

2025-11

Fuzzy Hyper and Pythagorean Fuzzy Composite Structures on BCL and LiuB Algebra

Abebe, Asmamaw

<http://ir.bdu.edu.et/handle/123456789/16867>

Downloaded from DSpace Repository, DSpace Institution's institutional repository



BAHIR DAR UNIVERSITY
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

**Fuzzy Hyper and Pythagorean Fuzzy Composite Structures on
BCL-Algebra and Liu^B-Algebra**

By

Asmamaw Abebe Biabeyin

NOVEMBER 03, 2025

BAHIR DAR, ETHIOPIA

BAHIR DAR UNIVERSITY
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

**Fuzzy Hyper and Pythagorean Fuzzy Composite Structures on BCL–Algebra
and Liu^B–Algebra**

A Dissertation Submitted to the Department of Mathematics, College of Science,
Bahir Dar University in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy in Mathematics

By

Asmamaw Abebe Biabeyin

Supervisor: Berhanu Assaye Alaba (Ph.D., Professor)
(Bahir Dar University)

Co-supervisor: Yohannes Gedamu Wondifraw (Ph.D., Associate Prof.)
(Bahir Dar University)

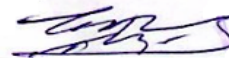
Bahir Dar University
College of Science
Department of Mathematics

Approval of Dissertation for Oral Defense

We hereby certify that we have supervised, read, and evaluated this dissertation entitled, “**Fuzzy Hyper and Pythagorean Fuzzy Composite Structures on BCL–Algebra and Liu^B–Algebra**” by **Asmamaw Abebe Biabeyin** prepared under our guidance. We recommend the dissertation to be submitted for oral defense

Berhanu Assaye Alaba (Ph.D., Prof.)

Supervisor’s name



Signature

_____ Date

Yohannes Gedamu Wondifraw (Ph.D., Assoc. Prof.)

Co-supervisor’s name



Signature




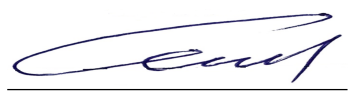
_____ Date

Bahir Dar University
College of Science
Department of Mathematics

Approval of Dissertation for Defense Result

As members of the board of examiners, we examined this dissertation entitled “**Fuzzy Hyper and Pythagorean Fuzzy Composite Structures on BCL–Algebra and Liu^B–Algebra**” by **Asmamaw Abebe Biabeyin**. We hereby certify that the dissertation is accepted for fulfilling the requirements for the award of the degree of doctor of philosophy in mathematics.

Board of Examiners

<u>V. Krishna Moorthy (Ph.D., Professor.)</u>		_____
External examiner's name 1	Signature	Date
<u>S. Ravikumar (Ph.D., Assoc. Professor)</u>		_____
External examiner's name 2	Signature	Date
<u>Gerima Tefera Dejen (Ph.D., Assoc. Prof.)</u>		_____
Internal examiner's name	Signature	Date
<u>Hunegnaw Dessie (Ph.D., Assoc. Prof.)</u>		_____
Chair person's name	Signature	Date

Declaration

This is to certify that the dissertation entitled “**Fuzzy Hyper and Pythagorean Fuzzy Composite Structures on BCL–Algebra and Liu^B–Algebra**”, submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, to Department of Mathematics, College of Science, Bahir Dar University, is a record of original work carried out by me and has never been submitted to this or any other institution to get any other degree or certificates. The assistance and help I received during the course of this investigation have been duly acknowledged.

Asmamaw Abebe Biabeyin

Name of Ph.D. Student



Signature

Date

Certificate

This is to certify that the dissertation entitled "FUZZY HYPER AND PYTHAGOREAN FUZZY COMPOSITE STRUCTURES ON BCL-ALGEBRA AND LIU^B-ALGEBRAS" submitted to Bahir Dar University in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy in Mathematics is a record of genuine research work carried out by Asmamaw Abebe Biabeyin under our supervision and guidance during the period of his study in the Department of Mathematics at Bahir Dar University. The results presented in this dissertation have not been submitted to any other university or institute for the award of any degree, diploma, or fellowship.

Berhanu Assaye Alaba (Ph.D., Professor)
Supervisor

Yohannes Gedamu Wondifraw (Ph.D., Associate Professor)
Co-Supervisor

Dedication

I dedicate this dissertation to my family, who have always been my strong support: To my dear wife, W/ro Workie Andargie Hailu, for her endless patience, sacrifices, and quiet strength during the many challenges of this journey, to my beloved sons, Deacon Seifeslasie Asmamaw and Petros Asmamaw, whose encouraging words and constant belief in me have given the courage to keep going, even when things were tough.

Additionally, this work is also dedicated to Berhanu Assay Alaba (Ph.D., Professor) and Asrat Atsedeweyin Andargie (Ph.D.), whose visionary mentor ship and immeasurable guidance opened the doors to Bahir Dar University and charted the course for this doctoral odyssey. Their wisdom, generosity, and unwavering faith in my potential transformed aspiration into achievement.

Publications

Published Papers

1. **Asmamaw Abebe Biabeyin**, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; Pythagorean fuzzy deductive system of BCL-algebra, **14** (2025); F1000Research..
2. **B. A. Asmamaw**, A. A. Berhanu, W. G. Yohannes, Pythagorean Fuzzy Ideal of LBA. **14** (2) (2025), Palestine Journal of Mathematics.

Accepted papers

1. **Asmamaw Abebe Biabeyin** , Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; Pythagorean fuzzy Liu^B – subalgebra of Liu^B - algebra. F1000Research.
2. **Yohannes Gedamu Wondifraw**, Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba; Pythagorean Fuzzy Subalgebra of BCL-algebra. Journal of Discrete Mathematical Science & Cryptography.

Under Review Papers

1. **Yohannes Gedamu Wondifraw**, Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba; Pythagorean Fuzzy Deductive System of Liu^B-algebra. Journal of Discrete Mathematical Science and Cryptography. Submitted on Mar. 02, 2025.
2. **Yohannes Gedamu Wondifraw**, Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba; Fuzzy Hyper Structures of Hyper BCL-algebra. Asian-European Journal of Mathematics. Submitted on Mar. 30, 2025.
3. **Yohannes Gedamu Wondifraw**, Berhanu Assaye Alaba, Asmamaw Abebe Biabeyin; Pythagorean Fuzzy Ideal of BCL–algebra. International Journal of Fuzzy Systems. (Sep. 02, 2025)

Communicated Papers

1. **Yohannes Gedamu Wondifraw**, Berhanu Assaye Alaba, Asmamaw Abebe Biabeyin; Hyper Structures of Hyper BCL-algebra. Journal of PLoS ONE. (Sept. 03, 2025)

Conference Presentations

1. Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; *Pythagorean Fuzzy Deductive System of BCL-Algebra*. **13th International Zeugma Conference on Scientific Research held February 24-26, 2025 /Gaziantep, Turkiye.**
2. Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; *Pythagorean Fuzzy Deductive System of Liu^B-Algebra*. **8th International Ankara Multidisciplinary Studies Congress held on line and in-person on March 16-18, 2025 / Ankara, Türkiye.**
3. Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; *Pythagorean Fuzzy Deductive System of BCL-Algebra*. **8th International Ankara Multidisciplinary Studies Congress held on line and in-person on March 16-18, 2025 / Ankara, Türkiye.**
4. Asmamaw Abebe Biabeyin; "Presentation as a Keynote Speaker"; *Pythagorean Fuzzy Deductive System of BCL-Algebra*. **8th International Virtual Conference on Physical Health, Public Health & Health care Management, Held During March 22-23, 2025 — Virtual Meeting.**
5. Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; *Pythagorean Fuzzy Deductive System of Liu^B-Algebra*. **5th African Graduate Students conference (AGSC V-BDU-2025), held on May 12-13, 2025, Bahir Dar University, Bahir Dar, Ethiopia.**
6. Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; *Fuzzy Hyper Ideals of Hyper BCL-Algebra*. **11th International Biltek Congress on Current Developments in Science, Technology and Social Sciences, held on May 25-26, 2025/ Ankara, Türkiye.**
7. Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; *Pythagorean Fuzzy Deductive System of BCL-Algebra*. **The 13th National Conference on Recent Trends (The 13th NCRTSR-2025), held 20-21, June 2025, Science College, Bahir Dar University, Bahir Dar, Ethiopia.**

Seminar Presentation

Asmamaw Abebe Biabeyin, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw; *Fuzzy Hyper Ideals of Hyper BCL-Algebra*, **Friday Seminar**: Presented on May 30, 2025, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.

Acknowledgment

Above all, I extend my deepest gratitude to Almighty God and His mother, St. Mary, for their boundless grace, unwavering faithfulness, and enduring love throughout my academic journey, culminating in the completion of this doctoral dissertation.

It is a great pleasure for me to express my heartfelt gratitude to my supervisors: Berhanu Assaye Alaba (Ph.D., Professor) and Yohannes Gedamu Wondifraw (Ph.D., Associate Professor) for their generous support, invaluable advice, and patient guidance throughout my graduate studies and the preparation of this dissertation. It has been an honor to be their student. Their vast knowledge and extensive experience have been a constant source of inspiration throughout my Ph.D. journey.

I would like to take this opportunity to thank the Department of Mathematics, College of Science at Bahir Dar University. In particular, my thanks go to Dr. Mihert Alemneh, Dr. Hunegnaw Dessie, Dr. Gtachew Muhabie and Mrs. Emebet Wondimu, the secretary of the Department and Dr. Beza Lamesgin in Debre Markos University for their valuable moral support and comments. Moreover, I would also like to thank ANRS Bureau of Labor and Skill, Debre Tabor Polytechnic College and Bahir Dar University for giving me this opportunity to pursue my Ph.D. study. Furthermore, my gratitude goes to Dr. Alachew Amaneh and Alemayehu Girum for their brotherly support that they made me competent enough to accomplish my study and their all rounded support.

I am grateful for the moral support and friendship I have received through my Ph.D study from my friends and colleagues at Bahir Dar University: Asfaw Tsegaye, Yimesgen Mehari, Amare Worku, Selamawit Hunie, Abebe Worku, Dr. Masresha Wassie, Dr. Zenaw Asnake, Dr. Nibret Mellese, Deacon Hailemariam Eshetie, Arega Deneke, Berihun Gizachew, Feleke Getnet, Adeladlew Dires, Estifanos Wudu for their friendly support. In addition, I thank my friends Belay Muchie, Kindie Getahun, Yitayal Ewunetu, Alem Zelalem, Henok Worku at Debre Tabor Polytechnic College.

Furthermore, words cannot express my gratitude to Bantideru Mihret, Fikiru Belay, Abriham Kindu and Abay Kindu for their moral and financial supports throughout my Ph.D. study.

Last, and most importantly, I could not have undertaken this journey without my wife, Woizero Workie Andargie, my sons Deacon Seifelasie Asmamaw and Petros Asmamaw, my family Abebe Setiye, Muluken Setiye, Fasika Fetene, Tigist Wodaje, Belet Wodaje, Engdaw Sisay, Mastewal Wodaje, Demeke Altaseb, Zebegnash Alene, Fenta Mihretu, Wassie Baye, Shiretie Demlie, Misganaw Zewudu and Abriham Setiye who remembered me in their prayers for my ultimate success and for their unreserved support and encouragement in every journey of my way.

Asmamaw Abebe Biabeyin

Abstract

This dissertation presents detailed and thorough research on the notions of hyper BCL–algebra, fuzzy substructures in hyper BCL–algebra, fuzzy substructures in BCL–algebra, Pythagorean fuzzy substructures in BCL–algebra, fuzzy substructures in Liu^B–algebra and Pythagorean fuzzy substructures in Liu^B–algebra. It provides new theoretical understanding, well refined definitions and rigorous characterizations that contribute to the development of algebraic knowledge about these structures. The research begins by introducing the hyper BCL–algebraic structures, supported by proven properties, under which (weak, strong) hyper subalgebras, (weak, strong) hyper deductive systems and (weak, strong) hyper ideals of hyper BCL–algebra are defined and several relevant properties are investigated. The relations among strong hyper subalgebras, weak hyper subalgebras and hyper subalgebras, as well as among strong hyper ideals, weak hyper ideals and hyper ideals of hyper BCL–algebra are clearly demonstrated in the context of hyper BCL–algebras. In hyper BCL–algebras, the relationship between hyper deductive systems and weak deductive systems is established. The intersection and union of corresponding (weak, strong) hyper substructures of hyper BCL–algebras are shown to be conserved. Following the introduction of hyper substructures, the notions of fuzzy hyper algebras are introduced, characterized as fuzzy (weak, strong) hyper subalgebras, fuzzy (weak, strong) hyper deductive systems and fuzzy (weak, strong) hyper ideals. The conservation of intersections of corresponding substructures is established; however, unions of such substructures are generally not conserved justified by examples. The relationships among fuzzy (weak, strong) hyper substructures are also explored. After investigating additional relevant properties, we introduce the notions of fuzzy subalgebra, fuzzy deductive system and fuzzy ideal of BCL–algebra. Moreover, we prove that the complements of fuzzy substructures of BCL–algebra and its characteristic functions correspond to fuzzy substructures of BCL–algebra. In the BCL–algebra, we also prove that intersections of fuzzy subalgebras, fuzzy deductive systems and fuzzy ideals are fuzzy subalgebra, fuzzy deductive system and fuzzy ideal of BCL–algebra, respectively; however, unions of such substructures are not conserved justified by counter examples. Several additional properties of fuzzy substructures of BCL–algebra are also demonstrated. Following the introduction and investigation of fuzzy subsets of BCL–algebra, we extend these notions to Pythagorean fuzzy substructures of BCL–algebra;

namely, Pythagorean fuzzy subalgebra, Pythagorean fuzzy deductive system and Pythagorean fuzzy ideal of BCL–algebra along with an investigation of their relevant properties. After introducing the Pythagorean fuzzy substructures of BCL–algebra, we present new definitions of Liu^{B} –algebra based on BCL–algebra combined with a semi–group. These definitions are illustrated with examples and their properties are explored. Following these investigations, we introduce fuzzy substructures of Liu^{B} –algebras, including fuzzy subalgebras, fuzzy deductive systems and fuzzy ideals. In relation to substructures of Liu^{B} –algebra, we extend these notions to Pythagorean fuzzy subalgebra, Pythagorean fuzzy deductive system and Pythagorean fuzzy ideal and relevant results are derived, with their properties explored in detail. Finally, the fuzzy subsets of BCL–algebra and Liu^{B} –algebra and Pythagorean fuzzy sets of BCL–algebra and Liu^{B} –algebra are described by making use of some basic tools. These include the Cartesian products of fuzzy subalgebras in BCL–algebra, the level sets in Pythagorean fuzzy deductive systems of BCL–algebra, and the homomorphisms of Pythagorean fuzzy deductive systems of Liu^{B} –algebra. In addition, Liu^{B} –algebra is also described through the Pythagorean (β, θ) –fuzzy ideal, and related properties arising from each of these descriptions are carefully examined.

