2025-11

Hesitant Fuzzy Algebraic Structures on Pseudo-TM Algebra with Multicriteria Decision Making

Girum, Alemayehu

BAHIR DAR UNIVERSITY
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

Hesitant Fuzzy Algebraic Structures on Pseudo-TM Algebra with Multicriteria Decision Making

By

Alemayehu Girum Melaku

NOVEMBER, 2025
BAHIR DAR, ETHIOPIA

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

Hesitant Fuzzy Algebraic Structures on Pseudo-TM Algebra with Multicriteria Decision Making

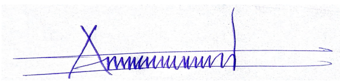
A Dissertation Submitted to the Department of Mathematics in Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

By
Alemayehu Girum Melaku

Supervisor: Berhanu Assaye Alaba (Ph.D. , Professor.)
Co-supervisor: Bekalu Tarekegn Bitew (Ph.D. , Assi. Professor.)

Declaration of Authorship

This is to certify that the dissertation entitled "**Hesitant Fuzzy Algebraic Structures on Pseudo-TM Algebra with Multicriteria Decision Making** ", submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, Department of Mathematics, College of Science, Bahir Dar University, is a record of original work carried out by me. and then it has not been submitted to this or any other institution for the award any other degree or certificates. The assistance and help I received during the course of this research have been duly acknowledged.



Alemayehu Girum Melaku		
Ph.D Candidate	Signature	Date

Bahir Dar University

College of Science

Department of Mathematics

Approval of Dissertation for Oral Defense

We hereby certify that we have supervised, read, and evaluated this dissertation entitled "**Hesitant Fuzzy Algebraic Structures on Pseudo-TM Algebra with Multicriteria Decision Making** " by **Alemayehu Girum Melaku** prepared under our guidance. We recommend the dissertation to be submitted for oral defense (mock-viva and viva voce).

Berhanu Assaye (Prof.)	______________	______________
Supervisor's name	Signature	Date
Bekalu Tarekegn (Dr.)	______________	______________
Co-supervisor's name	Signature	Date
Hungnaw Dessie(Dr.)	______________	______________
Department Head's name	Signature	Date

Bahir Dar University
Collage of Science
Department of Mathematics

Approval of Dissertation for Defense Result

As members of the board of examiners, we examined this dissertation entitled "**Hesitant Fuzzy Algebraic Structures on Pseudo-TM Algebra with Multicriteria Decision Making** " by **Alemayehu Girum Melaku**. We hereby certify that the dissertation is accepted for fulfilling the requirements for the award of the degree of doctor of philosophy in mathematics.

Board of Examiners

Sagi Srinivasa Raju (Ph.D., Professor.)		
External examiner's name 1	Signature	Date
Natnael Teshale (Ph.D., Assoc. Professor.)		
External examiner's name 2	Signature	Date
Yeshiwas Mebrat (Ph.D., Assoc. Professor.)		
Internal examiner's name	Signature	Date
Hunegnaw Dessie (Ph.D., Assoc. Professor.		
Chair person's name	Signature	Date

Dedication

I dedicate my dissertation work to my family. A special feeling of gratitude to my loving parents, Girum Melaku, Denkayehu Sinshaw, and my wife Asechalech Taye whose words of encouragement and push for tenacity ring in my ears.

Publications

This dissertation is based on the following works.

Published Paper

1. Melaku, A.G., Alaba, B.A., Bitew, B.T. & Derseh, B.L. Hesitant Fuzzy T-ideals of TM-Algebra. Palestine Journal of Mathematics (2025).

Accepted papers

1. Melaku, A.G., Alaba, B.A., Bitew, B.T. & Derseh, B.L. Fuzzy Pseudo-ideal in Pseudo-TM Alegbra. F1000Research (2024).
2. Melaku, A.G., Alaba, B.A., Bitew, B.T. & Derseh, B.L. Fuzzy PTM subalgebra of PTM - Algebra. F1000Research (2025).

Under Review Papers

1. Melaku, A.G., Alaba, B.A., Bitew, B.T. & Derseh, B.L. Hesitant fuzzy TM-subalgebra of TM-Algebra. Korean Journal of Mathematics. Submitted on Apr. 21, 2025
2. Melaku, A.G., Alaba, B.A., Bitew, B.T. & Derseh, B.L. Fuzzy Congruence Relation on Pseudo-TM Algebra. Thai Journal of Mathematics. Submitted on Dec. 12, 2024.
3. Melaku, A.G., Alaba, B.A., Bitew, B.T. & Derseh, B.L. Hesitant Fuzzy Pseudo Ideal of a Pseudo -TM algebra. Indian Journal of Pure and Applied Mathematics. Submitted on May 21, 2025.
4. Melaku, A.G., Alaba, B.A., Bitew, B.T. & Derseh, B.L. Hesitant Fuzzy Pseudo TM-Subalgebra of a pseudo TM-algebra. Discrete Mathematics Letters. Submitted on May 21, 2025.

Communicated Paper

1. Melaku, A.G., Alaba, B.A., Bitew, B.T. & Derseh, B.L. Bipolar Hesitant Fuzzy Soft Set in TM Algebra with Multicriteria Decision Making. Journal of Fuzzy Extension and Applications.

Conference Presentation

1. A.Girum. B. Assaye, B. Tarekgn and B. Lamesegn; Fuzzy PTM Subalgebra of PTM - Algebra. 5^{th} **African graduate students conference (AGSC IV–BDU–2025), which held from May 12–13, 2025, Bahir Dar University, Bahir Dar, Ethiopia.**

2. Alemayehu G. Berhanu A., Bekalu T. and Beza L.; Hesitant Fuzzy Pseudo TM-subalgebra of a Pseudo TM-Algebra. 11^{th} **International Aegean Congress on Innovation Technologies and Engineering April 4-6, 2025 Izmir, Türkiye.**

3. Alemayehu G. Berhanu A., Bekalu T. and Beza L.; Fuzzy PTM Subalgebra of PTM-Algebra. 11^{th} **International Aegean Congress on Innovation Technologies and Engineering April 4-6, 2025 Izmir, Türkiye.**

4. Alemayehu G. Berhanu A. Bekalu T& Beza L.; Fuzzy PTM Subalgebra of PTM - Algebra.. 1^{st} **INTERNATIONAL CYPRUS CONGRESS OF SCIENTIFIC RESEARCH 21-23 March 2025 / Near East University, Nicosia, TRNC**

5. Alemayehu G. Berhanu A. Bekalu T. and Beza L.; Hesitant Fuzzy TM Subalgebra of TM -Algebra. 2^{nd} **International Conference on Mathematics Applied in Life Sciences (ICMALS 2025) will be held virtually on July 3–4, 2025.**

6. Alemayehu G. Berhanu A. Bekalu T. and Beza L.; Fuzzy Pseudo-ideal in Pseudo-TM Algebra . **ATLAS 14^{th} INTERNATIONAL CONGRESS ON ADVANCED SCIENTIFIC STUDIES AND INTERDISCIPLINARY RESEARCH February 17-18, 2025 / Erzurum, Türkiye.**

7. Alemayehu G. Berhanu A. Bekalu T. and Beza L.; Hesitant Fuzzy Pseudo TM-Subalgebra of a Pseudo TM-Algebra 11th INTERNATIONAL AEGEAN CONGRESSES ON INNOVATION TECHNOLOGIES AND ENGINEERING, April 4-6, 2025 - Izmir, Türkiye.

Seminar Presentation

Alemayehu G. Berhanu A. Bekalu T.& Beza L. Hesitant fuzzy TM Subalgebra of TM Alegbra. **Friday seminar Presented on May 30, 2025, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.**

Acknowledgment

First and foremost, I extend my deepest gratitude to the Almighty God and His Blessed Mother, St. Mary, for their boundless grace, unwavering faithfulness, and unconditional love. Their divine guidance has accompanied me from the beginning of my academic journey to the successful completion of this doctoral dissertation.

This dissertation would not have been possible without the help, encouragement, and support of many individuals. While it is difficult to mention everyone, I am honored to express my heartfelt thanks to those who played a significant role in this journey.

I am profoundly grateful to my supervisors, Professor Birhanu Assaye, Dr. Bekalu Tarekegn, and Dr. Beza Lamesegen, for their exceptional mentorship, insightful guidance, and enduring patience. It has been a privilege to be under their supervision. Their vast knowledge and rich experience inspired and guided me throughout my Ph.D. study and dissertation preparation.

My sincere appreciation also goes to the Department of Mathematics, College of Science at Bahir Dar University, for creating an academic environment that nurtured my growth. I am especially thankful to Dr. Mihert Alemneh, Dr. Yohannes Gedamu , Dr. Yeshiwas Mebrat, Dr. Hunegnaw Dessie, Dr. Alachew Mechderso, Mr. Tarekegn Mitiku , and most warmly, Ms. Emebet, the department secretary, for her consistent kindness and support.

I would like to extend my gratitude to the Bahir Dar University for providing me with the opportunity to pursue this Ph.D. program. I am also indebted to Asmamaw Abebea Ph.D student at Bahir Dar University Department of mathematics and Dr. Alachew Mechderso, for his brotherly encouragement that helped me embark on this academic path and for the continued moral support he offered along the way. I am grateful for the friendship and moral support I received from my colleagues and friends at Bahir Dar University, including Zenaw Asenake, Abebe Worku, Dr. Mezgebu Manmekto, Dr.Nibirt Melese, Yimesgen Mehari, Asfaw Tsegaye, Dr. Tegegn Getachew, and Amare Worku. Finally, and most importantly, I owe everything to my beloved parents, family, and friends, whose prayers and constant encouragement sustained me. Without the prayers and love of my dear mother Denkayehu Sinshaw, my father Girum Melaku, My wife Asechalech Taye , my sisters Emebet Girum, Fentaye Girum, Zewuditu Girum, Berhanie Girum, Tihtena Girum and my child Bezawite Worku, Eyosiyas Alemayehu, and Mikyas Alemayehu, this journey and indeed every journey in my life would not have been possible.

Abstract

This dissertation presents an in-depth study of fuzzy algebraic structures, specifically focusing on TM-algebras and pseudo-TM algebras by using various extensions of fuzzy set theory such as hesitant fuzzy sets, and bipolar hesitant fuzzy soft sets. These theories help to deal with uncertainty, hesitation, and vagueness in mathematical modeling. In this study, we also define and investigate fuzzy subsets within pseudo-TM algebras. Important structures like fuzzy pseudo-TM subalgebras and fuzzy pseudo-TM ideals are introduced. Their properties are discussed using operations such as Cartesian product and homomorphism. It is shown that the intersection of two fuzzy pseudo-TM-subalgebras is also a fuzzy pseudo-TM subalgebra, but their union may not be. The study deals with the idea of fuzzy congruence relations more deeply. A fuzzy congruence relation is a fuzzy equivalence relation that respects the algebraic structure. It is shown how such relations can preserve the fuzzy structure under mappings and how they can be used to simplify the algebra into equivalent classes. The connection between fuzzy pseudo-ideals and fuzzy congruence relations is also discussed, providing a strong algebraic framework for fuzzy systems. The study moves from fuzzy sets to hesitant fuzzy sets. It introduces hesitant fuzzy TM-subalgebras, hesitant fuzzy T-ideals, hesitant fuzzy pseudo-TM subalgebras, and hesitant fuzzy pseudo-ideals. These structures allow multiple degrees of membership for each element, capturing hesitation in decision-making. Various properties of these structures are analyzed. It is shown that Cartesian products and homomorphic images of hesitant fuzzy ideals preserve the structure, under certain conditions. This provides a useful tool for modeling uncertain systems in mathematics and applications. The notions of bipolar hesitant fuzzy soft sets in TM-algebras are introduced. The combination of bipolarity (positive and negative views), hesitation (multiple values), and soft sets (parameter-based uncertainty) creates a powerful structure. These structures are applied to decision-making problems, especially when both satisfaction and dissatisfaction need to be considered. A numerical example is provided on selecting the best alcoholic drink based on multiple criteria, such as taste, health impact, and cost. Each criterion is evaluated with both positive and negative opinions, along with hesitation. The bipolar hesitant fuzzy soft set model is used to combine these opinions and find the best option. This shows the practical usefulness of the theoretical framework developed in this work. This research not only advances the theoretical understanding of fuzzy and hesitant fuzzy structures in algebra but also offers practical tools for modeling complex decision-making problems. The findings have potential applications in artificial intelligence, computer science, medical diagnosis, and other fields where human hesitation, dual opinions, and uncertainty are common.

Table of Contents

List of Symbols and Abbreviations

ψ, ω, ξ etc.	Fuzzy set, Fuzzy soft set, Bipolar fuzzy set,
h, σ, ξ_h	Hesitant fuzzy set, Hesitant fuzzy soft set , Bipolar hesitant fuzzy soft set
ϕ	Congurence Relation
ρ, θ, θ_1, θ_2, θ_3 etc	Fuzzy Relations
$\psi \times \omega$	Cartesian product of ψ and ω
$(\xi_h, A) \times (\xi_h, B)$	Cartesian product of hesitant fuzzy set, hesitant fuzzy soft set
$\psi(x)$	Degree of membership of the element x in the fuzzy set ψ
$\phi^{[x]}$	Neighborhood of a congurence relation ϕ on x
$\theta^{[x]}$	Neighborhood of a fuzzy congurence relation θ on x
$\xi_h^+(x)$	Positive degree of membership of the element x in the bipolar hesitant fuzzy soft set ξ_h
$\xi_h^-(x)$	Negative degree of membership of the element x in the bipolar hesitant fuzzy soft set ξ_h
h^c	Complement of h
χ_S	Fuzzy Characteristic function of S
$h_{\chi S}$	Characteristic hesitant fuzzy set of S
χ_{ξ_h}	Characteristic hesitant fuzzy soft set
$\cup$	Union
$\widetilde{\cup}$	Extended Union
$\cap$	Intersection
$\widetilde{\cap}$	Extended Intersection
$\subseteq$	Subset
$\in$	Belongs to
$\notin$	Doesn't belongs to
$\forall$	For all
$\mathbb{Z}$	The set of integer
$\mathbb{Q}$	The set of rational number

$U(\psi, t)$	Upper t-level set
$L(\psi, s)$	Lower s-level set
$U(h, \Lambda)$	Upper Λ-level set
$L(h, \Lambda)$	Lower Λ-level set
$U(\psi^+, t)$	Positive t-level set
$L(\psi^-, s)$	Negative s-level set
$\widetilde{BH}$	Bipolar hesitant fuzzy soft set
$f(\psi)$	Homomorphic image of ψ
$f^{-1}(\psi)$	Homomorphic inverse image of ψ
inf	Infimum
sup	Supremum
iff	If and only if
$Ker(f)$	Kernel of a function f
$Con(X)$	The set of all congruence on X
PTM	Pseudo-TM Algebra
FS	Fuzzy set
$FPTMA$	Fuzzy Pseudo TM-Algebra
$FPTMSA$	Fuzzy Pseudo TM-Subalgebra
$FPTMI$	Fuzzy Pseudo TM-Ideal
HFS	Hesitant Fuzzy set
$HFPTMA$	Hesitant Fuzzy Pseudo TM-Algebra
$HFPTMSA$	Hesitant Fuzzy Pseudo TM-Subalgebra
$CHFS$	Characteristic hesitant fuzzy set
$HFPTMSA$	Hesitant Fuzzy Pseudo TM-Subalgebra
$BHFS$	Bipolar hesitant fuzzy set

Introduction

Mathematics is a powerful tool used in many areas of life, from natural sciences to technology, economics, and decision-making. Classical mathematics is built on the idea that things are either true or false, or belong to a set or not. The study of classical set theory was first introduced by Georg Cantor in 1874 [12]. In classical set theory, each element of a set is either a member or not a member of the set. This type of set is called a crisp set (also known as a Cantor set). It is defined by a membership function that assigns each element a value of either 0 or 1 where 1 means the element belongs to the set, and 0 means it does not. However, in real-life situations, we often deal with uncertainty, hesitation, or vague conditions. To handle such situations, researchers introduced the idea of fuzzy sets. A fuzzy set allows an element to partly belong to a set with a membership value between 0 and 1. This concept was introduced by Zadeh in 1965 [72] and has since become a major topic of research in mathematics and applied sciences. Molodtsov in 2010 [31] introduced the concept of a soft set, which serves as a novel mathematical tool for addressing uncertainty. One of the key features of soft set theory is that it eliminates the need to define a membership function, making it highly adaptable and applicable across various fields. Soft set theory is a mathematical framework to deal with uncertainty and vagueness.

Later, different generalizations of fuzzy sets were developed to deal with more complex uncertainties. Atanassov's in 1986 [6] introduced intuitionistic fuzzy sets (IFS), Mendel in 2017 [45] introduced type-2 fuzzy sets (T2FS), Turksen in 1992 [66] introduced interval-valued fuzzy sets (IVFS) and Miyamoto in 2000 [46] introduced fuzzy multisets. Also , more generalized and flexible extension of the classical fuzzy set theory which allows for more expressive modeling of uncertainty called hesitant fuzzy set which is introduced by Torra in 2010 see [1, 56, 64, 65]. A hesitant fuzzy set allows multiple possible membership values for a single element, which models hesitation in the degree of belonging. The concept of hesitant fuzzy soft set was first introduced by Babitha and John in 2013 [7]. This model combines the ideas of soft sets and hesitant fuzzy sets to represent uncertain information in cases where there is hesitation in the membership degrees of elements with respect to certain parameters.

Real world decision making often involves both positive and negative judgments simultaneously. To capture the dual aspect of human thinking, the concept of bipolar fuzzy set was introduced by Zhang in 1994 [75]. In a bipolar fuzzy set, each element is associated with two membership degrees: A positive membership value in the interval $[0,1]$, representing the degree of satisfaction or agreement. A negative membership value in the interval $[-1,0]$, representing the degree of dissatisfaction or disagreement. This framework allows us to model complex situations where a decision-maker may have both support and opposition toward a particular element. To integrate soft set theory with bipolar fuzzy logic, researchers developed the concept of bipolar fuzzy soft set introduced by Abdullaha in 2014 [2]. This model provides a more flexible way to deal with parameter based decision-making problems that include both positive and negative information. In this structure, each parameter is associated with a bipolar fuzzy set over the universe. Further advancing this field, Zhang in 2013 [73] introduced the bipolar hesitant fuzzy soft set, which combines the ideas of bipolar fuzzy sets, hesitant fuzzy sets (which allow multiple membership degrees), and soft sets. In this model, each element under a given parameter is associated with a set of possible positive membership degrees from $[0,1]$ and a set of negative membership degrees from $[-1,0]$. This representation is especially useful in situations where the decision-maker hesitates among several degrees of satisfaction and dissatisfaction at the same time. These models bipolar fuzzy sets, bipolar fuzzy soft sets, and bipolar hesitant fuzzy soft sets have been widely applied in fields such as artificial intelligence, multicriteria decision making, medical diagnosis, and expert systems, where uncertainty, hesitation, and bipolar judgments frequently appear together. In addition to fuzzy structures, algebraic structures are another important area of mathematics. Algebraic systems like groups, rings, and algebras are used to understand logical operations and relationships among elements. Tanaka in 1978 [24] inroduced BCK-algebras and BCI-algebras introduced by Iséki in 1980 [25]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers in 2001 [51] introduced Q-algebras which is a generalization of BCK / BCI-algebras and several results are obtained. Megalai in 2010 [44] introduced a class of abstract algebras: TM-algebras which is a generalization of Q / BCK / BCI / BCH- algebras. TM-algebra can provide an algebraic framework for certain types of decision-making processes or learning algorithms. In artificial intelligence (AI), particularly in machine learning algorithms, understanding the algebraic properties of decision-making processes (e.g., decision trees, reinforcement learning) may involve the use of Turing machine models. Megalai in 2011 [43] introduced and explored various properties of fuzzy TM-subalgebras and fuzzy TM-ideals within TM-algebras. Prabpayak in 2017 [54] also presented the concept of homomorphisms in fuzzy TM-algebra and established several properties.

Georgescu in 2001 [19] proposed the concept of pseudo-BCK algebra as a generalization of BCK-algebra. Later, Dudek in 2008 [15] introduced the notion of pseudo-BCI algebra, which naturally extends both BCI-algebra and pseudo-BCK algebra. Jun in 2006 [27] introduced the notion of pseudo-BCI ideal (filter) of pseudo-BCI algebras. Romano in 2020 [57] introduced the concept of pseudo-UP ideals and pseudo-UP filters and derived basic properties. Nouri in 2019 [52] introduced the notion of Pseudo TM-Algebra which is an extension of TM-algebra. A pseudo-TM algebra is a generalization of TM-algebra, itself a generalization of several algebraic systems like BCK, BCI, Q-algebra, and BCH-algebra [See [52],[24], [25]].

The exploration of fuzzy subsets and their integration into various mathematical fields has led to the emergence of what is now known as fuzzy mathematics. One of the significant branches of this discipline is fuzzy algebra, which deals with the fuzzification of classical algebraic structures. These fuzzified forms are particularly relevant to the development of this thesis. Various researchers have contributed to this area, and key developments in the literature are summarized as follows: The

foundation of fuzzy algebraic structures was laid by Rosenfeld in 1971 [58], who introduced the concept of fuzzy subgroups using the idea of fuzzy sets within the context of group theory. He extended traditional group theory into the fuzzy setting by defining fuzzy subgroups and proving several important results. This work marked the beginning of fuzzifying classical algebraic systems. Following Rosenfeld's pioneering contribution, numerous scholars have devoted their efforts to extending the principles and results of abstract algebra into the fuzzy domain. However, it should be noted that not all classical results in group and ring theory can be directly translated or fuzzified in a consistent manner (see [30], [34]). Liu in 1982 [38] made a notable contribution by applying fuzzy set theory to ring theory . This was followed by Xi in 1991 [69], who extended fuzzy set concepts to BCK-algebras and introduced the idea of fuzzy ideals within these structures.
Murali in 1991 [50] introduced the concept of a fuzzy congruence relation, which generalizes classical congruence by assigning a degree of equivalence between elements of an algebraic structure, rather than relying on a strict binary classification. This approach facilitates the preservation of algebraic structure in a fuzzy or approximate sense and is particularly valuable in situations where rigid equivalence is too restrictive. Fuzzy congruences have been investigated in several standard algebraic systems, extending their classical analogs. In classical group theory, a congruence corresponds to a normal subgroup, leading to the construction of quotient groups. In the fuzzy case, fuzzy normal subgroups define equivalence classes with degrees of belonging, allowing for fuzzy quotient groups. Similarly, in semigroups, fuzzy congruences help identify fuzzy ideals and partition the semigroup into fuzzy equivalence classes, as explored by Mondal in 2012 [47]. In ring theory, congruence relations correspond to ideals. Fuzzy ideals and fuzzy congruences have been applied to extend the construction of quotient rings under uncertainty. These ideas also carry over to modules and vector spaces, where fuzzy linear subspaces and congruence relations help model vagueness in algebraic linear systems. In lattice theory, congruences help decompose complex structures. Fuzzy congruences in lattices play an important role in fuzzy logic, fuzzy ordering, and database theory, where elements may not be completely comparable or belong to a class with absolute certainty. In non-classical and logical algebraic structures like BCK and BCI-Algebras fuzzy congruences are linked with fuzzy ideals, and various studies have explored their behavior under fuzzy relations, especially for constructing fuzzy quotient BCK-algebras. In Pseudo-BCK and Pseudo-BCI-Algebras these are non-commutative generalizations of BCK/BCI-algebras, accommodating asymmetric or non-standard logical operations. Fuzzy congruence relations in these algebras involve intricate compatibility conditions due to relaxed axioms. They are important in non-classical logics and soft computing frameworks. Intuitionistic fuzzy congruences involve a pair of functions representing degrees of membership and non-membership, respectively. These relations are especially useful in settings with incomplete information. Bipolar fuzzy congruences incorporate both positive and negative degrees, ideal for modeling contradictory or dual-nature data. Such congruences can help study algebraic structures with opposing tendencies. Hesitant fuzzy congruences allow multiple membership degrees for a single pair, capturing the idea of hesitation or multiple expert opinions. These are particularly useful in decision-support systems and social network modeling. These advanced models support the construction of generalized fuzzy

quotient structures, allowing complex systems to be abstracted and analyzed under multifaceted uncertainty. Building upon this, Ahamed in 1993 [3] formulated the concept of fuzzy BCI-algebras, establishing their fundamental properties. Further developments include the work of Mostafa in 1995 [49], who defined fuzzy KU-ideals in KU-algebras and examined key properties, including the behavior of these ideals under homomorphic images and inverse images. He also demonstrated that the Cartesian product of fuzzy KU-ideals in the Cartesian product of KU-algebras results in another fuzzy KU-ideal. Somjanta in 2016 [62] introduced the concepts of fuzzy UP-subalgebras and fuzzy UP-ideals within the framework of UP-algebras, thereby enriching the field with new applications. The study of pseudo algebraic structures has also made progress in the fuzzy context. For example, Jun in 2003 [28] introduced the concept of fuzzy pseudo-ideals in pseudo BCK-algebras. Later, Lee in 2004 [37] extended this idea by fuzzifying pseudo ideals in pseudo BCI-algebras, offering further generalization of classical ideals into fuzzy settings. These foundational studies collectively form the basis for ongoing research in fuzzy algebra, enabling the application of fuzzy logic and set theory to more generalized algebraic structures. Jun in 2016 [29] introduced Hesitant fuzzy set theory applied to BCK/BCI-algebras. Bo in 2018 [74] investigated the lattice generated by hesitant fuzzy filters in pseudo-BCI algebras. Shao in 2018 [60] introduced neutrosophic hesitant fuzzy subalgebras and filters in pseudo-BCI algebras. Mechderso in 2023 [41] investigated the concept of fuzzy pseudo-UP ideals of pseudo-UP algebra. To the best of our knowledge, no research has yet explored the hesitant fuzzy structures of pseudo-TM algebra. This gap in the literature motivated us to initiate a study on hesitant fuzzy algebraic structures within the framework of pseudo -TM algebras.

The main goal of this dissertation is to study and develop hesitant fuzzy algebraic structures on pseudo-TM algebras. The study focuses on defining new types of fuzzy subalgebras, ideals, and congruence relations in the setting of pseudo-TM algebras and analyzing their properties. Furthermore, the study explores their generalizations to hesitant fuzzy, bipolar fuzzy, and soft set frameworks. These extended fuzzy algebraic structures provide better tools to handle uncertainty, hesitation, and decision-making problems.

This research is motivated by the need to combine algebraic logic with uncertain data. In many real-world problems, decisions need to be made with incomplete or vague information. For example, in selecting a suitable product, treatment, or plan, people often face multiple criteria with unclear values. Classical logic does not work well in these cases, but fuzzy logic and soft set theory offer better solutions. By building fuzzy structures on algebraic systems like pseudo-TM algebra and TM-algebera, we can model these problems more effectively.

The dissertation is organized into five chapters. Each chapter contributes to building a comprehensive theory of hesitant fuzzy algebraic structures on Pseudo-TM algebra with multicriteria decision making .

The first chapter presents the preliminary concepts, including definitions and properties of fuzzy sets, hesitant fuzzy sets, soft sets, and hesitant fuzzy soft sets. It also reviewes TM-algebras, pseudo TM-algebra and the newly introduced pseudo-TM algebras. Furthermore, subalgebras, ideals, homomorphisms, and Cartesian productsare provided to establish the background for the subsequent chapters.

In the second chapter focuses on fuzzy subsets in pseudo-TM algebras. Here, we define fuzzy pseudo-TM subalgebras and fuzzy pseudo-TM ideals. Their behavior under operations like Cartesian product and homomorphisms is studied. For instance, it is shown that the intersection of two fuzzy pseudo-TM subalgebras is again a fuzzy pseudo-TM subalgebra, but their union may not be. Fuzzy congruence relations are also introduced, and their link with fuzzy ideals is explained. This sections provides a foundational understanding of fuzzy logic in pseudo-TM algebraic systems.

In the third chapter develops the idea of fuzzy congruence relations on pseudo-TM algebras. It introduces new types of congruence relations in fuzzy settings and proves theorems about their properties. The study explains how these fuzzy congruence relations relate to fuzzy ideals and how they preserve algebraic structures under mappings. This is important because congruence relations help classify elements in an algebra based on equivalence, there by simplifying complex structures into more manageable parts.

In the fourth chapter introduces hesitant fuzzy TM-subalgebras, hesitant fuzzy T-ideals, and hesitant fuzzy pseudo-TM subalgebras. These are extensions of fuzzy sets by allowing hesitation in membership values. The properties of these new structures are explored using level subsets, Cartesian products, and homomorphisms. The hesitant fuzzy structures are shown to be more flexible in modeling situations where multiple opinions or data values exist. This chapter demonstrates the advantage of using hesitant fuzzy logic in algebraic systems.

In the fifth chapter applies the idea of hesitant fuzzy soft sets to TM-algebra. In particular, bipolar hesitant fuzzy soft sets are introduced, which consider both positive and negative membership values. These sets are powerful tools in decision-making since they capture both satisfaction and dissatisfaction levels. The chapter includes multicriteria a decision-making problem—selecting an alcoholic drink based on different attributes. A numerical example is provided to show how the proposed model can be used in real-world decision-making. This demonstrates the practical benefit of the theoretical framework developed in the earlier chapters.

Chapter 1
Preliminaries

In this chapter, we present a review of fundamental definitions and results that are essential for our study. The chapter is organized into six sections. Section 1 introduces basic concepts and results from fuzzy set theory. Section 2 reviews the foundational definitions and results related to hesitant fuzzy sets. Section 3 covers the core concepts of hesitant fuzzy soft sets and reintroduces the definition of hesitant fuzzy soft sets for clarity using examples. Section 4 outlines the definitions pertinent to bipolar hesitant fuzzy soft sets. Section 5 summarizes key definitions and results from algebraic structures such as BCK/BCI, Q, TM, and related algebras. Finally, Section 6 discusses the fundamental definitions and results associated with pseudo-TM algebras.

1.1. Fuzzy Set Theory

In this section, we review and introduce some basic concepts and results from fuzzy set theory.

Definition 1.1.1. *[72] Let ψ be a fuzzy subset of a nonempty set X. This means that ψ is a function $\psi : X \to [0, 1]$ called a membership function. For each element $x \in X$, the value $\psi(x)$ represents the degree of membership of x in the fuzzy subset defined by ψ. The complement of ψ denoted by ψ^c, where $\psi^c = 1 - \psi(x)$ for all $x \in X$.*

Definition 1.1.2. *[72] Let ψ and ω be any two fuzzy subset of X. Then we have the following:*

(i). $\psi \subseteq \omega$ *if and only if* $\psi(x) \leqslant \omega(x)$, *for all* $x \in X$,

(ii). $\psi = \omega$ *if and only if* $\psi(x) = \omega(x)$, *for all* $x \in X$,

(iii). $(\psi \cap \omega)(x) = \min\{\psi(x), \omega(x)\}$. *for all* $x \in X$,

(iv). $(\psi \cup \omega)(x) = \max\{\psi(x), \omega(x)\}$, *for all* $x \in X$.

Definition 1.1.3. *[71] Let ψ be a fuzzy set in X. For a fixed $t \in [0,1]$, the set $U(\psi, t) = \psi_t = \{x \in X/\psi(x) \geqslant t\}$ is called an upper level of ψ. And , $L(\psi, t) = \psi_t = \{x \in X/\psi(x) \leqslant t\}$ is called a lower level of ψ. Also, we can define upper-strong and lower-strong level subset of ψ respectively as:*
$U^+(\psi, t) = \psi_t = \{x \in X/\psi(x) > t\}$ *and* $L^-(\psi, t) = \psi_t = \{x \in X/\psi(x) < t\}$.

Remark 1.1.1. *[72] If* $t_1 \leqslant t_2$*, then* $U(\psi, t_2) \subseteq U(\psi, t_1)$ *and* $L(\psi, t_1) \subseteq L(\psi, t_2)$*, for all* $t_1, t_2 \in [0,1]$.

Definition 1.1.4. *[61] Let* $f : X \to Y$ *be a mapping where* X *and* Y *are non-empty subset. If* ψ *and* ω *be a fuzzy subset of* X *and* Y *respectively. Then the image of* ψ *under* f *denoted by* $f(\psi)$ *is a fuzzy subset of* Y *such that*

$$f(\psi)(y) = \begin{cases} \sup\{\psi(x)/x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \quad \text{,where } y \in Y$$

The inverse image of ω *under* f *is denoted by* $f^{-1}(\omega)$ *is a fuzzy subset of* Y *defined by* $f^{-1}(\omega)(x) = \omega(f(x))$*, for all* $x \in X$.

Definition 1.1.5. *[4] Let* f *be a function from* X *to* Y*, and* ψ*,* ω *be fuzzy subset of* X*. Then*

1. $f(f^{-1}(\psi)) = \psi$.
2. $\psi \subseteq f^{-1}(f(\psi))$.
3. $\psi \subseteq \omega \Rightarrow f(\psi) \subseteq f(\omega)$.

Definition 1.1.6 ([72])**.** *Let* $\{\psi_i : i \in I\}$ *be an indexed collection of fuzzy subsets of* X*. Then, the fuzzy subsets* $\cap_{i\in I}\psi_i$ *and* $\cup_{i\in I}\psi_i$ *are defined by:*

1. $(\cap_{i\in I}\psi_i)(x) = \inf_{i\in I}\{\psi_i(x)\}$, *for all* $x \in X$.
2. $(\cup_{i\in I}\psi_i)(x) = \sup_{i\in I}\{\psi_i(x)\}$, *for all* $x \in X$.

Lemma 1.1.1. *[71] Let* ψ *be any fuzzy subset of* X*. Then the following holds true for all* $x, y \in X$.

1. $1 - \min\{\psi(x), \psi(y)\} = \max\{1 - \psi(x), 1 - \psi(y)\}$.
2. $1 - \max\{\psi(x), \psi(y)\} = \min\{1 - \psi(x), 1 - \psi(y)\}$.

Definition 1.1.7. *[9] Let* ψ *and* ω *be any fuzzy subsets of* X *and* Y*, respectively. Then the Cartesian product* $\psi \times \omega$ *with membership function* $\psi \times \omega : X \times Y \to [0,1]$ *is defined by* $(\psi \times \omega)(x,y) = \min\{\psi(x), \omega(y)\}$*, for all* $(x,y) \in X \times Y$.

Definition 1.1.8. *[9] A fuzzy relation on a nonempty set* X *is a fuzzy set* θ *with a membership function* $\theta : X \times X \to [0,1]$.

Definition 1.1.9. *[9] Let* ψ *be a fuzzy subset on a set* X*, then the strongest fuzzy relation on* X*, that is, a fuzzy relation* θ *on* ψ *and whose membership function* $\theta : X \times X \to [0,1]$ *is given by* $\theta(x,y) = \min\{\psi(x), \psi(y)\}$*, for all* $x, y \in X$.

Definition 1.1.10. *[58] A fuzzy subset* ψ *in* X *is said to have the sup-property if for any subset* $T \subseteq X$ *there exist* $x_0 \in T$ *such that* $\psi(x_0) = \sup_{t\in T} \psi(t)$.

Definition 1.1.11. *[58] Let $f: X \to Y$ be a mapping. A fuzzy subset ψ of X is said to be f-invariant, if $f(x) = f(y)$ implies $\psi(x) = \psi(y)$, where $x, y \in X$.*

Definition 1.1.12. *[23] Let θ be an equivalence relation on the set X. Then θ is called congruence relation on X if it satisfies the following axioms: for all $x, y, z \in X$*

1. $(x,y) \in \theta \Longrightarrow x \star z,\ y \star z \in \theta$*(right compatible),*

2. $(x,y) \in \theta \Longrightarrow z \star x,\ z \star y \in \theta$*(left compatible.)*

Definition 1.1.13. *[4] A fuzzy relation θ on X is said to be:*

1. *Reflexive, if $\theta(x,x) = 1$, for all $x \in X$.*

2. *Symmetric, if $\theta(x,y) = \theta(y,x)$, for all $x,\ y \in X$.*

3. *Transitive, if $\theta(x,z) \geqslant \mathrm{Sup}_{z \in X}\{\min\{\theta(x,y), \theta(y,z)\}\}$, for all $x,\ y,\ z \in X$.*

Definition 1.1.14. *[4] Suppose θ_1 and θ_2 are two fuzzy relations on X. Then their composition denoted by $\theta_1 \circ \theta_2$ is defined as*

$$(\theta_1 \circ \theta_2)(x,z) = \sup_{y \in X}\{\min\{\theta_1(x,y), \theta_2(y,z)\}\}$$

If $\theta_1 = \theta_2 = \theta$ and $\theta \circ \theta \subseteq \theta$, then the fuzzy relation ψ is called transitive.

A fuzzy relation θ on X is said to be a fuzzy equivalence relation if ψ is reflexive, symmetric and transitive.

1.2. Hesitant Fuzzy Set

In this section, we recall basic definitions and concepts of hesitant fuzzy sets of a non-empty set X.

Definition 1.2.1. *[33] Let X be a reference set, a hesitant fuzzy set h on X denoted by (HFS(X)) is defined in terms of a function h that when applied to X returns a subset of $[0,1]$.*

Definition 1.2.2. *[48] Let h be a hesitant fuzzy set on X. The hesitant fuzzy set h^c defined by $h^c(x) = [0,1] - h(x)$, for all $x \in X$ is said to be the complement of h on X. For all hesitant fuzzy set h on X we have $(h^c)^c = h$.*

[70] expressed an HFS(X) by the following mathematical expression:

$$H = \{\langle x, h(x)\rangle / x \in X\}$$

where $h(x)$ is a set of some values in $[0,1]$ denoting the possible membership degrees of the element $x \in X$ to the set H. For agreements, [70] called $h(x)$ a hesitant fuzzy element HFE(X).
In H as special cases: $h^0 = \{\langle x, \{0\}\rangle \mid \forall x \in X\}$ is the empty hesitant set, $h^1 = \{\langle x, \{1\}\rangle \mid \forall x \in X\}$ is the full hesitant set, $h^{[0,1]} = \{\langle x, [0,1]\rangle \mid \forall x \in X\}$ is the set to represent complete ignorance for $x \in X$ and $h^{\emptyset} = \{\langle x, \emptyset\rangle \mid \forall x \in X\}$ is the non-sense set [53].

Definition 1.2.3. *[64] Let* h, h_1 *and* $h_2 \in HFS(X)$. *Then for all* x *in* X. *We have:*

1. *Empty set:* $h(x) = \{0\}$.
2. *Full set:* $h(x) = \{1\}$.
3. *Complete ignorance (all is possible):* $h(x) = [0,1]$.
4. *Set for a non-sense* $x : h(x) = \emptyset$.
5. *Lower bound:* $h^-(x) = min\, h(x)$.
6. *Upper bound:* $h^+(x) = max\, h(x)$.
7. *Involutive:* $(h^c(x))^c = h(x)$.
8. *α-upper bound:* $h^+_\alpha(x) = \{h \in h(x) \mid h \geqslant \alpha\}$
9. *α-lower bound:* $h^-_\alpha(x) = \{h \in h(x) \mid h \leqslant \alpha\}$. *α-upper bound and α-lower bounds are a crucial requirement for determining the union and intersection of HFS(X). That is*
10. $(h_1 \cup h_2)(x) = \left\{h \in (h_1(x) \cup h_2(x)) \mid h \geqslant \max\left(h_1^-, h_2^-\right)\right\} = (h_1(x) \cup h_2(x))^+_\alpha$ *for* $\alpha = \max\left(h_1^-(x), h_2^-(x)\right)$.
11. $(h_1 \cap h_2)(x) = \left\{h \in (h_1(x) \cap h_2(x)) \mid h \leqslant \min\left(h_1^+, h_2^+\right)\right\} = (h_1(x) \cap h_2(x))^-_\alpha$ *for* $\alpha = \min\left(h_1^+(x), h_2^+(x)\right)$.
12. $h_1 \otimes h_2 = \bigcup_{\substack{\gamma_1 \in h_1 \\ \gamma_2 \in h_2}} \{\gamma_1 \gamma_2\}$.
13. $h_1 \oplus h_2 = \bigcup_{\substack{\gamma_1 \in h_1 \\ \gamma_2 \in h_2}} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$

Lemma 1.2.1. *[45] Let* $H = \{(x, h(x)) | x \in X\}$ *be a hesitant fuzzy set on* X. *Then the following statements hold: for all* $x, y \in X$,

(1). $[0,1] - (h(x) \cup h(y)) = ([0,1] - h(x)) \cap ([0,1] - h(y))$,

(2). $[0,1] - (h(x) \cap h(y)) = ([0,1] - h(x)) \cup ([0,1] - h(y))$.

Definition 1.2.4. *[32] Let* $h_A = (x, h_A(x))$ *and* $h_B = (x, h_B(x))$ *be a hesitant fuzzy sets on* X *and* Y, *respectively. Then the Cartesian product* $h_A \times h_B = (x, h(x))$ *defined by* $h(x, y) = h_A(x) \cap h_B(y)$ *, where A and B are a subset of X and Y respectively and* $h : X \times Y \to \mathcal{P}[0,1]$ *for all* $x \in X$ *and* $y \in Y$.

Definition 1.2.5. *[32] Let* $(h_i)_{i \in I} \subseteq HFS(X)$ *be an indexed collection of finite hesitant fuzzy sets. Then*

1. *the union of* $(h_i)_{i\in I}$ *denoted by* $\bigcup_{i\in I} h_i$*, is a hesitant fuzzy set in* X *defined as follows: for all* $x \in X$*:*

$$\left(\bigcup_{i\in I} h_i\right)(x) = \bigcup_{\gamma_i \in h_i(x)} \bigvee_{i\in I} \gamma_i.$$

2. *the intersection of* $(h_i)_{i\in I}$ *denoted by* $\bigcap_{i\in I} h_i$ *is a hesitant fuzzy set in* X *defined as follows: for all* $x \in X$

$$\left(\bigcap_{i\in I} h_i\right)(x) = \bigcup_{\gamma_i \in h_i(x)} \bigwedge_{i\in I} \gamma_i.$$

[32] expressed images and pre-images of HFS(X) and HFS(Y) in the following ways:
Let X *and* Y *be a non-empty sets. For any* $h_X \in \mathrm{HFS}(X)$ *and* $h_Y \in \mathrm{HFS}(Y))$ *and* $f : X \to Y$ *be a mapping. Then*

1. *the image of* h_X *under* f *denoted by* $f(h_X)$ *is a hesitant fuzzy set in* Y *defined as follows: for each* $y \in Y$*,*

$$f(h_X)(y) = \begin{cases} \bigcup_{x\in f^{-1}(y)} h_X(x) & \text{if } f^{-1}(y) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

2. *the pre-image of* h_Y *under* f *denoted by* $f^{-1}(h_Y)$ *is a hesitant fuzzy set in* Y *defined as follows: for all* $x \in X$

$$f^{-1}(h_Y)(x) = h_Y \circ f(x).$$

Example 1.2.1. *[53] Let* $X = \{x_1, x_2, x_3\}$ *be the reference set and let* $h_1, h_2 \in \mathrm{HFS}(X)$ *be two hesitant fuzzy sets such that*

$$h_1 = \{\langle x_1, \{0.3, 0.4\}\rangle, \langle x_2, \{0.6, 0.8\}\rangle, \langle x_3, \{0.3, 0.4, 0.5, 0.7\}\rangle\},$$

and

$$h_2 = \{\langle x_1, \{0.5, 0.6\}\rangle, \langle x_2, \{0.4, 0.5\}\rangle, \langle x_3, \{0.2, 0.3, 0.4, 0.6\}\rangle\}.$$

Then we have

1. $(h_1)^-(x_1) = \min\{0.3, 0.4\} = 0.3, (h_1)^+(x_1) = \max\{0.3, 0.4\} = 0.4$
2. $(h_1)^+_{0.3}(x_1) = \{\gamma \in h_1(x_1) \mid \gamma \geqslant 0.3\} = \{0.3, 0.4\} = h_1(x_1)$*;*
3. $(h_1)^-_{0.4}(x_1) = \{\gamma \in h_1(x_1) \mid \gamma \leqslant 0.4\} = \{0.3, 0.4\} = h_1(x_1)$
4. $(h_1)^-_{0.45}(x_3) = \{\gamma \in h_1(x_3) \mid \gamma \leqslant 0.45\} = \{0.3, 0.4\}$*;*
5. $(h_1)^+_{0.45}(x_3) = \{\gamma \in h_1(x_3) \mid \gamma \geqslant 0.45\} = \{0.5, 0.7\}$*;*
6. $(h_1)^c(x_2) = \cup_{\gamma \in h_1(x_2)}\{1-\gamma\} = \{1-0.6, 1-0.8\} = \{0.4, 0.2\}$

7. $(h_1 \cup h_2)(x_3) = \{\gamma \in h_1(x_3) \cup h_2(x_3) \mid \gamma \geqslant \max\{(h_1)^-(x_3), (h_2)^-(x_3)\}\}$ $= \{\gamma \in h_1(x_3) \cup h_2(x_3) \mid \gamma \geqslant \max\{0.3, 0.2\}\} = \{0.3, 0.4, 0.5, 0.6, 0.7\}$;

8. $(h_1 \cap h_2)(x_3) = \{\gamma \in h_1(x_3) \cup h_2(x_3) \mid \gamma \leqslant \min\{(h_1)^+(x_3), (h_2)^+(x_3)\}\} = \{\gamma \in h_1(x_3) \cup h_2(x_3) \mid \gamma \leqslant \min\{0.7, 0.6\}\} = \{0.2, 0.3, 0.4, 0.5, 0.6\}$;

9. $(h_1 \oplus h_2)(x_1) = \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 \mid \gamma_1 \in h_1(x_1), \gamma_2 \in h_2(x_1)\} = \{0.65, 0.72, 0.7, 0.76\}$;

10. $(h_1 \otimes h_2)(x_1) = \{\gamma_1\gamma_2 \mid \gamma_1 \in h_1(x_1), \gamma_2 \in h_2(x_1)\} = \{0.15, 0.18, 0.2, 0.24\}$.

1.3. Hesitant Fuzzy Soft Set

Now, we recall some fundamental definitions from hesitant fuzzy soft set theory. Molodtsov [31] introduced the concept of a soft set, which serves as a novel mathematical tool for addressing uncertainty. One of the key features of soft set theory is that it eliminates the need to define a membership function, making it highly adaptable and applicable across various fields.

Definition 1.3.1. *[11] Let $\mathcal{U}$ be refer to an initial universe, E be a set of parameters, $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$, and $A \subseteq E$. A pair (F, A) is called a soft set over $\mathcal{U}$ where F is a mapping given by $F : A \to \mathcal{P}(\mathcal{U})$.*

Definition 1.3.2. *[11] A fuzzy soft set ξ_A over $\mathcal{P}(\mathcal{U})$ is a set defined by a function ξ_A representing a mapping*

$$\xi_A : E \to F(U) \text{ such that } \xi_A(x) = \emptyset \text{ if } x \notin A.$$

Here, ξ_A is called fuzzy approximate function of the fuzzy soft set ξ_A, and the value $\xi_A(x)$ is a set called x-element of the fuzzy set for all $x \in E$. Thus, a fuzzy set ξ_A over $\mathcal{U}$ can be represented by the set of ordered pairs

$$\xi_A = \{(x, \xi_A(x)) : x \in E,\ \xi_A(x) \in \mathcal{P}(\mathcal{U})\}$$

Example 1.3.1. *[11] Let the universe of discourse be $\mathcal{U} = \{apple,\ banana,\ orange\}$ and the set of parameters be $E = \{sweet,\ ripe,\ cheap\}$.*
The subset of parameters under consideration be $A = \{sweet,\ cheap\} \subseteq E$
Now, define the fuzzy soft set $\xi_A : E \to F(U)$ such that:
For $x \in A$, $\xi_A(x)$ is a fuzzy subset of $\mathcal{U}$ and for $x \notin A$, $\xi_A(x) = \emptyset$.
Then, we can define:

$$\xi_A(sweet) = \{(apple,\ 0.9),\ (banana,\ 0.8),\ (orange,\ 0.6)\}$$
$$\xi_A(cheap) = \{(apple,\ 0.5),\ (banana,\ 0.7),\ (orange,\ 0.4)\}$$
$$\xi_A(ripe) = \emptyset$$

We can representation of the Fuzzy Soft Set as follows:

$$\xi_A = \left\{ \begin{array}{l} (sweet, \{(apple, 0.9), (banana, 0.8), (orange, 0.6)\}), \\ (cheap, \{(apple, 0.5), (banana, 0.7), (orange, 0.4)\}) \end{array} \right\}$$

The fuzzy soft set ξ_A describes the fuzziness of fruits with respect to some parameters (like sweet and cheap). For parameters not in A (like ripe), the fuzzy value is empty. Each fuzzy set value represents degree of membership of elements in $\mathcal{U}$ under a particular parameter.

Definition 1.3.3. *[39] For two fuzzy soft sets (ξ_1, A) and (ξ_2, B) in a soft class ($\mathcal{U}, E$), we say that*

1. *(ξ_1, A) is a fuzzy soft subset of (ξ_2, B), if*

   (i). $A \subseteq B$

   (ii). $\xi_1(\epsilon) \subseteq \xi_2(\epsilon)$ *and is written as* $(\xi_1, A) \subseteq (\xi_2, B)$, *for all* $\epsilon \in A$.

2. *The extended union of two fuzzy soft sets (ξ_1, A) and (ξ_2, B) in a soft class ($\mathcal{U}, E$) is a fuzzy soft set (ξ, C), where $C = A \cup B$ and, for all $\epsilon \in C$,*

$$\xi(\epsilon) = \begin{cases} \xi_1(\epsilon), & \text{if } \epsilon \in A - B \\ \xi_2(\epsilon), & \text{if } \epsilon \in B - A \\ \xi_1(\epsilon) \cup \xi_2(\epsilon), & \text{if } \epsilon \in A \cap B \end{cases}$$

   and is written as $(\xi, A) \widetilde{\cup} (\xi_2, B) = (\xi, C)$.

3. *The extended intersection of two fuzzy soft sets (ξ_1, A) and (ξ_2, B) in a soft class ($\mathcal{U}, E$) is a fuzzy soft set (ξ, C) where $C = A \cap B$ and $\forall \epsilon \in C$,*

$$\xi(\epsilon) = \begin{cases} \xi_1(\epsilon), & \text{if } \epsilon \in A - B \\ \xi_2(\epsilon), & \text{if } \epsilon \in B - A \\ \xi_1(\epsilon) \cap \xi_2(\epsilon), & \text{if } \epsilon \in A \cap B \end{cases}$$

   and is written as $(\xi_1, A) \widetilde{\cap} (\xi_2, B) = (\xi, C)$.

4. *(ξ_1, A) AND (ξ_2, B) is a fuzzy soft set denoted by $(\xi_1, A) \wedge (\xi_2, B)$ and is defined by $(\xi_1, A) \wedge (\xi_2, B) = (H, A \times B)$, where $H(x, y) = \xi_1(x) \cap \xi_2(y)$, for all $x \in A$ and, for all $y \in B$, where $\cap$ is the operation intersection of two fuzzy sets.*

5. *(ξ_1, A) OR (ξ_2, B) is a fuzzy soft set denoted by $(\xi_1, A) \vee (\xi_2, B)$ and is defined by $(\xi_1, A) \vee (\xi_2, B) = (K, A \times B)$, where $K(x, y) = \xi_1(x) \cup \xi_2(y)$, for all $x \in A$ and, for all $y \in B$, where $\cup$ is the operation union of two fuzzy sets.*

6. *The complement of a fuzzy soft set (ξ_1, A) is denoted by $(\xi_1, A)^c$ and is defined by $(\xi_1, A)^c = (\xi_1^c, A)$, where $\xi_1^c : A \to \widetilde{P}(U)$ is a mapping given by $\xi_1^c(\epsilon) = [\xi_1(\epsilon)]^c$ for all $\epsilon \in A$.*

Definition 1.3.4. *[26] A pair (ξ_h, A) is said to be a hesitant fuzzy soft set over a reference set U , where ξ is a mapping given by*

$$\xi_h : A \to HFS(U)$$

Example 1.3.2. *A hesitant fuzzy soft set (HFSS) in a decision-making problem, such as choosing the best job candidate. A company wants to hire a new software developer. There are 3 candidates: C_1: Alice, C_2: Bob and C_3: Charlie. The decision criteria (parameters) are: e_1: Technical Skills, e_2: Communication Skills and e_3: Teamwork. Because different managers have varying opinions (hesitations) on candidates' ratings, we use hesitant fuzzy soft sets to model these uncertainties.*
Hesitant Fuzzy Soft Set Representation: Let $U = \{C_1, C_2, C_3\}$ be the universe of candidates, and $E = \{e_1, e_2, e_3\}$ the set of parameters. Now define a hesitant fuzzy soft set F is a mapping $F : E \to \xi(\mathcal{U})$, where $\xi(\mathcal{U})$ denotes hesitant fuzzy subsets of U. For example:
For e_1 (Technical Skills):
$F(e_1) = \{(C_1,\{0.8,0.9\}),(C_2,\{0.6,0.7\}),(C_3,\{0.4,0.5\})\}$.
For e_2 (Communication Skills):
$F(e_2) = \{(C_1,\{0.6,0.7\}),(C_2,\{0.8\}),(C_3,\{0.5,0.6\})\}$.
For e_3 (Teamwork): $F(e_3) = \{(C_1,\{0.7,0.8\}),(C_2,\{0.5,0.6\}),(C_3,\{0.8,0.9\})\}$
Decision Making:
To decide the best candidate:
1. Aggregate the hesitant values for each candidate under all parameters.
2. Defuzzify or rank them using score functions (e.g., average, max-min). Let's compute average scores for simplicity:
Candidate C_1:
Tech: Avg(0.8, 0.9) = 0.85
Comm: Avg(0.6, 0.7) = 0.65
Team: Avg(0.7, 0.8) = 0.75
Overall Avg = (0.85 + 0.65 + 0.75) / 3 = 0.75
Candidate C_2:
Tech: Avg(0.6, 0.7) = 0.65
Comm: 0.8
Team: Avg(0.5, 0.6) = 0.55
Overall Avg = (0.65 + 0.8 + 0.55) / 3 = 0.667
Candidate C_3:
Tech: Avg(0.4, 0.5) = 0.45
Comm: Avg(0.5, 0.6) = 0.55
Team: Avg(0.8, 0.9) = 0.85
Overall Avg = (0.45 + 0.55 + 0.85) / 3 = 0.617.
Therefore; candidate C_1 (Alice) has the highest average score (0.75), so she is the best choice based on this hesitant fuzzy soft set model.

Definition 1.3.5. *[39] Let* $f : X \to Y$ *and* $g : A \to B$ *be two functions,* A *and* B *are parametric sets from the crisp sets* X *and* Y*, respectively. Then the pair (* f, g *) is called a bipolar fuzzy soft function from* X *to* Y*.*

Definition 1.3.6. *[39] Let (* ξ_{1h}, A *) and (* ξ_{2h}, B *) be two hesitant fuzzy soft sets over* X *and* Y*, respectively and let* (f, g) *be a hesitant fuzzy soft function from a TM-algebra X to a TM-algebra Y.*

1) *The image of (* ξ_{1h}, A *) under a hesitant fuzzy soft function (* f, g *), denoted by* $(f, g)(\xi_{1h}, A)$*, is a hesitant fuzzy soft set on* Y *defined by* $(f, g)(\xi_{1h}, A) = (f(\xi_{1h}), g(A))$*, where for all* $k \in g(A), y \in Y$

$$h_{f(\xi_{1h})_k}(y) = \begin{cases} \bigvee_{f(x)=y} \bigvee_{g(a)=k} \xi_{1h}(x) & \text{if } x \in g^{-1}(y), \\ \emptyset & \text{otherwise,} \end{cases}$$

2) *The pre-image of (* ξ_{2h}, B *) under a hesitant fuzzy soft function (* f, g *), denoted by* $(f, g)^{-1}(\xi_{2h}, B)$*, is a hesitant fuzzy soft set over* X *defined by* $(f, g)^{-1}(\xi_{2h}, B) = \big(f^{-1}(\xi_{2h}), g^{-1}(B)\big)$*, where for all* $a \in g^{-1}(A)$*, for all* $x \in X$*,*

Definition 1.3.7. *[26] Let (* f, g *) be a hesitant fuzzy soft function from* X *to* Y*. If* f *is a homomorphism from* X *to* Y *then (* f, g *) is said to be a hesitant fuzzy soft homomorphism. If* f *is a isomorphism from* X *to* Y *and* g *is one-to-one mapping from* A *onto* B *then (* f, g *) is said to be hesitant fuzzy soft isomorphism.*

1.4. Bipolar Hesitant Fuzzy Soft Set

Now, we recall some basic concepts from the theory of bipolar hesitant fuzzy soft sets. In classical fuzzy soft sets, only the membership degree of elements is taken into account. However, Zhang and Lee[73] extended this idea by introducing the concept of bipolar hesitant fuzzy soft sets, where both positive membership values (indicating satisfaction or agreement) and negative membership values (indicating dissatisfaction or disagreement) are considered simultaneously. This approach allows for a more refined representation of uncertainty by capturing both supportive and opposing evaluations in decision-making scenarios.

Definition 1.4.1. *[59] A triplet* (ξ_1, ξ_2, A) *is called a bipolar soft set over* $\mathcal{U}$*, where* ξ_1 *and* ξ_2 *are mappings, given by* $\xi_1 : A \to \mathcal{P}(U)$ *and* $\xi_2 : \neg A \to \mathcal{P}(U)$ *such that* $\xi_1(\epsilon) \cap \xi_2(\neg\epsilon) = \emptyset$*, for all* $\epsilon \in A$*.*

Definition 1.4.2. *[36] A bipolar fuzzy set in a non-empty set* X *is an object having the form,* $\xi = \{(x, \xi^+(x), \xi^-(x)) : x \in X\}$*, where* $\xi^+ : X \to [0,1]$*,* $\xi^- : X \to [-1,0]$*. So* ξ_A^+ *denote for positive information and* ξ^- *denote for negative information.*

Definition 1.4.3. *[35] For every two bipolar fuzzy sets* $\xi_1 = (\xi_1^+, \xi_1^-)$ *and* $\xi_2 = (\xi_2^+, \xi_2^-)$ *in* X*, then the following operations hold:*

1. $\xi_1 \subseteq \xi_2$ *if and only if* $\xi_1^+(x) \leqslant \xi_2^+(x)$ *and* $\xi_1^-(x) \geqslant \xi_2^-(x)$, *for all* $x \in X$,

2. $(\xi_1 \cup \xi_2)(x) = (\max\{\xi_1^+(x), \xi_2^+(x)\}, \min\{\xi_1^-(x), \xi_2^-(x)\})$, *for all* $x \in X$

3. $(\xi_1 \cap \xi_2)(x) = (\min\{\xi_1^+(x), \xi_2^+(x)\}, \max\{\xi_1^-(x), \xi_2^-(x)\})$, *for all* $x \in X$.

4. $(\xi)^c(x) = (1 - \xi^+(x), -1 - \xi^-(x))$, *for all* $x \in X$.

Definition 1.4.4. *[75] Let* ξ *be a bipolar fuzzy set on* X. *For* $(s,t) \in [-1,0] \times [0,1]$ *then*

$$U(\xi^+, t) = \{x \in X : \xi^+(x) \geqslant t\}$$
$$\text{and}$$
$$L(\xi^-, s) = \{x \in X : \xi^-(x) \leqslant s\}$$

$U(\xi^+, t)$ *is called positive* t*-level and* $L(\xi^-, s)$ *is called negative* s*- level set of* $\xi = (\xi^+, \xi^-)$, *respectively. The set* $A(\xi, (t,s)) = U(\xi^+, t) \cap L(\xi^-, s)$ *is called* (t,s)*-level set of* $\xi = (\xi^+, \xi^-)$.

Definition 1.4.5. *[75] Let* ξ_1 *and* ξ_2 *be any two bipolar fuzzy sets defined on universes of discourse* X *and* Y, *respectively. We call a mapping* $\xi_1 \times \xi_2 = (\xi_1^+ \times \xi_2^+, \xi_1^- \times \xi_2^-) : X \times Y \to [0,1] \times [-1,0]$, *a bipolar fuzzy relation from* ξ_1 *to* ξ_2 *such that*

$$(\xi_1^+ \times \xi_2^+)(x,y) = \min\{\xi_1^+(x), \xi_2^+(y)\}$$
$$\text{and}$$
$$(\xi_1^- \times \xi_2^-)(x,y) = \max\{\xi_1^-(x), \xi_2^-(y)\}.$$

In fuzzy set theory, fuzzy subsets are assumed to satisfy sup-property. Analogously, in the following the notion sup-inf property is defined for a bipolar fuzzy set ξ of X.

Definition 1.4.6. *[40] A bipolar fuzzy set* $\xi = (\xi^+, \xi^-)$ *in a set* X *with positive membership* $\xi^+ : X \to [0,1]$ *and negative membership* $\xi^- : X \to [-1,0]$ *is indicated to have sup-inf property, if for every nonempty subset* T *of* X, *there exists* $x_0 \in T$ *such that*

$$\xi^+(x_0) = \sup_{t \in T} \xi^+(t) \text{ and } \xi^-(x_0) = \inf_{t \in T} \xi^-(t).$$

Definition 1.4.7. *[40] Let* $f : X \to Y$ *be a function and let* $\xi = (\xi^+, \xi^-)$ *and* $\sigma = (\sigma^+, \sigma^-)$ *be the bipolar fuzzy sets of X and Y , respectively. Then*

1. *The image of* ξ *under* f *is defined as* $f(\xi) = (f(\xi)^+, f(\xi)^-)$ *such that*

$$f(\xi)^+(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \xi^+(x) & \text{if } f^{-1}(y) = \{x \in X : f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

And

$$f(\xi)^{-}(y)=\begin{cases}\inf_{x\in f^{-1}(y)}\xi^{-}(x) & \textit{if } f^{-1}(y)=\{x\in X: f(x)=y\}\neq\phi\\ 0 & \textit{otherwise}\end{cases}$$

2. *The inverse image of* σ *under* f *is defined as* $f^{-1}(\sigma)=\left(f^{-1}(\sigma)^{+}, f^{-1}(\sigma)^{-}\right)$ *such that* $f^{-1}(\sigma)^{+}(x)=\sigma^{+}(f(x))$ *and* $f^{-1}(\sigma)^{-}(x)=\sigma^{-}(f(x))$*, for all* $x\in X$.

Definition 1.4.8. *[2] Let* $\mathcal{U}$ *be a universe,* E *a set of parameters and* $A\subset E$*. Define* $F: A\to BF$*, where* BF *is the collection of all bipolar fuzzy subsets of* $\mathcal{U}$*. Then (* F, A *) is said to be a bipolar fuzzy soft set over a universe* $\mathcal{U}$*. It is defined by*

1. $(F, A)=F(e_i)$
2. $F(e_i)=(x_i,\ \xi^{+}(x_i),\ \xi^{-}(x_i))$*, for all* $x_i\in\mathbb{U}$, $e_i\in A$

Definition 1.4.9. *[14] Let* $\mathcal{U}$ *be a universe,* E *a set of parameters and* $A\subseteq E$*. Define* $\xi: A\to BF$*, where* BF *is the collection of all bipolar fuzzy subsets of* $\mathcal{U}$*. Then,* (ξ, A) *is said to be a bipolar fuzzy soft set over a universe* $\mathcal{U}$*. It is defined by*

$$(\xi, A)=\left\{\left(x,\ \xi_e^{+}(x),\ \xi_e^{-}(x)\right), \textit{for all}\quad x\in\mathcal{U}, e\in A\right\}.$$

Definition 1.4.10. *[68] A bipolar hesitant fuzzy set (BHFS)* ξ_h *on the universe* $\mathcal{U}$ *is defined as*

$$\xi_h=\left\{\left(x,\ \xi_h^{+},\ \xi_h^{-}\right)\,\middle|\, x\in\mathcal{U}\right\},$$

where:

(i). ξ_h^{+} *is called the hesitant fuzzy positive membership, and it is a finite subset of* $[0,1]$ *representing the possible degrees of satisfaction of the element* $x\in\mathcal{U}$ *with respect to the property described by* ξ_h.

(ii). ξ_h^{-} *is called the hesitant fuzzy negative membership, and it is a finite subset of* $[-1,0]$ *representing the possible degrees of satisfaction of the element* $x\in\mathcal{U}$ *with respect to the opposite (or counter) property of* ξ_h.

(iii). The pair $\xi_h(x)=\left(\xi_h^{+},\ \xi_h^{-}\right)$ *is referred to as the bipolar hesitant fuzzy element, which assigns a set of membership values from* $[0,1]\times[-1,0]$ *, for all* $x\in\mathcal{U}$.

Definition 1.4.11. *[68] A pair* (ξ_h, A) *is called a bipolar hesitant fuzzy soft set on* $\mathcal{U}$*, where* ξ_h *is a mapping given by* $\xi_h: A\longrightarrow \mathrm{BHFS}(\mathcal{U})$ *and* $\xi_h(\epsilon)=\{\langle x, \xi_{\epsilon h}(x)\rangle\mid x\in\mathcal{U}\}$*, for all* $\epsilon\in A$.

