^L(x * y), \phi^U(x * y)] &= \tilde{\phi}(x * y) \\
&\geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\} \\
&= r\min\{[\phi^L(x), \phi^U(x)], [\phi^L(y), \phi^U(y)]\} \\
&= [\min\{\phi^L(x), \phi^L(y)\}, \min\{\phi^U(x), \phi^U(y)\}].
\end{aligned}$$

This implies:

$$\begin{aligned}
\phi^L(x * y) &\geqslant \min\{\phi^L(x), \phi^L(y)\} \quad \text{and} \\
\phi^U(x * y) &\geqslant \min\{\phi^U(x), \phi^U(y)\}.
\end{aligned}$$

Thus ϕ^L and ϕ^U are fuzzy JU-subalgebras of X. □

Theorem 5.1.2. *If* $\tilde{\phi}$ *is an I-V fuzzy JU-subalgebra of* X*, then* $\tilde{\phi}(1) \geqslant \tilde{\phi}(x)$*, for all* $x \in X$.

Proof. Suppose $\tilde{\phi}$ is an I-V fuzzy JU-subalgebra of X. Then for any $x \in X$:

$$\begin{aligned}
\tilde{\phi}(1) &= \tilde{\phi}(x * x) \quad \text{(By Lemma 1.6.1)} \\
&\geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x)\} \\
&= r\min\{[\phi^L(x), \phi^U(x)], [\phi^L(x), \phi^U(x)]\} \\
&= [\min\{\phi^L(x), \phi^L(x)\}, \min\{\phi^U(x), \phi^U(x)\}]
\end{aligned}$$

$$
\begin{aligned}
&= [\phi^L(x), \phi^U(x)] \\
&= \tilde{\phi}(x).
\end{aligned}
$$

□

Theorem 5.1.3. *The intersection of any two I-V fuzzy JU-subalgebras of X is also an I-V fuzzy JU-subalgebra of X.*

Proof. Suppose $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are two I-V fuzzy JU-subalgebra of X and $\forall x, y \in X$.Then,

$$
\begin{aligned}
(\tilde{\phi}_1 \cap \tilde{\phi}_2)(x*y) &= \min\{\tilde{\phi}_1(x*y), \tilde{\phi}_2(x*y)\} \\
&= \min\{[\phi_1^L(x*y), \phi_1^U(x*y)], [\phi_2^L(x*y), \phi_2^U(x*y)]\} \\
&= \min\{[\phi_1^L(x*y), \phi_2^L(x*y), \phi_1^U(x*y), \phi_2^U(x*y)]\} \\
&\geqslant \Big[\min\big\{\min\{\phi_1^L(x), \phi_1^L(y)\}, \min\{\phi_2^L(x), \phi_2^L(y)\}\big\}, \\
&\quad \min\big\{\min\{\phi_1^U(x), \phi_1^U(y)\}, \min\{\phi_2^U(x), \phi_2^U(y)\}\big\}\Big] \\
&= \Big[\min\big\{\min\{\phi_1^L(x), \phi_2^L(x)\}, \min\{\phi_1^L(y), \phi_2^L(y)\}\big\}, \\
&\quad \min\big\{\min\{\phi_1^U(x), \phi_2^U(x)\}, \min\{\phi_1^U(y), \phi_2^U(y)\}\big\}\Big] \\
&= \Big[\min\big\{(\phi_1^L \cap \phi_2^L)(x), (\phi_1^L \cap \phi_2^L)(y)\big\}, \\
&\quad \min\big\{(\phi_1^U \cap \phi_2^U)(x), (\phi_1^U \cap \phi_2^U)(y)\big\}\Big] \\
&= \min\big\{(\tilde{\phi}_1 \cap \tilde{\phi}_2)(x), (\tilde{\phi}_1 \cap \tilde{\phi}_2)(y)\big\}
\end{aligned}
$$

□

Corollary 5.1.4. *Let $\{\tilde{\phi}_i | i \in \Lambda\}$ be a family of I-V fuzzy JU-subalgebras of X. Then $\bigcap_{i\in\Lambda} \tilde{\phi}_i$ is also an I-V fuzzy JU-subalgebra of X.*

Proof. Let $x, y \in X$ be arbitrary. For each $i \in \Lambda$, since $\tilde{\phi}_i$ is an I-V fuzzy JU-subalgebra, we have
$\tilde{\phi}_i(x*y) \geqslant \min\{\tilde{\phi}_i(x), \tilde{\phi}_i(y)\}$
Define the intersection $\tilde{\phi} : X \to C[0,1]$ by

$$
\tilde{\phi}(x) = [\inf_{i\in\Lambda} \tilde{\phi}_i(x), \inf_{i\in\Lambda} \tilde{\phi}_i(x)].
$$

Then,

$$
\begin{aligned}
\tilde{\phi}(x*y) &= [\inf_{i\in\Lambda} \tilde{\phi}_i(x*y), \inf_{i\in\Lambda} \tilde{\phi}_i(x*y)] \\
&\geqslant [\inf_{i\in\Lambda} \min\{\tilde{\phi}_i(x), \tilde{\phi}_i(y)\}, \inf_{i\in\Lambda} \min\{\tilde{\phi}_i(x), \tilde{\phi}_i(y)\}] \\
&= [\min\{\inf_{i\in\Lambda} \tilde{\phi}_i(x), \inf_{i\in\Lambda} \tilde{\phi}_i(y)\}, \min\{\inf_{i\in\Lambda} \tilde{\phi}_i(x), \inf_{i\in\Lambda} \tilde{\phi}_i(y)\}] \\
&= \min\{\tilde{\phi}(x), \tilde{\phi}(y)\}.
\end{aligned}
$$

Therefore, $\bigcap_{i\in\Lambda} \tilde{\phi}_i$ is an I-V fuzzy JU-subalgebra of X. □

Remark 5.1.1. *The union of any two I-V fuzzy JU-subalgebras of a JU-algebra X may not be an I-V fuzzy JU-subalgebra of X.*

Example 5.1.2. *Let X = {1,2,3,4} in which ∗ is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $(X;*,1)$ *is a JU-algebra. Define* $\tilde{\phi}_1(x)$ *and* $\tilde{\phi}_2(x)$ *as follows:*

$$\tilde{\phi}_1(x) = \begin{cases} [0.8,0.9] & \textit{if } x \in \{1,3\} \\ [0.4,0.5] & \textit{otherwise} \end{cases}$$

and

$$\tilde{\phi}_2(x) = \begin{cases} [0.9,1] & \textit{if } x = 1 \\ [0.6,0.8] & \textit{if } x = 2 \\ [0.2,0.3] & \textit{otherwise} \end{cases}$$

$$\begin{aligned} \textit{Then, } (\tilde{\phi}_1 \cup \tilde{\phi}_2)(2*3) &= r\max\{\tilde{\phi}_1(2*3), \tilde{\phi}_2(2*3)\} \\ &= r\max\{\tilde{\phi}_1(4), \tilde{\phi}_2(4)\} \\ &= r\max\{[0.4,0.5],[0.2,0.3]\} = [0.4,0.5]\}\cdots(*) \end{aligned}$$

And,

$$\begin{aligned} r\max\{\tilde{\phi}_1(2*3), \tilde{\phi}_2(2*3)\} &\geqslant r\max\{r\min\{\tilde{\phi}_1(2), \tilde{\phi}_1(3)\}, r\min\{\tilde{\phi}_2(2), \tilde{\phi}_2(3)\}\} \\ &= r\max\{r\min\{[0.4,0.5],[0.8,0.9]\}\}, r\min\{[0.6,0.8],[0.2,0.3]\}\} \\ &= [r\min\{\max\{0.4,0.8\}, \max\{0.5,0.9\}, r\min\{\max\{0.6,0.2\}, \max\{0.8,0.3\}\}] \\ &= r\min\{[0.8,0.9],[0.6,0.8]\} \\ &= [\min\{0.8,0.6\}, \min\{0.9,0.8\}] = [0.6,0.8]\cdots(**) \end{aligned}$$

From $(*)$ *and* $(**)$ *we get* $[0.4,0.5] \geqslant [0.6,0.8]$ *which is false.*
Therefore, we verfied that the union of any two I-V fuzzy JU-subalgebras of a JU-algebra X may not be an I-V fuzzy JU-subalgebra of X.

Theorem 5.1.5. *Let* $U(\tilde{\phi} : [t_1,t_2]) = \{x \in X | \tilde{\phi}(x) \geqslant [t_1,t_2]\}$ *be nonempty set and* $\forall [t_1,t_2] \in C[0,1]$. *A fuzzy subset* $\tilde{\phi}$ *of a JU-algebra X is an interval Valued fuzzy JU-subalgebra of X, if and only if* $U(\tilde{\phi} : [t_1,t_2])$ *is JU-subalgebra of X.*

Proof. Suppose $\tilde{\phi}$ is an interval Valued fuzzy JU-subalgebra of X. Let $[t_1,t_2] \in C[0,1]$ and $U(\tilde{\phi} : [t_1,t_2]) \neq \emptyset$. Let $x,y \in U(\tilde{\phi} : [t_1,t_2])$ implies that $\tilde{\phi}(x) \geqslant [t_1,t_2]$ and $\tilde{\phi}(y) \geqslant [t_1,t_2]$. Then, $\tilde{\phi}(x*y) \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(y)\} \geqslant r\min\{[t_1,t_2],[t_1,t_2]\} = [t_1,t_2]$
This implies that $x*y \in U(\tilde{\phi} : [t_1,t_2])$ and hence $U(\tilde{\phi} : [t_1,t_2])$ is a JU-subalgebra of X.
Conversely, suppose that $U(\tilde{\phi} : [t_1,t_2])$ is a JU-subalgebra of X for any $[t_1,t_2] \in C[0,1]$ and that $U(\tilde{\phi} : [t_1,t_2]) \neq \emptyset$. Assume that $\tilde{\phi}$ is not an interval valued fuzzy JU-subalgebra of X. Then there exist some $z_0, z_1 \in X$ such that

$$\tilde{\phi}(z_0 * z_1) < r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_1)\}.$$

Let $\tilde{\phi}(z_0) = [\beta_1,\beta_2]$, $\tilde{\phi}(z_1) = [\beta_3,\beta_4]$, and $\tilde{\phi}(z_0 * z_1) = [t_1,t_2]$. Then

$$[t_1,t_2] < r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4]\}.$$

Take

$$[\alpha_1,\alpha_2] = \frac{1}{2}\big(\tilde{\phi}(z_0 * z_1) + r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_1)\}\big) = \frac{1}{2}\big([t_1,t_2] + r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4]\}\big)$$

$$= \left(\frac{1}{2}(t_1 + \min\{\beta_1,\beta_3\}), \frac{1}{2}(t_2 + \min\{\beta_2,\beta_4\})\right).$$

Hence,

$$\min\{\beta_1,\beta_3\} > \alpha_1 = \frac{1}{2}(t_1 + \min\{\beta_1,\beta_3\}) > t_1,$$

and

$$\min\{\beta_2,\beta_4\} > \alpha_2 = \frac{1}{2}(t_2 + \min\{\beta_2,\beta_4\}) > t_2.$$

Therefore,

$$(\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}) > [\alpha_1,\alpha_2] > [t_1,t_2] = \tilde{\phi}(z_0 * z_1).$$

This implies that $z_0 * z_1 \notin U(\tilde{\phi} : [\alpha_1,\alpha_2])$.
Moreover,

$$\tilde{\phi}(z_0) = [\beta_1,\beta_2] > (\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}) > [\alpha_1,\alpha_2],$$

and

$$\tilde{\phi}(z_1) = [\beta_3,\beta_4] > (\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}) > [\alpha_1,\alpha_2].$$

Hence $z_0, z_1 \in U(\tilde{\phi} : [\alpha_1,\alpha_2])$, which contradicts the assumption that $U(\tilde{\phi} : [\alpha_1,\alpha_2])$ is a JU-subalgebra of X.
Therefore,

$$\tilde{\phi}(z_0 * z_1) \geqslant r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_1)\}, \quad \forall z_0, z_1 \in X.$$

□

Theorem 5.1.6. *Let $\tilde{\phi}$ be an interval valued fuzzy subset of a JU-algebra X. Then $\tilde{\phi}^c$ is an interval valued fuzzy JU-subalgebra of X if and only if $L(\tilde{\phi},[t_1,t_2])$ is either empty or a JU-subalgebra of*

X *for* $[t_1,t_2] \in C[0,1]$.

Proof. Let $\tilde{\phi}$ be an interval valued fuzzy subset of a JU-algebra X. Suppose $\tilde{\phi}^c$ is an interval valued fuzzy JU-subalgebra of X and $L(\tilde{\phi},[t_1,t_2]) \neq \emptyset$. We need to prove that $L(\tilde{\phi},[t_1,t_2])$ is a JU-subalgebra of X. Since $\tilde{\phi}^c$ is an interval valued fuzzy JU-subalgebra of X, we have

$$\tilde{\phi}^c(x*y) \geqslant r\min\{\tilde{\phi}^c(x),\tilde{\phi}^c(y)\}, \quad \text{for all } x,y \in X \text{ and } [t_1,t_2] \in C[0,1].$$

It follows that

$$\begin{aligned}\tilde{1}-\tilde{\phi}(x*y) &\geqslant r\min\{\tilde{1}-\tilde{\phi}(x),\tilde{1}-\tilde{\phi}(y)\}\\ &= \tilde{1}-r\max\{\tilde{\phi}(x),\tilde{\phi}(y)\}, \quad \text{by Lemma 1.1.1}\end{aligned}$$

Hence,

$$\begin{aligned}&\tilde{1}-\tilde{\phi}(x*y) \geqslant \tilde{1}-r\max\{\tilde{\phi}(x),\tilde{\phi}(y)\}\\ \Rightarrow \quad &\tilde{\phi}(x*y) \leqslant r\max\{\tilde{\phi}(x),\tilde{\phi}(y)\} \leqslant r\max\{[t_1,t_2],[t_1,t_2]\} = [t_1,t_2].\end{aligned}$$

Thus, $\tilde{\phi}(x*y) \leqslant [t_1,t_2]$, implies that $x*y \in L(\tilde{\phi},[t_1,t_2])$. Hence, $L(\tilde{\phi},[t_1,t_2])$, is a JU-subalgebra of X.

Conversely, assume that $L(\tilde{\phi},[t_1,t_2])$ is a nonempty JU-subalgebra of X. We need to show that $\tilde{\phi}^c$ is an interval valued fuzzy JU-subalgebra of X.
Let $\tilde{\phi}(x),\tilde{\phi}(y) \in C[0,1]$ for $x,y \in X$ and $[t_1,t_2] \in C[0,1]$. Choose

$$[t_1,t_2] = r\max\{\tilde{\phi}(x),\tilde{\phi}(y)\}.$$

Then $\tilde{\phi}(x) \leqslant [t_1,t_2]$ and $\tilde{\phi}(y) \leqslant [t_1,t_2]$ imply that $x,y \in L(\tilde{\phi},[t_1,t_2]) \neq \emptyset$. Since $L(\tilde{\phi},[t_1,t_2])$ is a JU-subalgebra of X, we have $x*y \in L(\tilde{\phi},[t_1,t_2])$.
Thus,

$$\tilde{\phi}(x*y) \leqslant [t_1,t_2] = r\max\{\tilde{\phi}(x),\tilde{\phi}(y)\},$$

which gives

$$\begin{aligned}\tilde{\phi}^c(x*y) &= \tilde{1}-\tilde{\phi}(x*y)\\ &\geqslant \tilde{1}-r\max\{\tilde{\phi}(x),\tilde{\phi}(y)\}\\ &= r\min\{\tilde{1}-\tilde{\phi}(x),\tilde{1}-\tilde{\phi}(y)\}\\ &= r\min\{\tilde{\phi}^c(x),\tilde{\phi}^c(y)\}, \quad \text{by Lemma 1.1.1}\end{aligned}$$

Hence,

$$\tilde{\phi}^c(x*y) \geqslant r\min\{\tilde{\phi}^c(x),\tilde{\phi}^c(y)\}.$$

Therefore, $\tilde{\phi}^c$ is an interval valued fuzzy JU-subalgebra of X. $\square$

Theorem 5.1.7. *If $\tilde{\phi}$ is an I-V fuzzy JU-subalgebra of X, then the set $X_{\tilde{\phi}} := \{x \in X | \tilde{\phi}(x) = \tilde{\phi}(1)\}$ is a JU-subalgebra of X.*

Proof. Let $x, y \in X_{\tilde{\phi}}$. Then $\tilde{\phi}(x) = \tilde{\phi}(1) = \tilde{\phi}(y)$, and so

$\tilde{\phi}(x * y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\} = r\min\{\tilde{\phi}(1), \tilde{\phi}(1)\}$.

$\Rightarrow \tilde{\phi}(x * y) \geqslant \tilde{\phi}(1)$ but $\tilde{\phi}(x * y) \leqslant \tilde{\phi}(1), \forall x, y \in X$ (by Theorem 5.1.1.1)

$\Rightarrow \tilde{\phi}(x * y) = \tilde{\phi}(1)$

$\Rightarrow x * y \in X_{\tilde{\phi}}$.

Hence $X_{\tilde{\phi}}$ is a JU-subalgebra of X. $\square$

Theorem 5.1.8. *Every interval valued fuzzy JU-subalgebra of a JU-algebra X can be realized as a level JU-subalgebra of an interval valued fuzzy JU-subalgebra of X.*

Proof. Let S be a JU-subalgebra of X and let $\tilde{\phi}$ be an interval valued fuzzy set in X defined by

$$\tilde{\phi}(x) = \begin{cases} [t_1, t_2], & \text{if } x \in S, \\ [0,0], & \text{otherwise,} \end{cases}$$

where $[t_1, t_2] \in C[0,1]$ with $t_1 \leqslant t_2$.

We claim that $\tilde{\phi}$ is an interval valued fuzzy JU-subalgebra of X. Let $x, y \in X$. Consider the following cases:

Case 1. If $x, y \in S$, then $x * y \in S$, and so

$$\tilde{\phi}(x * y) = [t_1, t_2] = r\min\{[t_1, t_2], [t_1, t_2]\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}.$$

Case 2. If $x, y \notin S$, then $\tilde{\phi}(x) = [0,0] = \tilde{\phi}(y)$ and thus

$$\tilde{\phi}(x * y) = [0,0] = r\min\{[0,0], [0,0]\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}.$$

Case 3. If $x \in S$ and $y \notin S$, then

$$\tilde{\phi}(x * y) = [0,0] = r\min\{[t_1, t_2], [0,0]\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}.$$

Similarly, for the case $x \notin S$ and $y \in S$, we get

$$\tilde{\phi}(x * y) = r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}.$$

Therefore, $\tilde{\phi}$ is an interval valued fuzzy JU-subalgebra of X. $\square$

Theorem 5.1.9. *Let $\{S_i\}$ be any family of JU-subalgebras of JU-algebra X such that $S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n = X$. Then there exist an I-V fuzzy JU-subalgebras which are exactly the JU-subalgebras of this chain $\{S_i\}$.*

Proof. Consider a set of numbers $[t_0,t_0'] > [t_1,t_1'] > [t_2,t_2'] > \cdots > [t_n,t_n']$ where each $[t_i t_i'] \in C[0,1]$. Let $\tilde{\phi}: X \to C[0,1]$ be an I-V fuzzy set defined by

$$\tilde{\phi}(x) = \begin{cases} [t_0,t_0'], & \text{if } x \in S_0 \\ [t_i,t_i'], & \text{if } x \in S_i - S_{i-1},\ 0 < i \leqslant n. \end{cases}$$

Now, we claim that $\tilde{\phi}$ is an I-V fuzzy JU-subalgebra of X and $U(\tilde{\phi},[t_i,t_i']) = S_i$ for $0 < i \leqslant n$. Let $x, y \in X$, then we consider the following cases:

Case i. If $x, y \in S_i - S_{i-1}$ for some i, then $x, y \in S_i$. Therefore, by definition of $\tilde{\phi}$, we have $\tilde{\phi}(x) = [t_i,t_i'] = \tilde{\phi}(y)$. Since S_i is a JU-subalgebra of X, it follows that $x * y \in S_i$ and so either $x * y, \in S_i - S_{i-1}$ or $x * y, \in S_{i-1}$. Implies that $\tilde{\phi}(x) \geqslant [t_i,t_i']$ or $\tilde{\phi}(x) \geqslant [t_i - 1, t_i' - 1] > [t_i,t_i']$. In any case we get $\tilde{\phi}(x * y) \geqslant [t_i,t_i'] = \min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$

Case ii. For $i > j$, $[t_j,t_j'] > [t_i,t_i']$ and $S_j \subset S_i$. Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$ implies $\tilde{\phi}(x) = [t_i,t_i']$, $\tilde{\phi}(y) = [t_j,t_j'] \geqslant [t_i,t_i']$. Since S_i is a JU-subalgebra of X and $S_j \subset S_i$, then $x * y \in S_i$. Now, $\tilde{\phi}(x * y) \geqslant [t_i,t_i'] = \min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$. Thus $\tilde{\phi}$ is an I-V fuzzy JU-subalgebra of X. Also, from the definition of $\tilde{\phi}$, it follows that $\text{Im}(\tilde{\phi}) = \{[t_0,t_0'],[t_1,t_1'],[t_2,t_2'],\ldots,[t_n,t_n']\}$. Hence the upper level JU-subalgebras of $\tilde{\phi}$ are given by the chain of JU-subalgebras:

$$U(\tilde{\phi},[t_0,t_0']) \subset U(\tilde{\phi},[t_1,t_1']) \subset U(\tilde{\phi},[t_2,t_2']) \subset \cdots \subset U(\tilde{\phi},[t_n,t_n']) = X, \quad \text{for } 0 < i \leqslant n.$$

Now, $U(\tilde{\phi},[t_0,t_0']) = \{x \in X : \tilde{\phi}(x) \geqslant [t_0,t_0']\} = S_0$. Finally, we prove that $U(\tilde{\phi},[t_i,t_i']) = S_i$, for $0 < i \leqslant n$. Now let $x \in S_i$, then $\tilde{\phi}(x) \geqslant [t_i,t_i']$. This implies $x \in U(\tilde{\phi},[t_i,t_i']$. Hence, $S_i \subseteq U(\tilde{\phi},[t_i,t_i'])$. If $x \in U(\tilde{\phi},[t_i,t_i'])$, then $\tilde{\phi}(x) \geqslant [t_i,t_i']$. Which implies that $x \notin S_j$ for $j > i$. Otherwise, if $x \in S_j$, then $\tilde{\phi}(x) \geqslant [t_j,t_j']$, which implies $[t_i,t_i'] \geqslant \tilde{\phi}(x) \geqslant [t_j,t_j']$. This contradicts the assumption that $x \in U(\tilde{\phi},[t_i,t_i'])$. Hence $\tilde{\phi}(x) \in \{[t_0,t_0'],[t_1,t_1'],[t_2,t_2'],\ldots,[t_n,t_n']\}$. So, $x \in S_k$ for some $k \leqslant i$. As $S_k \subseteq S_i$, it follows that $x \in S_i$. Hence $U(\tilde{\phi},[t_i,t_i']) \subseteq S_i$. Therefore, $U(\tilde{\phi},[t_i,t_i']) = S_i$, for $0 < i \leqslant n$.

□

Theorem 5.1.10. *Let S be a subset of X and let $\tilde{\phi}$ be an I-V fuzzy set on X, given by*

$$\tilde{\phi}(x) = \begin{cases} [a_1,a_2] & \text{if } x \in S, \\ [b_1,b_2] & \text{otherwise}, \end{cases}$$

for all $[a_1,a_2],[b_1,b_2] \in C[0,1]$ with $a_1 \geqslant b_1$ and $a_2 \geqslant b_2$. Then ϕ is an I-V fuzzy JU-subalgebra of X if and only if S is a JU-subalgebra of X. Moreover, in this case, $X_{\tilde{\phi}} = S$.

Proof. Let $\tilde{\phi}$ be an I-V fuzzy JU-subalgebra of X, and let $x, y \in S$ then,

$\tilde{\phi}(x) = [a_1,a_2] = \tilde{\phi}(y)$

$\tilde{\phi}(x * y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\} = r\min\{[a_1,a_2],[a_1,a_2]\} = [a_1,a_2]$

$\Rightarrow \tilde{\phi}(x*y) \geqslant [a_1,a_2]$ but $\tilde{\phi}(x*y) \leqslant [a_1,a_2]$
$\Rightarrow \tilde{\phi}(x*y) = [a_1,a_2]$
$\Rightarrow x*y \in S$.
$\Rightarrow$ S is a JU-subalgebra of X.

Conversely, suppose that S is a JU-subalgebra of X, let $x,y \in X$. Let us consider the following cases.