Table of Contents

Declaration		ii
Certificate		iii
Publications		v
Acknowledgments		vii
Abstract		viii
List of Symbols and Abbreviations		xii
Introduction		1
1 Preliminaries		5
1.1	Some Types of Algebras, Substructures and BCL–Algebra	5
1.2	Some Types of Hyper Algebras	9
1.3	Basic Concepts of Fuzzy Sets	11
1.4	Basic Concepts of Pythagorean Fuzzy Sets	14
1.5	Basic Concepts of Some Fundamental Structures	16
2 Hyper Structures of BCL–Algebra		18
2.1	Hyper BCL–Algebra	18
2.2	Hyper Subalgebras of Hyper BCL–Algebra	27
2.3	Hyper Deductive Systems of Hyper BCL–Algebra	32
2.4	Hyper Ideals of Hyper BCL–Algebra	38
3 Fuzzy Hyper Structures of BCL–Algebra		44
3.1	Fuzzy Hyper Subalgebras of Hyper BCL–Algebra	44
3.2	Fuzzy Hyper Deductive Systems of Hyper BCL–algebra	51
3.3	Fuzzy Hyper Ideals of Hyper BCL–algebra	59

4	Pythagorean Fuzzy Structures of BCL–Algebra	71
4.1	Fuzzy Substructures of BCL–Algebra	71
4.1.1	Fuzzy Subalgebra of BCL–Algebra	76
4.1.2	Fuzzy Deductive Systems of BCL–Algebra	80
4.1.3	Fuzzy Ideal of BCL–Algebra	85
4.2	Pythagorean Fuzzy Substructures of BCL–Algebra	93
4.2.1	Pythagorean Fuzzy Subalgebras of BCL–Algebra	93
4.2.2	Pythagorean Fuzzy Deductive Systems of BCL–Algebra	107
4.2.3	Pythagorean Fuzzy Ideals of BCL–Algebra	115
5	Pythagorean Fuzzy Structures of Liu^B–Algebra	130
5.1	Basic Concepts of Liu ^B –Algebra and Its Substructures	130
5.2	Fuzzy Substructures of Liu ^B –Algebra	139
5.2.1	Fuzzy Subalgebra of Liu ^B –Algebra	140
5.2.2	Fuzzy Deductive System of Liu ^B –Algebra	143
5.2.3	Fuzzy Ideal of Liu ^B –Algebra	147
5.3	Pythagorean Fuzzy Substructures of Liu ^B –Algebra	153
5.3.1	Pythagorean Fuzzy Subalgebra of Liu ^B –Algebra	154
5.3.2	Pythagorean Fuzzy Deductive System of Liu ^B –Algebra	161
5.3.3	Pythagorean Fuzzy Ideal of Liu ^B –Algebra	169
6	Fundamental Structures in (Pythagorean) Fuzzy Composite Structures	184
6.1	Cartesian Products in Fuzzy Subalgebra of BCL–Algebra	184
6.2	Level Sets in Pythagorean Fuzzy Deductive System of BCL–Algebra	189
6.3	Homomorphism of Pythagorean Fuzzy Deductive System of Liu ^B –algebra.	199
6.4	Pythagorean (β, θ)–Fuzzy Ideal of Liu ^B –Algebra	209
	Conclusion and Future Works	222
	Bibliography	227

List of Symbols and Abbreviations

R, B, L	Non-empty sets in algebras like, BCI-algebra., BCL-algebra. Liu ^B -algebra
LBA	Abbreviations for Liu ^B -algebra
η, τ	Fuzzy sets
$\bar{\eta}$	Complement of the fuzzy set η
$\bar{\bar{\eta}}$	Square deviation of the fuzzy set η
$\eta_1 \times \eta_2$	Cartesian product of η_1 and η_2
$\eta(m)$	Degree of membership of the element m in the fuzzy set η
$\tau(m)$	Degree of non-membership of the element m in the fuzzy set τ
χ	Characteristic function of set M
\cup	Union of sets
\cap	Intersection of sets
\subseteq	Subset
\in	Belongs to
\notin	Doesn't belong to
\forall	For all
\exists	There exist(s) or for some
$f(\eta)$	Homomorphic image of η
$f^{-1}(\eta)$	Homomorphic inverse image of η
\inf	Infimum
\sup	Supremum
\otimes	binary operation
$@$	hyper operation on hyper algebras
$P(B)$	Power set of a non-empty set B
\mathbb{B}	$P(B) - \{\emptyset\}$
\prec, \preceq, \succeq	Different hyper relations on BCL-algebra of sets and/or elements

Introduction

Around 300 BC, the Greek philosopher Aristotle formalized the law of asserting that any proposition must be either true or false which is a binary logic that dominated different thoughts. This principle underpinned Georg Cantor's inspiring work in [13] on classical set theory, where elements crisply belong or do not belong to a set. Such crisp sets, defined by binary membership functions, became the bedrock of mathematical reasoning. However, their rigidity clashes with the pervasive ambiguity of real-world phenomena. Concepts like "tall," "intelligent," or "beautiful" resist binary classification, revealing a critical gap that classical sets cannot model partial truths or graded membership.

A transformative shift began with Iseki and Tanaka's introduction of BCK-algebras in [28] and Iseki's subsequent development of BCI-algebras in [27]. These structures abstracted logical implication and set-theoretic difference into algebraic axioms, enabling formal study of deductive systems. This sparked decades of generalization: BCH-algebras developed by Hu and Li [24] broadened BCI-algebras, BCC-algebras introduced by Dudek in [21]) and BH-algebras by Jun in [30]) further expanded the landscape, KU-algebras explored by Prabayak and Leerawat in [60] unified BCK/BCI/BCC systems, UP-algebras by Iampan in [25]) generalized KU/BCI/BCK-algebras, Romano in [65]) extended UP-algebras to multi-operational settings. Each innovation addressed structural limitations, enriching algebra's capacity to model complex logical relationships. Yet, they remained confined to deterministic frameworks.

Lotfi Zadeh's [81] breakthrough fuzzy sets revolutionized mathematics by introducing degrees of membership. In a fuzzy set, elements belong to a value in $[0,1]$, where 0 represents non-membership and 1 represents full membership. This continuum captures vagueness like "warm weather" or "high risk", etc. far better than crisp sets. Rosenfeld in [66] soon applied fuzzy sets to group theory, catalyzing "fuzzy algebra". Fuzzy ideals, subalgebras, and homomorphisms were defined for BCK/BCI-algebras by Meng and Guo in [52], ideals in KU-algebras by Mostafa in [53], and UP-algebras by Somjanta [69]. B. Oavvaz in [16] defined fuzzy subhypergroup of a hypergroup or fuzzy H-groups and then proved some results.

Despite their utility, fuzzy sets use a single membership function, ignoring non-membership explicitly. Krassimir Atanassov addressed this in [7] with intuitionistic fuzzy sets, assigning each element, a membership degree $\eta(m) \in [0, 1]$ and a non-membership degree $\tau(m) \in [0, 1]$ with the condition

$0 \leq \eta(m) + \tau(m) \leq 1$. Intuitionistic fuzzy sets better modeled hesitation like voting with "abstain" options. Yet, the constraint $0 \leq \eta(m) + \tau(m) \leq 1$ proved limiting.

R. R. Yager's Pythagorean fuzzy sets in [78, 79] solved a major limitation that was present in intuitionistic fuzzy sets. This was done by allowing the sum of the squares of membership and non-membership degrees to be at most 1. This means we have a bigger range to describe uncertainty. Specifically, Pythagorean fuzzy sets can include cases where the sum of the squares of membership and non-membership degrees is greater than the linear sum of the membership and non-membership degrees but still less than or equal to 1. This makes Pythagorean fuzzy sets more flexible and useful in many real-world problems with the condition that $0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1$. Because of their rich ability to handle uncertainty, Pythagorean fuzzy sets are helpful in decision making and further mathematical structures.

Parallel developments arose in hyper-algebra, where operations map to sets of outcomes $m @ n$ yields a set with at least one element, which may not be singleton set. Initiated by Marty [49], hyper-structures model non-deterministic systems, such as parallel computing or biological networks. Y. Jun [30, 31] applied hyper-operations to BCK-algebras, defining hyper-BCK/BCI-algebras and ideals. Later, Xiao [75] refined hyper-ideal theory, linking hyper-algebras to group theory.

In the middle of these new developments and advances, Y. H. Liu [42] - [46] introduced notions of algebras different but related algebras as BCL-algebra, BCL^+ -algebra and Liu-algebra with their partial orders and different algebraic structures as subalgebras, deductive systems and ideals. Liu's later BCL^+ -algebras, using a constant 1 instead of 0, and Liu-algebras, integrating semi-groups with BCL^+ -algebra further broadened the field where these algebras provided a very helpful framework or setting to combine hyper-structures with fuzzy concepts and see how they work together, however having the gap of combining semi-group with BCL-algebra.

Having investigated and gone through each end every literatures introduced and developed so far, we observed the following research gaps and Motivation: Despite progress, critical voids persisted, hyper BCL-algebras lacked systematic study, fuzzy hyper substructures, merging hyper-operations with fuzzy sets were underdeveloped, Pythagorean fuzzy algebra, especially for substructures like subalgebras, deductive systems and ideals of BCL-algebra and Liu^B -algebra, remained unexplored. Liu^B -algebras (LBA), a recent extension of BCL-algebra, to fill the gap of combining semi-group with BCL-algebra that Liu left out with no fuzzy/Pythagorean fuzzy framework are seen unexplored.

This dissertation bridges these gaps by unifying three powerful paradigms: Hyper-structures for non-determinism, fuzzy sets for graded membership, and Pythagorean fuzzy sets for independent membership versus non-membership grades within BCL-algebras and Liu^B -algebras.

In this dissertation, therefore, we aim to: Formalize Hyper BCL-Algebras by defining hyper subalgebras, hyper deductive systems, and hyper ideals, establishing their core properties within hyper BCL-algebra, by developing Fuzzy Hyper Structures in Hyper BCL-Algebra as constructing fuzzy hyper subalgebras, fuzzy hyper deductive systems and fuzzy hyper ideals for Hyper BCL-algebra. Pioneering Pythagorean fuzzy algebra, we define Pythagorean fuzzy subalgebras, Pythagorean fuzzy deductive systems and Pythagorean fuzzy ideals for BCL-algebras.

Furthermore, we Extend Pythagorean fuzzy sets to Liu^B-algebras, introducing Liu^B-algebra along with corresponding substructures, analyze fundamental structures as homomorphisms of Pythagorean fuzzy deductive systems, level-set representations linking Pythagorean fuzzy sets to classical substructures, Cartesian products of fuzzy subalgebras of BCL-algebra and Pythagorean (β, θ) -fuzzy ideals of Liu^B-algebra with refined membership constraints.

Within the key innovations, we explored comprehensive framework for Pythagorean fuzzy ideals in logical algebras, syn dissertation of hyper-operations with fuzzy and Pythagorean fuzzy logic, characterization theorems and equivalence conditions for novel substructures, hierarchical relationships between classical, hyper, fuzzy, and Pythagorean fuzzy substructures.

Working out this much, we want to address implications and applications of this work which advances abstract algebra by enhancing uncertainty modeling, Pythagorean fuzzy substructures capture complex real-world hesitancy like ethical dilemmas in AI. To generalizing Prior Work, fuzzy hyper ideals subsume classical ideals and Pythagorean fuzzy frameworks extend intuitionistic fuzzy algebra. **The dissertation structure is of the following way:**

Chapter 1: Preliminaries:- revising literatures related to our work and those on which our work bases like BCL-algebra, hyper-structures, fuzzy/Pythagorean fuzzy set-theory by reviewing the main ideas from previous works that we use in this dissertation. It explains BCL-Algebra, hyper-structures, fuzzy sets, Pythagorean fuzzy sets, and related mathematical concepts. This forms a strong base for developing new structures and methods in subsequent chapters.

Chapter 2: Hyper BCL-algebras and their substructures: - introducing hyper BCL-Algebra, a new way to combine elements with set operations instead of standard operations. It also studies hyper subalgebras, hyper deductive systems, and hyper ideals of hyper BCL-algebra.

Chapter 3: Fuzzy hyper substructures in Hyper BCL-algebras: - focusing on fuzzy hyper substructures within hyper BCL-Algebra. Fuzzy subsets help us handle uncertainty, while hyper structures enable us to deal with multiple outcomes. This combination forms a rich mathematical framework for analyzing complex or imprecise information

Chapter 4: Pythagorean fuzzy substructures for BCL–algebra: - after exploring different notions of fuzzy BCL–algebra, we introduce Pythagorean fuzzy substructures for BCL-Algebra. Pythagorean fuzzy sets are more flexible than ordinary fuzzy sets, allowing us to represent uncertainty in a more realistic way. This lets us describe relationships and properties in BCL-Algebra more accurately under uncertainty.

Chapter 5: Pythagorean fuzzy substructures for Liu^B–algebras: - applying Pythagorean fuzzy sets to Liu^B-Algebra, a new algebra that combines group-like operations with logical structures. The Pythagorean fuzzy substructures enable us to handle uncertainty while preserving mathematical properties, adding depth and robustness to these algebraic models

Chapter 6: Fundamental structures like homomorphisms, level sets, Cartesian products, and (β, θ) -ideals) - covering homomorphisms, level sets, Cartesian products, and (β, θ) -ideals related to Pythagorean fuzzy structures. These structures enable us to connect different algebras, transfer properties from one to another, and combine them in a systematic way. This forms a strong mathematical framework for further applications and study.

In finalizing the introduction, the dissertation explored the wide range concepts by unifying hyperstructures, fuzzy logic, and Pythagorean fuzzy sets within BCL–algebra and Liu^B–algebra, and therefore, this dissertation constructs a robust algebraic toolkit for the era of uncertainty. It lays groundwork for **future explorations**, from computational logic to ethical AI—demonstrating how abstract algebra evolves to address the ambiguities of human experience.

This dissertation **concludes** by providing a comprehensive conclusion that summarizes all the main results, contributions, and findings of the study including future work. After that, a complete list of references and sources used in this research is presented in the **bibliography** at the end.

Chapter 1

Preliminaries

This chapter presents a summary of foundational concepts established in prior literature, forming the basis for the developments in this dissertation. It begins with well-known algebraic structures defined through axioms involving binary operations and constants, along with commonly studied substructures such as subalgebras, deductive systems, and ideals. The discussion extends to hyperstructures, where operations are generalized to set-valued mappings, leading to the notions of hyper subalgebras, hyper deductive systems, and hyper ideals. Additionally, fuzzy set theory and its generalization via Pythagorean fuzzy sets are revisited, offering novel ideas for modeling uncertainty. These concepts underpin fuzzy substructures like fuzzy subalgebras, fuzzy deductive systems, fuzzy ideals, and advanced notions including (β, θ) -fuzzy ideals, all of which have been investigated in existing works.

We also use " \exists " for "there exist(s)" or "for some" and " \forall " represents "for every" or "for each" or "for all", for the whole of this dissertation.

1.1. Some Types of Algebras, Substructures and BCL–Algebra

This section restates various algebraic structures along with their defining axioms, substructures, subalgebras, ideals and deductive systems, providing foundational concepts for further algebraic exploration.

Definition 1.1.1. [27] *A BCI–algebra is an algebra $(R; \otimes, 0)$ of type $(2, 0)$ satisfying the following conditions; $\forall m, n, w \in R$:*

- (i) $m \otimes m = 0$,
- (ii) $((m \otimes n) \otimes (m \otimes w)) \otimes (w \otimes n) = 0$,
- (iii) $(m \otimes (m \otimes n)) \otimes n = 0$
- (iv) $m \otimes n = 0$ and $n \otimes m = 0 \Rightarrow m = n$.

Definition 1.1.2. [33] An algebra $(R; \otimes, I)$ with a non-empty set R , a constant I and a binary operation “ \otimes ” is called a GE–algebra if it satisfies the following axioms, $\forall m, n, w \in R$:

- (i) $m \otimes m = I$,
- (ii) $I \otimes m = m$,
- (iii) $m \otimes (n \otimes w) = m \otimes (n \otimes (m \otimes w))$.

Definition 1.1.3. [14] A Hilbert algebra is an algebra $(R; \otimes, I)$ satisfying the following conditions:

- (i) $m \otimes (n \otimes m) = I$,
- (ii) $(m \otimes (n \otimes w)) \otimes ((m \otimes n) \otimes (m \otimes w)) = I$,
- (iii) $m \otimes n = n \otimes m = I \Rightarrow m = n$.

Definition 1.1.4. [80] A nonempty subset S of a pseudo–UP algebra R is called a pseudo–UP sub-algebra of R if it satisfies the following axioms:

- (i) $0 \in S$,
- (ii) $m \otimes n, m \odot n \in S, \forall m, n \in S$.

Definition 1.1.5. [12] A nonempty subset D of GE–algebra R is called a deductive system of R if it satisfies the following axioms, $\forall m, n, w \in R$:

- (i) $R \otimes D := \{m \otimes a : a \in D\} \subseteq D$,
- (ii) $m, n \in D \Rightarrow (m \otimes (n \otimes w)) \otimes w \in D$.

Definition 1.1.6. [18] A subset D of Hilbert algebra R is called deductive system of R if it satisfies:

- (i) $I \in D$,
- (ii) $m, m \otimes n \in D \Rightarrow n \in D$.

Definition 1.1.7. [56] Let R be a BCI–algebra and let I be a non empty subset of R . Then I is called an ideal of R if, $\forall m, n \in R$:

- (i) $0 \in I$,
- (ii) $m \otimes n \in I$ and $n \in I \Rightarrow m \in I$.

Definition 1.1.8. [42] An algebra $(B; \otimes, 0)$ of type $(2, 0)$ is said to be a BCL–algebra if and only if the following conditions hold, $\forall m, n, w \in B$:

- (i) $m \otimes m = 0$,
- (ii) $m \otimes n = 0$ and $n \otimes m = 0 \Rightarrow m = n$,
- (iii) $[((m \otimes n) \otimes w) \otimes ((m \otimes w \otimes n))] \otimes ((w \otimes n) \otimes m) = 0$.