Example 1.4.1. *Let the universe of discourse be* $\mathcal{U}=\{$*coffee, tea, juice*$\}$*, and the set of parameters be* $E=\{$*bitter, healthy, cheap*$\}$ *with* $A=\{$*bitter, cheap*$\}\subseteq E$.
Define the mapping $\xi_h: A\to \mathrm{BHFS}(\mathcal{U})$ *as follows:*
Case 1: For $\epsilon=$ *bitter. Then*

$\xi_h(bitter) = \Big\{ (coffee, (\{0.9\}, \{-0.1\})), \quad (tea, (\{0.6, 0.7\}, \{-0.2\})), (juice, (\{0.2\}, \{-0.6, -0.7\})) \Big\}$
Case 2: For $\epsilon = cheap$. *Then*
$\xi_h(cheap) = \Big\{ (coffee, (\{0.4, 0.5\}, \{-0.4\})), \quad (tea, (\{0.7\}, \{-0.3\})), (juice, (\{0.6, 0.8\}, \{-0.1\})) \Big\}$
Thus, the pair (ξ_h, A) *forms a bipolar hesitant fuzzy soft set on* $\mathcal{U}$.

1.5. Some Types of Algebras

In this section, we recall the definitions of different types of algebras that arise from propositional and implicational calculi.

Definition 1.5.1. *[24] A BCK-algebra* $(X; \star, 0)$ *is a nonempty set* X *with a constant 0 and a binary operation* $\star$ *satisfying the following axioms:*

(i). $((x \star y) \star (x \star z)) \star (z \star y) = 0$,

(ii). $(x \star (x \star y)) \star y = 0$,

(iii). $x \star x = 0$,

(iv). $x \star y = 0$ *and* $y \star x = 0$ *imply* $x = y$,

(v). $0 \star x = 0$, *for all* $x, y, z \in X$.

Definition 1.5.2. *[25] A BCI-algebra* $(X; \star, 0)$ *is a nonempty set* X *with a constant 0 and a binary operation* $\star$ *satisfying the following axioms:*

(i). $((x \star y) \star (x \star z)) \star (z \star y) = 0$,

(ii). $(x \star (x \star y)) \star y = 0$,

(iii). $x \star x = 0$,

(iv). $x \star y = 0$ *and* $y \star x = 0$ *imply* $x = y$, *for all* $x, y, z \in X$.

Definition 1.5.3. *[51] A Q-algebra* $(X; \star, 0)$ *is a nonempty set* X *with a constant 0 and a binary operation* $\star$ *satisfying axioms:*

(i). $x \star x = 0$,

(ii). $x \star 0 = x$,

(iii). $(x \star y) \star z = (x \star z) \star y$, *for all* $x, y, z \in X$.

Definition 1.5.4. *[13] An algebra* $(X; \star, 0)$ *of type* $(2, 0)$ *is called a BCH-algebra if the following conditions are satisfied:*

(i). $x \star x = 0$,

(ii). $x \star y = 0 = y \star x \Rightarrow x = y$,

(iii). $(x \star y) \star z = (x \star z) \star y$; *for all* $x, y, z \in X$.

Definition 1.5.5. *[19] A pseudo-*BCK *algebra is a structure* $(X;\ \odot, \star, 0)$, *where "* $\odot$ *" and "* $\star$ *" are binary operations on* X *and 0 is an element of* X, *for all* $x, y, z \in X$, *satisfying the axioms:*

(i). $(x \odot y) \star (x \odot z) \star (z \odot y) = 0, \quad (x \star y) \odot (x \star z) \odot (z \star y) = 0$,

(ii). $x \odot (x \star y) \odot y = 0, \quad x \star (x \odot y) \star y = 0$,

(iii). $x \odot x = 0 = x \star x$,

(iv). $x \odot 0 = x = x \star 0$,

(v). $x \leqslant y,\ y \leqslant x \Rightarrow x = y$,

(vi). $x \leqslant y \Leftrightarrow x \odot y = 0 \Leftrightarrow x \star y = 0$.

Definition 1.5.6. *[15] A pseudo-*BCI *algebras is a structure* $(X;\ \odot, \star, 0)$, *where, "* $\odot$ *" and "* $\star$ *" are binary operations on* X *and 0 is an element of* X *which satisfys the following axioms: for all* $x, y, z \in X$

(i). $(x \odot y) \star (x \odot z) \star (z \odot y) = 0, \quad (x \star y) \odot (x \star z) \odot (z \star y) = 0$,

(ii). $x \odot (x \star y) \odot y = 0, \quad x \star (x \odot y) \star y = 0$,

(iii). $x \odot x = 0 = x \star x$,

(iv). $x \leqslant y$ *and* $y \leqslant x \Rightarrow x = y$,

(v). $x \leqslant y \Leftrightarrow x \odot y = 0 \Leftrightarrow x \star y = 0$.

Definition 1.5.7. *[16] An algebra* $(X;\ \star, 0)$ *of type* $(2, 0)$ *is called an BCC-algebra if it satisfies the following axioms: for all* $x, y, z \in X$

(i). $(y \star z) \star ((x \star y) \star (x \star z)) = 0$,

(ii). $0 \star x = x$,

(iii). $x \star 0 = 0$,

(iv). $x \star y = 0$ *and* $y \star x = 0 \Rightarrow x = y$.

Definition 1.5.8. *[44] A TM-algebra* $(X;\ \star, 0)$ *is a non-empty set X with a constant 0 and a binary operation* $\star$ *satisfying the following axioms : for all* $x, y, z \in X$

(i). $x \star 0 = x$,

(ii). $(x \star y) \star (x \star z) = z \star y$.

Example 1.5.1. *[5] Let* $X = \{0,1,2,3,4,5\}$ *and a binary operations* $\star$ *defined by the following Cayley tables:*

$\star$	0	1	2	3	4	5
0	0	3	4	1	2	5
1	1	0	2	3	5	4
2	2	4	0	5	1	3
3	3	1	5	0	4	2
4	4	5	3	2	0	1
5	5	2	1	4	3	0

Table 1.1

See [43] $(X; \star, 0)$ is a TM-algebra.

Lemma 1.5.1. *[18] The following properties hold in a* TM*-algebra* X.

1. $x \star x = 0$,
2. $(x \star y) \star x = 0 \star y$,
3. $x \star (x \star y) = y$,
4. $(x \star y) \star z = (x \star z) \star y$,
5. $x \star 0 = 0 \Rightarrow x = 0$. *In other words* $x \leqslant 0 \Rightarrow x = 0$,
6. $(x \star y) \star y = x$,
7. $0 \star (x \star y) = y \star x = (0 \star x) \star (0 \star y)$,
8. $(x \star z) \star (y \star z) = (x \star y)$, *for all* $x, y, z \in X$.

Definition 1.5.9. *[18] In any* TM*-algebra* X*, we define a partial order* $\leqslant$ *by putting* $x \leqslant y$ *if and only if* $x \star y = 0$.

Definition 1.5.10. *[63] A non-empty subset* S *of a TM-algebra* X *is called a TM-subalgebra of* X *if* $x \star y \in S$*, for all* $x, y \in S$.

Definition 1.5.11. *[55] Let* $(X; \star, 0)$ *be a TM-algebra. A non-empty subset I of X is called an ideal of X if it satisfies the following conditions:*

(i). $0 \in I$,

(ii). $x \star y \in I$ *and* $y \in I \Rightarrow x \in I$*, for all* $x, y \in X$.

Definition 1.5.12. *[55] Let (* $X; \star, 0$ *) be a TM-algebra. A non-empty subset I of X is called a T-ideal of X if it satisfies the following conditions* *(i).* $0 \in I$,

(ii). $(x \star y) \star z \in I$ *and* $y \in I \Rightarrow x \star z \in I$, *for all* $x, y, z \in X$.

Definition 1.5.13. *[55] An ideal I of a TM-algebra X is said to be closed if* $0 \star x \in I$, *for all* $x \in I$.

Definition 1.5.14. *[17] Let* $(X; \star, 0)$ *and* $(Y; \star, 0)$ *be two TM algebras. The direct product* $X \times Y$ *is also a TM-algebra with the binary operation* $\star$ *defined as* $(x_1, y_1) \star (x_2, y_2) = (x_1 \star x_2, y_1 \star y_2)$ *for all* $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Definition 1.5.15. *[21] Let* $(X; \star, 0)$ *and* $(Y; \star, 0)$ *be two* TM*-algebras. A mapping* $f : X \to Y$ *is called a homomorphism from* X *into* Y, *if* $f(x \star y) = f(x) \star f(y)$, *for all* $x, y \in X$.

(i.) A homomorphism f is called an monomorphism if it is injuctive,

(ii.) A homomorphism f is called an epimorphism if it is surjective,

(iii.) A bijective homomorphism is called an isomorphism.

Let $f : X \to Y$ *be a homomorphism of TM-algebras. Then the set* $\{x \in X / f(x) = 0_Y\}$ *is called the kernel of f and denote by* $\mathrm{Ker}(f)$. *Moreover the set* $\{f(x) \in Y \mid x \in X\}$ *is called the image of f and denote by* $\mathrm{Im}(f)$.

Proposition 1.5.2. *[22] Let* $(X; \star, 0_X)$ *and* $(Y; \star, 0_Y)$ *be any two TM-algebras. If* $f : X \to Y$ *is a homomorphism, then*

(i). $f(0_X) = 0_Y$,

(ii). If $x \star y = 0$ *for all* $x, y \in X$, *then* $f(x) \star f(y) = 0_Y$,

(iii). $\mathrm{Ker}(f)$ *is an ideal of* X.

Definition 1.5.16. *[42] Let* $f : X \to X$ *be an endomorphism and* ψ *a fuzzy set in* X. *Define a new fuzzy set* ψ^f *in* X *by* $\psi^f(x) = \psi(f(x))$ *for all* x *in* X.

Definition 1.5.17. *[43] A fuzzy set* $A = (X, \psi_A)$ *of a TM-algebra* X *is called a fuzzy TM-subalgebra of* X *if* $\psi_A(x \star y) \geqslant \min\{\psi_A(x), \psi_A(y)\}$. *for all* $x, y \in X$.

Definition 1.5.18. *[20] A fuzzy subset* ψ *in a TM-algebra X is called a fuzzy ideal of X , if:*

(i). $\psi(0) \geqslant \psi(x)$,

(ii). $\psi(x) \geqslant \min\{\psi(x \star y), \psi(y)\}$, *for all* $x, y, z \in X$.

Definition 1.5.19. *[20] A fuzzy subset* ψ *in a TM-algebra X is called a fuzzy T-ideal of X, if:*

(i). $\psi(0) \geqslant \psi(x)$,

(ii). $\psi(x \star z) \geqslant \min\{\psi((x \star y) \star z), \psi(y)\}$, *for all* $x, y, z \in X$.

Example 1.5.2. *Consider the set X={0,1,2,3} with Cayley Table 1.2*

$\star$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 1.2

See [43]$(X;\star,0)$ *is a TM algebra.*
Define a fuzzy set ψ *in* X *as by* $\psi(0) = 0.8, \psi(1) = 0.7, \psi(2) = \psi(3) = 0.3$. *It is easy to verify that* ψ *is a fuzzy* T*-ideal of* X.

1.6. Basic Concepts of Pseudo-TM Algebra

In this section, we present the formal definition of recently introduced algebraic structure called pseudo-TM algebra. This structure originally proposed by [52] is formulated based on the principles of propositional calculus.

Definition 1.6.1. *[52] A pseudo-TM algebra is algebra* $(X; \odot, \star, 0)$ *of type (2, 2, 0) which satisfies the following axioms: for all* x, y*, and* $z \in X$.

(i). $x \odot 0 = x \star 0 = x,$

(ii). $(x \odot y) \star (x \odot z) = z \odot y$ *and* $(x \star y) \odot (x \star z) = z \star y.$

In a pseudo-TM algebra X define a binary operation $\leqslant$ by $x \leqslant y$ if and only if $x \odot y = 0$ and $x \star y = 0$.
Every pseudo-TM algebra X satisfying $x \odot y = x \star y$, for all $x, y \in X$ is a TM algebra.

Example 1.6.1. *[52] Let* $X = \{0, 1, 2\}$ *be a set with the following Cayley tables:*

$\odot$	0	1	2
0	0	1	2
1	1	1	1
2	2	2	2

$\star$	0	1	2
0	0	1	2
1	1	2	2
2	2	1	1

Table 1.3

See [52]($X; \odot, \star, 0$) *is a pseudo-TM algebra.*

Lemma 1.6.1. *[52] In a pseudo-TM algebra* X *the following holds: for all* $x, y \in X$.

1. $x \odot x = x \star x = 0,$
2. $y \odot (x \star y) = y = y \star (x \odot y),$

3. $(x \star y) \odot x = 0 \star y$ *and* $(x \odot y) \star x = 0 \odot y$,

4. $x \odot (x \star (x \star y)) = x \star y, \quad x \star (x \odot (x \odot y)) = x \odot y$,

5. $0 \odot (x \star y) = y, \quad 0 \star (x \odot y) = y$,

6. $0 \odot (y \star x) = x \star y, \quad 0 \star (y \odot x) = x \odot y$,

7. $(x \odot (x \star y)) \odot y = 0, \quad (x \star (x \odot y)) \star y = 0$,

8. $x \odot (x \star y) = y, \quad x \star (x \odot y) = y$,

9. $x \odot (y \star x) = x, \quad x \star (y \odot x) = x$,

10. $y \odot (y \star x) = x, \quad y \star (y \odot x) = x$.

Definition 1.6.2. *[52] A non-empty subset* I *of pseudo-TM algebra* X *is called a pseudo ideal of* X *if*

(i). $0 \in I$,

(ii). $x \odot y$, $x \star y \in I$ *and* $y \in I \Rightarrow x \in I$, *for all* $x, y \in X$.

Chapter 2
Fuzzy Subsets on Pseudo-TM Algebra

In this chapter, the algebraic structure known as pseudo-TM algebra is extended to the fuzzy setting, there by introducing and studying new fuzzy counterparts of this structure. In particular, the notions of fuzzy pseudo-TM subalgebras and fuzzy pseudo-TM ideals are introduced and systematically examined. The definitions of these fuzzy structures are formulated with reference to their level subsets within the underlying pseudo-TM algebra. In addition to foundational definitions, the chapter presents new theoretical results concerning the behavior and properties of fuzzy pseudo-TM subalgebras and fuzzy pseudo-TM ideals. Special attention is given to the preservation and transformation of these structures under homomorphisms and Cartesian product operations. Through this investigation, the chapter contributes to a deeper understanding of fuzzy generalizations of pseudo-TM algebras and provides a basis for further algebraic and fuzzy logical studies.

2.1. Fuzzy Pseudo-TM Subalgebra of Pseudo-TM algebra

In this section, we study a fuzzy pseudo TM-subalgebra of pseudo-TM algebra and study some basic properties of fuzzy pseudo-TM subalgebra of pseudo-TM algebra. Let X and Y denote a pseudo-TM algebras unless otherwise specified throughout this and the following section.

Definition 2.1.1. *A fuzzy subset* $\psi : X \to [0,1]$ *of a pseudo-TM algebra* X *is called fuzzy pseudo-TM subalgebra of* X *if*

1. $\psi(x \odot y) \geqslant \min\{\psi(x), \psi(y)\}$ *and*
2. $\psi(x \star y) \geqslant \min\{\psi(x), \psi(y)\}$, *for all* $x \in X$.

Example 2.1.1. *Let* $X = \{0,1,2,3\}$ *be a set with two binary operations* $\odot$ *and* $\star$ *which are given by table.*

$\odot$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	2	0

$\star$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	1	0

Table 2.1

See [52] $(X;\odot,\star,0)$ *is a pseudo -TM algebra. Let* ψ *and* ω *be a fuzzy subsets of* X *defined by*

$$\psi(x)=\begin{cases}1, & \textit{if } x=0;\\ 0.3, & \textit{if } x=3;\\ 0. & \textit{otherwise.}\end{cases} \quad \textit{and } \omega(x)=\begin{cases}0.7, & \textit{if } x=0;\\ 0.5, & \textit{if } x=3;\\ 0. & \textit{otherwise.}\end{cases}$$

By routine calculation, ψ *and* ω *are a fuzzy pseudo-TM subalgebra of* X.

Lemma 2.1.1. *If* ψ *is a fuzzy pseudo-TM subalgebra of* X*, then* $\psi(0)\geqslant\psi(x)$*, for all* $x\in X$.

Proof. Suppose that ψ is a fuzzy pseudo-TM subalgebra of X. By Lemma 1.6.1 we have

$$\begin{aligned}\psi(0)&=\psi(x\odot x)\\ &\geqslant \min\{\psi(x),\psi(x)\}\\ &=\psi(x)\text{ and}\\ \psi(0)&=\psi(x\star x)\\ &\geqslant \min\{\psi(x),\psi(x)\}\quad=\psi(x).\end{aligned}$$

Hence, $\psi(0)\geqslant\psi(x)$, for all $x\in X$. □

Theorem 2.1.2. *Let* ψ *be a fuzzy pseudo-TM subalgebra of* X*. Then*

(i.) $\psi(x\odot y)=\psi(y)$ *if and only if* $\psi(x)=\psi(0)$*, for all* $x,y\in X$.

(ii.) $\psi(x\star y)=\psi(y)$ *if and only if* $\psi(x)=\psi(0)$*, for all* $x,y\in X$.

Proof. Suppose that $\psi(x\odot y)=\psi(y)$, for all $x,y,\in X$. We need to prove that $\psi(x)=\psi(0)$, for all $x\in X$. By Definition 1.6.1 we have $x\odot 0=0$ and $x\star 0=0$, for all $x\in X$. Since $(x\odot 0)\odot 0=x\odot 0=x$ and $(x\star 0)\star 0=x\star 0=x$. Then

$$\begin{aligned}\psi(x)&=\psi((x\odot 0)\odot 0)\\ &\geqslant \min\{\psi(x\odot 0),\psi(0)\}\\ &=\min\{\psi(0),\psi(0)\}\\ &=\{\psi(0)\}\text{ and }\psi(x)=\psi((x\star 0)\star 0)\\ &\geqslant \min\{\psi(x\star 0),\psi(0)\}\\ &=\min\{\psi(0),\psi(0)\}\\ &=\psi(0).\text{ By Lemma 2.1.1}\\ \psi(0)&\geqslant\psi(x).\end{aligned}$$

Hence,$\psi(x)=\psi(0)$.
Conversely, assume that $\psi(x)=\psi(0)$, for all $x\in X$. We need to show that $\psi(x\odot y)=\psi(y)$ and

$\psi(x \star y) = \psi(y)$. By Lemma 2.1.1 we have $\psi(0) \geqslant \psi(y)$ for any $y \in X$. Then

$$\begin{aligned}
\psi(x) &\geqslant \psi(y) \quad \text{but} \\
\psi(x \odot y) &\geqslant \min\{\psi(x), \psi(y)\} \\
&= \psi(y) \text{ and} \\
\psi(x \star y) &\geqslant \min\{\psi(x), \psi(y)\} \\
&= \psi(y) \text{ also} \\
\psi(y) &= \psi(0 \odot (x \star y) \\
&\geqslant \min\{\psi(0), \psi(x \star y)\} \\
&= \min\{\psi(x), \psi(x \star y)\} \\
&= \psi(x \star y) \text{ and} \\
\psi(y) &= \psi(0 \star (x \odot y) \\
&\geqslant \min\{\psi(0), \psi(x \odot y)\} \\
&= \min\{\psi(x), \psi(x \odot y)\} \\
&= \psi(x \odot y).
\end{aligned}$$

Hence, $\psi(x \star y) = \psi(y)$ and $\psi(x \odot y) = \psi(y)$ □

Theorem 2.1.3. *If ψ and ω be two fuzzy pseudo-TM subalgebras of X. Then $\psi \cap \omega$ is also a fuzzy pseudo-TM subalgebra of X.*

Proof. Let ψ and ω be two fuzzy pseudo-TM subalgebras of X. Then we need to prove that $\psi \cap \omega$ is a fuzzy pseudo-TM subalgebra of X. Let $x, y \in X$, we have

$$\begin{aligned}
(\psi \cap \omega)(x \odot y) &= \min\{\psi(x \odot y), \omega(x \odot y)\} \\
&\geqslant \min\{\min\{\psi(x), \psi(y)\}, \min\{\omega(x), \omega(y)\}\} \\
&= \min\{\min\{\psi(x), \omega(x)\}, \min\{\psi(y), \omega(y)\}\}. \\
&\Rightarrow (\psi \cap \omega)(x \odot y) \\
&\geqslant \min\{(\psi \cap \omega)(x), (\psi \cap \omega)(y)\} \text{ and} \\
(\psi \cap \omega)(x \star y) &= \min\{\psi(x \star y), \omega(x \star y)\} \\
&\geqslant \min\{\min\{\psi(x), \psi(y)\}, \min\{\omega(x), \omega(y)\}\} \\
&= \min\{\min\{\psi(x), \omega(x)\}, \min\{\psi(y), \omega(y)\}\}. \\
&\Rightarrow (\psi \cap \omega)(x \odot y) \geqslant \min\{(\psi \cap \omega)(x), (\psi \cap \omega)(y)\}
\end{aligned}$$

Therefore, $\psi \cap \omega$ is also a fuzzy pseudo-TM subalgebra of X. □

Corollary 2.1.4. *Let $\{\psi_i / i \in I)$ be a family of fuzzy pseudo-TM subalgebra of X. Then $\cap_{i \in I} \psi_i$ is also a fuzzy pseudo-TM subalgebra of X.*

Proof. Suppose that $\{\psi_i / i \in I)$ be a family of fuzzy pseudo-TM subalgebra of X. Recall that the intersection of fuzzy sets is taken pointwise, i.e. for every $x \in X$ $\left(\bigcap_{i \in I} \psi_i\right)(x) = \inf_{i \in I} \psi_i(x)$. We

must show the two defining inequalities hold for this pointwise infimum. For any $x, y \in X$. For each $i \in I$, since ψ_i is a fuzzy pseudo-TM subalgebra we have:
$\psi_i(x \odot y) \geqslant \min\{\psi_i(x), \psi_i(y)\}$. Taking the infimum over (i) on both sides gives $\inf_{i\in I} \psi_i(x \odot y) \geqslant \inf_{i\in I} \min\{\psi_i(x), \psi_i(y)\}$. Now, $\inf_{i\in I} \min\{\psi_i(x), \psi_i(y)\} \geqslant \min\{\inf_{i\in I} \psi_i(x), \inf_{i\in I} \psi_i(y)\}$.
Combining the two displayed inequalities gives
$\left(\bigcap_{i\in I} \psi_i\right)(x \odot y) = \inf_i \psi_i(x \odot y) \geqslant \min\{\inf_i \psi_i(x), \inf_i \psi_i(y)\} = \min\{(\bigcap_i \psi_i)(x), (\bigcap_i \psi_i)(y)\}$
.
Also, $\left(\bigcap_{i\in I} \psi_i\right)(x \star y) = \inf_i \psi_i(x \star y) \geqslant \min\{\inf_i \psi_i(x), \inf_i \psi_i(y)\} = \min\{(\bigcap_i \psi_i)(x), (\bigcap_i \psi_i)(y)\}$
.
Thus $(\bigcap_{i\in I} \psi_i)$ satisfies both conditions of Definition (2.1.1), so it is a fuzzy pseudo-TM subalgebra of (X). □

Remark 2.1.1. *The union of any two fuzzy pseudo-TM subalgebras of X is not necessarily a fuzzy pseudo-TM subalgebras of X.*

Example 2.1.2. *Let $\mathbb{Q}$ be the set of all rational numbers. Let $\odot$ and $\star$ be two binary operations on $\mathbb{Q}$ defined by*

$$x \odot y = x - y \quad \textit{and} \quad x \star y = x - y, \quad \textit{for all } x, y \in \mathbb{Q},$$

where "$-$" is the usual subtraction of $\mathbb{Q}$. Then $(\mathbb{Q};\ \odot, \star, 0)$ is a pseudo-TM algebra.
Let ψ and ω be any fuzzy subsets of the set $\mathbb{Q}$ defined by

$$\psi(x) = \begin{cases} 0.9, & \textit{if } x \in \langle 5 \rangle, \\ 0.2, & \textit{otherwise,} \end{cases} \quad \textit{and} \quad \omega(x) = \begin{cases} 0.8, & \textit{if } x \in \langle 4 \rangle, \\ 0.1, & \textit{otherwise.} \end{cases}$$

By routine calculation, $\psi(x)$ and $\omega(x)$ are fuzzy pseudo-TM subalgebras of $\mathbb{Q}$.
Now, define $(\psi \cup \omega)(x)$ by

$$(\psi \cup \omega)(x) = \begin{cases} 0.9, & \textit{if } x \in \langle 5 \rangle, \\ 0.8, & \textit{if } x \in \langle 4 \rangle \setminus \langle 5 \rangle, \\ 0.2, & \textit{if } x \notin \langle 5 \rangle \textit{ and } x \notin \langle 4 \rangle. \end{cases}$$

Let $x = 5$ and $y = 4$. Then

$$(\psi \cup \omega)(x) = (\psi \cup \omega)(5) = 0.9, \qquad (\psi \cup \omega)(y) = (\psi \cup \omega)(4) = 0.8.$$

Now,

$$\begin{aligned} &(\psi \cup \omega)(x \odot y) \\ &\qquad = \max\{\psi(x \odot y), \omega(x \odot y)\} \\ &\qquad = \max\{\psi(x - y), \omega(x - y)\} \\ &\qquad = \max\{\psi(5 - 4), \omega(5 - 4)\} \\ &\qquad = \max\{0.2, 0.1\} = 0.2, \text{ and} \end{aligned}$$

$$\begin{aligned}(\psi \cup \omega)(x \star y) &= \max\{\psi(x \star y), \omega(x \star y)\}\\ &= \max\{\psi(x - y), \omega(x - y)\}\\ &= \max\{0.2, 0.1\}\\ &= 0.2.\end{aligned}$$

Hence,

$$(\psi \cup \omega)(5 \odot 4) = 0.2 = (\psi \cup \omega)(5 \star 4).$$

Now consider

$$\min\{(\psi \cup \omega)(x), (\psi \cup \omega)(y)\} = \min\{(\psi \cup \omega)(5), (\psi \cup \omega)(4)\} = \min\{0.9, 0.8\} = 0.8.$$

Therefore, $0.2 \geqslant 0.8$, *which is not true. This implies that the union of two fuzzy pseudo-TM subalgebras may not be a fuzzy pseudo-TM subalgebra of* X.

Theorem 2.1.5. *Let* ψ *is a fuzzy pseudo-TM subalgebra of* X. *Then the set* $G_\psi = \{x \in X/\psi(x) = \psi(0)\}$ *is a pseudo-TM subalgebra of* X.

Proof. Suppose that ψ is a fuzzy pseudo-TM subalgebra of X. Let $x, y \in G_\psi$. Then

$$\begin{aligned}\psi(x) &= \psi(0) = \psi(y). \text{ Now}\\ \psi(x \odot y) &\geqslant \min\{\psi(x), \psi(y)\}\\ &= \psi(0).\\ &\Rightarrow \psi(x \odot y) \geqslant \psi(0). \text{ by Lemma 2.1.1,}\\ \psi(0) &\geqslant \psi(x \odot y), \text{ for all} x, y \in X. \text{ It follows that}\end{aligned}$$

$$\begin{aligned}\psi(x \odot y) &= \psi(0).\\ &\Rightarrow x \odot y \in G_\psi \text{ and,}\\ \psi(x \star y) &\geqslant \min\{\psi(x), \psi(y)\}\\ &= \psi(0).\\ &\Rightarrow \psi(x \star y) \geqslant \psi(0). \text{By Lemma 2.1.1}\\ \psi(0) &\geqslant \psi(x \star y), \text{ for all } x, y \in X.\end{aligned}$$

It follows that $\psi(x \star y) = \psi(0)$. It implies that $x \star y \in G_\psi$.
Hence, G_ψ is a pseudo-TM subalgebra of X. □

Theorem 2.1.6. *A fuzzy subset* ψ *of a pseudo-TM algebra* X *is a fuzzy pseudo-TM subalgebra if and only if for all* $t \in [0, 1]$ *the non-empty level set* $U(\psi, t)$ *is a pseudo-TM subalgebra of X.*

Proof. Assume that ψ be a fuzzy pseudo-TM subalgebra of X. We need to show that the level subset $U(\psi, t)$ is either empty or a pseudo-TM subalgebra of X. Suppose that the level subset $U(\psi, t) \neq \emptyset$. For all $x, y \in U(\psi, t)$, we have $\psi(x) \geqslant t$ and $\psi(y) \geqslant t$. Then

$$\psi(x \odot y) \quad \geqslant \quad \min\{\psi(x), \psi(y)\} \geqslant \min\{t, t\} = t \quad \text{and}$$

$$\psi(x\star y) \geqslant \min\{\psi(x),\psi(y)\} \geqslant \min\{t,t\} = t.$$

Hence, $x\odot y, x\star y \in U(\psi,t)$. Therefore, $U(\psi,t)$ is a pseudo-TM subalgebra of X.
Conversely, assume that $U(\psi,t)$ is a pseudo-TM subalgebra of X. We need to show that ψ is a fuzzy pseudo- TM subalgebra of X.
For all $x,y \in X$. Take $t = \min\{\psi(x),\psi(y)\}$. Then $x\odot y$, $x\star y \in U(\psi,t)$ which implies that $\psi(x\odot y) \geqslant t = \min\{\psi(x),\psi(y)\}$ and $\psi(x\star y) \geqslant t = \min\{\psi(x),\psi(y)\}$.
$\text{Hence}, x\odot y, x\star y \in \psi$.
Therefore, ψ is a pseudo-TM subalgebra of X. □

Corollary 2.1.7. *Let $(X,\odot,\star,0)$ be a pseudo TM-algebra and let $\psi : X \to [0,1]$ be a fuzzy subset. Then ψ is a fuzzy pseudo-TM subalgebra of X if and only if the non-empty upper level set*

$$U(\psi,t) = \{x \in X \mid \psi(x) \geqslant t\},$$

for $t \in [0,1]$, is a pseudo-TM subalgebra of X for $t = 1$.

Proof. Suppose that ψ is a fuzzy pseudo-TM algebra subalgebra of X. We need to show that the non-empty level subset $U(\psi,1)$ is a pseudo-TM subalgebra of X. Assume that $U(\psi,1)$ is non-empty. Then there exists $x,y \in U(\psi,1)$ such that $\psi(x) = \psi(y) = 1$. Since ψ is a fuzzy pseudo-TM subalgebra of X. Then $\psi(x\odot y) \geqslant \min\{\psi(x),\psi(y)\} = 1$ and $\psi(x\star y) \geqslant \min\{\psi(x),\psi(y)\} = 1$.
Hence, $x\odot y$, $x\star y \in U(\psi,1)$.
Therefore, $U(\psi,t)$ is a pseudo-TM algebra of X.
Conversely, the non-empty upper level set $U(\psi,t) = \{x \in X \mid \psi(x) \geqslant t\}$, for $t \in [0,1]$, is a pseudo-TM subalgebra of X for $t = 1$. We need to show that ψ is a fuzzy pseudo-TM subalgebra of X.
For all $x,y \in X$. Take $t = 1$. Then $x\odot y$, $x\star y \in U(\psi,1)$ which implies that $\psi(x\odot y) \geqslant 1 \geqslant \min\{\psi(x),\psi(y)\}$ and $\psi(x\star y) \geqslant 1 \geqslant \min\{\psi(x),\psi(y)\}$.$\text{Hence}, x\odot y, x\star y \in \psi$.
Therefore, ψ is a pseudo-TM subalgebra of X. □

Proposition 2.1.8. *If ψ is a fuzzy pseudo-TM subalgebra of X, then we have the following results*

(i). $\psi(0\odot x) \geqslant \psi(x)$ *and* $\psi(0\star x) \geqslant \psi(x)$.

(ii). If there exists a sequence x_k in X such that $\lim_{x\to\infty}\psi(x_k) = 1$, *then* $\psi(0) = 1$.

Proof. Suppose that ψ is a fuzzy pseudo-TM subalgebra of X. We need to prove that $\psi(0\odot x) \geqslant \psi(x)$ and $\psi(0\star x) \geqslant \psi(x)$.

(i). Since $\psi(0\odot x) \geqslant \min\{\psi(0),\psi(x)\}$. It implies that $\psi(0\odot x) \geqslant \psi(x)$
and $\psi(0\star x) \geqslant \min\{\psi(0),\psi(x)\} \Rightarrow \psi(0\star x) \geqslant \psi(x)$.

(ii). Suppose that ψ is a fuzzy pseudo-TM subalgebra of X. By Lemma 2.1.1 $\psi(0) \geqslant \psi(x)$, for all $x \in X$. If x_k is a sequence in X, then $1 \geqslant \psi(0) = \psi(x_k\odot x_k) \geqslant \min\{\psi(x_k),\psi(x_k)\} = \psi(x_k)$. As $k \to \infty$, we have $\lim_{x\to\infty}\psi(1) \geqslant \lim_{x\to\infty}\psi(0) \geqslant \lim_{x\to\infty}\psi(x_k)$.

Hence, $\psi(0)=1$ and $1 \geqslant \psi(0)=\psi(x_k \star x_k) \geqslant \min\{\psi(x_k),\psi(k_k)\}=\psi(x_k)$. As $k \to \infty$. It follows that $\lim_{x\to\infty}\psi(1) \geqslant \lim_{x\to\infty}\psi(0) \geqslant \lim_{x\to\infty}\psi(x_k)$.
Hence, $\psi(0)=1$.

□

Theorem 2.1.9. *Let* X *be a pseudo-TM algebra. If* $\{B_i\}$ *is any sequence of a pseudo-TM subalgebras of* X *such that* $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_m = X$*, then there exists a fuzzy pseudo-TM subalgebra* ψ *of* X *whose level a pseudo-subalgebras are exactly the pseudo-subalgebras* $\{B_i\}$*.*

Proof. Assume that $s_0 > s_1 > \cdots > s_m$ be a set of numbers, where each $s_i \in [0,1]$. Let ψ be a fuzzy set defined by $\psi(B_0)=s_0$ and $\psi(B_i - B_{i-1}) = s_i, 0 < i \leqslant m$. We need to show that ψ is a fuzzy pseudo-TM subalgebra of X.

Case 1. If $x,y \in B_i - B_{i-1}$, for all $x,y \in X$. Then, $x,y \in B_i \Rightarrow x \odot y \in B_i$ and $x \star y \in B_i$. Also $x,y \in B_i - B_{i-1} \Rightarrow \psi(x)=s_i=\psi(y) \Rightarrow \min\{\psi(x),\psi(y)\}=s_i$. Now $x \odot y \in B_i$ and $x \star y \in B_i \Rightarrow x \odot y$ and $x \star y \in B_i - B_{i-1}$ or $x \odot y$ and $x \star y \in B_{i-1}$. $\Rightarrow \psi(x \odot y) = \psi(x \star y) = s_i \Rightarrow \psi(x \odot y) = \psi(x \star y) \geqslant s_i$. Thus $\psi(x \star y) \geqslant s_i = \min\{\psi(x),\psi(y)\}$ and $\psi(x \odot y) \geqslant s_i = \min\{\psi(x),\psi(y)\}$.

Case 2. For $i > j \Rightarrow s_j > s_i \Rightarrow B_j \subset B_i$. Let $x \in B_i - B_{i-1}$ and $y \in B_j - B_{j-1}$. Then $\psi(x) = s_i$ and $\psi(y) = s_j > t_i$.
Hence $\min\{\psi(x),\psi(y)\} = \min\left\{s_i,s_j\right\} = s_i$. Adationally, , $y \in B_j - B_{j-1} \Rightarrow y \in B_j \subset B_i \Rightarrow x,y \in B_i$. Since B_i is a pseudo-TM subalgebra of $X, x \odot y \in B_i$ and $x \star y \in B_i$. Therefore, $\psi(x \odot y) \geqslant t_i = \min\{\psi(x),\psi(y)\}$ and $\psi(x \star y) \geqslant t_i = \min\{\psi(x),\psi(y)\}$ Thus in both the cases, ψ is a fuzzy pseudo- TM subalgebra of X. From the definition of ψ, it follows that $\text{Im}(\psi) = \{s_0, s_1, \cdots, s_m\}$. Hence $\psi_{s_i} = \{x \in X/\psi(x) \geqslant s_i\}$, for $0 \leqslant i \leqslant m$ are the level pseudo-TM subalgebras of X. By Theorem 2.1.6 $\{\psi_{s_i}\}$ the sequence of level pseudo-TM subalgebras of ψ of the form $\psi_{s_0} \subset \psi_{s_1} \subset \cdots \psi_{t_m} = X$ Now $\psi_{t_0} = \{x \in X/\psi(x) \geqslant t_0\} = A_0$. Finally we prove $\psi_{t_i} = A_i$, for $0 < i \leqslant n$. Clearly, $A_i \subseteq \psi_{t_i}$. If $x \in \psi_{s_i}$, then $\psi(x) \geqslant s_i$ which implies $\psi(x) \in \{s_1, s_2, \cdots s_n\}$. Hence $x \in B_0$ or B_1 or $\cdots$ or B_i. It implies that $x \in B_i$.

Therefore $\psi_{s_i} = B_i$ for $0 \leqslant i \leqslant n$. Hence the level pseudo-TM subalgebras of ψ are a pseudo TM-subalgebras of X. □

Theorem 2.1.10. *Let* B *be any non-empty subset of a pseudo-TM algebra* X *and* ψ *is a fuzzy subset of* X *defined by*

$$\psi(x) = \begin{cases} u, & \text{if } x \in B; \\ v, & \text{if } x \notin B, \text{ for all } u,v \in [0,1] \text{ with } \quad u \geqslant v. \end{cases}$$

Then ψ *is a fuzzy pseudo-TM subalgebra of X if and only if* B *is a pseudo-TM subalgebra of X.*

Proof. Assume that ψ is a fuzzy pseudo-TM subalgebra of X. Then we need to show that B is a pseudo-TM subalgebra of X.
Let $x, y \in B$. Since ψ is a fuzzy pseudo-TM subalgebra of X, we have

$$\begin{aligned} \psi(x \odot y) &\geqslant \min\{\psi(x), \psi(y)\} = \min\{u, u\} = u \quad \text{and} \\ \psi(x \star y) &\geqslant \min\{\psi(x), \psi(y)\} = \min\{u, u\} = u. \end{aligned}$$

Hence, $x \odot y$, $x \star y \in B$.
Therefore, B is a fuzzy pseudo-TM subalgebra of X.
Conversely, assume that B is a pseudo-TM subalgebra of X. We need to show that ψ is a fuzzy pseudo-TM subalgebra of X. Consider the following cases

Case 1. If $x, y \in B$, then $x \odot y$, $x \star y \in B$.

$$\begin{aligned} \psi(x \odot y) = u &\geqslant \min\{\psi(x), \psi(y)\} \quad \text{and} \\ \psi(x \star y) = u &\geqslant \min\{\psi(x), \psi(y)\}. \end{aligned}$$

Hence, $x \odot y$, $x \star y \in \psi$.

Case 2. If $x \in B$ and $y \notin B$, then $\psi(x) = u \quad \text{and} \quad \psi(y) = v$.Thus

$$\begin{aligned} \psi(x \odot y) &\geqslant \min\{u, v\} = \min\{\psi(x), \psi(y)\} \quad \text{and} \\ \psi(x \star y) &\geqslant \min\{u, v\} = \min\{\psi(x), \psi(y)\}. \end{aligned}$$

Hence, $x \odot y, x \star y \in \psi$.

Case 3. If $x \notin B$ and $y \in B$, then interchanging the role of x and y in case(2) yields similar results.

$$\begin{aligned} \psi(x \odot y) &\geqslant \min\{v, u\} = \min\{\psi(y), \psi(x)\} = \min\{\psi(x), \psi(y)\} \quad \text{and} \\ \psi(x \star y) &\geqslant \min\{v, u\} = \min\{\psi(y), \psi(x)\} = \min\{\psi(y), \psi(x)\}. \end{aligned}$$

Hence, $x \odot y, x \star y \in \psi$.

Case 4. If $x \notin B, y \notin B$, then $\psi(x) = v \quad \text{and} \quad \psi(y) = v$.Thus

$$\begin{aligned} \psi(x \odot y) &\geqslant \min\{v, v\} = \min\{\psi(x), \psi(y)\} \quad \text{and} \\ \psi(x \star y) &\geqslant \min\{v, v\} = \min\{\psi(x), \psi(y)\}. \end{aligned}$$

Hence, $x \odot y,\ x \star y \in \psi$.
Therefore, by all cases we have ψ is a fuzzy pseudo-TM subalgebra of X. □

Lemma 2.1.11. *Any pseudo-TM subalgebra of a pseudo-TM algebra X can be realized as a level pseudo-TM subalgebra of some fuzzy pseudo-TM subalgebra of X.*

Proof. Let B be a pseudo-TM subalgebra of a pseudo-TM algebra X. For $\alpha \in [0,1]$. Let ψ be a fuzzy subset of X defined by

$$\psi(x) = \begin{cases} \alpha, & \text{if } x \in B; \\ 0, & \text{otherwise.} \end{cases}$$

If $x,\ y \in B$, then $x \odot y,\ x \star y \in B$. Then by definition $\psi(x) = \psi(y) = \psi(x \odot y) = \psi(x \star y) = \alpha$.

$$\begin{aligned} \psi(x \odot y) = \alpha &\geqslant \min\{\psi(x), \psi(y)\} \quad and \\ \psi(x \star y) = \alpha &\geqslant \min\{\psi(x), \psi(y)\}. \end{aligned}$$

Hence, $x \odot y,\ x \star y \in \psi$.
and, if $x,\ y \notin B$, then $\psi(x) = \psi(y) = 0$.

$$\begin{aligned} \psi(x \odot y) &\geqslant 0 = \min\{\psi(x), \psi(y)\} \quad and \\ \psi(x \star y) &\geqslant 0 = \min\{\psi(x), \psi(y)\} \end{aligned}$$

Hence, $x \odot y,\ x \star y \in \psi$.
If at least one of $x, y \in B$, then $\psi(x)$ or $\psi(y)$ is equal to 0.

$$\begin{aligned} \psi(x \odot y) &\geqslant 0 = \min\{\psi(x), \psi(y)\} \quad and \\ \psi(x \star y) &\geqslant 0 = \min\{\psi(x), \psi(y)\}. \end{aligned}$$

Hence, $x \odot y,\ x \star y \in \psi$.
This shows that B is level pseudo-TM subalgebra of X corresponding to a fuzzy pseudo-TM subalgebra of X. □

Corollary 2.1.12. *Let* B *be a subset of a pseudo-TM algebra* X. *Then the characterstics function defined by*

$$\chi_B(x) = \begin{cases} 1, & \textit{if } x \in B, \\ 0, & x \notin B. \end{cases}$$

is a fuzzy pseudo-TM subalgebra of X *if and only if* B *is a pseudo-TM subalgebra of* X.

Proof. Suppose that B is a pseudo- TM subalgebra of X. We need to show that χ_B is a fuzzy pseudo-TM subalgebra of X. Let $x,\ y \in X$. Then

Case 1. If $x, y \in B$, since B is a TM-subalgebra of X, $x \odot y,\ x \star y \in B$.
Hence, $\chi_B(x) = 1$, $\chi_B(y) = 1$, $\chi_B(x \odot y) = 1$ and $\chi_B(x \star y) = 1$. Then

i. $\chi_B(x \odot y) \geqslant \min\{\chi_B(x), \chi_B(y)\}$, for all $x, y \in X$.

ii. $\chi_B(x \star y) \geqslant \min\{\chi_B(x), \chi_B(y)\}$, for all $x, y \in X$.

Case 2. If both $x,\ y \notin B$, then $\chi_B(x) = 0$, $\chi_B(y) = 0$. In this case,

i. $\chi_B(x \odot y) \geqslant 0 = \min\{0,0\} = \min\{\chi_B(x), \chi_B(y)\}$, for all $x, y \in X$.

ii. $\chi_B(x \star y) \geqslant 0 = \min\{0,0\} = \min\{\chi_B(x), \chi_B(y)\}$, for all $x, y \in X$..

Case 3. If $x \in B, y \notin B, \chi_B(x) = 1, \chi_B(y) = 0$. Then

i. $\chi_B(x \odot y) \geqslant 0 = \min\{0,1\} = \min\{\chi_B(x), \chi_B(y)\}$, for all $x, y \in X$..

ii. $\chi_B(x \star y) \geqslant 0 = \min\{0,1\} = \min\{\chi_B(x), \chi_B(y)\}$, for all $x, y \in X$.

Case 4. Interchanging the roles of x and y in case 3 we can prove that χ_B is a fuzzy pseudo-TM subalgebra of X when $x \notin B$ and $y \in B$.

Conversely, suppose that χ_B be a fuzzy pseudo-TM algebra of X. For all x, $y \in B, \chi_B(x) = 1 = \chi_B(y)$. Therefore, $\chi_B(x \odot y) \geqslant \min\{\chi_B(x), \chi_B(y)\} = \min\{1,1\} = 1$. Hence, $\chi_B(x \odot y) = 1 \Rightarrow x \odot y \in B$. Also, $\chi_B(x \star y) \geqslant \min\{\chi_B(x), \chi_B(y)\} = \min\{1,1\} = 1$. Hence, $\chi_B(x \star y) = 1 \Rightarrow x \star y \in B$ Therefore, B is a pseudo-TM subalgebra of X. □

Theorem 2.1.13. *Let ψ be a fuzzy pseudo-TM subalgebra of a pseudo-TM algebra X such that $\psi(0) \neq 0$. Let ω be a fuzzy pseudo-TM algebra defined by $\omega(x) = \frac{\psi(x)}{\psi(0)}$, for all $x \in X$. Then ω is a fuzzy pseudo-TM subalgebra of X.*

Proof. Suppose that ψ be a fuzzy pseudo-TM subalgebra of a pseudo-TM algebra X such that $\psi(0) \neq 0$ and let ω be a fuzzy pseudo-TM algebra defined by $\omega(x) = \frac{\psi(x)}{\psi(0)}$, for all $x \in X$. Then we need to show that ω is a fuzzy pseudo-TM subalgebra of X.Let $x, y \in X$. Then

$$\begin{aligned}
\omega(x \odot y) &= \frac{\psi(x \odot y)}{\psi(0)} \\
&\geqslant \frac{1}{\psi(0)} \min\{\psi(x), \psi(y)\} \\
&= \min\left\{\frac{\psi(x)}{\psi(0)}, \frac{\psi(y)}{\psi(0)}\right\} \\
&= \min\{\omega(x), \omega(y)\} \text{ and} \\
\omega(x \star y) &= \frac{\psi(x \star y)}{\psi(0)} \\
&\geqslant \frac{1}{\psi(0)} \min\{\psi(x), \psi(y)\} \\
&= \min\left\{\frac{\psi(x)}{\psi(0)}, \frac{\psi(y)}{\psi(0)}\right\} \\
&= \min\{\omega(x), \omega(y)\}.
\end{aligned}$$

Therefore, ω is a fuzzy pseudo TM-subalgebra of X. □

2.2. Homomorphism On Fuzzy Pseudo-TM Subalgebra

In this section, we discuss on fuzzy pseudo TM-subalgebras in a pseudo-TM algebra under homomorphism. We examine the homomorphic image and inverse image of fuzzy pseudo-TM subalgebras of a pseudo- TM algebra. Also, we disscus a compostion of two epimorphic image and inverse image of a fuzzy pseudo-TM subalgebras of a pseudo-TM algebra ,as well as other results, are examined.

Theorem 2.2.1. *Let* $(X; \odot, \star, 0)$ *and* $(Y; \odot, \star, 0)$ *be two pseudo TM-algebras. Let* $f : X \to Y$ *be an epimorphism of a pseudo TM-algebras. If* ψ *is a fuzzy pseudo-TM subalgebra of* X *with sup-property, then* $f(\psi)$ *is a fuzzy pseudo-TM subalgebra of* Y.

Proof. Let $f : X \to Y$ be an epimorphism of a pseudo-TM algebras. Assume that ψ be a fuzzy pseudo-TM subalgebra of X with sup-property. Then we need to show that $f(\psi)$ is a fuzzy pseudo-TM subalgebra of Y. Let $a, b \in Y$, since f is an epimorphism then there exists $x, y \in X$ such that $f(x) = a$, $f(y) = b$. By Definition 1.1.10 we have $\psi(x) = Sup_{t \in f^{-1}(a)} \psi(t)$ and $\psi(y) = Sup_{t \in f^{-1}(b)} \psi(t)$, So

$$
\begin{aligned}
f(\psi(a \odot b)) &= Sup_{t \in f^{-1}(a \odot b)} \psi(t) \text{ and} \\
f(\psi(a \star b)) &= Sup_{t \in f^{-1}(a \star b)} \psi(t). \text{ Now,} \\
f(\psi(a \odot b)) &= Sup_{t \in f^{-1}(a \odot b)} \psi(t) \\
&= \psi(x \odot y) \\
&\geqslant \min\{\psi(x), \psi(y)\} \\
&= \min\left\{ Sup_{t \in f^{-1}(a)} \psi(t), Sup_{t \in f^{-1}(b)} \psi(t) \right\} \\
&= \min\{f(\psi)(x), f(\psi)(y)\} \text{ and,} \\
f(\psi(a \star b)) &= Sup_{t \in f^{-1}(a \star b)} \psi(t) \\
&= \psi(x \star y) \\
&\geqslant \min\{\psi(x), \psi(y)\} \\
&= \min\left\{ Sup_{t \in f^{-1}(a)} \psi(t), Sup_{t \in f^{-1}(b) \psi(t)} \right\} \\
&= \min\{f(\psi)(x), f(\psi)(y)\}, \text{for some} x, y \in X.
\end{aligned}
$$

Therefore, $f(\psi)$ is a fuzzy pseudo-TM subalgebra of X. □

Theorem 2.2.2. *Let* $(X; \odot, \star, 0) and (Y; \odot, \star, 0)$ *be two pseudo-TM algebras. Let* $f : X \to Y$ *be an homomorphism of a pseudo-TM algebra. If* ω *is a fuzzy pseudo-TM subalgebra of* Y *, then* $f^{-1}(\omega)$ *is a fuzzy pseudo-TM subalgebra of* X.

Proof. Assume that ω be a fuzzy pseudo-TM subalgebra of Y. We need to show that $f^{-1}(\omega)$ is a fuzzy pseudo-TM subalgebra of X. For all $x,y \in X$, we have

$$\begin{aligned}
f^{-1}(\omega(x \odot y)) &= \omega(f(x \odot y)) \\
&= \omega(f(x)f(y)) \\
&\geqslant \min\{\omega(f(x)), \omega(f(y))\} \\
&= \min\left\{f^{-1}(\omega)(x), f^{-1}(\omega)(y)\right\} \text{ and,} \\
f^{-1}(\omega(x \star y)) &= \omega(f(x \star y)) \\
&= \omega(f(x) \star f(y)) \\
&\geqslant \min\{\omega(f(x)), \omega(f(y))\} \\
&= \min\left\{f^{-1}(\omega)(x), f^{-1}(\omega)(y)\right\}
\end{aligned}$$

Therefore, $f^{-1}(\omega)$ is a fuzzy pseudo-TM subalgebra of X. □

Theorem 2.2.3. *Let $f: X \to Y$ be an epimorphism and ω be a fuzzy set in Y. If $f^{-1}(\omega)$ is a fuzzy pseudo-TM subalgebra of X, then ω is a fuzzy pseudo-TM subalgebra of Y.*

Proof. Suppose that f is an epimorphism and $f^{-1}(\omega)$ be a fuzzy pseudo-TM subalgebra of X. We need to show that ω is a fuzzy pseudo-TM subalgebra of Y. Let $y_1, y_2 \in Y$. Since f is an epimorphism then there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now,

$$\begin{aligned}
\omega(y_1 \odot y_2) &= \omega(f(x_1) \odot f(x_2)) \\
&= \omega(f(x_1 \odot x_2)) \\
&= f^{-1}(\omega)(x_1 \odot x_2) \\
&\geqslant \min\left\{f^{-1}(\omega)(x_1), f^{-1}(\omega)(x_2)\right\} \\
&= \min\{\omega(f(x_1)), \omega(f(x_2))\} \\
&= \min\{\omega(y_1), \omega(y_2)\} \text{ and} \\
\omega(y_1 \star y_2) &= \omega(f(x_1) \star f(x_2)) \\
&= \omega(f(x_1 \star x_2)) \\
&= f^{-1}(\omega)(x_1 \star x_2) \\
&\geqslant \min\left\{f^{-1}(\omega)(x_1), f^{-1}(\omega)(x_2)\right\} \\
&= \min\{\omega(f(x_1)), \omega(f(x_2))\} \\
&= \min\{\omega(y_1), \omega(y_2)\}.
\end{aligned}$$

Therefore, ω is a fuzzy pseudo-TM subalgebra of Y. □

Theorem 2.2.4. *Let $f: X \to Y$ be an epimorphism of a pseudo-TM algebra. Then the fuzzy set ψ is a fuzzy pseudo-TM subalgebra of Y if and only if ψ^f is a fuzzy pseudo-TM subalgebra of X.*

Proof. Assume that ψ is a fuzzy pseudo TM- subalgebra of Y. We need to show that ψ^f is a fuzzy pseudo TM-subalgebra of X. Let $x, y \in X$. By Definition 1.5.16 we have

$$\begin{aligned}
\psi^f(x \odot y) &= \psi(f(x \odot y)) \\
&= \psi(f(x) \odot f(y)) \\
&\geqslant \min\{\psi(f(x)), \psi(f(y))\} \\
&= \min\left\{\psi^f(x), \psi^f(y)\right\} \quad \text{and} \\
\psi^f(x \star y) &= \psi(f(x \star y)) \\
&= \psi(f(x) \star f(y)) \\
&\geqslant \min\{\psi(f(x)), \psi(f(y))\} \\
&= \min\left\{\psi^f(x), \psi^f(y)\right\}
\end{aligned}$$

Hence, ψ^f is fuzzy pseudo-TM subalgebra of X.
Conversely, let ψ^f is a fuzzy pseudo-TM subalgebra in X. We need to show that ψ is a fuzzy pseudo-TM subalgebra of Y. Let $y_1, y_2 \in Y$. Since f is an epimorphism then there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$

$$\begin{aligned}
\psi(y_1 \odot y_2) &= \psi(f(x_1) \odot f(x_2)) \\
&= \psi(f(x_1 \odot x_2)) \\
&= (\psi)^f(x_1 \odot x_2) \\
&\geqslant \min\left\{\psi^f(x_1), \psi^f(x_2)\right\} \\
&= \min\{\psi(f(x_1)), \psi(f(x_2))\} \\
&= \min\{\psi(y_1), \psi(y_2)\} \text{ and} \\
\psi(y_1 \star y_2) &= \psi(f(x_1) \star f(x_2)) \\
&= \psi(f(x_1 \star x_2)) \\
&= (\psi)^f(x_1 \star x_2) \\
&\geqslant \min\left\{\psi^f(x_1), \psi^f(x_2)\right\} \\
&= \min\{\psi(f(x_1)), \psi(f(x_2))\} \\
&= \min\{\psi(y_1), \psi(y_2)\}
\end{aligned}$$

Therefore, ψ is a fuzzy pseudo-TM subalgebra of Y. □

Proposition 2.2.5. *Let $f : X \to Y$ and $g : Y \to Z$ be an epimorphism of a pseudo-TM algebras and ψ be a fuzzy pseudo-TM subalgebra. Then $(gof)^{-1}(\psi)$ is a fuzzy pseudo-TM subalgebra of X if and only if $(gof)(\psi)$ is a fuzzy pseduo-TM subalgebra of Z, provided that sup-property holds.*

Proof. Suppose that $(gof)^{-1}(\psi)$ is a fuzzy pseudo-TM subalgebra of X . Let ψ be a fuzzy pseudo-TM subalgebra of X. We need to prove that $(gof)(\psi)$ is a fuzzy pseudo-TM subalgebra of Z. By Definition 1.1.14 we have for all $z, w \in Z$, there exist $x, y \in X$ such that $(g \circ f)(x) = z$ and $(g \circ f)(y) = w$. Since ψ provided sup properties we have $\psi(x) = \sup_{t \in (g \circ f)^{-1}(z)} \psi(t)$ and $\psi(y) =$

$\sup_{t\in(g\circ f)^{-1}(w)}\psi(t)$ for some $x,y \in X$. But, f,g are surjective. It implies that $g\circ f$ is surjective. Therefore, $x\odot y \in f^{-1}(z\odot w)$ and $x\star y\in f^{-1}(z\star w)$. Hence,

$$\begin{aligned}(g\circ f)(\psi)(z\odot w) =& \sup_{t\in(g\circ f)^{-1}(z\odot w)}\psi(t)\\ =&\psi(x\odot y)\\ \geqslant&\min\{\psi(x),\psi(y)\}\\ =&\min\left\{\sup_{t\in(g\circ f)^{-1}(z)}\psi(t),\sup_{t\in(g\circ f)^{-1}(w)}\psi(t)\right\}\\ =&\min\{(g\circ f)(\psi)(z),(g\circ f)(\psi)(w) \text{ and}\end{aligned}$$

$$\begin{aligned}(g\circ f)(\psi)(z\star w) =& \sup_{t\in(g\circ f)^{-1}(z\star w)}\psi(t)\\ =&\psi(x\odot y)\\ \geqslant&\min\{\psi(x),\psi(y)\}\\ =&\min\left\{\sup_{t\in(g\circ f)^{-1}(z)}\psi(t),\sup_{t\in(g\circ f)^{-1}(w)}\psi(t)\right\}\\ =&\min\{(g\circ f)(\psi)(z),(g\circ f)(\psi)(w)\}.\end{aligned}$$

Therefore, $(g\circ f)(\psi)$ is a fuzzy pseudo-TM subalgebra of Z.
Conversely, suppose that $(g\circ f)(\psi)$ is a fuzzy pseduo-TM subalgebra of Z. We need to prove that $(g\circ f)^{-1}(\psi)$ is a fuzzy pseudo-TM subalgebra of X.
Let ψ be a fuzzy pseudo-TM subalgebra of Z. For $x,y\in X$. We have

$$\begin{aligned}(g\circ f)^{-1}(\psi)(x\odot y) &= \psi(g\circ f)(x\odot y)\\ &= \psi\left((g\circ f)(x)\odot(g\circ f)(y)\right)\\ &\geqslant \min\{\psi(g\circ f)(x),\psi(g\circ f)(y)\}\\ &= \min\left\{(g\circ f)^{-1}(\psi)(x),(g\circ f)^{-1}(\psi)(y)\right\} \text{ and}\end{aligned}$$

$$\begin{aligned}(g\circ f)^{-1}(\psi)(x\star y) &= \psi((g\circ f)(x\star y))\\ &= \psi((g\circ f)(x)\star(g\circ f)(y))\\ &\geqslant \min\{\psi((g\circ f)(x)),\psi((g\circ f)(y))\}\\ &= \min\left\{(g\circ f)^{-1}(\psi)(x),(g\circ f)^{-1}(\psi)(y)\right\}\end{aligned}$$

Therefore, $(g\circ f)^{-1}(\psi)$ is a fuzzy pseudo-TM subalgebra of X. □

2.3. Cartesian Product of Fuzzy Pseudo-TM Subalgebra

In this section, we discuss the concept of cartesian product under level subset of a fuzzy pseudo-TM subalgebras. We prove that the cartesian product of two fuzzy pseudo-TM subalgebra is again a fuzzy pseudo-TM subalgebra and some other results are also investigated.

Lemma 2.3.1. *Let* ψ_1 *and* ψ_2 *be any two fuzzy pseudo-TM subalgebra of* X *and* Y *respectively. Then* $(\psi_1 \times \psi_2)(0,0) \geqslant (\psi_1 \times \psi_2)(x,y)$, *for all* $(x,y) \in X \times Y$.

Proof. Suppose that ψ_1 and ψ_2 be any two fuzzy pseudo-TM subalgebra of X and Y respectively. Then we need to show that $(\psi_1 \times \psi_2)(0,0) \geqslant (\psi_1 \times \psi_2)(x,y)$, for all $(x,y) \in X \times Y$.
For all $(x,y) \in X \times Y$. By Definition 1.1.7 we have

$$\begin{aligned}
(\psi_1 \times \psi_2)(0,0) &= (\psi_1 \times \psi_2)(x \odot x, y \odot y) \\
&= \min\{\psi_1(x \odot x), \psi_2(y \odot y)\} \\
&\geqslant \min\{\min\{\psi_1(x), \psi_1(x)\}, \min\{\psi_2(y), \psi_2(y)\}\} \\
&= \min\{\psi_1(x), \psi_2(y)\} \\
&= (\psi_1 \times \psi_2)(x,y) \quad \text{and} \\
(\psi_1 \times \psi_2)(0,0) &= (\psi_1 \times \psi_2)(x \star x, y \star y) \\
&= \min\{\psi_1(x \star x), \psi_2(y \star y)\} \\
&\geqslant \min\{\min\{\psi_1(x), \psi_1(x)\}, \min\{\psi_2(y), \psi_2(y)\}\} \\
&= \min\{\psi_1(x), \psi_2(y)\} \\
&= (\psi_1 \times \psi_2)(x,y)
\end{aligned}$$

Hence, $(\psi_1 \times \psi_2)(0,0) \geqslant (\psi_1 \times \psi_2)(x,y)$, for all $(x,y) \in X \times Y$. □

Theorem 2.3.2. *If* ψ_1 *and* ψ_1 *be a fuzzy pseudo-TM subalgebra of* X *and* Y *respectively, then* $\psi = \psi_1 \times \psi_2$ *is a fuzzy pseudo-TM subalgebra of* $X \times Y$.

Proof. Assume that ψ_1 and ψ_2 be a fuzzy pseudo-TM subalgebra of X and Y respectively. We need to show that $\psi = \psi_1 \times \psi_2$ is a fuzzy pseudo-TM subalgebra of $X \times Y$.
For all (x_1, x_2) and $(y_1, y_2) \in X \times Y$.

$$\begin{aligned}
\psi((x_1,x_2) \odot (y_1,y_2)) &= \psi(x_1 \odot y_1, x_2 \odot y_2) \\
&= (\psi_1 \times \psi_2)(x_1 \odot y_1, x_2 \odot y_2) \\
&= \min\{\psi_1(x_1 \odot y_1), \psi_2(x_2 \odot y_2))\} \\
&\geqslant \min\{\min\{\psi_1(x_1), \psi_1(y_1)\}, \min\{\psi_2(x_2), \psi_2(y_2)\}\} \\
&= \min\{\min\{\psi_1(x_1), \psi_2(x_2)\}, \min\{\psi_1(y_1), \psi_2(y_2)\}\} \\
&= \min\{(\psi_1 \times \psi_2)(x_1,x_2), (\psi_1 \times \psi_2)(y_1,y_2)\} \\
&= \min\{\psi(x_1,x_2), \psi(y_1,y_2)\} \quad \text{and} \\
\psi((x_1,x_2) \star (y_1,y_2)) &= \psi(x_1 \star y_1, x_2 \star y_2) \\
&= (\psi_1 \times \psi_2)(x_1 \star y_1, x_2 \star y_2) \\
&= \min\{\psi_1(x_1 \star y_1), \psi_2(x_2 \star y_2)\} \\
&\geqslant \min\{\min\{\psi_1(x_1), \psi_1(y_1)\}, \min\{\psi_2(x_2), \psi_2(y_2)\}\} \\
&= \min\{\min\{\psi_1(x_1), \psi_2(x_2)\}, \min\{\psi_1(y_1), \psi_2(y_2)\}\}
\end{aligned}$$

$$
\begin{aligned}
&= \min\{(\psi_1 \times \psi_2)(x_1,x_2),(\psi_1 \times \psi_2)(y_1,y_2)\} \\
&= \min\{\psi(x_1,x_2),\psi(y_1,y_2)\}
\end{aligned}
$$

Therefore, $\psi = \psi_1 \times \psi_2$ is a fuzzy pseudo-TM subalgebra of $X \times Y$. □

Theorem 2.3.3. *Let $\{X_i\}_{i=1}^{n}$ be a finite collection of a pseudo-TM algebras and $X = \prod_{i=1}^{n} X_i$ is the direct product psedo-TM algebra of $\{X_i\}$. Let ψ_i be a fuzzy psedo-TM subalgebra of x_i,where $1 \leqslant i \leqslant n$.*
Then $\psi = \prod_{i=1}^{n} \psi_i$ defined by $X = \prod_{i=1}^{n} \psi_i(x_1,x_2,...,x_n) = ((\psi_1)(x_1),(\psi_2)(x_2),...,(\psi_n)(x_n))$ is a fuzzy psedo-TM subalgebra of X.