Case 1. If $x,y \in S$ then $x*y \in S$, thus

$$\tilde{\phi}(x*y) = [a_1,a_2] = r\min\{\tilde{\phi}(x),\tilde{\phi}(y)\}$$

Case 2. If $x \notin S$ or $y \notin S$, then

$$\tilde{\phi}(x*y) \geqslant [b_1,b_2] = r\min\{\tilde{\phi}(x),\tilde{\phi}(y)\}$$

Case 3. If $x \in S$ or $y \notin S$, then

$$\tilde{\phi}(x*y) \geqslant [b_1,b_2] = r\min\{[a_1,a_2],[b_1,b_2]\} = r\min\{\tilde{\phi}(x),\tilde{\phi}(y)\}$$

Similarly, If $x \notin S$ or $y \in S$, then

$$\tilde{\phi}(x*y) \geqslant [b_1,b_2] = r\min\{[b_1,b_2],[a_1,a_2]\} = r\min\{\tilde{\phi}(x),\tilde{\phi}(y)\}$$

Thus, $\tilde{\phi}$ is an I-V fuzzy JU-subalgebra.
Moreover, we have

$$X_{\tilde{\phi}} = \{x \in X \mid \tilde{\phi}(x) = \tilde{\phi}(1)\} = \{x \in X \mid \tilde{\phi}(x) = [a_1,a_2]\} = S$$ □

Theorem 5.1.11. *Let* $(X,*,1)$ *and* $(Y,*',1')$ *be JU- algebras and a map* $g: X \to Y$ *be an onto homomorphism and let* $\tilde{\phi}$ *be an I-V fuzzy subset of X. If* $\tilde{\phi}$ *be an I-V fuzzy JU-subalgebra of X, then the image* $g(\tilde{\phi})$*, is an I-V fuzzy JU-subalgebra of Y with sup property.*

Proof. Let $g: X \to Y$ be an onto homomorphism of JU- algebra and let $\tilde{\phi}$ be an I-V fuzzy JU-subalgebra of X. We went to show that $g(\tilde{\phi})$ is an I-V fuzzy JU-subalgebra in Y. Let $y_1,y_2 \in Y$. Since g is onto, there exist $x_1,x_2 \in X$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$.
Also g is a homomorphism, $g(x_1*x_2) = g(x_1)*'g(x_2) = y_1*'y_2$. Now

$$\begin{aligned}
g(\tilde{\phi})(y_1*y_2) &= \sup_{x_1*x_2 \in g^{-1}(y_1*y_2)}\{\tilde{\phi}(x_1*x_2)\} \\
&= [\sup_{x_1*x_2 \in g^{-1}(y_1*y_2)}\{\phi^L(x_1*x_2),\phi^U(x_1*x_2)\}].\text{(by Lemma 1.4.1)} \\
&\geqslant [\min\{\sup_{x_1 \in g^{-1}(y_1)}\{\phi^L(x_1),\sup_{x_2 \in g^{-1}(y_2)}\{\phi^L(x_2)\}\},\min\{ \\
&\quad \sup_{x_1 \in g^{-1}(y_1)}\{\phi^U(x_1),\sup_{x_2 \in g^{-1}(y_2)}\phi^U(x_2)\}\}]. \\
&= [\min\{\sup_{x_1 \in g^{-1}(y_1)}\{\phi^L(x_1),\sup_{x_1 \in g^{-1}(y_1)}\{\phi^U(x_1)\}\},\min\{\sup_{x_2 \in g^{-1}(y_2)} \\
&\quad \{\phi^L(x_2),\sup_{x_2 \in g^{-1}(y_2)}\phi^U(x_2)\}\}]. \\
&= r\min\{\sup_{x_1 \in g^{-1}(y_1)}\{\tilde{\phi}(x_1)\},\sup_{x_2 \in g^{-1}(y_2)}\{\tilde{\phi}(x_2)\}\}. \\
&= r\min\{g(\tilde{\phi})(y_1)\},g(\tilde{\phi})(y_2)
\end{aligned}$$ □

Theorem 5.1.12. *Let* $(X,*,1)$ *and* $(Y,*',1')$ *be JU-algebras and a map* $g: X \to Y$ *be homomorphism. Let* $\tilde{\phi}$ *be an I-V fuzzy subset of Y. If* $\tilde{\phi}$ *be an I-V fuzzy JU-subalgebra of Y, then the Per-image* $g^{-1}(\tilde{\phi})$ *is an I-V fuzzy JU-subalgebras of X.*

Proof. Let $g : X \to Y$ be a homomorphism of JU- algebra and let $\tilde{\phi}$ be an I-V fuzzy JU-subalgebra of Y. We went to show that $g^{-1}(\tilde{\phi})$ is an I-V fuzzy JU-subalgebra in X.
For any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Then,

$$\begin{aligned}
g^{-1}(\tilde{\phi})(x_1 * x_2) &= [g^{-1}(\phi^L)(x_1 * x_2), g^{-1}(\phi^U)(x_1 * x_2)] \text{ (by Lemma 1.4.1)}\\
&= [\phi^L(g(x_1 * x_2)), \phi^U(g(x_1 * x_2))]\\
&= [\phi^L(g(x_1) *' g(x_2)), \phi^U(g(x_1) *' g(x_2))]\\
&= [\phi^L(y_1 *' y_2), \phi^U(y_1 *' y_2)]\\
&\geqslant [\min\{\phi^L(y_1), \phi^L(y_2)\}, \min\{\phi^U(y_1), \phi^U(y_2)\}]\\
&= [\min\{\phi^L(y_1), \phi^U(y_1)\}, \min\{\phi^L(y_2), \phi^U(y_2)\}]\\
&= r\min\{\tilde{\phi}(y_1), \tilde{\phi}(y_2)\}\\
&= r\min\{\tilde{\phi}(g(x_1), \tilde{\phi}(g(x_2))\}\\
&= r\min\{g^{-1}(\tilde{\phi})(x_1), g^{-1}(\tilde{\phi})(x_2)\}
\end{aligned}$$

□

Theorem 5.1.13. *Let X, Y, Z be JU-algebras. Then the function $f : X \to Y$ and $g : Y \to Z$ be JU-homomorphism then the following holds:*

(a) If f, g are surjective then $(g \circ f)(\tilde{\phi})$ is an I-V fuzzy JU-subalgebra of Z, for any I-V fuzzy JU-subalgebra ϕ of X, provided that sup property holds.

(b) $(g \circ f)^{-1}(\psi)$ is an I-V fuzzy JU-subalgebra of X, for any I-V fuzzy JU-subalgebra $\tilde{\psi}$ of Z.

Proof. (a) Let $\tilde{\phi}$ be an I-V fuzzy JU-subalgebra of X, for any $z_1, z_2 \in Z$. there exist $x_1, x_2 \in X$ such that $(g \circ f)(x_1) = z_1, (g \circ f)(x_2) = z_2$. Then

$\tilde{\phi}(s) = \sup_{r \in (g \circ f)^{-1}(z_1)}\{\tilde{\phi}(r)\}$ and $\tilde{\phi}(t) = \sup_{r \in (g \circ f)^{-1}(z_2)}\{\tilde{\phi}(r)\}$ for some $s, t \in X$. Also f,g are homomorphism, so is $g \circ f$, and hence $s * t \in (g \circ f)^{-1}(z_1 * z_2)$. Now,

$$\begin{aligned}
(g \circ f)(\tilde{\phi})(z_1 * z_2) &\geqslant \tilde{\phi}(s * t)\\
&= [\phi^L(s * t), \phi^U(s * t)]\\
&\geqslant [\min\{\sup_{r \in (g \circ f)^{-1}(z_1)}\{\phi^L(r)\}, \sup_{r \in (g \circ f)^{-1}(z_2)}\{\phi^L(r)\}, \min\{\\
&\quad \sup_{r \in (g \circ f)^{-1}(z_1)}\{\phi^U(r)\}, \sup_{r \in (g \circ f)^{-1}(z_2)}\{\phi^U(r)\}\}]\\
&= [\min\{\phi^L(s), \phi^L(t)\}, \min\{\phi^U(s), \phi^U(t)\}]\\
&= [\min\{\phi^L(s), \phi^U(s)\}, \min\{\phi^L(t), \phi^U(t)\}]\\
&= r\min\{\tilde{\phi}(s)\}, \tilde{\phi}(t)\}.\\
&= r\min(g \circ f)(\tilde{\phi})(z_1), (g \circ f)(\tilde{\phi})(z_2)\}
\end{aligned}$$

(b) Let $\tilde{\psi}$ be a fuzzy JU-subalgebra of Z and for any $x, y \in X$. Then,

$$\begin{aligned}
(g \circ f)^{-1})(\tilde{\psi})(x * y) &= \tilde{\psi}((g \circ f)(x * y))\\
&= [\psi^L((g \circ f)(x * y)), \psi^U((g \circ f)(x * y))]\\
&\geqslant \min\{\psi^L((g \circ f)(x)), \psi^L((g \circ f)(y))\}, \min\{\psi^U(g \circ f)(x)),
\end{aligned}$$

$$
\begin{aligned}
&\psi^U(g\circ f)(y))\} \\
&= \min\{\psi^L((g\circ f)(x)), \psi^U(g\circ f)(x)\}, \min\{\psi^L(g\circ f)(y), \\
&\quad \psi^U((g\circ f)(y))\} \\
&= \min\{(g\circ f)^{-1}(\psi^L)(x), (g\circ f)^{-1}(\psi^U)(x)\}, \min\{(g\circ f)^{-1}(\psi^L)(y), \\
&\quad (g\circ f)^{-1}(\psi^U)(y)\} \\
&= r\min\{(g\circ f)^{-1}(\tilde{\psi}))(x), (g\circ f)^{-1}(\tilde{\psi}))(y)\}
\end{aligned}
$$

□

Theorem 5.1.14. *The Cartesian product of any two I-V fuzzy JU-subalgebras of* X *are also I-V fuzzy JU-subalgebra of* X.

Proof. Suppose $\tilde{\phi}_1$ and $\tilde{\phi}_2$ be any two I-V fuzzy JU-subalgebras of X. Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then

$$
\begin{aligned}
\tilde{\phi}_1 \times \tilde{\phi}_2((x_1,x_2)*(y_1,y_2)) &= [\phi_1^L \times \phi_2^L((x_1,x_2)*(y_1,y_2)), \phi_1^U \times \phi_2^U((x_1,x_2)*(y_1,y_2))] \\
&= [\phi_1^L \times \phi_2^L((x_1*y_1),(x_2*y_2)), \phi_1^U \times \phi_2^U((x_1*y_1),(x_2*y_2))] \\
&= [\min\{\phi_1^L(x_1*y_1), \phi_2^L(x_2*y_2)\}, \min\{\phi_1^U(x_1*y_1), \phi_2^U(x_2*y_2)\}] \\
&\geqslant [\min\{\min\{\phi_1^L(x_1), \phi_1^L(y_1)\}, \min\{\phi_2^L(x_2), \phi_2^L(y_2)\}\}, \\
&\quad \min\{\min\{\phi_1^U(x_1), \phi_1^U(y_1)\}, \min\{\phi_2^U(x_2), \phi_2^U(y_2)\}\}] \\
&= [\min\{\min\{\phi_1^L(x_1), \phi_2^L(x_2)\}, \min\{\phi_1^L(y_1), \phi_2^L(y_2)\}\}, \\
&\quad \min\{\min\{\phi_1^U(x_1), \phi_2^U(x_2)\}, \min\{\phi_1^U(y_1), \phi_2^U(y_2)\}\}] \\
&= [\min\{\phi_1^L \times \phi_2^L(x_1,x_2), \phi_1^L \times \phi_2^L(y_1,y_2)\}, \\
&\quad \min\{\phi_1^U \times \phi_2^U(x_1,x_2), \phi_1^U \times \phi_2^U(y_1,y_2)\}] \\
&= [\min\{\phi_1^L \times \phi_2^L(x_1,x_2), \phi_1^U \times \phi_2^U(x_1,x_2)\}, \\
&\quad \min\{\phi_1^L \times \phi_2^L(y_1,y_2), \phi_1^U \times \phi_2^U(y_1,y_2)\}] \\
&= r\min\{\tilde{\phi}_1 \times \tilde{\phi}_2(x_1,x_2), \tilde{\phi}_1 \times \tilde{\phi}_2(y_1,y_2)\}
\end{aligned}
$$

□

Corollary 5.1.15. *Let* $\{X_j\}_{j=1}^n$ *be the finite family of JU-algebras and* $X = \Pi_{j=1}^n X_j$ *the Cartesian product of JU-algebras of* $\{X_j\}$. *Let* $\tilde{\phi}_j$ *be I-V fuzzy JU-subalgebras of* X_j, *where* $1 \leqslant j \leqslant n$. *Then* $\tilde{\phi} = \Pi_{j=1}^n \tilde{\phi}_j$ *is defined by:*

$$
\begin{aligned}
\tilde{\phi}(x_1, x_2, \ldots, x_n) &= (\Pi_{j=1}^n \tilde{\phi}_j)(x_1, x_2, \ldots, x_n) \\
&= r\inf\{\tilde{\phi}_1(x_1), \tilde{\phi}_2(x_2), \ldots, \tilde{\phi}_n(x_n)\}
\end{aligned}
$$

is also an I-V fuzzy JU-subalgebra of X.

Proof. We verify the defining properties of an I–V fuzzy JU–subalgebra for $\tilde{\phi}$, working componentwise and using that each $\tilde{\phi}_j$ satisfies the same properties on X_j.

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in X$. The product in X is componentwise: $x * y = (x_1 *_1 y_1, \dots, x_n *_n y_n)$, where $*_j$ is the operation on X_j.
Using the definition of $\tilde{\varphi}$ and the fact that each $\tilde{\varphi}_j$ is an I–V fuzzy JU–subalgebra, we have for every component j

$$\tilde{\varphi}_j(x_j *_j y_j) \geqslant r\inf\{\tilde{\varphi}_j(x_j), \tilde{\varphi}_j(y_j)\}.$$

Taking the r-infimum across components and using the monotonicity of $r\inf$, we get

$$\begin{aligned}
\tilde{\varphi}(x * y) &= r\inf\{\tilde{\varphi}_1(x_1 *_1 y_1), \dots, \tilde{\varphi}_n(x_n *_n y_n)\} \\
&\geqslant r\inf\{r\inf\{\tilde{\varphi}_1(x_1), \tilde{\varphi}_1(y_1)\}, \dots, r\inf\{\tilde{\varphi}_n(x_n), \tilde{\varphi}_n(y_n)\}\} \\
&= r\inf\{\tilde{\varphi}_1(x_1), \dots, \tilde{\varphi}_n(x_n), \tilde{\varphi}_1(y_1), \dots, \tilde{\varphi}_n(y_n)\} \\
&= r\inf\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}.
\end{aligned}$$

Thus $\tilde{\varphi}(x * y) \geqslant r\inf\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}$. Therefore $\tilde{\varphi}$ is an I-V fuzzy JU-subalgebra on X. □

5.1.1 $\tilde{s}$-Interval Valued Fuzzy JU-Subalgebras

Definition 5.1.2. *Let $\tilde{\varphi}_{\tilde{S}}$ be an $\tilde{S}$-interval valued fuzzy set in a JU-algebra X and let $\tilde{S} \in C[0,1]$. Then $\tilde{\varphi}_{\tilde{S}}$ of X is called the $\tilde{S}$-interval valued fuzzy JU-subalgebra of X if*

$$\tilde{\varphi}_{\tilde{S}}(x * y) \geqslant r\min\{\tilde{\varphi}_{\tilde{S}}(x), \tilde{\varphi}_{\tilde{S}}(y)\}, \quad \forall x, y \in X.$$

The $\tilde{S}$-interval valued fuzzy JU-subalgebra of X extends the concept of an interval valued fuzzy JU-subalgebra of X, and an interval valued fuzzy JU-subalgebra of X is a special case of an $\tilde{S}$-interval valued fuzzy JU-subalgebra of X when S equal to 1.

Example 5.1.1.1. *Let X = $\{1,2,3,4\}$ in which $*$ is defined by the following Cayley table:*

$*$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

See [3] $(X;*,1)$ is a JU-algebra. Let $\tilde{s} = [1,1]$ and let us define $\tilde{s}$-interval valued fuzzy subset $\tilde{\varphi}_{\tilde{s}} : X \to C[0,1]$ by

$$\tilde{\varphi}_{\tilde{s}}(x) = \begin{cases} r\min\{\tilde{s}, [0.6, 0.7]\} & \text{if } x \in \{1,2\} \\ r\min\{\tilde{s}, [0.2, 0.3]\} & \text{otherwise} \end{cases}$$

We now verify that $\tilde{\varphi}_{\tilde{s}}$ is $\tilde{s}$-interval valued fuzzy JU-subalgebra of X. Now, let we take $x = 1$ and $y = 3$, then $\tilde{\varphi}_{\tilde{s}}(1 * 3) = \tilde{\varphi}_{\tilde{s}}(3) = [0.2, 0.3] \cdots (*)$
and $r\min\{\tilde{\varphi}_{\tilde{s}}(1), \tilde{\varphi}_{\tilde{s}}(3)\} = r\min\{[0.6, 0.7], [0.2, 0.3]\} = [0.2, 0.3] \cdots (**)$
From (*) and (**) we have $[0.2, 0.3] \geqslant [0.2, 0.3]$, which is true.

Similarly, the result holds for all values of x and y in X.
Therefore, $\tilde{S}$*-interval valued fuzzy JU-subalgebra of X.*

Theorem 5.1.1.1. *If* $\tilde{\phi}_{\tilde{S}}$ *is an* $\tilde{s}$ *interval valued fuzzy JU-subalgebra of* X*, then* $\tilde{\phi}_{\tilde{S}}(1) \geqslant \tilde{\phi}_{\tilde{S}}(x)$*, for all* $x \in X$ *and* $\tilde{S} \in C[0,1]$.

Proof. Suppose $\tilde{\phi}_{\tilde{S}}$ is an $\tilde{s}$ interval valued fuzzy JU-subalgebra of X. Then for any $x \in X$ and $\tilde{S} \in C[0,1]$

$$\begin{aligned} \tilde{\phi}_{\tilde{S}}(1) &= \tilde{\phi}_{\tilde{S}}(x * x) \\ &\geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x)\} \\ &= \tilde{\phi}_{\tilde{S}}(x). \end{aligned}$$

□

Theorem 5.1.1.2. *The intersection of any two* $\tilde{s}$ *interval valued fuzzy JU-subalgebra of X is also* $\tilde{s}$ *interval valued fuzzy JU-subalgebra of X.*

Proof. Suppose $\tilde{\phi}_{\tilde{S_1}}$ and $\tilde{\phi}_{\tilde{S_2}}$ are two $\tilde{s}$ interval valued fuzzy JU-subalgebra of X and $\forall x, y \in X$ and $\tilde{S} \in C[0,1]$.Then,

$$\begin{aligned} (\tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}})(x * y) &= \min\{\tilde{\phi}_{\tilde{S_1}}(x * y), \tilde{\phi}_{\tilde{S_2}}(x * y)\} \\ &\geqslant \min\big\{\min\{\tilde{\phi}_{\tilde{S_1}}(x), \tilde{\phi}_{\tilde{S_1}}(y)\}, \min\{\tilde{\phi}_{\tilde{S_2}}(x), \tilde{\phi}_{\tilde{S_2}}(y)\}\big\} \\ &= \min\big\{\min\{\tilde{\phi}_{\tilde{S_1}}(x), \tilde{\phi}_{\tilde{S_2}}(x)\}, \min\{\tilde{\phi}_{\tilde{S_1}}(y), \tilde{\phi}_{\tilde{S_2}}(y)\}\big\} \\ &= \min\big\{(\tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}})(x), \tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}}(y)\big\} \end{aligned}$$

Thus, $(\tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}})(x * y) \geqslant \min\big\{(\tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}})(x), \tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}}(y)\big\}$ □

Corollary 5.1.1.3. *Let* $\{\tilde{\phi}_{\tilde{S_i}} | i \in \Lambda\}$ *be a family of* $\tilde{s}$*-interval valued fuzzy JU-subalgebra of X. Then* $\bigcap_{i\in\Lambda} \tilde{\phi}_{\tilde{S_i}}$ *is also an* $\tilde{s}$*-interval valued fuzzy JU-subalgebra of X.*

Proof. Define

$$\tilde{\phi}(x) = \bigcap_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}(x) = r\inf_{i\in\Lambda}\{\tilde{\phi}_{\tilde{S}_i}(x)\}, \quad \forall x \in X.$$

We show that $\tilde{\phi}$ is $\tilde{s}$-interval valued fuzzy JU-subalgebra of X.
Since each $\tilde{\phi}_{\tilde{S_i}}$ is $\tilde{s}$-interval valued fuzzy JU-subalgebra of X, we have for all $x, y \in X$,

$$\tilde{\phi}_{\tilde{S}_i}(x * y) \geqslant r\min\big\{\tilde{\phi}_{\tilde{S}_i}(x), \tilde{\phi}_{\tilde{S}_i}(y), \tilde{s}\big\}.$$

Now for any $x, y \in X$,

$$\tilde{\phi}(x) = r\inf_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}(x), \qquad \tilde{\phi}(y) = r\inf_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}(y),$$

and

$$\tilde{\phi}(x*y) = r\inf_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}(x*y).$$

Applying the JU-subalgebra condition for each $i \in \Lambda$, we have

$$\tilde{\phi}_{\tilde{S}_i}(x*y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}_i}(x), \tilde{\phi}_{\tilde{S}_i}(y), \tilde{s}\}.$$

Taking $r\inf_{i\in\Lambda}$ on both sides yields

$$r\inf_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}(x*y) \geqslant r\inf_{i\in\Lambda}\big(r\min\{\tilde{\phi}_{\tilde{S}_i}(x), \tilde{\phi}_{\tilde{S}_i}(y), \tilde{s}\}\big).$$

Since $r\inf$ preserves order and distributes over $r\min$, we get

$$r\inf_{i\in\Lambda}\big(r\min\{\tilde{\phi}_{\tilde{S}_i}(x), \tilde{\phi}_{\tilde{S}_i}(y), \tilde{s}\}\big) = r\min\{r\inf_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}(x),\ r\inf_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}(y),\ \tilde{s}\}.$$

Thus,

$$\tilde{\phi}(x*y) = r\inf_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}(x*y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y), \tilde{s}\}.$$

Therefore, $\bigcap_{i\in\Lambda} \tilde{\phi}_{\tilde{S}_i}$ is $\tilde{s}$-interval valued fuzzy JU-subalgebra of X. □

Theorem 5.1.16. *Let* $U(\tilde{\phi}_{\tilde{S}} : [t_1,t_2]) = \{x \in X \mid \tilde{\phi}_{\tilde{S}}(x) \geqslant [t_1,t_2]\}$ *be a nonempty set for all* $[t_1,t_2] \in C[0,1]$. *A fuzzy subset* $\tilde{\phi}_{\tilde{S}}$ *of a JU-algebra* X *is an interval valued fuzzy JU-subalgebra of* X *if and only if* $U(\tilde{\phi}_{\tilde{S}} : [t_1,t_2])$ *is a JU-subalgebra of* X.

Proof. Suppose $\tilde{\phi}_{\tilde{S}}$ is an interval valued fuzzy JU-subalgebra of X. Let $[t_1,t_2] \in C[0,1]$ and assume $U(\tilde{\phi}_{\tilde{S}} : [t_1,t_2]) \neq \emptyset$. Let $x,y \in U(\tilde{\phi}_{\tilde{S}} : [t_1,t_2])$. Then $\tilde{\phi}_{\tilde{S}}(x) \geqslant [t_1,t_2]$ and $\tilde{\phi}_{\tilde{S}}(y) \geqslant [t_1,t_2]$. Hence,

$$\tilde{\phi}_{\tilde{S}}(x*y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(y)\} \geqslant r\min\{[t_1,t_2],[t_1,t_2]\} = [t_1,t_2].$$

Thus, $x*y \in U(\tilde{\phi}_{\tilde{S}} : [t_1,t_2])$, and therefore $U(\tilde{\phi}_{\tilde{S}} : [t_1,t_2])$ is a JU-subalgebra of X.

Conversely, suppose that $U(\tilde{\phi}_{\tilde{S}} : [t_1,t_2])$ is a JU-subalgebra of X for every $[t_1,t_2] \in C[0,1]$ and $U(\tilde{\phi}_{\tilde{S}} : [t_1,t_2]) \neq \emptyset$. Assume that $\tilde{\phi}_{\tilde{S}}$ is not an interval valued fuzzy JU-subalgebra of X. Then there exist some $z_0, z_1 \in X$ such that

$$\tilde{\phi}_{\tilde{S}}(z_0 * z_1) < r\min\{\tilde{\phi}_{\tilde{S}}(z_0), \tilde{\phi}_{\tilde{S}}(z_1)\}.$$

Let

$$\tilde{\phi}_{\tilde{S}}(z_0) = [\beta_1,\beta_2], \quad \tilde{\phi}_{\tilde{S}}(z_1) = [\beta_3,\beta_4], \quad \tilde{\phi}_{\tilde{S}}(z_0 * z_1) = [t_1,t_2].$$

Then

$$[t_1,t_2] < r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4]\}.$$

Now take

$$[\alpha_1,\alpha_2] = \frac{1}{2}\big(\tilde{\phi}_{\tilde{S}}(z_0 * z_1) + r\min\{\tilde{\phi}_{\tilde{S}}(z_0), \tilde{\phi}_{\tilde{S}}(z_1)\}\big),$$

which gives

$$[\alpha_1,\alpha_2] = \frac{1}{2}\big([t_1,t_2] + r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4]\}\big) = \left[\frac{1}{2}(t_1+\min\{\beta_1,\beta_3\}), \frac{1}{2}(t_2+\min\{\beta_2,\beta_4\})\right].$$

Hence,

$$\min\{\beta_1,\beta_3\} > \alpha_1 = \frac{1}{2}(t_1+\min\{\beta_1,\beta_3\}) > t_1,$$

and

$$\min\{\beta_2,\beta_4\} > \alpha_2 = \frac{1}{2}(t_2+\min\{\beta_2,\beta_4\}) > t_2.$$

Thus,

$$(\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}) > [\alpha_1,\alpha_2] > [t_1,t_2] = \tilde{\phi}_{\tilde{S}}(z_0 * z_1).$$

This implies that $z_0 * z_1 \notin U(\tilde{\phi}_{\tilde{S}} : [\alpha_1,\alpha_2])$, while

$$\tilde{\phi}_{\tilde{S}}(z_0) = [\beta_1,\beta_2] > (\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}) > [\alpha_1,\alpha_2],$$

and

$$\tilde{\phi}_{\tilde{S}}(z_1) = [\beta_3,\beta_4] > (\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}) > [\alpha_1,\alpha_2].$$

Hence $z_0,z_1 \in U(\tilde{\phi}_{\tilde{S}} : [\alpha_1,\alpha_2])$, which contradicts the assumption that $U(\tilde{\phi}_{\tilde{S}} : [\alpha_1,\alpha_2])$ is a JU-subalgebra of X. Therefore, $\tilde{\phi}_{\tilde{S}}(z_0 * z_1) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(z_0),\tilde{\phi}_{\tilde{S}}(z_1)\}$, $\forall z_0,z_1 \in X$. Hence, $\tilde{\phi}_{\tilde{S}}$ is $\tilde{S}$-interval valued fuzzy JU-subalgebra of X. □

Theorem 5.1.1.4. *Let $\tilde{\phi}_{\tilde{S}}$ is $\tilde{S}$-interval valued fuzzy set in a JU-algebra X. Then if $\tilde{\phi}^c_{\tilde{S}}$ is $\tilde{S}$-interval valued fuzzy JU-subalgebra of X if and only if $L(\tilde{\phi}_{\tilde{S}},\tilde{t'})$ is either empty or a JU-subalgebra of X for $\tilde{t'} \in C[0,1]$ and $\tilde{S} \in [0,1]$.*

Proof. Suppose $\tilde{\phi}^c_{\tilde{S}}$ is a $\tilde{s}$-interval valued fuzzy JU-subalgebra of X and $L(\tilde{\phi}_{\tilde{S}},\tilde{t'}) \neq \emptyset$. We need to prove that $L(\tilde{\phi}_{\tilde{S}},\tilde{t'})$ is a JU-subalgebra of X. Since $\tilde{\phi}^c_{\tilde{S}}$ is a $\tilde{s}$-interval valued fuzzy JU-subalgebra of X, then

$$\tilde{\phi}^c_{\tilde{S}}(x*y) \geqslant r\min\{\tilde{\phi}^c_{\tilde{S}}(x),\tilde{\phi}^c_{\tilde{S}}(y)\}, \quad \text{for all } x,y \in X \text{ and } \tilde{S} \in C[0,1].$$

It follows that

$$\begin{aligned} \tilde{1} - \tilde{\phi}_{\tilde{S}}(x*y) &\geqslant r\min\{\tilde{1} - \tilde{\phi}_{\tilde{S}}(x), \tilde{1} - \tilde{\phi}_{\tilde{S}}(y)\} \\ &= \tilde{1} - r\max\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(y)\}, \qquad \text{(By Lemma 1.1.1)} \end{aligned}$$

hence

$$\begin{aligned} & \tilde{1} - \tilde{\phi}_{\tilde{S}}(x*y) \geqslant \tilde{1} - r\max\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(y)\} \\ \Rightarrow \quad & \tilde{\phi}_{\tilde{S}}(x*y) \leqslant r\max\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(y)\} \leqslant r\max\{\tilde{t'},\tilde{t'}\} = \tilde{t'}. \end{aligned}$$

Thus, $\tilde{\phi}_{\tilde{S}}(x*y) \leqslant \tilde{t'}$, implies that $x*y \in L(\tilde{\phi}_{\tilde{S}},\tilde{t'})$. Hence, $L(\tilde{\phi}_{\tilde{S}},\tilde{t'})$ is a JU-subalgebra of X.