Remark 1.1.1. [44] Let $m \leq n$ if and only if $m \otimes n = 0$, then Definition 1.1.8 for BCL–algebra B , can be written as follows, $\forall m, n, w \in B$:

- (i) $m \leq m$,
- (ii) $m \leq n$ and $n \leq m \Rightarrow m = n$,
- (iii) $[(m \otimes n) \otimes w] \otimes [(m \otimes w) \otimes n] \leq [(w \otimes n) \otimes m]$.

Let the binary relation " \leq " on B be such that $m \leq n \Leftrightarrow m \otimes n = 0$, for any $m, n \in B$, then the BCL–ordering " \leq " is called ordered relation on B .

Example 1.1.1. [42] Let $B = \{0, p, q, r\}$ and define a binary operation \otimes on B by the table below:

\otimes	0	p	q	r
0	0	0	0	0
p	p	0	r	p
q	q	r	0	q
r	r	0	0	0

Table 1.1: A table that defines a binary operation \otimes on BCL–algebra, B

From the Table 1.1 above, $(B; \otimes, 0)$ is a BCL–algebra.

Definition 1.1.9. [43] An algebra $(B; \otimes, 1)$ of type $(2, 0)$ is said to be a BCL^+ –algebra if and only if the following conditions hold, $\forall m, n, w \in B$:

- (i) $m \otimes m = 1$,
- (ii) $m \otimes n = 1$ and $n \otimes m = 1 \Rightarrow m = n$,
- (iii) $[((m \otimes n) \otimes w) \otimes ((m \otimes w \otimes n))] = ((w \otimes n) \otimes m)$.

Let the relation " \leq " on B be defined by $m \leq n$ if and only if $m \otimes n = 1$, $\forall m, n \in B$. Then, the BCL–ordering " \leq " is called an ordered relation on B .

Definition 1.1.10. [44] Let $(B; \otimes, 1)$ be a BCL^+ -algebra. Any non-empty subset S of the BCL^+ -algebra B , is called subalgebra of B if $m \otimes n \in S, \forall m, n \in S$.

Definition 1.1.11. [45] If D is a nonempty subset of a BCL^+ -algebra $(B; \otimes, 1)$, then we say that D is a deductive system of B if the following axioms are satisfied, $\forall m, n \in R$:

- (i) $1 \in D$,
- (ii) $m \in D$ and $m \otimes n \in D \Rightarrow n \in D$.

Definition 1.1.12. [46] A Liu algebra $L = (L; \otimes, \odot, 1)$, where L is a nonempty set, two binary operations \otimes and \odot are defined on L and 1 is fixed element of L is called Liu-algebra if the following axioms hold $\forall m, n, w \in L$:

- (i) $(L; \otimes, 1)$ is a BCL^+ algebra,
- (ii) $(L; \odot)$ is a semi-group,
- (iii) $m \odot (n \otimes w) = (m \odot n) \otimes (m \odot w)$,
- (iv) $(n \otimes m) \odot w = (n \odot w) \otimes (m \odot w)$.

Definition 1.1.13. [46] Let $(L; \otimes, \odot, 1)$ be a Liu algebra. A nonempty set $S \subseteq L$ is a subalgebra if S is closed under the two binary operations \otimes and \odot in L ; alternatively, $\forall m, n \in S, m \otimes n, m \odot n \in S$ (of course, $(S; \otimes, \odot, 1)$ is also a Liu-algebra).

Definition 1.1.14. [46] Let I be a nonempty subset of BCL^+ - algebra $(B; \otimes, 1)$. We say that I is called an ideal of B if, the following are satisfied, $\forall m, n, w$ in B :

- (i) $1 \in I$,
- (ii) $m \otimes n, n \in I \Rightarrow m \in I$,
- (iii) $(m \otimes (n \otimes w)) \otimes w \in I$.

Definition 1.1.15. [46] Let I be a nonempty subset of Liu-algebra $(L; \otimes, \odot, 1)$, I is called an ideal of L if, $\forall m, n \in L$:

- (i) $m \otimes n, n \in I \Rightarrow m \in I$
- (ii) $n \odot m, m \odot n \in I \Rightarrow m \in I$.

Definition 1.1.16. [46] If D is a nonempty subset of a Liu–algebra $(L; \otimes, \odot, 1)$, then D is said to be a deductive system of L if the following two axioms are satisfied, $\forall m, n \in L$:

- (i) $1 \in D$ and
- (ii) $m \in D, m \otimes n \in D, m \odot n \in D \Rightarrow n \in D$.

1.2. Some Types of Hyper Algebras

In this section, we recall fundamental definitions related to hyper–algebraic structures. Key concepts such as hyper–operations, hyper ordered relations and various forms of substructures including strong hyper, weak hyper and hyper substructures are presented to establish the groundwork for our exploration. For this subsection, we denote $P(R)$ to be power set of $R \neq \emptyset$ and $P^*(R) = P(R) \setminus \{\emptyset\}$.

Definition 1.2.1. [75] An algebra $(R; @, 0)$ is called hyper BCI–algebra if R is a non-empty set endowed with a hyper-operation $@$, a constant "0" and a hyper relation \preceq satisfying the axioms below, $\forall m, n, w \in R$

- (i) $(m @ w) @ (n @ w) \preceq (m @ n)$,
- (ii) $(m @ n) @ w = (m @ w) @ n$,
- (iii) $m \preceq m$,
- (iv) $m \preceq n$ and $n \preceq m \Rightarrow m = n$,
- (v) $0 @ (0 @ m) \preceq m, m \neq 0$;

where $m \preceq n$ is defined by $0 \in m @ n$ and for every $A, B \subseteq R, A \preceq B$ is defined by for all $a \in A$, there exists $b \in B$ such that $a \preceq b$. In such a case, " \preceq " is called the hyper ordered relation in R .

Definition 1.2.2. [36] Let R be a non-empty set with a constant "0" and " $@$ " be a hyper operation defined on R . Then $(R; @, 0)$ is said to be a hyper BCH–algebra if the following axioms are satisfied, $\forall m, n, w \in R$:

- (i) $m \preceq m$,
- (ii) $(m @ n) @ w = (m @ w) @ n$,
- (iii) $m \preceq n$ and $n \preceq m \Rightarrow m = n$,

where the hyper ordered relation \preceq is such that $m \preceq n$ is defined by $0 \in m @ n$.

Definition 1.2.3. [75] Let R be a nonempty set and $@ : R \times R \rightarrow P^*(R)$ be a hyper-operation. Then $(R; @, 1)$ is called a hyper BE–algebra, if it satisfies the following axioms, $\forall m, n, w \in L$:

- (i) $m \preceq 1$ and $m \preceq m$,

- (ii) $m @ (n @ w) = n @(m @ w)$,
- (iii) $m \in I @ m$,
- (iv) $I \preceq m \Rightarrow m = I$,

where the hyper relation " \preceq " is defined by $m \preceq n \Leftrightarrow I \in m @ n$ and for any two nonempty subsets A and B of R , we define $A \preceq B$ if and only if there exist $m \in A$ and $n \in B$ such that $m \preceq n$, and $A @ B = \bigcup_{\substack{m \in A, \\ n \in B}} (m @ n)$.

Definition 1.2.4. [63] An algebra $(R; @, I)$, where $@ : R \times R \rightarrow P^*(R)$ is a hyper-operation and I is a constant, is called a hyper CI-algebra, if for all $m, n, w \in R$, it satisfies:

- (i) $m @ (n @ w) = n @(m @ w)$,
- (ii) $I \in m @ m$,
- (iii) $m \in I @ m$.

The relation " \preceq " is defined by $m \preceq n \Leftrightarrow I \in m @ n$. Let G and K be two non-empty subsets of R .

Define $G @ K = \bigcup_{\substack{m \in G, \\ n \in K}} (m @ n)$ and $G \preceq K \Leftrightarrow$ there exist $m \in G$ and $n \in K$ such that $m \preceq n$.

Definition 1.2.5. [62] Let R be hyper BE-algebra. A hyper-operation " $@$ " on R is a function from $R \times R \rightarrow P^*(R)$ satisfying the following conditions:

- (i) $m @ B = \bigcup_{n \in B} (m @ n)$, where m is fixed element of R and $B \subseteq R$,
- (ii) $A @ n = \bigcup_{m \in A} (m @ n)$, where n is fixed element of R and $A \subseteq R$.

Definition 1.2.6. [34] A hyper MV-algebra is a non-empty set M endowed with a hyper operation " $@$ ", a unary operation " $*$ " and a constant " 0 " satisfying the following axioms, $\forall m, n, w \in M$:

- (i) $m @ (n @ w) = (m @ n) @ w$,
- (ii) $m @ n = n @ m$,
- (iii) $(m^*)^* = m$,
- (iv) $(m^* @ n)^* @ n = (n^* @ m)^* @ m$,
- (v) $0^* \in m @ 0^*$,
- (vi) $0^* \in m @ m^*$.

Definition 1.2.7. [62] A non-empty subset S of a hyper BE-algebra R is said to be a hyper subalgebra of R , if $m @ n \subseteq S$, for all $m, n \in S$.

Definition 1.2.8. [35] A nonempty subset D of M is called a hyper MV-deductive system of M if it satisfies the following axioms:

- (i) $0 \in D$,
- (ii) $(\forall m, n \in M)(m^* @ n)^* \preceq D, n \in D \Rightarrow m \in D$.

Definition 1.2.9. [75] Let L be a non-empty subset of a hyper BCI-algebra R . Then L is said to be a hyper BCI-ideal of R if, $\forall m, n \in R$:

- (i) $0 \in L$,
- (ii) $m @ n \preceq L$ and $n \in L \Rightarrow m \in L$.

Definition 1.2.10. [75] Let L be a non-empty subset of a hyper BCI-algebra R . Then L is said to be a strong hyper BCI-ideal of R if, $\forall m, n \in R$:

- (i) $0 \in L$,
- (ii) $(m @ n) \cap I \neq \emptyset$ and $n \in L \Rightarrow m \in L$.

Definition 1.2.11. [62] Let I be a non-empty subset of a hyper BE-algebra R , \preceq be hyper relation such that $0 \in I$, then:

- (i) I is a hyper BE-ideal of R if for all $m, n \in R$, $m @ n \preceq I$ and $n \in I \Rightarrow m \in I$,
- (ii) I is a weak hyper BE-ideal of R if for all $m, n \in R$, $m @ n \subseteq I$ and $n \in I \Rightarrow m \in I$,
- (iii) I is a strong hyper BE-ideal of R if, $\forall m, n \in R$, $(m @ n) \cap I \neq \emptyset$ and $n \in I \Rightarrow m \in I$.

1.3. Basic Concepts of Fuzzy Sets

This section recalls fundamental definitions of fuzzy structures, including fuzzy sets, operations, and relations, as well as fuzzy subalgebras, fuzzy deductive systems, and fuzzy ideals, laying the groundwork for further exploration that follows the preliminary. Besides, we restate fuzzy hyper substructures, defining various fuzzy ideals, subalgebras, and deductive systems in different hyper algebras using order-based and other defining conditions.

Definition 1.3.1. [81] A fuzzy set A of a non-empty set R is the set of ordered pairs defined as $A = \{ \langle m, \eta_A(m) \rangle : m \in R \}$, where $\eta_A : R \rightarrow [0, 1]$ is a function called the membership function of A which assigns to every element m of R , a degree of membership $\eta_A(m)$ in the fuzzy set A . The complement of η_A is the fuzzy subset denoted by $\bar{\eta}_A(m)$ is the fuzzy subset in R with $\bar{\eta}_A(m) = 1 - \eta_A(m)$, $\forall m \in R$.

Definition 1.3.2. [39] Let $R \neq \emptyset$, $\eta_1 : R \rightarrow [0, 1]$ and $\eta_2 : R \rightarrow [0, 1]$ be fuzzy sets in R , then:

- (i) $\eta_1 \subseteq \eta_2 \Leftrightarrow \eta_1(m) \leq \eta_2(m), \forall m \in R$, where \subseteq is the inclusion,
- (ii) $\eta_1 \subset \eta_2 \Leftrightarrow \eta_1(m) < \eta_2(m), \forall m \in R$, where \subset is the strict inclusion,
- (iii) $\eta_1 = \eta_2 \Leftrightarrow \eta_1(m) = \eta_2(m), \forall m \in R$, where $=$ is the equality,
- (iv) $(\eta_1 \cup \eta_2)(m) = \max\{\eta_1(m), \eta_2(m)\}, \forall m \in R$,
- (v) $(\eta_1 \cap \eta_2)(m) = \min\{\eta_1(m), \eta_2(m)\}, \forall m \in R$,
- (vi) $\eta_1(m) \vee \eta_2(n) = \max\{\eta_1(m), \eta_2(n)\}$ and $\eta_1(m) \wedge \eta_2(n) = \min\{\eta_1(m), \eta_2(n)\}, \forall m, n \in R$,

Definition 1.3.3. [39] Let $R \neq \emptyset$ and $\{\eta_i : i \in I\}$ be a family of fuzzy subsets in R , then:

- (i) $\bigcup_{i \in I} (\eta_i(m)) = \sup_{i \in I} \{\eta_i(m)\}, \forall m \in R$,
- (ii) $\bigcap_{i \in I} (\eta_i(m)) = \inf_{i \in I} \{\eta_i(m)\}, \forall m \in R$,

Definition 1.3.4. [57] Fuzzy subset η in BCI–algebra R is called fuzzy ideal of R if:

- (i) $\eta(0) \geq \eta(m), \forall m \in R$,
- (ii) $\eta(m) \geq \min\{\eta(m \otimes n), \eta(n)\}, \forall m, n \in R$.

Definition 1.3.5. [73] A fuzzy subset η is called a fuzzy subalgebra of a CI–algebra R if it satisfies, $\forall m, n \in R$, $\eta(m \otimes n) \geq \eta(m) \wedge \eta(n)$.

Definition 1.3.6. [56] A fuzzy subset η in a hyper BCI–algebra R with a hyper relation \preceq is said to be a fuzzy hyper BCI–ideal of R if the axioms hereunder hold, $\forall m, n \in R$:

- (i) $m \preceq n \Rightarrow \eta(m) \geq \eta(n)$
- (ii) $\eta(m) \geq \min\left\{ \inf_{w \in (m \otimes n)} \eta(w), \eta(n) \right\}$

Definition 1.3.7. [6] Let $(R; @, 0)$ be a hyper AT–algebra. A fuzzy subset η in R is called a fuzzy hyper subalgebra of R if, $\forall m, n \in R$; $\inf_{w \in (m \otimes n)} \eta(w) \geq \min\{\eta(m), \eta(n)\}$

Definition 1.3.8. [34] A fuzzy subset η of M is called a fuzzy hyper-MV-subalgebra of M if the following hold, $\forall m, n \in M$:

(i) $\eta(m^*) \geq \eta(m)$, where $*$ is a unary operation such that $(m^*)^* = m$

(ii) $\inf_{u \in (m @ n)} \eta(u) \geq \min\{\eta(m), \eta(n)\}$

Definition 1.3.9. [34] A fuzzy subset η of M is called a fuzzy hyper-MV-deductive system of M if the following hold, $\forall m, n \in M$:

(i) $\eta(m) \geq \min\left\{\inf_{w \in (m^* @ n)^*} (\eta(w), \eta(n))\right\}$

(ii) $m \preceq n \Rightarrow \eta(m) \geq \eta(n)$

Definition 1.3.10. [34] A fuzzy subset η of M is called a fuzzy weak hyper-MV-deductive system of M if the following hold, $\forall m, n \in M$:

(i) $\eta(0) \geq \eta(m)$,

(ii) $\eta(m) \geq \min\left\{\inf_{w \in (m^* @ n)^*} (\eta(w), \eta(n))\right\}$

Definition 1.3.11. [34] A fuzzy subset η of M is called a fuzzy strong hyper-MV-deductive system of M if the following hold, $\forall m, n \in M$: $\inf_{u \in (m^* @ m)^*} \eta(u) \geq \eta(m) \geq \min\left\{\sup_{w \in (m^* @ n)^*} \eta(w), \eta(n)\right\}$

Definition 1.3.12. [45] Let $(H; @, 0)$ be a hyper BCI-algebra, then a fuzzy subset η is called be hyper fuzzy BCI-subalgebra of H , if $\forall m, n, u \in H$: $\inf_{w \in m @ n} \{\eta(w)\} \geq \{\eta(m), \eta(n)\}$

Definition 1.3.13. [45] Let $(R; @, 0)$ be a hyper BCI-algebra, then a fuzzy subset η is said to be hyper fuzzy BCI-ideal of R if, $\forall m, n, u \in R$

(i) $\eta(0) \geq \eta(m)$,

(ii) $m \ll n \Rightarrow \eta(n) \geq \eta(m)$,

(iii) $\eta(m) \geq \min\left\{\inf_{u \in m @ n} \eta(u), \eta(n)\right\}$

Definition 1.3.14. [47] Let $(R; @, 0)$ be a hyper BCK-algebra. Then $\forall m, n, w \in R$, a fuzzy subset η in R is said to be:

(i) a fuzzy weak hyper p -ideal of R if:

$$\eta(0) \geq \eta(m) \geq \min\left\{\inf_{u \in (m @ w) @ (n @ w)} \eta(u), \eta(n)\right\},$$

(ii) a fuzzy hyper \mathfrak{p} -ideal of R if

$$m \preceq n \Rightarrow \eta(m) \geq \eta(n) \text{ and } \eta(m) \geq \min \left\{ \inf_{u \in (m@w)@(n@w)} \eta(u), \eta(n) \right\},$$

(iii) a fuzzy strong hyper \mathfrak{p} -ideal of R if:

$$\inf_{u \in (m@m)} \eta(u) \geq \eta(m) \geq \min \left\{ \sup_{w \in (m@w)@(n@w)} \eta(w), \eta(n) \right\}.$$

1.4. Basic Concepts of Pythagorean Fuzzy Sets

This section recalls Pythagorean fuzzy sets as a generalization of intuitionistic fuzzy sets. Also, we recall related functions like score function, accuracy function, and the degree indeterminacy, and compare them through examples and tabular analysis.