Proof. Suppose that ψ_i be a fuzzy psedo-TM subalgebra of x_i,where $1 \leqslant i \leqslant n$. Then we need to show that $\psi = \prod_{i=1}^{n} \psi_i$ defined by $X = \prod_{i=1}^{n} \psi_i(x_1,x_2,...,x_n) = ((\psi_1)(x_1),(\psi_2)(x_2),...,(\psi_n)(x_n))$ is a fuzzy psedo-TM subalgebra of X. Let $x = (x_1,x_2,...,x_n)$ and $x = (y_1,y_2,...,y_n)$ be any fuzzy elements of $X = \prod_{i=1}^{n} x_i$. Then

$$
\begin{aligned}
\psi(x \odot y) &= \psi(x_1 \odot y_1,x_2 \odot y_2,...,x_n \odot y_n) \\
&= \prod_{i=1}^{n} \psi_i(x_1 \odot y_1,x_2 \odot y_2,...,x_n \odot y_n) \\
&= (\psi_1(x_1 \odot y_1),\psi_2(x_2 \odot y_2),...,\psi_n(x_n \odot y_n)) \\
&\geqslant \min\{\psi_1(x_1),\psi_1(x_1),\psi_2(x_2),\psi_2(x_2),...,\psi_n(x_n),\psi_n(x_n)\} \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\psi(x \star y) &= \psi(x_1 \star y_1,x_2 \star y_2,...,x_n \star y_n) \\
&= \prod_{i=1}^{n} \psi_i(x_1 \star y_1,x_2 \star y_2,...,x_n \star y_n) \\
&= (\psi_1(x_1 \star y_1),\psi_2(x_2 \star y_2),...,\psi_n(x_n \star y_n)) \\
&\geqslant \min\{\psi_1(x_1),\psi_1(x_1),\psi_2(x_2),\psi_2(x_2),...,\psi_n(x_n),\psi_n(x_n).\}
\end{aligned}
$$

Therefore, $\psi = \prod_{i=1}^{n} \psi_i$ is a fuzzy psudo-TM subalgebra of X. □

Theorem 2.3.4. *Let ψ_1 and ψ_2 be two fuzzy subset of X and Y respectively. Then $\psi_1 \times \psi_2$ is a fuzzy pseudo-TM subalgebra of $X \times Y$ if and only if the non-empty upper t-level subset $U(\psi_1 \times \psi_2,t)$ is a fuzzy pseudo-TM subalgebra of $X \times Y$.*

Proof. Let $\psi_1 \times \psi_2$ is a fuzzy pseudo-TM subalgebra of $X \times Y$. We need to show that $U(\psi_1 \times \psi_2,t)$ is a fuzzy pseudo-TM subalgebra of $X \times Y$. Let $(x_1,x_2),(y_1, y_2) \in X \times Y$ such that $(x_1, x_2), (y_1, y_2) \in U(\psi_1 \times \psi_2,t)$, for all $t \in [0,1]$. Then $(\psi_1 \times \psi_2)(x_1,x_2) \geqslant t$ and $(\psi_1 \times \psi_2)(y_1,y_2) \geqslant t$. Since

$\psi_1 \times \psi_2$ is a fuzzy pseudo-TM subalgebra of $X \times Y$,we have

$$\begin{aligned}(\psi_1 \times \psi_2)((x_1,x_2) \odot (y_1,y_2)) &= (\psi_1 \times \psi_2)(x_1 \odot y_1, x_2 \odot y_2)\\ &= \min\{\psi_1(x_1 \odot y_1), \psi_2(x_2 \odot y_2)\}\\ &\geqslant \min\{\min\{\psi_1(x_1), \psi_1(y_1)\}, \min\{\psi_2(x_2), \psi_2(y_2)\}\}\\ &= \min\{\min\{\psi_1(x_1), \psi_2(x_2)\}, \min\{\psi_1(y_1), \psi_2(y_2)\}\}\\ &= \min\{(\psi_1 \times \psi_2)(x_1,x_2), (\psi_1 \times \psi_2)(y_1,y_2)\}\\ &\geqslant \min\{t,t\} = t \text{ and}\end{aligned}$$

Hence, $(x_1,x_2) \odot (y_1,y_2) \in U(\psi_1 \times \psi_2, t)$

$$\begin{aligned}(\psi_1 \times \psi_2)((x_1,x_2) \star (y_1,y_2)) &= (\psi_1 \times \psi_2)(x_1 \star y_1, x_2 \star y_2)\\ &= \min\{\psi_1(x_1 \odot y_1), \psi_2(x_2 \star y_2)\}\\ &\geqslant \min\{\min\{\psi_1(x_1), \psi_1(y_1)\}, \min\{\psi_2(x_2), \psi_2(y_2)\}\}\\ &= \min\{\min\{\psi_1(x_1), \psi_2(x_2)\}, \min\{\psi_1(y_1), \psi_2(y_2)\}\}\\ &= \min\{(\psi_1 \times \psi_2)(x_1,x_2), (\psi_1 \times \psi_2)(y_1,y_2)\}\\ &\geqslant \min\{t,t\} = t.\end{aligned}$$

Hence, $(x_1,x_2) \star (y_1,y_2) \in U(\psi_1 \times \psi_2, t)$
Therefore, $U(\psi_1 \times \psi_2, t)$ is a fuzzy pseudo-TM subalgebra of X.
Conversely, suppose that $U(\psi_1 \times \psi_2, t)$ be a fuzzy pseudo-TM subalgebra of X. We need to show that $\psi_1 \times \psi_2$ is a fuzzy pseudo-TM subalgebra of $X \times Y$. Assume that $\psi_1 \times \psi_2$ is not a fuzzy pseudo-TM subalgebra of $X \times Y$. Then there exists $(x_1,x_2),(y_1,y_2) \in X \times Y$ such that $(\psi_1 \times \psi_2)((x_1,x_2) \odot (y_1,y_2)) < \min\{(\psi_1 \times \psi_2)(x_1,x_2), (\psi_1 \times \psi_2)(y_1,y_2)\}$. Consider $t_0 = \frac{1}{2}\{(\psi_1 \times \psi_2)((x_1,x_2) \odot$
We have $(\psi_1 \times \psi_2)((x_1,x_2) \odot (y_1,y_2)) < t_0 < \min\{(\psi_1 \times \psi_2)(x_1,x_2), (\psi_1 \times \psi_2)(y_1,y_2)\}$. So $(x_1,x_2) \odot (y_1,y_2) \notin U(\psi_1 \times \psi_2, t_0)$. But $(x_1,x_2),(y_1,y_2) \in U(\psi_1 \times \psi_2, t_0)$. And, $(\psi_1 \times \psi_2)((x_1,x_2) \star (y_1,y_2)) < \min\{(\psi_1 \times \psi_2)(x_1,x_2), (\psi_1 \times \psi_2)(y_1,y_2)\}$.
Consider $t_0 = \frac{1}{2}\{(\psi_1 \times \psi_2)((x_1,x_2) \star (y_1,y_2)) + \min\{(\psi_1 \times \psi_2)(x_1,x_2), (\psi_1 \times \psi_2)(y_1,y_2)\}\}$. We have $(\psi_1 \times \psi_2)(x_1,x_2) \star (y_1,y_2) < t_0 < \min\{(\psi_1 \times \psi_2)(x_1,x_2), (\psi_1 \times \psi_2)(y_1,y_2)\}$. So $(x_1,x_2) \star (y_1,y_2) \notin U(\psi_1 \times \psi_2, t_0)$. But $(x_1,x_2),(y_1,y_2) \in U(\psi_1 \times \psi_2, t_0)$. Which implies that $U(\psi_1 \times \psi_2, t_0)$ is not a fuzzy pseudo-TM algebra of $X \times Y$ which is a contradiction.
Therefore, $\psi_1 \times \psi_2$ is a fuzzy pseudo-TM subalgebra of $X \times Y$. □

2.4. Fuzzy Pseudo ideal in Pseudo-TM Algebra

In this section, we study the basic definition of a fuzzy pseudo ideal in a pseudo-TM algebra. Also, we discuss the basic properties of fuzzy pseudo ideals in a pseudo-TM algebra are investigate and the relationship between fuzzy pseudo ideal of a pseudo- TM algebra and their level set of a pseudo-TM algebra. Let X and Y denote a pseudo-TM algebras unless otherwise specfied throughout this and the following section.

Definition 2.4.1. *A fuzzy subset* $\psi : X \to [0,1]$ *is called a fuzzy Pseudo ideal of* X *if it satisfies*

1. $\psi(0) \geqslant \psi(x)$, *for all* $x \in X$
2. $\psi(x) \geqslant \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\}$, *for all* $x, y \in X$.

Example 2.4.1. *Let* $X = \{0,1,2,3,4,5,6\}$ *be a set with two binary operations* $\odot$ *and* $\star$ *which are given by table.*

$\odot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	2	0	0	0	0	0
3	3	2	2	0	0	0	0
4	4	0	0	0	0	0	0
5	5	0	0	0	0	0	0
6	6	0	0	0	0	0	0

$\star$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	2	0	0	0	0	0
3	3	3	3	0	0	0	0
4	4	0	0	0	0	0	0
5	5	0	0	0	0	0	0
6	6	0	0	0	0	0	0

Table 2.2

See[57] $(X; \odot, \star, 0)$ is a pseudo-TM algebra.

Let ψ and ω be a fuzzy subsets of X defined by

$$\psi(x) = \begin{cases} 0.6, & \text{if } x = 0; \\ 0.3, & \text{if } x = 1,2,3,4,5,6. \end{cases} \quad \text{and } \omega(x) = \begin{cases} 0.7, & \text{if } x=0; \\ 0.5, & \text{if } x=1,2,3,4,5,6. \end{cases}$$

By routine calculation, ψ and ω are a fuzzy pseudo-ideal of X.

Example 2.4.2. *Consider a pseudo-TM algebra* X *in Example 2.4.1. Let* $\psi : X \to [0,1]$ *be a fuzzy set in* X *defined by*

$$\psi(x) = \begin{cases} 1, & \textit{if } x = 0, \\ 0.1, & \textit{if } x = 1, \\ 0.4, & \textit{if } x = 2,3,4,5,6. \end{cases}$$

By routine calculation, ψ is a fuzzy pseudo-ideal of X.

Theorem 2.4.1. *If* ψ *and* ω *are fuzzy pseudo-ideal of a pseudo-TM algebra* X*. Then* $\psi \cap \omega$ *is also fuzzy pseudo-ideal of* X.

Proof. Suppose that ψ and ω are fuzzy pseudo-ideal of a pseudo-TM algebra X. Then we need to show that $\psi \cap \omega$ is also fuzzy pseudo-ideal of X. Let $x, y \in X$. Then $(\psi \cap \omega)(0) = \min\{\psi(0), \omega(0)\}$ $\geqslant \min\{\psi(x), \omega(x)\} = (\psi \cap \omega)(x)$.

These implies that $(\psi\cap\omega)(0) \geqslant (\psi\cap\omega)(x)$.
Again, $(\psi\cap\omega)(x) = \min\{\psi(x),\omega(x)\}$

$$\begin{aligned}
&\geqslant \min\{\min\{\psi(x\odot y),\psi(x\star y),\psi(y)\},\min\{\omega(x\odot y),\omega(x\star y),\psi(y)\}\}\\
&= \min\{\min\{\psi(x\odot y),\omega(x\odot y))\},\min\{\psi(x\star y),\omega(x\star y))\},\min\{\psi(y),\omega(y)\}\}\\
&= \min\{(\psi\cap\omega)(x\odot y),(\psi\cap\omega)(x\star y),(\psi\cap\omega)(y)\}.\\
&\Rightarrow (\psi\cap\omega)(x) \geqslant \min\{(\psi\cap\omega)(x\odot y),(\psi\cap\omega)(x\star y),(\psi\cap\omega)(y)\}.
\end{aligned}$$

Hence, $\psi\cap\omega$ is a fuzzy pseudo-ideal of a pseudo-TM algebra X. □

Corollary 2.4.2. *Let $\{\psi_i \mid i\in I\}$ be a family of a fuzzy pseudo ideal of a pseudo-TM algebra X. Then $\cap_{i\in I}\psi_i$ is a fuzzy pseudo-ideal of X.*

Proof. Let $\{\psi_i \mid i\in I\}$ be a family of fuzzy pseudo-ideal of X. We need to show that $\cap_{i\in I}\psi_i$ is a fuzzy pseudo-ideal of X.
1) Let $x\in X$. Then $\bigcap_{i\in I}\psi_i(0) = \inf\{\psi_i(0)\mid i\in I\}) \geqslant \psi_i(x)$. Since $\psi_i(0)\geqslant\psi_i(x) \quad \forall i\in I$.
2) Let $x,y\in X$. Then, we have $\bigcap_{i\in I}\psi_i(x) = \inf\{\psi_i(x)\mid i\in I\} \geqslant \inf\{\psi_i(x\odot y),\psi_i(x\star y),\psi_i(y)\}$.
Because $\psi_i(x) = \inf\{\psi_i(x)\mid i\in I\} \geqslant \inf\{\psi_i(x\odot y),\psi_i(x\star y),\psi_i(y)\}, \forall i\in I$
$\Rightarrow \bigcap_{i\in I}\psi_i \geqslant \inf\{\psi_i(x\odot y),\psi_i(x\star y),\psi_i(y)\}$.
Therefore, $\bigcap_{i\in I}\psi_i$ is a fuzzy pseudo-ideal of X. □

Proposition 2.4.3. *Every fuzzy pseudo-ideal of a pseudo-TM algebra X is order reversing.*

Proof. If $x\leqslant y$, then $x\odot y = 0$ and $x\star y = 0\in I$ where I is Pseudo-ideal of X and,

$$\begin{aligned}
\text{So}, \psi(x) &\geqslant \min\{\psi(x\odot y),\psi(x\star y),\psi(y)\}\\
&= \min\{\psi(0),\psi(0),\psi(y)\}\\
&= \min\{\psi(0),\psi(y)\}\\
&= \psi(y), \quad \text{since} \quad \psi(0)\geqslant\psi(y).
\end{aligned}$$

Thus, $x\leqslant y$ implies $\psi(x)\geqslant\psi(y)$. This shows that ψ is order reversing. □

Remark 2.4.1. *The union of any two fuzzy pseudo- ideal of a pseudo-TM algebra X is not necessarily a fuzzy pseudo-ideal of X.*

Example 2.4.3. *Let $X = \{0,1,2,3\}$ be a set with binary operatiom $\odot$ and $\star$ by the following Cayley table.*
See [57] $(X,\odot,\star,0)$ is a valid pseudo-TM algebra.
Define a fuzzy Pseudo-Ideals ψ_1 and ψ_2 as follows:

$$\psi_1(x) = \begin{cases} 1 & \text{if } x=0,\\ 0.5 & \text{if } x=1,\\ 0.3 & \text{if } x=2,\\ 0.1 & \text{if } x=3, \end{cases} \qquad \psi_2(x) = \begin{cases} 0.3 & \text{if } x=0,\\ 0.3 & \text{if } x=1,\\ 0.3 & \text{if } x=2,\\ 0.3 & \text{if } x=3. \end{cases}$$

$\odot$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\star$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 2.3

It is easly verify that ψ_1 and ψ_2 are fuzzy pseudo-ideals. Then taking the union $\psi = \psi_1 \cup \psi_2$ defined by:

$$\psi(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.3 & \text{if } x = 3. \end{cases}$$

Let $x = 3$ and $y = 1$. Then

$$\begin{aligned} \psi(3) = 0.3 &\geqslant \min\{\psi(3 \odot 1), \psi(3 \star 1), \psi(1)\} \\ &= \min\{\psi(2), \psi(2), 0.5\} \\ &= \min\{0.6, 0.6, 0.5\} = 0.5. \textit{ But}, 0.3 \not\geqslant 0.5 \ \textit{ which is not true.} \end{aligned}$$

Therefore, the union of two fuzzy pseudo-ideals in a pseudo-TM algebra is not necessarily a fuzzy pseudo-ideal in a pseudo-TM algebra.

Theorem 2.4.4. *Assume that X is a pseudo-TM algebra and let ψ be a fuzzy subset of X. Then ψ is a fuzzy pseudo-ideal of X if and only if the non-empty upper level set $U(\psi, t)$ is a pseudo-ideal of X, for all $t \in [0,1]$.*

Proof. Suppose that ψ is a fuzzy pseudo-ideal of X. We need to show that $U(\psi, t)$ is a pseudo-ideal of X or empty of X, for all $t \in [0,1]$. Let ψ be a fuzzy pseudo-ideal of a pseudo-TM algebra X and $U(\psi, t) \neq \emptyset$ for every $t \in [0,1]$. Obviously, $0 \in U(\psi, t)$. Since $\psi(0) \geqslant t$. Assume $x, y \in X$ such that $(x \odot y)$, $(x \star y)$ and $y \in U(\psi, t)$, then $\psi(x \odot y) \geqslant t$, $\psi(x \star y) \geqslant t$ and $\psi(y) \geqslant t$.
It follows that $\psi(x) \geqslant \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\} \geqslant t$. It implies that $x \in U(\psi, t)$.
Therefore, $U(\psi, t)$ is a pseudo-ideal of X.
Conversely, that $U(\psi, t)$ is a pseudo-ideal of X or empty of X, for all $t \in [0,1]$. We need to show that ψ is a fuzzy pseudo-ideal of X. For every $t \in [0,1]$ and $U(\psi, t)$ a pseudo-ideal, $\forall x \in X$.
Let $\psi(x) = t$ then $x \in U(\psi, t)$ and $U(\psi, t) \neq \emptyset$. Since $U(\psi, t)$ is a pseudo-ideal of X which implies that $0 \in U(\psi, t)$.
Hence, $\psi(0) \geqslant t = \psi(x)$, for all $x \in X$.
Suppose that for every $x, y \in X$ such that $\psi(x) < \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\}$.
Assume that $t_1 = \frac{1}{2}(\psi(x) + \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\})$,
then $\psi(x) < t_1 < \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\}$.
Hence, $x \notin U(\psi, t_1)$ and $\psi(x \odot y), \psi(x \star y), \psi(y) \in U(\psi, t_1)$.This is impossible.

Therefore, ψ is a fuzzy pseudo-ideal of a pseudo-TM algebra X. $\square$

Theorem 2.4.5. *If ψ is a fuzzy pseudo -ideal of a pseudo-TM algebra,then the set*

$$J = \{x \in X \mid \psi(x) = \psi(0)\}$$

is a pseudo -ideal of X.

Proof. Suppose that ψ is a fuzzy pseudo -ideal of a pseudo-TM algebra. We need to show that $J = \{x \in X \mid \psi(x) = \psi(0)\}$ is a pseudo -ideal of X. Since ψ is a fuzzy pseudo -ideal of a pseudo-TM algebra. So $0 \in J$.
Let $x, y \in X$ be such that $x \odot y,\ x \star y \in J$ and $y \in J$. Then $\psi(x \odot y) = \psi(0) = \psi(x \star y)$ and $\psi(y) = \psi(0)$ which implies that $\psi(x) \geqslant \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\} = \psi(0)$. But ψ is a fuzzy pseudo -ideal of a pseudo-TM algebra. Hence $\psi(0) \geqslant \psi(x)$. It follows that $\psi(x) = \psi(0)$.
Therefore, $x \in J$, and thus J is a pseudo-ideal of X. $\square$

Proposition 2.4.6. *If ψ is a fuzzy pseudo-ideal of a pseudo-TM -algebra X, then the set* $J = \{x \in X \mid \psi(x) \geqslant \psi(b)\}$ *is a pseudo -ideal of* X *for any* $b \in X$.

Proof. Suppose that ψ is a fuzzy pseudo-ideal of X.We need to show that J is a pseudo-ideal of X. Let $x, y \in X$ such that $x \odot y \in J$, $x \star y \in J$ and $y \in J$. Then

$$\begin{aligned} \psi(x \odot y) \geqslant \psi(b), \psi(x \star y) \geqslant \psi(b) \ \text{ and} \\ \psi(y) \geqslant \psi(b) \\ \Rightarrow \psi(x) \geqslant \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\} \geqslant \psi(b). \end{aligned}$$

It follows that $\psi(x) \geqslant \psi(b)$. $\square$

Hence, $x \in J$, and thus J is a pseudo-ideal of X.

Theorem 2.4.7. *Let ψ be any fuzzy subset of X. Then the fuzzy subset $\psi^\star$ of X defined by* $\psi^\star = \sup\{t \in [0,1] \mid x \in \langle U(\psi, t)\rangle\}$ *for all $x \in X$. Then $\psi^\star$ is the least fuzzy pseudo-ideal of X that contains ψ and $\langle U(\psi, t)\rangle$ is the least pseudo-ideal contains* $U(\psi, t)$.

Proof. Suppose that ψ is a fuzzy subset of X and $\psi^\star = \sup\{t \in [0,1] \mid x \in \langle U(\psi, t)\rangle\}$.
We need to prove that $\psi^\star$ is a fuzzy pseudo-ideal of X. Let $k \in \operatorname{Im}(\psi^\star)$. For any $\epsilon > 0$ such that $s = k - \epsilon$.
If $x \in U(\psi^\star, K)$, then $\psi^\star(x) \geqslant k$.
It implies that $\sup\{t \in [0,1] \mid x \in \langle U(\psi; t)\rangle\} \geqslant k > k - \epsilon = s$.
$\Rightarrow s \leqslant K$.
Hence, there exists $t_1 \in \{t \in [0,1] \mid x \in \langle U(\psi, t)\rangle\}$ such that $t_1 > s$.

$$\Rightarrow U(\psi, t_1) \subseteq U(\psi, s)$$
$$x \in \langle U(\psi, t_1) \rangle$$
$$\subseteq \langle U(\psi, s) \rangle$$
$$\Rightarrow x \in \langle U(\psi, s) \rangle .$$
$$\Rightarrow U(\psi^\star, K) \subseteq \langle U(\psi, s) \rangle \text{ and}$$
$$\text{Let } x \in \langle U(\psi, s) \rangle, \text{ then}$$
$$s \in \{t \in [0,1] \mid x \in \langle U(\psi, t) \rangle\}$$
$$\Rightarrow s \leqslant Sup\{t \in [0,1] \mid x \in \langle U(\psi, t) \rangle\} = \psi^\star(x)$$
$$\Rightarrow s \leqslant \psi^\star(x)$$
$$\Rightarrow U(\psi^\star, K) = U(\psi, s) .$$

Hence, $U(\psi^\star, K)$ is a pseudo-ideal of X.
Therefore, $\psi^\star$ is a fuzzy pseudo-ideal of X.
Again we need to show that $\psi^\star$ contains ψ.
Suppose that for any $x \in X$. Let $k \in \{t \in [0,1] \mid x \in U(\psi, t)\}$.
Then $x \in U(\psi, k)$ and that $x \in \langle U(\psi, t) \rangle$.
So, $k \in \{t \in [0,1] \mid x \in \langle U(\psi, t) \rangle\}$.
This implies that $t \in [0,1] \mid x \in U(\psi, t) \subseteq t \in [0,1] \mid x \in \langle U(\psi, t) \rangle$.
$\Rightarrow \psi(x) = Sup\{t \in [0,1] \mid x \in U(\psi, t)\} \leqslant Sup\{t \in [0,1] \mid x \in \langle U(\psi, t) \rangle\} = \psi^\star \leqslant \psi^\star$
$\Rightarrow \psi \subseteq \psi^\star$.
Finally, let ω be a fuzzy pseudo-ideal of X containing ψ. For any $x \in X$.

Case 1. If $\psi^\star = 0$ implies $\psi^\star(x) \leqslant \omega(x)$.

Case 2. Assume that $\psi^\star(x) = k \neq 0$.Then $x \in U(\psi^\star, k) = \langle U(\psi, s) \rangle$.
$\Rightarrow x \in \langle U(\psi, s) \rangle$
$\Rightarrow \omega(x) \geqslant \psi(x) \geqslant s = k - \epsilon$
$\Rightarrow \omega(x) \geqslant s = \psi^\star(x)$.
It follows that $\psi^\star$ is contained in ω.

□

Theorem 2.4.8. *Every fuzzy pseudo-ideal of a pseudo-TM algebra is a fuzzy pseudo-subalgebra.*

Proof. Let ψ be a fuzzy pseudo-ideal of a pseudo-TM algebra X. We want to show that ψ is a fuzzy pseudo-TM subalgebra, i.e., for all $x, y \in X$,

(i) $\psi(x \odot y) \geqslant \min\{\psi(x), \psi(y)\}$,

(ii) $\psi(x \star y) \geqslant \min\{\psi(x), \psi(y)\}$.

We first prove (i). Since ψ is a fuzzy pseudo-ideal, for any $x, y \in X$, we have

$$\psi(x \odot y) \geqslant \min\{\psi((x \odot y) \odot y), \psi((x \odot y) \star y), \psi(y)\}.$$

By Lemma 1.5.1 we have

$$\psi(x \odot y) \geqslant \min\{\psi(x), \psi((x \odot y) \star y), \psi(y)\}. \tag{2.4.1}$$

Now, consider $\psi((x \odot y) \star y)$. Using the fuzzy pseudo-ideal condition again with x replaced by $(x \odot y) \star y$ and y replaced by x, we get

$$\psi((x \odot y) \star y) \geqslant \min\{\psi(((x \odot y) \star y) \odot x), \psi(((x \odot y) \star y) \star x), \psi(x)\}. \tag{2.4.2}$$

Assume, without loss of generality $((x \odot y) \star y) \odot x = 0$ and $((x \odot y) \star y) \star x = 0$. Therefore, since $\psi(0) \geqslant \psi(z)$ for all $z \in X$, we have $\psi(((x \odot y) \star y) \odot x) = \psi(0)$ and $\psi(((x \odot y) \star y) \star x) = \psi(0)$. Hence, (2.4.2) becomes $\psi((x \odot y) \star y) \geqslant \min\{\psi(0), \psi(0), \psi(x)\} = \psi(x)$. So, $\psi((x \odot y) \star y) \geqslant \psi(x)$. Substituting this into (2.4.1) yields $\psi(x \odot y) \geqslant \min\{\psi(x), \psi(x), \psi(y)\} = \min\{\psi(x), \psi(y)\}$. This proves (i). Similarly, for any $x, y \in X$, using the fuzzy pseudo-ideal condition with x replaced by $x \star y$ and y replaced by y, we have $\psi(x \star y) \geqslant \min\{\psi((x \star y) \odot y), \psi((x \star y) \star y), \psi(y)\}$. Since $(x \star y) \star y = x$ in a pseudo-TM algebra, this becomes $\psi(x \star y) \geqslant \min\{\psi(x), \psi((x \star y) \odot y), \psi(y)\}$. Now, consider $\psi((x \star y) \odot y)$. By Lemma 1.5.1 and y replaced by x, we get $\psi((x \star y) \odot y) \geqslant \min\{\psi(((x \star y) \odot y) \odot x), \psi(((x \star y) \odot y) \star x), \psi(x)\}$. Assume, without loss of generality $((x \star y) \odot y) \odot x = 0$ and $((x \star y) \odot y) \star x = 0$. So then $\psi((x \star y) \odot y) \geqslant \min\{\psi(0), \psi(0), \psi(x)\} = \psi(x)$. Thus, $\psi((x \star y) \odot y) \geqslant \psi(x)$, and hence $\psi(x \star y) \geqslant \min\{\psi(x), \psi(x), \psi(y)\} = \min\{\psi(x), \psi(y)\}$. This proves (ii). Therefore, ψ is a fuzzy pseudo-TM subalgebra of X. □

Remark 2.4.2. *The converse of Theorem 2.4.8 is may not be necessary true.*

Example 2.4.4. *Let the set* $X = \{0, 1, 2\}$, *and define two binary operations* $\odot$ *and* $\star$, *and constant 0 such that:*

$\odot$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\star$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

Table 2.4

See [57]$(X; \odot, \star, 0)$ *is a pseudo TM-algebra. Define a fuzzy subset* $\psi : X \to [0, 1]$ *by*

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.3 & \text{if } x = 1, \\ 0.2 & \text{if } x = 2. \end{cases}$$

Then ψ is a fuzzy pseudo-TM subalgebra. *We must check* $\psi(x \odot y) \geqslant \min\{\psi(x), \psi(y)\}$ *and* $\psi(x \star y) \geqslant \min\{\psi(x), \psi(y)\}$

Now, ψ is not a fuzzy pseudo-ideal. *By Lemma 2.1.1* $\psi(0) \geqslant \psi(x)$ *for all* $x \in X$. *Here* $\psi(0) = 0$, *but* $\psi(1) = 0.3 > 0$. *Hence* $\psi(0) \geqslant \psi(1)$ *fails, so* ψ *is* not *a fuzzy pseudo-ideal.*
Therefore, every fuzzy pseudo-TM subalgebra need not be a fuzzy pseudo-ideal.

2.5. Homomorphism of Fuzzy Pseudo ideal of Pseudo-TM Algebra

In this section, we give several characterizations for fuzzy pseudo-ideals and fuzzy homomorphisms of a pseudo-TM algebra.

Theorem 2.5.1. *Let* $(X, \odot, \star, 0)$ *and* $(Y, \odot, \star, 0)$ *be a pseudo-TM algebras and let* $f : X \to Y$ *be a homomorphism and* ψ *be a fuzzy pseudo-ideal of* Y. *Then* $f^{-1}(\psi)$ *is a fuzzy pseudo-ideal of* X.

Proof. Suppose that ψ be a fuzzy pseudo-ideal of Y. We need to show that $f^{-1}(\psi)$ is a fuzzy pseudo-ideal of X. Let $x \in X$. Then $f(x) \in Y$ and ψ is a fuzzy pseudo-ideal of Y,we have
$f^{-1}(\psi)(0) = \psi(f(0)) \geqslant \psi(f(x))$. Thus we get $f^{-1}(\psi)(0) \geqslant f^{-1}(\psi)(x)$, for all $x \in X$.
Let $x, y \in X$. Since ψ is a fuzzy pseudo-ideal of Y, we have

$$\begin{aligned}
f^{-1}(\psi(x)) = \psi(f(x)) &\geqslant \min\{\psi(f(x) \odot f(y)), \psi(f(x) \star f(y)), \psi(f(y))\} \\
&= \min\{\psi(f(x \odot y)), \psi(f(x \star y)), \psi(f(y))\} \\
&= \min\left\{f^{-1}(\psi)(x \odot y), f^{-1}(\psi)(x \star y), f^{-1}(\psi)(y)\right\} \\
\Rightarrow f^{-1}(\psi(x)) &\geqslant \min\left\{f^{-1}(\psi)(x \odot y), f^{-1}(\psi)(x \star y), f^{-1}(\psi)(y)\right\}
\end{aligned}$$

That is $f^{-1}(\psi)$ satisfies Definition 2.4.1.
Therefore, $f^{-1}(\psi(x))$ is a fuzzy pseudo-ideal of X. □

Lemma 2.5.2. *Let* $(X, \odot, \star, 0)$ *and* $(Y, \odot, \star, 0)$ *be a pseudo-TM algebras and let* $f : X \to Y$ *be a homomorphism and* ψ *be a fuzzy pseudo-ideal of* X. *If* ψ *is constant on* $\mathrm{Ker}(f) = f^{-1}(0)$, *then* $f^{-1}(f)(\psi) = \psi$.

Proof. Suppose that ψ be a fuzzy pseudo-ideal of X and if ψ is constant on $\mathrm{Ker}(f) = f^{-1}(0)$. We need to show that that $f^{-1}(f)(\psi) = \psi$.
Let $x \in X$ and $f(x) = y$. Hence

$$\begin{aligned}
f^{-1}(f)(\psi)(x) &= f(\psi)f(x) \\
&= f(\psi)(y) \\
&= \mathrm{Sup}\left\{\psi(b) : b \in f^{-1}(y)\right\}
\end{aligned}$$

For all $b \in f^{-1}(y)$, we have $f(x) = f(b)$. Hence, $f(b \odot x) = f(b \star x) = 0$
$\Rightarrow b \odot x, b \star x \in \mathrm{Ker}(f)$. It follows that $\psi(b \odot x) = \psi(0) = \psi(b \star x)$.Then
$\psi(b) \geqslant \min\{\psi(b \odot x), \psi(b \star x), \psi(x)\} = \min\{\psi(0), \psi(x)\} = \psi(x)$

We get,$\psi(b) \geqslant \psi(x)$. Since by Proposition 2.4.6 $\psi(x) \geqslant \psi(b)$.
Hence, $\psi(x) = \psi(b)$.
Thus ,$\left(f^{-1}(f)(\psi)\right)(x) = \text{Sup}\left\{\psi(b) : b \in f^{-1}(y)\right\} = \psi(x)$.
Therefore, $f^{-1}(f)(\psi) = \psi$. □

Theorem 2.5.3. *Let* $(X, \odot, \star, 0)$ *and* $(Y, \odot, \star, 0)$ *be a pseudo-TM algebras and let* $f : X \to Y$ *be an epimorphism and* ψ *be a fuzzy pseudo-ideal of* X *such that* ψ *is constant on* $Ker(f)$*. Then* $f(\psi)$ *is a fuzzy pseudo-ideal of* Y*, provided that the sup property holds.*

Proof. Suppose that ψ be a fuzzy pseudo-ideal of X such that ψ is constant on $Ker(f)$. We need to show that $f(\psi)$ is a fuzzy pseudo-ideal of Y, provided that the sup property holds.
Let $0 \in Y$, then there exist $0 \in X$ such that $f(0) = 0$. Now, $f(\psi)(0) = \left\{\sup \psi(k) \mid k \in f^{-1}(0)\right\} = \psi(0) \geqslant \psi(x)$, for all $x \in X$. Let $y \in Y$. Since f is an epimorphism, we have $f^{-1}(y) \neq \emptyset$ and $\psi(0) \geqslant \psi(k) \mid k \in f^{-1}(y)$. Which implies that $f(\psi)(0) \geqslant \left\{\sup \psi(k) \mid k \in f^{-1}(y)\right\} = f(\psi)(y)$, for all $y \in Y$. Hence, $f(\psi)(0) \geqslant f(\psi)(y), \forall y \in Y$.
Thus $f(\psi)$ satisfies Definition 2.4.1.
Assume that $(f(\psi))(a) < \min\{(f(\psi))(a \odot b), (f(\psi))(a \star b), (f(\psi))(b)\}$ for some $a, b \in Y$. Since f is an epimorphism there exists $x, y \in X$ such that $f(x) = a$ and $f(x) = b$.
Hence,$(f(\psi))(f(x)) < \min\{(f(\psi))(f(x \odot y)), (f(\psi))(f(x \star y)), (f(\psi))(f(y))\}$.
Then $\left(f^{-1}(f(\psi))\right)(x) < \min\left\{\left(f^{-1}(f(\psi))\right)(x \odot y), \left(f^{-1}(f(\psi))\right)(x \star y), \left(f^{-1}(f(\psi))\right)(y)\right\}$.
But ψ is constant on $Ker(f)$. Then by Lemma 4.5.3, we get $\psi(x) < \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\}$ which is a contradiction with the fact that ψ is a fuzzy pseudo-ideal of X.
Therefore, $f(\psi)$ is a fuzzy pseudo-ideal of Y. □

2.6. Cartesian product of Fuzzy Pseudo ideal of Pseudo-TM Algebra

In this section, we introduced the notion of fuzzy pseudo-ideals in pseudo-TM algebra in terms of Cartesian product of any two pseudo-TM algebra.

Theorem 2.6.1. *Let* $(X, \odot, \star, 0)$ *and* $(Y, \odot, \star, 0)$ *be a pseudo-TM algebras and let* ψ *and* ω *be any two fuzzy pseudo-ideal of* X *and* Y *respectively. Then* $\psi \times \omega$ *is a fuzzy pseudo-ideals of* $X \times Y$*.*

Proof. Suppose that ψ and ω be a pseudo-TM algebras of X and Y respectively. We need to show that $\psi \times \omega$ is a fuzzy pseudo-ideals of $X \times Y$.

(i) Let $(x, y) \in X \times Y$. Then $(\psi \times \omega)(0, 0) = \min\{\psi(0), \omega(0)\} \geqslant \min\{\psi(x), \omega(y)\}$.
Hence, $(\psi \times \omega)(0, 0) \geqslant (\psi \times \omega)(x, y)$.

(ii) Let $(x_1, x_2), (y_1, y_2) \in X \times Y$.

Then

$$(\psi \times \omega)(x_1, x_2) = \min\{\psi(x_1), \omega(x_2)\}$$

$$\geqslant \min\Big\{ \min\{\psi(x_1 \odot y_1), \psi(x_1 \star y_1), \psi(y_1)\},$$
$$\min\{\omega(x_2 \odot y_2), \omega(x_2 \star y_2), \omega(y_2)\} \Big\}$$
$$= \min\Big\{ \min\{\psi(x_1 \odot y_1), \omega(x_2 \odot y_2)\},$$
$$\min\{\psi(x_1 \star y_1), \omega(x_2 \star y_2)\},$$
$$\min\{\psi(y_1), \omega(y_2)\} \Big\}$$
$$= \min\Big\{ (\psi \times \omega)\,((x_1,x_2) \odot (y_1,y_2)),$$
$$(\psi \times \omega)\,((x_1,x_2) \star (y_1,y_2)),$$
$$(\psi \times \omega)(y_1,y_2) \Big\}$$

Hence, $(\psi \times \omega)\,(x_1,x_2) \geqslant \min\{(\psi \times \omega)\,(x_1,x_2) \odot (y_1,y_2), (\psi \times \omega)\,(x_1,x_2) \star (y_1,y_2), (\psi \times \omega)\,(y_1,y_2)\}$.
By (i) and (ii) above, we have that $\psi \times \omega$ is a fuzzy pseudo-ideal of X. □

Theorem 2.6.2. *Let* $(X,\odot,\star,0)$ *and* $(Y,\odot,\star,0)$ *is a pseudo-TM algebras. If* $\psi \times \omega$ *is a fuzzy pseudo-ideal of* $X \times Y$ *for any fuzzy subsets* ψ *and* ω *of a pseudo-TM algebra of* X *and* Y *respectively, then either* ψ *is a fuzzy pseudo-ideal of* X *or* ω *is a fuzzy pseudo-ideal of* Y.

Proof. Suppose that $\psi \times \omega$ is a fuzzy pseudo-ideal of $X \times Y$ such that ψ and ω is a fuzzy subset of X and Y respectively. We need to show that either ψ is a fuzzy pseudo-ideal of X or ω is a fuzzy pseudo-ideal of Y. Then $\psi \times \omega\,(0,0) = \min\{\psi(0), \omega(0)\} \geqslant \min\{\psi(x), \omega(y)\}$, for all $(x,y) \in X \times Y$.
Suppose that $\psi(x) > \psi(0)$ and $\omega(y) > \omega(0)$, for some $(x,y) \in X \times Y$.
Now, $(\psi \times \omega)(x,y) = \min\{\psi(x), \omega(y)\} > \min\{\psi(0), \omega(0)\} = (\psi \times \omega)(0,0)$ which is a contradiction.
Thus, $\psi(0) \geqslant \psi(x)$ or $\omega(0) \geqslant \omega(x)$, for all $(x,y) \in X \times Y$.
Let $(x_1,x_2), (y_1,y_2) \in X \times Y$. Then

$$\begin{aligned}
(\psi \times \omega)\,(x_1,x_2) &= \min\{\psi(x_1), \omega(x_2)\} \\
&\geqslant \min\Big\{ (\psi \times \omega)\,((x_1,x_2) \odot (y_1,y_2)), (\psi \times \omega)\,((x_1,x_2) \star (y_1,y_2)), (\psi \times \omega)\,(y_1,y_2) \Big\} \\
&= \min\Big\{ \min\{\psi(x_1 \odot y_1), \psi(x_1 \star y_1), \psi(y_1)\}, \min\{\omega(x_2 \odot y_2), \omega(x_2 \star y_2), \omega(y_2)\} \Big\} \\
&= \min\Big\{ \min\{\psi(x_1 \odot y_1), \omega(x_2 \odot y_2)\}, \min\{\psi(x_1 \star y_1), \omega(x_2 \star y_2)\}, \\
&\qquad \min\{\psi(y_1), \omega(y_2)\} \Big\} \\
&\Rightarrow \psi(x_1) \geqslant \min\{\psi(x_1 \odot y_1), \psi(x_1 \star y_1), \psi(y_1)\} \\
&\qquad \text{or } \omega(x_2) \geqslant \min\{\omega(x_2 \odot y_2), \omega(x_2 \star y_2), \omega(y_2)\}.
\end{aligned}$$

Hence, either $\psi(x_1) \geqslant \min\{\psi(x_1 \odot y_1), \psi(x_1 \star y_1), \psi(y_1)\}$
or $\omega(x_2) \geqslant \min\{\omega(x_2 \odot y_2), \omega(x_2 \star y_2), \omega(y_2)\}$.

$\square$

Remark 2.6.1. *The converse of Theorem 2.6.2 is may not be necessarly true.*

Example 2.6.1. *Let* $X = \{0,1,2,3\}$ *with the binary operations* $\odot$ *and* $\star$ *given by the Cayley tables defined in 2.4.3. Define the fuzzy subset* $\psi : X \to [0,1]$ *by*

$$\psi(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.6 & \text{if } x = 1, \\ 0.5 & \text{if } x = 2,3. \end{cases}$$

We verify the two conditions of a fuzzy pseudo-ideal.

(1) $\psi(0) \geqslant \psi(x)$ *for all* $x \in X$*:*

$$\psi(0) = 1 \geqslant 0.6 = \psi(1), \quad 1 \geqslant 0.5 = \psi(2), \quad 1 \geqslant 0.5 = \psi(3).$$

Thus condition (1) holds.

(2) For all $x, y \in X$,

$$\psi(x) \geqslant \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\}.$$

Note that the image of ψ *is the set* $\{1, 0.6, 0.5\}$*. Hence for any* x, y *the right-hand side* $\min\{\psi(x \odot y), \psi(x \star y), \psi(y)\}$ *is one of* $1, 0.6, 0.5$*. Since every value* $\psi(x)$ *is at least* 0.5 *(indeed* $\psi(x) \in \{1, 0.6, 0.5\}$*), we have* $\psi(x) \geqslant 0.5$ *for all* x*. Therefore for every* x, y *the inequality* $\psi(x) \geqslant \min\{\psi(x \odot y), \psi(x \star y), \psi(y)\}$ *holds. Consequently,* ψ *is a fuzzy pseudo-ideal of X.*

Define the fuzzy subset $\omega : X \to [0,1]$ *by*

$$\omega(x) = \begin{cases} 0.1 & \text{if } x = 0, \\ 0.6 & \text{if } x = 1,2,3. \end{cases}$$

Now, consider for any $(x,y) \in X \times Y$*. Then* $(\psi \times \omega)(0,0) = \min\{\psi(0), \omega(0)\} \geqslant \min\{\psi(x), \omega(y)\}$*. Taking* $x = 0$ *and* $y = 2$ *implies* $\min\{1, 0.1\} = 0.1 \ngeqslant \min\{\psi(x), \omega(y)\} = \min\{\psi(0), \omega(2)\} = \min\{1, 0.6\} = 0.6$*. Hence the converse of Theorem 2.6.2 is may not be necessary true.*

Definition 2.6.1. *Let* $(X, \odot, \star, 0)$ *and* $(Y, \odot, \star, 0)$ *be a pseudo-TM algebras and let* ψ *and* ω *be any two fuzzy subset of* X *and* Y *respectively. Then* $\psi \times \omega$ *is a fuzzy pseudo-ideals of* $X \times Y$*. For* $t \in [0,1]$*, the set* $U(\psi \times \omega, t) = \{(x,y) \in X \times Y \mid (\psi \times \omega)(x,y) \geqslant t\}$ *is called the upper* t*-level set of* $\psi \times \omega$.

Theorem 2.6.3. *Let* $(X; \odot, \star, 0)$ *and* $(Y; \odot, \star, 0)$ *be a pseudo-TM algebras and let* ψ *and* ω *be any two fuzzy pseudo-ideal of* X *and* Y *respectively. Then* $\psi \times \omega$ *is a fuzzy pseudo-ideals of* $X \times Y$ *if and only if the non-empty upper* $t-$*level set* $U(\psi \times \omega, t)$ *is a pseudo-ideal of* $X \times Y$ *for any* $t \in [0,1]$.

Proof. Let $\psi \times \omega$ be a fuzzy pseudo-ideals of $X \times Y$ such that $U(\psi \times \omega, t) \neq \emptyset$ for all $t \in [0,1]$. We need to show that $U(\psi \times \omega, t)$ is a pseudo-ideal of $X \times Y$ for any $t \in [0,1]$. Now, there exists $(x,y) \in U(\psi \times \omega, t)$. Thus $(\psi \times \omega)(x,y) \geqslant t$. Since $\psi \times \omega$ is a fuzzy pseudo-ideal of $X \times Y$. For all $(x,y) \in X \times Y$. We have

$$\begin{aligned}(\psi \times \omega)(0,0) &\geqslant (\psi \times \omega)(x,y)\\ &\Rightarrow (\psi \times \omega)(0,0)\\ &\geqslant (\psi \times \omega)(x,y) \geqslant t\\ &\Rightarrow (\psi \times \omega)(0,0) \geqslant t\\ &\Rightarrow (0,0) \in U(\psi \times \omega, t).\end{aligned}$$

Suppose that $(x_1,y_1),(x_2,y_2) \in U(\psi \times \omega, t)$ such that $(x_1,y_1) \odot (x_2,y_2), (x_1,y_1) \star (x_2,y_2), (x_2,y_2) \in U(\psi \times \omega, t)$
$\Rightarrow (\psi \times \omega)((x_1,y_1) \odot (x_2,y_2)) \geqslant t, (\psi \times \omega)((x_1,y_1) \star (x_2,y_2)) \geqslant t$ and $(\psi \times \omega)((x_2,y_2) \geqslant t$.
Since $\psi \times \omega$ is a fuzzy pseudo-ideal of $X \times Y$. We have

$$\begin{aligned}(\psi \times \omega)(x_1,y_1) &\geqslant \min\{(\psi \times \omega)((x_1,y_1) \odot (x_2,y_2)), (\psi \times \omega)((x_1,y_1) \star (x_2,y_2)), (\psi \times \omega)(x_2,y_2)\}\\ &\geqslant \min\{t,t,t\} = t\\ &\Rightarrow (\psi \times \omega)(x_1,y_1) \geqslant t\\ &\Rightarrow (x_1,y_1) \in U(\psi \times \omega, t)\end{aligned}$$

Hence, $U(\psi \times \omega, t)$ is a pseudo-ideal of $X \times Y$.
Conversely, assume that the set $U(\psi \times \omega, t)$ be a pseudo-ideal of $X \times Y$, for each $t \in [0,1]$.
Let $(x,y) \in X \times Y$ such that $(\psi \times \omega) = t$. Then $(x,y) \in U(\psi \times \omega, t)$. Since $(\psi \times \omega)$ is a fuzzy pseudo-ideal of $X \times Y$, then $(0,0) \in U(\psi \times \omega, t)$.
Hence, $(\psi \times \omega)(0,0) \geqslant t = (\psi \times \omega)(x,y)$ for all $(x,y) \in X \times Y$. Assume that
$t_1 = \frac{1}{2}\{(\psi \times \omega)(x,x) + \min\{(\psi \times \omega)((x,x) \odot (x_1,y_1)), (\psi \times \omega)((x,x) \star (x_1,y_1)), (\psi \times \omega)(x_1,y_1)\}\}$.
Then $(\psi \times \omega)(x,x) < t_1 < \min\{(\psi \times \omega)((x,x) \odot (x_1,y_1)), (\psi \times \omega)((x,x) \star (x_1,y_1)), (\psi \times \omega)(x_1,y_1)\}$.
It follows that $(x,x) \neq U(\psi \times \omega, t_1)$. Also, $(\psi \times \omega)((x,x) \odot (x_1,y_1)), (\psi \times \omega)((x,x) \star (x_1,y_1)) \in U(\psi \times \omega, t_1)$ and $(\psi \times \omega)(x_1,y_1) \in U(\psi \times \omega, t_1)$ which is a contradiction. Since $U(\psi \times \omega, t_1)$ is a pseudo-ideal of $X \times Y$. Therefore, $\psi \times \omega$ is a fuzzy pseudo-ideal of $X \times Y$. □

Theorem 2.6.4. *Let* $(X; \odot, \star, 0)$ *and* $(Y; \odot, \star, 0)$ *be a pseudo-TM algebras and let* ψ *and* ω *be any two fuzzy pseudo-ideal of* X *and* Y *respectively such that* $\psi \times \omega$ *is a fuzzy pseudo-ideals of* $X \times Y$.*Then*

(i) Either $\psi(0) \geqslant \psi(x)$ *or* $\omega(0) \geqslant \omega(x)$*, for all* $x \in X$.

(ii) If $\psi(0) \geqslant \psi(x)$ *for all* $x \in X$*, then either* $\omega(0) \geqslant \psi(x)$ *or* $\omega(0) \geqslant \omega(x)$

(iii) If $\omega(0) \geqslant \psi(x)$ *for all* $x \in X$*, then either* $\psi(0) \geqslant \psi(x)$ *or* $\psi(0) \geqslant \omega(x)$.

Proof. let ψ and ω be any two fuzzy pseudo-ideal of X and Y respectively such that $\psi \times \omega$ is a fuzzy pseudo-ideals of $X \times Y$.

(i) If $\psi(0) < \psi(x)$ and $\omega(0) < \omega(y)$ for some $x, y \in X$. Then $(\psi \times \omega)(x,y) = \min\{\psi(x), \omega(y)\} > \min\{\psi(0), \omega(0)\} = (\psi \times \omega)(0,0)$, which is a contradiction. Hence, either $\psi(0) \geqslant \psi(x)$ or $\omega(0) \geqslant \omega(x)$ for all $x \in X$.

(ii) Let $\psi(0) \geqslant \psi(x)$ for all $x \in X$. Assume there exist $x, y \in X$ such that $\omega(0) < \psi(x)$ and $\omega(0) < \omega(y)$.
Then, $(\psi \times \omega)(0,0) = \min\{\psi(0), \omega(0)\} = \omega(0)$, and

$$(\psi \times \omega)(x,y) = \min\{\psi(x), \omega(y)\} > \omega(0) = (\psi \times \omega)(0,0).$$

Therefore, $(\psi \times \omega)(x,y) > (\psi \times \omega)(0,0)$, which is a contradiction.
Hence, either $\omega(0) \geqslant \omega(x)$ or $\psi(0) \geqslant \psi(x)$ for all $x \in X$.

(iii) This result is obtained by interchanging the roles of ψ and ω in part (ii).

□

Chapter 3
Fuzzy Congruence Relation on Pseudo-TM Algebra

In algebra, a congruence relation is a special type of equivalence relation that is compatible with the operations of an algebraic structure, such as a group, ring, or lattice. This means if two elements are related by a congruence, their algebraic combinations with other elements will also be related. Congruence relations play a central role in constructing quotient structures and studying homomorphisms. Extending this concept, a fuzzy congruence relation arises from fuzzy set theory, which allows elements to be related to varying degrees, rather than in an absolute true/false manner. In a fuzzy congruence, elements of an algebraic structure are said to be congruent to each other with a degree of membership in the interval $[0,1]$. This generalization supports the study of algebraic systems under uncertainty or partial information and is useful in fields like computer science, decision-making, and control systems.
In this chapter, we apply the idea of congruence relation on a pseudo-TM algebra. We introduce the notion congruence relation in pseudo-TM algebras, considering the concept of fuzzy congruence relation in pseudo-TM algebra and investigate some associated properties. Moreover, we investigate the relationship between fuzzy pseudo-ideals and fuzzy congruence relations of pseudo-TM algebra and obtain some important results. Moreover, we discussed the relationship between fuzzy permutable congruence relations and fuzzy pseudo-ideals of a pseudo TM-algebra are investigate.

3.1. Congrunce Relation On Pseudo-TM Algebra

In this section we introduce the notion of congruence relations and quotients on pseudo-TM algebra and investigate some of its properties. Also, we prove that the quotient algebra is a pseudo-TM algebra and the quotient algebra is isomorphic to the image of pseudo-TM algebra under some conditions.

Definition 3.1.1. *Let* $(X, \odot, \star, 0)$ *be a pseudo-TM algebra . An equivalence relation* ϕ *is said to be congurence relation on* X. *If*

(i) $(x,y) \in \phi \Rightarrow (z \odot x, z \odot y) \in \phi$ *and* $(z \star x, z \star y) \in \phi$, *for all* $z \in X$.

(ii) $(x,y) \in \phi \Rightarrow (x \odot z, y \odot z) \in \phi$ *and* $(x \star z, y \star z) \in \phi$, *for all* $z \in X$.

(iii) If for any $(x,y),(z,w) \in \phi$. We have $(x \odot z, y \odot w) \in \phi$ and $(x \star z, y \star w) \in \phi$.

Example 3.1.1. *Let $X = \{0,1,2,3,4,5\}$ be a set with two binary operations $\odot$ and $\star$ which are given by table.*

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	0
2	2	2	0	0	0	0
3	3	2	2	0	0	0
4	4	0	0	0	0	0
5	5	0	0	0	0	0

$\star$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	0
2	2	2	0	0	0	0
3	3	3	1	0	0	0
4	4	0	0	0	0	0
5	5	0	0	0	0	0

Table 3.1

See [57] $(X,\odot,\star,0)$ is a pseudo-TM algebra. Consider the following $\phi = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)$ is a congruence relation on X. If we define $\phi = \{(0,0),(1,1),(2,2),(3,3),(0,2),(2,0),(2,3),(3,2)\}$, then ϕ is satisfied congruence relation on Definition 3.1.1(i), but it is not congruence relation on Definition 3.1.1(ii) because if $(3,2) \in \phi$, but $(3 \star 2, 2 \star 2) = (1,0) \notin \phi$. Let $Con(X)$ denote the set of all congruence on X. For $\phi \in Con(X)$ write $\phi^{[x]}$ for the equivalence class containing x, such that $\phi^{[x]} = \{y \in X : y \sim x\}$. The preservation conditions make it possible to define operations of $\ddot{\odot}$ and $\ddot{\star}$ on the set of equivalence classes: for any two equivalence classes $\phi^{[x]}$ and $\phi^{[y]}$ we have $\phi^{[x\odot y]} = \phi^{[x]} \odot \phi^{[y]}$ and $\phi^{[x\star y]} = \phi^{[x]} \star \phi^{[y]}$

Theorem 3.1.1. *Let ϕ be a congruence relation on a pseudo-TM algebra X. Then the set $\phi^{[0]} = \{x \in X : x \sim 0\}$ is a pseudo ideals of a pseudo-TM algebra X.*

Proof. Suppose that ϕ be a congruence relation on a pseudo-TM algebra X. Since ϕ is a reflexive relation, we see that $(0,0) \in \phi$. Hence $0 \sim 0$. Let $x,y \in X$ such that $x \odot y, x \star y \in \phi^{[x]}$ and $y \in \phi^{[x]}$. So, $x \odot y \sim 0, x \star y \sim 0$ and $y \sim 0$. From $y \sim 0$ it follows that $x \odot y \sim x \odot 0, x \star y \sim x \star 0$. Thus, $x \odot y \sim x$, $x \star y \sim x$. Since $x \odot y \sim 0$ and $x \odot y \sim x$. Also, $x \star y \sim 0$ and $x \star y \sim x$. In both sides we have $0 \sim x$. But ϕ is symmetric we have $x \sim 0$. Therefore, $\phi^{[0]}$ is a pseudo ideals of X. $\square$

Definition 3.1.2. *Let $\phi \in Con(X)$. Define $f : X \to X/\phi$ be such that $f(x) = \phi^{[x]}$ for all $x \in X$. Then the mapping f is called the canonical epimorphism, or natural epimorphism from $X \to X/\phi$.*

Theorem 3.1.2. *Let $\phi \in Con(X)$. Then $\left(X/\phi;\ \odot,\ \star,\ \phi^{[0]}\right)$ is a pseudo -TM algebra.*

Proof. Suppose that $\phi^{[x]}, \phi^{[y]}, \phi^{[z]} \in X/\phi$, for all $x,y,z \in X$, then we have

1)

$$\begin{aligned} \phi^{[x]} \odot \phi^{[0]} &= \phi^{[x\odot 0]} \\ &= \phi^{[x]} \quad \text{and} \\ \phi^{[x]} \star \phi^{[0]} &= \phi^{[x\star 0]} = \phi^{[x]} \end{aligned}$$

2)

$$\left(\phi^{[x]} \odot \phi^{[y]}\right) \star \left(\phi^{[x]} \odot \phi^{[z]}\right) = \phi^{[x\odot y]} \star \phi^{[x\odot z]}$$
$$= \phi^{[(x\odot y)\star(x\odot z)]}$$
$$= \phi^{[z\odot y]} \text{ and}$$
$$\left(\phi^{[x]} \star \phi^{[y]}\right) \odot \left(\phi^{[x]} \odot \phi^{[z]}\right) = \phi^{[x\odot y]} \odot \phi^{[x\odot z]}$$
$$= \phi^{[(x\star y)\odot(x\star z)]}$$
$$= \phi^{[z\star y]}$$

Therefore, $\left(X/\phi;\ \odot,\ \star,\ \phi^{[0]}\right)$ is a pseudo -TM algebra. □

Corollary 3.1.3. *The kernel of a pseudo-TM homomorphism of pseudo-TM algebras* X *is always a pseudo-ideal of* X.

Proof. Obviously $0 \in$ kerf, since $f(0) = 0_y$. Let $x \star y,\ x \odot y \in \ker f$, and $y \in$ kerf. So $f(x \star y) = 0_y$, $f(x \odot y) = 0_y$ and $f(y) = 0_y$. That is $f(x) \star f(y) = 0_y$ and $f(x) \odot f(y) = 0_y$

$$\Rightarrow f(x) \star 0_y = 0_y \text{ and } f(x) \odot 0_y = 0_y$$
$$\Rightarrow f(x) = 0_y$$

Hence $x \in \ker f$, So $\ker f$ is a pseudo ideal of X. □

Theorem 3.1.4. *Let* $f : X \to Y$ *be a pseudo-TM epimorphism . If* I *is a pseudo-ideal of a pseudo-TM algebra* X*, then* $f(I)$ *is a pseudo-ideal of* X.

Proof. Suppose that I is a pseudo-ideal of X. We need to show that $f(I)$ is a pseudo-ideal of X. Since I is a pseudo-ideal of X. Then $0 \in I$. It implies that $f(0)$ is in $f(I)$ and therefore $f(I)$ is non-empty

Let $x, y \in Y$ such that $x \odot y, x \star y \in f(I)$ and $y \in f(I)$. Since f is an epimorphism. Then there exists $a \in X$ and $b \in I$ such that $f(a) = x$ and $f(b) = y$.

Thus, $x \odot y = f(a) \odot f(b) = f(a \odot b) \in f(I)$ and $x \star y = f(a) \star f(b) = f(a \star b) \in f(I)$. So, $a \odot b, a \star b \in I$. Since I is a pseudo-ideal of X and $b \in I$. It follows that $a \in I$. Hence $x = f(a) \in f(I)$. Therefore, $x \in f(I)$. □

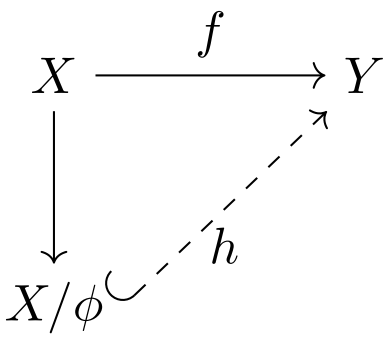
Theorem 3.1.5. *Let* f *be a pseudo-TM homomorphism from* X *onto* Y*. If* $\phi = \{(x,y) | f(x) = f(y)\}$ *is a congruence relation on* X*, then* $X/\phi \cong f(X)$.

Proof. Let h be the mapping from $X/\phi \to f(X)$ defined by $h(\phi^{[x]}) = f(x)$, for all $\phi^{[x]} \in X/\phi$. See the diagram below.

We need to show that h is well defined. Suppose that $\phi^{[x]} = \phi^{[y]}$ for all $x, y \in X$, then $(x, y) \in \phi$. Hence $f(x) = f(y)$. Therefore, $h(\phi^{[x]}) = h(\phi^{[y]})$. Also, we need to show that h is a monomorphism.

Now

$$\mathrm{Ker}(h) = \left\{\phi^{[x]} | h(\phi^{[x]}) = 0\right\}$$



$$
\begin{aligned}
&= \left\{\phi^{[x]} | h(\phi^{[x]}) = f(x) = 0 = f(0)\right\} \\
&= \left\{\phi^{[x]} | (x,0) \in \phi\right\} = \phi^{[0]}.
\end{aligned}
$$

Thus, h is a monomorphism and f maps X onto Y. It implies that h is an epimorphism. The proof that h preserves $\odot$ and $\star$ makes use of the definitions of $\odot$ and $\star$ in X/ϕ the definition of h, and the homomorphism properties of f.

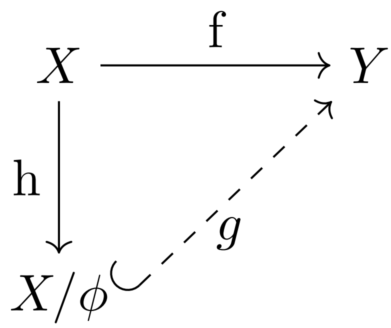
Let $\phi^{[x]}, \phi^{[y]} \in X/\phi$. Then $h(\phi^{[x]} \odot \phi^{[y]}) = h(\phi^{[x \odot y]}) = f(x \odot y) = f(x) \odot f(y) = h(\phi^{[x]}) \odot h(\phi^{[y]})$ and $h(\phi^{[x]} \star \phi^{[x]}) = h(\phi^{[x \star y]}) = f(x \star y) = f(x) \star f(y) = h(\phi[x]) \star h(\phi^{[y]})$. Thus h is a homomorphism.

Therefore, $X/\phi \cong f(X)$. □

Theorem 3.1.6. *Let* $f : X \to Y$ *be a pseudo-TM homomorphism between pseudo-TM algebras with congruence relation* ϕ*, and the canonical map from* $h : X \to X/\phi$*. Then there exist a unique monomorphism* $g : X/\phi \to Y$ *such that* $f = goh$.

Proof. Suppose that Y_0 be the image of X under f. Then the quotients X/ϕ is isomorphic to Y_0 via an isomorphism g that maps each costs X/ϕ to $f(x)$ by Theorem 3.1.5 and the commutative diagram below g is monomorphism of X/ϕ in to Y. The composition goh coincides with f because it maps each elements $x \in X$ to the element $g(h(x)) = g(X/\phi) = f(x)$. Assume that k be any homomorphism from X/ϕ into Y for which the composition koh coincides with f, then $k(h(x)) = k(X/\phi) = f(x)$.

Hence k coincides with g. Therefore, g is unique. □



Corollary 3.1.7. *Let* I *be a pseudo-ideal of a pseudo-TM algebra* X*. Then there exist an epimorphism* $f : X \to X/I$ *and* $\mathsf{Ker} f = I$.

Proof. Suppose that I be a pseudo-ideal of a pseudo-TM algebra X. We need to show that there exist an epimorphism $f : X \to X/I$ and $\mathsf{Ker} f = I$.
To prove this statement, first, construct the quotient algebra X/I and the canonical projection map $f : X \to X/I$, which is an epimorphism by definition of a quotient structure and a pseudo-ideal. Then, prove that the kernel of this map, $\text{Ker}(f)$, is precisely the pseudo-ideal I by showing two inclusions: $I \subseteq \text{Ker}(f)$ and $\text{Ker}(f) \subseteq I$. The first inclusion is true by the definition of the kernel and the quotient structure's relation to the pseudo-ideal, and the second inclusion follows from the properties of the quotient map and the definition of a pseudo-ideal.
First constructing an epimorphism map. Define X/I for a pseudo-up algebra X and a pseudo-ideal I, the quotient algebra X/I is formed by considering the equivalence classes of X under the relation $x \sim y$ if and only if $x \star y \in I$, $x \odot y \in I$ and $y \in I$. Now, define the canonical projection map $f : X \to X/I$ by $f(x) = [x]$, where $[x]$ is the equivalence class of x in X/I. We need to prove f is an epimorphism: By the construction of the quotient algebra and the definition of f, the map f is surjective (onto), meaning for every element in X/I, there's an element in X that maps to it. This makes f an epimorphism (a surjective homomorphism). Next we need to show that $\text{Ker}(f) = I$. We show how $I \subseteq \text{Ker}(f)$. An element x is in $\text{Ker}(f)$ if $f(x) = 0_{X/I}$ (the zero equivalence class in X/I). This means that $x \sim 0$, which implies $x \star 0 \in I$, $x \odot 0 \in I$ and $0 \in I$. Since $0 \in I$ and I is a pseudo-ideal, these conditions imply that $x \in I$. Therefore, $I \subseteq \text{Ker}(f)$. Show $\text{Ker}(f) \subseteq I$. Let $x \in \text{Ker}(f)$. Then, $f(x) = 0_{X/I}$, which means $x \sim 0$. By the definition of the equivalence relation for a quotient algebra, $x \sim 0$ if and only if $x \star 0 \in I$, $x \odot 0 \in I$ and $0 \in I$. Since $0 \in I$ is an element of a pseudo-ideal and I satisfies the properties of a pseudo-ideal (specifically, the closure under the pseudo-operations), $x * 0 \in I$, $x \odot 0 \in I$ and $0 \in I$ implies $x \in I$. Thus, $\text{Ker}(f) \subseteq I$. Therefore, $\text{Ker}(f) = \mathbf{I}$. □

3.2. Fuzzy Congruence Relations on Pseudo-TM Algebra

In this section, we define the notion of fuzzy congruence relations on pseudo-TM algebra and discuss basic properties of fuzzy congruence relations on pseudo-TM algebras. Characterizations of fuzzy pseudo-ideals of a pseudo-TM algebra are given. Let X be a pseudo-TM algebra unless otherwise specified.