Conversely, assume that $L(\tilde{\phi}_{\tilde{S}}, \tilde{t'})$ is a nonempty JU-subalgebra of X. We need to show that $\tilde{\phi}^c_{\tilde{S}}$ is a $\tilde{s}$-interval valued fuzzy JU-subalgebra of X.
Let $\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(y) \in C[0,1]$ for $x, y \in X$ and $\tilde{S} \in C[0,1]$. Choose

$$\tilde{t'} = r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(y)\}.$$

Then $\tilde{\phi}_{\tilde{S}}(x) \leqslant \tilde{t'}$ and $\tilde{\phi}_{\tilde{S}}(y) \leqslant \tilde{t'}$ imply that $x, y \in L(\tilde{\phi}_{\tilde{S}}, \tilde{t'}) \neq \emptyset$. Since $L(\tilde{\phi}_{\tilde{S}}, \tilde{t'})$ is a JU-subalgebra of X, we have $x * y \in L(\tilde{\phi}_{\tilde{S}}, \tilde{t'})$.
Thus,

$$\tilde{\phi}_{\tilde{S}}(x * y) \leqslant \tilde{t'} = r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(y)\},$$

which gives

$$\begin{aligned}
\tilde{\phi}^c_{\tilde{S}}(x * y) &= \tilde{1} - \tilde{\phi}_{\tilde{S}}(x * y) \\
&> \tilde{1} - r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(y)\} \\
&= r\min\{\tilde{1} - \tilde{\phi}_{\tilde{S}}(x), \tilde{1} - \tilde{\phi}_{\tilde{S}}(y)\} \quad \text{By Lemma 1.1.1} \\
&= r\min\{\tilde{\phi}^c_{\tilde{S}}(x), \tilde{\phi}^c_{\tilde{S}}(y)\}.
\end{aligned}$$

Hence,

$$\tilde{\phi}^c_{\tilde{S}}(x * y) > r\min\{\tilde{\phi}^c_{\tilde{S}}(x), \tilde{\phi}^c_{\tilde{S}}(y)\}.$$

Therefore, $\tilde{\phi}^c_{\tilde{S}}$ is a $\tilde{S}$-interval valued fuzzy JU-subalgebra of X. □

Theorem 5.1.1.5. *If $\tilde{\phi}_{\tilde{s}}$ is $\tilde{s}$-interval valued fuzzy JU-subalgebra of X, then the set $X_{\tilde{\phi}_{\tilde{s}}} = \{x \in X | \tilde{\phi}_{\tilde{s}}(x) = \tilde{\phi}_{\tilde{s}}(1)\}$ is a JU-subalgebra of X.*

Proof. Let $x, y \in X_{\tilde{\phi}_{\tilde{s}}}$. Then $\tilde{\phi}_{\tilde{s}}(x) = \tilde{\phi}_{\tilde{s}}(1) = \tilde{\phi}_{\tilde{s}}(y)$, and so
$\tilde{\phi}_{\tilde{s}}(x * y) \geqslant r\min\{\tilde{\phi}_{\tilde{s}}(x), \tilde{\phi}_{\tilde{s}}(y)\} = r\min\{\tilde{\phi}_{\tilde{s}}(1), \tilde{\phi}_{\tilde{s}}(1)\}$.
$\Rightarrow \tilde{\phi}_{\tilde{s}}(x * y) \geqslant \tilde{\phi}_{\tilde{s}}(1)$ but $\tilde{\phi}_{\tilde{s}}(x * y) \leqslant \tilde{\phi}_{\tilde{s}}(1), \forall x, y \in X$ (by Theorem 5.1.1.1)
$\Rightarrow \tilde{\phi}_{\tilde{s}}(x * y) = \tilde{\phi}_{\tilde{s}}(1)$
$\Rightarrow x * y \in X_{\tilde{\phi}_{\tilde{s}}}$.
Hence $X_{\tilde{\phi}_{\tilde{s}}}$ is a JU-subalgebra of X. □

Theorem 5.1.1.6. *Let $(X, *, 1)$ and $(Y, *', 1')$ be JU-algebras and a map $g: X \to Y$ be an onto homomorphism and let $\tilde{\phi}_{\tilde{s}}$ be an $\tilde{s}$-interval valued fuzzy subset of X. If $\tilde{\phi}_{\tilde{s}}$ be an $\tilde{s}$-interval valued fuzzy JU-subalgebra of X, then the image $g(\tilde{\phi}_{\tilde{s}})$, is an $\tilde{s}$-interval valued fuzzy JU-subalgebra of Y with sup property.*

Proof. Let $g: X \to Y$ be an onto homomorphism of JU-algebra and let $\tilde{\phi}_{\tilde{s}}$ be an $\tilde{s}$-interval valued fuzzy JU-subalgebra of X. We want to show that $g(\tilde{\phi}_{\tilde{s}})$ is an $\tilde{s}$-interval valued fuzzy JU-subalgebra in Y. Let $y_1, y_2 \in Y$. Since g is onto, there exist $x_1, x_2 \in X$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$.

Also g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Now

$$\begin{aligned} g(\tilde{\phi}_{\tilde{s}})(y_1 * y_2) &= \sup_{(x_1*x_2)\in g^{-1}(y_1*y_2)} \tilde{\phi}_{\tilde{s}}(x_1 * x_2) \\ &\geqslant r\min\left\{ \sup_{x_1\in g^{-1}(y_1)} \tilde{\phi}_{\tilde{s}}(x_1), \sup_{x_2\in g^{-1}(y_2)} \tilde{\phi}_{\tilde{s}}(x_2) \right\} \\ &= r\min\left\{ g(\tilde{\phi}_{\tilde{s}})(y_1), g(\tilde{\phi}_{\tilde{s}})(y_2) \right\} \end{aligned}$$

□

Theorem 5.1.1.7. *Let $(X,*,1z)$ and $(Y,*',1')$ be JU-algebras and a map $g : X \to Y$ be homomorphism. Let $\tilde{\phi}_{\tilde{s}}$ be an $\tilde{s}$-interval valued fuzzy subset of Y. If $\tilde{\phi}_{\tilde{s}}$ be an $\tilde{s}$-interval valued fuzzy JU-subalgebra of Y, then the pre-image $g^{-1}(\tilde{\phi}_{\tilde{s}})$ is an $\tilde{s}$-interval valued fuzzy JU-subalgebras of X.*

Proof. Let $g : X \to Y$ be a homomorphism of JU-algebra and let $\tilde{\phi}_{\tilde{s}}$ be an $\tilde{s}$-interval valued fuzzy JU-subalgebra of Y. We want to show that $g^{-1}(\tilde{\phi}_{\tilde{s}})$ is an $\tilde{s}$-interval valued fuzzy JU-subalgebra in X.
For any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Then,

$$\begin{aligned} g^{-1}(\tilde{\phi}_{\tilde{s}})(x_1 * x_2) &= \tilde{\phi}_{\tilde{s}}(g(x_1 * x_2)) \\ &= \tilde{\phi}_{\tilde{s}}(g(x_1) *' g(x_2)) \\ &= \tilde{\phi}_{\tilde{s}}(y_1 *' y_2) \\ &\geqslant r\min\{\tilde{\phi}_{\tilde{s}}(y_1), \tilde{\phi}_{\tilde{s}}(y_2)\} \\ &= r\min\{\tilde{\phi}_{\tilde{s}}(g(x_1)), \tilde{\phi}_{\tilde{s}}(g(x_2))\} \\ &= r\min\{g^{-1}(\tilde{\phi}_{\tilde{s}})(x_1), g^{-1}(\tilde{\phi}_{\tilde{s}})(x_2)\} \end{aligned}$$

□

Theorem 5.1.1.8. *The Cartesian product of any two $\tilde{s}$-interval valued fuzzy JU-ideals of X is also $\tilde{s}$-interval valued fuzzy JU-ideal of X.*

Proof. Suppose $\tilde{\phi}_1$ and $\tilde{\phi}_2$ be any two $\tilde{s}$-interval valued fuzzy JU- ideals of X. Let $(x_1,x_2),(y_1,y_2) \in X \times X$. Then,

$$\begin{aligned} (\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})((x_1,x_2) * (y_1,y_2)) &= (\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})((x_1 * y_1),(x_2 * y_2)) \\ &= r\min\{\tilde{\phi}_{\tilde{s_1}}(x_1 * y_1), \tilde{\phi}_{\tilde{s_2}}(x_2 * y_2)\} \\ &\geqslant r\min\{r\min\{\tilde{\phi}_{\tilde{s_1}}(x_1), \tilde{\phi}_{\tilde{s_1}}(y_1)\}, r\min\{\tilde{\phi}_{\tilde{s_2}}(x_2), \tilde{\phi}_{\tilde{s_2}}(y_2)\}\} \\ &= r\min\{r\min\{\tilde{\phi}_{\tilde{s_1}}(x_1), \tilde{\phi}_{\tilde{s_2}}(x_2)\}, r\min\{\tilde{\phi}_{\tilde{s_1}}(y_1), \tilde{\phi}_{\tilde{s_2}}(y_2)\}\} \\ &= r\min\{(\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})((x_1,x_2)), (\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})((y_1,y_2))\} \end{aligned}$$

$\square$

5.1.2 Interval Valued $(\tilde{E},\tilde{F})$-Fuzzy JU-Subalgebras

Definition 5.1.2.1. *Let $\tilde{E},\tilde{F} \in C[0,1]$ and $\tilde{0} < \tilde{E} < \tilde{F} \leqslant \tilde{1}$. An interval valued fuzzy subset $\tilde{\phi}$ of X is called an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X if*

$$r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\},\ \forall x,y \in X.$$

In what follows, let $\tilde{E},\tilde{F} \in C[0,1]$ be such that $\tilde{0} < \tilde{E} < \tilde{F} \leqslant \tilde{1}$.

Example 5.1.2.1. *Let X = {1,2,3,4} in which $*$ is defined by the following Cayley table:*

$*$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

See [3] $X;*,1)$ is a JU-algebra. Define $\tilde{\phi}(x)$ as follows:

$$\tilde{\phi}(x) = \begin{cases} [0.7,0.9] & \mathrm{if} x \in \{1,2\} \\ [0.4,0.5] & \mathrm{otherwise} \end{cases}$$

Let $\tilde{E} = [0.2,0.3]$ and $\tilde{F} = [0.6,0.7]$. Then we claim that $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. We verify that for all $x,y \in X$, $r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\}$.
For $x = 1$ and $y = 2$, then
$r\max\{\tilde{\phi}(x*y),\tilde{E}\} = r\max\{\tilde{\phi}(1*2),\tilde{E}\} = r\max\{\tilde{\phi}(2),\tilde{E}\} = r\max\{[0.7,0.9],[0.2,0.3]\} = [0.7,0.9]\cdots(*)$
And, $r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\} = r\min\{\tilde{\phi}(1),\tilde{\phi}(2),\tilde{F}\} = r\min\{[0.7,0.9],[0.7,0.9],[0.6,0.7]\} = [0.6,0.7].\cdots(**)$
From (*) and (**) we get $[0.7,0.9] \geqslant [0.6,0.7]$
Similarly, the result holds for all values of x and y in X. Therefore, we conclude that $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X.

Theorem 5.1.2.1. *If $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X, then $r\max\{\tilde{\phi}(1),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{F}\}$, $\forall x \in X$.*

Proof. Suppose $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X.

$$\begin{aligned} \text{Since, } r\max\{\tilde{\phi}(1),\tilde{E}\} &= r\max\{\tilde{\phi}(x*x),\tilde{E}\} \\ &\geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(x),\tilde{F}\} \text{ (By Lemma 1.6.1)} \\ &= r\min\{\tilde{\phi}(x),\tilde{F}\} \end{aligned}$$

$\square$

Theorem 5.1.2.2. *The intersection of any two interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebras of X is also interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X.*

Proof. Suppose $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are any two interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebras of X and let $x,y \in X$. Then, we have

$$\begin{aligned} r\max\{(\tilde{\phi}_1 \cap \tilde{\phi}_2)(x*y),\tilde{E}\} &= r\max\{r\min\{\tilde{\phi}_1(x*y),\tilde{\phi}_2(x*y)\},\tilde{E}\} \\ &= r\min\{r\max\{\tilde{\phi}_1(x*y),\tilde{E}\},r\max\{\tilde{\phi}_2(x*y),\tilde{E}\}\} \\ &\geqslant r\min\{r\min\{\tilde{\phi}_1(x),\tilde{\phi}_1(y),\tilde{F}\},r\min\{\tilde{\phi}_2(x),\tilde{\phi}_2(y),\tilde{F}\}\} \\ &= r\min\{r\min\{\tilde{\phi}_1(x),\tilde{\phi}_2(x)\},r\min\{\tilde{\phi}_1(y),\tilde{\phi}_2(y)\},\tilde{F}\} \\ &= r\min\{(\tilde{\phi}_1 \cap \tilde{\phi}_2)(x),(\tilde{\phi}_1 \cap \tilde{\phi}_2)(y),\tilde{F}\}. \end{aligned}$$

Hence, $\tilde{\phi}_1 \cap \tilde{\phi}_2$ is also an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. □

Corollary 5.1.2.3. *Let $\{\phi_i | i \in \Lambda\}$ be interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. Then $\bigcap_{i\in\Lambda} \phi_i = r\inf_{i\in\Lambda}\{x \in X | \tilde{K}_{\phi_i}(x)\}$, is also interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X.*

Proof. Let $x,y \in X$. Since each $\tilde{\phi}_i$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra, we have

$$r\max\{\tilde{\phi}_i(x*y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}_i(x),\tilde{\phi}_i(y),\tilde{F}\}, \quad \forall i \in \Lambda.$$

Define $\tilde{\phi} = \bigcap_{i\in\Lambda} \tilde{\phi}_i = r\inf_{i\in\Lambda}\{\tilde{\phi}_i(x)\}$. Then, by properties of $r\max$ and $r\min$, we get

$$\begin{aligned} r\max\{\tilde{\phi}(x*y),\tilde{E}\} &= r\max\left\{r\inf_{i\in\Lambda}\tilde{\phi}_i(x*y),\tilde{E}\right\} \\ &\geqslant r\inf_{i\in\Lambda} r\max\{\tilde{\phi}_i(x*y),\tilde{E}\} \\ &\geqslant r\inf_{i\in\Lambda} r\min\{\tilde{\phi}_i(x),\tilde{\phi}_i(y),\tilde{F}\} \\ &= r\min\left\{r\inf_{i\in\Lambda}\tilde{\phi}_i(x),r\inf_{i\in\Lambda}\tilde{\phi}_i(y),\tilde{F}\right\} \\ &= r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\}. \end{aligned}$$

Hence,

$$r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\}, \quad \forall x,y \in X,$$

which proves that $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. □

Remark 5.1.2.1. *The union of any two interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebras of a JU-algebra X may not be an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X.*

Example 5.1.2.2. *Let X = $\{1,2,3,4\}$ in which $*$ is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $(X;*,1)$ is a JU-algebra. Define $\tilde{\phi}_1(x)$ and $\tilde{\phi}_2(x)$ as follows:

$$\tilde{\phi_1}(x) = \begin{cases} [0.8,0.9] & \textit{if } x \in \{1,2\} \\ [0.4,0.5] & \textit{otherwise} \end{cases}$$

and

$$\tilde{\phi_2}(x) = \begin{cases} [0.9,1] & \textit{if } x = 1 \\ [0.6,0.8] & \textit{if } x = 2 \\ [0.2,0.3] & \textit{otherwise} \end{cases}$$

Let $(\tilde{E},\tilde{F}) = ([0.2,0.3],[0.6,0.7])$. *Then,*

$$\begin{aligned} r\max\{\phi_1 \tilde{\cup} \phi_2(1*3),\tilde{E}\} &= r\max\{r\max\{\tilde{\phi_1}(1*3),\tilde{\phi_2}(1*3)\},\tilde{E}\} \\ &= r\max\{r\max\{\tilde{\phi_1}(3),\tilde{\phi_2}(3)\},\tilde{E}\} \\ &= r\max\{r\max\{[0.4,0.5],[0.2,0.3]\},[0.2,0.3]\} \\ &= [0.4,0.5] \cdots (*) \end{aligned}$$

And,

$$\begin{aligned} r\max\{r\max\{\tilde{\phi_1}(1*3),\tilde{\phi_2}(1*3)\},\tilde{E}\} &= r\max\{r\max\{\tilde{\phi_1}(1*3),\tilde{E}\},r\max\{\tilde{\phi_2}(1*3),\tilde{E}\}\} \\ &\geqslant r\max\{r\min\{\tilde{\phi_1}(1),\tilde{\phi_1}(3),\tilde{F}\},r\min\{\tilde{\phi_2}(1),\tilde{\phi_1}(3),[\tilde{F}]\}\} \\ &= r\min\{r\max\{\tilde{\phi_1}(1),\tilde{\phi_2}(1),\tilde{F}\},r\max\{\tilde{\phi_1}(3),\tilde{\phi_2}(3),\tilde{F}\}\} \\ &= r\min\{r\max\{[0.8,0.9],[0.9,1],[0.6,0.7]\},r\max\{[0.4,0.5], \\ &\quad [0.2,0.3],[0.6,0.7]\}\} \\ &= r\min\{[0.9,1],[0.6,0.7]\} \\ &= [0.6,0.7] \cdots (**) \end{aligned}$$

From $(*)$ *and* $(**)$ *we get* $[0.4,0.5] \geqslant [0.6,0.7]$ *which is false.*

Theorem 5.1.2.4. *If an interval valued fuzzy subset* $\tilde{\phi}$ *is an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-subalgebra of X, then the set* $\tilde{\phi}_{\tilde{E}} = \{x \in X | \tilde{\phi}(x) \geqslant \tilde{E}\}$ *is a JU-subalgebra of X.*

Proof. Let $\tilde{\phi}$ be an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X and let $x,y \in X$ such that $x,y \in \tilde{\phi}_{\tilde{E}}$ implies that $\tilde{\phi}(x) \geqslant \tilde{E}$ and $\tilde{\phi}(y) \geqslant \tilde{E}$. Then

$r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\} > r\min\{\tilde{E},\tilde{E},\tilde{F}\} = \tilde{E}$.

$\Rightarrow r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant \tilde{E}$

$\Rightarrow \tilde{\phi}(x*y) \geqslant \tilde{E}$.

$\Rightarrow x*y \in \tilde{\phi}_{\tilde{E}}$.

Hence, $\tilde{\phi}_{\tilde{E}}$ is a JU-subalgebra of X. □

Theorem 5.1.2.5. *A nonempty subset S of X is a JU-subalgebra of X if and only if the interval valued* $(\tilde{E},\tilde{F})$*-fuzzy subset* $\tilde{\phi}$ *of X defined as follows:*

$$\tilde{\phi}(x) = \begin{cases} \tilde{F} & \textit{if } x \in S \\ \tilde{E} & \textit{if } x \notin S, \end{cases}$$

is an interval valued $(\tilde{E},\tilde{F})$*-fuzzy JU-subalgebra of X.*

Proof. Suppose that S is a JU-subalgebra of X and let $x,y \in X$. Then consider the following cases.

Case 1. If $x,y \in S$, then $x*y \in S$, thus,

$$r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant \tilde{F} = r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\}$$

Case 2. If $x \notin S$ or $y \notin S$, then

$$r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant \tilde{E} = r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\}.$$

Case 3. If $x \notin S$ or $y \in S$, then

$$r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant \tilde{E} = r\min\{\tilde{E},\tilde{F},\tilde{F}\} = r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\}.$$

Or, If $x \in S$ or $y \notin S$, then

$$r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant \tilde{E} = r\min\{\tilde{F},\tilde{E},\tilde{F}\} = r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\}.$$

Consequently $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X.

Conversely, suppose that $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. Let $x,y \in S$, then $\tilde{\phi}(x) \geqslant \tilde{F}$, $\tilde{\phi}(y) \geqslant \tilde{F}$. Now,

$r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\} \geqslant r\min\{\tilde{F},\tilde{F},\tilde{F}\} = \tilde{F}$.

$\Rightarrow r\max\{\tilde{\phi}(x*y),\tilde{E}\} \geqslant \tilde{F}$. Since $\tilde{\phi}(x*y) \geqslant \tilde{E}$ by Theorem 5.1.2.4.

$\Rightarrow \tilde{\phi}(x*y) \geqslant \tilde{F}$

$\Rightarrow x*y \in S$, hence S is a JU-subalgebra of X. □

Remark 5.1.2.2. *An interval valued fuzzy JU-subalgebra is an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-subalgebra of X with* $\tilde{E} = [0,0]$ *and* $\tilde{F} = [1,1]$.

Theorem 5.1.2.6. *Let* $U(\tilde{\phi}:[t_1,t_2]) = \{x \in X | \tilde{\phi}(x) \geqslant [t_1,t_2]\}$ *be nonempty set and* $\forall [t_1,t_2] \in C[0,1]$. *A fuzzy subset* $\tilde{\phi}$ *of a JU-algebra X is an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-subalgebra of X , if and only if* $U(\tilde{\phi}:[t_1,t_2])$ *is JU-subalgebra of X, for* $\tilde{E} < [t_1,t_2] \leqslant \tilde{F}$.

Proof. Suppose $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. Let $[t_1,t_2] \in C[0,1]$ and $U(\tilde{\phi}:[t_1,t_2]) \neq \emptyset$. Let $x,y \in U(\tilde{\phi}:[t_1,t_2])$ implies that $\tilde{\phi}(x) \geqslant [t_1,t_2]$ and $\tilde{\phi}(y) \geqslant [t_1,t_2]$. Then,

$$\begin{aligned} r\max\{\tilde{\phi}(x*y),\tilde{E}\} &\geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(y),\tilde{F}\} \\ &\geqslant r\min\{[t_1,t_2],[t_1,t_2],\tilde{F}\} \\ &= [t_1,t_2] \end{aligned}$$

$\Rightarrow \tilde{\phi}(x*y) \geqslant [t_1,t_2]$ implies that $x*y \in U(\tilde{\phi}:[t_1,t_2])$

Hence $U(\tilde{\phi}:[t_1,t_2])$ is a JU-subalgebra of X for $\tilde{E} < [t_1,t_2] \leqslant \tilde{F}$.

Conversely, Suppose that $U(\tilde{\phi}:[t_1,t_2])$ is a JU-subalgebra of X for any $[t_1,t_2] \in C[0,1]$ and $U(\tilde{\phi}:[t_1,t_2]) \neq \emptyset$. Assume that $\tilde{\phi}$ is not an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. Then there exist some $z_0,z_1 \in X$ such that $r\max\{\tilde{\phi}(z_0*z_1),\tilde{E}\} < r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_1),\tilde{F}\}$.