Definition 1.4.1. [7] An intuitionistic fuzzy set I in a non-empty set R is an object having the form: $\{\langle m, \eta(m), \tau(m) \rangle : m \in R\}$, where the functions: $\eta : R \rightarrow [0, 1]$ and $\tau(m) : R \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership, respectively satisfying the condition:

$$0 \leq \eta(m) + \tau(m) \leq 1$$

Definition 1.4.2. [78, 84] A Pythagorean fuzzy set P in a non-empty set R is an object having the form: $P = \{\langle m, \eta(m), \tau(m) \rangle : m \in R\}$ or simply $P = (\eta, \tau)$, where the functions $\eta : R \rightarrow [0, 1]$ and $\tau : R \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership, respectively satisfying the condition: $0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1$. In such cases where P is a Pythagorean fuzzy set, there is a degree of indeterminacy of $m \in R$ to P defined by:

$$\pi_p(m) = \sqrt{1 - [(\eta(m))^2 + (\tau(m))^2]} \text{ and hence } \pi(m) \in [0, 1]$$

Definition 1.4.3. [79, 84] Let P be Pythagorean fuzzy set. Then, the score function s of P is defined by: $s_p(m) = (\eta(m))^2 - (\tau(m))^2$, and hence $s_p(m) \in [-1, 1]$.

Definition 1.4.4. [79] Let P be a Pythagorean fuzzy set. Then, the accuracy function α of P is defined by: $\alpha_p(m) = (\eta(m))^2 + (\tau(m))^2$ and hence $\alpha_p(m) \in [0, 1]$

Example 1.4.1. Let P and Q be in $R \neq \emptyset$ such that η_p, τ_p, η_Q and τ_Q are defined as follows:

Mappings	m	n
η_p	0.6	0.5
τ_p	0.4	0.4
$\eta_p + \tau_p$	1	0.9
$(\eta_p)^2$	0.36	0.25
$(\tau_p)^2$	0.16	0.16
$(\eta_p)^2 + (\tau_p)^2$	0.52	0.41

Mappings	m	n
η_Q	0.7	0.5
τ_Q	0.5	0.4
$\eta_Q + \tau_Q$	1.2	0.9
$(\eta_Q)^2$	0.49	0.25
$(\tau_Q)^2$	0.25	0.16
$(\eta_Q)^2 + (\tau_Q)^2$	0.74	0.41

Table 1.2: Tables comparing intuitionistic and Pythagorean fuzzy sets

From the above tables, it is observed that:

$P = \{ \langle m, \eta_p(m), \tau_p(m) \rangle : m \in R \}$ is both Pythagorean and intuitionistic fuzzy sets.

However, $Q = \{ \langle m, \eta_Q(m), \tau_Q(m) \rangle : m \in R \}$ is a Pythagorean fuzzy set but not an intuitionistic fuzzy set and:

$$a_p(m) = 0.36 + 0.16 = 0.52,$$

$$a_p(n) = 0.25 + 0.16 = 0.41,$$

$$s_p(m) = 0.36 - 0.16 = 0.2,$$

$$s_p(n) = 0.25 - 0.16 = 0.09,$$

$$\pi_p(m) = 1 - 0.36 - 0.16 = 1 - 0.52 = 0.48,$$

$$\pi_p(n) = 1 - 0.25 - 0.16 = 1 - 0.41 = 0.59$$

and similarly one can follow similar calculations to determine the accuracy function, the score function and the indeterminacy degrees for Q .

Remark 1.4.1. (1) *Pythagorean fuzzy set is the generalization of intuitionistic fuzzy set as the preceding example, for instance, indicates how Pythagorean fuzzy set generalizes the intuitionistic fuzzy set as particular case.*

(2) *As X. Peng & J. Dai [58] in 2017 and [59] in 2018 explored descriptions of the three basic concepts: accuracy function, score function and degree of indeterminacy, the concepts meant as follows:*

(i) *Accuracy Function: It determines the intensity of expressed judgment whether it is large or small in expression. It is an indicator of the general conviction or determination of the assessment. The higher the accuracy function, the lower the hesitation in making a decision; and the lower the accuracy, the higher the hesitation,*

- (ii) *Score Function*: It measures the net dip or slant of a judgment, that is, in favor or against a given option. Its uses are to rank or compare alternatives. The more the score the higher the support and the less the score the higher the opposition,
- (iii) *Degree of Indeterminacy*: It is the level of indeterminacy reflecting the extent of doubt in a Pythagorean fuzzy assessment. It determines the amount of the information that remains undecided-neither obviously to the affirmative, nor to the negation. The degree of indeterminacy shows less clearness in judgment and the judgment is more hesitant. The higher the degree of indeterminacy, the greater the doubt; the lower the degree, the more decisive the judgment.

1.5. Basic Concepts of Some Fundamental Structures

(Homomorphisms, Level sets, Cartesian Products and (β, θ) -Fuzzy Structures)

This section reviews key background concepts including homomorphisms, level sets, and Cartesian products of fuzzy sets, as well as the framework of (β, θ) -fuzzy structures, all foundational in earlier studies of fuzzy algebraic systems.

Definition 1.5.1. [9] A mapping $f: R \rightarrow Y$ of BE-algebras $(R; \otimes, 1)$ is called a homomorphism if $f(m \otimes n) = f(m) \otimes f(n), \forall m, n \in R$.

Definition 1.5.2. [15] Let η be a fuzzy subset in a non-empty set S . For $t \in [0, 1]$, then the set $A = \{m \in S : \eta(m) \geq t\}$ is called a level subset of the set S .

Definition 1.5.3. [74] Let R be PMS-algebra and $A = (\eta, \tau)$ be intuitionistic fuzzy PMS-ideal of R . For $t, s \in [0, 1]$, the ideals $U(\eta, t) = \{m \in R : \eta \geq t\}$ and $L(\tau, t) = \{m \in R : \tau \leq s\}$ are called the upper and lower level PMS-ideals of R , respectively.

Definition 1.5.4. [51, 61, 71] Let η be fuzzy sets of an algebra R . Then the Cartesian product $\eta \times \eta : R \times R \rightarrow [0, 1]$ is defined by $(\eta \times \eta)(m, n) = \min\{\eta(m), \eta(n)\}, \forall m, n \in R$, and then $(m_1, n_1) \otimes (m_2, n_2) = (m_1 \otimes m_2, n_1 \otimes n_2), \forall m_1, m_2, n_1, n_2 \in B$, so that for fuzzy sets $\eta_1, \eta_2 :$
 $(\eta_1 \times \eta_2)(m, n) = \min\{\eta_1(m), \eta_2(n)\}, \forall m, n \in R$ and
 $(\eta_1 \times \eta_2)(m_1, m_2) \otimes (\eta_1, \eta_2)(n_1, n_2) = (\eta_1 \times \eta_2)(m_1 \otimes n_1, m_2 \otimes n_2), \forall m_1, m_2, n_1, n_2 \in B$.

Definition 1.5.5. [8] Let η be any fuzzy subsets of R and Y . Then the Cartesian product $\eta \times \eta$ with membership function $\eta \times \eta : R \times Y \rightarrow [0,1]$ is defined by:

$$(\eta \times \eta)(m, n) = \min\{\eta(m), \eta(n)\}, \quad \forall (m, n) \in R \times Y.$$

Definition 1.5.6. [73] A fuzzy subset η is called a (β, θ) -fuzzy subalgebra of CI-algebra R if it satisfies, $\forall m, n \in R$ and $0 \leq \beta < \theta \leq 1$:

$$\eta(m \otimes n) \vee \beta \geq \eta(m) \wedge \eta(n) \wedge \theta$$

Definition 1.5.7. [73] An intuitionistic fuzzy set $A = (\eta, \tau)$ of CI-algebra R is called the intuitionistic (β, θ) -fuzzy subalgebra of R if it satisfies: for all $m, n \in R$

$$(i) \quad \eta(m \otimes n) \vee \beta \geq \eta(m) \wedge \eta(n) \wedge \theta$$

$$(ii) \quad \tau(m \otimes n) \wedge \theta \leq \tau(m) \vee \tau(n) \vee \beta$$

Example 1.5.1. [73] Let $R = \{1, p, q, r\}$ in which \otimes is defined by the table below:

\otimes	1	p	q	r
1	1	p	q	r
p	1	1	q	q
q	1	p	1	p
r	1	1	r	1

Table 1.3: A table of CI-algebra, $(R; \otimes, 1)$

Define membership and non-membership fuzzy sets η_1 and τ_1 respectively as follows:

$$\eta_1(m) = \begin{cases} 1, & \text{if } m = 1, q, r \\ 0, & \text{if } m = p \end{cases} \quad \text{and} \quad \tau_1(m) = \begin{cases} 0, & \text{if } m = 1, q, r \\ 1, & \text{if } m = p \end{cases}$$

Then $A = (\eta_1, \tau_1)$ is an intuitionistic (β, θ) -fuzzy subalgebra of R .

But if we define membership and non-membership fuzzy sets η_2 and τ_2 respectively as follows:

$$\eta_2(m) = \begin{cases} 0.4, & \text{if } m = 1, q \\ 0.6, & \text{if } m = p, r \end{cases} \quad \text{and} \quad \tau_2(m) = \begin{cases} 0.6, & \text{if } m = 1, q \\ 0.4, & \text{if } m = p, r \end{cases}$$

Then $B = (\eta_2, \tau_2)$ is NOT an intuitionistic (β, θ) -fuzzy subalgebra of R .

Chapter 2

Hyper Structures of BCL–Algebra

2.1. Hyper BCL–Algebra

Under this section, we introduce different notions of hyper BCL–algebra, different types of these hyper substructures like hyper subalgebra, weak hyper subalgebra, strong hyper subalgebra, hyper deductive system, weak hyper deductive system, strong hyper deductive system, hyper ideal, weak hyper ideal and strong hyper ideal of hyper BCL–algebra, state and prove theorems and properties of the hyper BCL–algebra and its hyper substructures.

By (weak, strong) hyper substructure, we mean weak hyper substructure, strong hyper substructure, hyper substructure, and by weak/strong hyper substructure, we mean weak hyper substructure and strong hyper substructure.

At this stage, since the entire dissertation is based on the BCL–algebra, we discuss some new and relevant properties of the BCL–algebra defined in Definition 1.1.8, which have not been discussed before. These properties are important because we will use each of them in different subsections and sections of the study.

Proposition 2.1.1. *If $(B; \otimes, 0)$ is BCL–algebra then the following statements hold, $\forall m \in B$.*

- (1) $0 \otimes (0 \otimes m) = 0$,
- (2) $0 \otimes (0 \otimes (0 \otimes m)) = 0$ (and generally, $\dots 0 \otimes (0 \otimes (0 \otimes (0 \otimes m))) = 0$),
- (3) $0 \otimes m = 0$,
- (4) $m \otimes 0 = 0 \Rightarrow m = 0$,
- (5) $m \otimes 0 = m$.

Proof. Let B be a BCL–algebra and suppose $m \in B$.

(1) Then we need to prove that $0 \otimes (0 \otimes m) = 0, \forall m \in B$.

From Definition 1.1.8 (3), we have the following:

$$[((m \otimes m) \otimes m) \otimes ((m \otimes m) \otimes m)] \otimes ((m \otimes m) \otimes m) = 0 \Rightarrow 0 \otimes (0 \otimes m) = 0, \text{ since for:}$$

$k = (m \otimes m) \otimes m$, we have $k \otimes k = 0$ and $m \otimes m = 0$ by Definition 1.1.8

Hence $0 \otimes (0 \otimes m) = 0, \forall m \in B$.

(2) Here we need to show $0 \otimes (0 \otimes (0 \otimes m)) = 0, \forall m \in B$.

$$\Rightarrow 0 \otimes (0 \otimes (0 \otimes m)) = 0 \otimes (0 \otimes 0) = 0 \otimes 0 = 0, \forall m \in B, \text{ since } 0 \otimes (0 \otimes m) = 0, \forall m \in B, \text{ by (1)}$$

above, and it holds for such similar successive equations.

(3) Under this proposition, we need to prove $0 \otimes m = 0$, by (1) above

Suppose: $0 \otimes m \neq 0$ and let $0 \otimes m = n$, where $n \neq 0$. Now, $0 \otimes (0 \otimes m) = 0, \forall m \in B$

$$\Rightarrow 0 \otimes n = 0, \forall m \in B, \forall n = 0 \otimes m \text{ and } n \neq 0 \text{ which is a contradiction to our assumption:}$$

$0 \otimes m = n \neq 0 \Rightarrow 0 \otimes m = 0, \forall m \in B$. Again, by the definition of binary relation on B, we have:

$0 \otimes m = 0$, if and only if $0 \leq m, \forall m \in B$ so that it implies that 0 is the least element of B.

(4) Supposing $m \otimes 0 = 0$, then we need show that $m = 0$

From $0 \otimes m = 0$ and $m \otimes 0 = 0$, we get $m = 0$ by Definition 1.1.8 (2).

(5) To show that $m \otimes 0 = m$, first we need show $(m \otimes 0) \otimes m = 0$ and $m \otimes (m \otimes 0) = 0$ to get

$m \otimes 0 = m$ by Definition 1.1.8 (2).

(i) We claim $(m \otimes 0) \otimes m = 0$. Then from Definition 1.1.8 (3), by letting $n = 0$ and $w = m$,

we have: $[(m \otimes 0) \otimes m] \otimes ((m \otimes m) \otimes 0) \otimes ((m \otimes 0) \otimes m) = 0$ and by letting

$p = (m \otimes 0) \otimes m$, we get: $(p \otimes 0) \otimes p = 0, \forall p \in B$, as $m \otimes m = 0 \Rightarrow (m \otimes m) \otimes 0 = 0 \forall m \in B$.

(ii) We claim $p \otimes (p \otimes 0) = 0$.

From Definition 1.1.8 (3), by letting $m, p, w = 0$ and then $n = p \otimes 0$, we have:

$$[((p \otimes (p \otimes 0)) \otimes 0) \otimes ((p \otimes 0) \otimes (p \otimes 0))] \otimes ((0 \otimes (p \otimes 0)) \otimes p) = 0$$

$$\Rightarrow [((p \otimes (p \otimes 0)) \otimes 0) \otimes ((p \otimes 0) \otimes (p \otimes 0))] \otimes 0 = 0, \text{ by (3) above}$$

$$\Rightarrow ((p \otimes (p \otimes 0)) \otimes 0) \otimes ((p \otimes 0) \otimes (p \otimes 0)) = 0, \text{ by (4) above}$$

$\Rightarrow ((p \circledast (p \circledast 0)) \circledast 0) \circledast 0 = 0$, since $(p \circledast 0) \circledast (p \circledast 0) = 0$, by Definition 1.1.8 (1)

$\Rightarrow (p \circledast (p \circledast 0)) \circledast 0 = 0$ and then $p \circledast (p \circledast 0) = 0, \forall p \in B$, by (4) above.