Definition 3.2.1. *A fuzzy relation θ on X is called a fuzzy congruence relation on a pseudo-TM algebra of X, if it satisfies the following axioms: for any $x,y,z \in X$*

1. $\theta(x,x) = 1$ *(fuzzy reflexive)*
2. $\theta(x,y) = \theta(y,x)$ *(fuzzy symmetric)*
3. $\theta(x,z) \geqslant \sup_{y\in X}\{\min\{\theta(x,y),\theta(y,z)\}\}$ *(fuzzy transitive)*
4. *θ is called fuzzy right compatible, if $\theta(x \odot z, y \odot z) \geqslant \theta(x,y)$ and $\theta(x \star z, y \star z) \geqslant \theta(x,y)$*
5. *θ is called fuzzy left compatible, if $\theta(z \odot x, z \odot y) \geqslant \theta(x,y)$ and $\theta(z \star x, z \star y) \geqslant \theta(x,y)$.*
6. *θ is called compatible, if $\theta(x \odot u, y \odot v) \geqslant \min\{\theta(x,y),\theta(u,v)\}$ and $\theta(x \star u, y \star v) \geqslant \min\{\theta(x,y),\theta(u,v)\}$, for all $x,y,u,v \in X$.*

Example 3.2.1. *Consider a pseudo-TM algebra defined in Example 3.1.1.*

Let $\theta : X \times X \to [0,1]$ be a fuzzy relation on X defined by

θ	0	1	2	3	4	5
0	0.5	0.3	0.3	0.3	0.3	0.3
1	0.3	0.5	0.3	0.3	0.3	0.3
2	0.3	0.3	0.5	0.3	0.3	0.3
3	0.3	0.3	0.3	0.5	0.3	0.3
4	0.3	0.3	0.3	0.3	0.5	0.3
5	0.3	0.3	0.3	0.3	0.3	0.5

Table 3.2

By routine calculation θ is a fuzzy congruence relation on X.

Definition 3.2.2. *A fuzzy equivalence relation θ of X is a fuzzy congruence relation on a pseudo-TM algebra X if for all $x,y,z,w \in X$ satisfying $\theta(x \odot z, y \odot w) \geqslant \min\{\theta(x,y),\theta(z,w)\}$ and $\theta(x\star z, y \star w) \geqslant \min\{\theta(x,y),\theta(z,w)\}$.*

Theorem 3.2.1. *If θ is a fuzzy congruence relation on a pseudo-TM algebra X. Then*

1. $\theta(0,0) \geqslant \theta(x,y)$
2. $\theta(x \odot y, 0) \geqslant \theta(x,y)$, *and* $\theta(x \star y, 0) \geqslant \theta(x,y)$, *for all $x,y \in X$.*

Proof. Suppose that θ is a fuzzy congruence relation on a pseudo-TM algebra X. We need to show that

1. $\theta(0,0) \geqslant \theta(x,y)$

2. $\theta(x \odot y, 0) \geqslant \theta(x,y)$, and $\theta(x \star y, 0) \geqslant \theta(x,y)$, for all $x,y \in X$.

Since θ is a fuzzy congruence relation on X. By Definition 3.2.1, we have $\theta(0,0) = 1 = \theta(x,x) \geqslant \min\{\theta(x,y), \theta(y,x)\} = \min\{\theta(x,y), \theta(x,y)\} = \theta(x,y)$.
Hence, $\theta(0,0) \geqslant \theta(x,y)$.
Similarly, θ is a fuzzy congruence relation on X. Definition 3.2.1, we have $\theta(x \odot y, 0) = \theta(x \odot y, y \odot y) \geqslant \theta(x,y)$.
Also, $\theta(x \star y, 0) = \theta(x \star y, y \star y) \geqslant \theta(x,y)$.

□

Theorem 3.2.2. *If θ and ρ are fuzzy congruence relations on X. Then $\theta \cap \rho$ is a fuzzy congruence relation on X.*

Proof. Suppose that θ and ρ are fuzzy congruence relations on X. Then we need to show that $\theta \cap \rho$ is a fuzzy congruence relation on X.

1. Let $x \in X$. Then,

$$\begin{aligned}(\theta \cap \rho)(x,x) &= \min\{\theta(x,x), \rho(x,x)\}\\ &= \min\{1,1\}\\ &= 1.\end{aligned}$$

2. Let $x,y \in X$.

$$\begin{aligned}(\theta \cap \rho)(x,y) &= \min\{\theta(x,y), \rho(x,y)\}\\ &= \min\{\theta(y,x), \rho(y,x)\}\\ &= (\theta \cap \rho)(y,x).\end{aligned}$$

3. Let $x,y,z \in X$.

$$\begin{aligned}(\theta \cap \rho)(x,z) &= \min\{\theta(x,z), \rho(x,z)\}\\ &\geqslant \min\{\min\{\theta(x,y), \theta(y,z)\}, \min\{\rho(x,y), \rho(y,z)\}\}\\ &\geqslant \min\{\min\{\theta(x,y), \rho(x,y)\}, \min\{\theta(y,z), \rho(y,z)\}\}\\ &\geqslant \min\{(\theta \cap \rho)(x,y), (\theta \cap \rho)(y,z)\}.\end{aligned}$$

4. Let $x,y,z,k \in X$.

$$\begin{aligned}&(\theta \cap \rho)(\,(x,y) \star (z,k))\\ &\quad= \min\{\theta(x \star z, y \star k), \rho(x \star z, y \star k)\}\\ &\quad\geqslant \min\{\min\{\theta(x,y), \theta(z,k)\},\ \min\{\rho(x,y), \rho(z,k)\}\}\\ &\quad= \min\{\min\{\theta(x,y), \rho(x,y)\},\ \min\{\theta(z,k), \rho(z,k)\}\}\\ &\quad= \min\{(\theta \cap \rho)(x,y), (\theta \cap \rho)(z,k)\} \text{ and}\end{aligned}$$

For $x,y,z,k \in X$.

$$
\begin{aligned}
(\theta \cap \rho)((x,y) \odot (z,k)) \\
&= \min\{\theta(x \odot z, y \odot k), \rho(x \odot z, y \odot k)\} \\
&\geqslant \min\{\min\{\theta(x,y), \theta(z,k)\}, \min\{\rho(x,y), \rho(z,k)\}\} \\
&= \min\{\min\{\theta(x,y), \rho(x,y)\}, \min\{\theta(z,k), \rho(z,k)\}\} \\
&= \min\{(\theta \cap \rho)(x,y), (\theta \cap \rho)(z,k)\}.
\end{aligned}
$$

Therefore, $\theta \cap \rho$ is a fuzzy congruence relation on X.

□

Corollary 3.2.3. *If $\{\theta_i : i \in I\}$ is a family of fuzzy congruence relation on a pseudo-TM algebra X. Then $\cap_{i \in I}\theta_i$ is a fuzzy congruence relation on X.*

Proof. Suppose that $\{\theta_i : i \in I\}$ is a family of fuzzy congruence relation on a pseudo-TM algebra X. Then we need to show that $\cap_{i \in I}\theta_i$ is a fuzzy congruence relation on X.

1. Let $x \in X$ and suppose that $\cap_{i\in I}\theta_i(x,x) = \inf_{i\in I}(\theta_i(x,x))$. Then $\inf_{i\in I}(\theta_i(x,x)) = \inf_{i\in I}(1) = \cap_{i\in I}(1) = 1$. Hence, $\cap_{i\in I}\theta_i(x,x) = 1$

2. For $x,y \in X$, $\cap_{i\in I}\theta_i(x,y) = \inf_{i\in I}\theta_i(x,y) = \inf_{i\in I}\theta_i(y,x) = \cap_{i\in I}\theta_i(y,x)$.

3. For any $x,y,z \in X$, $\cap_{i\in I}\theta_i(x,z) = \inf_{i\in I}\theta_i(x,z) \geqslant \inf_{i\in I}\min\{\theta_i(x,y),\theta_i(y,z)\}$ $= \min\{\inf_{i\in I}\ \theta_i(x,y), \inf_{i\in I}\theta_i(y,z)\} = \min\{\cap_{\in I}\theta_i(x,y), \cap_{i\in I}\theta_i\ (y,z)\}$.

4. For any $x,y,z \in X$, $\cap_{i\in I}\theta(x \odot z, y \odot z = \inf_{i\in I}(\theta_i(x\odot z, y \odot z)) \geqslant \inf_{i\in I}\theta_i(x,y) = \cap_{i\in I}\theta_i(x,y)$ and $\cap_{i\in I}\theta(x \star z, y \star z) = \inf_{i\in I}(\theta_i(x \star z, y \star z)) \geqslant \inf_{i\in I}\theta_i(x,y) = \cap_{i\in I}\theta_i(x,y)$. Similarly, $\cap_{i\in I}\theta(z \odot x, z \odot y = \inf_{i\in I}(\theta_i(z \odot x, z \odot y)) \geqslant \inf_{i\in I}\theta_i(x,y) = \cap_{i\in I}\theta_i(x,y)$ and $\cap_{i\in I}\theta(z \star x, z \star y) = \inf_{i\in I}(\theta_i(z \star x, z \star y)) \geqslant \inf_{i\in I}\theta_i(x,y) = \cap_{i\in I}\theta_i(x,y)$.

Therefore, $\cap_{i\in I}\theta_i$ is a fuzzy congruence relation on X. □

Remark 3.2.1. *The union of any two fuzzy congruence relation on X is not necessarily a fuzzy congruence relation on X.*

Example 3.2.2. *Let $X = \{0,1,2,3\}$ be a finite pseudo-TM algebra with operations $\odot$ and $\star$ defined by:*

$\odot$	0	1	2	3
0	0	1	2	3
1	1	0	0	0
2	2	0	0	0
3	3	0	0	0

$\star$	0	1	2	3
0	0	1	2	3
1	2	0	0	0
2	2	0	0	0
3	2	0	0	0

Table 3.3

See[57] $(X;\ \odot,\star,0)$ is a pseudo-TM algebra. Now define two fuzzy relations θ_1 and θ_2 on $X \times X$:

θ_1	0	1	2	3
0	0.6	0.1	0.1	0.1
1	0.1	0.6	0.2	0.2
2	0.1	0.2	0.6	0.2
3	0.1	0.2	0.2	0.6

θ_2	0	1	2	3
0	0.4	0.2	0.2	0.2
1	0.2	0.4	0.3	0.3
2	0.2	0.3	0.4	0.2
3	0.1	0.2	0.3	0.4

It's easy to check that both θ_1 and θ_2 satisfy the fuzzy congruence axioms individually. Now consider the union $(\theta_1 \cup \theta_2)(x,y) = \max\{\theta_1(x,y), \theta_2(x,y)\}$. Then by routine calculation the fuzzy transitivity condition fails.
Therefore, $\theta_1 \cup \theta_2$ is not a fuzzy congruence relation.

Theorem 3.2.4. *Let θ and ρ be fuzzy congruences on a pseudo-TM algebra X. If $\theta \circ \rho$ is a fuzzy congruence, then $\theta \circ \rho = \rho \circ \theta$.*

Proof. Let θ and ρ be fuzzy congruences on X. Assume that $\theta \circ \rho$ is a fuzzy congruences. We need to show that $\theta \circ \rho = \rho \circ \theta$.
Let $(a,b) \in X \times X$. Then, since ρ and θ are fuzzy congruences, we have

$$\begin{aligned}(\theta \circ \rho)(a,b) &= \sup_{z \in X}\{\min\{\theta(a,z), \rho(z,b)\}\} \\ &= \sup_{z \in X}\{\min\{\rho(b,z), \theta(z,a)\}\} \\ &= (\rho \circ \theta)(b,a) \\ &= (\rho \circ \theta)(a,b), \text{ since fuzzy congruences.}\end{aligned}$$

Thus we obtain that $\rho \circ \theta = \theta \circ \rho$. □

Corollary 3.2.5. *Let $\{\theta_i : i \in I\}$ be a family of a fuzzy congruence relations on X. If $\theta_1 \circ \theta_2 \circ \theta_3 \circ \ldots \circ \theta_n$ be a fuzzy congruence relations on a pseudo-TM algebra X, then $\theta_1 \circ \theta_2 \circ \theta_3 \circ \ldots \circ \theta_n = \sup\{\theta_1, \theta_2, \theta_3, \ldots, \theta_n\}$.*

Proof. Suppose that $\theta_1 \circ \theta_2 \circ \theta_3 \circ \ldots \circ \theta_n$ be a fuzzy congruence relations on a pseudo-TM algebra X. Then we need to show that $\theta_1 \circ \theta_2 \circ \theta_3 \circ \ldots \circ \theta_n = \sup\{\theta_1, \theta_2, \theta_3, \ldots, \theta_n\}$. We will prove this by establishing both inequalities. First, we show that $\theta_1 \circ \theta_2 \circ \theta_3 \circ \cdots \circ \theta_n \leqslant \sup\{\theta_1, \theta_2, \theta_3, \ldots, \theta_n\}$.
By definition of the composition of fuzzy relations, for any $x, z \in X$. We have
$(\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n)(x,z) = \sup_{y_1, \ldots, y_{n-1} \in X}\{\min\{\theta_1(x,y_1), \theta_2(y_1,y_2), \ldots, \theta_n(y_{n-1},z)\}\}$.
For any choice of $y_1, \ldots, y_{n-1} \in X$, we have
$\min\{\theta_1(x,y_1), \theta_2(y_1,y_2), \ldots, \theta_n(y_{n-1},z)\} \leqslant \sup\{\theta_1, \theta_2, \ldots, \theta_n\}(x,z)$.
Since each $\theta_i(x,y) \leqslant \sup\{\theta_1, \theta_2, \ldots, \theta_n\}(x,y)$ for all $x, y \in X$.
Taking supremum over all $y_1, \ldots, y_{n-1} \in X$ which implies that
$(\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n)(x,z) \leqslant \sup\{\theta_1, \theta_2, \ldots, \theta_n\}(x,z)$. Now, we prove the reverse inequality $\sup\{\theta_1, \theta_2, \ldots, \theta_n\} \leqslant \theta_1 \circ \theta_2 \circ \cdots \circ \theta_n$. Since $\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n$ is a fuzzy congruence relation, it is fuzzy reflexive.
Therefore, for any $x, z \in X$ $\theta_i(x,z) \leqslant (\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n)(x,z)$ for each $i = 1, 2, \ldots, n$. This is

because we can take $y_1 = y_2 = \cdots = y_{n-1} = z$ when $i = 1$, $y_1 = x, y_2 = \cdots = y_{n-1} = z$ when $i = 2$, and similarly for other indices, using the fuzzy reflexivity property $\theta_j(a,a) = 1$ for all j. To show $\theta_1(x,z) \leqslant (\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n)(x,z)$, take $y_1 = z$ and $y_2 = \cdots = y_{n-1} = z$. Then $\min\{\theta_1(x,z), \theta_2(z,z), \theta_3(z,z), \ldots, \theta_n(z,z)\} = \min\{\theta_1(x,z), 1, 1, \ldots, 1\} = \theta_1(x,z)$. Thus, $(\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n)(x,z) \geqslant \theta_1(x,z)$. Similarly, we can show this for each θ_i.
Hence, for each $i = 1,2,\ldots,n$ and for all $x,z \in X$ $\theta_i(x,z) \leqslant (\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n)(x,z)$, which implies $\sup\{\theta_1, \theta_2, \ldots, \theta_n\}(x,z) \leqslant (\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n)(x,z)$.
Therefore, $\theta_1 \circ \theta_2 \circ \theta_3 \circ \cdots \circ \theta_n = \sup\{\theta_1, \theta_2, \theta_3, \ldots, \theta_n\}$. □

Theorem 3.2.6. *A fuzzy relation θ is fuzzy congruence on a pseudo-TM algebra X if and only if for each $t \in [0,1]$, the non-empty subset $U(\theta,t)$ is a congruence relations.*

Proof. Assume that θ is a fuzzy congruence on X. We need to show that for each $t \in [0,1]$, the set

$$U(\theta,t) = \{(x,y) \in X \times X \mid \theta(x,y) \geqslant t\}$$

is a congruence relation on X, i.e., an equivalence relation that is compatible with $\odot$ and $\star$.

(i) Reflexivity. Let $t \in [0,1]$ be such that $U(\theta,t) \neq \emptyset$. Let $x,y \in X$ such that $(x,y) \in U(\theta,t)$. Then $\psi(x,y) \geqslant t$. Since θ is fuzzy congruence relation, then we have $t \leqslant \psi(x,y) \leqslant 1 = \psi(x,x)$. Which implies that $(x,x) \in U(\theta,t), \forall t \in [0,1]$. Thus $U(\theta,t)$ is reflexive.

(ii) Symmetry. If $(x,y) \in U(\theta,t)$, then $\theta(x,y) \geqslant t \Rightarrow \theta(y,x) = \theta(x,y) \geqslant t \Rightarrow (y,x) \in U(\theta,t)$. So symmetry holds.

(iii) Transitivity. If $(x,y),(y,z) \in U(\theta,t)$, then $\theta(x,y) \geqslant t, \quad \theta(y,z) \geqslant t$. From fuzzy transitivity $\theta(x,z) \geqslant \min\{\theta(x,y), \theta(y,z)\} \geqslant t \Rightarrow (x,z) \in U(\theta,t)$. So transitivity holds.

(iv) Compatibility with operations. Let $(x_1,y_1),(x_2,y_2) \in U(\theta,t)$, so $\theta(x_1,y_1) \geqslant t, \quad \theta(x_2,y_2) \geqslant t$. Since θ is a fuzzy congruence, we have $\theta(x_1 \odot x_2, y_1 \odot y_2) \geqslant \min\{\theta(x_1,y_1), \theta(x_2,y_2)\} \geqslant t$. So, $(x_1 \odot x_2, y_1 \odot y_2) \in U(\theta,t)$ and $\theta(x_1 \star x_2, y_1 \star y_2) \geqslant \min\{\theta(x_1,y_1), \theta(x_2,y_2)\} \geqslant t$. So, $(x_1 \star x_2, y_1 \star y_2) \in U(\theta,t)$. Hence, for each $t \in [0,1]$, $U(\theta,t)$ is a crisp congruence relation on X.

Conversely, assume a level subset $U(\theta,t)$ is a congruence relation on X . We need to show that θ is a fuzzy congruence relation on X. Let $t \in [0,1]$ be such that $U(\theta,t) \neq \emptyset$. Let $x \in X$ such that $(x,x) \in U(\theta,t)$. Then $\theta(x,x) \geqslant t$ and take $t = \theta(x,x)$. Since $U(\theta,t)$ is a congruence relation on X, we have $1 \in U(\theta,t)$ such that $1 \geqslant t = \theta(x,x)$. Similarly, we have $\theta(x,x) \geqslant 1$. Hence, $\theta(x,x) = 1$. Let $\theta(x,y) = t$, then $(x,y) \in U(\theta,t)$ and $(y,x) \in U(\theta,t)$. Because $U(\theta,t)$ is a congruence relation on X. Thus $\theta(y,x) \geqslant t = \theta(x,y)$. Similarly, we have $\theta(x,y) \geqslant \theta(y,x)$. Hence, $\theta(x,y) = \theta(y,x)$. Let $(x,y),(y,z) \in U(\theta,t)$, then $\theta(x,y) = t_1$ and $\theta(y,z) = t_2$ and take $t = \min\{t_1,t_2\}$. Since $U(\theta,t)$ is a congruence relation, we have $(x,z) \in U(\theta,t)$. Then $\theta(x,z) \geqslant t = \min\{t_1,t_2\} = \min\{\theta(x,y), \theta(y,z)\}$. Assume that $\theta(x,y) = t$, then $(x,y) \in U(\theta,t)$. Since, $U(\theta,t)$ is a congruence relation on X , we have $(x \odot z, y \odot z)$ and $(x \star z, y \star z) \in U(\theta,t)$. Then

$\theta(x \odot z, y \odot z) \geqslant t = \theta(x,y)$ implies that $\theta(x \odot z, y \odot z) \geqslant \theta(x,y)$ and $\theta(x \star z, y \star z) \geqslant t = \theta(x,y)$ implies that $\theta(x \star z, y \star z) \geqslant \theta(x,y)$. Similarly, $\theta(z \odot x, z \odot y) \geqslant t = \theta(x,y)$ implies that $\theta(z \odot x, z \odot y) \geqslant \theta(x,y)$ and $\theta(z \star x, z \star y) \geqslant t = \theta(x,y)$ implies that $\theta(z \star x, z \star y) \geqslant \theta(x,y)$.Therefore, θ is a fuzzy congruence relation on pseudo-TM algebra of X. □

Theorem 3.2.7. *A congruence relation θ is a fuzzy congruence relation on X if and only if its characteristic function χ_θ is a fuzzy congruence relation on X.*

Proof. Suppose that θ is a fuzzy congruence relation on X. We need to show that the characteristic function χ_θ is a fuzzy congruence relation on X. Let $\chi_\theta : X \times X \to \{0,1\}$ be the characteristic function of θ, defined as:

$$\chi_\theta(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \theta, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose θ is a congruence relation. We show that χ_θ is a fuzzy congruence relation. To do this, we verify that χ_θ satisfies the conditions of a fuzzy congruence:

(i) Fuzzy reflexivity. Since θ is reflexive, $(x,x) \in \theta \Rightarrow \chi_\theta(x,x) = 1$. Thus, $\chi_\theta(x,x) = 1$ for all $x \in X$.

(ii) Fuzzy symmetry. If $\chi_\theta(x,y) = 1$, then $(x,y) \in \theta \Rightarrow (y,x) \in \theta \Rightarrow \chi_\theta(y,x) = 1$. So $\chi_\theta(x,y) = \chi_\theta(y,x)$.

(iii) Fuzzy transitivity. If $\chi_\theta(x,y) = \chi_\theta(y,z) = 1$, then $(x,y),(y,z) \in \theta \Rightarrow (x,z) \in \theta \Rightarrow \chi_\theta(x,z) = 1$. Hence, $\chi_\theta(x,z) \geqslant \min\{\chi_\theta(x,y), \chi_\theta(y,z)\}$.

(iv) Compatibility with operations $\odot$ and $\star$. Let $\chi_\theta(x_1,y_1) = \chi_\theta(x_2,y_2) = 1 \Rightarrow (x_1,y_1),(x_2,y_2) \in \theta$. Since θ is a congruence, $(x_1 \odot x_2, y_1 \odot y_2)$, $(x_1 \star x_2, y_1 \star y_2) \in \theta \Rightarrow \chi_\theta(x_1 \odot x_2, y_1 \odot y_2) = \chi_\theta(x_1 \star x_2, y_1 \star y_2) = 1$. So, $\chi_\theta(x_1 \odot x_2, y_1 \odot y_2) \geqslant \min\{\chi_\theta(x_1,y_1), \chi_\theta(x_2,y_2)\}$, and $\chi_\theta(x_1 \star x_2, y_1 \star y_2) \geqslant \min\{\chi_\theta(x_1,y_1), \chi_\theta(x_2,y_2)\}$. Hence, χ_θ is a fuzzy congruence relation. Conversely, suppose χ_θ is a fuzzy congruence relation. We must show that $\theta = \{(x,y) \in X \times X \mid \chi_\theta(x,y) = 1\}$ is a crisp congruence relation.

(i) Reflexivity. For all $x \in X$, $\chi_\theta(x,x) = 1 \Rightarrow (x,x) \in \theta$.

(ii) Symmetry. If $(x,y) \in \theta \Rightarrow \chi_\theta(x,y) = 1 \Rightarrow \chi_\theta(y,x) = 1 \Rightarrow (y,x) \in \theta$.

(iii) Transitivity. If $(x,y),(y,z) \in \theta \Rightarrow \chi_\theta(x,y) = \chi_\theta(y,z) = 1 \Rightarrow \chi_\theta(x,z) \geqslant \min\{1,1\} = 1 \Rightarrow (x,z) \in \theta$.

(iii) Compatibility with operations. If $(x_1,y_1),(x_2,y_2) \in \theta \Rightarrow \chi_\theta(x_1,y_1) = \chi_\theta(x_2,y_2) = 1$. Then $\chi_\theta(x_1 \odot x_2, y_1 \odot y_2) \geqslant \min\{1,1\} = 1 \Rightarrow (x_1 \odot x_2, y_1 \odot y_2) \in \theta$, and $\chi_\theta(x_1 \star x_2, y_1 \star y_2) \geqslant \min\{1,1\} = 1 \Rightarrow (x_1 \star x_2, y_1 \star y_2) \in \theta$.

Hence, θ is a ongruence relation. □

Definition 3.2.3. *Let* θ *be a fuzzy congruence relation on a pseudo-TM algebra* X*. For each* $x \in X$*, define a fuzzy subset by* $\theta^{[x]}(y) = \theta(x,y)$ *for any* $y \in X$ *is called a fuzzy congruence relation containing* x*. Hence,* $X/\theta^{[x]}$ *be the set of all fuzzy congruence classes of* $\theta^{[x]}$.

Definition 3.2.4. *Let* θ *be a fuzzy congruence relation on a pseudo-TM algebra* X*. Then the set* $X/\theta = \left\{\theta^{[x]}/x \in X\right\}$ *is called quotient pseudo-TM algebra.*

Definition 3.2.5. *Let* $f : X \to Y$ *be a mapping between pseudo-TM algebras. Let* θ *and* ρ *be fuzzy congruence relations of* X *and* Y*, respectively. Then, the inverse image* $f^{-1}(\rho)$ *is a fuzzy subset on* X *defined by*

$$f^{-1}(\rho)(x,y) = \rho(f(x), f(y)),$$

for all $x \in X$*. The image* $f(\theta)$ *of* θ *is a fuzzy relation on* Y *defined by*

$$f(\theta)(x,y) = \begin{cases} \sup_{(a,b)\in f^{-1}(x,y)} \theta(a,b) & \textit{if } f^{-1}(x,y) \neq \emptyset, \\ 0 & \textit{in all other cases} \end{cases}$$

for all $x, y \in Y, x_i, y_i \in X$.

Remark 3.2.2. *For any fuzzy congruence* θ *on a pseudo-Tm algebra* X *satisfying* $\theta \subseteq f^{-1}(f(\theta))$*. If* f *is injective, then* $\theta = f^{-1}(f(\theta))$.

Theorem 3.2.8. *Let* f *be a homomorphism from a pseudo-TM algebra* X *to a pseudo-TM algebra* Y*. If* θ *is a fuzzy congruence relation on* Y*, then the inverse image* $f^{-1}(\theta)$ *is a fuzzy congruence relation on* X.

Proof. Suppose that θ is a fuzzy congruence relation on Y. We need to show that $f^{-1}(\theta)$ is a fuzzy congruence relation on X.

1) Let $x \in X$. We have

$$\begin{aligned} f^{-1}(\theta)(x,x) &= \theta(f(x), f(x)) \\ &= 1 \end{aligned}$$

Hence $f^{-1}(\theta)$ is reflexive.

2) Let $x, y \in X$. We have

$$\begin{aligned} f^{-1}(\theta)(x,y) &= \theta(f(x), f(y)) \\ &= \theta(f(y), f(x)) \\ &= f^{-1}(\theta(y,x)). \end{aligned}$$

Hence $f^{-1}(\theta)$ is symmetric.

3) Let $x, y \in X$. We have

$$\begin{aligned} f^{-1}(\theta)(x,y) &= \theta(f(x),f(y)) \\ &\geqslant \sup_{z\in X}\{\min\{\theta(f(x),f(z)),\theta(f(z),f(y))\}\} \\ &= \sup_{z\in X}\left\{\min\left\{f^{-1}(\theta)(x,z),f^{-1}(\theta)(z,y)\right\}\right\} \end{aligned}$$

Hence $f^{-1}(\theta)$ is transitive

4) Let $x, y, z, w \in X$,it follows from Definition (3.2.2) that

$$\begin{aligned} f^{-1}(\theta)(x\odot z, y\odot w) &= \theta(f(x\odot z, y\odot w)) \\ &= \theta(f(x\odot z), f(y\odot w)) \\ &= \theta(f(x)\odot f(z), f(y)\odot f(w)) \\ &\geqslant \min\{\theta(f(x)\odot f(y)), \theta(f(z)\odot f(w))\} \\ &= \min\{\theta(f(x\odot y)), \theta(f(z\odot w))\} \\ &= \min\left\{f^{-1}(\theta)(x\odot y), f^{-1}(\theta)(z\odot w)\right\}. \\ \text{and, } f^{-1}(\theta)(x\star z, y\star w) &= \theta(f(x\star z, y\star w)) \\ &= \theta(f(x\star z), f(y\star w)) \\ &= \theta(f(x)\star f(z), f(y)\star f(w)) \\ &\geqslant \min\{\theta(f(x)\star f(y)), \theta(f(z)\star f(w))\} \\ &= \min\{\theta(f(x\star y)), \theta(f(z\star w))\} \\ &= \min\left\{f^{-1}(\theta)(x\star y), f^{-1}(\theta)(z\star w)\right\}. \end{aligned}$$

Therefore, the relation $f^{-1}(\theta)$ is a fuzzy congruence relation on X. □

Theorem 3.2.9. *Let f be an epimorphism from a pseudo-TM algebra X to a pseudo-TM algebra Y. If θ is a fuzzy congruence on X, then the image $f(\theta)$ is a fuzzy congruence relation on Y provided that the sup property holds.*

Proof. Let f be an epimorphism on a pseudo-TM algebra X into a pseudo-TM algebra Y . Suppose that θ is a fuzzy congruence relation on X. We need to show that $f(\theta)$ is a fuzzy congruence relation on Y. Let $y \in Y$, then

$$\begin{aligned} f(\theta)(y,y) &= \sup\left\{\theta(x,x), x \in f^{-1}(y) \neq \emptyset\right\} \\ &= 1 \end{aligned}$$

Let $y_1, y_2 \in Y$. Then

$$\begin{aligned} f(\theta)(y_1,y_2) &= \sup\left\{\theta(x_1,x_2), \quad x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\right\} \\ &= \sup\left\{\theta(x_2,x_1), \quad x_2 \in f^{-1}(y_2), x_1 \in f^{-1}(y_1)\right\} \\ &= f(\theta)(y_2,y_1) \\ \Rightarrow & f(\theta)(y_1,y_2) \\ &= f(\theta)(y_2,y_1) \end{aligned}$$

Next, we prove that $f(\theta)$ is fuzzy transitive. For any $y_1, y_2 \in Y$.

$$\begin{aligned} (f(\theta)\circ f(\theta))(y_1,y_2) &= \sup_{y_3\in Y}\{\min\{f(\theta)(y_1,y_3),\ f(\theta)(y_3,y_2)\}\} \\ &= \sup_{y_3\in Y}\left\{\min\left\{\sup_{x_1\in f^{-1}(y_1)}\theta(x_1,x_3),\ \sup_{x_3\in f^{-1}(y_3)}\theta(x_3,x_2)\right\}\right\} \\ &\leqslant \sup_{y_3\in Y}\left\{\min\left\{\sup_{x_1\in f^{-1}(y_1)}\{\theta(x_1,x_3),\theta(x_3,x_2)\}\right\}\right\} \\ &\leqslant \sup_{\substack{x_1\in f^{-1}(y_1)\\ x_2\in f^{-1}(y_2)}}\left\{\sup_{x_3\in f^{-1}(y_3)}\min\{\theta(x_1,x_3),\theta(x_3,x_2)\}\right\} \\ &\leqslant \sup_{\substack{x_1\in f^{-1}(y_1)\\ x_2\in f^{-1}(y_2)}}\theta(x_1,x_2) = f(\theta)(y_1,y_2). \\ &\Rightarrow \quad (f(\theta)\circ f(\theta))(y_1,y_2) \leqslant f(\theta)(y_1,y_2). \end{aligned}$$

Next, we need to show that $f(\theta)$ is left and right fuzzy compatible. let $h_1, h_2, k_1, k_2 \in Y$. Since f is an epimorphism then there exists $f_1, f_2, g_1, g_2 \in X$ such that $f(f_i) = h_i$ and $f(g_i) = k_i$ for all $i = 1,2$.
Also θ satisfies sup property,we have :

$$\begin{aligned} \theta(a,b) &= \sup\left\{\theta(t_1,t_2)/(t_1,t_2) \in f^{-1}(h_1,h_2)\right\} \\ \theta(c,d) &= \sup\left\{\theta(t_1,t_2)/(t_1,t_2) \in f^{-1}(k_1,k_2)\right\}. \end{aligned}$$

It follows that
$f(\theta(h_1 \odot k_1, h_2 \odot k_2)) \geqslant \theta((a,b)\odot(c,d)) = \theta(a\odot c, b\odot d)$

$$\begin{aligned} &\geqslant \min\{\theta(a,b),\theta(c,d)\} \\ &= \min\left\{\sup\left\{\theta(t_1,t_2)/(t_1,t_2) \in f^{-1}(h_1,h_2)\right\}, \sup\left\{\theta(t_1,t_2)/(t_1,t_2) \in f^{-1}(k_1,k_2)\right\}\right\} \\ &= \min\{f(\theta(h_1,h_2)), f(\theta(k_1,k_2))\}. \end{aligned}$$

and $f(\theta(h_1 \star k_1, h_2 \star k_2)) \geqslant \theta((a,b) \star (c,d)) = \theta(a \star c, b \star d)$

$$\begin{aligned}
&\geqslant \min\{\theta(a,b), \theta(c,d)\} \\
&= \min\left\{\sup\left\{\theta(t_1,t_2)/(t_1,t_2) \in f^{-1}(h_1,h_2)\right\}, \sup\left\{\theta(t_1,t_2)/(t_1,t_2) \in f^{-1}(k_1,k_2)\right\}\right\} \\
&= \min\{f(\theta(h_1,h_2)), f(\theta(k_1,k_2))\}.
\end{aligned}$$

Hence, $f(\theta)$ is a fuzzy congruence relation on Y. □

Theorem 3.2.10. *Let* f *be an epimorphism from a pseudo-TM algebra* X *to a pseudo-TM algebra* Y. *If* θ *is a fuzzy congruence on* Y, *then* $X/f^{-1}(\theta) \cong Y/\theta$.

Proof. Suppose that θ is a fuzzy congruence on Y. We need to show that $X/f^{-1}(\theta) \cong Y/\theta$. By Theorem 3.2.8 we have $f^{-1}(\theta)$ is a fuzzy congruence on Y. Also, $X/f^{-1}(\theta)$ and Y/θ are pseudo-TM algebras. Now, define a map $h : X/f^{-1}(\theta) \longrightarrow Y/\theta$ by $h(f^{-1}(\theta)(x)) = \theta(f(x))$ for $x \in X$. We need to show that

(i) h is well defined and is one to one.

(ii) h is a homomorphism.

(iii) h is onto.

Let $x, y \in X$ such that $f^{-1}(\theta)(x) = f^{-1}(\theta)(y)$. Then

(i)

$$\begin{aligned}
f^{-1}(\theta)(x) = f^{-1}(\theta)(y) &\Leftrightarrow f^{-1}(\theta)(x,y) = f^{-1}(\theta)(0,0) \\
&\Leftrightarrow f^{-1}(\theta)(x,y) = f^{-}(\theta)(0,0) \\
&\Leftrightarrow \theta(f(x), f(y)) = \theta(f(0), f(0)) \\
&\Leftrightarrow \theta(f(x)) = \theta(f(y)) \\
&\Leftrightarrow h(f^{-1}(\theta)(x)) = h(f^{-1}(\theta)(y))
\end{aligned}$$

This shows that h is well defined and one-to-one.

(ii) To show h is homomorphism; let $f^{-1}(\theta)(x), f^{-1}(\theta)(y) \in X/f^{-1}(\theta)$. Then,

$$\begin{aligned}
h\left(f^{-1}(\theta)(x) \odot f^{-1}(\theta)(y)\right) &= h\left(f^{-1}(\theta)(x \odot y)\right) \\
&= \theta(f(x \odot y)) \\
&= \theta(f(x) \odot f(y)) \\
&= \theta(f(x)) \odot \theta(f(y)) \\
&= h(f^{-1}(\theta)(x)) \odot h(f^{-1}(\theta)(y))
\end{aligned}$$

and

$$\begin{aligned} h\left(f^{-1}(\theta)(x)\star f^{-1}(\theta)(y)\right) &= h\left(f^{-1}(\theta)\,(x\star y)\right) \\ &= \theta\,(f\,(x\star y)) \\ &= \theta\,(f(x)\star f(y)) \\ &= \theta(f(x))\star\theta(f(y)) \\ &= h(f^{-1}(\theta)(x))\star h(f^{-1}(\theta)(y) \end{aligned}$$

Hence, $h : X/f^{-1}(\theta) \longrightarrow Y/\theta$ is homomorthetasm.

(iii) To show h is onto. Let $\theta(y) \in Y/\theta$ for some $y \in Y$. Since, f is onto there exists $x \in X$ such that $f(x) = y$. Then, $h(f^{-1}(\theta)(x)) = \theta(f(x)) = \theta(y)$. This shows that h is onto. Hence, h is an isomorthetasm.
Therefore, $X/f^{-1}(\theta) \cong Y/\theta$.

□

Theorem 3.2.11. *Let* f *be an endomorphism of* X. *If* θ_1 *be a fuzzy congurence relation on a pseudo TM-algebra X, then* θ *is defined by* $\theta(x,y) = \theta_1(f(x), f(y))$ *is a fuzzy congurence relation on X.*

Proof. Suppose that θ_1 be a fuzzy congurence relation on a pseudo TM-algebra X. We need to show that $\theta(x,y) = \theta_1(f(x), f(y))$ is a fuzzy congurence relation on X. By routine calculation θ is well-defined. Let $x, y, z, u, v \in X$.

1. $\theta(x,x) = \theta_1(f(x), f(x)) = 1$

2. $\theta(x,y) = \theta_1(f(x), f(y)) = \theta_1(f(y), f(x)) = \theta(y,x)$

3. $\theta(x,y) = \theta_1(f(x), f(y)) \geqslant \min[\theta_1(f(x), f(z)), \theta_1(f(z), f(y))] = \min[\theta(x,z), \theta(z,y)]$

4. $\theta(x \odot u, y \odot u) = \theta_1(f(x \odot u), f(y \odot u))$
   $= \theta_1(f(x) \odot f(u), f(y) \odot f(u))$
   $\geqslant \theta_1(f(x), f(y)) = \theta(x,y)$
   Similarly, $\theta(v \odot x, v \odot y) = \theta_1(f(v \odot x), f(v \odot y))$
   $= \theta_1(f(v) \odot f(x), f(v) \odot f(y))$
   $\geqslant \theta_1(f(x), f(y)) = \theta(x,y)$

5. $\theta(x \star u, y \star u) = \theta_1(f(x \star u), f(y \star u))$
   $= \theta_1(f(x) \star f(u), f(y) \star f(u))$
   $\geqslant \theta_1(f(x), f(y)) = \theta_1(x,y)$.
   Similarly, $\theta(v \star x, v \star y) = \theta_1(f(v \star x), f(v \star y))$
   $= \theta_1(f(v) \star f(x), f(v) \star f(y))$
   $\geqslant \theta_1(f(x), f(y)) = \theta(x,y)$.

□

Theorem 3.2.12. *Let* θ *and* ρ *be a fuzzy congruence relations on a pseudo-TM algebra* X *and assume that* $\theta \subseteq \rho$. *Then* $\frac{X}{\theta}/\frac{\rho}{\theta} \cong \frac{X}{\rho}$.

Proof. Suppose that θ and ρ be a fuzzy congruence relations on a pseudo-TM algebra X and assume that $\theta \subseteq \rho$. To show that $\frac{X}{\theta}/\frac{\rho}{\theta} \cong \frac{X}{\rho}$. Now, we have consider the following conditions. Consider, the map $h : X/\theta \to X/\rho$ by $h\left(\theta^{[x]}\right) = \rho^{[x]}$, for all $x \in X$.

(i) We need to show that h is well defined.
Assume that $\theta^{[x]} = \theta^{[y]}$, for all $x, y \in X$, then $\theta(x, y) = 0$. By assumption $\theta \subseteq \rho$, thus $\rho(x, y) \geqslant \theta(x, y) = 0$, so $\rho(x, y) = 0$. It implies that $\rho^{[x]} = \theta^{[y]}$.
Hence h is well defined.

(ii) To show that h is homomorphism.
Let $x, y \in X$, we have: $h(\theta^{[x]} \odot \theta^{[y]}) = h(\theta^{[x \odot y]}) = \rho^{[x \odot y]} = \rho^{[x]} \odot \rho^{[y]} = h(\theta^{[x]}) \odot h(\theta^{[y]})$, and $h(\theta^{[x]} \star \theta^{[y]}) = h(\theta^{[x \star y]}) = \rho^{[x \star y]} = \rho^{[x]} \star \rho^{[y]} = h(\theta^{[x]}) \star h(\theta^{[y]})$.
Hence, h is a homomorphism.

(iii) To show that h is an epimorphism.
For any $\rho^{[x]} \in X/\rho$, there exist $\theta^{[x]} \in X/\theta$ such that $h(\theta^{[x]}) = \rho^{[x]}$, so h is an epimorphism.

(iv) To show ker $(h) = \rho/\theta$. Now,

$$\begin{aligned}
\ker(h) &= \left\{\theta^{[x]} \in X/\theta \mid h\left(\theta^{[x]}\right) = \rho^{[0]}\right\} \\
&= \left\{\theta^{[x]} \in X/\theta \mid \rho^{[x]} = \rho^{[0]}\right\} \\
&= \left\{\theta^{[x]} \in X/\phi \mid \rho(x, 0) = 1\right\} \\
&= \left\{\theta^{[x]} \in X/\theta \mid \rho(0, x) = 1\right\} \\
&= \left\{\theta^{[x]} \in X/\theta \mid \rho^{[x]} \in X/\rho\right\} \\
&= \rho/\theta.
\end{aligned}$$

Therefore,

$$\left(\frac{X}{\theta}\right) / \left(\frac{\rho}{\theta}\right) \cong \frac{X}{\rho}.$$

□

3.3. Fuzzy Congruence Relations Induced by Fuzzy Pseudo-ideal

In this section, we discuss the new notions concerning the relationship between fuzzy pseudo-ideals and fuzzy congruence relations of pseudo-TM algebras and investigate some important results. Moreover, we discuss the relationship between fuzzy permutable congruence relations and fuzzy pseudo-ideals of a TM-algebra.

Theorem 3.3.1. *If θ be a fuzzy congruence relation in a pseudo-TM algebra, then $\theta^{[0]}$ is a fuzzy pseudo-ideal of X.*

Proof. Suppose that θ be a fuzzy congruence relation in a pseudo-TM algebra. We need to show that $\theta^{[0]}$ is a fuzzy pseudo-ideal of X. Let $x,y \in X$. Since by Theorem (3.2.1) we have $\theta(0,0) = 1 \geqslant \theta(x,y)$. It implies that $\theta^{[0]}(0) \geqslant \theta(x,y)$. If $y = 0$, then $\theta^{[0]}(0) \geqslant \theta(x,0) = \theta(0,x) = \theta^{[0]}(x)$. It follows that $\theta^{[0]}(0) \geqslant \theta^{[0]}(x)$. On the other hands

$$\begin{aligned}
\theta^{[0]}(x) = \theta(0,x) \geqslant \{\theta(0,y),\theta(y,x)\} \\
&= \min\left\{\theta^{[0]}(y),\theta(y\star x,0),\theta(y\odot x,0\right\}) \\
&= \min\left\{\theta^{[0]}(y),\theta(0,y\star x),\theta(0,y\odot x\right\}) \\
&= \min\left\{\theta^{[0]}(y),\theta[0](y\star x),\theta^{[0]}(y\odot x\right\}) \\
&= \min\left\{\theta^{[0]}(y),\theta^{[0]}(x\star y),\theta_0(x\odot y\right\})
\end{aligned}$$

Therefore, $\theta^{[0]}(x) \geqslant \min\left\{\theta^{[0]}(y),\theta^{[0]}(x\star y),\theta^{[0]}(x\odot y\right\})$ □

Definition 3.3.1. *Let θ and ρ be a fuzzy congruence relations on a pseudo-TM algebra X. Then θ and ρ are said to be permutable fuzzy congruence relation if $\theta \circ \rho = \rho \circ \theta$.*

Definition 3.3.2. *A fuzzy congruence relation of a pseudo-TM algebra X is called permutable if every pair of congruence relation are permutable.*

Example 3.3.1. *Let $X = \{0,1,2\}$ be a finite pseudo-TM algebra with operations $\odot$ and $\star$ defined by:*

$\odot$	0	a	b
0	0	a	b
a	a	a	b
b	b	a	b

$\star$	0	a	b
0	0	0	0
a	a	0	0
b	b	0	0

Table 3.4

See[57] $(X;\ \odot,\star,0)$ is a pseudo TM-algebra. Now define two fuzzy congruence relations θ_1 and θ_2 on $X \times X$:

θ_1	0	a	b
0	1	0.1	0
a	0	1	0.5
b	0	0.5	1

θ_2	0	a	b
0	1	0	0.5
a	0	1	0
b	0.5	0	1

It can be verified that both θ_2 and θ_1 are valid fuzzy congruence relations for this pseudoalgebra. To check if θ_2 and θ_1 permute, we must compute their compositions and verify that $\theta_2 \circ \theta_1 = \theta_1 \circ \theta_2$.

Theorem 3.3.2. *let θ_1 and θ_2 be any two fuzzy pseudo-ideals of X. If θ_1 and θ_2 are a fuzzy congruence permutable relation on X, then $\theta_1 \circ \theta_2$ is a fuzzy congruence relation on X.*

Proof. Suppose that θ_1 and θ_2 be fuzzy congruence permutable relations on X. We need to show that $\theta_1 \circ \theta_2$ be a fuzzy congruence relation on X.

1) Let $x \in X$. We have

$$\begin{aligned}(\theta_1 \circ \theta_2)(x,x) &= \sup_{z\in X}\min\{\theta_1(x,x),\theta_2(x,x)\}\\ &\geqslant \min\{1,1\}\\ &= 1.\end{aligned}$$

It follows that $(\theta_1 \circ \theta_2)(x,x) \geqslant 1$. But $(\theta_1 \circ \theta_2)(x,x) \leqslant 1$.
Hence, $(\theta_1 \circ \theta_2)(x,x) = 1$.
Therefore, $\theta_1 \circ \theta_2$ is reflexive.

2) Let $x,y \in X$. We have

$$\begin{aligned}(\theta_1 \circ \theta_2)(x,y) &= \sup_{z\in X}\min\{\theta_1(x,y),\theta_2(x,y)\}\\ &= \sup_{z\in X}\min\{\theta_2(z,y),\theta_1(x,z)\}\\ &= (\theta_2 \circ \theta_1)(y,x)\\ &= (\theta_1 \circ \theta_2)(y,x) \quad by \quad permutablity.\end{aligned}$$

Hence, $\theta_1 \circ \theta_2$ is symmetric.

3) Let $x,y,z \in X$. We have

$$\begin{aligned}(\theta_1 \circ \theta_2)(x,z) &= \sup_{y\in X}\min\{\theta_1(x,y),\theta_2(y,z)\}\\ &\geqslant \sup_{y\in X}\min\{\min\{\theta_1(x,z),\theta_1(z,y)\},\min\{\theta_2(y,y),\theta_1(y,z)\}\}\\ &= \sup_{y\in X}\min\{\min\{\theta_1(x,z),\theta_2(y,z)\},\min\{\theta_1(z,y),\theta_2(y,y)\}\}\\ &= \sup_{y\in X}\min\{\min\{\theta_1(x,z),\theta_2(y,z)\},\min\{\theta_1(y,z),\theta_2(y,y)\}\}\\ &\geqslant \sup_{y\in X}\min\{\min\{\theta_1(x,z),\theta_2(y,z)\},\min\{\theta_1(y,z),\theta_2(z,y)\}\}\\ &= \sup_{y\in X}\min\{\sup_{z\in X}\min\{\theta_1(x,z),\theta_2(y,z)\},\sup_{z\in X}\min\{\theta_1(y,z),\theta_2(z,y)\}\}\\ &= \sup_{y\in X}\min\{\theta_1 \circ \theta_2(x,y),\theta_1 \circ \theta_2(y,y)\}\\ &\geqslant \min\{\theta_1 \circ \theta_2(x,y),\theta_1 \circ \theta_2(y,y)\}\\ &\geqslant \min\{\theta_1 \circ \theta_2(x,y),\theta_1 \circ \theta_2(y,z)\}\end{aligned}$$

Hence, $(\theta_1 \circ \theta_2)(x,z) \geqslant \min\{\theta_1 \circ \theta_2(x,y),\theta_1 \circ \theta_2(y,z)\}$
Therefore, $\theta_1 \circ \theta_2$ is transitive.

4) Let $x, y, z, w \in X$. We have

$$\begin{aligned}(\theta_1 \circ \theta_2)(x \odot z, y \odot w) &= \sup_{u \in X} \min\{\theta_1(x \odot z, u), \theta_2(u, y \odot w)\} \\ &\geqslant \sup_{a,b \in X} \min\{\theta_1(x \odot z, a \odot b), \theta_2(a \odot b, y \odot w)\} \\ &\geqslant \sup_{a,b \in X} \min\{\min\{\theta_1(x,a), \theta_1(z,b)\}, \min\{\theta_2(a,y), \theta_2(b,w)\}\} \\ &\geqslant \min\{\sup_{a \in X} \min\{\theta_1(x,a), \theta_2(a,y)\}, \sup_{b \in X} \min\{\theta_2(z,b), \theta_2(b,w)\}\} \\ &= \min\{\theta_1 \circ \theta_2(x,y), \theta_1 \circ \theta_2(z,w)\}\end{aligned}$$

and

$$\begin{aligned}(\theta_1 \circ \theta_2)(x \star z, y \star w) &= \sup_{z \in X} \min\{\theta_1(x \star z, u), \theta_2(u, y \star w)\} \\ &\geqslant \sup_{a,b \in X} \min\{\theta_1(x \star z, a \star b), \theta_2(a \star b, y \star w)\} \\ &\geqslant \sup_{a,b \in X} \min\{\min\{\theta_1(x,a), \theta_1(z,b)\}, \min\{\theta_2(a,y), \theta_2(b,w)\}\} \\ &\geqslant \min\{\sup_{a \in X} \min\{\theta_1(x,a), \theta_2(a,y)\}, \sup_{b \in X} \min\{\theta_2(z,b), \theta_2(b,w)\}\} \\ &= \min\{\theta_1 \circ \theta_2(x,y), \theta_1 \circ \theta_2(z,w)\}\end{aligned}$$

Therefore, $\theta_1 \circ \theta_2$ is a fuzzy congruence relation on a pseudo-TM algebra X.

□

Theorem 3.3.3. *Let θ_1, θ_2, and θ_3 be a fuzzy congruence relation on a pseudo TM-algebra X. Then $\theta_1 \circ (\theta_2 \cap \theta_3) \leqslant (\theta_3 \circ \theta_2) \cap (\theta_1 \circ \theta_3)$.*

Proof. Assume that θ_1, θ_2, and θ_3 be a fuzzy congruence relation on a pseudo TM-algebra X. We need to show $\theta_1 \circ (\theta_2 \cap \theta_3) \leqslant (\theta_3 \circ \theta_2) \cap (\theta_1 \circ \theta_3)$. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}[\theta_1 \circ (\theta_2 \cap \theta_3)](x,y) &= \sup_{z \in X}\{\min\{\theta_1(x,z), (\theta_2 \cap \theta_3)(z,y)\}\} \\ &= \sup_{z \in X}\{\min\{\theta_1(x,z), \min\{\theta_2(z,y), \theta_3(z,y)\}\}\} \\ &\leqslant \min\left\{\sup_{z \in X}\{\min\{\theta_1(x,z), \theta_2(z,y)\}\}, \sup_{z \in X}\{\min[\theta_1(x,z), \theta_3(z,y)]\}\right\} \\ &= \min\{(\theta_1 \circ \theta_2)(x,y), (\theta_1 \circ \theta_3)(x,y)\} \\ &= \{(\theta_1 \circ \theta_2) \cap (\theta_1 \circ \theta_3)\}(x,y).\end{aligned}$$

Therefore, $\theta_1 \circ (\theta_2 \cap \theta_3) \leqslant (\theta_3 \circ \theta_2) \cap (\theta_1 \circ \theta_3)$ □

Chapter 4
Hesitant Fuzzy Subset on Pseudo-TM algebra

In the study of algebraic structures, a TM algebra is a specific type of algebraic system characterized by certain operations and properties. A subalgebra of a TM algebra is a subset that itself forms a TM algebra under the same operations. When uncertainty or hesitation is involved in determining membership of elements in these subalgebras, classical fuzzy sets may not fully capture this ambiguity. In such cases, hesitant fuzzy sets provide a more suitable framework. Hesitant fuzzy set theory is applied to many scientific and engineering fields, and also applied to algebraic structures. Unlike standard fuzzy sets, hesitant fuzzy sets allow the membership degree of an element to be represented by a set of possible values rather than a single number. This approach models situations where there is hesitation or indecision about the exact membership degree. A hesitant fuzzy TM-subalgebra is thus a generalization of fuzzy TM-subalgebra defined with hesitant fuzzy sets. It assigns to each element a set of membership degrees reflecting hesitation, providing a richer and more flexible framework to handle uncertainty in TM algebraic structures. Studying hesitant fuzzy TM-subalgebras helps in better understanding and modeling complex systems where ambiguity or multiple perspectives on membership are natural.

In this section, the notions of hesitant fuzzy TM-subalgebra of a TM-algebras, hesitant fuzzy T-ideals of TM algebras, the concept of hesitant fuzzy sets(HFS) applied to pseudo-TM subalgebra(PTMS) in pseudo-TM algebra(PTMA), the notions of hesitant fuzzy pesudo ideal of a pseudo TM-algebras, and some intersting properties are introduced.

4.1. Hesitant Fuzzy TM-Subalgebra of TM-algebra

In this section, we introduce the concepts of hesitant fuzzy TM-subalgebras of TM algebras and presents some interesting properties. Let X, Y denote a TM-algebras, let $\mathcal{P}([0,1])$ denote the power set of $[0,1]$, let A and B are a non-empty asubset of a TM-algebras X and Y respectively unless otherwise specfied throughout this and the following section.

Definition 4.1.1. *Let* X *be a* TM*-algebra. Given a non-empty subset* S *of* X*, a hesitant fuzzy set on* X *satisfying*

$$H_X = \{(x, h_X(x)) \mid x \in X\} \text{ and}$$

$$h_X(x) = \emptyset \textit{ for all } x \notin S$$

this is called a hesitant fuzzy set related to S *(briefly, S-hesitant fuzzy set) on* X*, and is represented by* $H_S = \{(x, h_S(x)) \mid x \in X\}$*, where* h_S *is a mapping from* X *to* $\mathcal{P}[0,1]$ *with* $h_S(x) = \emptyset$ *for all* $x \notin S$*.*

Definition 4.1.2. *A hesitant fuzzy set* $H = \{(x, h(x)) | x \in X\}$ *is called a hesitant fuzzy TM-subalgebra of a TM algebra* X *if*

$$(\forall x, y \in X)\,(h(x) \cap h(y)) \subseteq h(x \star y).$$

Definition 4.1.3. *A non-empty subset* S *of a TM-algebra* X *is called an A hesitant fuzzy subalgebra of X related to* S*,* $H_S = \{(x, h_S(x)) | x \in X\}$ *if it satisfies the following condition:*

$$(\forall x, y \in S)\,(h_S(x) \cap h_S(y)) \subseteq h_S(x \star y).$$

An S-hesitant fuzzy subalgebra of X *with* $S = X$ *is called a hesitant fuzzy subalgebra of* X*.*

Example 4.1.1. *Let* $X = \{0, 1, 2, 3, 4\}$ *with the following Cayley table:*

$\star$	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Table 4.1

See [43] $(X, \star, 0)$ is a TM algebra. We define the hesitant fuzzy set $H = \{(x, h(x))\}$ on X as follows $h(0) = \{0.1, 0.3\}, h(1) = h(2) = h(3) = h(4) = \{0.3\}$.
Then h is a hesitant fuzzy TM-subalgebra of X.

Theorem 4.1.1. *Let* $H = \{(x, h(x)) | x \in X\}$ *be a hesitant fuzzy TM-subalgebra of a TM-algebra* X*, then the following property holds:*

$$(\forall x \in X)\, h(x) \subseteq h(0).$$

Proof. Suppose that $H = \{(x, h(x)) | x \in X\}$ be a hesitant fuzzy TM-subalgebra of a TM algebra X. We need to show that $(\forall x \in X)\, h(x) \subseteq h(0)$. For any $x \in X$, we have

$$h(0) = h(x \star x) \supseteq h(x) \cap h(x) = h(x).$$

□

Corollary 4.1.2. *Let X be a TM-algebra. If* $H = \{(x, h_(x)) | x \in X\}$ *be a hesitant fuzzy TM-subalgebra of a TM -algebra* X*, then for all x,y* $\in X | h(x) \cap h(y) \subseteq h(x \star (0 \star y))$*.*

Proof. Suppose that $H = \{(x, h_(x)) | x \in X\}$ be a hesitant fuzzy TM-subalgebra of a TM -algebra X. We need to show that for all x,y $\in X | h(x) \cap h(y) \subseteq h(x \star (0 \star y))$. Now,

$$\begin{aligned} h(x \star (0 \star y)) &\supseteq h(x) \cap h(0 \star y) \\ &\supseteq h(x) \cap h(0) \cap h(y) \\ &\supseteq h(x) \cap h(y) \end{aligned}$$

□

Definition 4.1.4. *The characteristic hesitant fuzzy set (CHFS) of a subset* S *of a set* X *is defined to be the structure* $\chi_S = (x, h_{\chi_S})$*, where*

$$h_{\chi_S}(x) = \begin{cases} [0,1] & \textit{if } x \in S \\ \emptyset & \textit{otherwise} \end{cases}$$

Corollary 4.1.3. *In a non-empty subset* S *of* X*,* $h_{\chi_S}(0) \supseteq h_{\chi_S}(x)$*, for all* $x \in X$*.*

Proof. Suppose that S is a non-empty subset of X. If $0 \in S$, then $h_{\chi_S}(0) = [0,1]$. Thus, $h_{\chi_S}(0) = [0,1] \supseteq h_{\chi_S}(x)$ for all $x \in X$. Also, if $0 \notin S$ $h_{\chi_S}(0) = \emptyset$. Then $h_{\chi_S}(0) = \emptyset \subseteq h_{\chi_S}(x)$ for all $x \in X$. □

Theorem 4.1.4. *A non-empty subset* S *of a TM-algebra* $(X, \star, 0)$ *is a TM-subalgebra of* X *if and only if the (CHFS* χ_S*) is a hesitant fuzzy TM-subalgebra of* X*.*

Proof. Assume that S is a subalgebra of a TM-algebra X. We need to show that(CHFS χ_S) is a hesitant fuzzy TM-subalgebra of X. Let $x, y \in X$.

Case 1: If $x, y \in S$, then $h_{\chi S}(x) = [0,1]$ and $h_{\chi S}(y) = [0,1]$. Thus, $h_{\chi S}(x) \cap h_{\chi S}(y) = [0,1]$. Since S is a subalgebra of X. Then $x \star y \in S$ and thus, $h_{\chi S}(x \star y) = [0,1]$. Then $h_{\chi S}(x \star y) = [0,1] \supseteq [0,1] = h_{\chi_S}(x) \cap h_{\chi S}(y)$.

Case 2: If $x \in S$ and $y \notin S$, then $h_{\chi S}(x) = [0,1]$ and $h_{\chi S}(y) = \emptyset$. Thus, $h_{\chi S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \star y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$.

Case 3: If $x \notin S$ and $y \in S$, then $h_{\chi S}(x) = \emptyset$ and $h_{\chi S}(y) = [0,1]$. Thus, $h_{\chi S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi S}(x \star y) \supseteq \emptyset = h_{\chi S}(x) \cap h_{\chi_S}(y)$.

Case 4: If $x \notin S$ and $y \notin S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = \emptyset$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. So, $h_{\chi_S}(x \star y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$.

Therefore, χ_S is a hesitant fuzzy TM-subalgebra of X.

Conversely, assume that χ_S is a hesitant fuzzy TM-subalgebra of X. We need to show that S is a TM-subalgebra of X. Let $x, y \in S$. Then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = [0,1]$. Thus, $h_{\chi_S}(x \star y) \supseteq h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0,1]$, so $h_{\chi_S}(x \star y) = [0,1]$.

Hence, $x \star y \in S$. Therefore, S is a subalgebra of X.

□

Theorem 4.1.5. *If* $H = \{(x, h(x)) | x \in X\}$ *is a hesitant fuzzy TM-subalgebra of a TM-algebra* $(X, \star, 0)$*, then the set* $X_h = \{x \in X \mid h(x) = h(0)\}$ *is a subalgebras of* X.

Proof. Suppose that $H = \{(x, h(x)) | x \in X\}$ is a hesitant fuzzy TM-subalgebra of a TM-algebra $(X, \star, 0)$. We need to show that $X_h = \{x \in X \mid h(x) = h(0)\}$ is a subalgebras of X. Our aim is we need to show that if $x, y \in X_h$, then $x \star y \in X_h$. We need to show that $0 \in X_h$. Now, $h(0) = h(0) \Rightarrow 0 \in X_h$. We need to show that X_h is closed under $\star$. Take any $x, y \in X_h$, i.e., $h(x) = h(0)$ and $h(y) = h(0)$. We want to show that $h(x \star y) = h(0)$. We use the hesitant fuzzy subalgebra condition $h(x \star y) \supseteq h(x) \cap h(y)$. But, $h(x) = h(0)$ and $h(y) = h(0)$. So, $h(x) \cap h(y) = h(0) \Rightarrow h(x \star y) \supseteq h(0)$. Also, by the property of the hesitant fuzzy subalgebra $h(x \star y) \subseteq h(0)$. So, we have both $h(x \star y) \subseteq h(0)$ and $h(x \star y) \supseteq h(0)$. Therefore, $h(x \star y) = h(0) \Rightarrow x \star y \in X_h$. □

Corollary 4.1.6. *Let X be a TM-algebra. Let* $H = \{(x, h(x)) | x \in X\}$ *be a hesitant fuzzy TM-subalgebra of a TM -algebra* X*. Then* $h(y) \subseteq h(x \star y))$ *if and only if* $h(x) = h(0)$.

Definition 4.1.5. *Let* $h : X \to \mathcal{P}[0,1]$*. For any* $\Lambda \in \mathcal{P}[0,1]$*, the sets* $\mathcal{U}(h, \Lambda) = \{x \in X \mid h(x) \supseteq \Lambda\}$ *and* $\mathcal{U}^+(h, \Lambda) = \{x \in X \mid h(x) \supset \Lambda\}$ *are called an upper* Λ*-level subset and an upper* Λ*-strong level subset of* h *respectively. The sets* $\mathcal{L}(h, \Lambda) = \{x \in X \mid h(x) \subseteq \Lambda\}$ *and* $\mathcal{L}^-(h, \Lambda) = \{x \in X \mid h(x) \subset \Lambda\}$ *are called a lower* Λ*-level subset and a lower* Λ*-strong level subset of* h *respectively. The set* $\mathcal{E}(h, \Lambda) = \{x \in X \mid h(x) = \Lambda\}$ *is called an equal* Λ*-level subset of* h*. Then* $\mathcal{U}(h, \Lambda) = \mathcal{U}^+(h, \Lambda) \cup \mathcal{E}(h, \Lambda)$ *and* $\mathcal{L}(h, \Lambda) = \mathcal{L}^-(h, \Lambda) \cup \mathcal{E}(h, \Lambda)$.

Theorem 4.1.7. *A hesitant fuzzy set* $H = \{(x, h(x)) | x \in X\}$ *on a TM-algebra* $(X, \star, 0)$ *is a hesitant fuzzy TM-subalgebra of* X *if and only if for all* $\Lambda \in \mathcal{P}[0,1]$*, the non-empty subset* $\mathcal{U}(h, \Lambda)$ *of* X *is a subalgebra.*

Proof. Suppose that H is a hesitant fuzzy TM-subalgebra of X. We need to show that $\mathcal{U}(h, \Lambda)$ of X is a subalgebra. Let $\Lambda \in \mathcal{P}[0,1]$ such that $\mathcal{U}(h, \Lambda) \neq \emptyset$ and let $x, y \in \mathcal{U}(h, \Lambda)$. Then $h(x) \supseteq \Lambda$ and $h(y) \supseteq \Lambda$. Since H is a hesitant fuzzy TM-subalgebra of X, we have $h(x \star y) \supseteq h(x) \cap h(y) \supseteq \Lambda$ and thus $x \star y \in \mathcal{U}(h, \Lambda)$. Thus, $\mathcal{U}(h, \Lambda)$ is a subalgebra of X.
Conversely, assume that for every $\Lambda \in \mathcal{P}[0,1]$, the nonempty subset $\mathcal{U}(h, \Lambda)$ is subalgebra of X. We need to show that H is a hesitant fuzzy TM-subalgebra of X. Let $x, y \in X$. Choose $\Lambda = h(x) \cap h(y) \in \mathcal{P}[0,1]$. Then $h(x) \supseteq \Lambda$ and $h(y) \supseteq \Lambda$. Thus, $x, y \in \mathcal{U}(h, \Lambda) \neq \emptyset$. By assumption, $\mathcal{U}(h, \Lambda)$ is a subalgebra of X and thus $x \star y \in \mathcal{U}(h, \Lambda)$. So, $h(x \star y) \supseteq \Lambda = h(x) \cap h(y)$. Therefore, H is a hesitant fuzzy TM-subalgebra of X. □

Theorem 4.1.8. *Let* h_1 *and* h_2 *be two hesitant fuzzy TM-subalgebra of a TM-algebra* X*. Then* $h_1 \cap h_2$ *is a hesitant fuzzy TM-subalgebra of* X.