Let $\tilde{\phi}(z_0) = [\beta_1,\beta_2]$, $\tilde{\phi}(z_1) = [\beta_3,\beta_4]$, $\tilde{\phi}(z_0*z_1) = [t_1,t_2]$, and $\tilde{E} = [e_1,e_2], \tilde{F} = [f_1,f_2]$. Then

$r\max\{[t_1,t_2],[e_1,e_2]\} < r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4],[f_1,f_2]\}$.

$$\begin{aligned}
\text{Take } [\alpha_1,\alpha_2] &= \frac{1}{2}\Big(r\max\{\tilde{\phi}(z_0 * z_1),\tilde{E}\} + r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_1),\tilde{F}\}\Big)\\
&= \frac{1}{2}\Big(\max\{t_1,e_1\}+\min\{\beta_1,\beta_3,f_1\}, \max\{t_2,e_2\}+\min\{\beta_2,\beta_4,f_2\}\Big)\\
\Rightarrow \min\{\beta_1,\beta_3,f_1\} > \alpha_1 &= \frac{1}{2}\big(\max\{t_1,e_1\}+\min\{\beta_1,\beta_3,f_1\}\big) > \max\{t_1,e_1\},\\
\min\{\beta_2,\beta_4,f_2\} > \alpha_2 &= \frac{1}{2}\big(\max\{t_2,e_2\}+\min\{\beta_2,\beta_4,f_2\}\big) > \max\{t_2,e_2\},
\end{aligned}$$

$$\Rightarrow (\min\{\beta_1,\beta_3,f_1\}, \min\{\beta_2,\beta_4,f_2\}) > [\alpha_1,\alpha_2] > (\max\{t_1,e_1\}, \max\{t_2,e_2\}) = r\max\{\tilde{\phi}(z_0 * z_1),\tilde{E}\},$$

$$\begin{aligned}
&\Rightarrow \tilde{\phi}(z_0 * z_1) = [t_1,t_2] < [\alpha_1,\alpha_2],\\
&\Rightarrow z_0 * z_1 \notin U(\tilde{\phi} : [\alpha_1,\alpha_2]),\\
&\tilde{\phi}(z_0) = [\beta_1,\beta_2] \geqslant (\min\{\beta_1,\beta_3,f_1\}, \min\{\beta_2,\beta_4,f_2\}) > [\alpha_1,\alpha_2],\\
&\tilde{\phi}(z_1) = [\beta_3,\beta_4] \geqslant (\min\{\beta_1,\beta_3,f_1\}, \min\{\beta_2,\beta_4,f_2\}) > [\alpha_1,\alpha_2],\\
&\Rightarrow z_0, z_1 \in U(\tilde{\phi} : [\alpha_1,\alpha_2])
\end{aligned}$$

Which is contradict to $U(\tilde{\phi} : [\alpha_1,\alpha_2])$ is a JU-subalgebra of X.
Hence, $r\max\{\tilde{\phi}(z_0 * z_1),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_1),\tilde{F}\}$ for any $z_0, z_1 \in X$. □

Theorem 5.1.2.7. *Let* $(X,*,1)$ *and* $(Y,*',1')$ *be JU- algebras and a map* $g : X \to Y$ *be an onto homomorphism and let* $\tilde{\phi}$ *be an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy subset of X. If* $\tilde{\phi}$ *be an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-subalgebra of X, then the image* $g(\phi)$*, is an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-subalgebra of Y with sup property.*

Proof. Let $g : X \to Y$ be an onto homomorphism of JU- algebra and let $\tilde{\phi}$ be an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. We went to show that $g(\phi)$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra in Y. Let $y_1, y_2 \in Y$. Since g is onto, there exist $x_1, x_2 \in X$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$.
Also g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Now

$$\begin{aligned}
r\max\{g(\tilde{\phi})(y_1 *' y_2),\tilde{E}\} &= r\max\{\sup\nolimits_{x_1 * x_2 \in g^{-1}(y_1 *' y_2)} \tilde{\phi}(x_1 * x_2),\tilde{E}\}\\
&\geqslant r\min\{\sup\nolimits_{x_1 \in g^{-1}(y_1)} \tilde{\phi}(x_1), \sup\nolimits_{x_2 \in g^{-1}(y_2)} \tilde{\phi}(x_2),\tilde{F}\}\\
&= r\min\{g(\tilde{\phi})(y_1), g(\tilde{\phi})(y_2)\}
\end{aligned}$$
□

Theorem 5.1.2.8. *Let* $(X,*,1)$ *and* $(Y,*',1')$ *be JU-algebras and a map* $g : X \to Y$ *be homomorphism. Let* $\tilde{\phi}$ *be an interval valued* $(\tilde{E},\tilde{F})$*- fuzzy subset of Y. If* $\tilde{\phi}$ *be an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-subalgebra of Y, then the Per-image* $g^{-1}(\phi)$ *is an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-subalgebras of X.*

Proof. Let $g : X \to Y$ be a homomorphism of JU- algebra and let $\tilde{\phi}$ be an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of Y. We went to show that $g^{-1}(\phi)$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra in X.

For any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Then,

$$\begin{aligned}
r\max\{g^{-1}(\tilde{\phi})(x_1 * x_2), \tilde{E}\} &= r\max\{\tilde{\phi}(g(x_1 * x_2)), \tilde{E}\} \\
&= r\max\{\tilde{\phi}(g(x_1) *' g(x_2)), \tilde{E}\} \\
&= r\max\{\tilde{\phi}(y_1 *' y_2), \tilde{E}\} \\
&\geqslant r\min\{\tilde{\phi}(y_1), \tilde{\phi}(y_2), \tilde{F}\} \\
&= r\min\{\tilde{\phi}(g(x_1)), \tilde{\phi}(g(x_2)), \tilde{F}\} \\
&= r\min\{g^{-1}(\tilde{\phi})(x_1), g^{-1}(\tilde{\phi})(x_2), \tilde{F}\}
\end{aligned}$$

□

Theorem 5.1.2.9. *The Cartesian product of any two interval valued (*$\tilde{E}$, $\tilde{F}$*)-fuzzy JU-subalgebras of X are also interval valued (*$\tilde{E}$, $\tilde{F}$*)-fuzzy JU-subalgebra of X.*

Proof. : Suppose $\tilde{\phi_1}$ and $\tilde{\phi_2}$ be two interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-subalgebras of X. Then let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then

$$\begin{aligned}
r\max\{\tilde{\phi_1} \times \tilde{\phi_1}((x_1, x_2) * (y_1, y_2)), \tilde{E}\} &= r\max\{\tilde{\phi_1} \times \tilde{\phi_1}((x_1 * y_1), (x_2 * y_2)), \tilde{E}\} \\
&= r\max\{r\min\{\tilde{\phi_1}(x_1 * y_1), \tilde{\phi_2}(x_2 * y_2)\}, \tilde{E}\} \\
&= r\min\{r\max\{\tilde{\phi_1}(x_1 * y_1), \tilde{E}\}, r\max\{\tilde{\phi_2}(x_2 * y_2), \\
&\quad \tilde{E}\}\} \\
&\geqslant r\min\{r\min\{\tilde{\phi_1}(x_1), \tilde{\phi_1}(y_1), \tilde{F}\}, r\min\{\tilde{\phi_2}(x_2), \\
&\quad \tilde{\phi_2}(y_2), \tilde{F}\} \\
&= r\min\{r\min\{\tilde{\phi_1}(x_1), \tilde{\phi_2}(x_2), \tilde{F}\}, r\min\{\tilde{\phi_1}(y_1), \\
&\quad \tilde{\phi_2}(y_2), \tilde{F}\} \\
&= r\min\{\tilde{\phi_1} \times \tilde{\phi_2}(x_1, x_2),\ \tilde{\phi_1} \times \tilde{\phi_1}(y_1, y_2), \tilde{F}\}
\end{aligned}$$

□

Corollary 5.1.2.10. *Let* $\{X_j\}_{j=1}^{n}$ *be the finite family of JU- algebras and* $X = \Pi_{j=1}^{n} X_j$ *the Cartesian product of JU- algebras of* $\{X_j\}$*. Let* $\tilde{\phi}_j$ *be interval valued* $(\tilde{E}, \tilde{F})$*-fuzzy JU-subalgebras of* X_j*, where* $1 \leqslant j \leqslant n$*. Then* $\tilde{\phi} = \Pi_{j=1}^{n} \tilde{\phi}_j$ *is defined by* $r\sup\{\tilde{\phi}(x_1, x_2, ..., x_n), \tilde{E}\} = r\sup\{\Pi_{j=1}^{n} \tilde{\phi}_j(x_1, x_2, ..., x_n), \tilde{E}\} \geqslant r\inf\{\tilde{\phi_1}(x_1), \tilde{\phi_2}(x_2), ..., \tilde{\phi_n}(x_n), \tilde{F}\}\}$ *is also interval valued* $(\tilde{E}, \tilde{F})$ *-fuzzy JU-subalgebras of X.*

Proof. Let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X$. Then the binary operation in X is defined componentwise as

$$(x_1, x_2, \ldots, x_n) * (y_1, y_2, \ldots, y_n) = (x_1 * y_1, x_2 * y_2, \ldots, x_n * y_n).$$

We need to show that

$$r\max\{\tilde{\phi}((x_1, x_2, \ldots, x_n) * (y_1, y_2, \ldots, y_n)), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x_1, x_2, \ldots, x_n), \tilde{\phi}(y_1, y_2, \ldots, y_n), \tilde{F}\}.$$

Now, by the definition of $\tilde{\phi}$, we have

$$\tilde{\phi}((x_1, x_2, \ldots, x_n) * (y_1, y_2, \ldots, y_n)) = r \inf_{1 \leqslant j \leqslant n} \tilde{\phi}_j(x_j * y_j).$$

Since each $\tilde{\phi}_j$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X_j, we have for all $x_j, y_j \in X_j$:

$$r\max\{\tilde{\phi}_j(x_j * y_j), \tilde{E}\} \geqslant r\min\{\tilde{\phi}_j(x_j), \tilde{\phi}_j(y_j), \tilde{F}\}.$$

Taking the r-infimum over all $j = 1,2,\ldots,n$, we get

$$r\max\left\{r\inf_{1\leqslant j\leqslant n}\tilde{\phi}_j(x_j * y_j), \tilde{E}\right\} \geqslant r\min\left\{r\inf_{1\leqslant j\leqslant n}\tilde{\phi}_j(x_j), r\inf_{1\leqslant j\leqslant n}\tilde{\phi}_j(y_j), \tilde{F}\right\}.$$

By the definition of $\tilde{\phi}$, this inequality becomes

$$r\max\{\tilde{\phi}((x_1,x_2,\ldots,x_n) * (y_1,y_2,\ldots,y_n)), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x_1,x_2,\ldots,x_n), \tilde{\phi}(y_1,y_2,\ldots,y_n), \tilde{F}\}.$$

Hence, $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-subalgebra of X. □

5.2. Interval valued Fuzzy JU-Ideals of JU-Algebras

In this section, we introduce the notion of interval valued fuzzy JU-ideals of JU-algebra, and some of their properties are discussed.

Definition 5.2.1. *Let $\tilde{\phi}$ be an I-V fuzzy set in X. The set $\tilde{\phi}$ is an I-V fuzzy JU-ideal of X, with the binary operation $*$ if it satisfies the following axioms:*
a) $\tilde{\phi}(1) \geqslant \tilde{\phi}(x)$,
b) $\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$, *for all* $x, y \in X$.

Example 5.2.1. *Let X =* $\{1,2,3,4,5,6\}$ *in which* $*$ *is defined by the following Cayley table:*

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	6
6	1	1	2	1	1	1

See [3] $(X;*,1)$ *is a JU-algebra. Define* $\tilde{\phi}(x)$ *as follows:*

$$\tilde{\phi}(x) = \begin{cases} [0.5, 0.8] & \textit{if } x \in \{1,2\} \\ [0.3, 0.4] & \textit{otherwise} \end{cases}$$

Let we verify $\tilde{\phi}$ *is an interval valued fuzzy JU-ideal of* X.

(i) $\tilde{\phi}(1) \geqslant \tilde{\phi}(x)$ *for all* $x \in X$.

Since $\tilde{\phi}(1) = [0.5,0.8]$, *and for any* $x \in X$, *either* $\tilde{\phi}(x) = [0.5,0.8]$ *(when* $x \in \{1,2\}$*) or* $\tilde{\phi}(x) = [0.3,0.4]$ *(when* $x \in \{3,4,5,6\}$*), we have in both cases:*

$$[0.5,0.8] \geqslant [0.5,0.8] \quad \textit{and} \quad [0.5,0.8] \geqslant [0.3,0.4].$$

Thus, condition (i) holds.

(ii) For all $x, y \in X$, *we have*

$$\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x*y)\}.$$

Let $x = 1$ *and* $y = 2$. *Then*

$$\tilde{\phi}(y) = \tilde{\phi}(2) = [0.5,0.8] \cdots (*)$$

and

$$r\min\{\tilde{\phi}(x), \tilde{\phi}(x*y)\} = r\min\{\tilde{\phi}(1), \tilde{\phi}(1*2)\} = r\min\{[0.5,0.8],[0.5,0.8]\} = [0.5,0.8]. \cdots (**)$$

From (*) and (**) we get $[0.5,0.8] \geqslant [0.5,0.8]$, *which is true.*
Similarly, the result holds for all values of x and y in X.
Therefore, we conclude that, $\tilde{\phi}$ *is an interval valued fuzzy JU-ideal of* X.

Theorem 5.2.1. *An I-V fuzzy set* $\tilde{\phi} = \{(x,[\phi^L(x), \phi^U(x)]), x \in X\}$ *is an I-V fuzzy JU-ideal of X if and only if* ϕ^L *and* ϕ^U *are fuzzy JU-ideal of X.*

Proof. Let ϕ^L and ϕ^U be fuzzy JU-ideals of X and let $x \in X$. Then,

$\tilde{\phi}(1) = [\phi^L(1), \phi^U(1)] \geqslant [\phi^L(x), \phi^U(x)] = \tilde{\phi}(x)$

And let $x, y \in X$.

$$\begin{aligned}\tilde{\phi}(y) = [\phi^L(y), \phi^U(y)] &\geqslant [\min\{\phi^L(x), \phi^L(x*y)\}, \min\{\phi^U(x), \phi^U(x*y)\}] \\ &= r\min\{[\phi^L(x), \phi^U(x)], [\phi^L(X*y), \phi^U(X*y)]\} \\ &= r\min\{\tilde{\phi}(x), \tilde{\phi}(x*y)\}.\end{aligned}$$

Conversely, suppose $\tilde{\phi}$ is an I-V fuzzy JU-ideal of X and $\forall x \in X$. Then

$[\phi^L(1), \phi^U(1)] = \tilde{\phi}(1) \geqslant \tilde{\phi}(x) = [\phi^L(x), \phi^U(x)]$

$\Rightarrow \phi^L(1) \geqslant \phi^L(x)$ and $\phi^U(1) \geqslant \phi^U(x)$

And $\forall x, y \in X$

$$\begin{aligned}[\phi^L(y), \phi^U(y)] = \tilde{\phi}(y) &\geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(X*y)\} \\ &= r\min\{[\phi^L(x), \phi^U(x)], [\phi^L(x*y), \phi^U(x*y)]\} \\ &= [\min\{\phi^L(x), \phi^L(x*y)\}, \min\{\phi^U(x), \phi^U(x*y)\}].\end{aligned}$$

$\Rightarrow \phi^L(y) \geqslant \min\{\phi^L(x), \phi^L(x*y)\}$ and $\phi^U(y) \geqslant \min\{[\phi^U(x), \phi^U(x*y)\}$.

Thus ϕ^L and ϕ^U are fuzzy JU-ideal of X. □

Theorem 5.2.2. *Every I-V fuzzy JU-ideals of X is order reversing.*

Proof. Let $\tilde{\phi}$ be an I-V fuzzy JU-ideals of X and for any $x, y \in X$ be such that $x \leqslant y$, then $y * x = 1$. Now

$$\begin{aligned}\tilde{\phi}(x) &\geqslant r\min\{\tilde{\phi}(y * x), \tilde{\phi}(y)\} \\ &= r\min\{\tilde{\phi}(1), \tilde{\phi}(y)\} \\ &= \tilde{\phi}(y)\end{aligned}$$

□

Theorem 5.2.3. *Let $\tilde{\phi}$ be an I-V fuzzy JU-ideals of X. Then $\tilde{\phi}((x * y) * y) \geqslant \tilde{\phi}(x)$.*

Proof. Since $(x * [(x * y) * y]) = 1$ (By Lemma 1.6.1)
Then, $\tilde{\phi}((x * y) * y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * [(x * y) * y])\}$
$= r\min\{\tilde{\phi}(x), \tilde{\phi}(1)\}$
$= \tilde{\phi}(x)$ □

Theorem 5.2.0.11. *Every I-V fuzzy JU-ideals of X is I-V fuzzy JU-subalgebra.*

Proof. Suppose $\tilde{\phi}$ is I-V fuzzy JU-ideals of X. That is:

a) $\tilde{\phi}(1) \geqslant \tilde{\phi}(x)$,

b) $\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$, for all $x, y \in X$.

We claim that $\tilde{\phi}$ is an I-V fuzzy JU-subalgebra. That is, for all $x, y \in X$
$\tilde{\phi}(x * y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$.
Assume that without loss of generality:
$\tilde{\phi}(x) \leqslant \tilde{\phi}(y) \Rightarrow r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\} = \tilde{\phi}(x)$
Now we consider two cases:

Case 1. If $\tilde{\phi}(x * y) \geqslant \tilde{\phi}(x)$, then

$\tilde{\phi}(x * y) \geqslant \tilde{\phi}(x) = r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$

$\Rightarrow \tilde{\phi}(x * y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$

The inequality is satisfied directly.

Case 2. If $\tilde{\phi}(x * y) < \tilde{\phi}(x)$, then from the assumption that $\tilde{\phi}$ is an I-V fuzzy JU-ideal, apply condition (b):

$\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\} = \tilde{\phi}(x * y)$

$\Rightarrow \tilde{\phi}(y) \geqslant \tilde{\phi}(x * y)$

Since we also know, $\tilde{\phi}(x * y) < \tilde{\phi}(x)$ and $\tilde{\phi}(x) \leqslant \tilde{\phi}(y)$, it follows that

$\tilde{\phi}(x * y) < \tilde{\phi}(x) \leqslant \tilde{\phi}(y) \Rightarrow \tilde{\phi}(x * y) < r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$.

Which is contradict the inequality $\tilde{\phi}(x * y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$

Thus, Case 1 must always hold, which implies:

$\tilde{\phi}(x * y) \geqslant \tilde{\phi}(x) = r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$

Hence, $\tilde{\phi}$ is I-V fuzzy JU-subalgebra of X.

□

Remark 5.2.1. *The converse of the above theorem may not be true.*

Example 5.2.2. *Let X = {1,2,3,4} in which ∗ is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $X;*,1)$ *is a JU-algebra. Define* $\tilde{\phi}(x)$ *as follows:*

$$\tilde{\phi}(x) = \begin{cases} [0.9,1] & \text{if } x = 1 \\ [0.7,0.8] & \text{if } x \in \{2,4\}, \\ [0.4,0.5] & \text{if } x = 3. \end{cases}$$

Since, $\tilde{\phi}$ *is an I-V fuzzy JU-subalgebra of X.*
The I-V fuzzy JU-ideal condition requires, $\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$*, for all* $x, y \in X$
Let $x = 2$, $y = 3 \Rightarrow x * y = 4$ *and*
Then, $\tilde{\phi}(y) = \tilde{\phi}(3) = [0.4, 0.5] \cdots (*)$
And, $r\min\tilde{\phi}(x), \tilde{\phi}(x * y)\} = r\min\tilde{\phi}(2), \tilde{\phi}(2 * 4)\} = r\min\{[0.7, 0.8], [0.7, 0.8]\} = [0.7, 0.8] \cdots (**)$
From (*) and (**) we get $[0.4, 0.5] \geqslant [0.7, 0.8]$ *which is false.*

Theorem 5.2.4. *The intersection of any two I-V fuzzy JU-ideals of X is also I-V fuzzy JU-ideal of X.*

Proof. Suppose ϕ_1 and ϕ_2 are any two I-V fuzzy JU-ideals of X and $\forall x, y \in X$.Then,

$$\begin{aligned}
\tilde{\phi_1} \cap \tilde{\phi_2}(1) &= \min\{\tilde{\phi_1}(1), \tilde{\phi_2}(1)\} \\
&= \min\{[\phi_1^L(1), \phi_1^U(1)]\}, [\phi_2^L(1), \phi_2^U(1)]\} \\
&\geqslant [\min\{\phi_1^L(x), \phi_1^U(x)\}, \min\{\phi_2^L(x), \phi_2^U(x)\}] \\
&= \tilde{\phi_1} \cap \tilde{\phi_2}(x).
\end{aligned}$$

And,

$$\begin{aligned}
\tilde{\phi}_1 \cap \tilde{\phi}_2(y) &= \min\{\tilde{\phi}_1(y), \tilde{\phi}_2(y)\} \\
&= \min\{[\phi_1^L(y), \phi_2^U(y)], [\phi_2^L(y), \phi_2^U(y)]\} \\
&= [\min\{\phi_1^L(y), \phi_2^L(y)\}, \min\{[\phi_1^U(y), \phi_2^U(y)\}] \\
&\geqslant [\min\{\min\{\phi_1^L(x), \phi_1^L(x * y)\}, \min\{\phi_2^L(x), \phi_2^L(x * y)\}\}, \min\{\min\{\phi_1^U(x), \\
&\quad \phi_1^U(x * y)\}, \min\{\phi_2^U(x), \phi_2^U(x * y)\}\}]
\end{aligned}$$

$$= [\min\{\min\{\phi_1^L(x), \phi_2^L(x)\}, \min\{\phi_1^L(x*y), \phi_2^L(x*y)\}, \min\{\min\{\phi_1^U(x), \phi_2^U(x)\}, \min\{\phi_1^U(x*y), \phi_2^U(x*y)\}\}]$$

$$= r\min\{\tilde{\phi_1} \cap \tilde{\phi_2}(x), \tilde{\phi_1} \cap \tilde{\phi_2}(x*y)\}$$ □

Corollary 5.2.5. *Let $\{\tilde{\phi_i} | i \in \Lambda\}$ be a family of I-V fuzzy JU-ideals of X. Then $\bigcap_{i\in\Lambda} \tilde{\phi_i}$ is also an I-V fuzzy JU-ideal of X.*

Proof. Let $\{\tilde{\phi_i} \mid i \in \Lambda\}$ be a family of interval valued fuzzy JU-ideals of X, and define

$$\tilde{\phi}(x) = \bigcap_{i\in\Lambda} \tilde{\phi_i}(x) = r\inf_{i\in\Lambda}\{\tilde{\phi_i}(x)\}, \quad \forall x \in X.$$

We shall show that $\tilde{\phi}$ satisfies the defining conditions of an interval valued fuzzy JU-ideal of X.

(i) For any $x \in X$, since each $\tilde{\phi_i}$ is an interval valued fuzzy JU-ideal, we have

$$\tilde{\phi_i}(1) \geqslant \tilde{\phi_i}(x), \quad \forall i \in \Lambda.$$

Taking the infimum over all $i \in \Lambda$, we get

$$r\inf_{i\in\Lambda}\{\tilde{\phi_i}(1)\} \geqslant r\inf_{i\in\Lambda}\{\tilde{\phi_i}(x)\},$$

which means that

$$\tilde{\phi}(1) \geqslant \tilde{\phi}(x), \quad \forall x \in X.$$

Hence, condition (i) of the definition of I-V fuzzy JU-ideal holds.

(ii) For all $x, y \in X$, since each $\tilde{\phi_i}$ is an interval valued fuzzy JU-ideal, we have

$$\tilde{\phi_i}(y) \geqslant r\min\{\tilde{\phi_i}(x), \tilde{\phi_i}(x*y)\}, \quad \forall i \in \Lambda.$$

Taking the infimum over $i \in \Lambda$ on both sides, we obtain

$$r\inf_{i\in\Lambda}\{\tilde{\phi_i}(y)\} \geqslant r\inf_{i\in\Lambda}\{r\min\{\tilde{\phi_i}(x), \tilde{\phi_i}(x*y)\}\}.$$

Since the infimum operator preserves the order relation under $r\min$, we can write

$$r\inf_{i\in\Lambda}\{\tilde{\phi_i}(y)\} \geqslant r\min\{r\inf_{i\in\Lambda}\{\tilde{\phi_i}(x)\}, r\inf_{i\in\Lambda}\{\tilde{\phi_i}(x*y)\}\}.$$

That is,

$$\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x*y)\}.$$

Thus, both axioms of an interval valued fuzzy JU-ideal are satisfied by $\tilde{\phi}$.