Then from (i) and (ii) above, we have the following, as a whole:

$(p \circledast 0) \circledast p = 0$ and $p \circledast (p \circledast 0) = 0 \Rightarrow (p \circledast 0) = p, \forall p \in B$, by Definition 1.1.8 (2) □

Definition 2.1.1. An algebra $B = (B; @, 0)$, with a hyper operation $@ : B \times B \rightarrow \mathfrak{B}$ and a constant 0 is called a hyper BCL–algebra, if $\forall m, n, w \in B$ and a hyper relation \preceq on a non-empty set B the following axioms are satisfied:

(1) $0 \preceq m, m \preceq m$,

(2) $m \preceq n$ and $n \preceq m \Rightarrow m = n$,

(3) $((m @ n) @ w) @ ((m @ w) @ n) \preceq (w @ n) @ m$,

where the hyper relation “ \preceq ” is defined by: $m \preceq n \Leftrightarrow 0 \in m @ n$ and $\mathfrak{B} = P(B) - \{\emptyset\}$,

$P(B)$ is power set of a non-empty set B .

Remark 2.1.1. Let $(B; @, 0)$ be a hyper BCL–algebra and $m \preceq n \Leftrightarrow 0 \in m @ n$ as defined in Definition 2.1.1. Then the axioms for hyper BCL–algebra can be written as follows, $\forall m, n, w \in B$:

(1) $0 \in 0 @ m, 0 \in m @ m$,

(2) $0 \in m @ n$ and $0 \in n @ m \Rightarrow m = n$,

(3) $0 \in [((m @ n) @ w) @ ((m @ w) @ n)] @ ((w @ n) @ m)$,

Definition 2.1.2. For non-empty subsets G and K of the hyper BCL–algebra B ; $a, b \in B$, and two hyper relations \prec and \succeq , the following are defined:

(1) (i) $G @ K = \bigcup_{\substack{g \in G, \\ k \in K}} (g @ k)$, (ii) $a @ K = \bigcup_{k \in K} (a @ k)$, (iii) $G @ b = \bigcup_{g \in G} (g @ b)$,

(2) $G \prec K \Leftrightarrow (\exists g \in G) (\exists k \in K)$ such that $g \preceq k$,

(3) $G \preceq K \Leftrightarrow (\exists g \in G) (\forall k \in K)$ such that $g \preceq k$,

(4) $G \succeq K \Leftrightarrow (\forall g \in G) (\exists k \in K)$ such that $g \preceq k$,

(5) $m @ n \subseteq G \Leftrightarrow \forall g \in m @ n \Rightarrow g \in G$

Example 2.1.1. If $(B; \otimes, 0)$ is a BCL–algebra and $@$ is a hyper operation on B defined by:
 $m @ n = \{m \otimes n\}, \forall m, n \in B$, then $(B; @, 0)$ is a hyper BCL–algebra as justified below:

Let $(B; \otimes, 0)$ be a BCL–algebra and $@$ be a hyper operation on B , then we need to show that $(B; @, 0)$ is a hyper BCL–algebra where $\{m \otimes n\} = m @ n$. In other words, we justify that the three axioms for Definition 2.1.1 are satisfied in the sense that $m @ n = \{m \otimes n\}, \forall m, n \in B$, as follows:

Definition 2.1.1 (i): From Definition 1.1.8 (1), we have, $m \otimes m = 0 \Rightarrow 0 \in \{m \otimes m\} = m @ m$

$$\Rightarrow 0 \in m @ m \Rightarrow m \preceq m, \text{ by definition of } "\preceq",$$

Again, as proved in Proposition 2.1.1 (3), we have: $0 \otimes m = 0$

$$\Rightarrow 0 \in \{0 \otimes m\} = 0 @ m \Rightarrow 0 \in 0 @ m \Rightarrow 0 \preceq m, \text{ by definition of } "\preceq".$$

Definition 2.1.1 (ii): From Definition 1.1.8 (2), $m \otimes n = 0$ and $n \otimes m = 0 \Rightarrow m = n$

$$\Rightarrow 0 \in \{m \otimes n\} = m @ n \text{ and } 0 \in \{n \otimes m\} = m @ n \Rightarrow m = n$$

$$\Rightarrow 0 \in m @ n \text{ and } 0 \in m @ n \Rightarrow m = n \text{ and}$$

$$0 \in m @ n \Leftrightarrow m \preceq n \text{ and } 0 \in n @ m \Leftrightarrow n \preceq m, \text{ by definition of } "\preceq"$$

$$\Rightarrow m \preceq n \text{ and } m \preceq n \Rightarrow m = n$$

Definition 2.1.1 (iii): From Definition 1.1.8 (3), $\forall m, n, w \in B$, we have:

Since $(B; \otimes, 0)$ is a BCL–algebra, by 2.1.1 (iii), for arbitrary $m, n, w \in B$, we have:

$$(((m \otimes n) \otimes w) \otimes ((m \otimes w) \otimes n)) \otimes ((w \otimes n) \otimes m) = 0 \text{ and then,}$$

$$\{(((m \otimes n) \otimes w) \otimes ((m \otimes w) \otimes n)) \otimes ((w \otimes n) \otimes m)\} = \{0\}$$

$$\Rightarrow 0 \in \{0\} = \{(((m \otimes n) \otimes w) \otimes ((m \otimes w) \otimes n)) \otimes ((w \otimes n) \otimes m)\}$$

$$= [((m @ n) @ w) @ ((m @ w) @ n)] @ ((w @ n) @ n) @ m$$

$$\Rightarrow ((m @ n) @ w) @ ((m @ w) @ n) \preceq (w @ n) @ n @ m, \text{ by definition of } "\preceq".$$

Example 2.1.2. Let $B = \{0, p, q, r\}$ and define a hyper operation $@$ on B by the following table:

@	0	p	q	r
0	{0}	{0, p}	{0}	{0, p, q}
p	{p, r}	{0, q}	{0, q, r}	{p, q, r}
q	{q}	{p, r}	{0, p}	{p, q}
r	{p, q, r}	{0}	{0, p, r}	{0, p}

Table 2.1: A te describing a hyper BCL–algebra, $B = (B; @, 0)$:

From Table 2.1 of Example 2.1.2 above, we have $(B; @, 0)$ is a hyper BCL–algebra.

Lemma 2.1.1. Let $(B; @, 0)$ be hyper BCL–algebra such that $\emptyset \neq C \subseteq B$.

If $m @ n \subseteq C, \forall m, n \in C$, then $0 \in C$.

Proof. Suppose $\emptyset \neq C \subseteq B$ where B is a hyper BCL–algebra such that $m @ n \subseteq C, \forall m, n \in C$

\Rightarrow Since $\emptyset \neq C, \exists m \in C$ such that $m@m \subseteq C$ then by Definition 2.1.1 (1), $0 \in m@m \subseteq C$

$\Rightarrow 0 \in C$. □

Lemma 2.1.2. Let $(B; @, 0)$ be a hyper BCL–algebra such that $\emptyset \neq C \subseteq B$.

(1) If $0 \neq m \in C$ then $0 \notin m @ 0, \forall m \in C$, and hence $0 @ 0 = \{0\}$.

(2) $m \in m @ 0, \forall m \in B$.

Proof. (1) Let $\emptyset \neq C \subseteq B$ such that $0 \neq m \in C$. Assume $0 \in m @ 0$. Then by Definition 2.1.1 (2),

and as $0 \in 0 @ m, \forall m \in B$ by Definition 2.1.1 (1); we get; $0 \in 0 @ m$ and $0 \in m @ 0 \Rightarrow m = 0$

by Definition 2.1.1 (2), which is a contradiction to the hypothesis $0 \neq m$.

Hence $0 \notin m @ 0, \forall m \in B$, but $m = 0 \Rightarrow 0 @ 0 = \{0\}$, by the next Remark 2.1.2.

(2) Since $@ : B \rightarrow \mathfrak{B}$ is hyper operation, we have, $m@0 \neq \emptyset, \forall m \in B$.

Using Definition 2.1.1 (3): $[((m@n)@w)@((m@w)@n)] \preceq ((w@n)@m), \forall m, n, w \in B$

$\Rightarrow [((m@0)@m)@((m@m)@0)] \preceq ((m@0)@m)$, when $n = 0$ and $w = m, \forall m \in B$

$\Rightarrow \exists a \in ((m@0)@m)@((m@m)@0) \preceq$ such that $a \preceq b, \forall b \in ((m@0)@m), \forall m \in B$

$\Rightarrow a \preceq b \Leftrightarrow 0 \in a@b, \forall b \in (m@0)@m, \forall m \in B \Rightarrow$ There is a possibility for $b = 0$

$$\Rightarrow 0 = b \in (m @ 0) @ m, \forall m \in B$$

$$\Rightarrow 0 \in m @ 0 \text{ or } m \in m @ 0, \forall m \in B, \text{ by Definition 2.1.1 (1)}$$

But $0 \in m @ 0, \forall m \in B \Rightarrow m = 0$, by Lemma 2.1.1, which is a contradiction since m is arbitrary

$$\Rightarrow 0 \notin m @ 0, \text{ for } m \neq 0 \text{ and hence } m \in m @ 0, \forall m \in B. \quad \square$$

Remark 2.1.2. (1) The Contra-positive form of Lemma 2.1.2 above, stated as: "For the hyper BCL-algebra, $(B; @, 0)$ and $\emptyset \neq C \subseteq B; 0 \in m @ 0 \Rightarrow m = 0, \forall m \in C$ " holds; in other words, $m \preceq 0 \Rightarrow m = 0$,

(2) In general, from Remark 2.1.2 (1) and by Lemma 2.1.2 above, we infer that $m \preceq 0$ if and only if $m = 0$.

Proposition 2.1.2. Let $(B; \otimes, 0)$ be a hyper BCL-algebra. Then, the following hold, $\forall m, n, w \in B$:

$$(i) \quad 0 \in 0 @ (m @ n),$$

$$(ii) \quad 0 \in (0 @ m) @ (0 @ n),$$

$$(iii) \quad 0 \in (0 @ m) @ (n @ w),$$

$$(iv) \quad 0 \in (m @ (m @ m)) @ m.$$

Proof. Let B be hyper BCL-algebra. Then by Definition 2.1.1 (1), $0 \in 0 @ m, \forall m \in B$ and then,

$$(i) \quad m @ n \subseteq B \Rightarrow 0 \in 0 @ (m @ n), \text{ by Definition 4.1.1 (i)}$$

$$(ii) \quad 0 \in 0 @ 0 \Rightarrow 0 \in (0 @ m) @ (0 @ n), \text{ by Definition 2.1.1 (1) and (i) above}$$

$$(iii) \quad 0 \in 0 @ (m @ n) \Rightarrow 0 \in (0 @ m) @ (n @ w), \text{ as } n @ w \subseteq B, \text{ by (i) above.}$$

$$(iv) \quad 0 \in 0 @ m, \forall m \in B, \text{ by Definition 2.1.1 (1)}$$

$$\Rightarrow 0 \in (m @ 0) @ m, \text{ as } m \in m @ 0, \forall m \in B, \text{ by Lemma 2.1.2 (2)}$$

$$\Rightarrow 0 \in (m @ (m @ m)) @ m, \forall m \in B, \text{ as } 0 \in m @ m, \text{ by Definition 2.1.1 (1)}$$

□

Proposition 2.1.3. Let $(B; @, 0)$ be a hyper BCL-algebra and $\emptyset \neq C \subseteq B$.

Then the following hold, $\forall m, n \in B$:

$$(i) \quad 0 \preceq 0 @ m, \quad [\text{and generally, } 0 \preceq \dots 0 @ (0 @ (0 @ m))]$$

$$(ii) \ 0 @ m \prec \{n\},$$

$$(iii) \ 0 @ m \preceq \{n\},$$

$$(iv) \ C @ \{0\} = \{0\} \Rightarrow C = \{0\}.$$

Proof. (i) By the axioms of Definition 2.1.1 (3) of the hyper BCL–algebra B and \preceq , we have:

$$0 \in [((m@m)@m)@((m@m)@m)]@((m@m)@m), \forall m \in B \text{ (translating definition of } \preceq)$$

$$\Rightarrow 0 \in [(0 @ m) @ (0 @ m)] @ (0 @ m) \Rightarrow 0 \in (0 @ 0) @ (0 @ m), \forall m \in B$$

$$\Rightarrow 0 \in 0 @ (0 @ m) \Rightarrow 0 \preceq 0 @ m, \forall m \in B$$

$$\text{Or as } 0 @ m \subseteq B, \text{ we have } 0 \in 0 @ (0 @ m) \Rightarrow 0 \preceq 0 @ m, \forall m \in B$$

Furthermore, along the steps of the above proof, it is observed that:

$$0 \in (0 @ (0 @ (0 @ m))) \text{ and hence } 0 \preceq 0 @ (m @ (n @ w)), \forall m, n, w \in B$$

$$(ii) \text{ Let } 0 \preceq m \Rightarrow \exists 0 \in 0 @ m \text{ such that } 0 \in (0 @ m) @ n \Rightarrow 0 @ m \prec \{n\}$$

(iii) Follows from (ii), since $\{n\}$ is singleton set.

$$(iv) \text{ Let } \emptyset \neq C \subseteq B \text{ and } C @ \{0\} = \bigcup_{\substack{m \in C, \\ 0 \in \{0\}}} (m @ 0) = \bigcup_{m \in C} (m @ 0) = \{0\}$$

$$\Rightarrow m @ 0 = \{0\}, \forall m \in C \Rightarrow m = 0 \Rightarrow C = \{0\} \text{ which is automatic by Remark 2.1.2.}$$

□

Proposition 2.1.4. $B = (B; @, 0)$ be hyper BCL–algebra. Then the following hold, for $\emptyset \neq C \subseteq B$, $\emptyset \neq D \subseteq B$, $\forall m \in B$:

- | | |
|---------------------------------------------------------------------|---------------------------------------------------------------------------------|
| 1. $C \preceq C, C \succeq C,$ | 7. $0 @ m \prec C,$ |
| 2. $C @ D \subseteq B,$ | 8. $C \cap D \neq \emptyset \Rightarrow C \prec D, D \prec C,$ |
| 3. $C \subseteq D \Rightarrow C \prec D, C \succeq D, D \preceq C,$ | 9. $C \preceq D \Rightarrow C \prec D,$ |
| 4. $\{0\} \preceq C, \{0\} \succeq C,$ | 10. $C \succeq D \Rightarrow C \prec D,$ |
| 5. $C \succeq \{0\} \Rightarrow C = \{0\},$ | 11. $C \cap D \prec C \cup D, \text{ for } C \cap D \neq \emptyset,$ |
| 6. $0 \in C \Rightarrow C \preceq D,$ | 12. $C \cap D \prec C, C \cap D \prec D, \text{ for } C \cap D \neq \emptyset.$ |

Proof. Let $(B; @, 0)$ be hyper BCL–algebra, $\emptyset \neq C, \emptyset \neq D$ such that $C, D \subseteq B$, $m, n, c, d, p, q \in B$:

- (1) Since $\forall m \in C; 0 \in m @ m$ hence $C \preceq C$ and $C \succeq C$, are straightforward,
- (2) $C @ D = \bigcup_{\substack{m \in C(\subseteq B), \\ n \in D(\subseteq B)}} (m @ n) \subseteq \bigcup_{\substack{p \in B, \\ q \in B}} (p @ q) = \bigcup_{p, q \in B} (p @ q) = B$,
- (3) Suppose $C \subseteq D$
 - (i) $\exists m \in C$ such that $m \in D \Rightarrow 0 \in m @ m \Rightarrow C \prec D$,
 - (ii) Let $m \in C \Rightarrow m \in D \Rightarrow 0 \in m @ m, \forall m \in C \Rightarrow C \succeq D$, and
 - (iii) $m \in C \Rightarrow m \in D \Rightarrow \exists m \in D$ such that $m \preceq m, \forall m \in C \Rightarrow D \preceq C$,
- (4) For $\emptyset \neq C \subseteq B, 0 \in 0 @ m, \forall m \in C$ by Definition 2.1.1 (1) and since $\{0\}$ is singleton set it follows than $\{0\} \preceq C$ and $\{0\} \succeq C$.
- (5) Suppose $C \succeq \{0\} \Rightarrow (\forall m \in C)(\exists 0 \in \{0\})$ such that $0 \in m @ 0$ and by Proposition 2.1.3 (ii) or by Remark 2.1.2 we have: $m = 0, \forall m \in C \Rightarrow C = \{0\}$,
- (6) $0 \in C \Rightarrow 0 \preceq m$ or $0 \in 0 @ m, \forall m \in D \Rightarrow C \preceq D$,
- (7) $\exists 0 \in (0 @ m)$ such that $0 \preceq n$ (or $0 \in 0 @ n$), for some $n \in C \Rightarrow 0 @ m \prec C$,
- (8) Since $C \cap D \neq \emptyset, \exists m \in C \cap D \Rightarrow m \in C$ and $m \in D \Rightarrow 0 \in m @ m \Rightarrow C \prec D$ and $D \prec C$,
- (9) $C \preceq D \Rightarrow \exists m \in C, \forall n \in D$ such that $0 \in m @ n \Rightarrow (\exists m \in C)(\exists n \in D)$ such that $0 \in m @ n \Rightarrow C \prec D$,
- (10) Suppose $C \succeq D \Rightarrow (\forall m \in C)(\exists n \in D)$ such that $0 \in m @ n \Rightarrow (\exists m \in C)(\exists n \in D)$ such that $0 \in m @ n \Rightarrow C \prec D$,
- (11) As $C \cap D \neq \emptyset, \exists m \in C \cap D$ and then $m \in C \cup D$ such that $0 \in m @ m \Rightarrow C \cap D \prec C \cup D$.
- (12) As $C \cap D \neq \emptyset, \exists m \in C \cap D$ and $m \in C$ such that $0 \in m @ m \Rightarrow C \cap D \prec C$, and
as $C \cap D \neq \emptyset, \exists m \in C \cap D$ and $m \in D$ such that $0 \in m @ m \Rightarrow C \cap D \prec D$.