Proof. Assume that h_1 and h_2 be a hesitant fuzzy TM-subalgebras of a TM-algebra X. We need to show that $(h_1 \cap h_2)(x \star y) \supseteq (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)$, for all $x, y \in X$.
We need to consider the following three cases:

Case (i). $h_1(x\star y)\subseteq h_2(x\star y)$. By Definition 1.2.3

$$\begin{aligned}(h_1\cap h_2)(x\star y) &= \left\{h\in(h_1(x\star y)\cap h_2(x\star y)) \mid h\leqslant \min\left(h_1^+(x\star y),h_2^+(x\star y)\right)\right\}\\ &= \left\{h\in h_1(x\star y) \mid h\leqslant \min\left(h_1^+(x\star y),h_1^+(x\star y)\right)\right\}\\ &\supseteq \left\{h\in(h_1(x)\cap h_1(y)) \mid h\leqslant \min\left(h_1^+(x\star y),h_1^+(x\star y)\right)\right\}\\ &\supseteq \left\{h\in((h_1\cap h_2)(x)\cap(h_1\cap h_2)(y)) \mid h\leqslant \min\left((h_1\cap h_2)^+(x),(h_1\cap h_2)^+(y)\right)\right\}\\ &= (h_1\cap h_2)(x)\cap(h_1\cap h_2)(y).\end{aligned}$$

Case (ii). $h_1(x\star y)\supseteq h_2(x\star y)$

$$\begin{aligned}(h_1\cap h_2)(x\star y) &= \left\{h\in(h_1(x\star y)\cap h_2(x\star y)) \mid h\leqslant \min\left(h_1^+(x\star y),h_2^+(x\star y)\right)\right\}\\ &= \left\{h\in h_2(x\star y) \mid h\leqslant \min\left(h_2^+(x\star y),h_2^+(x\star y)\right)\right\}\\ &\supseteq \left\{h\in(h_2(x)\cap h_2(y)) \mid h\leqslant \min\left(h_2^+(x),h_2^+(y)\right)\right\}\\ &\supseteq \left\{h\in((h_1\cap h_2)(x)\cap(h_1\cap h_2)(y)) \mid h\leqslant \min\left((h_1\cap h_2)^+(x),(h_1\cap h_2)^+(y)\right)\right\}\\ &= (h_1\cap h_2)(x)\cap(h_1\cap h_2)(y).\end{aligned}$$

Case (iii). $h_1(x\star y)\approx h_2(x\star y)$

$$\begin{aligned}(h_1\cap h_2)(x\star y) &= h_1(x\star y)\cap h_2(x\star y)\\ &\approx h_1(x\star y)\\ &\supseteq h_1(x)\cap h_1(y)\\ &\supseteq (h_1\cap h_2)(x)\cap(h_1\cap h_2)(y).\end{aligned}$$

□

Corollary 4.1.9. *If* $\{h_i \mid i\in I\}$ *is a family of hesitant fuzzy TM-subalgebra of a TM-algebra* $(X,\star,0)$*, then* $\cap_{i\in I}h_i$ *is a hesitant fuzzy TM-subalgebra of* X*.*

Remark 4.1.1. *The union of two hesitant fuzzy TM-subalgebra of a TM-algebra does not need to be a hesitant fuzzy TM-subalgebra. This is illustrated in the following example:*

Example 4.1.2. *Consider a TM-algebra defined in Example 4.1.1.*
Let h_1 *and* h_2 *be two hesitant fuzzy sets in* $\mathbb{Z}$ *defined by*

$$h_1(x)=\begin{cases}\{0.4,0.5,0.6\} & \text{if } x \text{ is odd}\\ \{0.7,0.8,0.9\} & \text{if } x \text{ is even,}\end{cases} \quad \text{and} \quad h_2(x)=\begin{cases}\{0.8,0.9\} & \text{if } x=7n, n\in\mathbb{Z}\\ \{0.4,0.5\} & \text{otherwise.}\end{cases}$$

By Definition 4.1.2, h_1 *and* h_2 *are a hesitant fuzzy TM-subalgebra of* $\mathbb{Z}$*. Now, taking* $x=7$ *and* $y=2$*. By Definition 1.2.3. We have*

$$
\begin{aligned}
(h_1 \cup h_2)(7-2) &= (h_1 \cup h_2)(5) \\
&= \left\{h \in (h_1(5) \cup h_2(5)) \mid h \geqslant \max\left(h_1^-(5), h_2^-(5)\right)\right\} \\
&= \{0.4, 0.5, 0.6\}. \textit{ But } (h_1 \cup h_2)(7) \cap (h_1 \cup h_2)(2) = \{0.8, 0.9\}.
\end{aligned}
$$

Therefore, $(h_1 \cup h_2)(7-2) \supsetneq (h_1 \cup h_2)(7) \cap (h_1 \cup h_2)(2)$. *Hence,* $h_1 \cup h_2$ *is not a hesitant fuzzy TM-subalgebra of* $\mathbb{Z}$.

Theorem 4.1.10. *Let* h_1 *and* h_2 *be two hesitant fuzzy TM-subalgebra of a TM-algebra* X. *Then* $h_1 \otimes h_2$ *and* $h_1 \oplus h_2$ *are a hesitant fuzzy TM-subalgebra of* X.

Proof. Assume that h_1 and h_2 be a hesitant fuzzy TM-subalgebras of a TM-algebra X. We need to show that $(h_1 \otimes h_2)(x \star y) \supseteq (h_1 \otimes h_2)(x) \cap (h_1 \otimes h_2)(y)$, for all $x, y \in X$. By Definition 1.2.3 we have

$$
\begin{aligned}
&(h_1 \otimes h_2)(x) \cap (h_1 \otimes h_2)(y) \\
&= \left\{\bigcup \gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\right\} \\
&\cap \left\{\bigcup \lambda_1\lambda_2 : \lambda_1 \in h_1(y), \lambda_2 \in h_2(y)\right\} \\
&= \{\alpha_1\alpha_2 : \alpha_1 \in h_1(x) \cap h_1(y), \alpha_2 \in h_2(x) \cap h_2(y)\} \\
&\subseteq \{\alpha_1\alpha_2 : \alpha_1 \in h_1(x \star y), \alpha_2 \in h_2(x \star y)\} \\
&= (h_1 \otimes h_2)(x \star y).
\end{aligned}
$$

Therefore, $h_1 \otimes h_2$ is a hesitant fuzzy TM-subalgebra of X.

and

$$
\begin{aligned}
&(h_1 \oplus h_2)(x) \cap (h_1 \oplus h_2)(y) \\
&= \left\{\bigcup \gamma_1 + \gamma_2 - \gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\right\} \\
&\cap \left\{\bigcup \lambda_1 + \lambda_2 - \lambda_1\lambda_2 : \lambda_1 \in h_1(y), \lambda_2 \in h_2(y)\right\} \\
&= \{\alpha_1 + \alpha_2 - \alpha_1\alpha_2 : \alpha_1 \in h_1(x) \cap h_1(y), \alpha_2 \in h_2(x) \cap h_2(y)\} \\
&\subseteq \{\alpha_1\alpha_2 : \alpha_1 \in h_1(x \star y), \alpha_2 \in h_2(x \star y)\} \\
&= (h_1 \oplus h_2)(x \star y).
\end{aligned}
$$

Therefore, $h_1 \oplus h_2$ is a hesitant fuzzy TM-subalgebra of X. □

Theorem 4.1.11. *If* $h : X \to \mathcal{P}[0,1]$ *is a hesitant fuzzy TM-subalgebra of a TM-algebra* X. *Then*
1) (α-upper bounded) h_α^+ *is a hesitant furzy TM-subalgebra of* X.
2) (α-lower bounded) h_α^- *is a hesitant fuzzy TM-subalgebra of* X.

Proof. Assume that h(x) is a hesitant TM-subalgebra of X. Let $x, y \in X$. By Definition(1.2.3) we have:

1) $h^+_\alpha(x\star y)=\{h\in h(x\star y):h\geqslant\alpha\}$

$$\begin{aligned}&\supseteq\{h\in h(x)\cap h\in h(y):h\geqslant\alpha\}\\&=\{h\in h(x):h\geqslant\alpha\cap h\in h(y):h\geqslant\alpha\}\\&=\{h\in h(x):h\geqslant\alpha\}\cap\{h\in h(y):h\geqslant\alpha\}\\&=h^+_\alpha(x)\cap h^+_\alpha(y).\end{aligned}$$

Therefore, $h^+_\alpha(x\star y)\supseteq h^+_\alpha(x)\cap h^+_\alpha(y)$.
2) $h^-_\alpha(x\star y)=\{h\in h(x\star y):h\leqslant\alpha\}$

$$\begin{aligned}&\supseteq\{h\in h(x)\cap h\in h(y):h\leqslant\alpha\}\\&=\{h\in h(x):h\leqslant\alpha\cap h\in h(y):h\leqslant\alpha\}\\&=\{h\in h(x):h\leqslant\alpha\}\cap\{h\in h(y):h\geqslant\alpha\}\\&=h^-_\alpha(x)\cap h^-_\alpha(y).\end{aligned}$$

Therefore, $h^-_\alpha(x\star y)\supseteq h^-_\alpha(x)\cap h^-_\alpha(y)$.

□

Theorem 4.1.12. *Let h be a hesitant fuzzy set on X. Then h^c is a hesitant fuzzy TM-subalgebra of X if and only if for all $\Lambda\in\mathcal{P}[0,1]$, a non-empty subset $\mathcal{L}(h,\Lambda)$ of X is a TM-subalgebra of X.*

Proof. Suppose that h^c is a hesitant fuzzy TM-subalgebra of X. We need to show that $\mathcal{L}(h,\Lambda)$ of X is a TM-subalgebra of X. Let $\Lambda\in\mathcal{P}[0,1]$ such that $\mathcal{L}(h,\Lambda)\neq\varnothing$ and let $x,y\in\mathcal{L}(h,\Lambda)$. Then $h(x)\subseteq\Lambda$ and $h(y)\subseteq\Lambda$. Since h^c is a hesitant fuzzy TM-subalgebra of X, we have $h^c(x\star y)\supseteq h^c(x)\cap h^c(y)$. By Definitions 1.2.2, we have $[0,1]-h(x\star y)\supseteq([0,1]-h(x))\cap([0,1]-h(y))=[0,1]-(h(x)\cup h(y))$. Hence, $h(x\star y)\subseteq h(x)\cup h(y)\subseteq\Lambda$. Thus, $x\star y\in\mathcal{L}(h,\Lambda)$. Therefore, $\mathcal{L}(h,\Lambda)$ is a TM-subalgebra of X.
Conversely, suppose that for all $\Lambda\in\mathcal{P}[0,1]$, a non-empty subset $\mathcal{L}(h,\Lambda)$ of X is a TM-subalgebra of X. Let $x,y\in X$. Choose $\Lambda=h(x)\cup h(y)\in\mathcal{P}[0,1]$. Then $h(x)\subseteq\Lambda$ and $h(y)\subseteq\Lambda$. Thus, $x,y\in\mathcal{L}(h,\Lambda)\neq\varnothing$. By assumption, we have $\mathcal{L}(h,\Lambda)$ is a TM-subalgebra of X and thus, $x\star y\in\mathcal{L}(h,\Lambda)$. Hence, $h(x\star y)\subseteq\Lambda=h(x)\cup h(y)$.
By Definitions 1.2.2, we have

$$\begin{aligned}h^c(x\star y)&=[0,1]-h(x\star y)\\&\supseteq[0,1]-(h(x)\cup h(y))\\&=([0,1]-h(x))\cap([0,1]-h(y))\\&=h^c(x)\cap h^c(y).\end{aligned}$$

Therefore, h^c is a hesitant fuzzy TM-subalgebra of X. □

Proposition 4.1.13. *For a hesitant fuzzy set* h *on* X*, let* $\tilde{h}$ *be a hesitant fuzzy set on* X *defined by*

$$\tilde{h}: X \to \mathcal{P}[0,1], x \mapsto \begin{cases} h(x) & \text{if } x \in \mathcal{U}(h,\Lambda), \\ \emptyset & \text{otherwise} \end{cases}$$

where $\Lambda \in \mathcal{P}[0,1] \backslash \{\emptyset\}$*. If* h *is a hesitant fuzzy TM-subalgebra of* X*, then* $\tilde{h}$ *is also a hesitant fuzzy TM-subalgebra of* X*.*

Proof. Assume that h is a hesitant fuzzy TM-subalgebra of X. By Theorem 4.1.7 $\mathcal{U}(h,\Lambda)$ is a subalgebra of X, for all $\Lambda \in \mathcal{P}[0,1]$ with $\mathcal{U}(h,\Lambda) \neq \emptyset$. Let $x, y \in X$.
Case 1: If $x \in \mathcal{U}(h,\Lambda)$ and $y \in \mathcal{U}(h,\Lambda)$, then $x \star y \in \mathcal{U}(h,\Lambda)$. Thus

$$\tilde{h}(x \star y) = h(x \star y) \supseteq h(x) \cap h(y) = \tilde{h}(x) \cap \tilde{h}(y).$$

Case 2: If $x \notin \mathcal{U}(h,\Lambda)$ or $y \notin \mathcal{U}(h,\Lambda)$, then $\tilde{h}(x) = \emptyset$ or $\tilde{h}(y) = \emptyset$.
Hence,

$$\tilde{h}(x \star y) \supseteq \emptyset = \tilde{h}(x) \cap \tilde{h}(y).$$

Therefore, $\tilde{h}$ is a hesitant fuzzy TM-subalgebra of X. □

Theorem 4.1.14. *If* $h_X = (x, h_X(x))$ *and* $h_Y = (y, h_Y(y))$ *are two hesitant fuzzy TM-subalgebras of TM-algebras* X *and* Y*, respectively, then the Cartesian product* $h_X \times h_Y$ *is also an hesitant fuzzy TM-subalgebra of* $X \times Y$*.*

Proof: Suppose that $h_X = (x, h_X(x))$ and $h_Y = (y, h_Y(y))$ are two hesitant fuzzy TM-subalgebras of TM-algebras X and Y, respectively. We need to show that $h_X \times h_Y$ is also an hesitant fuzzy TM-subalgebra of $X \times Y$. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. By Definition (1.2.4) we have

$$\begin{aligned} h((x_1,y_1) \star (x_2,y_2)) &= h((x_1 \star x_2),(y_1 \star y_2)) \\ &= h_A(x_1 \star x_2) \cap h_B(y_1 \star y_2) \\ &\supseteq (h_A(x_1) \cap h_A(x_2)) \cap (h_B(y_1) \cap h_B(y_2)) \\ &= (h_A(x_1) \cap h_B(y_1)) \cap (h_A(x_2) \cap h_B(y_2)) \\ &= h(x_1,y_1) \cap h(x_2,y_2). \end{aligned}$$

Hence, $h_A \times h_B$ is a hesitant fuzzy TM-subalgebra of $X \times Y$.

Theorem 4.1.15. *Let h be a hesitant fuzzy TM-subalgebra of a TM-algebra* X *and f be a homomorphism from a TM-algebra* X *to a TM-algebra* Y*. Then f*(h) *is a hesitant fuzzy TM-subalgebra over* Y*.*

Proof. Suppose that be a hesitant fuzzy TM-subalgebra of a TM-algebra X. We needc to show that $f(h)$ is a hesitant fuzzy TM-subalgebra over Y. Let $y_1, y_2 \in Y$. If $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$, then we have

$$f(h)(y_1) \cap f(h)(y_2) = \varnothing.$$

Therefore,

$$f(h)(y_1 \star y_2) \supseteq f(h)(y_1) \cap f(h)(y_2).$$

Hence, $f(h)$ is a hesitant fuzzy TM-subalgebra of Y.
If $f^{-1}(y_1) \neq \emptyset$ and $f^{-1}(y_2) \neq \emptyset$, then $f^{-1}(y_1 \star y_2) \neq \emptyset$. Let us assume that there exist $x_1, x_2 \in X$ such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. By Definition 1.2.5 we have

$$\begin{aligned} f(h)(y_1 \star y_2) &= \bigcup_{x \in f^{-1}(y_1 \star y_2)} h(x) \\ &\supseteq h(x_1 \star x_2) \\ &\supseteq [h(x_1) \cap h(x_2)]. \end{aligned}$$

Therefore, $f(h)$ is a hesitant fuzzy TM-subalgebra of Y. □

Theorem 4.1.16. *Let f be a homomorphism of* TM*-algebras. If* $H = (y, h(y)/y \in Y)$ *is a hesitant fuzzy* TM*-subalgebra of* Y*, then* $f^{-1}(H) = (x, (h \circ f)(x))$ *is a hesitant fuzzy* TM*-subalgebra of* X.

Proof. Assume that $H = (y, h(y))$ is a hesitant fuzzy TM-subalgebra of Y. We need to show that $f^{-1}(H) = (x, (h \circ f)(x))$ is a hesitant fuzzy TM-subalgebra of X.
Let $x, y \in X$. Then

$$\begin{aligned} (h \circ f)(x \star y) &= h(f(x \star y)) \\ &= h(f(x) \star f(y)) \\ &\supseteq h(f(x)) \cap h(f(y)) \\ &= (h \circ f)(x) \cap (h \circ f)(y), \end{aligned}$$

Therfore, $f^{-1}(h)$ is a hesitant fuzzy TM-subalgebra of X. □

4.2. Hesitant Fuzzy T-ideals of TM-algebra

In this section, we introduce the concepts of hesitant fuzzy TM-ideal of TM-algebras and present some interesting properties.

Definition 4.2.1. *A subset* h *of a TM-algebra* X *is called hesitant fuzzy ideal of* X *if it satisfies the following condition*

$$(\forall x, y \in X)\begin{pmatrix} h(0) \supseteq h(x) \\ h(x) \supseteq h(x \star y) \cap h(y) \end{pmatrix}$$

Definition 4.2.2. *A subset* h *of a TM-algebra* X *is called hesitant fuzzy TM-ideal of* X *if it satisfies the following condition*

$$(\forall x, y, z \in X)\begin{pmatrix} h(0) \supseteq h(x) \\ h(x \star y) \supseteq h(x \star z) \cap h(z \star y) \end{pmatrix}$$

Example 4.2.1. *Let* $X = \{0, 1, 2, 3\}$ *with the following Cayley table:*

$\star$	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	0	1
3	3	2	1	0

Table 4.2

See [43] $(X,\star,0)$ is a TM -algebra. We define the hesitant fuzzy set $H_X = \{(x,h_X(x)) \mid x \in X\}$ on X as follows:
$h(0) = [0,1], h(1) = \{0.1\}, h(2) = \emptyset, h(3) = \{0.2,0.3\}$. Then I is a hesitant fuzzy TM-ideal of X.

Definition 4.2.3. *A subset* h *of a TM-algebra* X *is called hesitant fuzzy T-ideal of* X *if it satisfies the following condition:*

$$(\forall x,y,z \in X)\begin{pmatrix} h(0) \supseteq h(x) \\ h(x \star z) \supseteq h((x \star y) \star z) \cap h(y) \end{pmatrix}$$

Example 4.2.2. *Let* $X = \{0,1,2,3\}$ *with the following Cayley table:*

$\star$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 4.3

See [43] $(X,\star,0)$ is a TM- algebra. We define the hesitant fuzzy set $H_X := \{(x,h_X(x)) \mid x \in X\}$ on X as follows:
$h(0) = \{1.0,\ 0.8\},\ h(1) = \{0.8\},\ h(2) = h(3) = \{0.8\}$. It is easly verify that h is a hesitant fuzzy T-ideal of X.

Theorem 4.2.1. *A hesitant fuzzy set* h *in a TM -algebra* X *is ahesitant fuzzy T-ideal if and only if it is a hesitant fuzzy ideal of* X.

Proof. Let h be a hesitant fuzzy T-ideal of X. Then $h(x \star z) \supseteq h((x \star y) \star z) \cap h(y), \forall x,y,z \in X$.
Assuming $z = 0$ we have $h(x) \supseteq h((x \star y) \cap h(y)$.
Also, $h(0) \supseteq h(x)$.
Hence, h is a hesitant fuzzy ideal of X.
Conversely, suppose that h is a hesitant fuzzy ideal of X. Then $h(x) \supseteq h((x \star y) \cap h(y), \forall x,y \in X$.
It follows that for all $x,y,z \in X$ we have $h(x \star z) \supseteq h((x \star y) \star z) \cap h(y)$. This complate the proof. □

Theorem 4.2.2. *A hesitant fuzzy ideal of a TM -algebra* X *is order reversing.*

Proof. Let $x, y, z \in X$ be such that $x \leqslant y$. Then $x \star y = 0$, and so $h(x) \supseteq h(x \star y) \cap h(y) = h(0) \cap h(y) = h(y)$.

□

Lemma 4.2.3. *The constant of a TM -algebra* X *is a non-empty subset S of* X *if and only if* $h_{\chi_S}(0) \supseteq h_{\chi_S}(x)$

Proof. By Definition 4.1.4 we have
If $0 \in S$, then $h_{\chi_S}(0) = [0,1]$. Thus $h_{\chi_S}(0) = [0,1] \supseteq h_{\chi_S}(x)$, for all $x \in X$.
Coversely, assume that $h_{\chi_S}(0) \supseteq h_{\chi_S}(x)$, for all $x \in X$. Since S is a non-empty subset of S, we have there exists $t \in S$ for some $t \in X$. Then $h_{\chi_S}(0) \supseteq h_S(t) = [0,1]$. So, $h_{\chi_S}(0) = [0,1]$.
Hence, $0 \in S$. □

Theorem 4.2.4. *A non-empty subset h of* X *is a TM-ideal of* X *if and only if the characteristic hesitant fuzzy set is a hesitant fuzzy TM-ideal of* X.

Proof. Assume that h is a TM-ideal of X. Since $0 \in h$ it follows from Lemma 4.2.3 that $h_h(0) = [0,1] \supseteq h_h(x)$ for all $x \in X$. Next, let $x, z \in X$.
Case 1: If $x, z \in h$, then $h_h(x) = [0,1]$ and $h_h(z) = [0,1]$.
Hence, $h_h(x) = [0,1] \supseteq h_h(x \star z) = h_h(x \star z) \cap h_h(z)$.
Case 2: If $x \in h$ and $z \notin h$, then $h_h(x) = [0,1]$ and $h_h(z) = \emptyset$.
Hence, $h_h(x) = [0,1] \supseteq \emptyset = h_h(x \star z) \cap h_h(z)$.
Case 3: If $x \notin h$ and $z \in h$, which is similar to case(2).
Case 4: If $x \notin h$ and $z \notin h$, then $h_h(x) = \emptyset$ and $h_h(z) = \emptyset$.
Hence, $h_h(x) \supseteq \emptyset = h_h(x \star z) \cap h_h(z)$.
Hence, h_h is an hesitant fuzzy ideal of X.
Conversely, assume that h_h is an hesitant fuzzy ideal of X. Since $h_h(0) \supseteq h_h(x)$ for all $x \in X$, it follows from Lemma 4.2.3 that $0 \in h$. Let $x, z \in X$ be such that $x \star z \in I$ and $z \in h$. Then, $h_h(x \star z) = h_h(z) = [0,1]$. Thus, $h_h(x) \supseteq h_h(x \star z) \cap h_h(z)$, so $h_h(x) = [0,1]$. Hence, $x \in I$ and so, h is an ideal of X. □

Definition 4.2.4. *Given a non-empty subset S of* X, *an S-hesitnat fuzzy set* $H_S = \{(x, h_S(x)) \mid x \in X\}$ *on* X *is called a hesitant fuzzy closed T-ideal of* X *related to S (brifely, S-hesitant fuzzy closed T-ideal of* X*) if it satisfies:*

$$(\forall x, y, z \in S) \begin{pmatrix} h_S(0 \star x) \supseteq h_S(x) \\ h_S(x \star z) \supseteq h_S((x \star y) \star z) \cap h_S(y) \end{pmatrix}$$

Theorem 4.2.5. *Let* $H_S = \{(x, h_X(x)) \mid x \in X\}$ *be a hesitant fuzzy closed T-ideal of a TM-algebra* X *if and only if the non-empty upper* Λ*-level set* $\mathcal{U}(h_S, \Lambda)$ *is a closed T-ideal of* X, *for any* $\Lambda \in \mathcal{P}[0,1]$.

Proof. Suppose that $H_X = \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy T-ideal of a TM-algebra X. We need to show that $\mathcal{U}(h_S, \Lambda)$ is a closed T-ideal of X, for any $\Lambda \in \mathcal{P}[0,1]$. For any $\Lambda \in \mathcal{P}[0,1]$. We define the sets $\mathcal{U}(h_S,\Lambda)=\{x\in X\supseteq\Lambda\}$. Since $\mathcal{U}(h_S,\Lambda)\neq\emptyset$, for $x\in\mathcal{U}(h_S,\Lambda)$. Then

$$\begin{aligned}&\Rightarrow h_S(x)\supseteq\Lambda\\&\Rightarrow h_S(0\star x)\supseteq h_S(0)\cap h_S(x)=h_S(x)\supseteq\Lambda\\&\Rightarrow h_S(0\star x)\supseteq\Lambda\\&\Rightarrow 0\star x\in\mathcal{U}(h_S,\Lambda).\end{aligned}$$

Assume that $(x\star y)\star z\in\mathcal{U}(h_S,\Lambda)$ and $y\in\mathcal{U}(h_S,\Lambda)$. It implies that $(h_S((x\star y)\star z))\supseteq\Lambda$ and $h_S(y)\supseteq\Lambda$. Since $h_S(x\star z)\supseteq h_S((x\star y)\star z)\cap h_S(y)\supseteq\Lambda\cap\Lambda=\Lambda$. Thus, $h_S(x\star z)\supseteq\Lambda$. Therefore, $\mathcal{U}(h_S,\Lambda)$ is a closed T-ideal of X.
Conversely, suppose that for any $\Lambda\in\mathcal{P}[0,1]$ the non-empty subset $\mathcal{U}(h_S,\Lambda)$ is a closed T-ideal of X. Let $x\in X$. Then $h_S(x)\in\mathcal{P}[0,1]$. Choose $\Lambda=h_S(x)\in\mathcal{P}[0,1]$. Then $h_S(x)\supseteq\Lambda$. So, $x\in\mathcal{U}(h_S,\Lambda)$ is a closed T-ideal of X and thus $0\in\mathcal{U}(h_S,\Lambda)$. Thus, $h_S(0)\supseteq\Lambda=h_S(x)$. Let $x,y,z\in X$. Then $h_S(y),h_S((x\star y)\star z)\in\mathcal{P}[0,1]$. Now, taking $\Lambda=h_S((x\star y)\star z)\cap h_S(y)\in\mathcal{P}[0,1]$. So, $h_S((x\star y)\star z)\supseteq\Lambda$ and $h_S(y)\supseteq\Lambda$. It implies that $y,(x\star y)\star z\in\mathcal{U}(h_S,\Lambda)\neq\emptyset$. Since $\mathcal{U}(h_S,\Lambda)$ is a closed T-ideal of X. Hence, $(x\star z)\in\mathcal{U}(h_S,\Lambda)$. Thus, $h_S(x\star z)\supseteq\Lambda=h_S((x\star y)\star z)\cap h_S(y)$. Therefore, H is a hesitant fuzzy T-ideal of a TM -algebra X. □

Theorem 4.2.6. $H_X=\{(x,h_S(x))\mid x\in X\}$ *be a hesitant fuzzy T-ideal of a TM-algebra* X *if and only if the non-empty upper* Λ*-level set* $\mathcal{U}(h_S,\Lambda)$ *is a T-ideal of* X*, for any* $\Lambda\in\mathcal{P}[0,1]$.

Proof. Suppose that $H_X=\{(x,h_X(x))\mid x\in X\}$ be a hesitant fuzzy T-ideal of a TM-algebra X. For any $\Lambda\in\mathcal{P}[0,1]$. We define the sets $\mathcal{U}(h_S,\Lambda)=\{x\in X\supseteq\Lambda\}$. Since $\mathcal{U}(h_S,\Lambda)\neq\emptyset$, for $x\in\mathcal{U}(h_S,\Lambda)$. Then

$$\begin{aligned}&\Rightarrow h_S(x)\supseteq\Lambda\\&\Rightarrow h_S(0)\supseteq h_A(x)\supseteq\Lambda\\&\Rightarrow h_S(0)\supseteq\Lambda\\&\Rightarrow 0\in\mathcal{U}(h_S,\Lambda).\end{aligned}$$

Assume that $(x\star y)\star z\in\mathcal{U}(h_S,\Lambda)$ and $y\in\mathcal{U}(h_S,\Lambda)$. It implies that $(h_S((x\star y)\star z))\supseteq\Lambda$ and $h_S(y)\supseteq\Lambda$. Since $h_S(x\star z)\supseteq h_S((x\star y)\star z)\cap h_S(y)\supseteq\Lambda\cap\Lambda=\Lambda$. Thus, $h_S(x\star z)\supseteq\Lambda$. Therefore, $\mathcal{U}(h_S,t1)$ is a T-ideal of X.
Conversely, suppose that for any $\Lambda\in\mathcal{P}[0,1]$ the non-empty subset $\mathcal{U}(h_S,\Lambda)$ is a T-ideal of X. Let $x\in X$. Then $h_S(x)\in\mathcal{P}[0,1]$. Choose $\Lambda=h_S(x)\in\mathcal{P}[0,1]$. Then $h_S(x)\supseteq\Lambda$. So, $x\in\mathcal{U}(h_S,\Lambda)$ is a T-ideal of X and thus $0\in\mathcal{U}(h_S,\Lambda)$. Thus, $h_S(0)\supseteq\Lambda=h_S(x)$. Let $x,y,z\in X$. Then $h_S(y),h_S((x\star y)\star z)\in\mathcal{P}[0,1]$. Now, taking $\Lambda=h_S((x\star y)\star z)\cap h_S(y)\in\mathcal{P}[0,1]$. So, $h_S((x\star y)\star z)\supseteq t$ and $h_S(y)\supseteq\Lambda$. It implies that $y,(x\star y)\star z\in\mathcal{U}(h_S,\Lambda)\neq\emptyset$. Since $\mathcal{U}(h_S,\Lambda)$ is a T-ideal of X. Hence, $(x\star z)\in\mathcal{U}(h_S,\Lambda)$. Thus, $h_S(x\star z)\supseteq\Lambda=h_S((x\star y)\star z)\cap h_S(y)$.
Therefore, H is a hesitant fuzzy T-ideal of a TM -algebra X. □

Theorem 4.2.7. *The intersection of two hesitant fuzzy T-ideals of X is also a hesitant fuzzy T-ideal of X.*

Proof. Let h_1 and h_2 be two hesitant fuzzy T-ideals of X. We need to show that $h_1 \cap h_2$ is a hesitant fuzzy T-ideal of X. Let $x, y, z \in X$. Then

$$\begin{aligned}(h_1 \cap h_2)(0) = (h_1 \cap h_2)(x \star x) &= \min\{h_1(0), h_2(0)\} \\ &\supseteq \min\{h_1(x), h_2(x)\} \\ &= (h_1 \cap h_2)(x)\end{aligned}$$

$$\begin{aligned}(h_1 \cap h_2)(x \star z) &= \min\{h_1(x \star z), h_2(x \star z)\} \\ &\supseteq \min\{h_1((x \star y) \star z) \cap h_1(y), h_2((x \star y) \star z) \cap h_2(y)\} \\ &= \min\{h_1((x \star y) \star z) \cap h_1(y), h_2((x \star y) \star z) \cap h_2(y)\} \\ &= \min\{\min\{h_1((x \star y) \star z), h_1(y)\}, \min\{h_2((x \star y) \star z), h_2(y)\}\} \\ &= \min\{\min\{h_1((x \star y) \star z), h_2((x \star y) \star z)\}, \min\{h_1(y), h_2(y)\}\} \\ &= \min\{(h_1 \cap h_2)((x \star y) \star z), (h_1 \cap h_2)(y)\} \\ &= (h_1 \cap h_2)((x \star y) \star z) \cap (h_1 \cap h_2)(y)\end{aligned}$$

Therefore, the intersection of two hesitant fuzzy T-ideals of X is also a hesitant fuzzy T-ideal of X. □

Definition 4.2.5. *Let* $\{H_i \mid i \in I\}$ *be a family of hesitant fuzzy sets on a reference set* X*. We define the hesitant fuzzy set* $\bigcap_{i\in I} H_i = \left(\bigcap_{i\in I} h_i\right)$ *by* $\left(\bigcap_{i\in I} h_i\right)(x) = \bigcap_{i\in I} h_i(x)$ *and for all* $x \in X$*, which is called the hesitant intersection of hesitant fuzzy sets.*

Theorem 4.2.8. *If* $\{h_i \mid i \in I\}$ *is a family of hesitant fuzzy* T*-ideal of* X*, then* $\bigcap_{i\in I} h_i$ *is an hesitant fuzzy T-ideal of* X*.*

Proof. Let $\{h_i \mid i \in I\}$ be a family of hesitant fuzzy T-ideal of X. Let $x \in X$. We need to show that $\bigcap_{i\in I} h_i$ is an hesitant fuzzy T-ideal of X. Then

$$\left(\bigcap_{i\in I} h_i\right)(0) = \bigcap_{i\in I} h_i(0) \supseteq \bigcap_{i\in I} h_i(x) = \left(\bigcap_{i\in I} h_i\right)(x)$$

.

Let $x, y, z \in X$. Then

$$\begin{aligned}\left(\bigcap_{i\in I} h_i\right)(x \star z) &= \bigcap_{i\in I} h_i(x \star z) \\ &\supseteq \bigcap_{i\in I} (h_i((x \star y) \star z) \cap h_i(y)) \\ &= \left(\bigcap_{i\in I} h_i((x \star y) \star z)\right) \cap \left(\bigcap_{i\in I} h_i(y)\right) \\ &= \left(\bigcap_{i\in I} h_i\right)((x \star y) \star z) \cap \left(\bigcap_{i\in I} h_i\right)(y)\end{aligned}$$

Hence, $\bigcap_{i\in I} h_i$ is hesitant fuzzy T-ideal of X. $\square$

Theorem 4.2.9. *Let* $h_A = \{(x, h_A(x)) \mid x \in X\}$ *and* $h_B = \{(z, h_B(z)) \mid z \in Y\}$ *are two hesitant fuzzy T-ideals on a TM-algebra* X *and* Y *respectively, then the Cartesian product* $h_A \times h_B$ *is also a hesitant fuzzy T-ideal of* $X \times Y$ *where A and B are a non-empty asubset of a TM-algebra X and Y respectively.*

Proof. Suppose that $h_A = \{(x, h_A(x)) \mid x \in X\}$ and $h_B = \{(z, h_B(z)) \mid z \in Y\}$ are two hesitant fuzzy T-ideals on a TM-algebra X and Y respectively. We need to show that $h_A \times h_B$ is also a hesitant fuzzy T-ideal of $X \times Y$. Let $(x,z),(w,y) \in X \times Y$. Then by Definition 1.2.4 we have

$$\begin{aligned}
h((x,z)\star(w,y)) &= h((x\star w, z\star y))\\
&= h_A(x\star w)\cap h_B(z\star y)\\
&\supseteq h_A((x\star m)\star w)\cap h_A(m)\cap h_B((z\star n)\cdot y)\cap h_B(n)\\
&= h_A((x\star m)\star w)\cap h_B((z\star n)\star y)\cap h_A(m)\cap h_B(n)\\
&= h((x\star m)\star w), z\star n)\star y))\cap h(m,n)\\
&= h((x,z)\star(m,n))\star(w,y)\cap h(m,n)
\end{aligned}$$

Again, let $(x,z) \in X \times Y$. Then

$$\begin{aligned}
h(0_X,0_Y) &= h_A(0_X)\cap h_B(0_Y)\\
&\supseteq h_A(x)\cap h_B(z)\\
&= h(x,z)
\end{aligned}$$

Therefore, the Cartesian product $h_A \times h_B$ is a hesitant fuzzy T-ideal of $X \times Y$. $\square$

Theorem 4.2.10. *Let* $h_A = \{(x, h_A(x)) \mid x \in X\}$ *and* $h_B = \{(y, h_B(y)) \mid y \in Y\}$ *are two hesitant fuzzy closed T-ideals on a TM-algebra* X *and* Y *respectively, then the Cartesian product* $h_A \times h_B$ *is also a hesitant fuzzy closed T-ideal of* $X \times Y$.

Proof. Let $(x,z),(w,y) \in X \times Y$. Then by Definition 1.2.4 . We have

$$\begin{aligned}
h((0,0)\star(x,z) &= h_A(0\star x)\cap h_B(0\star z)\\
&\supseteq h_A(x)\cap h_B(z)\\
&= h(x,z),\ \text{again}\\
h((x,z)\star(w,y)) &= h((x\star w, z\cdot y))\\
&= h_A(x\star w)\cap h_B(z\star y)\\
&\supseteq h_A((x\star m)\star w)\cap h_A(m)\cap h_B((z\star n)\cdot y)\cap h_B(n)\\
&= h_A((x\star m)\star w)\cap h_B((z\star n)\star y)\cap h_A(m)\cap h_B(n)\\
&= h((x\star m)\star w), z\star n)\star y))\cap h(m,n)\\
&= h((x,z)\star(m,n))\star(w,y)\cap h(m,n)
\end{aligned}$$

Therefore, the Cartesian product $h_A \times h_B$ is a hesitant fuzzy T-ideal of $X \times Y$. □

Definition 4.2.6. *Let f be a mapping from a TM-algebra* X *to* Y *and A be a subset of a TM-algebra* X. *If* $H = \{(x, h_A(x)/x \in X)\}$ *is a hesitant fuzzy set on* Y, *then the hesitant fuzzy set* $f^{-1}(H) = h_A \circ f$ *in* X *is called the pre-image of* H *under* f.

Theorem 4.2.11. *Let* f *be a homomorphism from a TM-algebra* X *to* Y *and A be a subset of a TM-algebra* X. *If* $H = \{(x, h_A(x)/x \in X)\}$ *is a hesitant fuzzy T-ideals of* Y, *then* $f^{-1}(H) = h_A of$ *in* X *is a hesitant fuzzy T-ideal of* X.

Proof. Suppose that $H = \{(x, h_A(x)/x \in X)\}$ is a hesitant fuzzy T-ideals of Y. We need to show that $f^{-1}(H) = h_A of$ in X is a hesitant fuzzy T-ideal of X. We know that $h_A(f(0_X)) = h_A(0_Y) \supseteq h_A(x)$, for all $x \in X$.
Let $x, y, z \in X$. Then

$$\begin{aligned}(h_A \circ f)(x \star z) &= h_A(f(x \star z)) \\ &= h_A(f(x) \star f(z)) \\ &\supseteq h_A((f(x) \star f(y)) \star f(z)) \cap h_A(f(y)) \\ &= h_A(f((x \star y) \star z)) \cap h_A(f(y)) \\ &= (h_A of)((x \star y) \star z) \cap (h_A of)(z)\end{aligned}$$

Therefore, $f^{-1}(H)$ is a hesitant fuzzy T-ideal of X. □

Proposition 4.2.12. *Let* f *be a homomorphism from a TM-algebra* X *to* Y *and A be a subset of a TM-algebra* X. *If* $H = \{(x, h_A(x)/x \in X)\}$ *is a hesitant fuzzy closed T-ideals of* Y, *then* $f^{-1}(H) = h_A of$ *in* X *is a hesitant fuzzy closed T-ideal of* X.

Proof. Suppose that $H = \{(x, h_A(x)/x \in X)\}$ is a hesitant fuzzy closed T-ideals of Y. We need to show that $f^{-1}(H) = h_A of$ in X is a hesitant fuzzy closed T-ideal of X. Since $h_A(f(0 \star x)) = h_A(f(0) \star f(x)) \supseteq h_A(f(x))$, for all $x \in X$.
Let $x, y, z \in X$. Then

$$\begin{aligned}(h_A of)(x \star z) &= h_A(f(x \star z)) \\ &= h_A(f(x) \star f(z)) \\ &\supseteq h_A((f(x) \star f(y)) \star f(z)) \cap h_A(f(y)) \\ &= h_A(f((x \star y) \star f(z))) \cap h_A(f(y)) \\ &= (h_A of)((x \star y) \star z) \cap (h_A of)(y)\end{aligned}$$

Therefore, $f^{-1}(H)$ is a hesitant fuzzy T-ideal of X. □

4.3. Hesitant Fuzzy Pseudo-TM Subalgebra (HFPTMSA)

In this section, we introduce the concepts of hesitant FPTMSAs of a pseudo TM -algebras and present some interesting properties.

Definition 4.3.1. *A hesitant fuzzy set* $H = \{(x, h(x)) \,|\, x \in X\}$ *is called a hesitant FPTMSA of a pseudo TM- algebra* X *if the following holds:*

$$(\forall x, y \in X)\begin{pmatrix} h(x \odot y) \supseteq h(x) \cap h(y) \\ h(x \star y) \supseteq h(x) \cap h(y) \end{pmatrix}.$$

Example 4.3.1. *Let* $X = \{0,1,2,3,4\}$ *be a set with two binary operations* $\odot$ *and* $\star$ *which are given by table.*

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	2	2	0	0
4	4	0	0	0	0

$\star$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	0	0	0	0

Table 4.4

See [52] $(X, \odot, \star, 0)$ is a pseudo TM-algebra. Let h be a fuzzy subsets of X defined by

$$h(x) = \begin{cases} 0.5, & \text{if } x = 0, \\ 0.2, & \text{if } x = 1, 2, 3, 4. \end{cases}$$

We define the hesitant fuzzy set $H = \{(x, h(x))\}$ on X as follows:
$h(0) = \{0.1, 0.3\}, h(1) = h(2) = h(3) = h(4) = \{0.3\}$.
Then h is a hesitant FPTMSA of X.

Proposition 4.3.1. *Let* $H = \{(x, h(x)) \,|\, x \in X\}$ *is a hesitant FPTMSA of a pseudo TM- algebra* X *then the following property holds:*

$$(\forall x \in X)\, h(x) \subseteq h(0).$$

Proof. Suppose that $H = \{(x, h(x)) \,|\, x \in X\}$ is a hesitant FPTMSA of a pseudo TM- algebra X. We need to show that $(\forall x \in X)\, h(x) \subseteq h(0)$. For any $x \in X$, we have

$$h(x) = h(x) \cap h(x) \subseteq h(x \odot x) = h(0) \text{ and}$$

$$h(x) = h(x) \cap h(x) \subseteq h(x \star x) = h(0).$$

□

Proposition 4.3.2. *Let X be a pseudo TM-algebra. If* $H = \{(x, h(x)) \,|\, x \in X\}$ *is a hesitant FPTMSA of a pseudo TM- algebra* X*. Then*
1. $h(x) \cap h(y) \subseteq h(x \odot (0 \odot y)$,
2. $h(x) \cap h(y) \subseteq h(x \star (0 \star y))$, *for all* $x, y \in X$.

Proof. Suppose that $H = \{(x,h(x))\,|x \in X\}$ is a hesitant FPTMSA of a pseudo TM- algebra X. Let $x,y \in X$. Since h is a hesitant fuzzy pseudo TM-subalgebra of X. Then

$$\begin{aligned} h(x \odot (0 \odot y) &\supseteq h(x) \cap h(0 \odot y) \supseteq h(x) \cap h(0) \cap h(y) \\ &= h(x) \cap h(y) \text{ and} \\ h(x \star (0 \star y) &\supseteq h(x) \cap h(0 \star y) \supseteq h(x) \cap h(0) \cap h(y) \\ &= h(x) \cap h(y) \end{aligned}$$

□

Lemma 4.3.3. *Let X be a TM-algebra. Let* $H = \{(x,h(x))\,|x \in X\}$ *is a hesitant FPTMSA of a pseudo TM- algebra* X. *Then*
1. $h(y) \subseteq h(x \odot y)$ *if and only if* $h(x) = h(0)$.
2. $h(y) \subseteq h(x \star y)$ *if and only if* $h(x) = h(0)$.

Proof. Assume that $H = \{(x,h(x))\,|x \in X\}$ is a hesitant FPTMSA of a pseudo TM- algebra X.
1) Suppose that $h(y) \subseteq h(x \odot y)$. We need to show that $h(x) = h(0)$. Since h is a hesitant fuzzy pseudo TM-subalgebra of X. Since X is a pseudo TM-algebra, we know that $x \odot x = 0$. Let's substitute $y = x$ in our assumption. We get $h(x) \subseteq h(x \odot x)$. This simplifies to $h(x) \subseteq h(0)$. Also, from the definition of a FPTMSA, we know that $h(x \odot y) \supseteq h(x) \cap h(y)$. Setting $y = 0$ in the FPTMSA property gives $h(x \odot 0) \supseteq h(x) \cap h(0)$. Using the pseudo TM-algebra property $x \odot 0 = x$, this becomes $h(x) \supseteq h(x) \cap h(0)$, which is a trivial truth. Let's consider the FPTMSA property on $h(x \odot 0)$. We have $h(x \odot 0) \supseteq h(x) \cap h(0)$. Since $x \odot 0 = x$, we have $h(x) \supseteq h(x) \cap h(0)$. Now consider the property $h(y) \subseteq h(x \odot y)$. Let's set $x = 0$. We have $h(y) \subseteq h(0 \odot y)$. By the pseudo TM-algebra property, $0 \odot y = 0$. So we have $h(y) \subseteq h(0)$. We have shown that $h(x) \subseteq h(0)$ for any $x \in X$ and $h(y) \subseteq h(0)$ for any $y \in X$. we have $h(x) \subseteq h(0)$ for any $x \in X$. Let's consider the case when x takes the value 0 . We have $\mathbf{h}(0) \subseteq \mathbf{h}(0)$, which is trivially true. Combining $h(x) \subseteq h(0)$ and the FPTMSA property $h(x \odot x) \supseteq h(x) \cap h(x) = h(x)$, we have $h(0) \supseteq h(x)$. This means that $h(x) = h(0)$ for all $x \in X$. Thus, if $h(y) \subseteq h(x \odot y)$ holds for all $x,y \in X$, then $h(x) = h(0)$ for all $x \in X$.
Conversely, assume $h(x) = h(0)$ we need to prove $h(y) \subseteq h(x \odot y)$. Since $h(x) = h(0)$ for all $x \in X$. By our assumption, $h(y) = h(0)$ and $h(x \odot y) = h(0)$. Since $h(y) = h(0)$ and $h(x \odot y) = h(0)$, it follows that $h(y) = h(x \odot y)$. From this, we trivially have $h(y) \subseteq h(x \odot y)$.
2) Suppose that $h(y) \subseteq h(x \star y)$. We need to show that $h(x) = h(0)$. Since h is a hesitant fuzzy pseudo TM-subalgebra of X. Since X is a pseudo TM-algebra, we know that $x \star x = 0$. Let's substitute $y = x$ in our assumption. We get $h(x) \subseteq h(x \star x)$. This simplifies to $h(x) \subseteq h(0)$. Also, from the definition of a FPTMSA, we know that $h(x \star y) \supseteq h(x) \cap h(y)$. Setting $y = 0$ in the FPTMSA property gives $h(x \star 0) \supseteq h(x) \cap h(0)$. Using the pseudo TM-algebra property $x \star 0 = x$, this becomes $h(x) \supseteq h(x) \cap h(0)$, which is a trivial truth. Let's consider the FPTMSA property on $h(x \star 0)$. We have $h(x \star 0) \supseteq h(x) \cap h(0)$. Since $x \star 0 = x$, we have $h(x) \supseteq h(x) \cap h(0)$. Now consider the property $h(y) \subseteq h(x \star y)$. Let's set $x = 0$. We have $h(y) \subseteq h(0 \star y)$. By the pseudo

TM-algebra property, $0 \star y = 0$. So we have $h(y) \subseteq h(0)$. We have shown that $h(x) \subseteq h(0)$ for any $x \in X$ and $h(y) \subseteq h(0)$ for any $y \in X$. we have $h(x) \subseteq h(0)$ for any $x \in X$. Let's consider the case when x takes the value 0 . We have $\mathbf{h}(0) \subseteq \mathbf{h}(0)$, which is trivially true. Combining $h(x) \subseteq h(0)$ and the FPTMSA property $h(x \star x) \supseteq h(x) \cap h(x) = h(x)$, we have $h(0) \supseteq h(x)$. This means that $h(x) = h(0)$ for all $x \in X$. Thus, if $h(y) \subseteq h(x \star y)$ holds for all $x, y \in X$, then $h(x) = h(0)$ for all $x \in X$.

Conversely, assume $h(x) = h(0)$ we need to prove $h(y) \subseteq h(x \star y)$. Since $h(x) = h(0)$ for all $x \in X$. By our assumption, $h(y) = h(0)$ and $h(x \star y) = h(0)$. Since $h(y) = h(0)$ and $h(x \star y) = h(0)$, it follows that $h(y) = h(x \star y)$. From this, we trivially have $h(y) \subseteq h(x \star y)$. □

Theorem 4.3.4. *A non-empty subset* S *of a pseudo TM -algebra* $(X, \odot, \star, 0)$ *is a TM-subalgebra of* X *if and only if the (CHFS* χ_S*) is a hesitant FPTMSA of* X*.*

Proof. Assume that S is a subalgebra of a TM-algebra X. Let $x, y \in X$. We need to show that χ_S is a hesitant FPTMSA of X.

Case 1. If $x, y \in S$, then $h_{\chi S}(x) = [0,1]$ and $h_{\chi S}(y) = [0,1]$. Thus, $h_{\chi S}(x) \cap h_{\chi S}(y) = [0,1]$. Since S is a subalgebra of X. Then $x \odot y \in S$ and thus, $h_{\chi S}(x \odot y) = [0,1]$. Then $h_{\chi S}(x \odot y) = [0,1] \supseteq [0,1] = h_{\chi_S}(x) \cap h_{\chi S}(y)$.

Case 2. If $x \in S$ and $y \notin S$, then $h_{\chi S}(x) = [0,1]$ and $h_{\chi S}(y) = \emptyset$. Thus, $h_{\chi S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \odot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$.

Case 3. If $x \notin S$ and $y \in S$, then $h_{\chi S}(x) = \emptyset$ and $h_{\chi S}(y) = [0,1]$. Thus, $h_{\chi S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi S}(x \odot y) \supseteq \emptyset = h_{\chi S}(x) \cap h_{\chi_S}(y)$.

Case 4. If $x \notin S$ and $y \notin S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = \emptyset$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. So, $h_{\chi_S}(x \odot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$.

Similarly, Case 1. If $x, y \in S$, then $h_{\chi S}(x) = [0,1]$ and $h_{\chi S}(y) = [0,1]$. Thus, $h_{\chi S}(x) \cap h_{\chi S}(y) = [0,1]$. Since S is a subalgebra of X. Then $x \star y \in S$ and thus, $h_{\chi S}(x \star y) = [0,1]$. Then $h_{\chi S}(x \star y) = [0,1] \supseteq [0,1] = h_{\chi_S}(x) \cap h_{\chi S}(y)$.

Case 2. If $x \in S$ and $y \notin S$, then $h_{\chi S}(x) = [0,1]$ and $h_{\chi S}(y) = \emptyset$. Thus, $h_{\chi S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \star y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$.

Case 3. If $x \notin S$ and $y \in S$, then $h_{\chi S}(x) = \emptyset$ and $h_{\chi S}(y) = [0,1]$. Thus, $h_{\chi S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi S}(x \star y) \supseteq \emptyset = h_{\chi S}(x) \cap h_{\chi_S}(y)$.

Case 4. If $x \notin S$ and $y \notin S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = \emptyset$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. So, $h_{\chi_S}(x \star y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$.

Therefore, χ_S is a hesitant fuzzy pseudo TM-subalgebra of X. Conversely, assume that χ_S is a hesitant fuzzy TM-subalgebra of X. Let $x, y \in S$. Then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = [0,1]$. Thus, $h_{\chi_S}(x \odot y) \supseteq h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0,1]$, so $h_{\chi_S}(x \odot y) = [0,1]$.

Therefore, $x \odot y \in S$, hence, S is a subalgebra of X and

Case 1. If $x, y \in S$, then $h_{\chi S}(x) = [0,1]$ and $h_{\chi S}(y) = [0,1]$. Thus, $h_{\chi S}(x) \cap h_{\chi S}(y) = [0,1]$. Since S is a subalgebra of X. Then $x \star y \in S$ and thus, $h_{\chi S}(x \star y) = [0,1]$. Then $h_{\chi S}(x \star y) = [0,1] \supseteq [0,1] = h_{\chi_S}(x) \cap h_{\chi S}(y)$.

Case 2. If $x \in S$ and $y \notin S$, then $h_{\chi S}(x) = [0,1]$ and $h_{\chi S}(y) = \emptyset$. Thus, $h_{\chi S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then

$h_{\chi_S}(x \star y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$.
Case 3. If $x \notin S$ and $y \in S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = [0,1]$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \star y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$.
Case 4. If $x \notin S$ and $y \notin S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = \emptyset$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. So, $h_{\chi_S}(x \star y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Therefore, χ_S is a hesitant fuzzy pseudo TM-subalgebra of X.
Conversely, assume that χ_S is a hesitant fuzzy TM-subalgebra of X. Let $x, y \in S$. Then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = [0,1]$. Thus, $h_{\chi_S}(x \star y) \supseteq h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0,1]$, so $h_{\chi_S}(x \star y) = [0,1]$. Therefore, $x \star y \in S$. Hence, S is a subalgebra of X.

□

Theorem 4.3.5. *A hesitant fuzzy set* $H = \{(x, h(x)) | x \in X\}$ *on a pseudo TM-algebra* $(X, \odot, \star, 0)$ *is a hesitant FPTMSA of* X *if and only if for all* $\Lambda \in \mathcal{P}[0,1]$, *the non-empty subset* $\mathcal{U}(h, \Lambda)$ *of* X *is a pseudo TM-subalgebra.*

Proof. Suppose that H is a hesitant FPTMSA of X. Let $\Lambda \in \mathcal{P}[0,1]$ be such that $\mathcal{U}(h, \Lambda) \neq \emptyset$ and let $x, y \in \mathcal{U}(h, \Lambda)$. Then $h(x) \supseteq \Lambda$ and $h(y) \supseteq \Lambda$. Since H is a hesitant fuzzy PTMASA of X, we have

$$(\forall x, y \in X) \begin{pmatrix} h(x \odot y) \supseteq h(x) \cap h(y) \supseteq \Lambda \\ h(x \star y) \supseteq h(x) \cap h(y) \supseteq \Lambda \end{pmatrix}.$$

Thus, $x \odot y$ and $x \star y \in \mathcal{U}(h, \Lambda)$. Hence, $\mathcal{U}(h, \Lambda)$ is a pseudo-TM subalgebra of X.
Conversely, assume that for every $t \in \mathcal{P}[0,1]$, the non-empty subset $\mathcal{U}(h, \Lambda)$ is a pseudo-TM subalgebra of X. Let $x, y \in X$. Choose $\Lambda = \Lambda_1 \cap \Lambda_2 \in \mathcal{P}[0,1]$. Then $h(x) = \Lambda_1$ and $h(y) = \Lambda_2$. Thus, $x, y \in \mathcal{U}(h, \Lambda) \neq \emptyset$. By assumption, $\mathcal{U}(h, \Lambda)$ is a pseudo-TM subalgebra of X and thus $x \odot y$ and $x \star y \in \mathcal{U}(h, \Lambda)$. So, $h(x \odot y) \supseteq \Lambda = \Lambda_1 \cap \Lambda_2 = h(x) \cap h(y)$ and $h(x \star y) \supseteq \Lambda = \Lambda_1 \cap h(x) \cap h(y)$.
Therefore, H is a hesitant FPTMSA of X. □

Proposition 4.3.6. *Let* h_1 *and* h_2 *be two hesitant FPTMSA of a pseudo TM-algebra* X. *Then* $h_1 \cap h_2$ *is a hesitant FPTMSA of* X.

Proof. Assume that h_1 and h_2 be a hesitant fuzzy PTMASAs of a pseudo TM-algebra X. We need to show that $(h_1 \cap h_2)(x \star y) \supseteq (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)$ and $(h_1 \cap h_2)(x \star y) \supseteq (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)$, for all $x, y \in X$.
We need to consider the following three cases:

Case (i). $h_1(x \odot y) \subseteq h_2(x \odot y)$. By Definition (1.2.3)

$$\begin{aligned}
(h_1 \cap h_2)(x \odot y) &= \left\{h \in (h_1(x \odot y) \cup h_2(x \odot y)) \mid h \leqslant \min\left(h_1^+(x \odot y), h_2^+(x \odot y)\right)\right\} \\
&= \left\{h \in h_2(x \odot y) \mid h \leqslant \min\left(h_2^+(x \odot y), h_2^+(x \odot y)\right)\right\} \\
&\supseteq \left\{h \in (h_2(x) \cap h_2(y)) \mid h \leqslant \min\left(h_2^+(x \odot y), h_2^+(x \odot y)\right)\right\} \\
&\supseteq \left\{h \in ((h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)) \mid h \leqslant \min\left((h_1 \cap h_2)^+(x), (h_1 \cap h_2)^+(y)\right)\right\} \\
&= (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y).
\end{aligned}$$

Case (ii). $h_1(x \odot y) \supseteq h_2(x \odot y)$

$$\begin{aligned}(h_1 \cap h_2)(x \odot y) &= \left\{h \in (h_1(x \odot y) \cup h_2(x \odot y)) \mid h \leqslant \min\left(h_1^+(x \odot y), h_2^+(x \odot y)\right)\right\} \\ &= \left\{h \in h_1(x \odot y) \mid h \leqslant \min\left(h_1^+(x \odot y), h_1^+(x \odot y)\right)\right\} \\ &\supseteq \left\{h \in (h_1(x) \cap h_1(y)) \mid h \leqslant \min\left(h_1^+(x), h_1^+(y)\right)\right\} \\ &\supseteq \left\{h \in ((h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)) \mid h \leqslant \min\left((h_1 \cap h_2)^+(x), (h_1 \cap h_2)^+(y)\right)\right\} \\ &= (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y).\end{aligned}$$

Case (iii). $h_1(x \odot y) \approx h_2(x \odot y)$

$$\begin{aligned}(h_1 \cap h_2)(x \odot y) &= h_1(x \odot y) \cup h_2(x \odot y) \\ &\approx h_1(x \odot y) \\ &\supseteq h_1(x) \cap h_1(y) \\ &\supseteq (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y) \text{ and}\end{aligned}$$

Case (i). $h_1(x \star y) \subseteq h_2(x \star y)$. By Definition (1.2.3)

$$\begin{aligned}(h_1 \cap h_2)(x \star y) &= \left\{h \in (h_1(x \star y) \cup h_2(x \star y)) \mid h \leqslant \min\left(h_1^+(x \star y), h_2^+(x \star y)\right)\right\} \\ &= \left\{h \in h_2(x \star y) \mid h \leqslant \min\left(h_2^+(x \star y), h_2^+(x \star y)\right)\right\} \\ &\supseteq \left\{h \in (h_2(x) \cap h_2(y)) \mid h \leqslant \min\left(h_2^+(x \star y), h_2^+(x \star y)\right)\right\} \\ &\supseteq \left\{h \in ((h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)) \mid h \leqslant \min\left((h_1 \cap h_2)^+(x), (h_1 \cap h_2)^+(y)\right)\right\} \\ &= (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y).\end{aligned}$$

Case (ii). $h_1(x \star y) \supseteq h_2(x \star y)$

$$\begin{aligned}(h_1 \cap h_2)(x \star y) &= \left\{h \in (h_1(x \star y) \cup h_2(x \star y)) \mid h \leqslant \min\left(h_1^+(x \star y), h_2^+(x \star y)\right)\right\} \\ &= \left\{h \in h_1(x \star y) \mid h \leqslant \min\left(h_1^+(x \star y), h_1^+(x \star y)\right)\right\} \\ &\supseteq \left\{h \in (h_1(x) \cap h_1(y)) \mid h \leqslant \min\left(h_1^+(x), h_1^+(y)\right)\right\} \\ &\supseteq \left\{h \in ((h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)) \mid h \leqslant \min\left((h_1 \cap h_2)^+(x), (h_1 \cap h_2)^+(y)\right)\right\} \\ &= (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y).\end{aligned}$$

Case (iii). $h_1(x \star y) \approx h_2(x \star y)$

$$\begin{aligned}(h_1 \cap h_2)(x \star y) &= h_1(x \star y) \cup h_2(x \star y) \\ &\approx h_1(x \star y) \\ &\supseteq h_1(x) \cap h_1(y) \\ &\supseteq (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y).\end{aligned}$$

Hence, in both cases $h_1 \cap h_2$ is a hesitant FPTMSA of X. □

Corollary 4.3.7. *If* $\{h_i \mid i \in I\}$ *is a family of hesitant FPTMSA of a pseudo TM-algebra* $(X, \odot, \star, 0)$, *then* $\cap_{i\in I} h_i$ *is a hesitant FPTMSA of* X.

Remark 4.3.1. *The union of two hesitant FPTMSA of a pseudo TM-algebra need not be a hesitant FPTMSA. This is illustrated by the following example:*

Example 4.3.2. *Consider a pseudo TM-algebra defined in example 4.3.1.*
Let h_1 *and* h_2 *be two HFS in* $\mathbb{Z}$ *defined by*

$$h_1(x) = \begin{cases} \{0.4, 0.5, 0.6\} & \text{if } x \text{ is odd} \\ \{0.6, 0.7, 0.8\} & \text{if } x \text{ is even,} \end{cases} \quad \text{and} \quad h_2(x) = \begin{cases} \{0.8, 0.9\} & \text{if } x = 9n, n \in \mathbb{Z} \\ \{0.3, 0.4\} & \text{otherwise.} \end{cases}$$

By Definition 4.3.1, h_1 *and* h_2 *are a hesitant FPTMSA of* $\mathbb{Z}$ *using the usual subtraction operation. Now, taking* $x = 9$ *and* $y = 4$. *By Definition 1.2.3, we have* $(h_1 \cup h_2)(9 \odot 2) = (h_1 \cup h_2)(5) = \left\{h \in (h_1(5) \cup h_2(5)) \mid h \geqslant \max\left(h_1^-(5), h_2^-(5)\right)\right\}$
$= \{0.4, 0.5, 0.6\}$. *But* $(h_1 \cup h_2)(9) \cap (h_1 \cup h_2)(4) = \{0.8, 0.9\}$. *Therefore,* $(h_1 \cup h_2)(9 \odot 4) \supsetneq (h_1 \cup h_2)(9) \cap (h_1 \cup h_2)(4)$

Hence, $h_1 \cup h_2$ is not a hesitant FPTMSA of $\mathbb{Z}$.
By routine calculation $(h_1 \cup h_2)(x \star y)$ is not a hesitant FPTMSA of $\mathbb{Z}$.

Theorem 4.3.8. *If* $h : X \to \mathcal{P}[0,1]$ *is a hesitant FPTMSA of a pseudo TM-algebra* X. *Then*
1) h_α^+ *is a hesitant FPTMSA of* X.
2) h_α^- *is a hesitant FPTMSA of* X.