Therefore, $\bigcap_{i\in\Lambda} \tilde{\phi_i}$ is an interval valued fuzzy JU-ideal of X. □

Remark 5.2.2. *The union of any two I-V fuzzy JU-ideals of a JU-algebra X may not be an I-V fuzzy JU-ideal of X.*

Example 5.2.3. *Let* $X = \{1,2,3,4,5,6\}$ *in which* $*$ *is defined by the following Cayley table:*

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	6
6	1	1	2	1	1	1

See [3] $(X;*,1)$ *is a JU-algebra. Let us define* $\tilde{\phi_1}(x)$ *and* $\tilde{\phi_1}(x)$ *by:*

$$\tilde{\phi_1}(x) = \begin{cases} [0.7,0.8] & \textit{if } x \in \{1,2\} \\ [0.4,0.5] & \textit{otherwise} \end{cases}$$

and

$$\tilde{\phi_2}(x) = \begin{cases} [0.9,1] & \textit{if } x \in \{1,2\} \\ [0.6,0.7] & \textit{if } x \in \{3,4\} \\ [0.1,0.3] & \textit{otherwise} \end{cases}$$

Then, $\tilde{K}_{\phi_1 \cup \phi_2}(3) = r\max\{\tilde{\phi_1}(3), \tilde{\phi_2}(3)\} = r\max\{[0.4,0.5],[0.6,0.7]\} = [0.6,0.7] \cdots (*)$

And, $r\max\{\tilde{\phi_1}(3), \tilde{\phi_2}(3)\} \geqslant r\max\{r\min\{\tilde{\phi_1}(2), \tilde{\phi_1}(2*3)\}, r\min\{\tilde{\phi_2}(2), \tilde{\phi_2}(2*3)\}\}$

$$= r\max\{r\min\{\tilde{\phi_1}(2), \tilde{\phi_1}(3)\}, r\min\{\tilde{\phi_2}(2), \tilde{\phi_2}(3)\}\}$$
$$= r\min\{r\max\{[0.7,0.8],[0.4,0.5]\}, r\max\{[0.9,1],[0.6,0.7]\}\}$$
$$= [r\min\{[0.7,0.8],[0.9,1]\}] = [0.7,0.8]$$

From $(*)$ *and* $(**)$ *we get* $[0.6,0.7] \geqslant [0.7,0.8]$ *which is false.*

Theorem 5.2.6. *Let* $U(\tilde{\phi} : [t_1,t_2]) = \{x \in X | \tilde{\phi}(x) \geqslant [t_1,t_2]\}$ *be nonempty set and* $\forall [t_1,t_2] \in C[0,1]$. *A fuzzy subset* $\tilde{\phi}$ *of a JU-algebra X is an I-V fuzzy JU-ideal of X , if and only if* $U(\tilde{\phi} : [t_1,t_2])$ *is JU-ideal of X.*

Proof. Suppose $\tilde{\phi}$ is an I-V fuzzy JU-ideal of X. Let $[t_1,t_2] \in C[0,1]$ and $U(\tilde{\phi} : [t_1,t_2]) \neq \emptyset$. For any $x \in U(\tilde{\phi} : [t_1,t_2])$ such that $\tilde{\phi}(x) \geqslant [t_1,t_2]$. Since $\tilde{\phi}$ is an I-V fuzzy JU-ideal of X, we have $\tilde{\phi}(1) \geqslant \tilde{\phi}(x)$, for all $x \in X$. Thus $\tilde{\phi}(1) \geqslant \tilde{\phi}(x) \geqslant [t_1,t_2]$. Therefore $1 \in U(\tilde{\phi} : [t_1,t_2])$.
And let $x, x*y \in U(\tilde{\phi} : [t_1,t_2])$ implies that $\tilde{\phi}(x) \geqslant [t_1,t_2]$ and $\tilde{\phi}(x*y) \geqslant [t_1,t_2]$. Then, $\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x*y)\} \geqslant r\min\{[t_1,t_2],[t_1,t_2]\} = [t_1,t_2]$
This implies that $y \in U(\tilde{\phi} : [t_1,t_2])$ and hence $U(\tilde{\phi} : [t_1,t_2])$ is a JU-ideal of X.
Conversely, Suppose that $U(\tilde{\phi} : [t_1,t_2])$ is a JU-ideal of X for any $[t_1,t_2] \in C[0,1]$ and $U(\tilde{\phi} : [t_1,t_2]) \neq \emptyset$. Assume that $\tilde{\phi}$ is not I-V fuzzy JU- ideal of X. Then there exist some $z_0 \in X$ such that $\tilde{\phi}(1) < \tilde{\phi}(z_0)$.
Take $[\beta_1,\beta_2] = \frac{1}{2}(\tilde{\phi}(1) + \tilde{\phi}(x))$

$$\Rightarrow \tilde{\phi}(1) < [\beta_1,\beta_2] < \tilde{\phi}(x)$$

$z_0 \in U(\tilde{\phi} : [\beta_1, \beta_2])$ and $1 \notin U(\tilde{\phi} : [\beta_1, \beta_2])$, which is contradict to our assumption that $U(\tilde{\phi} : [\beta_1, \beta_2])$ is a JU-ideal of X. Therefore, $\tilde{\phi}(1) \geqslant \tilde{\phi}(x)$, $\forall x \in X$.
And assume that $z_0, z_1 \in X$ such that
$\tilde{\phi}(z_1) < r\min\{\tilde{\phi}(z_0), \tilde{\phi}(z_0 * z_1)\}$
Let $\tilde{\phi}(z_0) = [\beta_1, \beta_2]$ and $\tilde{\phi}(z_0 * z_1) = [\beta_3, \beta_4]$ and $\tilde{\phi}(z_1) = [t_1, t_2]$. Then
$[t_1, t_2] < r\min\{[\beta_1, \beta_2], [\beta_3, \beta_4]\}$.
Take $[\alpha_1, \alpha_2] = \frac{1}{2}(\tilde{\phi}(z_1) + r\min\{\tilde{\phi}(z_0), \tilde{\phi}(z_0 * z_1)\})$
$= \frac{1}{2}([t_1, t_2] + r\min\{[\beta_1, \beta_2], [\beta_3, \beta_4]\}$
$= \frac{1}{2}(t_1 + \min\{\beta_1, \beta_3\}), \frac{1}{2}(t_2 + \min\{\beta_2, \beta_4\})$
It follows that $\min\{\beta_1, \beta_3\} > \alpha_1 = \frac{1}{2}(t_1 + \min\{\beta_1, \beta_3\}) > t_1$ and $\min\{\beta_2, \beta_4\} > \lambda_2 = \frac{1}{2}(t_2 + \min\{\beta_2, \beta_4\}) > t_2$, so that
$(\min\{\beta_1, \beta_3\}, \min\{\beta_2, \beta_4\}) > [\alpha_1, \alpha_2] > [t_1, t_2] = \tilde{\phi}(z_1)$.
$\Rightarrow z_1 \notin U(\tilde{\phi} : [\alpha_1, \alpha_2])$ and
$\tilde{\phi}(z_0) = [\beta_3, \beta_4] < [\alpha_1, \alpha_2] < (\min\{\beta_1, \beta_3\}, \min\{\beta_2, \beta_4\}) > [t_1, t_2]$, and
$\tilde{\phi}(z_0 * z_1) = [\beta_1, \beta_2] < [\alpha_1, \alpha_2] < (\min\{\beta_1, \beta_3\}, \min\{\beta_2, \beta_4\}) > [t_1, t_2]$.
$\Rightarrow z_0, z_0 * z_1 \in U(\tilde{\phi} : [\alpha_1, \alpha_2])$
Which is contradict to $U(\tilde{\phi} : [\alpha_1, \alpha_2])$ is a JU-ideal of X. Hence $\tilde{\phi}(z_1) \geqslant r\min\{\tilde{\phi}(z_0), \tilde{\phi}(z_0 * z_1)\}$ for any $z_0, z_1 \in X$. □

Theorem 5.2.7. *Let $\tilde{\phi}$ be I-V fuzzy subset of a JU-algebra X. Then $\tilde{\phi}^c$ is I-V fuzzy JU-ideal of X if and only if $L(\tilde{\phi}, \tilde{t'})$ is either empty or a JU-ideal of X for $\tilde{t'} \in C[0,1]$.*

Proof. Suppose $\tilde{\phi}^c$ is a I-V fuzzy JU-ideal of X and $L(\tilde{\phi}, \tilde{t'}) \neq \emptyset$. We need to prove that $L(\tilde{\phi}, \tilde{t'})$ is a JU-ideal of X. Then there exists $x_0 \in L(\tilde{\phi}, \tilde{t'})$ such that $\tilde{\phi}(x_0) \leqslant \tilde{t'}$. Since $\tilde{\phi}^c$ is an I-V fuzzy JU-ideal of X, we have $\tilde{\phi}^c(1) \geqslant \tilde{\phi}^c(x)$, for all $x \in X$. Then,
$\tilde{\phi}^c(1) \geqslant \tilde{\phi}^c()(x_0) \Rightarrow 1 - \tilde{\phi}(1) \geqslant 1 - \tilde{\phi}(x_0) \Rightarrow \tilde{\phi}(1) \leqslant \tilde{\phi}(\tilde{x_0}) \leqslant \tilde{t'}$, hence $1 \in L(\tilde{\phi}, \tilde{t'})$.
Next, for $x, y \in X$ such that $x, x * y \in L(\tilde{\phi}, \tilde{t'})$, we have $\tilde{\phi}(x) \leqslant \tilde{t'}$ and $\tilde{\phi}(x * y) \leqslant \tilde{t'}$. Then,

$$\tilde{\phi}^c(y) \geqslant r\min \tilde{\phi}^c(x), \tilde{\phi}^c(x * y)\}, \quad \text{for all } x, x * y \in X.$$

It follows that

$$\begin{aligned} \tilde{1} - \tilde{\phi}(y) &\geqslant r\min\{\tilde{1} - \tilde{\phi}(x), \tilde{1} - \tilde{\phi}(x * y)\} \\ &= \tilde{1} - r\max \tilde{\phi}(x), \tilde{\phi}(x * y)\}, \quad \text{By Lemma 1.1.1} \end{aligned}$$

hence

$$\begin{aligned} & \tilde{1} - \tilde{\phi}(y) \geqslant \tilde{1} - r\max \tilde{\phi}(x), \tilde{\phi}(x * y)\} \\ \Rightarrow \quad & \tilde{\phi}(y) \leqslant r\max \tilde{\phi}(x), \tilde{\phi}(x * y)\} \leqslant r\max\{\tilde{t'}, \tilde{t'}\} = \tilde{t'}. \end{aligned}$$

Thus, $\tilde{\phi}(y) \leqslant \tilde{t'}$, implies that $y \in L(\tilde{\phi}, \tilde{t'})$. Thus, $L(\tilde{\phi}, \tilde{t'})$ is a JU-ideal of X.

Conversely, assume that $L(\tilde{\phi}, \tilde{t'})$ is a nonempty JU-ideal of X. We need to show that $\tilde{\phi}^c$ is a I-V fuzzy JU-ideal of X.
Let $\tilde{\phi}(x), \tilde{\phi}(x * y) \in C[0,1]$ for $x, x * y \in X$. Choose

$$\tilde{t'} = r\max \tilde{\phi}(x), \tilde{\phi}(x * y)\}.$$

Then $\tilde{\phi}(x) \leqslant \tilde{t'}$ and $\tilde{\phi}(x * y) \leqslant \tilde{t'}$ imply that $x, x * y \in L(\tilde{\phi}, \tilde{t'}) \neq \emptyset$. Since $L(\tilde{\phi}, \tilde{t'})$ is a JU-ideal of X, we have $y \in L(\tilde{\phi}, \tilde{t'})$.
Thus,

$$\tilde{\phi}(y) \leqslant \tilde{t'} = r\max \tilde{\phi}(x), \tilde{\phi}(x * y)\},$$

which gives

$$\begin{aligned}
\tilde{\phi}^c(y) &= \tilde{1} - \tilde{\phi}(y) \\
&\geqslant \tilde{1} - r\max \tilde{\phi}(x), \tilde{\phi}(x * y)\} \\
&= r\min\{\tilde{1} - \tilde{\phi}(x), \tilde{1} - \tilde{\phi}(x * y)\} \quad \text{By Lemma 1.1.1} \\
&= r\min \tilde{\phi}^c(x), \tilde{\phi}^c(x * y)\}.
\end{aligned}$$

Hence,

$$\tilde{\phi}^c(y) \geqslant r\min \tilde{\phi}^c(x), \tilde{\phi}^c(x * y)\}.$$

Therefore, $\tilde{\phi}^c$ is a I-V fuzzy JU-ideal of X. □

Theorem 5.2.8. *If ϕ is an I-V fuzzy JU-ideal of X, then the set $X_{\tilde{\phi}} = \{x \in X | \tilde{\phi}(x) = \tilde{\phi}(1)\}$ is a JU-ideal of X.*

Proof. Let $x, x * y \in X_{\tilde{\phi}}$. Then $\tilde{\phi}(x) = \tilde{\phi}(1) = \tilde{\phi}(x * y)$, and so
$\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\} = r\min\{\tilde{\phi}(1), \tilde{\phi}(1)\}$.
$\Rightarrow \tilde{\phi}(y) \geqslant \tilde{\phi}(1)$ but $\tilde{\phi}(y) \leqslant \tilde{\phi}(1)$, $\forall y \in J$ (by Definition 5.2.1)
$\Rightarrow \tilde{\phi}(y) = \tilde{\phi}(1)$
$\Rightarrow y \in X_{\tilde{\phi}}$. Thus $X_{\tilde{\phi}}$ is a JU-ideal of X. □

Theorem 5.2.9. *Every I-V fuzzy JU-ideal of a JU-algebra X can be realized as a level JU-ideal of an I-V fuzzy JU-ideal of X.*

Proof. Let J be a JU-ideal of X and let ϕ be an I-V fuzzy set in X defined by

$$\tilde{\phi}(x) = \begin{cases} [t_1, t_2] & \text{if } x \in J \\ [0,0] & \text{other wise} \end{cases}$$

, where $[t_1, t_2] \in C[0,1]$ with $t_1 \leqslant t_2$.
We claim that ϕ is I-V fuzzy JU-ideal of X. Let $x, y \in X$. Then consider the following cases.

Case 1. If $x, x * y \in J$, then $y \in J$, and so

$$\tilde{\phi}(y) \geqslant [t_1, t_2] = r\min\{[t_1, t_2], [t_1, t_2]\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}.$$

Case 2. If $x, x * y \notin J$, then $\tilde{\phi}(x) = [0,0] = \tilde{\phi}(x * y)$ and thus

$$\tilde{\phi}(y) \geqslant [0,0] = r\min\{[0,0], [0,0]\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}.$$

Case 3. If $x \in J$ and $x * y \notin J$, then

$$\tilde{\phi}(y) \geqslant [0,0] = r\min\{[t_1, t_2], [0,0]\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}.$$

Similarly for the case $x \notin J$ and $x * y \in J$ we get $\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$.

Therefore, ϕ is an I-V fuzzy JU-ideal of X. □

Theorem 5.2.10. *Let* $\{J_i\}$ *be any family of JU-ideals of JU-algebra* X *such that* $J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = X$. *Then there exist an I-V fuzzy JU-ideals which are exactly the JU-ideals of this chain* $\{J_i\}$.

Proof. Consider a set of numbers $[t_0, t'_0] > [t_1, t'_1] > [t_2, t'_2] > \cdots > [t_n, t'_n]$ where each $[t_i, t'_i] \in C[0,1]$. Let $\tilde{\phi} : X \to C[0,1]$ be an I-V fuzzy set defined by

$$\tilde{\phi}(x) = \begin{cases} [t_0, t'_0], & \text{if } x \in J_0 \\ [t_i, t'_i], & \text{if } x \in J_i - J_{i-1},\ 0 < i \leqslant n. \end{cases}$$

Now, we claim that $\tilde{\phi}$ is an I-V fuzzy JU-ideal of X and $U(\tilde{\phi}, [t_i, t'_i]) = J_i$ for $0 < i \leqslant n$. Let $x, y \in X$, then we consider the following cases:

Case i). If $x, x * y \in J_i - J_{i-1}$ for some i, then $x, x * y \in J_i$. Therefore, by definition of $\tilde{\phi}$, we have $\tilde{\phi}(x) = [t_i, t'_i] = \tilde{\phi}(x * y)$. Since J_i is a JU-ideal of X, it follows that $y \in J_i$ and so either $y \in J_i - J_{i-1}$ or $y \in J_{i-1}$. Implies that $\tilde{\phi}(y) \geqslant [t_i, t'_i]$ or $\tilde{\phi}(y) \geqslant [t_i - 1, t'_i - 1] > [t_i, t'_i]$. In any case we get $\tilde{\phi}(y) \geqslant [t_i, t'_i] = \min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$

Case ii). For $i > j$, $[t_j, t'_j] > [t_i, t'_i]$ and $J_j \subset J_i$. Let $x \in J_i - J_{i-1}$ and $x * y \in J_j - J_{j-1}$ implies $\tilde{\phi}(x) = [t_i, t'_i]$, $\tilde{\phi}(x * y) = [t_j, t'_j] \geqslant [t_i, t'_i]$. Since J_i is a JU-ideal of X and $J_j \subset J_i$, then $y \in J_i$. Now, $\tilde{\phi}(y) \geqslant [t_i, t'_i] = \min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$. Thus $\tilde{\phi}$ is an I-V fuzzy JU-ideal of X. Also, from the definition of $\tilde{\phi}$, it follows that $\mathrm{Im}(\tilde{\phi}) = \{[t_0, t'_0], [t_1, t'_1], [t_2, t'_2], \ldots, [t_n, t'_n]\}$. Hence the upper level JU-ideals of $\tilde{\phi}$ are given by the chain of JU-ideals:

$$U(\tilde{\phi}, [t_0, t'_0]) \subset U(\tilde{\phi}, [t_1, t'_1]) \subset U(\tilde{\phi}, [t_2, t'_2]) \subset \cdots \subset U(\tilde{\phi}, [t_n, t'_n]) = X, \quad \text{for } 0 < i \leqslant n.$$

Now, $U(\tilde{\phi}, [t_0, t'_0]) = \{x \in X : \tilde{\phi}(x) \geqslant [t_0, t'_0]\} = J_0$. Finally, we prove that $U(\tilde{\phi}, [t_i, t'_i]) = J_i$, for $0 < i \leqslant n$. Now let $x \in J_i$, then $\tilde{\phi}(x) \geqslant [t_i, t'_i]$. This implies $x \in U(\tilde{\phi}, [t_i, t'_i]$. Hence, $J_i \subseteq U(\tilde{\phi}, [t_i, t'_i])$. If $x \in U(\tilde{\phi}, [t_i, t'_i])$, then $\tilde{\phi}(x) \geqslant [t_i, t'_i]$. Which implies that $x \notin J_j$ for $j > i$.

Otherwise, if $x \in J_j$, then $\tilde{\phi}(x) \geqslant [t_j, t_j']$ which implies $[t_i, t_i'] \geqslant \tilde{\phi}(x) \geqslant [t_j, t_j']$. This contradicts the assumption that $x \in U(\tilde{\phi}, [t_i, t_i'])$. Hence $\tilde{\phi}(x) \in \{[t_0, t_0'], [t_1, t_1'], [t_2, t_2'], \ldots, [t_n, t_n']\}$. So, $x \in J_k$ for some $k \leqslant i$. As $J_k \subseteq J_i$, it follows that $x \in J_i$. Hence $U(\tilde{\phi}, [t_i, t_i']) \subseteq J_i$. Therefore, $U(\tilde{\phi}, [t_i, t_i']) = J_i$, for $0 < i \leqslant n$.

□

Theorem 5.2.11. *Let* J *be a subset of* X *and let* $\tilde{\phi}$ *be an I-V fuzzy set on* X*, given by*

$$\tilde{\phi}(x) = \begin{cases} [a_1, a_2] & \text{if } x \in J, \\ [b_1, b_2] & \text{otherwise,} \end{cases}$$

for all $[a_1, a_2], [b_1, b_2] \in C[0,1]$ *with* $a_1 \geqslant b_1$ *and* $a_2 \geqslant b_2$*. Then* $\tilde{\phi}$ *is an I-V fuzzy JU-ideal of* X *if and only if* J *is a JU-ideal of* X*. Moreover, in this case,* $X_{\tilde{\phi}} = J$*.*

Proof. Let $\tilde{\phi}$ be an I-V fuzzy JU-ideal of X, and let $x, x * y \in J$ then,
$\tilde{\phi}(x) = [a_1, a_2] = \tilde{\phi}(x * y)$
$\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\} = r\min\{[a_1, a_2], [a_1, a_2]\} = [a_1, a_2]$
$\Rightarrow \tilde{\phi}(y) \geqslant [a_1, a_2]$ but $\tilde{\phi}(y) \leqslant [a_1, a_2]$
$\Rightarrow \tilde{\phi}(y) = [a_1, a_2]$
$\Rightarrow y \in J$. Thus, J is a JU-ideal of X.
Conversely, suppose that J is a JU-ideal of X, let $x, x * y \in X$. Let us consider the following cases.

Case 1. If $x, x * y \in J$ then $y \in J$, thus

$$\tilde{\phi}(y) = [a_1, a_2] = r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$$

Case 2. If $x \notin J$ or $x * y \notin J$, then

$$\tilde{\phi}(y) \geqslant [b_1, b_2] = r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$$

Case 3. If $x \in J$ or $x * y \notin J$, then

$$\tilde{\phi}(y) \geqslant [b_1, b_2] = r\min\{[a_1, a_2], [b_1, b_2]\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$$

Similarly, If $x \notin J$ or $x * y \in J$, then

$$\tilde{\phi}(y) \geqslant [b_1, b_2] = r\min\{[b_1, b_2], [a_1, a_2]\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y)\}$$

Thus, $\tilde{\phi}$ is an I-V fuzzy JU-ideal.
Moreover, we have
$X_{\tilde{\phi}} = \{x \in X \mid \tilde{\phi}(x) = \tilde{\phi}(1)\} = \{x \in X \mid \tilde{\phi}(x) = [a_1, a_2]\} = J$ □

Theorem 5.2.12. *Let* $(X, *, 1)$ *and* $(Y, *', 1')$ *be JU-algebras and a map* $g : X \to Y$ *be an onto homomorphism and let* $\tilde{\phi}$ *be an I-V fuzzy subset of X. If* $\tilde{\phi}$ *is an I-V fuzzy JU-ideal of X, then the image* $g(\tilde{\phi})$ *is an I-V fuzzy JU-ideal of Y with the sup property.*

Proof. Let $g : X \to Y$ be an onto homomorphism of JU-algebras and let $\tilde{\phi}$ be an I-V fuzzy JU-ideal of X. We want to show that $g(\tilde{\phi})$ is an I-V fuzzy JU-ideal in Y. Let $y \in Y$. Since g is onto, there exist $x \in X$ such that $g(x) = y$

Also g is a homomorphism, $g(1) = 1'$. Thus,

$g(\tilde{\phi})(1') = \sup_{1 \in g^{-1}(1')}\{\tilde{\phi}(1)\} \geqslant \sup_{x \in g^{-1}(y)} \tilde{\phi}(x)\} = g(\tilde{\phi})(y)$

And let $y_1, y_2 \in Y$. Since g is onto, there exist $x_1, x_2 \in X$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$. Also, since g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$.

Now,

$$\begin{aligned}
g(\tilde{\phi})(y_2) &= \sup_{x_2 \in g^{-1}(y_2)} \{\tilde{\phi}(x_2)\} \\
&= \sup_{x_2 \in g^{-1}(y_2)} \{[\phi^L(x_2), \phi^U(x_2)]\} \quad \text{(by Lemma 1.4.1)} \\
&\geqslant \left[\min\left\{ \sup_{x_1 \in g^{-1}(y_1)} \phi^L(x_1), \sup_{x_1 * x_2 \in g^{-1}(y_1 *' y_2)} \phi^L(x_1 * x_2) \right\}, \right. \\
&\quad \left. \min\left\{ \sup_{x_1 \in g^{-1}(y_1)} \phi^U(x_1), \sup_{x_1 * x_2 \in g^{-1}(y_1 *' y_2)} \phi^U(x_1 * x_2) \right\} \right] \\
&= r\min\left\{ g(\tilde{\phi})(y_1), g(\tilde{\phi})(y_1 *' y_2) \right\}.
\end{aligned}$$

□

Theorem 5.2.13. *Let* $(X, *, 1)$ *and* $(Y, *', 1')$ *be JU-algebras and a map* $g : X \to Y$ *be a homomorphism. Let* $\tilde{\phi}$ *be an I-V fuzzy subset of* Y. *If* $\tilde{\phi}$ *is an I-V fuzzy JU-ideal of* Y, *then the pre-image* $g^{-1}(\tilde{\phi})$ *is an I-V fuzzy JU-ideal of* X.

Proof. Let $g : X \to Y$ be a homomorphism of JU-algebras and let $\tilde{\phi}$ be an I-V fuzzy JU-ideal of Y. We want to show that $g^{-1}(\tilde{\phi})$ is an I-V fuzzy JU-ideal of X. For any $x \in X$ and $y \in Y$ such that $g(x) = y$. Since g is a homomorphism, $g(1) = 1'$. Then,

$g^{-1}(\tilde{\phi})(1) = \tilde{\phi}(g(1)) = \tilde{\phi}(1') \geqslant \tilde{\phi}(y) = \tilde{\phi}(g(x)) = g^{-1}(\tilde{\phi}(x))$

And for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Then,

$$\begin{aligned}
g^{-1}(\tilde{\phi})(x_2) &= [\phi^L(g(x_2)), \phi^U(g(x_2))] \\
&= [\phi^L(y_2), \phi^U(y_2)] \\
&\geqslant [\min\{\phi^L(y_1), \phi^L(y_1 * y_2)\}, \min\{\phi^U(y_1), \phi^U(y_1 *' y_2)\}] \\
&= [\min\{\phi^L(y_1), \phi^U(y_1)\}, \min\{\phi^L(y_1 *' y_2), \phi^U(y_1 *' y_2)\}] \\
&= r\min\{\tilde{\phi}(y_1), \tilde{\phi}(y_1 *' y_2)\} \\
&= r\min\{\tilde{\phi}(g(x_1)), \tilde{\phi}(g(x_1) *' g(x_2))\} \\
&= r\min\{\tilde{\phi}(g(x_1)), \tilde{\phi}(g(x_1 * x_2))\} \quad \text{Since } g \text{ is a homomorphism} \\
&= r\min\{g^{-1}(\tilde{\phi})(x_1), g^{-1}(\tilde{\phi})(x_1 * x_2)\}
\end{aligned}$$

$\square$

Theorem 5.2.14. *Let X, Y, Z be JU-algebras. Let $f : X \to Y$ and $g : Y \to Z$ be JU-homomorphisms. Then the following hold:*

(a) If f and g are surjective, then $(g \circ f)(\tilde{\phi})$ is an I-V fuzzy JU-ideal of Z, for any I-V fuzzy JU-ideal $\tilde{\phi}$ of X, provided that the sup property holds.

(b) $(g \circ f)^{-1}(\tilde{\psi})$ is an I-V fuzzy JU-ideal of X, for any I-V fuzzy JU-ideal $\tilde{\psi}$ of Z.

Proof. (a) Let $\tilde{\phi}$ be an I-V fuzzy JU-ideal of X. For $z \in Z$,there exist $x \in X$ such that,

$(g \circ f)(x) = z$. Then,

$(g \circ f)(\tilde{\phi})(1_z) = \sup_{1_x \in (g \circ f)^{-1}(1_z)}\{\tilde{\phi}(1_x)\} \geqslant \sup_{1_x \in (g \circ f)^{-1}(z)}\{\tilde{\phi}(x)\} = (g \circ f)(\tilde{\phi})(z)$

And for any $z_1, z_2 \in Z$, there exist $x_1, x_2 \in X$ such that $(g \circ f)(x_1) = z_1$, $(g \circ f)(x_2) = z_2$. Then, for some $s, t \in X$,

$$\tilde{\phi}(s) = \sup_{r \in (g \circ f)^{-1}(z_1)} \tilde{\phi}(r), \quad \tilde{\phi}(t) = \sup_{r \in (g \circ f)^{-1}(z_2)} \tilde{\phi}(r).$$

Also, $g \circ f$ is a homomorphism, so $s * t \in (g \circ f)^{-1}(z_1 * z_2)$. Now,

$$\begin{aligned}
(g \circ f)(\tilde{\phi})(z_2) &\geqslant \tilde{\phi}(t) \\
&= [\phi^L(t), \phi^U(t)] \\
&\geqslant \Bigg[\min\left\{\sup_{r \in (g \circ f)^{-1}(z_1)} \phi^L(r), \sup_{r \in (g \circ f)^{-1}(z_1 * z_2)} \phi^L(r)\right\}, \\
&\quad \min\left\{\sup_{r \in (g \circ f)^{-1}(z_1)} \phi^U(r), \sup_{r \in (g \circ f)^{-1}(z_1 * z_2)} \phi^U(r)\right\}\Bigg] \\
&= [\min\{\phi^L(s), \phi^L(s * t)\}, \min\{\phi^U(s), \phi^U(s * t)\}] \\
&= [\min\{\phi^L(s), \phi^U(s)\}, \min\{\phi^L(s * t), \phi^U(s * t)\}] \\
&= r\min\{\tilde{\phi}(s), \tilde{\phi}(s * t)\} \\
&= r\min\{(g \circ f)(\tilde{\phi})(z_1), (g \circ f)(\tilde{\phi})(z_1 * z_2)\}.
\end{aligned}$$

(b) Let $\tilde{\psi}$ be a fuzzy JU-ideal of Z. Let $x \in X$, $z \in Z$ such that $(g \circ f)(1_X) = 1_Z$. Then,

$(g \circ f)^{-1}(\tilde{\psi})(1_X) = \tilde{\psi}((g \circ f)(1_X)) \geqslant \tilde{\psi}((g \circ f)(x)) = g \circ f)^{-1}(\tilde{\psi}(x)$.