□

Remark 2.1.3. For the hyper BCL–algebra $(B; @, 0)$ if C and D are non-empty subsets of B then the following may not be necessarily true justified by the succeeding counter example:

$$(i) \quad C \preceq D \Leftrightarrow D \succeq C,$$

$$(ii) \quad C \cup D = C @ D, \quad C \cup D = \bigcup_{\substack{c \in C, \\ d \in D}} (c @ d), \text{ where } \bigcup \text{ represents the crisp union.}$$

$$(iii) \quad C \preceq D \text{ and } D \preceq C \text{ (or } C \succeq D \text{ and } D \succeq C) \Rightarrow C = D.$$

Example 2.1.3. Based on Table 2.1 of Example 2.1.2, we present counterexamples that support the assertions made in Remark 2.1.3.

(i) $\{r\} \preceq q @ p$ or $\{r\} \preceq \{p, r\}$ holds true but $q @ p \succeq \{r\}$ does not hold true and for the converse, $0 @ q \succeq p @ r$ or $\{0\} \succeq \{p, q, r\}$ holds true but $p @ r \preceq 0 @ q$ does not hold true and hence neither $C \preceq D \Rightarrow D \succeq C$ nor $D \succeq C \Rightarrow C \preceq D$

(ii) For $C = \{0\}$ and $D = \{0, r\}$, $C \cup D = \{0, r\}$ but

$$C @ D = \bigcup_{\substack{c \in C, \\ d \in D}} (c @ d) = (0 @ 0) \cup (0 @ r) = \{0\} \cup \{0, p, q\} = \{0, p, q\} :$$

$$\Rightarrow C \cup D = \{0, r\} \neq \{0, p, q\} = C @ D$$

(iii) $\{0\} \preceq \{0, p\}$ and $\{0, p\} \preceq \{0\}$ hold but $\{0\} = \{0, p\}$ does not hold.

Lemma 2.1.3. If the hyper BCL–algebra $(B; @, 0)$ is such that each column entry in column m is $\{0\}$, then $B = \{0\}$.

Proof. As B is hyper BCL–algebra, $0 \in 0 @ m$ by Definition 2.1.1 and by the hypothesis,

$$m @ 0 = \{0\}, \forall m \in B. \text{ Then by Definition 2.1.1 (2) } 0 \in 0 @ m \text{ and } 0 \in m @ 0, \forall m \in B$$

$$\Rightarrow m = 0, \forall m \in B \Rightarrow B = \{0\}. \quad \square$$

Lemma 2.1.4. In a hyper BCL–algebra $(B; @, 0)$, if B is such that all row entries in row- m are $\{0\}$, ($0 @ m = \{0\}$, for one element m of B), then $m = 0$ and the number of elements (cardinality) of B is reduced by one. Again since m and 0 are not distinct the cardinality is reduced by the number of elements that are the same as 0 , and for this.

Proof. Since $0 \in 0 @ m$, for $\forall m \in B$, by Definition 2.1.1 (2), and $m @ 0 = \{0\}$, by the hypothesis,

$$\text{we have } m = 0, \text{ by Definition 2.1.1 (1) and } m @ 0 = \{0\} \text{ by the hypothesis, it follows } m = 0$$

$$\text{by Definition 2.1.1 (2).} \quad \square$$

2.2. Hyper Subalgebras of Hyper BCL–Algebra

In this section, the idea of subalgebra is expanded to hyper subalgebras in a hyper BCL–algebra, where one operation can give several results. We show what conditions make a subset a hyper subalgebra and explain its main kinds: weak hyper subalgebra, strong hyper subalgebra, and hyper subalgebra. Their properties and differences are presented through clear definitions, examples, and well-proved theorems to show their correctness. The union and intersection of hyper subalgebras are also discussed and properly justified. Finally, the relationships among weak hyper subalgebra, strong hyper subalgebra, and hyper subalgebra are explained with convincing reasons.

Definition 2.2.1. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq S \subseteq B$. Then S is a hyper subalgebra of B if and only if the following is satisfied, $\forall m, n \in B: m, n \in S \Rightarrow m @ n \subseteq S$.

Definition 2.2.2. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq S \subseteq B$. Then S is weak hyper subalgebra of B if and only if the following is satisfied, $\forall m, n \in B: m, n \in S \Rightarrow m @ n \prec S$.

Definition 2.2.3. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq S \subseteq B$. Then S is strong hyper subalgebra of B if and only if the following is satisfied, $\forall m, n \in B: m, n \in S \Rightarrow m @ n \preceq S$.

Example 2.2.1. From Table 2.1 of Example 2.1.2 above, we have:

- (i) $\{0\}$, $\{0, q\}$ and B are hyper subalgebras of B ,
- (ii) $\{0\}$, $\{0, p\}$, $\{0, r\}$, $\{0, p, q\}$, $\{q\}$, $\{r\}$, $\{p, r\}$, B are weak hyper subalgebras of B ,
- (iii) $\{0\}$ is strong hyper subalgebra of B .

Example 2.2.2. Let $B = \{0, p, q, r\}$ and define a hyper operation $@$ on B by the table as follows:

@	0	p	q	r
0	{0}	{0, p}	{0}	{0, p, q}
p	{p, q}	{0, q}	{0, r}	{p, q, r}
q	{q}	{p, q, r}	{0}	{p, q}
r	{p, q, r}	{0}	{0, p, r}	{0, p}

Table 2.2: A table describing a hyper BCL–algebra, $B = (B; @, 0)$:

From Table 2.2 of Example 2.2.2 above, $(B; @, 0)$ is a hyper BCL–algebra. Furthermore:

- (i) $\{0\}$, $\{0, p, q\}$, B are hyper subalgebras.
- (ii) $\{0\}$, $\{0, q\}$, $\{0, r\}$, $\{0, p, q\}$, $\{0, q, r\}$, B are weak hyper subalgebras of B .
- (iii) $\{0\}$, $\{p\}$, $\{q\}$, $\{r\}$, $\{p, q\}$, $\{p, r\}$ is strong hyper subalgebra of B .

Proposition 2.2.1. *Let $(B; @, 0)$ be a hyper BCL–algebra. Then:*

- (1) $\{0\}$ is a (weak, strong) hyper subalgebra of B .
- (2) B is (weak) hyper subalgebra but not strong hyper subalgebra of B .

Proof. Let $(B; @, 0)$ be a hyper BCL–algebra.

- (1) Clearly, $0 \in B$. Now since $0@0 = \{0\}$, we have: $\{0\} = 0@0 \subseteq \{0\}$, $\{0\} = 0@0 \prec \{0\}$ and $\{0\} = 0 @ 0 \preceq \{0\}$ and hence $\{0\}$ is a hyper subalgebra, weak hyper subalgebra and strong hyper subalgebra of B .

- (2) The proof for B is (weak) hyper subalgebra:

Since the hyper operation $@ : R \times R \rightarrow \mathfrak{B}$, we have $m@n \subseteq B, \forall m, n \in B \Rightarrow B$ is hyper subalgebra of B .

Furthermore, $m@n \subseteq B, m@n \preceq B, \forall m, n \in B$ by Proposition 2.1.4 (3) and hence B is weak hyper subalgebra of B .

For B is not strong hyper subalgebra of B can be justified by considering $q@r$ of Example 2.2.2 for $B = \{0, p, q, r\}$ since there is no element say $k \in q@r$ such that $\forall m \in B, 0 \in k@m$ as $q@r = \{p, q\} \not\subseteq \{0, p, q, r\} = B$ in that $0 \notin p@0$ or $0 \notin q@r$ □

Lemma 2.2.1. *Let $(B; @, 0)$ be a hyper BCL–algebra. Then the following hold:*

- (1) If S_s is a strong hyper subalgebra of B , then $S_s = \{0\}$.
- (2) If S_h is hyper subalgebra of B , then, $0 \in S_h$.

Proof. Let $(B; @, 0)$ be a hyper BCL–algebra.

- (1) Let S_s be strong hyper subalgebra of B . Then since $S_s \neq \emptyset$ by definition of strong hyper subalgebra of B , $\exists m \in S_s$ such that $m@m \preceq S_s$ and then $0 \in m@m \preceq S_s$.

Then $\{0, m\} \preceq S_s$ or $\{0\} \preceq S_s$ if $m = 0$.

Now, we claim that $m = 0, \forall m \in S_s$ but since $0, m \in S_s, m @ 0 \preceq S_s$

$\Rightarrow (\exists p \in m@0) (\forall q \in S_s) (0 \in p@q)$

$\Rightarrow 0 \in p@0$, since q is arbitrary element of $S_s \Rightarrow p = 0$ by Remark 2.1.1

$\Rightarrow 0 \in m@0, \forall m \in S_s$

$\Rightarrow m = 0$ by Definition 2.1.1 (2), $\forall m \in S_s$ and therefore, $S_s = \{0\}$

(2) Let S_h be hyper subalgebra of B . Then by the definition of $S_h \neq \emptyset \Rightarrow \exists m \in S_h$ such that

$0 \in m@m \subseteq S_h \Rightarrow 0 \in S_h$.

□

Proposition 2.2.2. Let $(B; @, 0)$ be a hyper BCL–algebra. Then:

- (1) Every strong hyper subalgebra of B is hyper subalgebra of B ,
- (2) Every hyper subalgebra of B is weak hyper subalgebra of B .
- (3) Every strong hyper subalgebra of B is weak hyper subalgebra of B .

Proof. Let $(B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq S_s \subseteq B$.

- (1) Suppose S_s is strong hyper subalgebra of B . Then by Lemma 2.2.1 (1), $S_s = \{0\}$ and since for every hyper subalgebra S_h of B , $\{0\} \subseteq S_h$, by Proposition 2.2.1 and hence the proof.
- (2) Suppose S_h is a hyper subalgebra of B ; that is $m, n \in S_h, m @ n \subseteq S_h$. Then by Proposition 2.1.4 (3), $m @ n \prec S_h$. Hence S_h is weak hyper subalgebra of B .
- (3) Every hyper subalgebra of B is weak hyper subalgebra of B holds by transitivity of (1) to (2) above in this proposition.

Note: Generally, for S_s, S_h and S_w denoting the set of all strong hyper subalgebras, set of all hyper subalgebras and the set of all weak hyper subalgebras of a hyper BCL–algebra, respectively, we have: $S_s \subseteq S_h \subseteq S_w$.

□

Remark 2.2.1. The converses of each statement in Proposition 2.2.2 may not be necessary true as explicitly seen in Examples 2.2.2 as for instance that $\{0, q\}$ is hyper subalgebra but not strong hyper subalgebra of B .

Example 2.2.3. Let $(B; @, 0)$ be hyper BCL–algebra where $B = \{0, p, q, r\}$ and let $@$ be the hyper operation defined in Table 2.2 of Example 2.2.2 above. Then:

- (1) $\{0, q\}$ is hyper subalgebra of B (as listed in Example 2.2.2 (i))

but not strong hyper subalgebra of B , because of the following reasons:

$q @ 0 \preceq \{0, q\}$ is false as $q @ 0 = \{q\} \not\subseteq \{0, q\}$

or $(\exists s \in q@0) (\forall t \in \{0, q\}) (s \preceq t \text{ or } 0 \in s@t)$ is not true, by definition of \preceq .

Hence S is hyper subalgebra of B does not necessarily imply S is strong hyper subalgebra of B .

(2) $\{0, r\}$ is weak hyper subalgebra of B (as listed in Example 2.2.2 (ii))

but not hyper subalgebra of B , because of the following reasons:

$$r @ 0 \subseteq \{0, r\} \text{ is false as } r @ 0 = \{p, q, r\} \not\subseteq \{0, r\}$$

Hence S is weak hyper subalgebra of B does not necessarily imply S is hyper subalgebra of B .

(3) $\{0, r\}$ is weak hyper subalgebra of B (as listed in Example 2.2.2 (ii))

but not strong hyper subalgebra of B , because of the following reasons:

$$r @ 0 \preceq \{0, r\} \text{ is false as } r @ 0 = \{p, q, r\} \not\preceq \{0, r\} \text{ or}$$

$$(\exists s \in r @ 0) (\forall t \in \{0, q\}) (s \preceq t \text{ or } 0 \in \{0, r\} s @ t) \text{ is not true for } t = 0, \text{ by definition of } \preceq.$$

Hence S is weak hyper subalgebra of B does not imply S is strong hyper subalgebra of B .

Theorem 2.2.1. *Let $(B; @, 0)$ be a hyper BCL–algebra and S_1 and S_2 be (weak, strong) hyper subalgebras of the hyper BCL–algebra B . Then the non-empty intersection, $S_1 \cap S_2$, of any two (weak, strong) hyper subalgebras of B is also (weak, strong) hyper subalgebra of B , respectively.*

Proof. (1) Let S_1 and S_2 be any two hyper subalgebras of the hyper BCL–algebra B .

Suppose $m, n \in S_1 \cap S_2$, and we need to show that $m @ n \subseteq S_1 \cap S_2$.

Now, since $m, n \in S_1$, $m, n \in S_2$ and S_1, S_2 are both hyper subalgebras of B , we have:

$$m @ n \subseteq S_1 \text{ and } m @ n \subseteq S_2 \text{ and then } m @ n \subseteq S_1 \cap S_2.$$

Therefore, $S_1 \cap S_2$ is a hyper subalgebra of B .

(2) Let S_1 and S_2 be any two weak hyper subalgebras of B and suppose $m, n \in S_1 \cap S_2$.

Then we claim that $m @ n \prec S_1 \cap S_2$, meaning: $(\exists p \in m @ n)(p \in S_1 \cap S_2) (0 \in p @ p)$.

Now, since $m, n \in S_1$, $m, n \in S_2$ and S_1, S_2 are both weak hyper subalgebras of B , we have:

$$m @ n \subseteq S_1 \cap S_2 \text{ and then } \exists p \in m @ n \text{ and } p \in S_1 \text{ and } p \in S_2 \text{ such that } 0 \in p @ p$$

$$\Rightarrow (\exists p \in m @ n)(p \in S_1 \cap S_2) \text{ such that } 0 \in p @ p \Rightarrow m @ n \prec S_1.$$

Or alternatively, following $m @ n \subseteq S_1 \cap S_2$ (proved in (1) above) and by Proposition 2.1.4 (3),

we get $m @ n \prec S_1 \cap S_2$.

Therefore, in any ways, we get $S_1 \cap S_2$ is weak hyper subalgebra of B .

(3) Let S_1 and S_2 be any two strong hyper subalgebras of B . Then, by Lemma 2.2.1 (4),

$$S_1 = S_2 = \{0\} \text{ and hence } S_1 \cap S_2 = \{0\} \text{ which is also strong hyper subalgebra.}$$

□

Corollary 2.2.1. *The non-empty intersection, $\bigcap_{i \in I} S_i$, of any family of (weak, strong) hyper subalgebras $\{S_i : i \in I\}$ of hyper BCL–algebra B is (weak, strong) hyper subalgebra of the hyper BCL–algebra B , respectively.*

Proof. (1) Let $S_i, \forall i \in I$ be family of hyper subalgebras of the hyper BCL–algebra B such that

$$\bigcap_{i \in I} S_i \neq \emptyset. \text{ Suppose } m, n \in S_i, \forall i \in I, \text{ and we need to show that } m @ n \subseteq \bigcap_{i \in I} S_i.$$

Now, since $m, n \in \bigcap_{i \in I} S_i$ and $S_i, \forall i \in I$, are hyper subalgebras of B , we have: $m @ n \subseteq \bigcap_{i \in I} S_i$.