Proof. Assume that h is a hesitant FPTMSA of X. Let $x, y \in X$. By Definition 1.2.3 we have
1) $h_\alpha^+(x \odot y) = \{h \in h(x \odot y) : h \geqslant \alpha\}$

$$\begin{aligned} &\supseteq \{h \in h(x) \cap h \in h(y) : h \geqslant \alpha\} \\ &= \{h \in h(x) : h \geqslant \alpha \cap h \in h(y) : h \geqslant \alpha\} \\ &= \{h \in h(x) : h \geqslant \alpha\} \cap \{h \in h(y) : h \geqslant \alpha\} \\ &= h_\alpha^+(x) \cap h_\alpha^+(y). \end{aligned}$$

Therefore, $h_\alpha^+(x \odot y) \supseteq h_\alpha^+(x) \cap h_\alpha^+(y)$.
Similarly, $h_\alpha^+(x \star y) = \{h \in h(x \star y) : h \geqslant \alpha\}$

$$\begin{aligned} &\supseteq \{h \in h(x) \cap h \in h(y) : h \geqslant \alpha\} \\ &= \{h \in h(x) : h \geqslant \alpha \cap h \in h(y) : h \geqslant \alpha\} \\ &= \{h \in h(x) : h \geqslant \alpha\} \cap \{h \in h(y) : h \geqslant \alpha\} \\ &= h_\alpha^+(x) \cap h_\alpha^+(y). \end{aligned}$$

Therefore, $h_\alpha^+(x \star y) \supseteq h_\alpha^+(x) \cap h_\alpha^+(y)$.
2) Proof of (2) is similar to (1). □

Theorem 4.3.9. *Let* h *be a hesitant fuzzy set on* X*. Then* h^c *is a hesitant FPTMSA of* X *if and only if for all* $\Lambda \in \mathcal{P}[0,1]$*, a non-empty subset* $\mathcal{L}(h,\Lambda)$ *of* X *is a pseudo TM-subalgebra of* X*.*

Proof. Suppose that h^c is a hesitant FPTMSA of X. Let $\Lambda \in \mathcal{P}[0,1]$ be such that $\mathcal{L}(h,\Lambda) \neq \varnothing$ and let $x,y \in \mathcal{L}(h,\Lambda)$. Then $h(x) \subseteq \Lambda$ and $h(y) \subseteq \Lambda$. But h^c is a hesitant FPTMSA of X, we have:

$$(\forall x,y \in X)\begin{pmatrix} h^c(x \odot y) \supseteq h^c(x) \cap h^c(y) \\ h^c(x \star y) \supseteq h^c(x) \cap h^c(y) \end{pmatrix}.$$

By Lemma 1.2.1, we have

$$(\forall x,y \in X)\begin{pmatrix} [0,1]-h(x \odot y) \supseteq ([0,1]-h(x)) \cap ([0,1]-h(y)) = [0,1]-(h(x) \cup h(y)) \\ [0,1]-h(x \star y) \supseteq ([0,1]-h(x)) \cap ([0,1]-h(y)) = [0,1]-(h(x) \cup h(y)) \end{pmatrix}.$$

Hence,

$$(\forall x,y \in X)\begin{pmatrix} h(x \odot y) \subseteq h(x) \cup h(y) \subseteq \Lambda \\ h(x \star y) \subseteq h(x) \cup h(y) \subseteq \Lambda \end{pmatrix}.$$

Thus, $x \odot y$ and $x \star y \in \mathcal{L}(h,\Lambda)$.
Therefore, $\mathcal{L}(h,\Lambda)$ is a pseudo TM-subalgebra of X.
Conversely, suppose that for all $\Lambda \in \mathcal{P}[0,1]$, a non-empty subset $\mathcal{L}(h,\Lambda)$ of X is a pseudo TM-subalgebra of X. Let $x,y \in X$. Choose $t = h(x) \cup h(y) \in \mathcal{P}[0,1]$. Then $h(x) \subseteq \Lambda$ and $h(y) \subseteq \Lambda$. Thus, $x,y \in \mathcal{L}(h,\Lambda) \neq \varnothing$. By assumption, we have $\mathcal{L}(h,\Lambda)$ is a TM-subalgebra of X and thus, $x \star y \in \mathcal{L}(h,\Lambda)$. Hence, $h(x \star y) \subseteq \Lambda = h(x) \cup h(y)$.
By Lemma 1.2.1, we have:

$$\begin{aligned} h^c(x \star y) &= [0,1]-h(x \star y) \\ &\supseteq [0,1]-(h(x) \cup h(y)) \\ &= ([0,1]-h(x)) \cap ([0,1]-h(y)) \\ &= h^c(x) \cap h^c(y). \end{aligned}$$

Therefore, h^c is a hesitant fuzzy TM-subalgebra of X. □

Proposition 4.3.10. *For a hesitant fuzzy set* h *on* X*, let* $\tilde{h}$ *be a hesitant fuzzy set on* X *defined by*

$$\tilde{h} : X \to \mathcal{P}[0,1], x \mapsto \begin{cases} h(x) & \text{if } x \in \mathcal{U}(h,\Lambda), \\ \emptyset & \text{otherwise} \end{cases}$$

where $\Lambda \in \mathcal{P}[0,1] \backslash \{\emptyset\}$*. If* h *is a hesitant FPTMSA of* X*, then* $\tilde{h}$ *is also a hesitant FPTMSA of* X*.*

Proof. Assume that h is a hesitant FPTMSA of X. By Theorem 4.3.5 $\mathcal{U}(h,\Lambda)$ is a PTMASA of X for all $\Lambda \in \mathcal{P}[0,1]$ with $\mathcal{U}(h,\Lambda) \neq \emptyset$. Let $x,y \in X$.
Case 1: If $x \in \mathcal{U}(h,\Lambda)$ and $y \in \mathcal{U}(h,\Lambda)$, then $x \star y \in \mathcal{U}(h,\Lambda)$. Thus

$$\tilde{h}(x \star y) = h(x \star y) \supseteq h(x) \cap h(y) = \tilde{h}(x) \cap \tilde{h}(y).$$

Case 2: If $x \notin \mathcal{U}(h,\Lambda)$ or $y \notin \mathcal{U}(h,\Lambda)$, then $\tilde{h}(x) = \emptyset$ or $\tilde{h}(y) = \emptyset$. Hence,

$$\tilde{h}(x \star y) \supseteq \emptyset = \tilde{h}(x) \cap \tilde{h}(y) \text{ and}$$

$$\tilde{h}(x \odot y) \supseteq \emptyset = \tilde{h}(x) \cap \tilde{h}(y).$$

Therefore $\tilde{h}$ is a hesitant FPTMSA of X. □

Theorem 4.3.11. *If* $h_A = (x, h_A(x))$ *and* $h_B = (x, h_B(x))$ *are two hesitant FPTMSAs of pseudo TM-algebras* X *and* Y*, respectively, then the Cartesian product* $h_A \times h_B$ *is also an hesitant FPTMSA of* $X \times Y$.

Proof: Let $(x_1,y_1),(x_2,y_2) \in X \times Y$. By Definition 1.2.4 we have

$$\begin{aligned} h((x_1,y_1) \odot (x_2,y_2)) &= h((x_1 \star x_2),(y_1 \star y_2)) \\ &= h_A(x_1 \star x_2) \cap h_B(y_1 \odot y_2) \\ &\supseteq (h_A(x_1) \cap h_A(x_2)) \cap (h_B(y_1) \cap h_B(y_2)) \\ &= (h_A(x_1) \cap h_B(y_1)) \cap (h_A(x_2) \cap h_B(y_2)) \\ &= h(x_1,y_1) \cap h(x_2,y_2) \text{ and} \end{aligned}$$

$$\begin{aligned} h((x_1,y_1) \star (x_2,y_2)) &= h((x_1 \star x_2),(y_1 \star y_2)) \\ &= h_A(x_1 \star x_2) \cap h_B(y_1 \star y_2) \\ &\supseteq (h_A(x_1) \cap h_A(x_2)) \cap (h_B(y_1) \cap h_B(y_2)) \\ &= (h_A(x_1) \cap h_B(y_1)) \cap (h_A(x_2) \cap h_B(y_2)) \\ &= h(x_1,y_1) \cap h(x_2,y_2). \end{aligned}$$

Hence, $h_A \times h_B$ is a hesitant FPTMSA of $X \times Y$.

Theorem 4.3.12. *Let h be a hesitant FPTMSA of a pseudo TM-algebra* X *and f be a homomorphism from a TM-algebra* X *to a pseudo TM-algebra* Y*. Then f*(h) *is a hesitant FPTMSA over* Y.

Proof. Let $y_1,y_2 \in Y$. Then we have the following cases:
Case 1: If $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$, then we have

$$f(h)(y_1) \cap f(h)(y_2) = \varnothing.$$

Therefore,

$$f(h)(y_1 \star y_2) \supseteq f(h)(y_1) \cap f(h)(y_2).$$

Case 2: If $f^{-1}(y_1) \neq \emptyset$ and $f^{-1}(y_2) \neq \emptyset$, then $f^{-1}(y_1 \star y_2) \neq \emptyset$. Let us assume that there exist

$x_1, x_2 \in X$ such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Then by Definition 1.2.5

$$\begin{aligned} f(h)(y_1 \star y_2) &= \bigcup_{x \in f^{-1}(y_1 \star y_2)} h(x) \\ &\supseteq h(x_1 \star x_2) \\ &\supseteq h(x_1) \cap h(x_2). \end{aligned}$$

Without loss of generality, assume that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then we have:

$$\begin{aligned} f(h)(y_1 \star y_2) &\supseteq \left(\bigcup_{f(x_1)=y_1} h(x) \right) \cap \left(\bigcup_{f(x_1)=y_1} h(x) \right) \\ &= f(h)(y_1) \cap f(h)(y_2) \text{ and} \end{aligned}$$

If $f^{-1}(y_1) \neq \emptyset$ and $f^{-1}(y_2) \neq \emptyset$, then $f^{-1}(y_1 \odot y_2) \neq h$. Let us assume that there exist $x_1, x_2 \in X$ such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Then by Definition 1.2.5

$$\begin{aligned} f(h)(y_1 \odot y_2) &= \bigcup_{x \in f^{-1}(y_1 \odot y_2)} h(x) \\ &\supseteq h(x_1 \odot x_2) \\ &\supseteq h(x_1) \cap h(x_2). \end{aligned}$$

Without loss of generality, assume that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then we have:

$$\begin{aligned} f(h)(y_1 \odot y_2) &\supseteq \left(\bigcup_{f(x_1)=y_1} h(x) \right) \cap \left(\bigcup_{f(x_1)=y_1} h(x) \right) \\ &= f(h)(y_1) \cap f(h)(y_2). \end{aligned}$$

Therefore, $f(h)$ is a hesitant fuzzy TM-subalgebra of Y. □

Theorem 4.3.13. *Let $f: (X, \star, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of a pseudo* TM*-algebras. If* $H = \{(y, h(y))/y \in Y)\}$ *is a hesitant fuzzy pseudo* TM*-subalgebra of* Y*, then* $f^{-1}(H) = (x, (h \circ f)(x))$ *is a hesitant fuzzy pseudo* TM*-subalgebra of* X.

Proof. Assume that $H = (y, h(y))$ is a hesitant FPTMSA of Y.
Let $x, y \in X$. Then Similarly,

$$\begin{aligned} f^{-1}(h)(x \odot y) = (h \circ f)(x \odot y) &= h(f(x \odot y)) \\ &= h(f(x) \odot f(y)) \\ &\supseteq h(f(x)) \cap h(f(y)) \\ &= (h \circ f)(x) \cap (h \circ f)(y) \\ &= f^{-1}(h)(x) \cap f^{-1}(h)(y) \text{ and} \end{aligned}$$

$$
\begin{aligned}
f^{-1}(h)(x\star y) = (h\circ f)(x\star y) &= h(f(x\star y))\\
&= h(f(x)\star f(y))\\
&\supseteq h(f(x))\cap h(f(y))\\
&= (h\circ f)(x)\cap(h\circ f)(y)\\
&= f^{-1}(h)(x)\cap f^{-1}(h)(y).
\end{aligned}
$$

Therfore, $f^{-1}(H)$ is a hesitant fuzzy pseudo TM-subalgebra of X. □

Theorem 4.3.14. *Let $f\colon(X,\star,0_X)\to(Y,\star,0_Y)$ be an epimorphism of TM-algebras and A be a hesitant fuzzy set on Y. Then A is a hesitant FPTMSA of Y if and only if A^f is a hesitant FPTMSA of X.*

Proof. Suppose that A is a hesitant FPTMSA of Y.
Claim: A^f is a hesitant fuzzy TM-subalgebra of X.
Now, let $x,y\in X$. We have:

$$
\begin{aligned}
h_{A^f}(x)\cap h_{A^f}(y) &= h_Af(x)\cap h_Af(y)\\
&\subseteq h_A(f(x)\odot f(y))\\
&= h_A(f(x\odot y)\\
&= h_{A^f}(x\odot y)\ \text{and}
\end{aligned}
$$

$$
\begin{aligned}
h_{A^f}(x)\cap h_{A^f}(y) &= h_Af(x)\cap h_Af(y)\\
&\subseteq h_A(f(x)\star f(y))\\
&= h_A(f(x\star y)\\
&= h_{A^f}(x\star y).
\end{aligned}
$$

Therfore, A^f is a hesitant FPTMSA of X.
Conversely, suppose that A^f is a hesitant FPTMSA of X.
Claim: A is a hesitant FPTMSA of Y.
Since f is surjective, then there exists $x,y\in Y$ such that $f(u)=x\&f(v)=y$. We have

$$
\begin{aligned}
h_A(x)\cap h_A(y) &= h_Af(u)\cap h_Af(v)\\
&= h_{A^f}(u)\cap h_{A^f}(v)\\
&\subseteq h_{A^f}(u\odot v)\\
&= h_Af(u\odot v)\\
&= h_A(f(u)\odot f(v))\\
&= h_A(x\odot y)\ \text{and}
\end{aligned}
$$

$$
\begin{aligned}
h_A(x)\cap h_A(y) &= h_A f(u)\cap h_A f(v)\\
&= h_{A^f}(u)\cap h_{A^f}(v)\\
&\subseteq h_{A^f}(u\star v)\\
&= h_A f(u\star v)\\
&= h_A(f(u)\star f(v))\\
&= h_A(x\star y).
\end{aligned}
$$

Therfore, A is a hesitant fuzzy TM-subalgebra of y.

□

Theorem 4.3.15. *Let $f : X \to Y$ is homomorphism of a pseudo TM-algebras, let h_1 be a hesitant fuzzy pseudo TM- subalgebra of X and let h_2 be a hesitant fuzzy pseudo subalgebra of Y .*
(1) If h_1 has the sup-property, then $f(h_1)$ is a hesitant fuzzy pseudo subalgebra of Y.
(2) $f^{-1}(h_2)$ is also a hesitant fuzzy pseudo subalgebra of X .

Proof. Let h_1 be a hesitant fuzzy pseudo-TM subalgebra of X and by Definition 1.2.5(1). let $a, b \in Y$ with $x_0 \in f^{-1}(a)$, $\quad y_0 \in f^{-1}(b)$ such that $h_1(x_0) = \cup_{t\in f^{-1}(a)}\{h_1(t)\}$, $\quad h_1(y_0) = \cup_{t\in f^{-1}(b)}\{h_1(t)\}$. Now,

$$
\begin{aligned}
f(h_1)(a\odot b) &= \cup_{t\in f^{-1}(a\odot b)}\{h_1(t)\} \supseteq h_1(x_0\cdot y_0)\\
&\supseteq h_1(x_0)\cap h_1(y0)\\
&= \cup_{t\in f^{-1}(a)}\{h_1(t)\}\cap \cup_{t\in f^{-1}(b)}\{h_1(t)\}\\
&= f(h_1)(a)\cap f(h_1)(b).\\
f(h_1)(a\star b) &= \cup_{t\in f^{-1}(a\star b)}\{h_1(t)\} \supseteq h_1(x_0\star y_0)\\
&\supseteq h_1(x_0)\cap h_1(y_0)\\
&= \cup_{t\in f^{-1}(a)}\{h_1(t)\}\cap \cup_{t\in f^{-1}(b)}\{h_1(t)\}\\
&= f(h_1)(a)\cap f(h_1)(b).
\end{aligned}
$$

Hence, $f(h_1)$ is a hesitant fuzzy pseudo-TM subalgebra of Y and let h_2 be a hesitant fuzzy pseudo-TM subalgebra of Y and for all $x, y \in X$. Then,

$$\begin{aligned}
f^{-1}(h_2)(x \odot y) &= h_2(f(x \odot y)) \\
&= h_2(f(x \odot y)) \\
&= h_2(f(x) \odot f(y)) \\
&\supseteq h_2(f(x)) \cap h_2(f(y)) \\
&= f^{-1}(h_2)(x) \cap f^{-1}(h_2)(y). \\
\Longrightarrow f^{-1}(h_2)(x \odot y) &\supseteq f^{-1}(h_2)(x) \cap f^{-1}(h_2)(y). \\
f^{-1}(h_2)(x \star y) &= h_2(f(x \star y)) \\
&= h_2(f(x \star y)) \\
&= h_2(f(x) \star f(y)) \\
&\supseteq h_2(f(x)) \cap h_2(f(y)) \\
&= f^{-1}(h_2)(x) \cap f^{-1}(h_2)(y). \\
\Longrightarrow f^{-1}(h_2)(x \star y) &\supseteq f^{-1}(h_2)(x), f^{-1}(h_2)(y).
\end{aligned}$$

Hence, $f^{-1}(h_2)$ is a hesitant fuzzy pseudo-TM subalgebra of X.

□

Theorem 4.3.16. *Let h be a HFPTMS of a PTMA of* X *and f be surjective homomorphism from a pseudo TM-algebra* X *to a pseudo TM- algebra* Y*. Then f*(h) *is a hesitant fuzzy PTMASA of* Y.

Proof. Assume that h be a HFPTMS of a PTMA of X and f be surjective homomorphism from a pseudo TM-algebra X to a pseudo TM- algebra Y. We need to show that $f(h)$ is a hesitant fuzzy PTMASA of Y. Let $y_1, y_2 \in Y$. Let us assume that there exist $x_1, x_2 \in X$ such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. By definition 1.2.5

$$\begin{aligned}
f(h)(y_1 \odot y_2) &= \bigcup_{f(x)=y_1 \star y_2} h(x) \\
&\supseteq h(x_1 \cdot x_2) \\
&\supseteq h(x_1) \cap h(x_2).
\end{aligned}$$

Since for each $x_1, x_2 \in X$ satisfying $f(x_1) = y_1$ and $f(x_2) = y_2$, we have

$$\begin{aligned}
f(h)(y_1 \star y_2) &\supseteq \left[\bigcup_{f(x)=y_1} h(x)\right] \cap \left[\bigcup_{f(x)=y_2} h(x)\right] \\
&= f(h)(y_1) \cap f(h)(y_2).
\end{aligned}$$

Thus, $f(h)$ is a hesitant fuzzy TM-Subalgebra of X. □

4.4. Hesitant Fuzzy Pseudo-ideal of a pseudo-TM Algebra

In this section, we define the notion of a hesitant fuzzy pseudo-ideal in pseudo-TM algebras and study the properties of this concept.

Definition 4.4.1. *A hesitant fuzzy subset* $\mathtt{h}$ *of a pseudoTM -algebra X is called a hesitant fuzzy pseudo ideal, denoted by HF.pseudo I of X if it satisfies:for all x,y* $\in \mathrm{X}$

(i) $\mathtt{h}(0) \supseteq \mathtt{h}(x)$

(ii) $\mathtt{h}(x) \supseteq \mathtt{h}(x \odot y) \cap \mathtt{h}(x \star y) \cap \mathtt{h}(y)$.

Example 4.4.1. *Let* $X = \{0,1,2,3\}$ *be a pseudo TM with the following tables*

$\odot$	0	1	2	3
0	0	3	2	0
1	1	0	2	0
2	2	2	0	3
3	3	3	1	0

$\star$	0	1	2	3
0	0	1	0	0
1	2	0	0	0
2	2	2	0	3
3	3	3	3	0

Table 4.5

See [52] $(X;\odot,\star,)$ *is a pseudo TM-algebra.*

Define $\mathtt{h}$ *be a hesitant fuzzy subset of X by* $\mathtt{h}(x) = \begin{cases} 0.7 \text{ if } x = 0 \\ 0.6 \text{ otherwise} \end{cases}$

Then $\mathtt{h}$ *is a hesitant fuzzy pseudo-ideal of a TM -algebra X.*

Definition 4.4.2. *Let* $\mathtt{h}$ *and* σ *be hesitant fuzzy subsets of a pseudo-TM algebra X such that* $\mathtt{h} \subseteq \sigma$ *. Then* σ *is called a hesitant fuzzy pseudo-ideal of X related to* $\mathtt{h}$ *denoted by HF pseudo-ideal if it satisfies: for all* $x,y \in X$.

(i) $\sigma(0) \supseteq \sigma(x)$

(ii) $\sigma(x) \supseteq \mathtt{h}(x \odot y) \cap \mathtt{h}(x \star y) \cap \mathtt{h}(y)$.

Example 4.4.2. *Let* $X = \{0,1,2,3\}$ *be a pseudo TM -algebra with the following tables*

$\odot$	0	1	2	3
0	0	1	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

$\star$	0	1	2	3
0	0	1	0	1
1	1	0	3	0
2	2	2	0	3
3	3	3	2	0

Table 4.6

See [52] $(X;\odot,\star,0)$ *is a pseudo TM-algebra.*

Define h *and* σ *a hesitant fuzzy subsets of X by* $h(x) = \begin{cases} \{0.9\} \text{ if } x = 0 \\ \{0.4\} \text{ if } x = 1,2,3 \end{cases}$ and $\sigma(x) = \begin{cases} \{0.9, 0.4\} \text{ if } x = 0 \\ \{0.4\} \text{ if } x = 1,2,3 \end{cases}$
Clearly $h(x) \subseteq \sigma(x)$ *for every* $x \in X$*, so* $h \subseteq \sigma$*. We check the two defining properties of a hesitant fuzzy pseudo ideal related to h.*

(i) $\sigma(0) \supseteq \sigma(x)$ *for all x. Hence,* $\sigma(1) = \sigma(2) = \sigma(3) = \{0.4\} \subseteq \{0.9, 0.4\} = \sigma(0)$.

(ii) $\sigma(x) \supseteq h(x \odot y) \cap h(x \star y) \cap h(y)$ *for all* $x, y \in X$.
Note that for every $y \in 1,2,3$*. We have* $h(y) = 0.4$*, while* $h(0) = 0.9$*. Thus for any pair* (x, y) *there are only two possibilities for the triple* $h(x \odot y), h(x \star y), h(y)$ *either at least one of them equals h(0)=0.9 in which case the intersection* $h(x \odot y) \cap h(x \star y) \cap h(y) = \emptyset$*, and* $\emptyset \subseteq \sigma(x)$ *holds trivially or all three are equal to* 0.4*, in which case* $h(x \odot y) \cap h(x \star y) \cap h(y) = 0.4$*, and* $0.4 \subseteq \sigma(x)$ *holds because* $\sigma(x) = 0.4$ *for* $x = 1,2,3$ *or* $\sigma(0)$ *also contains (0.4). Concretely, using the Cayley tables one checks that whenever both* $x \odot y$ *and* $x \star y$ *lie in 1,2,3 and* $y \in 1,2,3$*, the intersection is 0.4 and is contained in* $\sigma(x)$*; otherwise the intersection is* $\emptyset$*. Thus (ii) holds for every* $x, y \in X$.

Therefore, σ *is a hesitant fuzzy pseudo ideal of X related to h.*

Example 4.4.3. *Let* $X = \{0,1,2,3\}$ *be a pseudo TM -algebra with the following tables*

$\odot$	0	1	2	3
0	0	0	0	0
1	2	0	3	1
2	2	3	0	2
3	3	3	3	0

$\star$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	0	0	0

Table 4.7

See [52] $(X; \odot, \star, 0)$ *is a pseudo TM-algebra.*
Define h *and* σ *fuzzy subsets of X by*

$$h(x) = \begin{cases} 0.7 \text{ if } x = 0 \\ 0.8 \text{ if } x = 2 \\ 0.5 \text{ if } x = 1,3 \end{cases} \qquad \sigma(x) = \begin{cases} 0.9 \text{ if } x = 0,2 \\ 0.6 \text{ if } x = 1,3 \end{cases}$$

Then σ *is a hesitant fuzzy pseudo ideal of* X *related to* h*, but* σ *is not a hesitant fuzzy pseudo ideal of TM -algebra related to* h *since* $\sigma(3) = 0.6 < \min\{h(3 \star 1), h(2)\} = 0.7$.

Example 4.4.4. *Let* $X = \{0,1,2,3\}$ *be a pseudo TM -algebra with the following tables*
See [52] $(X; \odot, \star,)$ *is a pseudo TM-algebra.*

$\odot$	0	1	2	3
0	0	3	3	3
1	1	0	3	3
2	2	3	0	3
3	3	3	1	0

$\star$	0	1	2	3
0	0	3	3	3
1	1	0	3	3
2	2	3	0	3
3	3	3	2	0

Table 4.8

and let

$$h_1(x)=\begin{cases}0.6 \textit{ if } x=0\\ 0.4 \textit{ if } x=1\\ 0.1 \textit{ if } x=2,3\end{cases} \quad \sigma_1(x)=\begin{cases}0.7 \textit{ if } x=0\\ 0.4 \textit{ if } x=1\\ 0.1 \textit{ if } x=2,3\end{cases}$$

$$h_2(x)=\begin{cases}0.8 \textit{ if } x=0\\ 0.6 \textit{ if } x=2\\ 0.1 \textit{ if } x=1,3\end{cases} \quad \sigma_2(x)=\begin{cases}0.8 \textit{ if } x=0\\ 0.5 \textit{ if } x=2\\ 0.3 \textit{ if } x=1,3\end{cases}$$

Then σ_1 and σ_2 are a hesitant fuzzy pseudo ideal of X related to h_1 and h_2 respectively.

$$h_1\cup h_2(x)=\begin{cases}0.6 \textit{ if } x=0\\ 0.4 \textit{ if } x=1,2\\ 0.1 \textit{ if } x=3\end{cases} \quad \sigma_1\cup\sigma_2(x)=\begin{cases}0.8 \textit{ if } x=0\\ 0.5 \textit{ if } x=1\\ 0.4 \textit{ if } x=2\\ 0.2 \textit{ if } x=3\end{cases}$$

But $\sigma_1\cup\sigma_2$ is not hesitant fuzzy pseudo ideal of X related to $h_1\cup h_2$, since $(\sigma_1\cup\sigma_2)(c)=0.3<h_1\cup h_2\,(3\odot 2)\,,h_1\cup h_2(3\star 2),h_1\cup h_2(2)=0.5.$

Theorem 4.4.1. *If $H=\{(x,h(x))\,|x\in X\}$ is a hesitant fuzzy pseudo-deal of a pseudo TM- algebra X. Then*

$$(\forall x,y\in X)\,(x\leqslant y\Rightarrow h(y)\subseteq h(x))\,.$$

Proof. Assume that $H=\{(x,h(x))\,|x\in X\}$ is a hesitant fuzzy pseudo ideal of a pseudo TM- algebra X. We need to show that $(\forall x,y\in X)\,(x\leqslant y\Rightarrow h(y)\subseteq h(x))$. Let $x,y\in X$ such that $x\leqslant y$. We have $x\odot y=0$ and $x\star y=0$. Since h is hesitant fuzzy pseudo ideal of a pseudo TM- algebra X. Then

$$\begin{aligned} h(x) &\supseteq h\,(x\odot y)\cap h(x\star y)\cap h(y)\\ &\supseteq h\,(0)\cap h(0)\cap h(y)\\ &= h(y)\end{aligned}$$

□

Theorem 4.4.2. *Let $\{h_i\mid i\in I\}$ be a family of hesitant fuzzy pseudo ideal of a pseudoTM -algebra X related to a hesitant fuzzy subset h of X. Then $\cap_{i\in I}h_i$ is a hesitant hesitant fuzzy pseudo-ideal of X related to h.*

Proof. Let $\{h_i \mid i \in I\}$ be a family of hesitant fuzzy pseudo ideals of X. We need to show that $\cap_{i\in I} h_i$ is a hesitant hesitant fuzzy pseudo ideal of X related to h.

(i) Let $x \in X$. Then $\cap_{i\in I} h_i(0) = \cap\{h_i(0) \mid i \in I\}) \supseteq h(x)$. Since $h_i(0) \supseteq h(x)$, for all $i \in I$.

(ii) Let $x, y \in X$. Then, we have $\cap_{i\in I} h_i(x) = \cap\{h_i(x) \mid i \in I\} \supseteq h(x \odot y) \cap h(x \star y) \cap h(y)$.
Since $h_i(x) = \cap\{h_i(x) \mid i \in I\} \supseteq h(x \odot y) \cap h(x \star y) \cap h(y), \forall i \in I$
$\Rightarrow \cap_{i\in I} h_i \supseteq h(x \odot y) \cap h(x \star y) \cap h(y)$.
Hence, $\cap_{i\in I} h_i$ is a hesitant fuzzy pseudo ideal of X related to h.

□

Theorem 4.4.3. *Let* X *be a pseudoTM -algebra* h *and* σ *be a hesitant fuzzy subsets of* X. *Then* σ *is a hesitant fuzzy pseudo ideal of* X *related to* h *if and only if the level subset* σ_Λ *is a pseudo ideal of X related to* $h_\Lambda, \forall \Lambda \in \mathcal{P}[0, \sigma(0)], \sigma(0) = \bigcup_{x\in X} \sigma(x)$.

Proof. Let σ be a hesitant fuzzy pseudo ideal of X related to h and $\Lambda \in \mathcal{P}[0, \sigma(0)]$. We need to show that σ_t is a pseudo ideal of X related to h_Λ.

(i) It is clear that $h_\Lambda \subseteq \sigma_\Lambda$ and $\sigma(0) \supseteq \Lambda \Rightarrow 0 \in \sigma_t$.

(ii) Let $x, y \in X$ such that $x \odot y, x \star y \in h_\Lambda$ and $y \in h_\Lambda$
$\Rightarrow h(x \odot y) \supseteq \Lambda, h(x \star y) \supseteq \Lambda$ and $h(y) \supseteq \Lambda$
$h(x \odot y) \cap h(x \star y) \cap h(y) \supseteq \Lambda$
But, $\sigma(x) \supseteq h(x \odot y) \cap h(x \star y) \cap h(y)$. Since σ is a hesitant fuzzy pseudo ideal of X related to h. It implies that $\sigma(x) \supseteq \Lambda \Rightarrow x \in \sigma_\Lambda$.
Hence, σ_Λ is a pseudo ideal of X related to h_Λ.

Conversely, let σ_t be a pseudo ideal of X related to h_Λ of $X, \forall \Lambda \in \mathcal{P}[0, \sigma(0)], \sigma(0) = \bigcup_{x\in X} h(x)$.
We need to show that σ is a hesitant fuzzy pseudo ideal of X related to h.

(i) It is clear that $\sigma(0) \supseteq \sigma(x), \forall x \in X$.

(ii) Let $x, y \in X$ such that $h(x \odot y) \cap h(x \star y) \cap h(y) = \Lambda$
$\Rightarrow h(x \odot y) \supseteq \Lambda, h(x \star y) \supseteq \Lambda$ and $h(y) \supseteq \Lambda$
$\Rightarrow x \odot y, x \star y \in h_\Lambda$ and $y \in h_\Lambda \Rightarrow x \in B_\Lambda$. Since σ_Λ is a pseudo ideal of X related to h_Λ
$\Rightarrow \sigma(x) \supseteq \Lambda \Rightarrow \sigma(x) \supseteq h(x \odot y) \cap h(x \star y) \cap h(y)$
Therefore, σ be a hesitant fuzzy pseudo ideal of X related to h of a pseudo-TM algebra X.

□

Theorem 4.4.4. *Assume that* X *is a pseudo-TM algebra and let* h *be a hesitant fuzzy subset of* X. *Then* h *is a hesitant fuzzy pseudo-ideal of* X *if and only if the upper level set* $U(h, \Lambda)$ *is a pseudo-ideal of* X *or empty of* $X, \forall \Lambda \in \mathcal{P}[0, 1]$.

Proof. Let h be a hesitant fuzzy pseudo-ideal of a pseudo-TM algebra X and $U(h,\Lambda) \neq \emptyset$ for every $\Lambda \in \mathcal{P}[0,1]$ Clearly $0 \in U(h,\Lambda)$ since $h(0) \supseteq \Lambda$. Suppose that $x, y \in X$ such that $x \odot y, x \star y$ and $y \in U(h,\Lambda)$, then $h(x \odot y) \supseteq \Lambda, h(x \star y) \supseteq \Lambda$ and $h(y) \supseteq \Lambda$.
It follows that $h(x) \supseteq h((x \odot y)) \cap h((x \star y)) \cap h(y) \supseteq \Lambda$. It implies that $x \in U(h,\Lambda)$.
Therefore, $U(h,\Lambda)$ is a pseudo-ideal of X.
Conversely, for every $\Lambda \in \mathcal{P}[0,1]$ and non-empty subset $U(h,\Lambda)$ is a pseudo-ideal of X, $\forall x \in X$.
Let $x \in X$. Then $h(x) \in \mathcal{P}[0,1]$. Then, $h(x) \supseteq \Lambda$. Thus, $x \in U(h,\Lambda) \neq \emptyset$. By assumption, we have $U(h,\Lambda)$ is a pseudo ideal of X and thus, $0 \in U(h,\Lambda_1)$. So, $h(0) \supseteq \Lambda = h(x)$. Next, let $x, y, \in X$. Then $h(x \odot y), h(x \star y), h(y) \in \mathcal{P}[0,1]$. Choose $\Lambda = h(x \odot y) \cap h(x \star y) \cap h(y) \in \mathcal{P}[0,1]$. Then $h(x \odot y) \supseteq \Lambda$, $h(x \star y) \supseteq \Lambda$ and $h(y) \supseteq \Lambda$. Thus $x \odot y, x \star y, y \in U(h;\Lambda) \neq \varnothing$. By assumption, we have $U(h;\Lambda)$ is a pseudo ideal of X. So $x \in U(h;\Lambda)$. Thus $h(x) \supseteq \Lambda = h(x \odot y) \cap h(x \star y) \cap h(y)$. ☐

Therefore, h is a fuzzy pseudo-ideal of a pseudo-TM algebra X.

Definition 4.4.3. *Let σ be a hesitant fuzzy subset of a pseudo-TM algebra X. The set $\{x \in X : \sigma(x) = \sigma(0)\}$ is denoted by X_σ which represents the level set of the hesitant fuzzy set σ where all elements in X_σ have the same membership value as the element 0 under σ.*

Theorem 4.4.5. *Let X be a pseudo-TM algebra X. If σ is a hesitant fuzzy pseudo ideal related to h such that $\sigma(0) = h(0)$.Then the set X_σ is a pseudo ideal related to X_h.*

Proof. Let σ be a hesitant fuzzy pseudo ideal related to h. We need to show that X_σ is a pseudo ideal related to X_h.

(i) Let $x \odot y, x \star y, y \in X_h \Rightarrow h(x \odot y) = h(x \star y) = h(y) = h(0)$
$\Rightarrow h(x \odot y) \cap h(x \star y) \cap h(y) = h(0)$
But $\sigma(x) \supseteq h(x \odot y) \cap h(x \star y) \cap h(y) = h(0)$
$\Rightarrow \sigma(x) = h(0) \Rightarrow \sigma(x) = \sigma(0) \Rightarrow x \in X_h$.
Hence X_σ is a pseudo ideal related to X_h.

☐

Definition 4.4.4. *Let X be a pseudo TM–algebra and let $\sigma = \{(x, \sigma(x)) : x \in X\}$ be a hesitant fuzzy subset of X, where $\sigma(x) \subseteq \mathcal{P}[0,1]$ denotes the set of all possible membership degrees of x in σ. Then the hesitant fuzzy complement or hesitant fuzzy derived set of σ, denoted by σ', is defined by $\sigma'(x) = \{\sigma(x) + 1 - \sigma(0)\} \cap [0,1], \quad \forall x \in X$, where the operations $(+)$ and $(-)$ are applied elementwise to all membership degrees. That is, for each $x \in X$, $\sigma'(x) = \{\alpha + 1 - \beta / \alpha \in \sigma(x), \beta \in \sigma(0)\} \cap [0,1]$.*

Example 4.4.5. *Let $X = \{0,\ 1,\ 2,\ 3\}$ be a pseudo TM–algebra Defined in 4.4.2. Define a hesitant fuzzy subset σ of X by $\sigma = (0,\ \{0.7, 0.8\}), (1,\ \{0.5, 0.6\}), (2,\ \{0.4\}), (3,\ \{0.3, 0.4\})$. We compute the hesitant fuzzy complement σ' using Definition 4.4.4 we have :*
$\sigma(0) = \{0.7, 0.8\}$.
Hence, for each $x \in X$, $\sigma'(x) = \{\alpha + 1 - \beta \mid \alpha \in \sigma(x), \beta \in \{0.7, 0.8\}\} \cap [0,1]$.

Now, compute each

$$\begin{aligned}
\sigma'(0) &= \{0.7+1-0.7,\ 0.7+1-0.8,\ 0.8+1-0.7,\ 0.8+1-0.8\} \cap [0,1] \\
&= \{0.9,\ 1.0\} \\
\sigma'(1) &= \{0.5+1-0.7,\ 0.5+1-0.8,\ 0.6+1-0.7,\ 0.6+1-0.8\} \cap [0,1] \\
&= \{0.7,\ 0.8,\ 0.9\} \\
\sigma'(2) &= \{0.4+1-0.7,\ 0.4+1-0.8\} \cap [0,1] = \{0.6,\ 0.7\} \\
\sigma'(3) &= \{0.3+1-0.7,\ 0.3+1-0.8,\ 0.4+1-0.7,\ 0.4+1-0.8\} \cap [0,1] \\
&= \{0.5,\ 0.6,\ 0.7\}.
\end{aligned}$$

Therefore, $\sigma' = (0, \{0.9, 1.0\}), (1, \{0.7, 0.8, 0.9\}), (2, \{0.6, 0.7\}), (3, \{0.5, 0.6, 0.7\})$.

Theorem 4.4.6. *Let* h *and* σ *be a hesitant fuzzy subsets of a pseudoTM* X *such that* $\sigma(0) = h(0)$. *Then* σ *is a hesitant fuzzy pseudo ideal of* X *related to* h *if and only if* $\sigma^{'}$ *is a hesitant fuzzy pseudo ideal of X related to* $h^{'}$.

Proof. Let σ be a hesitant fuzzy pseudo ideal of X related to h. We need to show that $\sigma^{'}$ is a hesitant fuzzy pseudo ideal of X related to $h^{'}$.

(i) $\sigma'(0) = \sigma(0) + 1 - \sigma(0) \Rightarrow \sigma'(0) = 1 \Rightarrow \sigma'(0) \supseteq \sigma'(x) \forall x \in X$.

(ii) Let $x, y \in X$. Then

$$\begin{aligned}
\sigma'(x) = \sigma(x) + 1 - \sigma(0) &\supseteq \{h(x \odot y) \cap h(x \star y) \cap h(y)\} + 1 - h(0) \\
&\supseteq \{h(x \odot y) + 1 - h(0) \cap h(x \star y) + 1 - h(0) \cap h(y) + 1 - h(0)\} \\
&\supseteq h'(x \odot y) \cap h'(x \star y) \cap h'(y) \\
&\Rightarrow \sigma'(x) \supseteq h'(x \odot y) \cap h'(x \star y) \cap h'(y)
\end{aligned}$$

$\Rightarrow h'$ is a hesitant fuzzy pseudo ideal of X related to h'.

Conversely,

(i) Let $x \in X$. Then $\sigma'(0) \supseteq \sigma'(x) \Rightarrow \sigma(0) \supseteq h(x)$.

(ii) Let $x, y \in X$. We have $\sigma'(x) \supseteq h'(x \odot y) \cap h'(x \star y) \cap h'(y)$
$\supseteq h(x \odot y) - 1 + h(0) \cap h(x \star y) - 1 + h(0) \cap h(y) - 1 + h(0)$
$\supseteq h(x \odot y) \cap h(x \star y) \cap h(y)\} - 1 + h(0)$.
$\Rightarrow \sigma(x) \supseteq h(x \odot y) \cap h(x \star y) \cap h(y)$
$\Rightarrow \sigma$ is a hesitant fuzzy pseudo ideal of X related to h.

□

Definition 4.4.5. *A hesitant fuzzy set* h *is upward closed* $\uparrow$ *with respect to a hesitant fuzzy set* σ *if for every* $x \in X$, *the membership degrees of* h *dominate those of* σ *in the following sense:*

$$\forall \gamma \in h_\sigma(x), \exists \beta \in h_h(x) \text{ such that } \beta \supseteq \gamma.$$

In other words, for every possible membership degree γ *in* $h_\sigma(x)$, *there exists a membership degree* β *in* $h_h(x)$ *that is greater than or equal to* γ.

Theorem 4.4.7. *Let* X *be a pseudo-TM algebra and* $h \in X$. *If* σ *is a hesitant fuzzy pseudo ideal of X related to* h *such that* $\sigma(h) = h(h)$. *Then* $\uparrow \sigma(h)$ *is a pseudo ideal of* X *related to* $\uparrow h(h)$.

Proof. Let σ be a hesitant fuzzy pseudo ideal of X related to h. We need to show that $\uparrow \sigma(h)$ is a pseudo ideal of X related to $\uparrow h(h)$.

(i) Since $\sigma(0) \supseteq \sigma(h) \Rightarrow 0 \in \uparrow \sigma(h)$.

(ii) Let $x \in X$ and $y \in \uparrow h(h)$ such that $(x \odot y), (x \star y) \in \uparrow h(h)$ and $y \in \uparrow h(h)$
$\Rightarrow h(x \odot y) \supseteq h(h), h(x \star y) \supseteq h(h)$ and $h(y) \supseteq h(h)$.
$\Rightarrow h(h) \subseteq h(x \odot y), h(h) \subseteq h(x \star y)$ and $h(h) \subseteq h(y) \Rightarrow h(h) \subseteq h(x \odot y) \cap h(x \star y) \cap (y)$.
But, $\sigma(x) \supseteq h(x \odot y) \cap h(x \star y) \cap h(y) \Rightarrow h(h) \subseteq \sigma(x) \Rightarrow h\sigma(h) \subseteq \sigma(x) \Rightarrow x \in \uparrow \sigma(h)$.
$\Rightarrow \uparrow \sigma(h)$ is a pseudo ideal of X related to $\uparrow h(h)$.

□

4.5. Homomorphism of Fuzzy Pseudo ideal of Pseudo-TM Algebra

In this section, we give the characterizations of hesitant fuzzy pseudo-ideals and hesitant fuzzy homomorphisms of a pseudo-TM algebra.

Theorem 4.5.1. *Let* $(X, \odot, \star, 0)$ *and* $(Y, \odot, \star, 0)$ *be a pseudo-TM algebras and let* $f : X \to Y$ *be a homomorphism and* h *be a hesitant fuzzy pseudo-ideal of* Y. *Then* $f^{-1}(h)$ *is a hesitant fuzzy pseudo-ideal of* X.

Proof. Assume that h be a hesitant fuzzy pseudo-ideal of Y. We need to show that $f^{-1}(h)$ is a hesitant fuzzy pseudo-ideal of X. Let $x \in X$. Then $f(x) \in Y$ and h is a hesitant fuzzy pseudo-ideal of Y, we have
$f^{-1}(h)(0) = h(f(0)) \supseteq h(f(x))$.Thus we get $f^{-1}(h)(0) \supseteq f^{-1}(h)(x)$ for any $x \in X$.
Let $x, y \in X$. Since $f(h)$ is a fuzzy pseudo-ideal of Y,we have:

$$\begin{aligned} f^{-1}(h(x)) = h(f(x)) &\supseteq h(f(x) \odot f(y)) \cap h(f(x) \star f(y)) \cap h(f(y)) \\ &= h(f(x \odot y)) \cap h(f(x \star y)) \cap h(f(y)) \\ &= f^{-1}(h)(x \odot y) \cap f^{-1}(h)(x \star y) \cap f^{-1}(h)(y) \\ \Rightarrow f^{-1}(h(x)) &\supseteq f^{-1}(h)(x \odot y) \cap f^{-1}(h)(x \star y) \cap f^{-1}(h)(y) \end{aligned}$$

Therefore, $f^{-1}(h(x))$ is a hesitant fuzzy pseudo-ideal of X. □

Corollary 4.5.2. *If* h *is a hesitant fuzzy pseudo-ideal of a pseudo-TM -algebra* X, *then the set* $H = \{x \in X \mid h(x) \supseteq h(b)\}$ *is a hesitant fuzzy pseudo -ideal of* X *for any* $b \in X$.

Proof. Suppose that h is a hesitant fuzzy pseudo-ideal of X. We need to show that H is a hesitant fuzzy pseudo-ideal of X.
Let $x, y \in X$ such that $x \odot y \in H$, $x \star y \in H$ and $y \in H$. Then

$$\begin{aligned} & h(x \odot y) \supseteq h(b),\ h(x \star y) \supseteq h(b) \text{ and } h(y) \supseteq h(b) \\ & \Rightarrow h(x) \supseteq h(x \odot y) \cap h(x \star y) \cap h(y)\} \supseteq h(b). \end{aligned}$$

Therefore, $h(x) \supseteq h(b)$. □

Lemma 4.5.3. *Let* $(X, \odot, \star, 0)$ *and* $(Y, \odot, \star, 0)$ *be a pseudo-TM algebras and let* $f : X \to Y$ *be a homomorphism and* h *be a hesitant fuzzy pseudo-ideal of* X. *If* h *is constant on* $\mathrm{Ker}(f) = f^{-1}(0)$, *then* $f^{-1}(f)(h) = h$.

Proof. Assume that h be a hesitant fuzzy pseudo-ideal of X and h is constant on $\mathrm{Ker(f)} = f^{-1}(0)$. We need to show that $f^{-1}(f)(h) = h$. Let $x \in X$ and $f(x) = y$. Hence

$$\begin{aligned} f^{-1}(f)(h)(x) &= f(h)f(x) \\ &= f(h)(y) \\ &= \bigcup \left\{ h(b) : b \in f^{-1}(y) \right\} \end{aligned}$$

For all $b \in f^{-1}(y)$, we have $f(x) = f(b)$.
Hence, $f(b \odot x) = f(b \star x) = 0$
$\Rightarrow b \odot x, b \star x \in \mathrm{Ker(f)}$. It follows that $h(b \odot x) = h(0) = h(b \star x)$. Then
$h(b) \supseteq h(b \odot x) \cap h(b \star x) \cap h(x) = h(0) \cap h(x) = h(x)$
We get, $h(b) \supseteq h(x)$. Since by Corollary 4.5.2 $h(x) \supseteq h(b)$. □

Hence, $h(x) = h(b)$.
Thus ,$\left(f^{-1}(f)(h)\right)(x) = \bigcup \left\{ h(b) : b \in f^{-1}(y) \right\} = h(x)$.
Therefore, $f^{-1}(f)(h) = h$.

Theorem 4.5.4. *Let* $(X, \odot, \star, 0)$ *and* $(Y, \odot, \star, 0)$ *be a pseudo-TM algebras and let* $f : X \to Y$ *be an epimorphism and* h *be a hesitant fuzzy pseudo-ideal of* X *such that* h *is constant on* $\mathrm{Ker}(f)$. *Then* $f(h)$ *is a hesitant fuzzy pseudo-ideal of* Y, *provided that the sup property holds.*

Proof. Suppose that h be a fuzzy pseudo-ideal of X such that h is constant on $\mathrm{Ker}(f)$.We need to show that $f(h)$ is a hesitant fuzzy pseudo-ideal of Y, provided that the sup property holds. Let $0 \in Y$, then there exist $0 \in X$ such that $f(0) = 0$. Now, $f(h)(0) = \bigcup h(k) \mid k \in f^{-1}(0) = h(0) \supseteq h(x), \forall x \in X$. Let $y \in Y$. Since f is an epimorphism, we have $f^{-1}(y) \neq \emptyset$ and $h(0) \supseteq h(k) \mid k \in f^{-1}(y)$. Which implies that $f(h)(0) \supseteq \bigcup h(k) \mid k \in f^{-1}(y) = f(h)(y), \forall y \in Y$. Hence, $f(h)(0) \supseteq f(h)(y), \forall y \in Y$. Thus $f(h)$ satisfies Definition 4.4.1 .
Assume that $f(h)(a) \subset f(h)(a \odot b) \cap f(h)(a \star b) \cap f(h)(b)$ for some $a, b \in Y$. Since f is an epimorphism there exists $x, y \in X$ such that $f(x) = a$ and $f(x) = b$.
Hence, $f(h)(f(x)) \subset f(h)(f(x \odot y)) \cap f(h)f(x \star y) \cap f(h)f(y)$.

Then $f^{-1}(f(h))(x) \subset f^{-1}(f(h))(x \odot y) \cap f^{-1}(f(h))(x \star y) \cap f^{-1}(f(h)(y))$.
But h is constant on $Kerf$.
Then by Lemma (4.5.3), we get $h(x) \subset h(x \odot y) \cap h(x \star y) \cap h(y)$ which is a contradiction with the fact that h is a hesitant fuzzy pseudo-ideal of X.
Therefore, $f(h)$ is a hesitant fuzzy pseudo-ideal of Y. □

4.6. Cartesian Product of Hesitant Fuzzy Pseudo ideal of Pseudo-TM Algebra

In this section, we introduce the notion of hesitant fuzzy pseudo-ideals in pseudo-TM algebras in terms of Cartesian product of any two pseudo-TM algebras.

Theorem 4.6.1. *Let* $(X,\odot,\star,0)$ *and* $(Y,\odot,\star,0)$ *be a pseudo-TM algebras and let* h *and* σ *be any two hesitant fuzzy pseudo-ideal of* X *and* Y *respectively. Then* $h \times \sigma$ *is a hesitant fuzzy pseudo-ideals of* $X \times Y$.

Proof. Suppose that h and σ be a hesitant fuzzy pseudo-TM algebras of X and Y respectively.
(a) Let $(x,y) \in X \times Y$. Then $(h \times \sigma)(0,0) = h(0) \cap \sigma(0) \supseteq h(x) \cap \sigma(y)$. Hence, $(h \times \sigma)(0,0) \supseteq (h \times \sigma)(x,y)$.
(b) Let $(x_1,x_2),(y_1,y_2) \in X \times Y$. Then

$$\begin{aligned}(h \times \sigma)(x_1,x_2) &= h(x_1) \cap \sigma(x_2)\\ &\supseteq h(x_1 \odot y_1) \cap h(x_1 \star y_1) \cap h(y_1) \cap \sigma(x_2 \odot y_2) \cap \sigma(x_2 \star y_2) \cap \sigma(y_2).\\ &= h(x_1 \odot y_1) \cap \sigma(x_2 \odot y_2) \cap h(x_1 \star y_1) \cap \sigma(x_2 \star y_2) \cap h(y_1) \cap \sigma(y_2).\\ &= (h \times \sigma)(x_1,x_2) \odot (y_1,y_2) \cap (h \times \sigma)(x_1,x_2) \star (y_1,y_2) \cap (h \times \sigma)(y_1,y_2)\end{aligned}$$

Hence, $(h \times \sigma)(x_1,x_2) \supseteq (h \times \sigma)(x_1,x_2) \odot (y_1,y_2) \cap (h \times \sigma)(x_1,x_2) \star (y_1,y_2) \cap (h \times \sigma)(y_1,y_2)$.
By (a) and (b) above, we have that $h \times \sigma$ is a hesitant fuzzy pseudo-ideal of X. □

Theorem 4.6.2. *Let* $(X,\odot,\star,0)$ *and* $(Y,\odot,\star,0)$ *be a pseudo-TM algebras. If* $h \times \sigma$ *be a hesitant fuzzy pseudo-ideal of* $X \times Y$ *for any hesitant fuzzy subset of* h *and* σ *of a pseudo-algebra of* X *and* Y *respectively, then either* h *is a hesitant fuzzy pseudo-ideal of* X *or* σ *be a hesitant fuzzy pseudo-ideal of* Y.

Proof. Suppose that $h \times \sigma$ is a hesitant fuzzy pseudo-ideal of $X \times Y$ such that h and σ be a hesitant fuzzy subset of X and Y respectively. Then $(h \times \sigma)(0,0) = h(0) \cap \sigma(0) \supseteq h(x) \cap \sigma(y)$ for all $(x,y) \in X \times Y$.
Suppose that $h(x) \supset h(0)$ and $\sigma(y) \supset \sigma(0)$ for some $(x,y) \in X \times Y$.
Now, $(h \times \sigma)(x,y) = min\{h(x),\sigma(y)\} \supset h(0) \cap \sigma(0) = (h \times \sigma)(0,0)$ which is a contradiction.
Thus, $h(0) \supseteq h(x)$ or $\sigma(0) \supseteq \sigma(x)$ for all $(x,y) \in X \times Y$.

Let $(x_1,x_2),(y_1,y_2) \in X\times Y$. Then

$$\begin{aligned}
(h\times\sigma)(x_1,x_2) &= h(x_1)\cap\sigma(x_2)\\
&\supseteq (h\times\sigma)(x_1,x_2)\odot(y_1,y_2)\cap(h\times\sigma)(x_1,x_2)\star(y_1,y_2)\cap(h\times\sigma)(y_1,y_2)\\
&= h(x_1\odot y_1)\cap h(x_1\star y_1)\cap h(y_1)\cap\sigma(x_2\odot y_2)\cap\sigma(x_2\star y_2)\cap\sigma(y_2).\\
&= h(x_1\odot y_1)\cap\sigma(x_2\odot y_2)\cap h(x_1\star y_1)\cap\sigma(x_2\star y_2)\cap h(y_1)\cap\sigma(y_2).\\
&\Rightarrow h(x_1)\supseteq h(x_1\odot y_1)\cap h(x_1\star y_1)\cap h(y_1)\\
&\text{or }\ \sigma(x_2)\supseteq\sigma(x_2\odot y_2)\cap\sigma(x_2\star y_2)\cap\sigma(y_2)
\end{aligned}$$

Hence, either $h(x_1)\supseteq h(x_1\odot y_1)\cap h(x_1\star y_1)\cap h(y_1)$
or $\sigma(x_2)\supseteq\sigma(x_2\odot y_2)\cap\sigma(x_2\star y_2)\cap\sigma(y_2)$. □

Theorem 4.6.3. *Let $(X;\odot,\star,0)$ and $(Y;\odot,\star,0)$ be a pseudo-TM algebras and let h and σ be any two hesitant fuzzy pseudo-ideal of X and Y respectively. Then $h\times\sigma$ is a hesitant fuzzy pseudo-ideals of $X\times Y$ if and only if the non-empty upper $\Lambda-$level set $U(h\times\sigma,t)$ is a pseudo-ideal of $X\times Y$ for any $\Lambda\in\mathcal{P}[0,1]$.*

Proof. Let $h\times\sigma$ be a fuzzy pseudo-ideals of $X\times Y$ such that $U(h\times\sigma,\Lambda)\neq\emptyset$, for all $\Lambda\in[0,1]$. We need to show that $U(h\times\sigma,t)$ is a pseudo-ideal of $X\times Y$ for any $\Lambda\in\mathcal{P}[0,1]$. Then there exists $(x,y)\in U(h\times\sigma,\Lambda)$. Thus $(h\times\sigma)(x,y)\supseteq\Lambda$. Since $h\times\sigma$ is a fuzzy pseudo-ideal of $X\times Y$. We have $(h\times\sigma)(0,0)\supseteq(h\times\sigma)(x,y)$ for all $(x,y)\in X\times Y$.
$\Rightarrow (h\times\sigma)(0,0)\supseteq(h\times\sigma)(x,y)\supseteq\Lambda$
$\Rightarrow (h\times\sigma)(0,0)\supseteq\Lambda$
$\Rightarrow (0,0)\in U(h\times\sigma,t)$
Suppose that $(x_1,y_1),(x_2,y_2)\in U(h\times\sigma,\Lambda)$ such that $(x_1,y_1)\odot(x_2,y_2),(x_1,y_1)\star(x_2,y_2),(x_2,y_2)\in U(h\times\sigma,\Lambda)$
$\Rightarrow (h\times\sigma)((x_1,y_1)\odot(x_2,y_2))\supseteq\Lambda,(h\times\sigma)((x_1,y_1)\star(x_2,y_2))\supseteq\Lambda$ and $(h\times\sigma)((x_2,y_2)\supseteq\Lambda$.
Since $h\times\sigma$ is a fuzzy pseudo-ideal of $X\times Y$. We have

$$\begin{aligned}
(h\times\sigma)(x_1,y_1) &\supseteq \min\{(h\times\sigma)((x_1,y_1)\odot(x_2,y_2)),(h\times\sigma)((x_1,y_1)\star(x_2,y_2)),(h\times\sigma)(x_2,y_2)\}\\
&\supseteq \min\{\Lambda,\Lambda,\Lambda\}=\Lambda\\
&\Rightarrow (h\times\sigma)(x_1,y_1)\supseteq\Lambda\\
&\Rightarrow (x_1,y_1)\in U(h\times\sigma,\Lambda)
\end{aligned}$$

Hence, $U(h\times\sigma,\Lambda)$ is a pseudo-ideal of $X\times Y$.
Conversely, assume that the set $U(h\times\sigma,\Lambda)$ be a pseudo-ideal of $X\times Y$, for each $\Lambda\in[0,1]$.
Let $(x,y)\in X\times Y$ such that $(h\times\sigma)=\Lambda$. Then $(x,y)\in U(h\times\sigma,\Lambda)$. Since $(h\times\sigma)$ is a fuzzy pseudo-ideal of $X\times Y$, then $(0,0)\in U(h\times\sigma,\Lambda)$.
Hence, $(h\times\sigma)(0,0)\supseteq\Lambda=(h\times\sigma)(x,y)$ for all $(x,y)\in X\times Y$. Assume that
$\Lambda_1=\frac{1}{2}\{(h\times\sigma)(x,x)+\min\{(h\times\sigma)((x,x)\odot(x_1,y_1)),(h\times\sigma)((x,x)\star(x_1,y_1)),(h\times\sigma)(x_1,y_1)\}\}$.
Then $(h\times\sigma)(x,x)\supset\Lambda_1<\min\{(h\times\sigma)((x,x)\odot(x_1,y_1)),(h\times\sigma)((x,x)\star(x_1,y_1)),(h\times\sigma)(x_1,y_1)\}$.

It follows that $(x,x) \neq U(h \times \sigma, t_1)$. Also, $(h \times \sigma)((x,x) \odot (x_1,y_1)), (h \times \sigma)((x,x) \star (x_1,y_1)) \in U(h \times \sigma, \Lambda_1)$ and $(h \times \sigma)(x_1,y_1) \in U(h \times \sigma, \Lambda_1)$ which is a contradiction. Since $U(h \times \sigma, \Lambda_1)$ is a pseudo-ideal of $X \times Y$. Therefore, $h \times \sigma$ is a fuzzy pseudo-ideal of $X \times Y$. □

Theorem 4.6.4. *Let* $(X;\odot,\star,0)$ *and* $(Y;\odot,\star,0)$ *be a pseudo-TM algebras and let* h *and* σ *be any two hesitant fuzzy pseudo-ideal of* X *and* Y *respectively such that* $h \times \sigma$ *is a hesitant fuzzy pseudo-ideals of* $X \times Y$*.Then:*
i) either $h(0) \supseteq h(x)$ *or* $\sigma(0) \supseteq \sigma(y)$*, for all* $(x,y) \in X \times Y$.
ii) If $(0) \supseteq h(x)$*, for all* $x \in X$*, then either* $\sigma(0) \supseteq \sigma(y)$ *or* $\sigma(0) \supseteq h(x)$*, for all* $(x,y) \in X \times Y$.
iii) If $\sigma(0) \supseteq \sigma(y)$*, for all* $y \in Y$*, then either* $h(0) \geqslant h(x)$ *or* $h(0) \supseteq \sigma(y)$*, for all* $(x,y) \in X \times Y$.

Proof. Let $h \times \sigma$ be a hesitant fuzzy pseudo- ideal of $X \times Y$. Then
i) Suppose that $h(0) \subset h(x)$, for some $x \in X$ and $\sigma(0) \subset \sigma(y)$, for some $y \in Y$. Then $(h \times \sigma)(x,y) = h(x) \cap \sigma(y) \supset h(0) \cap \sigma(0) = (h \times \sigma)(0,0)$. This is a contradiction to the fact that $h \times \sigma$ is a hesitant fuzzy pseudo ideal of $X \times Y$.
Therefore, either $h(0) \supseteq h(x)$ or $\sigma(0) \supseteq \sigma(y)$, for all $(x,y) \in X \times Y$.
ii) Suppose that $h(0) \supseteq h(x)$, for all $x \in X$. Assume that there exist $x \in X$ and $y \in Y$ such that $\sigma(0) \subset h(x)$ and $\sigma(0) \subset \sigma(y)$. Then $\sigma(0) \subset h(x) \subseteq h(0)$. Thus $(h \times \sigma)(x,y) = h(x) \cap \sigma(y) \supset \sigma(0) \cap \sigma(0) = \sigma(0) = h(0) \cap \sigma(0) = (h \times \sigma)(0,0)$. This is a contradiction since $h \times \sigma$ is a hesitant fuzzy pseudo ideal of $X \times Y$. Hence, either $\sigma(0) \supseteq \sigma(y)$, for all $y \in Y$ or $\sigma(0) \supseteq h(x)$, for all $x \in X$.
iii) Assume that $\sigma(0) \supseteq \sigma(y)$, for all $y \in Y$. Assume that there exist $x \in X$ and $y \in Y$ such that $h(0) \subset h(x)$ and $h(0) \subset \sigma(y)$. Then $h(0) \subset \sigma(y) \subseteq \sigma(y)$. Thus $(h \times \sigma)(x,y) = h(x) \cap \sigma(y) \subseteq h(0) \cap h(0) = h(0) = h(0) \cap \sigma(0) = (h \times \sigma)(0,0)$ which is a contradiction.
Hence, either $h(0) \supseteq h(x)$, for all $x \in X$ or $h(0) \supseteq \sigma(y)$, for all $y \in Y$. □

Chapter 5
Bipolar Hesitant Fuzzy Soft Set in TM-Algebra with Multicriteria Decision Making

In this chapter, we introduce bipolar hesitant fuzzy soft sets in TM-algebras and discuss their applications in real-world problems. The combination of bipolarity (positive and negative views), hesitation (multiple values), and soft sets (parameter-based uncertainty) creates a powerful framework. These structures are applied to decision-making problems, when both satisfaction and dissatisfaction need to be considered. A numerical example is provided for selecting the best alcoholic drink based on multiple criteria, such as taste, health impact, and cost. Each criterion is evaluated with both positive and negative opinions, along with hesitation. The bipolar hesitant fuzzy soft set model is used to combine these opinions and find the best option. This shows the practical usefulness of the theoretical framework developed in this work.

5.1. Hesitant Fuzzy Soft TM-Subalgebra of TM -algebra

In this section, we introduce the concepts of hesitant fuzzy soft TM-subalgebras of TM -algebras and provide some interesting properties. In what follows let U be a universal set, let E be a set of parameters and A and B are a subset of E and we take a TM-algebra X as a reference set unless otherwise specified.

Definition 5.1.1. *For a subset A of E, a hesitant fuzzy soft set (ξ_h, A) over X is called a hesitant fuzzy soft subalgebra based on $\epsilon \in A$ (briefly, ϵ-hesitant fuzzy soft subalgebra) over X if the hesitant fuzzy set*

$$\xi_h[\epsilon] = \left\{ \left(x, h_{\xi_h[\epsilon]}(x)\right) \mid x \in X \right\}$$

on X is a hesitant fuzzy subalgebra of X. If $(\xi_h[\epsilon], A)$ is an ϵ hesitant fuzzy soft subalgebra over X for all $\epsilon \in A$, we say that (ξ_h, A) is a hesitant fuzzy soft subalgebra.

Example 5.1.1. *Let $X = \{0,1,2\}$ with the binary operation "$\star$" defined by the following Cayley table:*

$\star$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	0

Table 5.1

See [43], where it is shown that $(X,\star,0)$ is a TM–algebra.
Let $E = \{e_1, e_2\}$ be a set of parameters and let $A = \{e_1, e_2\} \subseteq E$. Define a hesitant fuzzy soft set (ξ_h, A) over X by

$$\xi_h(e_1) = \left\{ \begin{matrix} (0,\{0.8,0.9\}), \\ (1,\{0.6,0.7\}), \\ (2,\{0.5,0.6\}) \end{matrix} \right\}, \qquad \xi_h(e_2) = \left\{ \begin{matrix} (0,\{0.9,1.0\}), \\ (1,\{0.7,0.8\}), \\ (2,\{0.6,0.7\}) \end{matrix} \right\}.$$

For the parameter $\epsilon = e_1$, the corresponding hesitant fuzzy set on X is

$$\xi_h[e_1] = \{(0,\{0.8,0.9\}),(1,\{0.6,0.7\}),(2,\{0.5,0.6\})\}.$$

We verify that $\xi_h[e_1]$ is a hesitant fuzzy subalgebra of X. For all $x,y \in X$,

$$h_{\xi_h[e_1]}(x \star y) \supseteq h_{\xi_h[e_1]}(x) \cap h_{\xi_h[e_1]}(y).$$

Now,

$$\begin{aligned} \text{Let} \quad x = 1, y = 2. \text{ Then} \quad & 1 \star 2 = 1, \text{ so} \\ & h_{\xi_h[e_1]}(1 \star 2) = \{0.6,0.7\}, \\ & h_{\xi_h[e_1]}(1) \cap h_{\xi_h[e_1]}(2) = \{0.6\} \subseteq \{0.6,0.7\}. \end{aligned}$$

All other combinations satisfy the same inclusion, so $\xi_h[e_1]$ is a hesitant fuzzy subalgebra of X.
Similarly, for $\epsilon = e_2$,

$$\xi_h[e_2] = \{(0,\{0.9,1.0\}),(1,\{0.7,0.8\}),(2,\{0.6,0.7\})\},$$

and by similar computation,

$$h_{\xi_h[e_2]}(x \star y) \supseteq h_{\xi_h[e_2]}(x) \cap h_{\xi_h[e_2]}(y), \quad \forall x,y \in X.$$

Hence, $\xi_h[e_2]$ is also a hesitant fuzzy subalgebra of X.
Therefore, since both $\xi_h[e_1]$ and $\xi_h[e_2]$ are hesitant fuzzy subalgebras of X, the hesitant fuzzy soft set (ξ_h, A) is an ϵ-hesitant fuzzy soft subalgebra of X for all $\epsilon \in A$. Thus, (ξ_h, A) is a hesitant fuzzy soft subalgebra of X.