And for any $x, y \in X$,

$$\begin{aligned}
(g \circ f)^{-1}(\tilde{\psi})(y) &= \tilde{\psi}((g \circ f)(y)) \\
&= [\psi^L((g \circ f)(y)), \psi^U((g \circ f)(y))] \\
&\geqslant [\min\{\psi^L((g \circ f)(x)), \psi^L((g \circ f)(x * y))\}, \\
&\quad \min\{\psi^U((g \circ f)(x)), \psi^U((g \circ f)(x * y))\}]
\end{aligned}$$

$$
\begin{aligned}
&= [\min\{\psi^L((g\circ f)(x)),\psi^U((g\circ f)(x))\},\\
&\quad \min\{\psi^L((g\circ f)(x*y)),\psi^U((g\circ f)(x*y))\}]\\
&= r\min\{(g\circ f)^{-1}(\tilde{\psi})(x),(g\circ f)^{-1}(\tilde{\psi})(x*y)\}.
\end{aligned}
$$

□

Theorem 5.2.15. *The Cartesian product of any two I-V fuzzy JU-ideals of X is also I-V fuzzy JU-ideal of X.*

Proof. : Suppose $\tilde{\phi}_1$ and $\tilde{\phi}_2$ be two I-V fuzzy JU- ideals of X. Then let $(x,y)\in X\times X$. Now

$$
\begin{aligned}
\tilde{\phi}_1\times\tilde{\phi}_2(1,1) &= [\phi_1^L\times\phi_2^L(1,1),\phi_1^U\times\phi_2^U(1,1)]\\
&= [\min\{\phi_1^L(1),\phi_2^L(1)\},\min\{\phi_1^U(1),\phi_2^U(1)\}]\\
&\geqslant [\min\{\phi_1^L(x),\phi_2^L(y)\},\min\{\phi_1^U(x),\phi_2^U(y)\}]\\
&= [\min\{\phi_1^L(x),\phi_2^U(x)\},\min\{\phi_1^L(y),\phi_2^U(y)\}]\\
&= r\min\{\tilde{\phi}_1\times\tilde{\phi}_2(x),\tilde{\phi}_1\times\tilde{\phi}_2(y)\}\\
&= \tilde{\phi}_1\times\tilde{\phi}_2(x,y)
\end{aligned}
$$

And, let Let $(x_1,x_2),(y_1,y_2)\in X\times X$. Then

$$
\begin{aligned}
\phi_1\times\phi_2(y_1,y_2) &= [\phi_1^L\times\phi_2^L(y_1,y_2),\phi_1^U\times\phi_2^U(y_1,y_2)]\\
&= [\min\{\phi_1^L(y_1),\phi_2^L(y_2)\},\min\{\phi_1^U(y_1),\phi_2^U(y_2)\}]\\
&\geqslant [\min\{\min\{\phi_1^L(x_1),\phi_1^L(x_1*y_1)\},\min\{\phi_2^L(x_2),\phi_2^L(x_2*y_2)\}\},\min\{\\
&\quad \min\{\phi_1^U(x_1),\phi_1^U(x_1*y_1)\},\min\{\phi_2^U(x_2),\phi_2^U(x_2*y_2)\}\}]\\
&= [\min\{\min\{\phi_1^L(x_1),\phi_2^U(x_1)\},\min\{\phi_1^L(x_1*y_1),\phi_2^U(x_1*y_1)\}\},\min\{\\
&\quad \min\{\phi_1^L(x_2),\phi_2^U(x_2)\},\min\{\phi_1^L(x_2*y_2),\phi_2^U(x_2*y_2)\}\}]\\
&= r\min\{\tilde{\phi}_1\times\tilde{\phi}_2(x_1,x_2),\tilde{\phi}_1\times\tilde{\phi}_2(x_1,x_2)*(y_1,y_2)\}
\end{aligned}
$$

□

Corollary 5.2.16. *Let* $\{X_j\}_{j=1}^n$ *be the finite family of JU- algebras and* $X=\Pi_{j=1}^n X_j$ *the Cartesian product of JU- algebras of* $\{X_j\}$. *Let* $\tilde{\phi}_j$ *be I-V fuzzy JU- ideals of* X_j, *where* $1\leqslant j\leqslant n$. *Then* $\tilde{\phi}=\Pi_{j=1}^n\tilde{\phi}_j$ *is defined by* $\tilde{\phi}(x_1,x_2,...,x_n)=(\Pi_{j=1}^n\tilde{\phi}_j)(x_1,x_2,...,x_n)=r\inf\{(\tilde{\phi_1}(x_1),\tilde{\phi_2}(x_2),...,\tilde{\phi_n}(x_n))$ *is also I-V fuzzy JU-ideals of X.*

Proof. We know that an interval valued (I-V) fuzzy set $\tilde{\phi}$ in a JU-algebra X is an I-V fuzzy JU-ideal if it satisfies the following axioms:

i. $\tilde{\phi}(1)\geqslant\tilde{\phi}(x)$ for all $x\in X$.

ii. $\tilde{\phi}(y)\geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(x*y)\}$ for all $x,y\in X$, where $*$ is the JU-algebra operation.

Now, consider the Cartesian product

$$
X=\prod_{j=1}^{n}X_j \quad\text{and}\quad \tilde{\phi}=\prod_{j=1}^{n}\tilde{\phi}_j.
$$

(i) Verification of the first condition:

$\tilde{\phi}(1,1,\ldots,1)=r\inf\{\tilde{\phi}_1(1),\tilde{\phi}_2(1),\ldots,\tilde{\phi}_n(1)\}\geqslant r\inf\{\tilde{\phi}_1(x_1),\tilde{\phi}_2(x_2),\ldots,\tilde{\phi}_n(x_n)\}=\tilde{\phi}(x_1,x_2,\ldots,x_n)$,

since each $\tilde{\phi}_j$ is an I-V fuzzy JU-ideal of X_j.
(ii) Verification of the second condition: For all $(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in X$, we have

$$\begin{aligned}\tilde{\phi}(y_1,\ldots,y_n) &= r\inf\{\tilde{\phi}_1(y_1),\ldots,\tilde{\phi}_n(y_n)\}\\ &\geqslant r\inf\left\{r\min\{\tilde{\phi}_1(x_1),\tilde{\phi}_1(x_1*y_1)\},\ldots,r\min\{\tilde{\phi}_n(x_n),\tilde{\phi}_n(x_n*y_n)\}\right\}\\ &= r\min\{\tilde{\phi}(x_1,\ldots,x_n),\tilde{\phi}((x_1*y_1),\ldots,(x_n*y_n))\},\end{aligned}$$

since each $\tilde{\phi}_j$ satisfies the second axiom.
Hence, $\tilde{\phi} = \prod_{j=1}^{n} \tilde{\phi}_j$, is also an I-V fuzzy JU-ideal of X □

5.2.1 $\tilde{S}$-Interval Valued Fuzzy JU-Ideals

Definition 5.2.2. *Let $\tilde{\phi}_{\tilde{S}}$ be an $\tilde{S}$-interval valued fuzzy set in a JU-algebra X and let $\tilde{S} \in C[0,1]$. Then $\tilde{\phi}_{\tilde{S}}$ of X is called the $\tilde{S}$-interval valued fuzzy JU-ideal of X it satisfies the following axioms:*
a) $\tilde{\phi}_{\tilde{S}}(1) \geqslant \tilde{\phi}_{\tilde{S}}(x)$,
b) $\tilde{\phi}_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(x*y)\}$, *for all* $x,y \in X$.
The $\tilde{S}$-interval valued fuzzy JU-ideal of X extends the concept of an interval valued fuzzy JU-ideal of X, and an interval valued fuzzy JU-ideal of X is a special case of an $\tilde{S}$-interval valued fuzzy JU-ideal of X when S equal to 1.

Example 5.2.1.1. *Let X =* $\{1,2,3,4,5,6\}$ *in which $*$ is defined by the following Cayley table:*

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	6
6	1	1	2	1	1	1

See [3] $(X;*,1)$ *is a JU-algebra. Let* $\tilde{S} = [1,1]$ *and let us define $\tilde{S}$- interval valued fuzzy subset* $\tilde{\phi}_{\tilde{S}} : X \to C[0,1]$ *by*

$$\tilde{\phi}_{\tilde{S}}(x) = \begin{cases} r\min\{\tilde{S},[0.7,0.9]\} & \textit{if} \quad x \in \{1,2\} \\ r\min\{\tilde{S},[0.4,0.5]\} & \textit{otherwise} \end{cases}$$

To verify that $\tilde{\phi}_{\tilde{S}}$ is an $\tilde{S}$-interval valued fuzzy JU-ideal of X, *we check the two conditions.*

(i) $\tilde{\phi}_{\tilde{S}}(1) \geqslant \tilde{\phi}_{\tilde{S}}(x)$ *for all* $x \in X$.
We have $\tilde{\phi}_{\tilde{S}}(1) = [0.7,0.9]$. *If* $x \in \{1,2\}$, *then* $\tilde{\phi}_{\tilde{S}}(x) = [0.7,0.9]$. *If* $x \in \{3,4,5,6\}$, *then* $\tilde{\phi}_{\tilde{S}}(x) = [0.4,0.5]$. *Thus,* $[0.7,0.9] \geqslant [0.7,0.9]$ *and* $[0.7,0.9] \geqslant [0.4,0.5]$. *Hence, condition (i) holds.*

(ii) For all $x,y \in X$, $\tilde{\phi}_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(x*y)\}$.
For $x = 1$, $y = 2$. *Then* $x*y = 2$.
Then, $\tilde{\phi}_{\tilde{S}}(y) = \tilde{\phi}_{\tilde{S}}(2) = [0.7,0.9]\cdots(*)$

And, $r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(x*y)\} = r\min\{\tilde{\phi}_{\tilde{S}}(1),\tilde{\phi}_{\tilde{S}}(1*2)\} = r\min\{[0.7,0.9],[0.7,0.9]\} = [0.7,0.9]\cdots(**)$.
From (*) and (**) we get $[0.7,0.9] \geqslant [0.7,0.9]$, *which is true. Hence, condition (ii) holds.*

Similarly, for $x = 2$, $y = 3$. *Then* $x*y = 3$.
Then, $\tilde{\phi}_{\tilde{S}}(y) = \tilde{\phi}_{\tilde{S}}(3) = [0.4,0.5]\cdots(***)$
And, $r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(x*y)\} = r\min\{\tilde{\phi}_{\tilde{S}}(2),\tilde{\phi}_{\tilde{S}}(2*3)\} = r\min\{[0.7,0.9],[0.4,0.5]\} = [0.4,0.5]\cdots(****)$.
From (***) and (****) we get $[0.4,0.5 \geqslant [0.4,0.5]$, *which is true.*
Similarly, the result holds for all values of x and y in X.
Therefore, we conclude that $\tilde{\phi}_{\tilde{S}}$ *is an* $\tilde{S}$*-interval valued fuzzy JU-ideal of X.*

Theorem 5.2.1.1. *Every* $\tilde{S}$ *- interval valued fuzzy JU-ideals of* X *is order reversing.*

Proof. Let $\tilde{\phi}_{\tilde{S}}$ be $\tilde{S}$-interval valued fuzzy JU-ideals of X and for any $x,y \in X$ be such that $x \leqslant y$, then $y*x = 1$. Now

$$\begin{aligned}\tilde{\phi}_{\tilde{S}}(x) &\geqslant r\min\{\tilde{\phi}_{\tilde{S}}(y*x),\tilde{\phi}_{\tilde{S}}(y)\}\\ &= r\min\{\tilde{\phi}_{\tilde{S}}(1),\tilde{\phi}_{\tilde{S}}(y)\}\\ &= \tilde{\phi}_{\tilde{S}}(y).\end{aligned}$$

□

Theorem 5.2.1.2. *Let* $\tilde{\phi}_{\tilde{S}}$ *be* $\tilde{S}$ *- interval valued fuzzy JU-ideals of* X*. Then* $\tilde{\phi}_{\tilde{S}}((x*y)*y) \geqslant \tilde{\phi}_{\tilde{S}}(x)$.

Proof. Since $(x*[(x*y)*y]) = 1$ (By Lemma 1.6.1)
Then, $\tilde{\phi}_{\tilde{S}}((x*y)*y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(x*[(x*y)*y])\}$

$$\begin{aligned}&= r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(1)\}\\ &= \tilde{\phi}_{\tilde{S}}(x)\end{aligned}$$

□

Theorem 5.2.1.3. *Every* $\tilde{S}$*-interval valued fuzzy JU-ideals of* X *is* $\tilde{S}$*-interval valued fuzzy JU-subalgebra.*

Proof. Suppose $\tilde{\phi}_{\tilde{S}}$ is $\tilde{S}$ - interval valued fuzzy JU-ideals of X. That is:

a) $\tilde{\phi}_{\tilde{S}}(1) \geqslant \tilde{\phi}_{\tilde{S}}(x)\}$,

b) $\tilde{\phi}_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(x*y)\}$, for all $x,y \in X$.

We claim that $\tilde{\phi}_{\tilde{S}}$ is $\tilde{S}$ - interval valued fuzzy JU-subalgebra. That is, for all $x,y \in X$

$$\tilde{\phi}_{\tilde{S}}(x*y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(y)\}\}.$$

Assume that without loss of generality:

$$\tilde{\phi}_{\tilde{S}}(x) \leqslant \tilde{\phi}_{\tilde{S}}(y) \Rightarrow r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(y)\} = \tilde{\phi}_{\tilde{S}}(x)$$

Now we consider two cases:

Case 1. If $\tilde{\phi}_{\tilde{S}}(x*y) \geqslant \tilde{\phi}_{\tilde{S}}(x)$, then

$$\tilde{\phi}_{\tilde{S}}(x*y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x)\} = r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(y)\}$$

$$\Rightarrow \tilde{\phi}_{\tilde{S}}(x*y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(y)\}$$

The inequality is satisfied directly.

Case 2. If $\tilde{\phi}_{\tilde{S}}(x*y) < \tilde{\phi}_{\tilde{S}}(x)$, then from the assumption that $\tilde{\phi}_{\tilde{S}}$ is $\tilde{S}$ - interval valued fuzzy JU-ideal, apply condition (b):

$\tilde{\phi}_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\} = r\min\{\tilde{\phi}_{\tilde{S}}(x*y)\}$

$\Rightarrow \tilde{\phi}_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x*y)\}$

Since we also know, $\tilde{\phi}_{\tilde{S}}(x*y) < \tilde{\phi}_{\tilde{S}}(x)$ and $\tilde{\phi}_{\tilde{S}}(x) \leqslant \tilde{\phi}_{\tilde{S}}(y)$, it follows that

$\tilde{\phi}_{\tilde{S}}(x*y) < \tilde{\phi}_{\tilde{S}}(x) \leqslant \tilde{\phi}_{\tilde{S}}(y) \Rightarrow r\max\{\tilde{\phi}_{\tilde{S}}(x*y) < r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(y)\}$.

Which is contradict the inequality $\tilde{\phi}_{\tilde{S}}(x*y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(y)\}$

Thus, Case 1 must always hold, which implies:

$\tilde{\phi}_{\tilde{S}}(x*y) \geqslant \tilde{\phi}_{\tilde{S}}(x) = r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(y)\}$

Hence, $\tilde{\phi}_{\tilde{S}}$ is $\tilde{S}$ - interval valued fuzzy JU-subalgebra of X.

□

Remark 5.2.1.1. *The converse of the above theorem may not be true.*

Example 5.2.1.2. *Let* $X = \{1,2,3,4\}$ *in which* $*$ *is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $X;*,1)$ *is a JU-algebra. Define* $\tilde{\phi}_{\tilde{S}}(x)$ *as follows:*

$$\tilde{\phi}_{\tilde{S}}(x) = \begin{cases} [0.9,1] & \text{if } x = 1 \\ [0.7,0.8] & \text{if } x \in \{2,4\} \\ [0.4,0.5] & \text{if } x = 3 \end{cases}$$

Since, $\tilde{\phi}_{\tilde{S}}$ *is* $\tilde{S}$*-interval valued fuzzy JU-subalgebra of X.*
The $\tilde{S}$ *- interval valued fuzzy JU-ideal condition requires:*
$\tilde{\phi}_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\}$, *for all* $x, y \in X$
Let $x = 2$, $y = 3 \Rightarrow x*y = 4$ *and*
Then, $\tilde{\phi}_{\tilde{S}}(y)\} = \tilde{\phi}_{\tilde{S}}(3)\} = [0.4,0.5], [0.2,0.3]\} = [0.4,0.5] \cdots (*)$
And, $r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\} = r\min\{\tilde{\phi}_{\tilde{S}}(2), \tilde{\phi}_{\tilde{S}}(2*4)\} = r\min\{[0.7,0.8],[0.7,0.8]\} = [0.7,0.8] \cdots (**)$
From $(*)$ *and* $(**)$ *we get* $[0.4,0.5] \geqslant [0.7,0.8]$ *which is false.*

Theorem 5.2.1.4. *The intersection of any two* $\tilde{S}$ *- interval valued fuzzy JU-ideals of X is also* $\tilde{S}$ *- interval valued fuzzy JU-ideals of X.*

Proof. Suppose $\tilde{\phi}_{\tilde{S_1}}$ and $\tilde{\phi}_{\tilde{S_2}}$ are any two $\tilde{S}$ - interval valued fuzzy JU-ideals of X and $\forall x, y \in X$. Then,

$\tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}}(1) = r\min\{\tilde{\phi}_{\tilde{S_1}}(1), \tilde{\phi}_{\tilde{S_2}}(1)\} \geqslant r\min\{\tilde{\phi}_{\tilde{S_1}}(x), \tilde{\phi}_{\tilde{S_1}}(x)\} = \tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}}(x)$

And,

$$\begin{aligned}
\tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}}(y) &= r\min\{\tilde{\phi}_{\tilde{S_1}}(y), \tilde{\phi}_{\tilde{S_2}}(y)\} \\
&\geqslant r\min\{r\min\{\tilde{\phi}_{\tilde{S_1}}(x), \tilde{\phi}_{\tilde{S_1}}(x*y)\}, r\min\{\tilde{\phi}_{\tilde{S_2}}(x), \tilde{\phi}_{\tilde{S_2}}(x*y)\}\} \\
&= r\min\{r\min\{\tilde{\phi}_{\tilde{S_1}}(x), \tilde{\phi}_{\tilde{S_2}}(x)\}, r\min\{\tilde{\phi}_{\tilde{S_1}}(x*y), \tilde{\phi}_{\tilde{S_2}}(x*y)\}\} \\
&= r\min\{\tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}}(x), \tilde{\phi}_{\tilde{S_1}} \cap \tilde{\phi}_{\tilde{S_2}}(x*y)\}
\end{aligned}$$

□

Corollary 5.2.1.5. *Let* $\{\tilde{\phi}_{\tilde{S_i}} | i \in \Omega\}$ *be a family of* $\tilde{S}$*-interval valued fuzzy JU-ideals of X. Then* $\bigcap_{i\in\Omega}\{\tilde{\phi}_{\tilde{S_i}} = rinf_{i\in\Lambda}\{x \in X|\{\tilde{\phi}_{\tilde{S_i}}(x)\}$ *is also* $\tilde{S}$*-interval valued fuzzy JU-ideals of X.*

Proof. Let $\{\tilde{\phi}_{\tilde{S_i}} \mid i \in \Omega\}$ be a family of $\tilde{S}$-interval valued fuzzy JU-ideals of X. Define

$$\tilde{\phi}(x) = \bigcap_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(x) = r\inf_{i\in\Omega}\{\tilde{\phi}_{\tilde{S_i}}(x)\}, \quad \forall x \in X.$$

We shall show that $\tilde{\phi}$ is also an $\tilde{S}$-interval valued fuzzy JU-ideal of X.

(i) For every $x \in X$,

$$\tilde{\phi}(1) = r\inf_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(1) \quad \text{and} \quad \tilde{\phi}(x) = r\inf_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(x).$$

Since each $\tilde{\phi}_{\tilde{S_i}}$ satisfies $\tilde{\phi}_{\tilde{S_i}}(1) \geqslant \tilde{\phi}_{\tilde{S_i}}(x)$, we have

$$r\inf_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(1) \geqslant r\inf_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(x),$$

Thus, $\tilde{\phi}(1) \geqslant \tilde{\phi}(x)$. Hence, the first condition holds.

(ii) For all $x, y \in X$,

$$\tilde{\phi}_{\tilde{S_i}}(y) \geqslant r\min\{\tilde{\phi}_{\tilde{S_i}}(x), \tilde{\phi}_{\tilde{S_i}}(x*y)\} \quad \text{for each } i \in \Omega.$$

$$\begin{aligned}
\tilde{\phi}(y) = \bigcap_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(y) &= r\inf_{i\in\Omega}\{\tilde{\phi}_{\tilde{S_i}}(y)\} \\
&\geqslant r\inf_{i\in\Omega} r\min\{\tilde{\phi}_{\tilde{S_i}}(x), \tilde{\phi}_{\tilde{S_i}}(x*y)\} \\
&= r\min\{r\inf_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(x), r\inf_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(x*y)\}.
\end{aligned}$$

Thus, $\tilde{\phi}(y) \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x*y)\}$. Hence, $\tilde{\phi}$ satisfies both conditions of $\tilde{S}$-interval valued fuzzy JU-ideal of X. Therefore, $\bigcap_{i\in\Omega} \tilde{\phi}_{\tilde{S_i}}(x) = r\inf_{i\in\Omega}\{\tilde{\phi}_{\tilde{S_i}}(x)\}$ is also an $\tilde{S}$-interval valued fuzzy JU-ideal of X. □

Theorem 5.2.1.6. *Let* $U(\tilde{\phi}_{\tilde{S}} : [t_1, t_2]) = \{x \in X | \tilde{\phi}_{\tilde{S}}(x) \geqslant [t_1, t_2]\}$ *be nonempty set and* $\forall [t_1, t_2], \tilde{S} \in$

$C[0,1]$. *A fuzzy subset $\tilde{\phi}_{\tilde{S}}$ of a JU-algebra X is an $\tilde{S}$-interval valued fuzzy JU-ideal of X , if and only if* $U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])$ *is JU-ideal of X.*

Proof. Suppose $\tilde{\phi}_{\tilde{S}}$ is an $\tilde{S}$-interval valued fuzzy JU-ideal of X. Let $[t_1,t_2]\in C[0,1]$ and $U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])\neq\emptyset$. Let $x,\in U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])$ implies that $\tilde{\phi}_{\tilde{S}}(x)\geqslant[t_1,t_2]$. Since $\tilde{\phi}_{\tilde{S}}$ is an $\tilde{S}$-interval valued fuzzy JU-ideal of X, then $\tilde{\phi}_{\tilde{S}}(1)\geqslant\tilde{\phi}_{\tilde{S}}(x)\geqslant[t_1,t_2]$.

$\Rightarrow\tilde{\phi}_{\tilde{S}}(1)\geqslant[t_1,t_2]$, implies that $1\in U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])$.

And, let $x,x*y\in U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])$ implies that $\tilde{\phi}_{\tilde{S}}(x)\geqslant[t_1,t_2]$ and $\tilde{\phi}_{\tilde{S}}(x*y)\geqslant[t_1,t_2]$. Then, $\tilde{\phi}_{\tilde{S}}(y)\geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x),\tilde{\phi}_{\tilde{S}}(x*y)\}\geqslant r\min\{[t_1,t_2],[t_1,t_2]\}=[t_1,t_2]$

This implies that $y\in U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])$ and hence $U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])$ is a JU-ideal of X.

Conversely, Suppose that $U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])$ is a JU-ideal of X for any $[t_1,t_2]\in C[0,1]$ and $U(\tilde{\phi}_{\tilde{S}}:[t_1,t_2])\neq\emptyset$. Assume that $\tilde{\phi}_{\tilde{S}}$ is not $\tilde{S}$-interval valued fuzzy JU-ideal of X. Then there exist some $z_0\in X$ such that $\tilde{\phi}_{\tilde{S}}(1)<\tilde{\phi}_{\tilde{S}}(z_0)$.

Take $[\beta_1,\beta_2]=\frac{1}{2}(\tilde{\phi}_{\tilde{S}}(1)+\tilde{\phi}_{\tilde{S}}(x))$

$\Rightarrow\tilde{\phi}_{\tilde{S}}(1)<[\beta_1,\beta_2]<\tilde{\phi}_{\tilde{S}}(x)$

$z_0\in U(\tilde{\phi}_{\tilde{S}}:[\beta_1,\beta_2])$ and $1\notin U(\tilde{\phi}_{\tilde{S}}:[\beta_1,\beta_2])$, which is contradict to our assumption that $U(\tilde{\phi}_{\tilde{S}}:[\beta_1,\beta_2])$ is a JU-ideal of X. Therefore, $\tilde{\phi}_{\tilde{S}}(1)\geqslant\tilde{\phi}_{\tilde{S}}(x),\forall x\in X$.

And assume that $z_0,z_1\in X$ such that

$\tilde{\phi}_{\tilde{S}}(z_1)<r\min\{\tilde{\phi}_{\tilde{S}}(z_0),\tilde{\phi}_{\tilde{S}}(z_0*z_1)\}$

Let $\tilde{\phi}_{\tilde{S}}(z_0)=[\beta_1,\beta_2]$ and $\tilde{\phi}_{\tilde{S}}(z_0*z_1)=[\beta_3,\beta_4]$ and $\tilde{\phi}_{\tilde{S}}(z_1)=[t_1,t_2]$. Then

$[t_1,t_2]<r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4]\}$.