Therefore, $\bigcap_{i \in I} S_i$ is a hyper subalgebra of B .

(2) Let $S_i, \forall i \in I$, be weak hyper subalgebras of B and suppose $m, n \in \bigcap_{i \in I} S_i$.

Then we claim that $m @ n \prec \bigcap_{i \in I} S_i$, meaning: $(\exists p \in m @ n)(\exists q \in \bigcap_{i \in I} S_i) (0 \in p @ q)$.

Now, since $m, n \in S_i, \forall i \in I$, and $S_i, \forall i \in I$, are all weak hyper subalgebras of B , we have:

$$m @ n \subseteq \bigcap_{i \in I} S_i \text{ and then } \exists p \in m @ n \text{ and } q = p \in S_i, \forall i \in I, \text{ such that } 0 \in p @ q = p @ p$$

$$\Rightarrow (\exists p \in m @ n)(\exists q \in \bigcap_{i \in I} S_i) \text{ such that } 0 \in p @ q \Rightarrow m @ n \prec S_i, \forall i \in I.$$

Or alternatively, following $m @ n \subseteq \bigcap_{i \in I} S_i$ (proved in (1) above) and by Proposition 2.1.4 (3),

$$\text{we get } m @ n \prec \bigcap_{i \in I} S_i.$$

Therefore, in any ways, we get $\bigcap_{i \in I} S_i$ is weak hyper subalgebra of B .

(3) Let $S_i, \forall i \in I$, be strong hyper subalgebras of B . Then, by Lemma 2.2.1 (4),

$$S_i = \{0\}, \forall i \in I, \text{ and hence } \bigcap_{i \in I} S_i = \{0\} \text{ which is also strong hyper subalgebra.} \quad \square$$

Remark 2.2.2. *The union of any two (weak, strong) hyper subalgebras of a hyper BCL–algebra may not always be a (weak, strong) hyper subalgebra of the hyper algebra B , respectively. We give a counterexample for one of the (weak, strong) hyper subalgebras as follows, and the others can be explained in a similar way.*

Example 2.2.4. *Let $(B; @, 0)$ be hyper BCL–algebra where $B = \{0, p, q, r\}$ and the hyper operation $@$ be as defined in Table 2.2 of Example 2.2.2 above. Then as listed in Example 2.2.2 (iii), $\{0\}$ and $\{p\}$ are strong hyper subalgebras of B but the unions, $\{0\} \cup \{p\} = \{0, p\}$ is not strong hyper subalgebra of B because of the following justifications:*

Taking $0, p \in \{0, p\}$ but $p @ 0 = \{p, q\} \not\subseteq \{0, p\}$, since there does not exist $k \in \{p, q\}$ such that $0 \in k @ t, \forall t \in \{p, q\}$ so that the union of strong hyper subalgebra of B is not necessarily strong hyper subalgebra of B .

2.3. Hyper Deductive Systems of Hyper BCL–Algebra

In this section, we define and explore hyper deductive systems in the setting of hyper BCL–algebras, where operations may produce sets of results. We discuss various kinds of hyper deductive systems as weak hyper deductive system, strong hyper deductive system, and hyper deductive systems and give examples to show how they behave differently and how they relate to other hyper substructures. *Under this section we use DS for deductive system.*

Definition 2.3.1. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq D \subseteq B$. Then D is a hyper deductive system (DS) of B if and only if the following are satisfied, $\forall m, n, u, w \in B$:

- (i) $m \in D \Rightarrow (m @ u) @ u \subseteq D$,
- (ii) $m, n \in D \Rightarrow m @ (n @ w) \subseteq D$.

Definition 2.3.2. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq D \subseteq B$. Then D is weak hyper DS of B if and only if the following are satisfied, $\forall m, n, u, w \in B$:

- (i) $m \in D \Rightarrow (m @ u) @ u \prec D$,
- (ii) $m, n \in D \Rightarrow m @ (n @ w) \prec D$.

Definition 2.3.3. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq D \subseteq B$. Then D is strong hyper DS of B if and only if the following are satisfied, $\forall m, n, u, w \in B$:

- (i) $m \in D \Rightarrow (m @ u) @ u \preceq D$,
- (ii) $m, n \in D \Rightarrow m @ (n @ w) \preceq D$.

Example 2.3.1. Let $B = \{0, p, q, r\}$ and define a hyper operation @ on B by the table as follows:

@	0	p	q	r
0	{0}	{0, q}	{0}	{0, r}
p	{p}	{0, p}	{p, r}	{p, r}
q	{q}	{q, r}	{0}	{q, r}
r	{r}	{0}	{0, p}	{0, r}

Table 2.3: A table describing a hyper BCL–algebra, $B = (B; @, 0)$:

(1) From Table 2.3, above, $B = (B; @, 0)$ is a hyper BCL–algebra. Furthermore:

- (i) B is the only hyper DS of B ,
- (ii) $\{0\}, \{0, p\}, \{0, q\}, \{0, r\}, \{0, p, q\}, \{0, p, r\}, \{0, q, r\}, B$ are weak hyper DSs of B ,
- (iii) $\{0\}, \{p\}, \{q\}, \{r\}$ are strong hyper DSs of B .

(2) From Table 2.1 of Example 2.1.2 above, we have the following:

- (i) B is the only hyper DS of B ,
- (ii) All subsets of B containing 0 are weak hyper DSs of B ,
- (iii) B is strong hyper DS of B .

Proposition 2.3.1. Let $B = (B; @; 0)$ be any hyper BCL–algebra. Then the following hold:

- (1) B is the only hyper DS of B .
- (2) If D is weak hyper DS of B , then $0 \in D$.
- (3) $\{0\}$ is weak hyper DS of B ,
- (4) B is weak hyper DS of B .

Proof. Suppose B is a hyper BCL–algebra:

(1) Suppose D is hyper DS of B . Then by its two axioms:

$m \in D \Rightarrow (m@u)@u \in D$ and $m, n \in D \Rightarrow m@(n@w) \in D$, for arbitrary u and $w \in B$ which could yield non-subset of D but as B is hyper BCL–algebra, the results are contained in B .

Then B is the only hyper DS of B .

(2) Suppose D is weak hyper DS of B

Then, since $D \neq \emptyset$, $\exists m \in D$ such that $(m @ m) @ m \prec D$, $m @ (m @ m) \prec D$,

$\Rightarrow 0 @ m \prec (m @ m) @ m \prec D$, $m @ 0 \preceq m @ (m @ m) \preceq D$,

$\Rightarrow 0 \in 0 @ m \prec D$, $0 \in m @ 0 \preceq D$, (and then by Definition 2.1.1 (2), we get the following:)

$\Rightarrow m = 0 \in D$, for any weak hyper DS D of B .

(3) $0 \in \{0\} \Rightarrow 0 \in (0 @ u) @ u$ and $0 \in 0 @ (0 @ w)$, $\forall u, w \in B$

$\Rightarrow 0 \in (0 @ u) @ u \prec \{0\}$ and $0 \in 0 @ (0 @ w) \prec \{0\}$, $\forall u, w \in B$

Therefore, $\{0\}$ is weak hyper DS of B

(4) Since $\exists 0 \in B$ such that $(0 @ u) @ u \prec B$ and $0 @ (0 @ w) \prec B$, $\forall u, w \in B$.

Therefore, B is weak hyper DS of B.

But, B may not be strong hyper DS of B could be explained as from Example 2.3.1:

$$r \in B \Rightarrow (r@0)@0 = \{r\}@0 = \{r\} \not\subseteq B \text{ as } \{r\} \not\subseteq r \text{ (or } 0 \notin r@r = r)$$

□

Proposition 2.3.2. *Let $(B; @, 0)$ be a hyper BCL–algebra. Then, every hyper DS of B is weak hyper DS of B.*

Proof. Let $(B; @, 0)$ be a weak hyper BCL–algebra and $\emptyset \neq D \subseteq B$ such that $0 \in D$.

Suppose D is hyper DS of B: Then:

$$m \in D \Rightarrow (m @ u) @ u \subseteq D \text{ and } m, n \in D \Rightarrow m @ (n @ w) \subseteq D, \forall u, w \in B.$$

Then by Proposition 2.1.4 (3), $m \in D \Rightarrow (m@u)@u \prec D$; $m, n \in D \Rightarrow m@(n@w) \prec D, \forall u, w \in B$.

Hence D is weak hyper DS of B.

□

Remark 2.3.1. *As shown in Example 2.3.2 below, the following are explained:*

(1) *The converses of the statement in Proposition 2.3.2 may not be necessary true.*

(2) *Strong hyper DS of B cannot be related to (weak) hyper DS of B.*

Example 2.3.2. *Let $(B; @, 0)$ hyper BCL–algebra where $B = \{0, p, q, r\}$ and the hyper operation @ is as defined in Table 2.3 above. then Remark 2.3.1 could be explained as follows:*

(1) *As described in the Example, $D = \{0, p\}$ is weak hyper DS of B.*

However, taking $0 \in D$ and $r \in B$ we have the following:

$$(0@r)@r = \{0, r\} @ r = (0@r) \cup (r@r) = \{0, r\} \cup \{0, r\} = \{0, r\} \not\subseteq D = \{0, p\}$$

Therefore, $D = \{0, p\}$ is weak hyper DS of B but not hyper DS of B.

(2) *To justify, "Strong hyper DS of B cannot be related to (weak) hyper DS of B", we give counter examples to assure the following four cases:*

(i) *D is strong hyper DS of B $\not\Rightarrow$ D is weak hyper DS of B,*

(ii) *D is strong hyper DS of B $\not\Rightarrow$ D is hyper DS of B,*

(iii) *D is weak hyper DS of B $\not\Rightarrow$ D is strong hyper DS of B,*

(iv) *D is hyper DS of B $\not\Rightarrow$ D is strong hyper DS of B,*

(i) *Let $D = \{p\}$, then as describe in Example 2.3.1, above, D is strong hyper DS of B. But taking*

$p \in D$ and $0, q \in B$, then, $(p@0)@q = \{p\}@q = \{q, r\} \not\subseteq D = \{p\}$ and hence D is not weak hyper DS of B.

- (ii) Let $D = \{0\}$, then as describe in Example 2.3.1, above, D is strong hyper DS of B . But taking $0 \in D$ and $0, p \in B$, then, $(0@0)@p = \{0\}@p = \{0, q\} \not\subseteq D$ and hence D is not hyper DS of B .
- (iii) Let $D = \{0, p\}$, then as describe in Example 2.3.1, above, D is weak hyper DS of B . But taking $p \in D$ and $0, q \in B$, then, $(p@0)@p = \{p\}@q = \{q, r\} \not\subseteq D$ and hence D is not strong hyper DS of B .
- (iv) Let $D = B$, then as describe in Example 2.3.1, above, D is hyper DS of B . But taking $q \in D$ and $0, r \in B$, then, $(q@0)@r = \{q\}@q = \{q, r\} \not\subseteq B$, since $r \in \{q, r\}$ and $0 \in B$ but $q \not\subseteq 0$. and therefore B is not strong hyper DS of B .

Theorem 2.3.1. *The non-empty intersection, $D_1 \cap D_2$, of any two (weak, strong) hyper DSs, D_1 and D_2 , of a hyper BCL–algebra $(B; @, 0)$ is also (weak, strong) hyper DS of B , respectively.*

Proof. (1) Let D_1 and D_2 be any two hyper DSs of B such that $D_1 \cap D_2 \neq \emptyset$. Then:

(i) Suppose $m \in D_1 \cap D_2$. We need to show that $(m @ u) @ u \subseteq D_1 \cap D_2, \forall u \in B$.

Now, since $m \in D_1 \cap D_2$, we have $m \in D_1$ and $m \in D_2$

$\Rightarrow (m @ u) @ u \subseteq D_1$ and $(m @ u) @ u \subseteq D_2, \forall u \in B$, as D_1 and D_2 are hyper DSs

$\Rightarrow (m @ u) @ u \subseteq D_1 \cap D_2, \forall u \in B$.

(ii) Suppose $m, n \in D_1 \cap D_2$. We need to show that $m @ (n @ w) \subseteq D_1 \cap D_2, \forall w \in B$

Since $m, n \in D_1 \cap D_2$, we have $m, n \in D_1$ and $m, n \in D_2$

$\Rightarrow m @ (n @ w) \subseteq D_1$ and $(m @ w) @ w \subseteq D_2, \forall w \in B$, as D_1 and D_2 are hyper DSs of B

$\Rightarrow m @ (n @ w) \subseteq D_1 \cap D_2, \forall w \in B$

Hence, by (i) and (ii) above, $D_1 \cap D_2$ is a hyper DS of B .

(2) Let D_1 and D_2 be any two weak hyper DSs of B .

(i) Suppose $m \in D_1 \cap D_2$. We need to show that $(m @ u) @ u \prec D_1 \cap D_2, \forall u \in B$

But from (1) above, we have: $(m @ u) @ u \subseteq D_1 \cap D_2, \forall u \in B$

Then by Proposition 2.1.4 (3), $(m @ u) @ u \prec D_1 \cap D_2, \forall u \in B$.

(ii) Similarly, $m, n \in D_1$ and $m, n \in D_2 \Rightarrow m @ (n @ w) \prec D_1 \cap D_2, \forall w \in B$.

Hence $D_1 \cap D_2$ is a weak hyper DS of B .

(3) Let D_1 and D_2 be any two strong hyper DSs of B . Then:

(i) Suppose $m \in D_1 \cap D_2$. We need to show that $(m @ u) @ u \preceq D_1 \cap D_2, \forall u \in B$.

Since $m \in D_1 \cap D_2$, we have $m \in D_1$ and $m \in D_2$

$\Rightarrow (m @ u) @ u \preceq D_1$ and $(m @ u) @ u \preceq D_2, \forall u \in B$, as D_1 and D_2 are strong hyper DSs of B

$\Rightarrow (\exists p \in ((m @ u) @ u)) (\forall q \in (D_1 \cap D_2)) (p \preceq q), \forall u \in B$, for $p = 0$ and $m = u$

$\Rightarrow (m @ u) @ u \preceq D_1 \cap D_2, \forall u \in B$.

(ii) Similarly, $m @ (n @ w) \preceq D_1 \cap D_2$, for $n = m$ and $w = 0$.

Hence, by (i) and (ii) above, $D_1 \cap D_2$ is a strong hyper DS of B . □

Corollary 2.3.1. *The non-empty intersection, $\bigcap_{i \in I} D_i$, of any family of (weak, strong) hyper DS $\{D_i : i \in I\}$ of BCL–algebra, B is (weak, strong) hyper deductive system of the hyper BCL–algebra B , respectively.*

Proof. (1) Let $\{D_i : i \in I\}$ be family of hyper DSs of B such that $\bigcap_{i \in I} D_i \neq \emptyset$. Then:

(i) Suppose $m \in \bigcap_{i \in I} D_i$. We need to show that $(m @ u) @ u \subseteq \bigcap_{i \in I} D_i, \forall u \in B$.

Now, since $m \in \bigcap_{i \in I} D_i$, we have $m \in D_i, \forall i \in I$

$\Rightarrow (m @ u) @ u \subseteq D_i, \forall i \in I, \forall u \in B$, as $D_i, \forall i \in I$, are hyper DSs

$\Rightarrow (m @ u) @ u \subseteq \bigcap_{i \in I} D_i, \forall u \in B$.

(ii) Suppose $m, n \in \bigcap_{i \in I} D_i$. We need to show that $m @ (n @ w) \subseteq \bigcap_{i \in I} D_i, \forall w \in B$

Since $m, n \in \bigcap_{i \in I} D_i$, we have $m, n \in D_i, \forall i \in I$.

$\Rightarrow m @ (n @ w) \subseteq D_i, \forall i \in I, \forall w \in B$, as $D_i, \forall i \in I$, are hyper DSs of B

$\Rightarrow m @ (n @ w) \subseteq \bigcap_{i \in I} D_i, \forall w \in B$

Hence, by (i) and (ii) above, $\bigcap_{i \in I} D_i$, is a hyper DS of B .

(2) Let $D_i, \forall i \in I$, be weak hyper DSs of B .

(i) Suppose $m \in \bigcap_{i \in I} D_i$. We need to show that $(m @ u) @ u \prec \bigcap_{i \in I} D_i, \forall u \in B$

But from (1) above, we have: $(m @ u) @ u \subseteq \bigcap_{i \in I} D_i, \forall u \in B$

Then by Proposition 2.1.4 (3), $(m @ u) @ u \prec \bigcap_{i \in I} D_i, \forall u \in B$.

(ii) Similarly, $m, n \in D_i, \forall i \in I \Rightarrow m @ (n @ w) \prec \bigcap_{i \in I} D_i, \forall w \in B$.