Example 5.1.2. *Let* $X = \{0, 1, 2, 3, 4\}$ *with the following Cayley table:*

$\star$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	1	1	1
2	2	1	2	2	2
3	3	1	2	3	3
4	4	1	2	3	4

Table 5.2

See [43] $(X, \star, 0)$ *is a TM -algebra. Define a hesitant fuzzy soft Set* $A = \{\epsilon_1, \epsilon_2\}$*. Now define the hesitant fuzzy set values for each element in* X*, under each parameter in* A*.*

$\xi_h[\epsilon]$	0	1	2	3	4
ϵ_1	$\{1.0\}$	$\{0.9, 1.0\}$	$\{0.8\}$	$\{0.8\}$	$\{0.7\}$
ϵ_2	$\{1.0\}$	$\{0.85\}$	$\{0.75, 0.8\}$	$\{0.7\}$	$\{0.6\}$

Hence, both $\xi_h[\epsilon_1]$ *and* $\xi_h[\epsilon_2]$ *are hesitant fuzzy subalgebras. The hesitant fuzzy soft set* (ξ_h, A) *has each* $\xi_h[\epsilon]$ *being a hesitant fuzzy subalgebra.*
Therefore, (ξ_h, A) *is a hesitant fuzzy soft subalgebra over* X*.*

Theorem 5.1.1. *If* (ξ_h, A) *is a hesitant fuzzy soft subalgebra over a TM-algebra* X*, then*

$$(\forall x \in X) \quad \left(h_{\xi_h[\epsilon]}(0) \supseteq h_{\xi_h[\epsilon]}(x)\right),$$

where ϵ *is any parameter in* A*.*

Proof. Assume that (ξ_h, A) is a hesitant fuzzy soft subalgebra of X and let $\epsilon \in A$ and $x \in X$. Then $\xi_h[\epsilon]$ is a hesitant fuzzy subalgebra of X. Thus,

$$h_{\xi_h[\epsilon]}(0) = h_{\xi_h[\epsilon]}(x \star x) \supseteq h_{\xi_h[\epsilon]}(x) \cap h_{\xi_h[\epsilon]}(x) h_{\xi_h[\epsilon]}(x).$$

□

Definition 5.1.2. *Let* (ξ_h, A) *be a hesitant fuzzy soft set over a universe X, where* $A \subseteq E$ *and E is a set of parameters. If* $\Omega \subseteq A$*, then the restriction of* (ξ_h, A) *to* Ω *is denoted by* $(\xi_h|_\Omega, \Omega)$ *and is defined as a mapping* $\xi_h|_\Omega : \Omega \longrightarrow HFS(X)$ *such that* $\xi_h|_\Omega(\epsilon) = \xi_h(\epsilon)$*, for all* $\epsilon \in \Omega$*.*

Proposition 5.1.2. *Let* (ξ_h, A) *be a hesitant fuzzy soft subalgebra over* X*. If* Ω *is a subset of* A*, then* ($\xi_h|_\Omega, \Omega$) *is a hesitant fuzzy soft subalgebra over* X*.*

Proof. Let X be a TM -algebra and A be a set of parameters. (ξ_h, A) be a hesitant fuzzy soft set over X, where

$$\xi_h : A \rightarrow HFS(X)$$

assigns to each parameter $\epsilon \in A$ a hesitant fuzzy subset $\xi_h(A)$ of X. Further, suppose that (ξ_h, A) is a hesitant fuzzy soft subalgebra, which means for each $A \in A$, $\xi_h(A)$ is a hesitant fuzzy subalgebra of X. Consider $\Omega \subseteq A$ we aim to show

$$(\xi_h|_\Omega, \Omega) \text{ is a hesitant fuzzy soft subalgebra over } X$$

Since for every $\epsilon \in \Omega \subseteq A$, we already have that $\xi_h(A)$ is a hesitant fuzzy subalgebra of X (because it was true for all elements of A). Then for every $A \in \Omega$, $\xi_h|_\Omega(A) = \xi_h(A)$ is also a hesitant fuzzy subalgebra of X.
Hence, the pair $(\xi_h|_\Omega, \Omega)$ satisfies the definition of a hesitant fuzzy soft subalgebra. □

Example 5.1.3. *Let* $X = \{0,1,2\}$ *with the following Cayley table:*

$\star$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	1

Table 5.3

See [43] $(x,\star,0)$ *is a TM -algebra.*
Define $A = \{\epsilon_1, \epsilon_2\}$, $\Omega = \{\epsilon_1\}$ *the hesitant fuzzy soft set* $\xi_h(\epsilon)$ *defined by*

$$\xi_h(\epsilon_1)(0) = \{0.8\}, \quad \xi_h(\epsilon_1)(1) = \{0.6\}, \quad \xi_h(\epsilon_1)(2) = \{0.5\}$$
$$\xi_h(\epsilon_2)(0) = \{0.2\}, \quad \xi_h(\epsilon_2)(1) = \{0.05\}, \quad \xi_h(\epsilon_2)(2) = \{0.1\}$$

Then $(\xi_h|_\Omega, \Omega)$ *is a hesitant fuzzy soft subalgebra but* (ξ_h, A) *is not because* ϵ_2 *fails the condition.*

Definition 5.1.3. *Let* (ξ_h, A) *be a hesitant fuzzy soft set over* X. *For each* $\Lambda \in \mathcal{P}[0,1]$, *the set* $\mathcal{U}(\xi_h, A)^\Lambda = \mathcal{U}\left(\xi_h^\Lambda, A\right)$ *is called an* Λ*-level soft set of* (ξ_h, A) *and is defined by* $\mathcal{U}(\xi_{\epsilon h}^\Lambda) = \{x \in X \mid \xi_{hA\epsilon}(x) \supseteq \Lambda,\ \xi_{hA\epsilon}(x) \subseteq \Lambda\}$ *for all* $\epsilon \in A$. *Clearly,* $(\xi_h, A)^\Lambda$ *is a soft set over* X.

Theorem 5.1.3. *Let* (ξ_h, A) *be a hesitant fuzzy soft subalgebra over* X. (ξ_h, A) *is a hesitant fuzzy soft subalgebra of* X *if and only if* $\mathcal{U}(\xi_h, A)^\Lambda$ *is a subalgebra over* X *for each* $\Lambda \in \mathcal{P}[0,1]$.

Proof. Suppose that (ξ_h, A) is a hesitant fuzzy soft subalgebra of X. For each $\Lambda \in \mathcal{P}[0,1]$, $\epsilon \in A$ and let $x, y \in \xi_h^\Lambda$. Then $h_{\xi_h[\epsilon]}(x) \supseteq \Lambda$ and $h_{\xi_h[\epsilon]}(y) \supseteq \Lambda$. Thus

$$h_{\xi_h[\epsilon]}(x \star y) \supseteq h_{\xi_h[\epsilon]}(x) \cap h_{\xi_h[\epsilon]}(y) \supseteq \Lambda.$$

This implies that $x \star y \in \mathcal{U}(\xi_h, A)^\Lambda$.
Therefore, $\mathcal{U}(\xi_h, A)^\Lambda$ is a subalgebra over X.
Conversely, assume that $\mathcal{U}(\xi_h, A)^\Lambda$ is a subalgebra over X for each $\Lambda \in \mathcal{P}[0,1]$. For each $\epsilon \in A$. Let $x, y \in X$. Assume that $h_{\xi_h[\epsilon]}(x) = \Lambda_x$ and $h_{\xi_h[\epsilon]}(y) = \Lambda_y$. Take $\Lambda = \Lambda_x \cap \Lambda_y$. Then $x, y \in \xi_h^\Lambda$

and so $x \star y \in \mathcal{U}((\xi_h, A))^{\Lambda}$.
Hence,

$$h_{\xi_h[\epsilon]}(x \star y) \supseteq \Lambda = \Lambda_x \cap \Lambda_y = h_{\xi_h[\epsilon]}(x) \cap h_{\xi_h[\epsilon]}(y)$$

Then by Definition 4.1.2 (ξ_h, A) is a hesitant fuzzy subalgebra over X.
Therefore, (ξ_h, A) is a hesitant fuzzy soft subalgebra of X. □

Definition 5.1.4. *Let* X *be a reference set. If* $\xi_h \subseteq X$, *the characteristic hesitant fuzzy soft set* χ_{ξ_h} *on* X *is a function of* X *into* $\mathcal{P}[0,1]$ *defined as for all* $x \in X$

$$\chi_{\xi_h}(x) = \begin{cases} [0,1], & \text{if } x \in \xi_h \\ \emptyset, & \text{otherwise} \end{cases}$$

Theorem 5.1.4. *A non-empty subset* ξ_h *of X is a subalgebra of* X *if and only if the characteristic hesitant fuzzy soft set* χ_{ξ_h} *is a hesitant fuzzy soft subalgebra of* X.

Proof. Assume that ξ_h is subalgebra of X. Assume that $x, y \in X$. We proceed by examining three distinct cases.

(i) Let $x, y \in \xi_h$.
Hence, $\chi_{\xi_h}(x) = [0,1]$ and $\chi_{\xi_h}(y) = [0,1]$. So, $\chi_{\xi_h}(x) \cap \chi_{\widetilde{\mathcal{H}}}(y) = [0,1]$. Since ξ_h is a subalgebra of X. We have $x \star y \in \xi_h$. As a result, $\chi_{\xi_h}(x \star y) = [0,1]$.
Therefore, $\chi_{\xi_h}(x \star y) \supseteq \chi_{\xi_h}(x) \cap \chi_{\xi_h}(y)$.

(ii) Let $x \in \xi_h$ and $y \notin \xi_h$. Then, $\chi_{\xi_h}(x) = [0,1]$ and $\chi_{\xi_h}(y) = \varnothing$. So, $\chi_{\xi_h}(x) \cap \chi_{\xi_h}(y) = \varnothing$.
Hence, $\chi_{\xi_h}(x \star y) \supseteq \varnothing$.
Therefore, $\chi_{\xi_h}(x \star y) \supseteq \chi_{\xi_h}(x) \cap \chi_{\widetilde{\mathcal{H}}}(y)$.

(iii) Let $x \notin \xi_h$ and $y \notin \xi_h$. Then, $\chi_{\xi_h}(x) = \varnothing$ and $\chi_{\xi_h}(y) = \varnothing$. So, $\chi_{\xi_h}(x) \cap \chi_{\xi_h}(y) = \varnothing$.
Hence, $\chi_{\xi_h}(x \star y) \supseteq \varnothing$.
Therefore, $\chi_{\xi_h}(x \star y) \supseteq \chi_{\xi_h}(x) \cap \chi_{\xi_h}(y)$.

Again, consider the following three cases.

(i) Let $x, y \in \xi_h$.
Hence, $\chi_{\xi_h}(x) = [0,1]$ and $\chi_{\xi_h}(y) = [0,1]$. So, $\chi_{\xi_h}(x) \cap \chi_H(y) = [0,1]$. Since ξ_h is a subalgebra of X. We have $x \star y \in \xi_h$. As a result, $\chi_{\xi_h}(x \star y) = [0,1]$.
Therefore, $\chi_{\xi_h}(x \star y) \supseteq \chi_{\xi_h}(x) \cap \chi_{\xi_h}(y)$.

(ii) Let $x \in \xi_h$ and $y \notin \xi_h$.
Then, $\chi_{\xi_h}(x) = [0,1]$ and $\chi_{\xi_h}(y) = \varnothing$. So, $\chi_{\xi_h}(x) \cap chi_{\xi_h}(y) = \varnothing$.
Hence, $\chi_{\xi_h}(x \star y) \supseteq \varnothing$.
Therefore, $\chi_{\xi_h}(x \star y) \supseteq \chi_{\xi_h}(x) \cap \chi_{\xi_h}(y)$.

(iii) Let $x \notin \xi_h$ and $y \notin \xi_h$.
Then, $\chi_{\xi_h}(x) = \varnothing$ and $\chi_{\xi_h}(y) = \varnothing$. So, $\chi_{\xi_h}(x) \cap \chi_{\xi_h}(y) = \varnothing$.
Hence, $\chi_{\xi_h}(x \star y) \supseteq \varnothing$. Therefore, $\chi_{\xi_h}(x \star y) \supseteq \chi_{\xi_h}(x) \cap \chi_{\xi_h}(y)$.

Conversely, assume that χ_{ξ_h} is a hesitant fuzzy soft subalgebra of X. We need to show that ξ_h is a subalgebra of X.

(i) We need to show that $0 \in \xi_h$. Since $\chi_{\xi_h}(0) \supseteq \chi_{\xi_h}(x)$, for all $x \in X$. By Theorem 5.1.1, we have $0 \in \xi_h$.

(ii) Let $x, y \in \xi_h$. Then, $\chi_{\xi_h}(x) = [0,1]$ and $\chi_{\xi_h}(y) = [0,1]$. Since χ_{ξ_h} is a hesitant fuzzy soft subalgebra of X. Then $\chi_{\xi_h}(x \star y) \supseteq \chi_{\xi_h}(x) \cap \chi_{\xi_h}(y) = [0,1]$. Thus, $\chi_{\xi_h}(x \star y) = [0,1]$.
Hence, $x \star y \in \xi_h$.

(iii) Let $x \in \xi_h$ and $y \notin \xi_h$.
Then, $\chi_{\xi_h}(x) = [0,1]$ and $\chi_{\xi_h}(y) = \varnothing$. So, $\chi_{\xi_h}(x) \cap \chi_{\xi_h}(y) = \varnothing$.
Hence, $\chi_{\xi_h}(x \star y) \supseteq \varnothing$.
Therefore, $x \star y \in \xi_h$.

(iv) Let $x \notin \xi_h$ and $y \notin \xi_h$.
Then, $\chi_{\xi_h}(x) = \varnothing$ and $\chi_{\xi_h}(y) = \varnothing$. So, $\chi_{\xi_h}(x) \cap \chi_{\xi_h}(y) = \varnothing$.
So, $\chi_{\xi_h}(x \star y) \supseteq \varnothing$. Hence, $x \star y \in \xi_h$.

Therefore, ξ_h is subalgebra of X. □

Definition 5.1.5. *Let* $(\xi_{1h}, A$ *) and* $(\xi_{2h}, B$ *) be two hesitant fuzzy soft sets over* X*, then* $(\xi_{1h}, A)\,\mathrm{AND}(\xi_{2h}, B)$ *is a hesitant fuzzy soft set denoted by* $(\xi_{1h}, A) \wedge (\xi_{2h}, B)$*, and is defined by* $(\xi_{1h}, A) \wedge (\xi_{2h}, B) = (\xi_{1h}, A \times B)$ *where,* $\xi_h(x, y) = \xi_h(x) \cap \xi_h(y)$*, for all* $(a, b) \in A \times B$*. Here* $\cap$ *is the operation of a hesitant fuzzy soft intersection.*

Definition 5.1.6. *Let* $(\xi_{1h}, A$ *) and* $(\xi_{2h}, B$ *) be two hesitant fuzzy soft sets over* X*. Then their extended intersection is a hesitant fuzzy soft set denoted by* (φ, C)*, where* $C = A \cup B$ *and*

$$\xi_h(\epsilon) = \begin{cases} \xi_{1h}(\epsilon) & \textit{if } \epsilon \in A - B, \\ \xi_{2h}(\epsilon) & \textit{if } \epsilon \in B - A, \\ \xi_{1h}(\epsilon) \cap \xi_{2h}(\epsilon) & \textit{if } \epsilon \in A \cap B, \end{cases}$$

for all $\epsilon \in C$*. This is denoted by* $(\xi_h, C) = (\xi_{1h}, A)\widetilde{\cap}(\xi_{2h}, B)$*.*

Theorem 5.1.5. *Let* (ξ_{1h}, A) *and* (ξ_{2h}, B) *be a hesitant fuzzy soft subalgebras over a TM -algebra, then* $(\xi_{1h}, A)\widetilde{\cap}(\xi_{2h}, B)$ *and* $(\xi_{1h}, A) \wedge (\xi_{2h}, B)$ *are a hesitant fuzzy soft TM-subalgebras over a TM-algebra.*

Proof. Suppose that (ξ_{1h}, A) and (ξ_{2h}, B) are hesitant fuzzy soft TM-subalgebras over a TM-algebra X. We need to show that (ξ_h, C) is a hesitant fuzzy soft subalgebra of X.

We proceed by examining three distinct cases.
Case 1: For $\epsilon \in A - B$ or $\epsilon \in B - A$, the values come directly from ξ_{1h} or ξ_{2h}, which are TM-subalgebras. So closure is preserved.
Case 2: For $\epsilon \in A \cap B$, and $x, y \in X$. We have $h_{\xi_h(\epsilon)}(x \star y) = h_{\xi_{1h}(\epsilon)}(x \star y) \cap h_{\xi_{2h}(\epsilon)}(x \star y)$ and

$$h_{\xi_h(\epsilon)}(x) = h_{\xi_h(\epsilon)}(x) \cap h_{\xi_{2h}(\epsilon)}(x), \quad h_{\xi_h(\epsilon)}(y) = h_{\xi_{1h}(\epsilon)}(y) \cap h_{\xi_{2h}(\epsilon)}(y)$$

Using the closure property of hesitant fuzzy subalgebras

$$h_{\xi_{ih}(\epsilon)}(x \star y) \supseteq h_{\xi_{ih}(\epsilon)}(x) \cap h_{\xi_{ih}(\epsilon)}(y),\ i = 1,2.$$

Then

$$h_{\xi_h(\epsilon)}(x \star y) \supseteq h_{\xi_{1h}(\epsilon)}(x) \cap h_{\xi_h(\epsilon)}(y)$$

So the closure is preserved in intersection.
Therefore, $(\xi_{1h}, A)\widetilde{\cap}(\xi_{2h}, B)$ is a hesitant fuzzy soft TM-subalgebra over X.
Similarly, it is easly prove that $(\xi_{1h}, A) \wedge (\xi_{2h}, B)$ is a hesitant fuzzy soft TM-subalgebra over X. □

Definition 5.1.7. *Let $(\xi_{1h}, A\ )$ and $(\xi_{2h}, B\ )$ be two hesitant fuzzy soft sets over X. Then their extended union denoted by (ξ_h, C), where $C = A \cup B$ and*

$$\xi_h(\epsilon) = \begin{cases} \xi_{1h}(\epsilon) & \text{if } \epsilon \in A - B, \\ \xi_{2h}(\epsilon) & \text{if } \epsilon \in B - A, \\ \xi_{1h}(\epsilon) \cup \xi_{2h}(\epsilon) & \text{if } \epsilon \in A \cap B, \end{cases}$$

for all $\epsilon \in C$. This is denoted by $(\xi_h, C) = (\xi_{1h}, A)\widetilde{\cup}(\xi_{1h}, B)$.

Theorem 5.1.6. *Let (ξ_{1h}, A) and (ξ_{2h}, B) be a hesitant fuzzy soft TM- subalgebras over X. If $A \cap B = \emptyset$, then the union of $(\xi_{1h}, A)\widetilde{\cup}(\xi_{2h}, B)$ is a hesitant fuzzy soft TM-subalgebra over X.*

Proof. Using 5.1.7, we can write $(\xi_h, C) = (\xi_{1h}, A)\tilde{\cup}(\xi_{2h}, B)$, where $C = A \cup B$ and for all $x \in C$,

$$\xi_{1h}(x) = \begin{cases} \xi_{1h}(x), \text{ if } x \in A - B \\ \xi_{2h}(x), \text{ if } x \in B - A \\ \xi_{1h}(x) \cup \xi_{2h}(x), \text{ if } x \in A \cap B \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A - B$ and $x \in B - A$ for all $x \in C$. We will consider the following cases:
Case 1: If $x \in A - B$, then $\widetilde{\mathcal{H}}_1(x)$ is a hesitant fuzzy soft TM-subalgebra of X because of (ξ_{1h}, A) is a hesitant fuzzy soft TM-subalgebra over X.
Case 2: If $x \in B - A$, then ξ_{2h} is a a hesitant fuzzy soft TM-subalgebra of X because of (ξ_{2h}, B) is a soft PMS-algebra over X. Hence $(\mathcal{H}, C) = (\xi_{1h}, A)\widetilde{\cup}(\xi_{2h}, B)$ is a hesitant fuzzy soft TM-subalgebra over X. □

Definition 5.1.8. *Let* (ξ_{1h}, A) *be a hesitant fuzzy soft set over* X *and* (ξ_{2h}, B) *be a hesitant fuzzy soft set over* Y*. The Cartesian product of (* ξ_{1h}, A *) and (* ξ_{2h}, B *) is defined as a hesitant fuzzy soft set* $(\xi_h, A \times B) = (\xi_{1h}, A) \times (\xi_{2h}, B)$*, where* $\xi_h(x, y) = \xi_{1h}(x) \times \xi_{2h}(y)$*, for all* $(x, y) \in A \times B$.

Theorem 5.1.7. *Let* (ξ_{1h}, A) *and* (ξ_{2h}, B) *be hesitant fuzzy soft TM-subalgebra over* X *and* Y*, respectively. Then the Cartesian product* $(\xi_{1h}, A) \times (\xi_{2h}, B)$ *is a hesitant fuzzy soft soft TM-subalgebra over* $X \times \mathcal{Y}$.

Proof. Suppose that $(\xi_{1h}, A) \times (\xi_{1h}, B) = (\xi_h, C)$, where $C = A \times B$ and $\xi_h(x, y) = \xi_{1h}(x) \times \xi_{2h}(y)$. Let $a = (x, y) \in \sup(\xi_h, C)$. Then $\xi_h(x, y) \neq \emptyset$ for all $(x, y) \in C$. Since (ξ_{1h}, A) and (ξ_{2h}, B) are hesitant fuzzy soft TM-subalgebra, then $\xi_{1h}(x)$ and $\xi_{1h}(y)$ are a hesitant fuzzy soft TM-subalgebras of X and Y rspectively, for all $x \in A$ and for all $y \in B$. So we have that $\xi_{1h}(x) \times \xi_{1h}(y)$ is a hesitant fuzzy soft TM-subalgebra of $X \times Y$.
Hence, the Cartesian product $(\xi_{1h}, A) \times (\xi_{2h}, B)$ is a hesitant fuzzy soft soft TM-subalgebra over $X \times Y$. □

Corollary 5.1.8. *Let* (ξ_{1h}, A) *be a hesitant fuzzy soft* TM*-subalgebra over* X *and let* $\{(\xi_{ih}, B_i) \mid i \in I\}$ *be a non-empty family of a hesitant fuzzy soft* X*-subalgebras of (* ξ_{1h}, A *). Then*

1) $\widetilde{\bigcap}_{i \in I}(\xi_{ih}, B_i)$ *is a hesitant fuzzy soft* TM*-subalgebra of* (ξ_{1h}, A)*,*

2) $\bigwedge_{i \in I}(\xi_{ih}, B_i)$ *is a hesitant fuzzy soft* TM*-subalgebra of* $\bigwedge_{i \in I}(\xi_{1h}, A)$*,*

Theorem 5.1.9. *Let* (ξ_{2h}, B) *be a hesitant fuzzy soft* TM*-subalgebra over a TM -algebra Y and let (* f, g *) be a hesitant fuzzy soft homomorphism from* X *to* Y*. Then* $(f, g)^{-1}(\xi_{2h}, B)$ *is a hesitant fuzzy soft TM-subalgebra over* X.

Proof. Assume that let (f, g) be a hesitant fuzzy soft homomorphism from X to Y. We need to show that $(f, g)^{-1}(\xi_{2h}, B)$ is a hesitant fuzzy soft TM-subalgebra over X. Let $x_1, x_2 \in X$, then

$$\begin{aligned} f^{-1}\left(h_{g\xi_{2h}}\right)(x_1 \star x_2) &= h_{\xi_{2h}}\left(f\left(x_1 \star x_2\right)\right) = h_{g\widetilde{\mathcal{H}}_2}\left(f\left(x_1\right) \star f\left(x_2\right)\right) \\ &\supseteq h_{g\xi_{2h}}\left(f\left(x_1\right)\right) \cap h_{g\xi_{2h}}\left(f\left(x_2\right)\right) \\ &= f^{-1}\left(h_{g\xi_{2h}}\right)(x_1) \cap f^{-1}\left(h_{g\xi_{2h}}\right)(x_2) \end{aligned}$$

Therefore, $(f, g)^{-1}(\xi_{2h}, B)$is a hesitant fuzzy soft TM-subalgebra over X. □

5.2. Hesitant fuzzy Soft T-ideals of TM -algebra

In this section, we introduce the concept of hesitant fuzzy soft T-ideals in TM-algebra and investigate their properties.

Definition 5.2.1. *A hesitant fuzzy set* $\xi_h = \left\{\left(x, h_{\xi_h}(x)\right) : x \in X\right\}$ *is called a hesitant fuzzy ideal of* X *if for every* $x, y \in X$ *satisfies*
(1) $h_{\xi_h}(0) \supseteq h_{\widetilde{\mathcal{H}}}(x)$ *and ,*
(2) $h_{\xi_h}(x) \supseteq h_{\xi_h}(x \star y) \cap h_{\xi_h}(y)$.

Definition 5.2.2. *A hesitant fuzzy set* $\xi_h = \{(x, h_{\xi_h}(x)) : x \in X\}$ *is called a hesitant fuzzy soft* T*-ideal of* X *if for every* $x, y, z \in X$ *satisfies*
(1) $h_{\xi_h}(0) \supseteq h_{\xi_h}(x)$ *and,*
(2) $h_{\xi_h}(x \star z) \supseteq h_{\xi_h}((x \star y) \star z) \cap h_{\xi_h}(y)$.

Example 5.2.1. *Let* $X = \{0, 1, 2, 3\}$ *with the following Cayley table:*

$\star$	0	1	2	3
0	0	1	2	3
1	2	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 5.4

See [42] $(X, \star, 0)$ is a TM -algebra. Now, define a hesitant Fuzzy ideal and T-Ideal in the following ways:
Define ξ_h with the following hesitant fuzzy set $h_{\xi_h}(0) = [0, 1]$ (full membership), $h_{\xi_h}(1) = [0, 0.6]$, $h_{\xi_h}(2) = [0, 0.4]$ and $h_{\xi_h}(3) = [0, 0.2]$. It is easly verify that the given example satisfyed on Definition (5.2.1) and Definition (5.2.2). Thus, ξ_h is both a hesitant fuzzy ideal and a hesitant fuzzy T-ideal.

Definition 5.2.3. *For a subset* A *of* E, *A hesitant fuzzy soft set (* ξ_h, A *) over* X. *We say that (* ξ_h, A *) is a hesitant fuzzy soft ideal if the hesitant fuzzy set* $\xi_h = \{(x, h_{\xi_h}(x)) : x \in X\}$ *and* $\epsilon \in A\}$ *is a hesitant fuzzy ideal over* X.

Definition 5.2.4. *Let (* ξ_h, A *) be a hesitant fuzzy soft set over* X *and* $A \subseteq E$. *We say that (* ξ_h, A *) is a hesitant fuzzy soft* T*-ideal if the hesitant fuzzy set* $\xi_h = \{(x, h_{\xi_h}(x)) : x \in X\}$ *and* $\epsilon \in A\}$ *is a hesitant fuzzy* T*-ideal over* X.

Example 5.2.2. *Consider the Caylay table defined in 5.2.1*

Now, let us construct a hesitant fuzzy soft ideal and a hesitant fuzzy soft T-Ideal in the following ways:
Let $E = \{\epsilon_1, \epsilon_2\}$ (set of parameters) and $A = \{\epsilon_1\} \subseteq E$. We define a hesitant fuzzy soft set (ξ_h, A). For $\epsilon_1 \in A$, $\xi_h(\epsilon_1)$ is a hesitant fuzzy set on X. Define $\xi_h(\epsilon_1)$ as: $h_{\xi_h(\epsilon_1)}(0) = [0, 1]$, $h_{\xi_h(\epsilon_1)}(1) = [0, 0.7]$, $h_{\xi_h(\epsilon_1)}(2) = [0, 0.5]$ and $h_{\xi_h(\epsilon_1)}(3) = [0, 0.3]$.
Thus, (ξ_h, A) is a hesitant fuzzy soft ideal. Similarly, it is also a hesitant fuzzy Soft T-Ideal Use the same $\xi_h(\epsilon_1)$ and verify the T-ideal condition.

Corollary 5.2.1. *Every hesitant fuzzy soft ideal over a TM-algebra is also a hesitant fuzzy soft T-ideal.*

Proof. Let $(\xi_h = (x, h_{\xi_h}(x))/x \in X)$ be a hesitant fuzzy soft ideal of the TM-algebra (X); so ξ_h satisfies Definition (5.2.1). We show ξ_h satisfies Definition (5.2.2), i.e. it is a hesitant fuzzy soft

T-ideal.

Condition (1) of Definition (5.2.2). Definition (5.2.1)(1) gives $h_{\xi_h}(0) \supseteq h_{\widetilde{\mathcal{H}}}(x)$ for every x. In particular (interpreting $h_{\widetilde{\mathcal{H}}}(x)$ as the hesitant-values of (x) used in the ideal-definition) we obtain $h_{\xi_h}(0) \supseteq h_{\xi_h}(x)$ for every $x \in X$. Thus Definition (5.2.2)(1) holds.

Condition (2) of Definition (5.2.2). From Definition (5.2.1)(2) we have, for every $u, v \in X$, $h_{\xi_h}(u) \supseteq h_{\xi_h}(u \star v) \cap h_{\xi_h}(v)$. Put $u = x \star z$ and $v = y$. Then $h_{\xi_h}(x \star z) \supseteq h_{\xi_h}\big((x \star z) \star y\big) \cap h_{\xi_h}(y)$. Now use the TM-algebra identity $(x \star z) \star y = (x \star y) \star z$ this equality holds in a TM-algebra by the TM-law/associativity–commutativity property of $\star$. Replacing $(x \star z) \star y$ by $(x \star y) \star z$ yields $h_{\xi_h}(x \star z) \supseteq h_{\xi_h}\big((x \star y) \star z\big) \cap h_{\xi_h}(y)$, which is exactly Definition (5.2.2)(2). Since both conditions of Definition (5.2.2) are satisfied, ξ_h is a hesitant fuzzy soft T-ideal. Therefore every hesitant fuzzy soft ideal over a TM-algebra is also a hesitant fuzzy soft T-ideal. □

Theorem 5.2.2. *If (ξ_{1h}, A) and (ξ_{2h}, B) are two hesitant fuzzy soft T-ideals of a TM-algebras X, then $\xi_{1h} \widetilde{\cap} \xi_{2h}$ is a hesitant fuzzy soft T-ideal of a TM-algebras X.*

Proof. Let $(X, \star, 0)$ be a TM-algebra, and let (ξ_{1h}, A) and (ξ_{2h}, B) be two hesitant fuzzy soft T-ideals over X. We need to show that $\xi_{1h} \widetilde{\cap} \xi_{2h}$ is a hesitant fuzzy soft T-ideal of a TM-algebras X. Define their intersection $(\xi_h, A \cap B) = (\xi_{1h}, A) \cap (\xi_{2h}, B)$ where for each $\epsilon \in A$ and $x \in X$:

$$h_{\xi_h(\epsilon)}(x) = h_{\xi_{1h}(\epsilon)}(x) \cap h_{\xi_{2h}(\epsilon)}(x).$$

We need to verify that the two condition for hesitant fuzzy soft T-ideal of a TM -algebra defined in 5.2.2.

(i) Since ξ_{1h} and ξ_{2h} are T-ideals, for all $\epsilon \in A$ and $x \in X$:

$$h_{\xi_{1h}(\epsilon)}(0) \supseteq h_{\xi_{1h}(\epsilon)}(x), \quad h_{\xi_{2h}(\epsilon)}(0) \supseteq h_{\widetilde{\mathcal{H}}_2(\epsilon)}(x).$$

Taking the intersection:

$$h_{\xi_h(\epsilon)}(0) = h_{\xi_{1h}(\epsilon)}(0) \cap h_{\xi_{2h}(\epsilon)}(0) \supseteq h_{\xi_{1h}(\epsilon)}(x) \cap h_{\xi_{2h}(\epsilon)}(x) = h_{\xi_h(\epsilon)}(x).$$

Thus, ξ_h satisfies 5.2.2(1).

(ii) For all $\epsilon \in A - B$ and $\epsilon \in B - A$ and $x, y, z \in X$, since ξ_{1h} and ξ_{2h} are hesitant fuzzy soft T-ideals:

$$h_{\xi_{1h}(\epsilon)}(x \star z) \supseteq h_{\xi_{1h}(\epsilon)}((x \star y) \star z) \cap h_{\xi_{1h}(\epsilon)}(y),$$

$$h_{\xi_{2h}(\epsilon)}(x \star z) \supseteq h_{\xi_{2h}(\epsilon)}((x \star y) \star z) \cap h_{\xi_{2h}(\epsilon)}(y).$$

Taking intersections:

$$h_{\xi_h(\epsilon)}(x \star z) \quad = \quad h_{\xi_{1h}(\epsilon)}(x \star z) \cap h_{\xi_{2h}(\epsilon)}(x \star z)$$

$$
\begin{aligned}
&\supseteq \left(h_{\xi_{1h}(\epsilon)}((x\star y)\star z)\cap h_{\xi_{1h}(\epsilon)}(y)\right)\cap\left(h_{\xi_{2h}(\epsilon)}((x\star y)\star z)\cap h_{\xi_{2h}(\epsilon)}(y)\right)\\
&= \left(h_{\xi_{2h}(\epsilon)}((x\star y)\star z)\cap h_{\xi_{2h}(\epsilon)}((x\star y)\star z)\right)\cap\left(h_{\xi_{2h}(\epsilon)}(y)\cap h_{\xi_{2h}(\epsilon)}(z)\right)\\
&= h_{\xi_h(\epsilon)}((x\star y)\star z)\cap h_{\xi_h(\epsilon)}(y).
\end{aligned}
$$

(iii) for $\epsilon\in A\cap B$ a hesitant fuzzy soft T-ideal of X.
Thus, ξ_h satisfies 5.2.2(2). Hence, $(\xi_h, A\cap B)$ satisfies definition 5.2.2.

Therefore, $(\xi_h, A\cap B)$ is also a hesitant fuzzy soft T-ideal.

□

Remark 5.2.1. *If (ξ_{1h}, A) and (ξ_{2h}, B) are two hesitant fuzzy soft T-ideals of a TM-algebras X, then $\xi_{1h}\widetilde{\cup}\xi_{2h}$ is not necessarily a hesitant fuzzy soft T-ideal of a TM-algebras X.*

Example 5.2.3. *Let $X=\{0,1,2\}$ with the following Cayley table:*

$\star$	0	1	2
0	0	1	2
1	2	0	2
2	2	1	2

Table 5.5

See [42] Then $(x,\star,0)$ is a TM -algebra.
Let $h_{\xi_h(1)}=\{0.3\}, h_{\xi_h}(2)=\{0.2\}, h_{\xi_h}(0)=\{0.3,0.2\}$, $h_{\xi_2h}(1)=\{0.5\}, h_{\xi_2h}(2)=\{0.6\}, h_{\xi_{2h}}(0)=\{0.5,0.6\}$. So, $h_{\xi_h}(1)=\{0.3,0.5\}, h_{\xi_h}(b)=\{0.2,0.6\}, h_{\xi_h}(0)=\{0.2,0.3,0.5,0.6\}$, Let's take $x=1, y=2, z=0$. Assume that $x\star z=1$, so $(x\star y)\star z=1\star 0=a$, $x\star y=1\star 2=2$. Then $h(x\star z)=h(2)=\{0.2,0.6\}$, $h((x\star y)\star z)\cap h(y)=h(1)\cap h(0)=\{0.3,0.5\}\cap\{0.2,0.3,0.5,0.6\}=\{0.3,0.5\}$. It implies that $\{0.2,\ 0.6\}\not\supseteq\{0.3,\ 0.5\}$. Thus, the union of two hesitant fuzzy soft T-ideals is not necessarily a hesitant fuzzy soft T-ideal in a TM-algebra.
Therefore, $\xi_{1h}\widetilde{\cup}\xi_{2h}$ is not necessarily a hesitant fuzzy soft T-ideal.

Definition 5.2.5. *Let (ξ_h, A) be a hesitant fuzzy soft set over X , then the complement of hesitant fuzzy soft set (ξ_h, A) is defined by $(\xi_h, A)^C=(\xi_h,\neg A)$, where $\xi_h^C:\neg A\to \mathsf{HF}(X)$ is a mapping given by $\xi_h^C[\epsilon]=\xi_h[\neg\epsilon]$ for all $\epsilon\in\neg A$.*

Definition 5.2.6. *Let $(\xi_h, A)^C$ be a hesitant furzy soft set over X and $A\subseteq E$. We say that $(\xi_h, A)^C$ is a hesitant fuzzy soft T-ideal of X if the hesitant fuzzy set $\xi_h^C[\epsilon]=\left\{\left(x, h_{\xi_h^C[\epsilon]}(x)x\in X \text{ and } \epsilon\in\neg A\right)\right\}$is a hesitant fuzzy T-ideal over X.*

Definition 5.2.7. *Let $(\xi_h, A)^C$ be a hesitant fuzzyt soft set over X and $A\subseteq E$. We say that $(\xi_h, A)^C$ is a hesitant fuzzy soft ideal if the hesitant fuzzy set $\xi_h^C[\epsilon]=\left\{\left(x, h_{\xi_h^C[\epsilon]}(x): x\in X \text{ and } \epsilon\in\neg A\right)\right\}$ is a hesitant fuzzy ideal over X.*

Definition 5.2.8. *Let m be any non-negative integer and given that a hesitant fuzzy soft set* (ξ_h, A)*, the power-*m *operation is defined as*

$$\xi_h^m[\epsilon] = \left\{ \left(x, h_{\xi_h^m[\epsilon]}(x) \right) : x \in X, \epsilon \in A \right\}$$

, where the membership are modified based on the parameter m *and*

$$h_{\xi_h^m[\epsilon]}(x) = \{\phi^m : \phi \in h_{\xi_h[\epsilon]}(x)\}$$

Remark 5.2.2.

1) For $m > 1$, it shrinks values less than one. It emphasizing lower certainty.

2) For $0 < m < 1$, it increases values less than one. It emphasizing uncertainty more softly.

3) $m = 1$ gives back the original HFSS.

Theorem 5.2.3. *If* (ξ_h, A) *is a hesitant fuzzy soft T-ideal over* X*, then* $(\xi_h, A)^C$ *is a hesitant fuzzy soft T-ideal over* X*.*

Proof. Suppose that (ξ_h, A) be a hesitant fuzzy soft T-ideal of X. We need to show that $(\xi_h, A)^C$ is a hesitant fuzzy soft T-ideal over X.
(i) Let $x \in X$ and $\epsilon \in \neg A$. Then

$$\begin{aligned} h_{\xi_h^C[\epsilon]}(0) &= h_{\xi_h[\neg\epsilon]}(0) \\ &= h_{\xi_h[\neg\epsilon]}(x * x). \\ &\supseteq h_{\xi_h[\neg\epsilon]}(x) \cap h_{\xi_h[\neg\epsilon]}(x) \\ &= h_{\xi_h[\neg\epsilon]}(x) \\ &= h_{\xi_h^C[\epsilon]}(x). \end{aligned}$$

(ii) For all $x, y, z \in X$ and $\epsilon \in \neg A$, we have

$$\begin{aligned} h_{\xi_h^C[\epsilon]}(x \star z) &= h_{\xi_h[\neg\epsilon]}(x \star z) \\ &\supseteq h_{\xi_h[\neg\epsilon]}((x \star y) \star z) \cap h_{\xi_h[\neg\epsilon]}(y) \\ &= h_{\xi_h^C[\epsilon]}((x \star y) \star z) \cap h_{\xi_h^C[\epsilon]}(y) \end{aligned}$$

Hence, $(\xi_h, A)^C$ is a hesitant fuzzy soft T-ideal of a TM -algebra X. □

Theorem 5.2.4. *If* (ξ_h, A) *is a hesitant fuzzey soft T-ideal over* X*, then* $(\xi_h, A)^m$ *is a hesitant fuzzy soft* T*-ideal over* X*.*

Proof. Suppose that (ξ_h, A) be a hesitant fuzzy soft T-ideal of X. We need to show that $(\xi_h, A)^m$ is a hesitant fuzzy soft T-ideal over X.

(i) Let $x \in X$ and $\epsilon \in A$. Then

$$\begin{aligned} h_{\xi_h^m[\epsilon]}(0) &= \left\{h_{\xi_h[\epsilon]}(0)\right\}^m \\ &= \left\{h_{\xi_h[\epsilon]}(x * x)\right\}^m. \\ &\supseteq \left\{h_{\xi_h[\epsilon]}(x) \cap h_{\xi_h[\epsilon]}(x)\right\}^m \\ &= \left\{h_{\xi_h[\epsilon]}(x)\right\}^m \\ &= h_{\xi_h^m[\epsilon]}(x). \end{aligned}$$

(ii) For every $x, y, z \in X$ and $\epsilon \in A$, we have

$$\begin{aligned} h_{\xi_h^m[\epsilon]}(x \star z) &= \left\{h_{\xi_h[\epsilon]}(x \star y)\right\}^m \\ &\supseteq \left\{h_{\xi_h[\epsilon]}((x \star y) \star z) \cap h_{\xi_h[\epsilon]}(y)\right\}^m \\ &\supseteq \left\{h_{\xi_h^m[\epsilon]}((x \star y) \star z)\right\}^m \cap \left\{h_{\xi_h^c[\epsilon]}(y)\right\}^m \end{aligned}$$

Hence, $(\xi_h, A)^m$ is a hesitant fuzzy soft T-ideal of a TM -algebra X. □

5.3. Bipolar Hesitant Fuzzy Soft Set in TM-Algebra

In this section, we present the bipolar hesitant fuzzy soft subalgebra and bipolar hesitant fuzzy soft ideal of a TM -algebra and examine some of their properties.

5.3.1 Bipolar Hesitant Fuzzy Soft Subalgebra of TM -algebra

Definition 5.3.1. *A bipolar hesitant fuzzy set (BHFS) of a TM-algebra* X *is defined as* $\xi_h = \left\{\left(x, \xi_h^+(x), \xi_h^-(x)\right) : x \in X\right\}$ *where* $\xi_h^+ : X \to [0,1]$ *and* $\xi_h^- : X \to [-1,0]$ *are the mappings. The positive membership function* $\xi_h^+(x)$ *denote the satisfaction degree of the element x to the property corresponding to the hesitant fuzzy set* $\xi_h = \left\{\left(x, \xi_h^+(x), \xi_h^-(x)\right) : x \in X\right\}$ *and the negative membership degree* $\xi_h^-(x)$ *denotes the satisfaction degree of an element x to some implicit counter part of* $\xi_h = \left\{\left(x, \xi_h^+(x), \xi_h^-(x)\right) : x \in X\right\}$.

Definition 5.3.2. *Let* U *be a universal set and* $A \subseteq E$ *be a set of parameters. A bipolar hesitant fuzzy soft set over* U *is a pair*

$$(\widetilde{BH}, A),$$

where

$$\widetilde{BH} : A \to BHF(U),$$

that is, for all $\epsilon \in A, \xi_{\epsilon h} = \{\langle x, \xi_{\epsilon h}^+(x), \xi_{\epsilon h}^-(x)\rangle : x \in U\}$, *with* $\xi_{\epsilon h}^+(x) \subseteq [0,1]$ *and* $\xi_{\epsilon h}^-(x) \subseteq [-1,0]$, *representing the sets of possible positive and negative membership degrees (i.e., hesitant values)* *for element* $x \in U$ *under parameter* $\epsilon \in A$.

Definition 5.3.3. *A Bipolar hesitant fuzzy soft set* $\widetilde{BH} = \left\{\left(x, \xi_h^+(x), \xi_h^-(x)\right) : x \in X\right\}$ *is called a bipolar hesitant fuzzy soft subalgebrea of X if it satisfies*

1. $\xi_h^+(x \star y) \supseteq \xi_h^+(x) \cap \xi_h^+(y)$,
2. $\xi_h^-(x \star y) \subseteq \xi_h^-(x) \cup \xi_h^-(y)$ *for all* $x, y \in X$.

Example 5.3.1. *Consider the set* $X = \{0,1,2,3,4,5\}$ *with the following Cayley table*

$\star$	0	1	2	3	4	5
0	0	3	4	1	2	5
1	1	0	2	3	5	4
2	2	4	0	5	1	3
3	3	1	5	0	4	2
4	4	5	3	2	0	1
5	5	2	1	4	3	0

Table 5.6

See [42] $(X, \star, 0)$ *is a TM-algebra.*
Define a bipolar hesitant fuzzy soft set $\widetilde{BH}$ *on* X *as follows:*

For the positive membership part ξ_h^+ *:*

$$\xi_h^+(0) = [0.8, 0.9],$$
$$\xi_h^+(1) = [0.3, 0.5],$$
$$\xi_h^+(2) = [0.3, 0.5],$$
$$\xi_h^+(3) = [0.3, 0.5],$$
$$\xi_h^+(4) = [0.3, 0.5],$$
$$\xi_h^+(5) = [0.3, 0.5].$$

For the negative membership part ξ_h^- *:*

$$\xi_h^-(0) = [-0.9, -0.7],$$
$$\xi_h^-(1) = [-0.5, -0.3],$$
$$\xi_h^-(2) = [-0.5, -0.3],$$
$$\xi_h^-(3) = [-0.5, -0.3],$$
$$\xi_h^-(4) = [-0.5, -0.3],$$
$$\xi_h^-(5) = [-0.5, -0.3].$$

We verify that $\widetilde{BH}$ *is a bipolar hesitant fuzzy soft subalgebra of* X.
Condition (1): $\xi_h^+(x \star y) \supseteq \xi_h^+(x) \cap \xi_h^+(y)$ *for all* $x, y \in X$. $\xi_h^+(x) \cap \xi_h^+(y)$ *equals* $[0.8, 0.9]$ *if* $x = 0$

and $y = 0$ $[0.3, 0.5]$ *otherwise*
We check several representative cases:
For $x = 0, y = 0$*:* $\xi_h^+(0 \star 0) = \xi_h^+(0) = [0.8, 0.9] \supseteq [0.8, 0.9]$
For $x = 0, y = 1$*:* $\xi_h^+(0 \star 1) = \xi_h^+(3) = [0.3, 0.5] \supseteq [0.3, 0.5]$
For $x = 1, y = 2$*:* $\xi_h^+(1 \star 2) = \xi_h^+(2) = [0.3, 0.5] \supseteq [0.3, 0.5]$
For $x = 2, y = 3$*:* $\xi_h^+(2 \star 3) = \xi_h^+(5) = [0.3, 0.5] \supseteq [0.3, 0.5]$
For $x = 4, y = 5$*:* $\xi_h^+(4 \star 5) = \xi_h^+(1) = [0.3, 0.5] \supseteq [0.3, 0.5]$
The pattern continues for all pairs: whenever the result contains element 0*, we get* $[0.8, 0.9]$ *which contains* $[0.3, 0.5]$*; otherwise we get equality. Thus condition 1 is satisfied.*
Condition (2): $\xi_h^-(x \star y) \subseteq \xi_h^-(x) \cup \xi_h^-(y)$ *for all* $x, y \in X$*. Note that* $\xi_h^-(x) \cup \xi_h^-(y)$ *equals* $[-0.9, -0.7]$ *if* $x = 0$ *or* $y = 0$ $[-0.5, -0.3]$ *otherwise. We check representative cases: For* $x = 0, y = 0$*:* $\xi_h^-(0 \star 0) = \xi_h^-(0) = [-0.9, -0.7] \subseteq [-0.9, -0.7]$
For $x = 0, y = 1$*:* $\xi_h^-(0 \star 1) = \xi_h^-(3) = [-0.5, -0.3] \subseteq [-0.9, -0.7]$
For $x = 1, y = 2$*:* $\xi_h^-(1 \star 2) = \xi_h^-(2) = [-0.5, -0.3] \subseteq [-0.5, -0.3]$
For $x = 2, y = 3$*:* $\xi_h^-(2 \star 3) = \xi_h^-(5) = [-0.5, -0.3] \subseteq [-0.5, -0.3]$
For $x = 4, y = 5$*:* $\xi_h^-(4 \star 5) = \xi_h^-(1) = [-0.5, -0.3] \subseteq [-0.5, -0.3]$.
The pattern continues whenever the result contains element 0*, we get* $[-0.9, -0.7]$ *which is contained in any union; otherwise we get equality. Thus Condition 2 is satisfied. Since both conditions are satisfied for all* $x, y \in X$.
Therfore, $\widetilde{BH}$ *is a bipolar hesitant fuzzy soft subalgebra of* X.

Lemma 5.3.1. *If* $\xi_h = \left(X;\ \xi_h^-, \xi_h^+\right)$ *is a bipolar hesitant fuzzy soft subalgebra of a TM -algebra* X*, then* $\xi^-(0) \subseteq \xi_h^-(x)$ *and* $\xi_h^+(0) \supseteq \xi^+(x)$*, for all* $x \in X$.

Proof. Suppose $\xi_h = \left(\xi_h^+, \xi_h^-\right)$ is a bipolar hesitant fuzzy soft subalgebra of X. We need to show that $\xi^-(0) \subseteq \xi_h^-(x)$ and $\xi_h^+(0) \supseteq \xi^+(x)$, for all $x \in X$. Since $x \star x = 0$ by Definition 5.3.3, we have

$$\xi_h^+(0) = \xi_h^+(x \star x) \supseteq \xi_h^+(x) \cap \xi_h^+(x) = \xi_h^+(x) \quad \text{and}$$
$$\xi_h^-(0) = \xi_h^-(x \star x) \subseteq \xi_h^-(x) \cup \xi_h^-(x) = \xi_h^-(x)$$

Hence, $\xi_h^+(0) \supseteq \xi_h^+(x)$ and $\xi_h^-(0) \subseteq \xi_h^-(x)$, for all $x \in X$. □

Definition 5.3.4. *Let (* ξ_{1h}, A *) and (* ξ_{2h}, B *) be a bipolar hesitant fuzzy soft subalgebras over a TM-algebra* X*. Then* $(\widetilde{BH}_1, A)$ *is a bipolar hesitant fuzzy soft subalgebra of* $(\widetilde{BH}_2, B)$ *if*

1) $A \subseteq B$ *and*

2) $\xi_{1h}(x)$ *is a bipolar hesitant fuzzy soft subalgebra of* $\xi_{2h}(x)$*, for all* $x \in A$.

Definition 5.3.5. *Let* (ξ_{1h}, A) *and* (ξ_{2h}, B) *be two bipolar hesitant fuzzy soft set* X*. Then* (ξ_{1h}, A) *AND* (ξ_{2h}, B) *, denoted by* $(\xi_{1h}, A) \wedge (\xi_{2h}, B)$ *is known as* $(\xi_{1h}, A) \wedge (\xi_{2h}, B) = (\xi_h, C)$ *where* $C = A \times B$ *and* $\xi_h(a, b) = \xi_{1h}(a) \cap \xi_{2h}(b), \forall (a, b) \in C = A \times B$.

Definition 5.3.6. *Let* (ξ_{1h}, A) *and* (ξ_{2h}, B) *be two bipolar hesitant fuzzy soft set* X. *Then* (ξ_{1h}, A) *OR* (ξ_{2h}, B), *denoted by*$(\xi_{1h}, A) \vee (\xi_{2h}, B)$ *is known as* $(\xi_{1h}, A) \vee (\xi_{2h}, B) = (\xi_h, W)$ *where* $C = A \times B$ *and* $(\xi_{1h}, A) \wedge (\xi_{2h}, B) = (\xi_h, W)$ *where* $W = A \times B$ *and* $\xi_h(a, b) = \xi_{1h}(a) \cup \xi_{2h}(b), \forall (a, b) \in C = A \times B$.

Theorem 5.3.2. *If* (ξ_{1h}, A) *and* (ξ_{2h}, B) *be two bipolar hesitant fuzzy soft TM-subalgebra over* X, *then* $(\xi_{1h}, A) \wedge (\xi_{2h}, B)$ *is also a bipolar hesitant fuzzy soft TM-subalgebra over X.*

Proof. Suppose that (ξ_{1h}, A) and (ξ_{2h}, B) be two bipolar hesitant fuzzy soft TM-subalgebra over X. We need to show that $(\xi_{1h}, A) \wedge (\xi_{2h}, B)$ is also a bipolar hesitant fuzzy soft TM-subalgebra over X. Let $x, y \in X$. Then by Definition 5.3.5 we have:

$$\begin{aligned}
\xi^+_{(a,b)h}(x \star y) &= \xi^+_{a1h}(x \star y) \cap \xi^+_{b2h}(x \star y) \supseteq \xi^+_{a1h}(x) \cap \xi^+_{a1h}(y) \\
&\cap \xi^+_{b2h}(x) \cap \xi^+_{b2h}(y) = \xi^+_{(a,b)h}(x) \cap \xi^+_{(a,b)h}(y) \quad \text{and} \\
\xi^-_{(a,b)h}(x \star y) &= \xi^-_{a1h}(x \star y) \cap \xi^-_{b2h}(x \star y) \subseteq \{\xi^-_{a1h}(x) \cup \xi^+_{a1h}(y)\} \\
&\cap \{\xi^+_{b2h}(x) \cup \xi^+_{b2h}(y)\} = \xi^+_{(a,b)h}(x) \cup \xi^+_{(a,b)h}(y)
\end{aligned}$$

Therefore, $(\xi_{1h}, A) \wedge (\xi_{2h}, B)$ is a bipolar hesitant fuzzy soft TM-subalgebra over X. □

Theorem 5.3.3. *If* (ξ_{1h}, A) *and* (ξ_{2h}, B) *be two bipolar hesitant fuzzy soft TM-subalgebra over* X, *then* $(\xi_{1h}, A) \vee (\xi_{2h}, B)$ *is also a bipolar hesitant fuzzy soft TM-subalgebra over X.*

Proof. Suppose that (ξ_{1h}, A) and (ξ_{2h}, B) be two bipolar hesitant fuzzy soTM-subalgebra over X. We need to show that $(\xi_{1h}, A) \vee (\xi_{2h}, B)$ is also a bipolar hesitant fuzzy soft TM-subalgebra over X. Let $x, y, z \in X$. Then by Definition 5.3.6 we have:

$$\begin{aligned}
\xi^+_{(a,b)h}(x \star y) &= \xi^+_{a1h}(x \star y) \cap \xi^+_{b2h}(x \star y) \supseteq \xi^+_{a1h}(x) \cap \xi^+_{a1h}(y) \\
&\cap \xi^+_{b2h}((x) \cap \xi^+_{b2h}(y) = \xi^+_{(a,b)h}((x) \cap \xi^+_{(a,b)h}(y) \quad \text{and} \\
\xi^-_{(a,b)h}(x \star y) &= \xi^-_{a1h}(x \star y) \cap \xi^-_{b2h}(x \star y) \subseteq \{\xi^-_{a1h}((x) \cup \xi^+_{a1h}(y)\} \\
&\cap \{\xi^+_{b2h}((x \star y) \star z) \cup \xi^+_{b2h}(y)\} = \xi^+_{(a,b)h}(x) \cup \xi^+_{(a,b)h}(y)
\end{aligned}$$

Therefore, $(\xi_{1h}, A) \wedge (\xi_{2h}, B)$ is a bipolar hesitant fuzzy soft TM-subalgebra over X. □

Theorem 5.3.4. *Let* $\xi_{1h} = \{(x, \xi^+_{1h}(x), \xi^-_{1h}(x)) : x \in X\}$ *and* $\xi_{2h} = \{(x, \xi^+_{2h}(x), \xi^-_{2h}(x)) : x \in X\}$ *be two bipolar hesitant fuzzy soft subalgebra of X . Then their extended intersection* $\xi_{1h} \widetilde{\cap} \xi_{2h}$ *is a bipolar hesitant fuzzy soft subalgebra of* X.

Proof. Suppose that (ξ_{1h}, A) and (ξ_{2h}, B) are bipolar hesitant fuzzy soft subalgebras over X. We need to show that $(\xi_{1h}, A) \widetilde{\cap} (\xi_{2h}, B)$ is also a bipolar hesitant fuzzy soft subalgebra of X. Let $(\xi_{1h}, A \cap B) = (\xi_{1h}, A) \widetilde{\cap} (\xi_{2h}, B)$ be the extended intersection of two bipolar hesitant fuzzy soft subalgebras. The extended union is defined as:

$$\xi_h(\varepsilon) = \begin{cases} \xi_{1h}(\varepsilon) & \text{if } \varepsilon \in A - B, \\ \xi_{2h}(\varepsilon) & \text{if } \varepsilon \in B - A, \\ \xi_{1h}(\varepsilon) \cap \xi_{2h}(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

Then, we consider the following cases: For any $x, y \in X$

Case 1: If $\epsilon \in A - B$, then

$$\begin{aligned} \xi_h^+(x \star y) &= \xi_{1h}^+(x \star y) \\ &\supseteq \xi_{1h}^+((x) \cap \xi_{1h}^+(y) \text{and} \\ \xi_h^-(x \star y) &= \xi_{1h}^-(x \star y) \\ &\subseteq \xi_{1h}^-(x) \cup \xi_{1h}^-(y) \end{aligned}$$

Case 2: If $\epsilon \in B - A$, then

$$\begin{aligned} \xi_h^+(x \star y) &= \xi_{2h}^+(x \star y) \\ &\supseteq \xi_{2h}^+((x) \cap \xi_{2h}^+(y) \text{and} \\ \xi_h^-(x \star y) &= \xi_{2h}^-(x \star y) \\ &\subseteq \xi_{2h}^-(x) \cup \xi_{2h}^-(y) \end{aligned}$$

Case 3: If $\epsilon \in A \cap B$, then

$$\begin{aligned} \xi_h^+(x \star y) &= \xi_{1h}^+(x \star y) \cap \xi_{2h}^+(x \star z) \\ &\supseteq \xi_{1h}^+(x) \cap \xi_{1h}^+(y) \cap \xi_{2h}^+(x) \cap \xi_{2h}^+(y) \\ &= \xi_h^+(x) \cap \xi_h^+(y) \\ \xi_h^-(x \star y) &= \xi_{1h}^-(x \star y) \cap \xi_{2h}^-(x \star y) \\ &\subseteq \xi_{1h}^-(x) \cup \xi_{1h}^-(y) \cup \xi_{2h}^-(x) \cup \xi_{2h}^+(y) \\ &= \xi_h^-(x) \cup \xi_h^-(y) \end{aligned}$$

Therefore, $(\xi_{1h}, A)\widetilde{\cap}(\xi_{2h}, B)$ is a bipolar hesitant fuzzy soft TM-subalgebra of X □

Theorem 5.3.5. *Let* (ξ_{1h}, A) *and* (ξ_{2h}, B) *be a bipolar hesitant fuzzy soft subalgebras over a TM-algebra* X. *If* $A \cap B = \emptyset$, *then* $(\xi_{1h}, A)\widetilde{\cup}(\xi_{1h}, B)$ *is a bipolar hesitant fuzzy soft subalgebra of* X *where* $A, B \subseteq E$*(the set of parameters).*

Proof. Suppose that (ξ_{1h}, A) and (ξ_{2h}, B) are bipolar hesitant fuzzy soft subalgebras over X with the property that $A \cap B = \emptyset$. We need to show that $(\xi_{1h}, A)\widetilde{\cup}(\xi_{2h}, B)$ is also a bipolar hesitant fuzzy soft subalgebra of X. Let $(\xi_{1h}, A \cup B) = (\xi_{1h}, A)\widetilde{\cup}(\xi_{2h}, B)$ be the extended union of two bipolar hesitant fuzzy soft subalgebras. Since $A \cap B = \emptyset$, the extended union is defined as:

$$\xi_h(\varepsilon) = \begin{cases} \xi_{1h}(\varepsilon) & \text{if } \varepsilon \in A - B, \\ \xi_{2h}(\varepsilon) & \text{if } \varepsilon \in B - A, \\ \xi_{1h}(\varepsilon) \cup \xi_{2h}(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

For $\varepsilon \in A - B$, $\xi_h(\varepsilon) = \xi_{2h}$. For $\varepsilon \in B - A$, $\xi_h(\varepsilon) = \xi_{2h}(\varepsilon)$. For every parameter $\varepsilon \in A \cup B$, and for all $x, y \in X$, the conditions $\xi_h^+(x \star y) \supseteq \xi_h^+(x) \cap \xi_h^+(y), \quad \xi_h^-(x \star y) \subseteq \xi_h^-(x) \cup \xi_h^-(y)$ hold for the corresponding hesitant membership functions.

Case 1: If $\varepsilon \in A - B$. Then by construction, $\xi_h = \xi_{1h}(\varepsilon)$. Since (ξ_{1h}, A) is a bipolar hesitant fuzzy soft subalgebra, for all $x, y \in X$, $\xi_{1h}^+(x \star y) \supseteq \xi_{1h}^+(x) \cap \xi_{1h}^+(y)$, and $\quad \xi_{1h}^-(x \star y) \subseteq \xi_{1h}^-(x) \cup \xi_{1h}^-(y)$ But $\xi_h^{\pm}(x) = \xi_{1h}^{\pm}(x)$, so the condition holds for $\xi_h(\varepsilon)$.

Case 2: If $\varepsilon \in B - A$. Then similarly, $\xi_h(\varepsilon) = \xi_{2h}(\varepsilon)$. Since (ξ_{2h}, B) is a bipolar hesitant fuzzy soft subalgebra, for all $x, y \in X$, $\xi_{2h}^+(x \star y) \supseteq \xi_{2h}^+(x) \cap \xi_{2h}^+(y)$, and $\quad \xi_{2h}^-(x \star y) \subseteq \xi_{2h}^-(x) \cup \xi_{2h}^-(y)$ Again, since $\xi_h^{\pm} = \xi_{2h}^{\pm}$ in this case, the condition holds.

Case 3: If $\varepsilon \in A \cap B$, then no justification is needed since $A \cap B = \emptyset$.

But, for each $\varepsilon \in A \cup B$, the corresponding bipolar hesitant fuzzy set $\xi_h(\varepsilon)$ satisfies the required conditions of Definition 5.3.3.

Therefore, $(\xi_{1h}, A)\widetilde{\cup}(\xi_{2h}, B)$ is a bipolar hesitant fuzzy soft subalgebra of the TM-algebra X. □

Definition 5.3.7. *Let ξ_h be a bipolar hesitant fuzzy soft set over* U. *For each $\Lambda_1 \in \mathcal{P}[0,1]$ and $\Lambda_2 \in \mathcal{P}[-1,0]$, the set $\mathcal{U}(\xi_h, \Lambda_1)$ is called an Λ_1-level soft set of (ξ_h, A), where* $\mathcal{U}(\xi_h, \Lambda_1) = \{x \in U \mid \xi_{\varepsilon h}^+(x) \supseteq \Lambda_1, \quad \xi_{\varepsilon h}^-(x) \subseteq \Lambda_2\}$, *for all* $\varepsilon \in A$.

Theorem 5.3.6. *Let (ξ_h, A) be a bipolar hesitant fuzzy soft set over X. (ξ_h, A) is a bipolar hesitant fuzzy soft TM-subalgebra if and only if $\mathcal{U}(\xi_h, \Lambda_1)$ is a soft TM-subalgebra over X for all $\Lambda_1 \in \mathcal{P}[0,1]$ and $\Lambda_2 \in \mathcal{P}[-1,0]$.*

Proof. Suppose that (ξ_h, A) is a bipolar hesitant fuzzy soft TM-subalgebra. For each $\Lambda_1 \in \mathcal{P}[0,1], \epsilon \in A$ and let $x_1, x_2 \in \mathcal{U}(\xi_h, \Lambda_1)$, then $\xi_{\epsilon h}^+(x_1) \supseteq \Lambda_1, \xi_{\epsilon h(x_2)}^+ \supseteq \Lambda_1$ and $\xi_{\epsilon h}^-(x_1) \subseteq \Lambda_2$, $\xi_{\epsilon h(x_2)}^- \subseteq \Lambda_2$. Since $\mathcal{U}(\xi_h, \Lambda_1)$ is a bipolar hesitant fuzzy TM-subalgebra over X. Thus $\xi_{\epsilon h}^+(x_1 \star x_2) \supseteq \xi_{\epsilon h}^+(x_1) \cap \xi_{\epsilon h}^+(x_2)$. It implies that $\xi_{\epsilon h}^+(x_1 \star x_2) \supseteq \Lambda_1$, and $\xi_{\epsilon h}^-(x_1 \star x_2) \subseteq \xi_{\epsilon h}^-(x_1) \cup \xi_{\epsilon h}^-(x_2)$. Implies $\xi_{\epsilon}^-(x_1 \star x_2) \subseteq \Lambda$. This implies that $x_1 \star x_2 \in \mathcal{U}(\xi_h, \Lambda_1)$. Therefore, $\mathcal{U}(\xi_h, \Lambda_1)$ is a TM-subalgebra over X. By Definition 5.3.7 $\mathcal{U}(\xi_h, \Lambda_2)$ is a soft TM-subalgebra over X for each $\Lambda \in \mathcal{P}[0,1]$.