Take $[\alpha_1,\alpha_2]=\frac{1}{2}(\tilde{\phi}_{\tilde{S}}(z_1)+r\min\{\tilde{\phi}_{\tilde{S}}(z_0),\tilde{\phi}_{\tilde{S}}(z_0*z_1)\})=\frac{1}{2}([t_1,t_2]+r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4]\}=\frac{1}{2}(t_1+\min\{\beta_1,\beta_3\}),\frac{1}{2}(t_2+\min\{\beta_2,\beta_4\})$

It follows that $\min\{\beta_1,\beta_3\}>\alpha_1=\frac{1}{2}(t_1+\min\{\beta_1,\beta_3\})>t_1$ and $\min\{\beta_2,\beta_4\}>\lambda_2=\frac{1}{2}(t_2+\min\{\beta_2,\beta_4\})>t_2$, so that

$(\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\})>[\alpha_1,\alpha_2]>[t_1,t_2]=\tilde{\phi}_{\tilde{S}}(z_1)$.

$\Rightarrow z_1\notin U(\tilde{\phi}_{\tilde{S}}:[\alpha_1,\alpha_2])$ and

$\tilde{\phi}_{\tilde{S}}(z_0)=[\beta_3,\beta_4]<[\alpha_1,\alpha_2]<(\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\})>[t_1,t_2]$, and

$\tilde{\phi}_{\tilde{S}}(z_0*z_1)=[\beta_1,\beta_2]<[\alpha_1,\alpha_2]<(\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\})>[t_1,t_2]$.

$\Rightarrow z_0,z_0*z_1\in U(\tilde{\phi}_{\tilde{S}}:[\alpha_1,\alpha_2])$, which is contradict to $U(\tilde{\phi}_{\tilde{S}}:[\alpha_1,\alpha_2])$ is a JU-ideal of X.

Hence $\tilde{\phi}_{\tilde{S}}(z_1)\geqslant r\min\{\tilde{\phi}_{\tilde{S}}(z_0),\tilde{\phi}_{\tilde{S}}(z_0*z_1)\}$ for any $z_0,z_1\in X$. □

Theorem 5.2.1.7. *Let $\tilde{\phi}_{\tilde{S}}$ be $\tilde{S}$-interval valued fuzzy subset of a JU-algebra X. Then $\tilde{\phi}^c_{\tilde{S}}$ is $\tilde{S}$-interval valued fuzzy JU-ideal of X if and only if $L(\tilde{\phi}_{\tilde{S}},\tilde{t'})$ is either empty or a JU-ideal of X for $\tilde{t'}\in C[0,1]$*

.

Proof. Suppose $\tilde{\phi}^c_{\tilde{S}}$ is a $\tilde{S}$-interval valued fuzzy JU-ideal of X and $L(\tilde{\phi}_{\tilde{S}},\tilde{t'})\neq\emptyset$. We need to prove that $L(\tilde{\phi}_{\tilde{S}},\tilde{t'})$ is a JU-ideal of X. Then there exists $x_0\in L(\tilde{\phi}_{\tilde{S}},\tilde{t'})$ such that $\tilde{\phi}_{\tilde{S}}(x_0)\leqslant\tilde{t'}$. Since $\tilde{\phi}^c_{\tilde{S}}$ is an $\tilde{S}$-interval valued fuzzy JU-ideal of X, we have $\tilde{\phi}^c_{\tilde{S}}(1)\geqslant\tilde{\phi}^c_{\tilde{S}}(x)$, for all $x\in X$. Then,

$\tilde{\phi}^c_{\tilde{S}}(1)\geqslant\tilde{\phi}^c_{\tilde{S}}(x_0)\Rightarrow 1-\tilde{\phi}_{\tilde{S}}(1)\geqslant 1-\tilde{\phi}_{\tilde{S}}(x_0)\Rightarrow\tilde{\phi}_{\tilde{S}}(1)\leqslant\tilde{\phi}_{\tilde{S}}(x_0)\leqslant\tilde{t'}$, hence $1\in L(\tilde{\phi}_{\tilde{S}},\tilde{t'})$.

Next, for $x, y \in X$ such that $x, x*y \in L(\tilde{\phi}_{\tilde{S}}, \tilde{t'})$, we have $\tilde{\phi}_{\tilde{S}}(x) \leqslant \tilde{t'}$ and $\tilde{\phi}_{\tilde{S}}(x*y) \leqslant \tilde{t'}$. Then,

$$\tilde{\phi}^c_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}^c_{\tilde{S}}(x), \tilde{\phi}^c_{\tilde{S}}(x*y)\}, \quad \text{for all } x, x*y \in X.$$

It follows that

$$\begin{aligned}\tilde{1} - \tilde{\phi}_{\tilde{S}}(y) &\geqslant r\min\{\tilde{1} - \tilde{\phi}_{\tilde{S}}(x), \tilde{1} - \tilde{\phi}_{\tilde{S}}(x*y)\} \\ &= \tilde{1} - r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\}, \quad \text{By Lemma 1.1.1}\end{aligned}$$

hence

$$\begin{aligned}&\tilde{1} - \tilde{\phi}_{\tilde{S}}(y) \geqslant \tilde{1} - r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\} \\ \Rightarrow \quad &\tilde{\phi}_{\tilde{S}}(y) \leqslant r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\} \leqslant r\max\{\tilde{t'}, \tilde{t'}\} = \tilde{t'}.\end{aligned}$$

Thus, $\tilde{\phi}_{\tilde{S}}(y) \leqslant \tilde{t'}$. Hence, $L(\tilde{\phi}_{\tilde{S}}, \tilde{t'})$ is a JU-ideal of X.
Conversely, assume that $L(\tilde{\phi}_{\tilde{S}}, \tilde{t'})$ is a nonempty JU-ideal of X. We need to show that $\tilde{\phi}^c_{\tilde{S}}$ is a $\tilde{S}$-interval valued fuzzy JU-ideal of X.
Let $\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y) \in C[0,1]$ for $x, x*y \in X$. Choose

$$\tilde{t'} = r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\}.$$

Then $\tilde{\phi}_{\tilde{S}}(x) \leqslant \tilde{t'}$ and $\tilde{\phi}_{\tilde{S}}(x*y) \leqslant \tilde{t'}$ imply that $x, x*y \in L(\tilde{\phi}_{\tilde{S}}, \tilde{t'}) \neq \emptyset$. Since $L(\tilde{\phi}_{\tilde{S}}, \tilde{t'})$ is a JU-ideal of X, we have $y \in L(\tilde{\phi}_{\tilde{S}}, \tilde{t'})$.
Thus,

$$\tilde{\phi}_{\tilde{S}}(y) \leqslant \tilde{t'} = r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\},$$

which gives

$$\begin{aligned}\tilde{\phi}^c_{\tilde{S}}(y) &= \tilde{1} - \tilde{\phi}_{\tilde{S}}(y) \\ &\geqslant \tilde{1} - r\max\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\} \\ &= r\min\{\tilde{1} - \tilde{\phi}_{\tilde{S}}(x), \tilde{1} - \tilde{\phi}_{\tilde{S}}(x*y)\} \quad \text{By Lemma 1.1.1} \\ &= r\min\{\tilde{\phi}^c_{\tilde{S}}(x), \tilde{\phi}^c_{\tilde{S}}(x*y)\}.\end{aligned}$$

Hence,

$$\tilde{\phi}^c_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}^c_{\tilde{S}}(x), \tilde{\phi}^c_{\tilde{S}}(x*y)\}.$$

Therefore, $\tilde{\phi}^c_{\tilde{S}}$ is a $\tilde{S}$-interval valued fuzzy JU-ideal of X. □

Theorem 5.2.1.8. *If ϕ is an $\tilde{S}$-interval valued fuzzy JU-ideal of X, then the set $X_{\tilde{\phi}_{\tilde{S}}} = \{x \in X | \tilde{\phi}_{\tilde{S}}(x) = \tilde{\phi}_{\tilde{S}}(1)\}$ is a JU-ideal of X.*

Proof. Let $x, x*y \in X_{\tilde{\phi}_{\tilde{S}}}$. Then $\tilde{\phi}_{\tilde{S}}(x) = \tilde{\phi}_{\tilde{S}}(1) = \tilde{\phi}_{\tilde{S}}(x*y)$, and so

$\tilde{\phi}_{\tilde{S}}(y) \geqslant r\min\{\tilde{\phi}_{\tilde{S}}(x), \tilde{\phi}_{\tilde{S}}(x*y)\} = r\min\{\tilde{\phi}_{\tilde{S}}(1), \tilde{\phi}_{\tilde{S}}(1)\} = \tilde{\phi}_{\tilde{S}}(1)$.

$\Rightarrow \tilde{\phi}_{\tilde{S}}(y) \geqslant \tilde{\phi}_{\tilde{S}}(1)$ but $\tilde{\phi}_{\tilde{S}}(y) \leqslant \tilde{\phi}_{\tilde{S}}(1)$, $\forall y \in J$ (by Definition 5.2.1)

$\Rightarrow \tilde{\phi}_{\tilde{S}}(y) = \tilde{\phi}_{\tilde{S}}(1)$

$\Rightarrow y \in X_{\tilde{\phi}_{\tilde{S}}}$, hence $X_{\tilde{\phi}_{\tilde{S}}}$ is a JU-ideal of X. □

Theorem 5.2.1.9. *Let $(X,*,1)$ and $(Y,*',1')$ be JU- algebras and a map $g : X \to Y$ be an onto homomorphism and let $\tilde{\phi}_{\tilde{S}}$ be an $\tilde{S}$-interval valued fuzzy subset of X. If $\tilde{\phi}_{\tilde{S}}$ be an $\tilde{S}$-interval valued fuzzy JU-ideal of X, then the image $g(\tilde{\phi}_{\tilde{S}})$, is an $\tilde{S}$-interval valued fuzzy JU-ideal of Y with sup property.*

Proof. Let $g : X \to Y$ be an onto homomorphism of JU- algebra and let $\tilde{\phi}_{\tilde{S}}$ be an $\tilde{S}$-interval valued fuzzy JU-ideal of X. We went to show that $g(\tilde{\phi}_{\tilde{S}})$ is an $\tilde{S}$-interval valued fuzzy JU-ideal in Y. Let $y_1, y_2 \in Y$. Since g is onto, there exist $x_1, x_2 \in X$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$.
Also g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$.
Since g is a homomorphism, $g(1) = 1'$. Thus:

$g(\tilde{\phi}_{\tilde{S}})(1') = \sup_{1 \in g^{-1}(1')}\{\tilde{\phi}_{\tilde{S}}(1)\} \geqslant \sup_{x \in g^{-1}(y)} \tilde{\phi}_{\tilde{S}}(x)\} = g(\tilde{\phi}_{\tilde{S}})(y)$. And,

$$\begin{aligned} g(\tilde{\phi}_{\tilde{S}})(y_2) &= \sup_{(x_2) \in g^{-1}(y_2)} \tilde{\phi}_{\tilde{S}}(x_2) \\ &\geqslant r\min\left\{ \sup_{x_1 \in g^{-1}(y_1)} \tilde{\phi}_{\tilde{S}}(x_1), \sup_{(x_1 * x_2) \in g^{-1}(y_1 * y_2)} \tilde{\phi}_{\tilde{S}}(x_1 * x_2) \right\} \\ &= r\min\left\{ g(\tilde{\phi}_{\tilde{S}})(y_1), g(\tilde{\phi}_{\tilde{S}})((y_1 * y_2) \right\} \end{aligned}$$

□

Theorem 5.2.1.10. *Let $(X,*,1)$ and $(Y,*',1')$ be JU-algebras and a map $g : X \to Y$ be homomorphism. Let $\tilde{\phi}_{\tilde{S}}$ be an $\tilde{S}$-interval valued fuzzy subset of Y. If $\tilde{\phi}_{\tilde{S}}$ be an $\tilde{S}$-interval valued fuzzy JU-ideal of Y, then the Per-image $g^{-1}(\tilde{\phi}_{\tilde{S}})$ is an $\tilde{S}$-interval valued fuzzy JU-ideals of X.*

Proof. Let $g : X \to Y$ be a homomorphism of JU-algebras and let $\tilde{\phi}_{\tilde{S}}$ be an $\tilde{S}$-interval valued fuzzy JU-ideal of Y. We want to show that $g^{-1}(\tilde{\phi}_{\tilde{S}})$ is an $\tilde{S}$-interval valued fuzzy JU-ideal of X. For any $x \in X$ and $y \in Y$ such that $g(x) = y$. Since g is a homomorphism, $g(1) = 1'$. Then,

$g^{-1}(\tilde{\phi}_{\tilde{S}})(1) = \tilde{\phi}_{\tilde{S}}(g(1)) = \tilde{\phi}_{\tilde{S}}(1') \geqslant \tilde{\phi}_{\tilde{S}}(y) = \tilde{\phi}_{\tilde{S}}(g(x) = g^{-1}(\tilde{\phi}_{\tilde{S}})(x)$

And for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Then,

$$\begin{aligned} g^{-1}(\tilde{\phi}_{\tilde{S}})(x_2) = \tilde{\phi}_{\tilde{S}}(g(x_2)) = \tilde{\phi}_{\tilde{S}}(y_2) & \\ &\geqslant r\min\{\tilde{\phi}_{\tilde{S}}(y_1), \tilde{\phi}_{\tilde{S}}(y_1 *' y_2)\} \\ &= r\min\{\tilde{\phi}_{\tilde{S}}(g(x_1)), \tilde{\phi}_{\tilde{S}}(g(x_1) *' g(x_2))\} \\ &= r\min\{\tilde{\phi}_{\tilde{S}}(g(x_1)), \tilde{\phi}_{\tilde{S}}(g(x_1 * x_2))\} \quad \text{Since g is a homomorphism} \\ &= r\min\{g^{-1}(\tilde{\phi}_{\tilde{S}})(x_1), g^{-1}(\tilde{\phi}_{\tilde{S}})(x_1 * x_2)\} \end{aligned}$$

$\square$

Theorem 5.2.1.11. *The Cartesian product of any two $\tilde{S}$-interval valued fuzzy JU-ideals of X are also $\tilde{S}$-interval valued fuzzy JU-ideal of X.*

Proof. Suppose $\tilde{\phi}_1^{\tilde{s}}$ and $\tilde{\phi}_2^{\tilde{s}}$ be any two $\tilde{S}$-interval valued fuzzy JU-ideals of X. Let $(x,y) \in X \times X$. Then

$$(\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})(1,1) = r\min\{\tilde{\phi}_{\tilde{s_1}}(1), \tilde{\phi}_{\tilde{s_2}}(1)\} \geqslant \min\{\tilde{\phi}_{\tilde{s_1}}(x), \tilde{\phi}_{\tilde{s_2}}(y)\} = (\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})(x,y).$$

And, Let $(x_1,x_2),(y_1,y_2) \in X \times X$. Then

$$\begin{aligned}
(\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})(y_1,y_2) &= r\min\{\tilde{\phi}_{\tilde{s_1}}(y_1), \tilde{\phi}_{\tilde{s_2}}(y_2)\} \\
&\geqslant r\min\{r\min\{\tilde{\phi}_{\tilde{s_1}}(x_1), \tilde{\phi}_{\tilde{s_1}}(x_1 * y_1)\}, r\min\{\tilde{\phi}_{\tilde{s_2}}(x_2), \tilde{\phi}_{\tilde{s_2}}(x_1 * y_2)\}\} \\
&= r\min\{r\min\{\tilde{\phi}_{\tilde{s_1}}(x_1), \tilde{\phi}_{\tilde{s_2}}(x_2)\}, r\min\{\tilde{\phi}_{\tilde{s_1}}(x_1 * y_1), \tilde{\phi}_{\tilde{s_2}}(x_2 * y_2)\}\} \\
&= r\min\{(\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})((x_1,x_2)), (\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})((x_1 * y_1),(x_2 * y_2))\} \\
&= r\min\{(\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})(x_1,x_2), (\tilde{\phi}_{\tilde{s_1}} \times \tilde{\phi}_{\tilde{s_2}})((x_1,x_2) * (y_1,y_2))\}
\end{aligned}$$

$\square$

5.2.2 Interval valued ($\tilde{E}$,$\tilde{F}$)-Fuzzy JU-Ideals

In this section, we introduce the notion of interval valued ($\tilde{E}$,$\tilde{F}$)-fuzzy subalgebras of JU-algebras, and some of their properties are discussed.

Definition 5.2.2.1. *Let $\tilde{\phi}$ be an interval valued fuzzy set in X. Then $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X, with the binary operation $*$ if it satisfies the following axioms:*

a) $r\max\{\tilde{\phi}(1), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{F}\}$,

b) $r\max\{\tilde{\phi}(y), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y), \tilde{F}\}$, *for all* $x,y \in X$.

Example 5.2.2.1. *Let X* = $\{1,2,3,4,5,6\}$ *in which $*$ is defined by the following Cayley table:*

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	6
6	1	1	2	1	1	1

See [3] $(X;*,1)$ *is a JU-algebra. Define* $\tilde{\phi}(x)$ *as follows:*

$$\tilde{\phi}(x) = \begin{cases} [0.9,1] & \text{if } x = 1 \\ [0.6,0.7] & \text{if } x = 2 \\ [0.3,0.4] & \text{otherwise} \end{cases}$$

,with $E = [0.3,0.4]$ *and* $F = [0.6,0.7]$.
Let we check condition (i) that is, $r\max\{\tilde{\phi}(1),E\} \geqslant r\min\{\tilde{\phi}(x),F\}, \quad \forall x \in X$
For $x = 2$*, then* $r\max\{\tilde{\phi}(1),E\} = r\max\{[0.9,1],[0.3,0.4]\} = [0.9,1]\cdots(*)$
And, $r\min\{\tilde{\phi}(x),F\} = r\min\{[0.6,0.7],[0.6,0.7]\} = [0.6,0.7]\cdots(**)$
From (*) and (**) we get $[0.9,1] \geqslant [0.6,0.7]$ *which is true. Hence,* $r\max\{\tilde{\phi}(1),E\} \geqslant r\min\{\tilde{\phi}(x),F\}, \quad \forall x \in X$*. Thus condition (i) holds.*
Likewise, for Condition (ii)

$$r\max\{\tilde{\phi}(y),E\} \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(x*y),F\}, \quad \forall x,y \in X$$

For $x = 4$*,* $y = 2$

$$r\max\{\tilde{\phi}(y),E\} = r\max\{\tilde{\phi}(2),E\} = r\max\{[0.6,0.7],[0.3,0.4]\} = [0.6,0.7]\cdots(*)$$

And,

$$\begin{aligned} r\min\{\tilde{\phi}(x),\tilde{\phi}(x*y),F\} &= r\min\{\tilde{\phi}(4),\tilde{\phi}(4*2),F\} \\ &= r\min\{\tilde{\phi}(4),\tilde{\phi}(1),F\} \\ &= r\min\{[0.3,0.4],[0.9,1],[0.6,0.7]\} = [0.3,0.4]\cdots(**) \end{aligned}$$

From (*) and (**) we get $[0.6,0.7] \geqslant [0.3,0.4]$ *which is true.*
Similarly, the result holds for all values of x and y in X.
Therefore, we conclude that $\tilde{\phi}$ *is an interval valued* $([0.3,0.4],[0.6,0.7])$*-fuzzy JU-ideal of X.*

Theorem 5.2.2.1. *Every interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-ideals of X is order reversing.*

Proof. Let $\tilde{\phi}$ be an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideals of X and for any $x,y \in X$ be such that $x \leqslant y$, then $y * x = 1$. Now

$$\begin{aligned} r\max\{\tilde{\phi}(x),\tilde{E}\} &\geqslant r\min\{\tilde{\phi}(y*x),\tilde{\phi}(y),\tilde{F}\} \\ &= r\min\{\tilde{\phi}(1),\tilde{\phi}(y),\tilde{F}\} \\ &= r\min\{\tilde{\phi}(y),\tilde{F}\}. \end{aligned}$$

□

Theorem 5.2.2.2. *Let* ϕ *be an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-ideals of X. Then* $r\max\{\tilde{\phi}((x*y)*y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{F}\}$.

Proof. Since $(x*[(x*y)*y]) = 1$ (By Lemma 1.6.1)
Then, $r\max\{\tilde{\phi}((x*y)*y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(x*[(x*y)*y]),\tilde{F}\}$

$$= r\min\{\tilde{\phi}(x), \tilde{\phi}(1), \tilde{F}\}$$
$$= r\min\{\tilde{\phi}(x), \tilde{F}\}$$
□

Theorem 5.2.2.3. *Every interval valued* $(\tilde{E}, \tilde{F})$*-fuzzy JU-ideals of* X *is interval valued* $(\tilde{E}, \tilde{F})$*-fuzzy JU-subalgebra.*

Proof. Suppose $\tilde{\phi}$ is interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-ideals of X. That is:

a) $r\max\{\tilde{\phi}(1), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{F}\}$,

b) $r\max\{\tilde{\phi}(y), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y), \tilde{F}\}$, for all $x, y \in X$.

We claim that $\tilde{\phi}$ is an interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-subalgebra. That is, for all $x, y \in X$
$r\max\{\tilde{\phi}(x * y), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}, \tilde{F}\}$.
Assume that without loss of generality:
$\tilde{\phi}(x) \leqslant \tilde{\phi}(y) \Rightarrow r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\} = \tilde{\phi}(x)$
Now we consider two cases:

Case 1. If $\tilde{\phi}(x * y) \geqslant \tilde{\phi}(x)$, then

$r\max\{\tilde{\phi}(x * y), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{F}\} = r\min\{\tilde{\phi}(x), \tilde{\phi}(y), \tilde{F}\}$

$\Rightarrow r\max\{\tilde{\phi}(x * y), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y), \tilde{F}\}$

The inequality is satisfied directly.

Case 2. If $\tilde{\phi}(x * y) < \tilde{\phi}(x)$, then from the assumption that $\tilde{\phi}$ is an interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-ideal, apply condition (b):

$r\max\{\tilde{\phi}(y), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(x * y), \tilde{F}\} = r\min\{\tilde{\phi}(x * y), \tilde{F}\}$

$\Rightarrow r\max\{\tilde{\phi}(y), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x * y), \tilde{F}\}$

Since we also know, $\tilde{\phi}(x * y) < \tilde{\phi}(x)$ and $\tilde{\phi}(x) \leqslant \tilde{\phi}(y)$, it follows that

$\tilde{\phi}(x * y) < \tilde{\phi}(x) \leqslant \tilde{\phi}(y) \Rightarrow r\max\{\tilde{\phi}(x * y), \tilde{E}\} < r\min\{\tilde{\phi}(x), \tilde{\phi}(y), \tilde{F}\}$.

Which is contradict the inequality $r\max\{\tilde{\phi}(x * y), \tilde{E}\} \geqslant r\min\{\tilde{\phi}(x), \tilde{\phi}(y), \tilde{F}\}$

Thus, Case 1 must always hold, which implies:

$\tilde{\phi}(x * y) \geqslant \tilde{\phi}(x) = r\min\{\tilde{\phi}(x), \tilde{\phi}(y)\}$

Hence, $\tilde{\phi}$ is interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-subalgebra of X.

□

Remark 5.2.2.1. *The converse of the above theorem may not be true.*

Example 5.2.2.2. *Let X* = {1,2,3,4} *in which* $*$ *is defined by the following table.*

$*$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

See [3] $X;*,1)$*, is a JU-algebra. Define* $\tilde{\phi}(x)$ *as follows:*

$$\tilde{\phi}(x) = \begin{cases} [0.9,1] & \text{if } x = 1 \\ [0.7,0.8] & \text{if } x \in \{2,4\} \\ [0.4,0.5] & \text{if } x = 3 \end{cases}$$

Since, $\tilde{\phi}$ *is an interval valued ([0.2, 0.3],[0.6, 0.7])-fuzzy JU-subalgebra of X.*
The interval valued $(\tilde{E},\tilde{F})$*-fuzzy JU-ideal condition requires:*

$r\max\{\tilde{\phi}(y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\}$, *for all* $x,y \in X$

Let $x = 2$, $y = 3 \Rightarrow x*y = 4$ *and*
Then, $r\max\{\tilde{\phi}(y),\tilde{E}\} = r\max\{\tilde{\phi}(3),\tilde{E}\} = r\max\{[0.4,0.5],[0.2,0.3]\} = [0.4,0.5]\cdots(*)$ *and,*

$$\begin{aligned} r\min\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\} &= r\min\tilde{\phi}(2),\tilde{\phi}(2*4),\tilde{F}\} \\ &= r\min\tilde{\phi}(2),\tilde{\phi}(3),\tilde{F}\} \\ &= r\min\{[0.7,0.8],[0.7,0.8],[0.6,0.7]\} \\ &= [0.6,0.7]\cdots(**) \end{aligned}$$

From $(*)$ *and* $(**)$ *we get* $[0.4,0.5] \geqslant [0.6,0.7]$ *which is false.*

Theorem 5.2.2.4. *The intersection of any two an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-ideals of X is also an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-ideals of X.*

Proof. Suppose $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are any two an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideals of X and $\forall x,y \in X$. Then,

$$\begin{aligned} r\max\{\tilde{\phi_1}\cap\tilde{\phi_2}(1),\tilde{E}\} &= r\max\{r\min\{\tilde{\phi_1}(1),\tilde{\phi_2}(1)\},\tilde{E}\} \\ &= r\min\{r\max\{\tilde{\phi_1}(1),\tilde{E}\},r\max\{\tilde{\phi_2}(1),\tilde{E}\}\} \\ &\geqslant r\min\{r\min\{\tilde{\phi_1}(x),\tilde{F}\},r\min\{\tilde{\phi_2}(x),\tilde{F}\}\} \\ &= r\min\{r\min\{\tilde{\phi_1}(x),\tilde{\phi_2}(x)\},\tilde{F}\}\} \\ &= r\min\{\tilde{\phi_1}\cap\tilde{\phi_2}(x),\tilde{F}\} \end{aligned}$$

And,

$$\begin{aligned} r\max\{\tilde{\phi_1}\cap\tilde{\phi_2}(y),\tilde{E}\} &= r\max\{r\min\{\tilde{\phi_1}(y),\tilde{\phi_2}(y)\},\tilde{E}\} \\ &= r\min\{r\max\{\tilde{\phi_1}(y),\tilde{E}\},r\max\{\tilde{\phi_2}(y),\tilde{E}\}\} \\ &\quad \geqslant \{r\min\{r\min\{\tilde{\phi_1}(x),\tilde{\phi_1}(x*y),\tilde{F}\},r\min\{\tilde{\phi_2}(x),\tilde{\phi_2}(x*y),\tilde{F}\}\} \\ &= r\min\{r\min\{\tilde{\phi_1}(x),\tilde{\phi_2}(x)\},r\min\{\tilde{\phi_1}(x*y),\tilde{\phi_2}(x*y)\},\tilde{F}\} \\ &= r\min\{\tilde{\phi_1}\cap\tilde{\phi_2}(x),\tilde{\phi_1}\cap\tilde{\phi_2}(x*y),\tilde{F}\} \end{aligned}$$

□

Corollary 5.2.2.5. *Let* $\{\tilde{\phi}_i | i \in \Lambda\}$ *be a family of interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-ideals of X. Then* $\bigcap_{i\in\Lambda}\tilde{\phi}_i = rinf_{i\in\Lambda}\{x \in X|\tilde{\phi}_i(x)\}$ *is also an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-ideals of X.*

Proof. Let $\tilde{\phi}_i$ be interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideals of X, satisfying the conditions:

(i) $r\max\{\tilde{\phi}_i(1),\tilde{E}\} \geqslant r\min\{\tilde{\phi}_i(x),\tilde{F}\}, \quad \forall x \in X,$

(ii) $r\max\{\tilde{\phi}_i(y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}_i(x),\tilde{\phi}_i(x*y),\tilde{F}\}, \quad \forall x,y \in X,$

Condition (i)

For all $x \in X$, consider

$$\tilde{\phi}(x) = \inf_{i\in\Lambda}\tilde{\phi}_i(x).$$

Then

$$r\max\{\tilde{\phi}(1),\tilde{E}\} = r\max\left\{\inf_{i\in\Lambda}\tilde{\phi}_i(1),\tilde{E}\right\}.$$

Since each $\tilde{\phi}_i$ satisfies condition (i), we have

$$r\max\{\tilde{\phi}_i(1),\tilde{E}\} \geqslant r\min\{\tilde{\phi}_i(x),\tilde{F}\}, \quad \forall i \in \Lambda.$$

Taking the infimum over $i \in \Lambda$ on the right-hand side gives

$$r\max\left\{\inf_{i\in\Lambda}\tilde{\phi}_i(1),\tilde{E}\right\} \geqslant \inf_{i\in\Lambda} r\min\{\tilde{\phi}_i(x),\tilde{F}\} = r\min\{\tilde{\phi}(x),\tilde{F}\}.$$

Hence, condition (i) is satisfied by $\tilde{\phi}$.