Conversely, assume that $\mathcal{U}(\xi_h, \Lambda_1)$ is a soft TM-subalgebra over X for each $\Lambda_1 \in \mathcal{P}[0,1]$. For all $\epsilon \in A$ and $x_1, x_2 \in X$, let $\Lambda_1 = \xi_{\epsilon h}^+(x_1) \cap \xi_{\epsilon h}^+(x_2)$ and let $\Lambda_2 = \xi_{\epsilon}^-(x_1) \cup \xi_{\epsilon h}^-(x_2)$, then $x_1, x_2 \in \mathcal{U}(\xi_{\epsilon h}, \Lambda_2)$. Since $\mathcal{U}(\xi_h, \Lambda_1)$ is a TM-subalgebra over X, then $x_1 \star x_2 \in \mathcal{U}(\xi_{\epsilon h}, \Lambda_1)$. This means that $\xi_{\epsilon h}^+(x_1 \star x_2) \subseteq \xi_{\epsilon h}^+(x_1) \cap \xi_{\epsilon h}^+(x_2)$, $\xi_{\epsilon h}^-(x_1 \star x_2) \subseteq \xi_{\epsilon h}^-(x_1) \cup \xi_{\epsilon h}^-(x_2)$, i.e., $\mathcal{U}(\xi_{\epsilon h}, \Lambda_1)$ is a bipolar hesitant fuzzy TM-subalgebra over X. According to Definition 5.3.7, (ξ_h, A) is a bipolar hesitant fuzzy soft TM-subalgebra over X. □

Definition 5.3.8. *Let $f : X \to Y$ and $g : A \to B$ be two functions, A and B are parametric sets from the crisp set X and Y, respectively. Then the pair (f, g) is called a bipolar hesitant fuzzy soft function from X to Y.*

Definition 5.3.9. *Let (ξ_{1h}, A) and (ξ_{2h}, B) be two bipolar hesitant fuzzy soft sets over X and Y, respectively and let (f, g) be a bipolar fuzzy soft function from X to Y.*

(i) The image of (ξ_{1h}, A) under the bipolar hesitant fuzzy soft function (f, g), denoted by $(f,g)(\xi_{1h}, A)$, is the bipolar hesitant fuzzy soft set on Y *defined by* $(f,g)(\xi_{1h}, A) = (f(\xi_{1h}), g(A))$, *where for all* $w \in g(A), y \in Y$

$$\xi^{+}_{f(\xi_{1h})_w}(y) = \begin{cases} \bigcup_{f(x)=y} \bigcup_{g(a)=w} \xi_{1ha(x)} & \text{if } x \in g^{-1}(y), \\ 1 & \text{otherwise} \end{cases}$$

$$\xi^{-}_{f(\xi_{1h})_w}(y) = \begin{cases} \bigcap_{f(x)=y} \bigcap_{g(a)=w} \xi_{1ha(x)} & \text{if } x \in g^{-1}(y), \\ -1 & \text{otherwise.} \end{cases}$$

(ii) The pre-image of (ξ_{2h}, B) *under the bipolar hesitant fuzzy soft function (f, g), denoted by* $(f,g)^{-1}(\xi_{2h}, B)$, *is the bipolar hesitant fuzzy soft set over* X *defined by* $(f,g)^{-1}(\xi_{1h}, B) = \left(f^{-1}(\xi_{1h}), g^{-1}(B)\right)$, *where for all* $a \in g^{-1}(A)$, *for all* $x \in X$,

$$\xi^{+}_{f^{-1}(\xi_{1h})_a}(x) = \xi^{+}_{\xi_{1hg(a)}}(f(x))$$
$$\xi^{-}_{f^{-1}(\xi_{1h})_a}(x) = \xi^{-}_{\xi_{1hg(a)}}(f(x))$$

Theorem 5.3.7. *Let* (ξ_{2h}, B) *be a bipolar hesitant fuzzy soft TM-subalgebra over Y and let* (f,g) *be a bipolar hesitant fuzzy soft homomorphism from X to Y. Then* $(f,g)^{-1}(\xi_{2h}, B)$ *is a bipolar hesitant fuzzy soft TM-subalgebra over X.*

Proof. Let $x_1, x_2 \in X$, then

$$\begin{aligned} f^{-1}\left(\xi^{+}_{\xi_{2hw}}\right)(x_1 \star x_2) &= \xi^{+}_{\xi_{2h}g(x)}(f(x_1 \star x_2)) = \xi^{+}_{\xi_{2h}g(x)}(f(x_1) \star f(x_2)) \\ &\supseteq \xi^{+}_{\xi_{2h}g(w)}(f(x_1)) \cap \xi^{+}_{\xi_{2h}g(w)}(f(x_2)) \\ &= f^{-1}\left(\xi^{+}_{g_w}\right)(x_1) \cap f^{-1}\left(\xi^{+}_{g_w}\right)(x_2) \\ f^{-1}\left(\xi^{-}_{\xi_{2h}w}\right)(x_1 \star x_2) &= \xi^{-}_{\xi_{2h}g(x)}(f(x_1 \star x_2)) = \xi^{-}_{\xi_{2h}g(x)}(f(x_1) \star f(x_2)) \\ &\subseteq \xi^{-}_{\xi_{2h}g(w)}(f(x_1)) \cup \xi^{-}_{\xi_{2h}g(w)}(f(x_2)) \\ &= f^{-1}\left(\xi^{-}_{g_k}\right)(x_1) \cup f^{-1}\left(\xi^{-}_{g_k}\right)(x_2) \end{aligned}$$

Hence $(f,g)^{-1}(\xi_{2h}, B)$ is a bipolar hesitant fuzzy soft TM-subalgebra over X. □

5.3.2 Bipolar Hesitant Fuzzy Soft T-ideal of TM -algebra

Definition 5.3.10. *A bipolar hesitant fuzzy soft set* $\xi_h = \left\{\left(x, \xi^{+}_{h}(x), \xi^{-}_{h}(x)\right) : x \in X\right\}$ *is called a bipolar hesitant fuzzy soft T-ideal of X if it satisfies: for all* $x, y, z \in X$.

1. $\xi^{+}_{h}(0) \supseteq \xi^{+}_{h}(x)$ *and* $\xi^{+}_{h}(0) \subseteq \xi^{-}_{h}(x)$;
2. $\xi^{+}_{h}(x \star z) \supseteq \xi^{+}_{h}((x \star y) \star z) \cap \xi^{+}_{h}(y)\}$ *and* $\xi^{-}_{h}(x \star z) \subseteq \xi^{-}_{h}((x \star y) \star z) \cup \xi^{-}_{h}(y)$

Theorem 5.3.8. *A bipolar hesitant fuzzy soft set* $\xi_h = \left\{\left(x, \xi^{+}_{h}(x), \xi^{-}_{h}(x)\right) : x \in X\right\}$ *is a bipolar hesitant fuzzy soft T-ideal of X if and only if* ξ_h *is a bipolar hesitant fuzzy soft ideal of* X.

Proof. ($\Rightarrow$) If we put $z = 0$ in Definition 5.3.10[2] , then $\xi_h^-(x) \subseteq \xi_h^-(x \star y) \cup \xi_h^-(y)$ and $\xi_h^+(x) \supseteq \xi_h^+(x \star y) \cap \xi_h^+(y)$. Hence, $\xi_h = \{(x, \xi_h^+(x), \xi_h^-(x))\}$ is a bipolar hesitant fuzzy soft set ideal of X.
($\Leftarrow$) Suppose that $\xi_h = \{(x, \xi_h^+(x), \xi_h^-(x))\}$ be a bipolar hesitant fuzzy soft set ideal of X, then $\xi_h^+(x \star z) \supseteq \xi_h^+((x \star y) \star z) \cap \xi_h^+(y)\}$ and $\xi_h^-(x \star z) \subseteq \xi_h^-((x \star y) \star z) \cup \xi_h^-(y)$. And by Definition 5.3.10[2], we get $\xi_h^+(x \star z) \supseteq \xi_h^+((x \star y) \star z) \cap \xi_h^+(y)\}$ and $\xi_h^-(x \star z) \subseteq \xi_h^-((x \star y) \star z) \cup \xi_h^-(y)$. Hence $\xi_h = \{(x, \xi_h^+(x), \xi_h^-(x))\}$ is a bipolar hesitant fuzzy soft set T-ideal of X of X. □

Theorem 5.3.9. *If* $\xi_h = \{(x, \xi_h^+(x), \xi_h^-(x)) : x \in X\}$ *is a bipolar hesitant fuzzy soft* T*-ideal of* X*, then the sets*

$$\xi_h = \{x \in X : \xi_h^+(x) = \xi_h^+(0) \text{ and } \xi_h^-(x) = \xi_h^-(0)\}$$

are T*-ideals of* X.

Proof. Assume that $\xi_h = \{(x, \xi_h^+(x), \xi_h^-(x)) : x \in X\}$ is a bipolar hesitant fuzzy soft T-ideal of X. We need to show that $\xi_h = \{x \in X : \xi_h^+(x) = \xi_h^+(0) \text{ and } \xi_h^-(x) = \xi_h^-(0)\}$. Since $0 \in X$, we have $\xi_h^+(0) = \xi_h^+(0)$ and $\xi_h^-(0) = \xi_h^-(0)$, which implies $0 \in \xi$. Therefore, $\xi \neq \emptyset$.

Let $(x \star y) \star z \in \xi$ and $y \in \xi$. This implies:

$$\xi_h^+((x \star y) \star z) = \xi_h^+(0) \quad \text{and} \quad \xi_h^+(y) = \xi_h^+(0).$$

Since

$$\xi_h^+(x \star z) \supseteq \min\{\xi_h^+((x \star y) \star z), \xi_h^+(y)\} = \xi_h^+(0),$$

we have

$$\xi_h^+(x \star z) \supseteq \xi_h^+(0).$$

But also,

$$\xi_h^+(x \star z) = \xi_h^+(0).$$

It follows that $(x \star z) \in \xi$ for all $x, y, z \in X$. Hence, ξ is a T-ideal of X.

Similarly, the condition $\xi_h^-(x) = \xi_h^-(0)$ also implies that ξ is a T-ideal of X with respect to the negative membership function.

□

Theorem 5.3.10. *Let* (ξ_{1h}, A) *and* (ξ_{2h}, B) *be two bipolar hesitant fuzzy soft T-ideal of X . Then their intersection* $(\xi_{1h}, A)\widetilde{\cap}(\xi_{2h}, B)$ *is a bipolar hesitant fuzzy soft T-ideal of* X.

Proof. Suppose that (ξ_{1h}, A) and (ξ_{2h}, B) are bipolar hesitant fuzzy soft subalgebras over X with the property that $A \cap B = \emptyset$. We need to show that $(\xi_{1h}, A)\widetilde{\cap}(\xi_{2h}, B)$ is also a bipolar hesitant fuzzy soft subalgebra of X. Let $(\xi_{1h}, A \cap B) = (\xi_{1h}, A)\widetilde{\cap}(\xi_{2h}, B)$ be the extended union of two bipolar hesitant fuzzy soft subalgebras. Since $A \cap B = \emptyset$, the extended union is defined as:

$$\xi_h(\epsilon) = \begin{cases} \xi_{1h}(\epsilon) & \text{if } \epsilon \in A - B, \\ \xi_{2h}(\epsilon) & \text{if } \epsilon \in B - A, \\ \xi_{1h}(\epsilon) \cap \xi_{2h}(\epsilon) & \text{if } \epsilon \in A \cap B, \end{cases}$$

Then we have the following cases: For any $x, y, z \in X$
Case 1: If $\epsilon \in A - B$, then

$$\xi_h = \xi_{1h}^+(0) \supseteq \xi_{1h}^+(x) \quad \text{and } \xi_h = \xi_{1h}^-(0) \subseteq \xi_{1h}^-(x)$$

$$\begin{aligned} \xi_h^+(x \star z) &= \xi_{1h}^+(x \star z) \\ &\supseteq \xi_{1h}^+((x \star y) \star z) \cap \xi_{1h}^+(y) \\ \xi_h^-(x \star z) &= \xi_{1h}^-(x \star z) \\ &\subseteq \xi_{1h}^-((x \star y) \star z) \cup \xi_{1h}^-(y) \end{aligned}$$

Case 2: If $\epsilon \in A - B$, then

$$\xi_h = \xi_{2h}^+(0) \supseteq \xi_{2h}^+(x) \quad \text{and } \xi_h = \xi_{1h}^-(0) \subseteq \xi_{2h}^-(x)$$

$$\begin{aligned} \xi_h^+(x \star z) &= \xi_{2h}^+(x \star z) \\ &\supseteq \xi_{2h}^+((x \star y) \star z) \cap \xi_{2h}^+(y) \\ \xi_h^-(x \star z) &= \xi_{2h}^-(x \star z) \\ &\subseteq \xi_{2h}^-((x \star y) \star z) \cup \xi_{2h}^-(y) \end{aligned}$$

Case 3: If $\epsilon \in A \cap B$, then

$$\begin{aligned} \xi_h^+(0) &= \xi_{1h}^+(0) \cap \xi_{2h}^+(0) \\ &\supseteq \xi_{1h}^+(x) \cap \xi_{2h}^+(x) = \xi^+(x) \quad \text{and} \\ \xi_h^-(0) &= \xi_{1h}^-(0) \cap \xi_{2h}^-(0) \\ &\subseteq \xi_{1h}^-(x) \cup \xi_{2h}^-(x) = \xi^-(x) \end{aligned}$$

$$\begin{aligned} \xi_h^+(x \star z) &= \xi_{1h}^+(x \star z) \cap \xi_{2h}^+(x \star z) \\ &\supseteq \xi_{1h}^+((x \star y) \star z) \cap \xi_{1h}^+(y) \cap \xi_{2h}^+((x \star y) \star z) \cap \xi_{2h}^+(y) \\ &= \xi_h^+((x \star y) \star z) \cap \xi_h^+(y) \\ \xi_h^-(x \star z) &= \xi_{1h}^-(x \star z) \cap \xi_{2h}^-(x \star z) \\ &\subseteq \xi_{1h}^-((x \star y) \star z) \cup \xi_{1h}^-(y) \cup \xi_{2h}^-((x \star y) \star z) \cup \xi_{2h}^+(y) \\ &= \xi_h^-((x \star y) \star z) \cup \xi_h^-(y) \end{aligned}$$

Therefore, $(\xi_{1h}, A)\widetilde{\cap}(\xi_{2h}, B)$ is a bipolar hesitant fuzzy soft T-ideal of X □

Theorem 5.3.11. *If (ξ_{1h}, A) and (ξ_{2h}, B) are two bipolar hesitant fuzzy soft T-ideals over X such that $(\xi_{1h}, A) \subseteq (\xi_{2h}, B)$, then $((\xi_{1h}, A)\widetilde{\cup}(\xi_{2h}, B)$ is also a bipolar hesitant fuzzy soft T-ideal over*

X.

Proof. Let (ξ_{1h}, A)) and (ξ_{2h}, B) be two bipolar hesitant fuzzy soft T-ideals over X. Then $(\xi_{1h}, A) \cup (\xi_{2h}, B) = (\xi_h, C)$, where $C = A \cup B$ and

$$\xi_h = \begin{cases} \xi_{1h} & \text{if } \epsilon \in A - B \\ \xi_{2h} & \text{if } \epsilon \in A - B \\ \xi_{1h} \cup \xi_{2h} & \text{if } \epsilon \in A \cap B \end{cases} \quad \text{for all } \epsilon \in C.$$

Then we have the following cases:
Case 1: If $\epsilon \in A - B$, then

$$\xi_h = \xi^+_{1h}(0) \supseteq \xi^+_{1h}(x) \quad \text{and } \xi_h = \xi^-_{1h}(0) \subseteq \xi^-_{1h}(x)$$

$$\begin{aligned} \xi^+_h(x \star z) &= \xi^+_{1h}(x \star z) \\ &\supseteq \xi^+_{1h}((x \star y) \star z) \cap \xi^+_{1h}(y) \\ \xi^-_h(x \star z) &= \xi^-_{1h}(x \star z) \\ &\subseteq \xi^-_{1h}((x \star y) \star z) \cup \xi^-_{1h}(y) \end{aligned}$$

Case 2: If $\epsilon \in A - B$, then

$$\xi_h = \xi^+_{2h}(0) \supseteq \xi^+_{2h}(x) \quad \text{and } \xi_h = \xi^-_{1h}(0) \subseteq \xi^-_{2h}(x)$$

$$\begin{aligned} \xi^+_h(x \star z) &= \xi^+_{2h}(x \star z) \\ &\supseteq \xi^+_{2h}((x \star y) \star z) \cap \xi^+_{2h}(y) \\ \xi^-_h(x \star z) &= \xi^-_{2h}(x \star z) \\ &\subseteq \xi^-_{2h}((x \star y) \star z) \cup \xi^-_{2h}(y) \end{aligned}$$

Case 3: If $\epsilon \in A \cap B$, then

$$\begin{aligned} \xi^+_h(0) &= \xi^+_{1h}(0) \cup \xi^+_{2h}(0) \\ &\supseteq \xi^+_{1h}(x) \cup \xi^+_{2h}(x) = \xi^+(x) \quad \text{and} \\ \xi^-_h(0) &= \xi^-_{1h}(0) \cup \xi^-_{2h}(0) \\ &\subseteq \xi^-_{1h}(x) \cup \xi^-_{2h}(x) = \xi^-(x) \end{aligned}$$

$$\begin{aligned} \xi^+_h(x \star z) &= \xi^+_{1h}(x \star z) \cup \xi^+_{2h}(x \star z) \\ &\supseteq \xi^+_{1h}((x \star y) \star z) \cap \xi^+_{1h}(y) \cup \xi^+_{2h}((x \star y) \star z) \cap \xi^+_{2h}(y) \\ &= \xi^+_h((x \star y) \star z) \cap \xi^+_h(y) \\ \xi^-_h(x \star z) &= \xi^-_{1h}(x \star z) \cup \xi^-_{2h}(x \star z) \\ &\subseteq \xi^-_{1h}((x \star y) \star z) \cup \xi^-_{1h}(y) \cup \xi^-_{2h}((x \star y) \star z) \cup \xi^+_{2h}(y) \\ &= \xi^-_h((x \star y) \star z) \cup \xi^-_h(y) \end{aligned}$$

Therefore, $(\xi_{1h},A)\widetilde{\cup}(\xi_{2h},B)$ is a bipolar hesitant fuzzy soft T-ideal of X. □

Definition 5.3.11. *Let* $f : X \to Y$ *be a hesitant fuzzy soft homomorphism, and let* $\widetilde{\Phi}$ *be a bipolar hesitant fuzzy soft set over* Y. *Then the inverse image of* $\widetilde{\Phi}$ *under* f, *denoted by* $f^{-1}(\widetilde{\Phi})$ *is a bipolar hesitant fuzzy soft set over* X *defined by:*

$$f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)(x) = \xi^{+}_{\widetilde{\Phi}h}(f(x)), \quad f^{-1}\left(\xi^{-}_{\widetilde{\Phi}h}\right)(x) = \xi^{-}_{\widetilde{\Phi}h}(f(x)), \quad \forall x \in X.$$

Conversely, let A *be a bipolar hesitant fuzzy soft set over* X. *The image of* A *under* f, *denoted by* $f(A)$ *is a bipolar hesitant fuzzy soft set over* Y *defined as follows:*

$$f(\xi^{+}_{h})(y) = \begin{cases} \bigcup\limits_{t \in f^{-1}(y)} \xi^{+}_{h}(t), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

$$f(\xi^{-}_{h})(y) = \begin{cases} \bigcap\limits_{t \in f^{-1}(y)} \xi^{-}_{h}(t), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall y \in Y.$$

where

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}.$$

Theorem 5.3.12. *Suppose that* $f : X \to Y$ *be an epimorphism mapping. let* $\widetilde{\Phi}$ *is a bipolar hesitant fuzzy soft T-ideal. Then* $f(\widetilde{\Phi})$ *is a bipolar hesitant fuzzy soft T-ideal of Y if and only if* $f^{-1}(\widetilde{\Phi})$ *is a is a bipolar hesitant fuzzy soft T-ideal of* X.

Proof. Suppose that $f(\widetilde{\Phi})$ is a bipolar hesitant fuzzy soft T-ideal of Y. We need to show that $f^{-1}(\widetilde{\Phi})$ is a is a bipolar hesitant fuzzy soft T-ideal of X. For any $y \in Y$ there exists $x \in X$ such that $f(x) = y$ we have

$$f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)(0) = \xi^{+}_{\widetilde{\Phi}h}(f(0)) = \xi^{+}_{\widetilde{\Phi}h}\left(0'\right) \supseteq \xi^{+}_{\widetilde{\Phi}h}(y) = \xi^{+}_{\widetilde{\Phi}h}(f(x)) = f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)(x)$$

and $\xi^{-}_{\widetilde{\Phi}h}(0) = \xi^{-}_{\widetilde{\Phi}h}(f(0)) = \xi^{-}_{\widetilde{\Phi}h}(0') \supseteq xi^{-}_{\widetilde{\Phi}h}(y) = \xi^{-}_{\widetilde{\Phi}h}(f(x)) = f^{-1}\left(\xi^{-}_{\widetilde{\Phi}h}\right)(x)$.
Let $x, y, z \in \mathcal{X}, \tau' \in Y$ then $\exists \tau \in X$ such that $f(\tau) = \tau'$. We have

$$\begin{aligned} f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)(x \star z) = \xi^{+}_{\widetilde{\Phi}h}\left(f(x \star z)\right) = \xi^{+}_{\widetilde{\Phi}h}(f(x) \star f(z)) &\supseteq \min\left\{\xi^{+}_{\widetilde{\Phi}h}\left(f(x) \star \tau'\right) \star f(z)\right), \vartheta^{+}(\tau)\right\} \\ &= \min\left\{\xi^{+}_{\widetilde{\Phi}h}((f(x) \star f(\tau)) \star f(z)), xi^{+}_{\widetilde{\Phi}h}(f(\tau))\right\} \\ &= \min\left\{f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)((x \star \tau) \star z), f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)(\tau)\right\} \end{aligned}$$

And

$$
\begin{aligned}
f^{-1}\left(\xi^{-}_{\widetilde{\Phi}h}\right)(x\star z) = \xi^{+}_{\widetilde{\Phi}h}(f(x\star z)) = \xi^{-}_{\widetilde{\Phi}h}(f(x)\star f(z))\\
&\subseteq \max\left\{\xi^{-}_{\widetilde{\Phi}h}\left(f(x)\star\tau'\right)\star f(z), \xi^{-}_{\widetilde{\Phi}h}(f(\tau))\right\}\\
&= \max\left\{\xi^{-}_{\widetilde{\Phi}h}(f(x)\star f(\tau)\star f(z)), \xi^{-}_{\widetilde{\Phi}h}(f(\tau))\right\}\\
&= \max\left\{f^{-1}\left(\xi^{-}_{\widetilde{\Phi}h}\right)((x\star\tau)\star z), f^{-1}\left(\xi^{-}_{\widetilde{\Phi}h}\right)(f(\tau))\right\}
\end{aligned}
$$

Hence $f^{-1}\left(\xi_{\widetilde{\Phi}h}\right)$is a bipolar hesitant fuzzy soft T-ideal of X. Conversely, since $f : X \to Y$ is an onto mapping, then for any $x,\tau,z \in Y$. It follows that, there exists $a,b,c \in X$ 'such that $f(a) = x, f(b) = \tau$ and $f(c) = z$. we have

$$
\begin{aligned}
\xi^{+}_{\widetilde{\Phi}h}(x\star z) = \xi^{+}_{\widetilde{\Phi}h}(f(a)\star f(c)) = \xi^{+}_{\widetilde{\Phi}h}(f(a\star c)) = f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)(a\star c)\\
&\supseteq \min\left\{f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)((a\star b)\star c), f^{-1}\left(\xi^{+}_{\widetilde{\Phi}h}\right)(b)\right\}\\
&= \min\left\{\xi^{+}_{\widetilde{\Phi}h}(f(a)\star f(b))\star f(c), \xi^{+}_{\widetilde{\Phi}h}(f(b))\right\} = \min\left\{\xi^{+}_{\widetilde{\Phi}h}((x\star\tau)\star z), \xi^{+}_{\widetilde{\Phi}h}(\tau)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\xi^{-}_{\widetilde{\Phi}h}(x\star z) = \xi^{-}_{\widetilde{\Phi}h}(f(a)\star f(c)) = \xi^{-}_{\widetilde{\Phi}h}(f(a\star c)) = f^{-1}\left(\xi^{-}_{\widetilde{\Phi}h}\right)(a\star c)\\
&\subseteq \max\left\{f^{-1}\left(\xi^{-}_{\widetilde{\Phi}h}\right)((a\star b)\star c), f^{-1}\left(\xi^{-}_{\widetilde{\Phi}h}\right)(b)\right\}\\
&= \max\left\{\xi^{-}_{\widetilde{\Phi}h}((f(a)\star f(b))\star f(c)), \xi^{-}_{\widetilde{\Phi}h}(f(b))\right\}\\
&= \max\left\{\xi^{-}_{\widetilde{\Phi}h}((x\star\tau)\star z), \xi^{-}_{\widetilde{\Phi}h}(\tau)\right\}
\end{aligned}
$$

Therefore $f(\widetilde{\Phi})$is a bipolar hesitant fuzzy soft T-ideal of Y. □

Definition 5.3.12. *Let $\xi_{1h} = \left\{\left(\xi^{+}_{1h}, \xi^{-}_{1h}\right)\right\}$ and $\xi_{2h} = \left\{\left(\xi^{+}_{2h}, \xi^{-}_{2h}\right)\right\}$ be two bipolar hesitant fuzzy soft set of X. Then the Cartesian product $\xi_{1h}\times\xi_{2h} = \left((x\times y), \xi^{+}_{1h}\times\xi^{+}_{2h}, \xi^{-}_{1h}\times\xi^{-}_{2h}\right)$ is defined by the following:*

1. $\left(\xi^{+}_{1h}\times\xi^{+}_{2h}\right)(x,y) = \min\left\{\xi^{+}_{1h}(x), \xi^{+}_{2h}(y)\right\}$
2. $\left(\xi^{-}_{1h}\times\xi^{-}_{2h}\right)(x,y) = \max\left\{\xi^{-}_{1h}(x), \xi^{-}_{2h}(y)\right\}$, *where* $\xi^{-}_{1h}\times\xi^{-}_{2h} : X\times Y \to [-1,0]$ *and* $\xi^{+}_{1h}\times\xi^{+}_{2h} : X\times Y \to [0,1], \forall (x,y) \in X\times Y$.

Theorem 5.3.13. *Let $\xi_{1h} = \left\{\left(\xi^{+}_{1h}, \xi^{-}_{1h}\right)\right\}$ and $\xi_{2h} = \left\{\left(\xi^{+}_{2h}, \xi^{-}_{2h}\right)\right\}$ be two bipolar hesitant fuzzy soft T-ideals of X and Y respectively, then $\xi_{1h}\times\xi_{2h}$ is a bipolar hesitant fuzzy soft T-ideal of $X\times Y$.*

Proof. Suppose that $\xi_{1h} = \left\{\left(\xi^{+}_{1h}, \xi^{-}_{1h}\right)\right\}$ and $\xi_{2h} = \left\{\left(\xi^{+}_{2h}, \xi^{-}_{2h}\right)\right\}$ be two bipolar hesitant fuzzy soft T-ideals of X and Y respectively. We need to show that $\xi_{1h}\times\xi_{2h}$ is a bipolar hesitant fuzzy soft T-ideal of $X\times Y$. For any $(x,y) \in X\times Y$, we have

$$
\left(\xi^{+}_{1h}\times\xi^{+}_{2h}\right)(0,0) = \min\left\{\xi^{+}_{1h}(0), \xi^{+}_{2h}(0)\right\} \supseteq \min\left\{\xi^{+}_{1h}(x), \xi^{+}_{2h}(y)\right\} = \left(\xi^{+}_{1h}\times\xi^{+}_{2h}\right)(x,y) \text{and}
$$

$$\left(\xi_{1h}^{-}\times\xi_2h^{-}\right)(0,0)=\max\left\{\xi_{1h}^{-}(0),\xi_{2h}^{-}(0)\right\}\subseteq\max\left\{\xi_{1h}^{-}(x),\xi_{2h}^{-}(y)\right\}=\left(\xi_{1h}^{-}\times\xi_{2h}^{-}\right)(x,y)$$

Let $(x_1,x_2),(y_1,y_2)$ and $(z_1,z_2)\in X\times Y$, then

$$\begin{aligned}
(\xi_{1h}^{+}\times\xi_{2h}^{+})(x_1\star z_1,x_2\star z_2) &= \min\{\xi_{1h}^{+}(x_1\star z_1),\,\xi_{2h}^{+}(x_2\star z_2)\}\\
&\supseteq \min\left\{\min\{\xi_{1h}^{+}((x_1\star y_1)\star z_1),\,\xi_{1h}^{+}(y_1)\},\min\{\xi_{2h}^{+}((x_2\star y_2)\star z_2),\,\xi_{2h}^{+}(y_2)\}\right\}\\
&= \min\left\{\min\{\xi_{1h}^{+}((x_1\star y_1)\star z_1),\,\xi_{2h}^{+}((x_2\star y_2)\star z_2)\},\min\{\xi_{1h}^{+}(y_1),\,\xi_{2h}^{+}(y_2)\}\right\}\\
&= \min\left\{(\xi_{1h}^{+}\times\xi_{2h}^{+})((x_1\star y_1)\star z_1,\,(x_2\star y_2)\star z_2),\,(\xi_{1h}^{+}\times\xi_{2h}^{+})(y_1,y_2)\right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
(\xi_{1h}^{-}\times\xi_{2h}^{-})(x_1\star z_1,x_2\star z_2) &= \max\{\xi_{1h}^{-}(x_1\star z_1),\,\xi_{2h}^{-}(x_2\star z_2)\}\\
&\subseteq \max\left\{\max\{\xi_{1h}^{-}((x_1\star y_1)\star z_1),\,\xi_{1h}^{-}(y_1)\},\max\{\xi_{2h}^{-}((x_2\star y_2)\star z_2),\,\xi_{2h}^{-}(y_2)\}\right\}\\
&= \max\left\{\max\{\xi_{1h}^{-}((x_1\star y_1)\star z_1),\,\xi_{2h}^{-}((x_2\star y_2)\star z_2)\},\max\{\xi_{1h}^{-}(y_1),\,\xi_{2h}^{-}(y_2)\}\right\}\\
&= \max\left\{(\xi_{1h}^{-}\times\xi_{2h}^{-})((x_1\star y_1)\star z_1,\,(x_2\star y_2)\star z_2),\,(\xi_{1h}^{-}\times\xi_{2h}^{-})(y_1,y_2)\right\}.
\end{aligned}$$

Therefore, $\xi_{1h}\times\xi_{2h}$ is a bipolar hesitant fuzzy soft T-ideal of $X\times Y$. □

5.3.3 Bipolar Hesitant Fuzzy Soft Sets with Multicriteria Decision Making

The bipolar hesitant fuzzy soft set (BHFSS) based decision-making model offers a robust mechanism to evaluate alternatives under hesitant, bipolar, and fuzzy conditions, making it highly applicable to real-world multi-criteria decision problems like selecting suitable products, services, or strategies.

The BHFSS is a powerful tool for handling uncertainty, vagueness, and bipolarity in decision-making problems. Real-life scenarios often involve conflicting and hesitant information, which cannot be effectively captured using classical models. In this section, we explore how the BHFSS framework can be applied to solve a prevailing multi-criteria decision-making problem: the selection of an appropriate alcoholic beverage based on various subjective and objective criteria. We employ the theoretical foundations of BHFSS to model the evaluation process of different alcoholic drinks across multiple critria, each possessing both positive and negative degrees of hesitation. A systematic approach is presented to construct the BHFSS model, followed by a decision-making algorithm that identifies the most suitable alternative. The methodology is validated through an illustrative example. The algorithm proposed provides a generalized set of rules to assist decision-makers in selecting the best alternative based on the input data (See Algorithm 5.3.17).

Definition 5.3.13. *(Score Function of a Bipolar Hesitant Fuzzy Element) Let* $\xi_h(x)=(\xi_h^{+}(x),\xi_h^{-}(x))$ *be a bipolar hesitant fuzzy element, where:*$\xi_h^{+}(x)\subseteq[0,1]$ *is the set of positive membership degrees,* $\xi_h^{-}(x)\subseteq[-1,0]$ *is the set of negative membership degrees. Then, the score function* $S(\xi_h(x))$ *of* *the bipolar hesitant fuzzy element is defined as:*

$$S(\xi_h(x)) = \left(\frac{1}{l^+(\xi_h(x))} \sum_{\gamma^+ \in \xi_h^+(x)} \gamma^+, \frac{1}{l^-(\xi_h(x))} \sum_{\gamma^- \in \xi_h^-(x)} \gamma^- \right)$$

where:

(i) $l^+(\xi_h(x))$ *is the number of elements in* $\xi_h^+(x)$*,*

(ii) $l^-(\xi_h(x))$ *is the number of elements in* $\xi_h^-(x)$*.*

Definition 5.3.14. *(Score Matrix of a Bipolar Hesitant Fuzzy Soft Set) Let* (F, A) *be a bipolar hesitant fuzzy soft set, where* $F : A \to \widetilde{BH}(U)$*, and* $\widetilde{BH}(U)$ *is the set of all bipolar hesitant fuzzy subsets of universe* U*. Then, the score matrix is a bipolar fuzzy soft set* (F_s, A)*, where each entry of* $F_s(\epsilon)(x)$ *is the score function of the corresponding bipolar hesitant fuzzy element* $F(\epsilon)(x)$*, i.e.,*

$$F_s(\epsilon)(x) = S(F(\epsilon)(x)) = \big(S^+(F(\epsilon)(x)), S^-(F(\epsilon)(x))\big), \quad \textit{for all} \quad x \in U, \forall \epsilon \in A.$$

Definition 5.3.15. *(Bipolar Decision Table) The bipolar decision table is created by listing the score values* $F_s(\epsilon)(d_j)$ *for every object* $d_j \in U$ *and for each parameter* $\epsilon \in A$*. The entries are tuples* (s^+, s^-) *showing both positive and negative scores. This decision table helps in determining the best alternative, allowing decision-makers to weigh both favorable and unfavorable aspects.*

Definition 5.3.16. *(Bipolar Comparison Table) Let* $d_1, d_2, \ldots, d_n$ *be objects in the universe. The bipolar comparison table is a square matrix of size* $n \times n$*, where each entry* ϵ_{ij} *represents the number of parameters* $\epsilon \in A$ *for which:*

$$S^+(F(\epsilon)(d_i)) \geqslant S^+(F(\epsilon)(d_j)) \quad \textit{and} \quad S^-(F(\epsilon)(d_i)) \geqslant S^-(F(\epsilon)(d_j)).$$

That is, ϵ_{ij} counts how many parameters indicate that d_i is at least as preferable as d_j based on both positive and negative scores.

5.3.4 Numerical Example: Alcoholic Drink Selection Using Bipolar Hesitant Fuzzy Soft Set

Let's consider a real-world decision-making scenario. Let $U = \{d_1, d_2, d_3, d_4\}$ be the set of four alcoholic drinks under consideration such as, whiskey, wine, beer, vodka. Let $E = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\}$, where: ϵ_1 = Expensive ϵ_2 = Taste ϵ_3 = Health Impact ϵ_4 = Brand Reputation ϵ_5 = Popularity be a set of parameters. Suppose a person, Mr. Y, wants to select an alcoholic drink based on a few parameters important to him.
Assume that Mr. Y's desired parameters (subset $A \subseteq E$) : $A = \{\epsilon_1, \epsilon_4, \epsilon_5\}$. Now, define the bipolar hesitant Fuzzy Soft Set.The Bipolar Hesitant Fuzzy Soft Set F is given below. For each selected parameter, the hesitant fuzzy values (positive and negative) for each drink are provided.

Drink	Positive Memberships	Negative Memberships
d_1	{0.9, 0.85}	{−0.3, −0.2}
d_2	{0.6, 0.5}	{−0.4, −0.35}
d_3	{0.7, 0.65}	{−0.6, −0.3}
d_4	{0.95, 0.8}	{−0.2, −0.1}

Table 5.7: $F(\epsilon_1)$– Expensive.

Drink	Positive Memberships	Negative Memberships
d_1	{0.7, 0.65}	{−0.4, −0.35}
d_2	{0.85, 0.8}	{−0.3, −0.25}
d_3	{0.6, 0.55}	{−0.5, −0.45}
d_4	{0.75, 0.7}	{−0.2, −0.15}

Table 5.8: $F(\epsilon_4)$ – Brand Reputation.

Drink	Positive Memberships	Negative Memberships
d_1	{0.8, 0.75}	{−0.35, −0.3}
d_2	{0.7, 0.6}	{−0.4, −0.3}
d_3	{0.65, 0.6}	{−0.45, −0.4}
d_4	{0.85, 0.8}	{−0.2, −0.15}

Table 5.9: $F(\epsilon_5)$– Popularity.

Definition 5.3.17. *(Algorithm)[See[8]]*

To compute the comparison tables for the positive information function and negative information function, we follow these steps:

Step 1 Extract Positive and Negative Membership Values

For each drink d_i and parameter ϵ_j, we have:

(i) Positive memberships (PM): Represent how well the drink satisfies the parameter.

(ii) Negative memberships (NM): Represent dissatisfaction or drawbacks.

Given $A = \{\epsilon_1, \epsilon_4, \epsilon_5\}$, we consider the tables for these parameters.

Algorithm : To find the best choice of drinks according to Mr. Y interest	
Step	**Description**
1	Define the set of drinks as U, the set of evaluation parameters as E, and the set of desired parameters as A.
2	Provide the bipolar hesitant fuzzy values for each drink with respect to each parameter.
3	Compute the score for each drink using the formula: Score = Average of positive membership degrees − Average of absolute negative membership degrees
4	Rank all the drinks based on their computed scores and select the best option.

Table 5.10: Algorithm to Select Best drinks Based on Mr. Y Preferences

Drink	ϵ_1	ϵ_4	ϵ_5
d_1	{0.9, 0.85}	{0.8, 0.95}	{0.8, 0.75}
d_2	{0.6, 0.5}	{0.85, 0.8}	{0.7, 0.6}
d_3	{0.7, 0.65}	{0.6, 0.55}	{0.65, 0.6}
d_4	{0.95, 0.8}	{0.75, 0.7}	{0.85, 0.8}

Table 5.11: Positive Membership Values (ξ_{ij}^{+}).

Drink	ϵ_1	ϵ_4	ϵ_5
d_1	{−0.3, −0.2}	{−0.4, −0.35}	{−0.35, −0.3}
d_2	{−0.4, −0.35}	{−0.3, −0.25}	{−0.4, −0.3}
d_3	{−0.6, −0.3}	{−0.5, −0.45}	{−0.45, −0.4}
d_4	{−0.2, −0.1}	{−0.2, −0.15}	{−0.2, −0.15}

Table 5.12: Negative Membership Values (ξ_{ij}^{-}).

Step 2 Compute the Positive Comparison Table

For each pair of drinks (d_i, d_j) the entries d_{ij} where d_{ij} = the number of parameters for which the value of d_i exceeds or equal to the value of d_j . Here, we compare the average of the positive memberships for each parameter.

Drink	ϵ_1	ϵ_4	ϵ_5
d_1	0.875	0.875	0.775
d_2	0.55	0.825	0.65
d_3	0.675	0.575	0.625
d_4	0.875	0.725	0.825

Table 5.13: Average Positive Memberships.

Positive Comparison Table (ξ^{+}):

For each (d_i, d_j), count how many parameters d_i is better than or equal to d_j.

	d_1	d_2	d_3	d_4
d_1	–	3	3	1
d_2	1	–	2	0
d_3	0	1	–	0
d_4	2	3	3	–

Table 5.14

Row Sums (R_i): d_1= 7, d_2= 3, d_3= 1, d_4= 8

Step 3 Compute the Negative Comparison Table

For each pair $(d_i, d_j)$the entries d_{ij} where d_{ij} = the number of parameters for which the value of d_i exceeds or equal to the value of d_j . (since less negative is better). Negative Comparison

Drink	ϵ_1	ϵ_4	ϵ_5
d_1	−0.25	−0.375	−0.325
d_2	−0.375	−0.275	−0.35
d_3	−0.45	−0.475	−0.425
d_4	−0.15	−0.175	−0.175

Table 5.15: Average Negative Memberships

Table (ξ^-) for each (d_i, d_j), count how many parameters d_i is better than or equal to d_j.

	d_1	d_2	d_3	d_4
d_1	–	2	3	0
d_2	1	–	3	0
d_3	0	0	–	0
d_4	3	3	3	–

Table 5.16

Row Sums (S_i): d_1= 5, d_2= 4, d_3= 0, d_4= 9

Step 4 Compute Total Scores ($t_i = r_i - s_i$)

$d_1 = 7 - 5 = 2$, $d_2 = 3 - 4 = -1$, $d_3 = 1 - 0 = 1$, $d_4 = 8 - 9 = -1$

Final Ranking:

(i) d_1 (Score= 2)

(ii) d_2 (Score= -1)

(iii) d_3 (Score= 1)

(iv) d_4 (Score= -1)

Finally, the most preferred drink is d_1 , while d_2, d_3, and d_4 are less favorable based on Mr. Y's criteria. It scores highest based on the combination of being expensive, reputable, and popular. This method uses the bipolar hesitant fuzzy soft set approach, allowing nuanced decision-making with both positive and negative uncertainty.

Conclusion and Future Works

In this section, we present what we have done in this dissertation and then indicate our future work.

I. Conclusion

Mathematics serves as a foundational tool in across a wide range of disciplines, including natural sciences, technology, economics, and decision-making. Traditionally, classical mathematics and set theory operate under binary logic, where elements either belong to a set or do not, as initially formalized by Georg Cantor in 1874 [12]. This led to the notion of crisp sets, characterized by membership values strictly in $\{0,1\}$. However, real-world situations are rarely black and white; they often involve ambiguity, vagueness, and hesitation. To model such uncertainties, various extensions of classical set theory have been developed. A major advancement in this regard was fuzzy set theory, introduced by Zadeh in 1965 [72], which permits partial membership of elements in a set with values in the interval $[0,1]$.

Parallel to this, soft set theory, introduced by Molodtsov in 2010 [31], offers a parameterized framework to model uncertainty without requiring a membership function. These two frameworks laid the groundwork for hybrid models such as hesitant fuzzy sets introduced by Torra in 2010 [1, 56, 64, 65], which allow multiple possible membership values for each element, capturing hesitation in assigning a precise degree of belonging. Further generalizations such as intuitionistic fuzzy sets by Atanassov in 1986 , interval-valued fuzzy sets by Turksen in 1992, type-2 fuzzy sets by Mendel in 2017, and fuzzy multisets by Miyamoto in 2000 [1, 6, 45, 46, 66] were developed to address increasingly complex types of uncertainty. Among these, hesitant fuzzy sets and soft sets have gained significant attention for their flexibility and applicability in decision-making environments.

Babitha in 2013 [7] introduced the hesitant fuzzy soft set merging hesitant fuzzy sets with soft sets. This model is particularly effective in situations where decisions are influenced by multiple, possibly conflicting, expert opinions or criteria. To further reflect the bipolar nature of real-life judgments where both positive and negative aspects may co-exist bipolar fuzzy sets were introduced by Zhang in 1994 [75]. These sets assign two membership degrees to each element: a positive one in $[0,1]$ and a negative one in $[-1,0]$. Abdullaha in 2014 [2] extended this idea into bipolar fuzzy soft sets , where bipolar assessments are parameterized by soft sets.

Building on this, Zhang in 2013 [73] proposed the bipolar hesitant fuzzy soft set , which combines bipolarity, hesitation, and parameterization. This model allows multiple degrees of positive and negative membership per element and has been applied in domains like AI, medical diagnosis, and multi-criteria decision-making. In parallel with fuzzy modeling, algebraic structures such as groups, rings, and algebras are central to abstract mathematical reasoning. Specialized algebras such as BCK-algebras and BCI-algebras, introduced by Tanaka in 1978 [24] and Iséki in 1980 [25] respectively and have been explored for their logical and computational significance. These were generalized into Q-algebras by Neggers in 2001 [51], and further into TM-algebras by Megalai and Tamilarasi [44], which unify properties from BCK, BCI, and Q-algebras.

TM-algebras provide an algebraic framework for modeling reasoning processes and decision-making algorithms, especially in AI contexts. Fuzzy variants, such as fuzzy TM-subalgebras and fuzzy TM-ideals, were introduced by Megalai in 2011 [43], while homomorphisms in fuzzy TM-algebras were studied by Prabpayak in 2017 [54]. Georgescu in 2001 [19] proposed the concept of pseudo-BCK algebra as a generalization of BCK-algebra. Later, Dudek in 2008 [15] introduced the notion of pseudo-BCI algebra this extends the classical structures by relaxing certain axioms, accommodating asymmetry in operations. Pseudo-BCI ideals and filters have been studied extensively by Jun in 2006 [27], and more recently, pseudo-TM algebras were introduced as a generalization of TM-algebras by Nouri in 2019 [52]. Fuzzy algebra, as a discipline, focuses on incorporating fuzziness into algebraic systems. The foundation was laid by Rosenfeld in 1971 [58], who introduced fuzzy subgroups, thereby fuzzifying group theory. Over time, this idea expanded to rings, modules, lattices, and logical algebras. While not all classical results carry over directly [30, 34], many have been successfully adapted. Notable developments include fuzzy ideals in rings Liu 1982 [38], BCK-algebras Xi 1991 [69], and fuzzy congruences Murali, 1991 [50]. Fuzzy congruence relations generalize classical congruence by allowing degrees of equivalence and have been investigated in various algebraic systems. In groups and rings, fuzzy congruences allow for the construction of fuzzy quotient structures, extending ideas like normal subgroups and ideals into the fuzzy domain. Similar extensions are seen in lattices, modules, semigroups, and logical algebras.

In more specialized settings, intuitionistic fuzzy congruences, bipolar fuzzy congruences, and hesitant fuzzy congruences have emerged. These advanced models address situations involving with incomplete, contradictory, or hesitant information, and they find applications in social networks, decision support systems, and soft computing. Building upon this, Ahamed introduced fuzzy BCI-algebras in 1993 [3], followed by Mostafa in 1995 [49] work on fuzzy KU-ideals, and Somjanta 2016 [62] study of fuzzy UP-subalgebras. Notably, Jun in 2016 [29] hesitant fuzzy structures have been applied to BCK/BCI-algebras, and even neutrosophic hesitant fuzzy subalgebras have been studied. Despite the rich development of fuzzy and hesitant fuzzy algebraic structures, a notable gap remains in the study of such structures in the context of pseudo-TM and TM-algebras. This dissertation aims to bridge that gap by developing a systematic framework for hesitant fuzzy algebraic structures on pseudo-TM and TM-algebras.

The primary objective of this research is to define and analyze new types of fuzzy subalgebras, ideals, and congruences , and to extend them into hesitant fuzzy, bipolar fuzzy, and soft set frameworks. These extensions are essential for modeling real-world problems that involve uncertainty, hesitation, and multiple criteria. The first major contribution of this dissertation is the development of fuzzy pseudo-TM subalgebras and fuzzy pseudo-TM ideals. In this context, the study defined fuzzy pseudo-TM subalgebras and examined their properties. It was shown that the intersection of fuzzy pseudo-TM subalgebras yields another fuzzy pseudo-TM subalgebra, whereas their union does not necessarily hold. The concept of fuzzy congruence relations was also introduced, and their relation with fuzzy ideals was analyzed. These results provide a solid theoretical foundation for extending classical algebraic properties into fuzzy environments.

Another important contribution of the dissertation is the introduction of homomorphisms and Cartesian product operations in these new fuzzy algebraic structures. Homomorphisms help in understanding how structure-preserving mappings behave under fuzzification. For example, it was proven that the image and pre-image of a fuzzy pseudo-ideal under a homomorphism are again fuzzy pseudo-ideals under some conditions. Similar results were found for hesitant fuzzy and hesitant fuzzy soft structures. The use of Cartesian products helps in constructing new fuzzy structures from existing ones and is particularly useful for building compound systems in applications. The study also explored the concept of fuzzy congruence relations in pseudo-TM algebras. These are fuzzy generalizations of classical equivalence relations. It was shown that such fuzzy congruences are closely connected to fuzzy ideals, and they help classify elements in fuzzy algebraic systems. This classification simplifies the structure and allows for modular construction of larger fuzzy systems.

Next, the research focused on hesitant fuzzy structures. Hesitant fuzzy TM-subalgebras and hesitant fuzzy T-ideals were defined for TM-algebras. Their behaviors were explored using level subsets and Cartesian products. One of the significant results is that the Cartesian product of two hesitant fuzzy T-ideals is again a hesitant fuzzy T-ideal. Demonstrating that the structure is stable under Cartesian operations. The behavior of these hesitant fuzzy sets under homomorphisms was also investigated, yielding important results regarding the preservation of images and pre-images of the algebraic structure. This part of the study shows that hesitant fuzzy logic provides a flexible and precise framework for modeling algebraic systems when there is hesitation in membership degrees.

Further, the study developed the concept of hesitant fuzzy soft TM-algebras, which combines hesitant fuzzy sets with soft set theory. In these systems, elements are evaluated with respect to several parameters, and for each parameter, there can be multiple hesitant membership values. This model is very suitable for decision-making scenarios where opinions vary, and decisions must be made under uncertainty and hesitation. The study defined hesitant fuzzy soft TM-subalgebras and soft T-ideals and investigated their algebraic properties.

In addition to theoretical development, the study provided bipolar hesitant fuzzy soft sets in a multicriteria decision-making. A numerical example was given on selecting an alcoholic drink based on different criteria. This example demonstrated how bipolar hesitant fuzzy soft sets can systematically compare and evaluate alternatives based on multiple criteria with both positive and negative hesitations. Such applications are essential because they show how abstract mathematical models can help in real-life decision problems such as product selection, medical diagnosis, risk evaluation, and so on.

The combination of fuzzy logic, hesitant fuzzy logic, soft sets, and bipolar evaluations allows handling complex, uncertain, and vague information. This makes the developed algebraic models suitable for various applications, including artificial intelligence, decision sciences, expert systems, and machine learning. Specifically, TM and pseudo-TM algebras, when enriched with fuzzy logic, provide a strong algebraic foundation to represent and process human-like reasoning and vague judgments.

In summary, the main contributions of the dissertation are:

1. Introduction and definition of fuzzy pseudo-TM subalgebras and fuzzy pseudo-TM ideals.
2. Study of their properties under intersection, union, homomorphism, and Cartesian product.
3. Development of fuzzy congruence relations and their connection with fuzzy pseudo-ideals.
4. Definition of hesitant fuzzy TM-subalgebras and hesitant fuzzy T-ideals, along with their theoretical analysis.
5. Extension of these concepts to hesitant fuzzy soft TM-algebras.
6. Establishment of key algebraic properties (homomorphism, Cartesian product, level sets) for all these structures.
7. Develope bipolar hesitant fuzzy soft sets to multicriteria decision-making problems.
8. Provision of numerical examples showing practical relevance.

In conclusion, this dissertation provides a deep theoretical understanding of fuzzy algebraic structures and introduces new tools for modeling uncertainty, hesitation, and decision-making in algebraic frameworks. The results form a strong foundation for both academic exploration and practical applications, especially in situations where decisions are made under vague or conflicting information.

II. Future Works

Although the dissertation achieved its objectives, it also opens new directions for further research:

1. Investigating interval-valued hesitant fuzzy soft subalgebras in TM-algebras and pseudo-TM algebras.
2. Exploring neutrosophic fuzzy and hesitant neutrosophic fuzzy algebraic structures on pseudo-TM algebra.
3. Intuitionistic fuzzy pseudo-TM algebra.
4. Developing decision-making algorithms based on hesitant fuzzy soft TM-ideals.

5. Applying the proposed models to medical diagnosis, risk assessment, and artificial intelligence.
6. Studying the computational complexity and implementation of the theoretical results in software tools.

Bibliography

[1] M.Y. Abbasi, A.F. Talee, S.A. Khan, & K. Hila, A Hesitant Fuzzy Set Approach to Ideal Theory in Γ-Semigroups. Advances in Fuzzy Systems, **1**(2018), 5738024.

[2] S. Abdullah, M. Aslam, and K. Ullah , Bipolar fuzzy soft sets and its applications in decision making problem, Journal of Intelligent and Fuzzy Systems, **27**(2)(2014), 729-742.

[3] M.M. Ahamed, Fuzzy BCI-algebras. Bull. Calcutta Math. Soc., **85**(3)(1993), 215–222.

[4] M.I. Ali, and F. Feng, A Note on (Î, Î Ú q)-Fuzzy Equivalence Relations and Indistinguishability Operators ABSTRACT FULL TEXT. Hacettepe Journal of Mathematics and Statistics, **40** (3) (2011), 383-400.

[5] M. Annalakshmi, and M. Chandramouleeswaran, Fuzzy topological subsystem on a TM-algebra, Int. J. Pure Appl. Math, **94** (3)(2014), 439-449.

[6] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, **20**(1986), 87–96.

[7] K. V. Babitha, and S.J. John, Hesitant fuzzy soft sets, Journal of New Results in Science, **2**(3)(2013).

[8] M. Balamurugan, N. Alessa, K. Loganathan, & M.M.S. Kumar, Bipolar intuitionistic fuzzy soft ideals of BCK/BCI-algebras and its applications in decision-making. Mathematics, **11**(21)(2023), 4471.

[9] P. Bhattacharya, and N.P. Mukherjee, Fuzzy relations and fuzzy groups. Information sciences, **36**(3)(1985), 267-282.

[10] M. Borah, T.J. Neog, and D.K. Sut, A study on some operations of fuzzy soft sets. International Journal of Modern Engineering Research (IJMER), **2**(2)(2012), 157-168.

[11] N. Cagman, S. Enginoglu, and F. Citak, Fuzzy soft set theory and its applications, Iranian journal of fuzzy systems, **8**(3) (2011), 137-147.

[12] G. Cantor, On a property of the class of all real algebraic numbers. Crelle's Journal for Mathematics, **77**(1874), 258-262.

[13] M. A. Chaudhry, and H. Fakhar-Ud-Din, On some classes of BCH-algebras, International Journal of Mathematics and Mathematical Sciences, **25**(3)(2001), 205-211.

[14] O. Dalkılıç,& N. Demirtaş, Bipolar fuzzy soft set theory applied to medical diagnosis, Turkish Journal of Mathematics and Computer Science, **16**(2)(2024), 314-324.

[15] W. A. Dudek, and Y.B. Jun, Pseudo-bci algebras, East Asian Mathematical Journal, **24** (2) (2008), 187-190.

[16] T. Gaketem, P. Khamrot, R. Prasertpong, and A. Iampan, A Novel Concept of Bipolar Fuzzy Sets in UP (BCC)-Algebras, Bipolar Fuzzy Implicative UP (BCC)-Filters, International Journal of Analysis and Applications, **23**(2025), 71-71.

[17] T. Ganeshkumar, M. Chandramouleeswaran, Symmetric biderivations on TM-algebra, Int. J. Pure Appl. Math, **87** (2013), 9–18.

[18] T. Ganeshkumar, and M. Chandramouleeswaran, t-derivations on TM-algebras, Int. J. Pure Appl. Math, **85**(1)(2013), 95-107.

[19] G. Georgescu, and A. Iorgulescu, Pseudo-BCK algebras, an extension of BCK algebras, In Combinatorics, Computability and Logic: Proceedings of the Third International Conference on Combinatorics, Computability and Logic,(DMTCS'01), (2001), 97-114.

[20] F.M. Ghlaim,F.F. Kareem, Interval valued fuzzy ideals of TM-algebra, Journal of Interdisciplinary Mathematics.

[21] M. Hamidi, Computation of Fundamental TM-algebras, (2014).

[22] A.H. Handam, Quotient TM-algebras, In Int. Math. Forum **8**(2013), 949-956).

[23] A. Iampan, and D.A. Romano, Congruence relations on pseudo-UP algebras. Computer Science, **17**(1)(2022),1-9.

[24] K. Iseki, and S.Tanaka , An introduction to the theory of BCK-algebras, Math Japonica, **23**(1978), 1- 20.

[25] K. Iseki, On BCI-algebras, In Math. Seminar Notes, **8**(1980), 125-130.

[26] Y. B. Jun, S.S. Ahn, and G. Muhiuddin, Hesitant Fuzzy Soft Subalgebras and Ideals in BCK/BCI-Algebras, The Scientific World Journal, **1**(2014), 763929.

[27] Y.B. Jun, H.S. Kim and J. Neggers, On pseudo-BCI ideals of pseudo-BCI algebras, Matematics Vesnik, **58**(1-2) (2006), 39-46.

[28] Y.B. Jun, Fuzzy pseudo-ideals in pseudo BCK-algebras. Inform. Sci., **150**(1–2) (2003), 49–60.

[29] Y. B. Jun, & S.S. Ahn, Hesitant fuzzy set theory applied to BCK/BCI-algebras, Journal of Computational Analysis & Applications, **20**(1)(2016).

[30] A.K. Katsaras, Fuzzy topological vector spaces, Fuzzy Sets and Systems, **12**(2)(1984), 143–154.

[31] O. S. M. A. N. Kazanci, S. Yilmaz, and S. U. L. T. A. N. Yamak, Soft sets and soft BCH-algebras, Hacettepe Journal of Mathematics and Statistics, **39** (2) (2010), 205-217.

[32] J. H. Kim, Y. B. Jun, P. K. Lim, J. G. Lee, and K. Hur, The category of hesitant H-fuzzy sets, Infinite Study, (2019).

[33] J. Kim, P. K. Lim, J. G. Lee, and K. Hur. Hesitant fuzzy subgroups and subrings, Ann. Fuzzy Math. Inform, **18** (2) (2019), 105-122.

[34] N. Kuroki, Fuzzy ideals and fuzzy subrings. Math. Japonica, **26**(2)(1981), 153–160.

[35] K.M. Lee, Bipolar-valued fuzzy sets and their operations, In Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, (2000), 307-312.

[36] K.M. Lee, Bipolarvalued fuzzy sets and their basic operations, Proceeding International Conference Bangkok Thailand **650**(2000), 307–317.

[37] K.J. Lee, Fuzzy pseudo ideals in BCI-algebras. Bull. Korean Math. Soc., **41**(1)(2004), 135–144.

[38] A. Liu, Fuzzy ideals in rings. Fuzzy Sets and Systems, **7**(3)(1982), 251–259.

[39] P. K. Maji, R. Biswas and A.R. Roy, Fuzzy Soft Sets, Journal of Fuzzy Mathematics, **9**(3)(2001),589-602.

[40] M. Marudai, and V. Rajendran, New Constructions on Bipolar Anti Q-fuzzy Groups and Bipolar Anti Q-fuzzy d-ideals under (t, s) norms. Advances in Fuzzy Mathematics, **6**(1)(2011),145-153.

[41] A.A. Mechdesro, B.A, Alaba, T.M. Munie, & T.G. Alemayehu, Fuzzy pseudo-UP ideal of pseudo-UP algebra, Research in Mathematics, **10**(1)(2023), 2234217.

[42] K. Megalai, A study on fuzzy TM algebra (Ph.D thesis, Anna University), (2014).

[43] K. Megalai, A. Tamilarasi, Fuzzy subalgebras and fuzzy T-ideals in TM-algebra, J. Math. Stat. **7** (2011), 107–111.

[44] K. Megalai, and A. Tamilarasi, TM-algebra-an introduction, International Journal of Computer Application, Special Issue on Computer Aided soft Computing Techniques for imaging and Biomedical Application **1** (2010), 17-23.

[45] J.M. Mendel, Type-2 fuzzy sets, Uncertain Rule-Based Fuzzy Systems, Introduction and New Directions, 2nd Edition. (2017), 259-306.

[46] S. Miyamoto, Multisets and fuzzy multisets, In Soft computing and human-centered machines, Tokyo, Springer Japan, (2000), 9-33.

[47] S.K. Mondal, &T.K. Pal, Fuzzy congruences in semigroups, Fuzzy Sets and Systems, **209**(2012), 56–70.

[48] P. Mosrijai, W. Kamti, A. Satirad, and A. Iampan, Hesitant fuzzy sets on UP-algebras, Manuscript submitted for publication, (2016).

[49] M.M. Mostafa, Fuzzy KU-ideals in KU-algebras, J. Fuzzy Math., **3**(4)(1995), 823–832.

[50] V. Murali, Fuzzy congruence relations, Fuzzy sets and systems, **41**(3)(1991), 359-369.

[51] J. Neggers, S.S. Ahan, and H.S. Kim, On Q-algebras, International Journal of Mathematics and Mathematical Sciences, **27**(12)(2001), 749-757.

[52] A. H. Nouri, M.G. Sabbar, Pseudo TM-algebra. International Journal of Algebra, **13**(8)(2019), 341-348.

[53] Z. Pei, and L. Yi, A note on operations of hesitant fuzzy sets. International Journal of Computational Intelligence Systems, **8** (2)(2015), 226-239.

[54] C. Prabpayak, Some Homomorphism Properties of TM-algebras, Applied Mechanics and Materials, **866**(2017), 406-409.

[55] T. Ramachandran, T. Priya, and M. Parimala, Anti-fuzzy T-ideals of TM-algebras and its lower level cuts. Int. J. Comp. Appl, **43**(2012), 17-22.

[56] R.M. Rodr´ıguez, L.Mart´ınez, V.Torra, Z.S.Xu, and F.Herrera, Hesitant fuzzy sets, state of the art and future directions, International Journal of Intelligent Systems, **29**(6)(2014), 495–524.

[57] D.A. Romano, Pseudo-UP ideals and pseudo-UP filters in pseudoUP algebras. Math. Sci. Appl. E-Notes, **8**(1)(2020), 155-158.

[58] A. Rosenfeld, Fuzzy groups. Journal of mathematical analysis and applications,**35**(3)(1971), 512-517.

[59] M. Shabir,& M. Naz, On bipolar soft sets. arXiv preprint arXiv, (2013), 1303-1344.

[60] S. Shao, X. Zhang, Bo. C., & F. Smarandache, Neutrosophic hesitant fuzzy subalgebras and filters in pseudo-BCI algebras. Symmetry, **10**(5) (2018), 174.

[61] P. Sharma, Homomorphism of intuitionistic fuzzy groups, In international Mathematical Forum, **6**(2011), 3169-3178.

[62] S. Somjanta, Fuzzy UP-ideals of UP-algebras. J. Math. Computer Sci., **16**(1)(2016), 87–96.

[63] J. Thomas, K. Indhira, and V.M. Chandrasekaran, T-normed fuzzy TM-subalgebra of TM-algebras, International Journal of Computational Intelligence Systems, **12**(2)(2019), 706-712.

[64] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems, **25**(6)(2010), 529–539.

[65] V. Torra, Y. Narukawa, On hesitant fuzzy sets and decision, in Proceedings of the IEEE International Conference on Fuzzy Systems, Jeju-do,Republic of Korea, (2009), 1378–1382,

[66] I.B. Turksen, Interval-valued fuzzy sets and compensatory AND, Fuzzy Sets and Systems, **51**(3) (1992), 295-307.

[67] R. Verma , and B. D. Sharma, New operations over hesitant fuzzy sets, Fuzzy Information and Engineering, **5**(2)(2013), 129-146.

[68] J.Y. Wang, and L. Liu, Hesitant Bipolar- Valued Fuzzy Soft Sets and Their Application in Decision Making, Complexity, **1** (2020), 6496030.

[69] Y. Xi, Fuzzy ideals in BCK-algebras. Kyungpook Math. J., **31**(1)(1991), 47–58.

[70] Z. S. Xu, and M. M. Xia, Distance and similarity measures for hesitant fuzzy sets, Inform. Sci. **181** (2011), 2128-2138.

[71] G.W. Yohannes, A.A. Berhanu , and D.L. Beza, Intuitionistic Fuzzy PMS-ideals in a PMS-algebra, Thai Journal of Mathematics, **20**(4)(2022), 1535-1548.

[72] L.A. Zadeh, Fuzzy sets, Information and Control, **8**(1965), 338-353.

[73] X. Zhang, & K.M. Lee, Bipolar Hesitant Fuzzy Soft Sets and Their Application in Decision Making, (2013).

[74] B.C. Zhang, X. Shao, and S. Park, The lattice generated by hesitant fuzzy filters in pseudo-BCI algebras. Journal of Intelligent & Fuzzy Systems, **35**(3)(2018), 3333-3345.

[75] W.R. Zhang, Bipolar fuzzy sets and relations, a computational framework forcognitive modeling and multiagent decision analysis, in Proceedings of the IEEE Conference, (2013), 305–309.