Condition (ii) Similarly, for all $x,y \in X$, we have

$$\tilde{\phi}(x) = \inf_{i\in\Lambda}\tilde{\phi}_i(x), \quad \tilde{\phi}(y) = \inf_{i\in\Lambda}\tilde{\phi}_i(y), \quad \tilde{\phi}(x*y) = \inf_{i\in\Lambda}\tilde{\phi}_i(x*y).$$

Since each $\tilde{\phi}_i$ satisfies condition (ii), we have

$$r\max\{\tilde{\phi}_i(y),\tilde{E}\} \geqslant r\min\{\tilde{\phi}_i(x),\tilde{\phi}_i(x*y),\tilde{F}\}, \quad \forall i \in \Lambda.$$

Taking the infimum over $i \in \Lambda$ yields

$$r\max\{\tilde{\phi}(y),\tilde{E}\} = r\max\left\{\inf_{i\in\Lambda}\tilde{\phi}_i(y),\tilde{E}\right\} \geqslant \inf_{i\in\Lambda} r\min\{\tilde{\phi}_i(x),\tilde{\phi}_i(x*y),\tilde{F}\} = r\min\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\}.$$

Hence, condition (ii) is also satisfied by $\tilde{\phi}$.

Therefore, $\tilde{\phi} = \inf_{i\in\Lambda}\tilde{\phi}_i$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X. □ □

Theorem 5.2.2.6. *If an interval valued fuzzy subset* $\tilde{\phi}$ *is an interval valued* $(\tilde{E},\tilde{F})$*-fuzzy JU-ideal of X, then the set* $\tilde{\phi}_{\tilde{E}} = \{x \in X|\tilde{\phi}(x) \geqslant \tilde{E}\}$ *is a JU-ideal of X.*

Proof. Let $\tilde{\phi}$ be an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X and let $x, y \in X$. Then,
For any $x \in \tilde{\phi}_{\tilde{E}}$ we have $\tilde{\phi}(x) \geqslant \tilde{E}$. Since $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X, then

$rmax\{\tilde{\phi}(1),\tilde{E}\} \geqslant rmin\{\tilde{\phi}(x),\tilde{F}\} \geqslant \tilde{E}$.

$\Rightarrow \tilde{\phi}(1) \geqslant \tilde{E}$, hence $1 \in \tilde{\phi}_{\tilde{E}}$.

And, for any $x, x*y \in \tilde{\phi}_{\tilde{E}}$ implies that $\tilde{\phi}(x) \geqslant \tilde{E}$ and $\tilde{\phi}(x*y) \geqslant \tilde{E}$. Then,

$rmax\{\tilde{\phi}(y),\tilde{E}\} \geqslant rmin\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\} \geqslant rmin\{\tilde{E},\tilde{E},\tilde{F}\} = \tilde{E}$.

$\Rightarrow rmax\{\tilde{\phi}(y),\tilde{E}\} \geqslant \tilde{E}$

$\Rightarrow \tilde{\phi}(y) \geqslant \tilde{E}$, implies that $y \in \tilde{\phi}_{\tilde{E}}$. Hence, $\tilde{\phi}_{\tilde{E}}$ is a JU-ideal of X. □

Theorem 5.2.2.7. *A nonempty subset J of X is a JU-ideal of X if and only if the interval valued fuzzy subset $\tilde{\phi}$ of X defined as follows:*

$$\tilde{\phi}(x) = \begin{cases} \tilde{F} & \text{if } x \in J \\ \tilde{E} & \text{if } x \notin J, \end{cases}$$ *is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X.*

Proof. Suppose that J is a JU-ideal of X and let $x, y \in X$. Then consider the following cases.

Case 1. If $x, x*y \in J$, then $y \in J$, thus,

$rmax\{\tilde{\phi}(y),\tilde{E}\} \geqslant \tilde{F} = rmin\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\}$

Case 2. If $x \notin J$ or $x*y \notin J$, then

$rmax\{\tilde{\phi}(y),\tilde{E}\} \geqslant \tilde{E} = rmin\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\}$.

Case 3. If $x \notin J$ or $x*y \in J$, then

$rmax\{\tilde{\phi}(y),\tilde{E}\} \geqslant \tilde{E} = rmin\{\tilde{E},\tilde{F},\tilde{F}\} = rmin\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\}$.

Or, If $x \in J$ or $x*y \notin J$, then

$rmax\{\tilde{\phi}(y),\tilde{E}\} \geqslant \tilde{E} = rmin\{\tilde{F},\tilde{E},\tilde{F}\} = rmin\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\}$.

Consequently $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X.

Conversely, suppose that $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X. Let $x, x*y \in J$, then $\tilde{\phi}(x) \geqslant \tilde{F}$, $\tilde{\phi}(x*y) \geqslant \tilde{F}$. Now,

$rmax\{\tilde{\phi}(y),\tilde{E}\} \geqslant rmin\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\} \geqslant rmin\{\tilde{F},\tilde{F},\tilde{F}\} = \tilde{F}$.

$\Rightarrow rmax\{\tilde{\phi}(y),\tilde{E}\} \geqslant \tilde{F}$. (Since $\tilde{\phi}(y) \geqslant \tilde{E}$ (by Theorem 5.2.2.6)

$\Rightarrow \tilde{\phi}(y) \geqslant \tilde{F}$

$\Rightarrow y \in J$, hence J is a JU-ideal of X. □

Remark 5.2.2.2. *An interval valued fuzzy JU-ideal is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X when $\tilde{E} = [0,0]$ and $\tilde{F} = [1,1]$.*

Theorem 5.2.2.8. *Let $U(\tilde{\phi} : [t_1,t_2]) = \{x \in X | \tilde{\phi}(x) \geqslant [t_1,t_2]\}$ be nonempty set and $\forall [t_1,t_2] \in C[0,1]$. A fuzzy subset $\tilde{\phi}$ of a JU-algebra X is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X , if and only if $U(\tilde{\phi} : [t_1,t_2])$ is JU-ideal of X, for $\tilde{E} < [t_1,t_2] \leqslant \tilde{F}$.*

Proof. Suppose $\tilde{\phi}$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X. Let $[t_1,t_2] \in C[0,1]$ and $U(\tilde{\phi} : [t_1,t_2]) \neq \emptyset$. Let $x \in U(\tilde{\phi} : [t_1,t_2])$ implies that $\tilde{\phi}(x) \geqslant [t_1,t_2]$. Then,

$r\max\{\tilde{\phi}(1),\tilde{E}\} \geqslant r\min\{\tilde{\phi}(x),\tilde{F}\} = r\min\{[t_1,t_2],\tilde{F}\} = [t_1,t_2]$

$\Rightarrow 1 \in U(\tilde{\phi} : [t_1,t_2])$

And let $x, x*y \in U(\tilde{\phi} : [t_1,t_2])$ implies that $\tilde{\phi}(x) \geqslant [t_1,t_2]$ and $\tilde{\phi}(x*y) \geqslant [t_1,t_2]$. Then

$$\begin{aligned} r\max\{\tilde{\phi}(y),\tilde{E}\} &\geqslant r\min\{\tilde{\phi}(x),\tilde{\phi}(x*y),\tilde{F}\} \\ &= r\min\{[t_1,t_2],[t_1,t_2],\tilde{F}\} \\ &= [t_1,t_2] \end{aligned}$$

$\Rightarrow r\max\{\tilde{\phi}(y),\tilde{E}\} \geqslant [t_1,t_2]$. (Since $\tilde{\phi}(y) \geqslant \tilde{E}$ (by Theorem 5.2.2.6)

$\Rightarrow \tilde{\phi}(y) \geqslant [t_1,t_2]$

$\Rightarrow y \in U(\tilde{\phi} : [t_1,t_2])$

Hence $U(\tilde{\phi} : [t_1,t_2])$ is a JU-ideal of X for $\tilde{E} < [t_1,t_2] \leqslant \tilde{F}$.

Conversely, Suppose that $U(\tilde{\phi} : [t_1,t_2])$ is a JU-ideal of X for any $[t_1,t_2] \in C[0,1]$ and $U(\tilde{\phi} : [t_1,t_2]) \neq \emptyset$. Assume that $\tilde{\phi}$ is not an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X. Then there exist some $z_0, z_1 \in X$ such that

$r\max\{\tilde{\phi}(z_1),\tilde{E}\} < r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_0*z_1),\tilde{F}\}$

Let $\tilde{\phi}(z_0) = [\beta_1,\beta_2]$, $\tilde{\phi}(z_0*z_1) = [\beta_3,\beta_4]$, $\tilde{\phi}(z_1) = [t_1,t_2]$, and $\tilde{E} = [e_1,e_2], \tilde{F} = [f_1,f_2]$. Then

$r\max\{[t_1,t_2],[e_1,e_2]\} < r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4],[f_1,f_2]\}$.

$$\begin{aligned} \text{Take } [\alpha_1,\alpha_2] &= \tfrac{1}{2}(r\max\{\tilde{\phi}(z_1),\tilde{E}\} + r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_0*z_1),\tilde{F}\}) \\ &= \tfrac{1}{2}(r\max\{[t_1,t_2],[e_1,e_2]\} + r\min\{[\beta_1,\beta_2],[\beta_3,\beta_4],[f_1,f_2]\}) \\ &= \tfrac{1}{2}(\max\{t_1,e_1\},\max\{t_2,e_2\} + \min\{\beta_1,\beta_3,f_1\},\min\{\beta_2,\beta_4,f_2\}) \\ &= \tfrac{1}{2}(\max\{t_1,e_1\} + \min\{\beta_1,\beta_3,f_1\}), \tfrac{1}{2}(\max\{t_2,e_2\} + \min\{\beta_2,\beta_4,f_2\}) \end{aligned}$$

$\Rightarrow \min\{\beta_1,\beta_3,f_1\} > \alpha_1 = \frac{1}{2}(\max\{t_1,e_1\} + \min\{\beta_1,\beta_3,f_1\}) > \max\{t_1,e_1\}$ and

$\min\{\beta_2,\beta_4,f_2\} > \alpha_1 = \frac{1}{2}(\max\{t_2,e_2\} + \min\{\beta_2,\beta_4,f_2\}) > \max\{t_2,e_2\}$

$\Rightarrow (\min\{\beta_1,\beta_3,f_1\},\min\{\beta_2,\beta_4,f_2\}) > [\alpha_1,\alpha_2] > (\max\{t_1,e_1\},\max\{t_2,e_2\}) = r\max\{\tilde{\phi}(z_1),\tilde{E}\}$.

$\Rightarrow \tilde{\phi}(z_1) = [t_1,t_2] < [\alpha_1,\alpha_2]$

$\Rightarrow z_1 \notin U(\tilde{\phi} : [\alpha_1,\alpha_2])$ and

$\tilde{\phi}(z_0) = [\beta_1,\beta_2] \geqslant (\min\{\beta_1,\beta_3,f_1\},\min\{\beta_2,\beta_4,f_2\}) > [\alpha_1,\alpha_2]$, and

$\tilde{\phi}(z_0*z_1) = [\beta_3,\beta_4] \geqslant (\min\{\beta_1,\beta_3,f_1\},\min\{\beta_2,\beta_4\},f_2) > [\alpha_1,\alpha_2]$.

$\Rightarrow z_0, z_0*z_1 \in U(\tilde{\phi} : [\alpha_1,\alpha_2])$

Which is contradict to $U(\tilde{\phi} : [\alpha_1,\alpha_2])$ is a JU-ideal of X.

Hence, $r\max\{\tilde{\phi}(z_1,\tilde{E}\} \geqslant r\min\{\tilde{\phi}(z_0),\tilde{\phi}(z_0*z_1),\tilde{F}\}$ for any $z_0, z_1 \in X$. □

Theorem 5.2.2.9. *Let $(X,*,1)$ and $(Y,*',1')$ be JU- algebras and a map $g : X \to Y$ be an onto homomorphism and let $\tilde{\phi}$ be an interval valued $(\tilde{E},\tilde{F})$-fuzzy subset of X. If $\tilde{\phi}$ be an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X, then the image $g(\tilde{\phi})$, is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of Y with sup property.*

Proof. Let $g : X \to Y$ be an onto homomorphism of JU- algebra and let $\tilde{\phi}$ be an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal of X. We went to show that $g(\tilde{\phi})$ is an interval valued $(\tilde{E},\tilde{F})$-fuzzy JU-ideal in Y. Let $y \in Y$. Since g is onto, there exist $x \in X$ such that $g(x) = y$. Also g is a homomorphism, $g(1) = 1'$. Thus,

$$\begin{aligned} r\max\{g(\tilde{\phi})(1'), \tilde{E}\} &= r\max\{\sup_{1 \in g^{-1}(1')} \tilde{\phi}(1), \tilde{E}\} \\ &\geqslant r\min\{\sup_{x \in g^{-1}(y)} \tilde{\phi}(x), \tilde{E}\}. \\ &= r\min\{g(\tilde{\phi})(y), \tilde{E}\}. \end{aligned}$$

And, let $y_1, y_2 \in Y$. Since g is onto, there exist $x_1, x_2 \in X$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$. Also g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Now

$$\begin{aligned} r\max\{g(\tilde{\phi})(y_2), \tilde{E}\} &= r\max\{\sup_{x_2 \in g^{-1}(y_2)} \tilde{\phi}(x_2), \tilde{E}\} \\ &\geqslant r\min\{\sup_{x_1 \in g^{-1}(y_1)} \tilde{\phi}(x_1), \sup_{x_1 * x_2 \in g^{-1}(y_1 *' y_2)} \tilde{\phi}(x_1 * x_2), \tilde{F}\} \\ &= r\min\{g(\tilde{\phi})(y_1), g(\tilde{\phi})(y_1 *' y_2), \tilde{F}\} \end{aligned}$$

□

Theorem 5.2.2.10. *Let $(X, *, 1)$ and $(Y, *', 1')$ be JU-algebras and a map $g : X \to Y$ be homomorphism. Let $\tilde{\phi}$ be an interval valued $(\tilde{E}, \tilde{F})$-fuzzy subset of Y. If $\tilde{\phi}$ be an interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-ideal of Y, then the Per-image $g^{-1}(\tilde{\phi})$ is an interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-ideals of X.*

Proof. Let $g : X \to Y$ be a homomorphism of JU- algebra and let $\tilde{\phi}$ be an interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-ideal of Y. We went to show that $g^{-1}(\tilde{\phi})$ is an interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-ideal in X. For any $x \in X$ and $y \in Y$ such that $g(x) = y$. Since g is a homomorphism, $g(1) = 1'$. Then,

$$\begin{aligned} r\max\{g^{-1}(\tilde{\phi})(1), \tilde{E}\} &= r\max\{\tilde{\phi}(g(1)), \tilde{E}\} \\ &= r\max\{\tilde{\phi}(1'), \tilde{E}\} \\ &\geqslant r\min\{\tilde{\phi}(y), \tilde{F}\} \\ &= r\min\{\tilde{\phi}(g(x)), \tilde{F}\} \\ &= r\min\{g^{-1}(\tilde{\phi})(x), \tilde{F}\} \end{aligned}$$

And, for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since g is a homomorphism, $g(x_1 * x_2) = g(x_1) *' g(x_2) = y_1 *' y_2$. Then,

$$\begin{aligned} r\max\{g^{-1}(\tilde{\phi})(x_2), \tilde{E}\} &= r\max\{\tilde{\phi}(g(x_2)), \tilde{E}\} \\ &= r\max\{\tilde{\phi}(y_2), \tilde{E}\} \\ &\geqslant r\min\{\tilde{\phi}(y_1), \tilde{\phi}(y_1 *' y_2), \tilde{F}\} \\ &= r\min\{\tilde{\phi}(g(x_1)), \tilde{\phi}(g(x_1) *' g(x_2)), \tilde{F}\} \\ &= r\min\{\tilde{\phi}(g(x_1)), \tilde{\phi}(g(x_1 * x_2)), \tilde{F}\} \\ &= r\min\{g^{-1}(\tilde{\phi})(x_1), g^{-1}(\tilde{\phi})(x_1 * x_2), \tilde{F}\} \end{aligned}$$

□

Theorem 5.2.2.11. *The Cartesian product of any two interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-ideals of X is also interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU-ideal of X.*

Proof. : Suppose $\tilde{\phi}_1$ and $\tilde{\phi}_2$ be any two interval valued $(\tilde{E}, \tilde{F})$-fuzzy JU- ideals of X. Then let $(x, y) \in X \times X$. Now

$$\begin{aligned} r\max\{\tilde{\phi_1} \times \tilde{\phi_2}(1, 1), \tilde{E}\} &= r\max\{r\min\{\tilde{\phi_1}(1), \tilde{\phi_2}(1), \tilde{E}\}\} \\ &= r\min\{r\max\{\tilde{\phi_1}(1), \tilde{E}\}, r\max\{\tilde{\phi_2}(1), \tilde{E}\}\} \\ &\geqslant r\min\{r\min\{\tilde{\phi_1}(x), \tilde{F}\}, r\min\{\tilde{\phi_2}(y), \tilde{F}\}\} \\ &= r\min\{r\min\{\tilde{\phi_1}(x), \tilde{\phi_2}(y), \tilde{F}\}\} \\ &= r\min\{\tilde{\phi_1} \times \tilde{\phi_2}(x, y), \tilde{F}\} \end{aligned}$$

And, let $(x_1,x_2),(y_1,y_2) \in X \times X$. Then

$$\begin{aligned}
r\max\{\tilde{\phi_1} \times \tilde{\phi_2}(y_1,y_2),\tilde{E}\} &= r\max\{r\min\{\tilde{\phi_1}(y_1),\tilde{\phi_1}(y_2),\tilde{E}\}\} \\
&= r\min\{r\max\{\tilde{\phi_1}(y_1),\tilde{E}\},r\max\{\tilde{\phi_2}(y_2),\tilde{E}\}\} \\
&\geqslant r\min\{r\min\{\tilde{\phi_1}(x_1),\tilde{\phi_1}(x_1 * y_1),\tilde{F}\},r\min\{\tilde{\phi_2}(x_2),\tilde{\phi_2}(x_2 \\
&\quad * y_2),\tilde{F}\}\} \\
&= r\min\{r\min\{\tilde{\phi_1}(x_1),\tilde{\phi_2}(x_2),\tilde{F}\},r\min\{\tilde{\phi_1}(x_1 * y_1),\tilde{\phi_2}(x_2 \\
&\quad * y_2),\tilde{F}\}\} \\
&= r\min\{\tilde{\phi_1} \times \tilde{\phi_2}(x_1,x_2),\tilde{\phi_1} \times \tilde{\phi_2}((x_1,x_2) * (y_1,y_2)),\tilde{F}\}
\end{aligned}$$

□

Conclusion and Future Works

In this section, we summarize the key contributions of this dissertation and outline potential directions for future research.

I. Conclusion

This dissertation has undertaken a comprehensive exploration of fuzzy algebraic structures within the framework of JU-algebras. We have extended the concepts of fuzzy sets, Q-fuzzy sets, T-fuzzy sets, and interval-valued fuzzy sets to JU-algebras. Many interesting and important results about level subset, homomorphism, and Cartesian product of JU-subalgebras, JU-ideals, doubt relation, Normal, p-multiplicative, and p-translation in JU-algebra have been investigated.

The concept of JU-subalgebra and JU-ideal of a JU-algebra have been extended to fuzzy JU-subalgebra and fuzzy JU-ideal. We have studied the relation between fuzzy JU-subalgebras and fuzzy JU-ideals and investigated their algebraic behavior under operations such as intersection, homomorphism, and Cartesian product. Notably, we demonstrated that while the intersection of fuzzy JU-subalgebras (ideals) are preserved but the unions are not necessarily preserved, as illustrated by counter examples. We have proved that if a function f is a homomorphism from a JU-algebra X to a JU-algebra Y and if ϑ is a fuzzy JU-subalgebra (ideal) of Y, then the pre-image $f^{-1}(\vartheta)$ is a fuzzy JU-subalgebra(ideal) of X, respectively, and if ϑ is a fuzzy JU-subalgebra (ideal) of X, then the homomorphic image of $f(\vartheta)$) is a fuzzy JU-subalgebra(ideal) of Y, respectively. Also, we have proved that the Cartesian product of fuzzy JU-Subalgebras (ideals) is a fuzzy JU-Subalgebra(ideal) in JU-algebra, respectively. We have also studied the relations between fuzzy sets and fuzzy p-multiplicative in JU-algebra. We proved that the lower and upper level subsets of a fuzzy JU-subalgebra or fuzzy JU-ideal are also a JU-subalgebra or JU-ideal Moreover, we have proved some theorems using the concepts of the strongest fuzzy relation on fuzzy JU-subalgebras (ideals) in JU-algebra.

Finally, the notions of Q-fuzzy JU-subalgebras and Q-fuzzy JU-ideals in a JU-algebra have been introduced. We proved that the lower and upper level subsets of a Q-fuzzy JU-subalgebra or Q-fuzzy JU-ideal are also a JU-subalgebra or JU-ideals Next, we have proved that every JU-subalgebra (ideal) of a JU-algebra is also the level JU-subalgebra (ideal) of a Q-fuzzy JU-subalgebra (ideal) of a JU-algebra. The relationships between doubt Q-fuzzy and normal Q-fuzzy JU-subalgebras and ideals were also examined. We have proved for a Q-fuzzy JU-subalgebra (ideal) of η of a JU-algebra X, we can generate as a normal Q-fuzzy JU-subalgebra (ideal) of X which contains η. Moreover, the JU-homomorphic images and inverse images of Q-fuzzy JU-subalgebras (ideals) are also shown to be Q-fuzzy JU-subalgebras (ideals), respectively. Further, we have proved that the Cartesian product of Q-fuzzy JU-subalgebras (ideals) is a Q-fuzzy JU-subalgebra (ideal) in a JU-algebra, respectively.

Furthermore, we have introduced the concept of T-fuzzy JU-subalgebras and T-fuzzy JU-ideas of JU-algebras and obtained important results. We demonstrated that the intersection of fuzzy JU-subalgebras (ideals) are preserved but the unions are not necessarily preserved, as illustrated by counter examples. Also We proved that if every T-fuzzy JU-ideals has a finite image, then every descending chain of JU-ideals converges at finite step and every ascending chain of JU-ideals converges at finite step if and only if the set of values of any T-fuzzy JU-ideals is a well ordered subset of $[0,1]$. Moreover, the Cartesian product of any two T-fuzzy JU-subalgebras and T-fuzzy JU-ideals of a JU-algebra are also a T-fuzzy JU-subalgebra and T-fuzzy JU-ideal respectively.

We have introduced the concept of interval valued fuzzy JU-subalgebras and interval valued fuzzy JU-ideals of JU-algebras and obtained important results. We proved that every interval valued fuzzy JU-ideals of a JU-algebra is an interval valued fuzzy JU-subalgebra but the converse is not hold, as illustrated by counter examples. We proved that the lower and upper level subsets of a interval valued fuzzy JU-subalgebra or interval valued fuzzy JU-ideal are also a JU-subalgebra or JU-ideals. Also, we proved that the image and inverse image of interval valued fuzzy JU-subalgebra and JU-ideal under homomorphism preserve their respective interval valued fuzzy properties. Moreover, we have demonstrated that the Cartesian product of any two interval valued fuzzy JU-subalgebras and JU-ideals of JU-algebras are also interval valued fuzzy JU-subalgebra and JU-ideal respectively. Furthermore, we have introduced interval valued fuzzy sets and their refined variants, such as $\tilde{s}$-interval valued and $(\tilde{E},\tilde{F})$-interval valued fuzzy JU-subalgebras and ideals. Their properties were analyzed with respect to level subsets, homomorphisms, and Cartesian products.

II. Future Works

For future research, our results can be applied to various algebraic structures, including areas such as topology, coding theory, and so on. In our upcoming study of JU-algebras, we aim to extend our work to:

1. Bipolar Fuzzy Ideals in JU-Algebras
2. Soft Neutrosophic JU-Algebras
3. Fuzzy congruence relation of JU-algebra
4. Fuzzy Uniform topology space on JU-algebra

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