Hence $\bigcap_{i \in I} D_i$ is a weak hyper DS of B.

(3) Let $D_i, \forall i \in I$, be strong hyper DSs of B. Then:

(i) Suppose $m \in \bigcap_{i \in I} D_i$. We need to show that $(m @ u) @ u \preceq \bigcap_{i \in I} D_i, \forall u \in B$.

Since $m \in \bigcap_{i \in I} D_i$, we have $m \in D_i, \forall i \in I$

$\Rightarrow (m @ u) @ u \subseteq D_i, \forall i \in I, \forall u \in B$, as $D_i, \forall i \in I$, are strong hyper DSs of B

$\Rightarrow (\exists p \in (m @ u) @ u) (\forall q \in \bigcap_{i \in I} D_i) (p \preceq q), \forall u \in B$, for $p = 0$ when $m = u$

$\Rightarrow (m @ u) @ u \preceq \bigcap_{i \in I} D_i, \forall u \in B$.

(ii) Similarly, $m @ (n @ w) \preceq \bigcap_{i \in I} D_i$, for $n = m$ and $w = 0$.

Hence, by (i) and (ii) above, $\bigcap_{i \in I} D_i$ is a strong hyper DS of B. □

Remark 2.3.2. *The union of any two (weak, strong) hyper DSs of a hyper BCL–algebra may not always be a (weak, strong) hyper DS of the hyper algebra B, respectively. We give a counterexample for one of the (weak or strong) hyper DSs below, and the others can be explained in a similar way.*

Example 2.3.3. *Let $(B; @, 0)$ be hyper BCL–algebra where $B = \{0, p, q, r\}$ and the hyper operation @ be as defined in Table 2.3 of Example 2.3.1 above.*

As listed in Example 2.3.1 (1) (iii), $\{p\}$ and $\{q\}$ are strong hyper DSs of B but $\{p\} \cup \{q\} = \{p, q\}$ is not strong hyper DS of B because of the following justifications:

Taking $p, q \in \{p\} \cup \{q\} = \{p, q\}$ and $q \in B$, we have:

$p @ (q @ q) = p @ \{0\} = \{p\} \not\subseteq \{p, q\}$ as $p \not\subseteq q$ (or $0 \notin p @ q$).

2.4. Hyper Ideals of Hyper BCL–Algebra

This section introduces hyper ideals in hyper BCL–algebra by defining weak hyper ideal, strong hyper ideal and hyper ideal and investigate their relationships. The section includes theorems and proofs that explain how (weak, strong) hyper ideals behave, especially corresponding intersections and limitations such as when a strong hyper ideal can only contain the zero element.

Definition 2.4.1. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq I \subseteq B$. Then I is called hyper ideal of B if and only if the following are satisfied, $\forall m, n, w \in B$:

- (i) $n \in I, m @ n \subseteq I \Rightarrow m \in I$;
- (ii) $m, n \in I \Rightarrow (m @ (n @ w)) @ w \subseteq I$.

Definition 2.4.2. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq I \subseteq B$. Then I is weak hyper ideal of B if and only if the following are satisfied, $\forall m, n, w \in B$:

- (i) $0 \in I$;
- (ii) $n \in I, m @ n \prec I \Rightarrow m \in I$;
- (iii) $m, n \in I \Rightarrow (m @ (n @ w)) @ w \prec I$.

Definition 2.4.3. Let $B = (B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq I \subseteq B$. Then I is strong hyper ideal of B if and only if the following are satisfied, $\forall m, n, w \in B$:

- (i) $0 \in I$;
- (ii) $n \in I, m @ n \preceq I \Rightarrow m \in I$;
- (iii) $m, n \in I \Rightarrow (m @ (n @ w)) @ w \preceq I$.

Example 2.4.1. Let $B_1 = \{0, p, q\}$ and define a hyper operation $@$ on B_1 by the table as follows:

@	0	p	q
0	{0}	{0}	{0}
p	{p}	{0}	{0, p}
q	{q}	{p, q}	{0}

Table 2.4: A table describing a hyper BCL–algebra, $(B_1; @, 0)$:

From Table 2.4 of Example 2.4.1 above, $(B_1; @, 0)$ is a hyper BCL–algebra. Furthermore:

- (i) $\{0\}$, $\{0, p\}$ and B_1 are hyper ideals of B_1 ,
- (ii) $\{0\}$ and B_1 are weak hyper ideals of B_1 ,
- (iii) $\{0\}$ is strong hyper ideal of B_1 .

Example 2.4.2. Let $(B_2; \otimes, 0)$ be a BCL–algebra where $B_2 = \{0, p, q, r\}$, \otimes is a binary operation on B_2 and define a hyper operation $@$ on B_2 by $m @ n = \{m \otimes n\}$, $\forall m, n \in B_2$, then $(B_2; @, 0)$ is a hyper BCL–algebra if \otimes is as defined in Table 1.1 of Example 1.1.1 as explained in Example 2.1.1. Then:

- (1) $\{0\}$, $\{0, r\}$, B_2 are hyper ideals of B_2 ,
- (2) $\{0\}$, $\{0, r\}$, B_2 are weak hyper ideals of B_2 ,
- (3) $\{0\}$ is strong hyper ideals of B_2 .

Lemma 2.4.1. Let $(B; @, 0)$ be a hyper BCL–algebra. If I is hyper ideal of B , then $0 \in I$.

Proof. Let $(B; @, 0)$ be a hyper BCL–algebra and suppose I is hyper ideal of B .

Then by the second axiom of definition of hyper ideal, we have:

$$m, n \in I \Rightarrow (m @ (n @ w)) @ w \subseteq I, \forall w \in B$$

As I is non-empty, take $m = n = w \Rightarrow (m @ (n @ w)) @ w$ and $m \in I \Rightarrow (m @ (m @ m)) @ m \subseteq I$

But $(m @ (m @ m)) @ m \subseteq I \Rightarrow (m @ 0) @ m \subseteq (m @ m) @ m \subseteq I$, since $0 \in m @ m$, (by Definition 2.1.1 (1)) $\Rightarrow m @ m \subseteq I \Rightarrow 0 \in m @ m \subseteq I \Rightarrow 0 \in I$

□

Proposition 2.4.1. For any hyper BCL–algebra, B ; the following hold:

- (1) $\{0\}$ is (weak, strong) hyper ideal of B ,
- (2) B is (weak) hyper ideal of B .

Proof. Suppose B is a hyper BCL–algebra and:

- (1) Suppose I is hyper ideal of B .

Then, since $0 \in I$ such that $(0 @ 0) @ 0 \subseteq I$, $(0 @ 0) @ 0 \preceq I$, $(0 @ 0) @ 0 \prec I$,

$$\Rightarrow 0 @ 0 \subseteq I, 0 @ 0 \preceq I, 0 @ 0 \prec I \Rightarrow \{0\} \subseteq I, \{0\} \preceq I, \text{ and } \{0\} \prec I,$$

Thus $\{0\}$ is (weak, strong) hyper ideal of B

- (2) Let $(B; @, 0)$ be hyper BCL–algebra and $I = B$, then to prove that I is (weak hyper ideal of B , we proceed to prove the same/similar axioms side by side:
- (i) Since B is hyper BCL–algebra, $0 \in B = I$ so that the first axioms for weak hyper ideals of B are satisfied,
 - (ii) $\forall m, n \in B$, let $n \in I (= B)$ and $(m@n \subseteq I, m@n \prec I) \Rightarrow m \in I = B$
(by definitions of (weak) hyper ideals of B), and therefore, the first axiom of hyper ideals and the second axioms for weak ideals of B are satisfied,
 - (iii) $\forall m, n, w \in B$, let $m, n \in I (= B)$ then we need to show:
 - (a) $(m@(n@w))@w \subseteq I$, (for hyper ideal)
 - (b) $(m@(n@w))@w \prec I$, (for weak hyper ideal)

Now, proving (a) guarantees that (b) is proved by Proposition 2.1.4 (3) or (9), and hence we only prove (a) as follows:

- (a) By definition of the hyper operation, $@ : B \times B \rightarrow \mathfrak{B} = P(B) - \emptyset$ (Definition 2.1.1), we have, $\forall m, n, w \in B$, $(m@(n@w))@w \subseteq B = I$,

Therefore, $I = B$ is (weak) hyper ideal of B .

□

Proposition 2.4.2. *Let $(B; @, 0)$ be a hyper BCL–algebra. Then:*

- (1) *Every hyper ideal of B is weak hyper ideal of B ,*
- (2) *Every strong hyper ideal of B is (weak) hyper ideal of B .*

Proof. Let $(B; @, 0)$ be a hyper BCL–algebra and $\emptyset \neq I \subseteq B$ such that $0 \in I$.

- (1) Suppose I is hyper ideal of B . Then:

$$m \in I \Rightarrow (m @ u) @ u \subseteq I \text{ and } m, n \in I \Rightarrow m @ (n @ w) \subseteq I, \forall u, w \in B$$

$$\text{By Proposition 2.1.4 (3), } (m @ u) @ u \prec I \text{ and } m, n \in I \Rightarrow m @ (n @ w) \prec I, \forall u, w \in B.$$

Hence I is weak hyper ideal of B .

- (2) Suppose I is strong hyper ideal of B :

$$(m \in I \Rightarrow (m @ u) @ u \preceq I \text{ and } m, n \in I \Rightarrow (m @ (n @ w)) @ w \preceq I, \forall u, w \in B)$$

$$\text{Then by Proposition 2.1.4 (9), } (m @ u) @ u \prec I \text{ and } \Rightarrow (m @ (n @ u)) @ u \prec I, \forall u, w \in B.$$

Hence, I is weak hyper ideal of B .

In addition, by Proposition 2.1.3 (ii) or by Remark 2.1.2, I is hyper ideal of B .

Note: Generally, if I_s , I_h and I_w denote the set of all strong hyper ideals, the set of all hyper ideals and the set of all weak hyper ideals of a hyper BCL–algebra, respectively then, $I_s \subseteq I_h \subseteq I_w$.

Remark 2.4.1. *The converses of each statement in Proposition 2.4.2 may not be necessary true as explicitly seen in the following Example.*

Example 2.4.3. *From Examples 2.4.1 and 2.4.2 $\{0, p\}$ and $\{0, q\}$ are hyper ideals but not strong hyper ideals of the hyper BCL–algebra B_1 and B_2 , respectively.*

Moreover, from Table 2.3 in Example 2.3.1, $\{0, r\}$ is weak hyper ideal but not hyper ideal of the corresponding hyper BCL–algebra $B = \{0, p, q, r\}$ because of the following justifications that:

$\{0, r\}$ is weak hyper ideal of B but not hyper ideal of; $r@0 = \{r\} \prec \{0, r\}$ since:

$r@0 = \{r\} \prec \{0, r\}$; $r@r = \{0, r\} \prec \{0, r\}$; $0@r = \{0, r\} \prec \{0, r\}$;

$0@0 = \{0\} \prec \{0, r\}$ but $r@0 = \{r\} \preceq \{0, r\}$ as $r \not\leq 0$ or $0 \notin r@0$.

Remark 2.4.2. *Let $(B; @, 0)$ be a hyper BCL–algebra and $S \subseteq B$. The following statements are justified by the example provided below:*

- (1) *S being a (weak) hyper subalgebra of B does not necessarily imply that S is (weak) hyper deductive system of B , respectively, and vice versa.*
- (2) *S being a (weak) hyper subalgebra of B does not necessarily imply that S is a (weak) hyper ideal of B , respectively, and vice versa..*
- (3) *S being a (weak) hyper DS of B does not necessarily imply that S is a (weak) hyper ideal of B , respectively, and vice versa..*
- (4) *Strong hyper ideal and strong hyper Ds could have elements other than 0 unlike strong hyper subalgebra, respectively.*

Example 2.4.4. *Let us justify some of these properties by the next example based on Table 2.4 of Example 2.4.1, above, and the others could be simply explained in similar ways:*

- (1) (i) *$\{0\}$ is (weak) hyper subalgebra of B but not (weak) hyper DS of B , respectively,*
(ii) *$\{0, p\}$ is (weak) hyper DS but not (weak) hyper subalgebra of B , respectively.*
- (2) *$\{0, p\}$ is (weak) hyper ideal but not (weak) hyper subalgebra of B , respectively.*
- (3) *$\{0\}$ is (weak) hyper ideal of B but not (weak) hyper DS of B , respectively.*
- (4) *$\{0, p, q\}$ is a strong hyper ideal with an element $p \neq 0$ or $q \neq 0$.*

Theorem 2.4.1. *The non-empty intersection, $I_1 \cap I_2$, of any two (weak, strong) hyper ideals I_1 and I_2 of a hyper BCL–algebra $(B; @, 0)$ is also (weak, strong) hyper ideal of B , respectively.*

Proof. (1) Suppose I_1 and I_2 are any two hyper ideals of B :

(i) Suppose $n \in I_1 \cap I_2$, $m @ n \subseteq I_1 \cap I_2$. Then We need to show that $m \in I_1 \cap I_2$

Since $n \in I_1 \cap I_2$, $m @ n \subseteq I_1 \cap I_2$, then, $n \in I_1$, $n \in I_2$, $m @ n \subseteq I_1$, $m @ n \subseteq I_2$

$\Rightarrow m \in I_1$ and $m \in I_2 \Rightarrow m \in I_1 \cap I_2$

(ii) Suppose $m, n \in I_1 \cap I_2$. We need to show that $(m @ (n @ w)) @ w \subseteq I_1 \cap I_2$

Now, since $m, n \in I_1 \cap I_2$, we have $m, n \in I_1$ and $m, n \in I_2$

$\Rightarrow (m @ (n @ w)) @ w \subseteq I_1$ and $(m @ (n @ w)) @ w \subseteq I_2 \Rightarrow (m @ (n @ w)) @ w \subseteq I_1 \cap I_2$

Hence $I_1 \cap I_2$ is a hyper ideal of B (by (i) – (ii) above).

(2) Suppose I_1 and I_2 are any two strong hyper ideals of B :

(i) As I_1 and I_2 are strong hyper ideals of B , we have: $0 \in I_1$ and $0 \in I_2 \Rightarrow 0 \in I_1 \cap I_2$.

(ii) Suppose $n \in I_1 \cap I_2$, $m @ n \preceq I_1 \cap I_2$. We need to show that $m \in I_1 \cap I_2$.

As $n \in I_1 \cap I_2$, $m @ n \preceq I_1 \cap I_2$, we have: $n \in I_1$, $n \in I_2$, $m @ n \preceq I_1$ and $m @ n \preceq I_2$.

$\Rightarrow m \in I_1$ and $m \in I_2 \Rightarrow m \in I_1 \cap I_2$

(iii) Suppose $m, n \in I_1 \cap I_2$. We need to show that $(m @ (n @ w)) @ w \preceq I_1 \cap I_2$.

Now, since $m, n \in I_1 \cap I_2$, we have $m, n \in I_1$ and $m, n \in I_2$.

$\Rightarrow (m @ (n @ w)) @ w \preceq I_1$, $(m @ (n @ w)) @ w \preceq I_2 \Rightarrow (m @ (n @ w)) @ w \preceq I_1 \cap I_2$.

Hence $I_1 \cap I_2$ is a strong hyper ideal of B (by (i) – (iii) above).

(3) Let I_1 and I_2 be any two weak hyper ideals of B .

Then since $I_1 \cap I_2$ is (strong) hyper ideal of B as proved above in (1) and (2), then following this

and by Proposition 2.1.4 (3) or by Proposition 2.4.2 (1), $I_1 \cap I_2$ is weak hyper ideal of B . □

Corollary 2.4.1. *The intersection, $\bigcap_{i \in J} I_i$, of any family of (weak, strong) hyper ideal $\{I_i : i \in J\}$ of hyper BCL–algebra, B is also (weak, strong) hyper ideal of B , respectively.*

Proof. (1) Let $I_i, \forall i \in J$ be family of hyper ideals of B :

(i) Suppose $n \in \bigcap_{i \in J} I_i$, $m @ n \subseteq \bigcap_{i \in J} I_i$. Then We need to show that $m \in \bigcap_{i \in J} I_i$

Since $n \in \bigcap_{i \in J} I_i$, $m @ n \subseteq \bigcap_{i \in J} I_i$, then, $n \in I_i, \forall i \in J$, $m @ n \subseteq I_i, \forall i \in J$

$\Rightarrow m \in I_i, \forall i \in J \Rightarrow m \in \bigcap_{i \in J} I_i$

(ii) Suppose $m, n \in \bigcap_{i \in J} I_i$. We need to show that $(m @ (n @ w)) @ w \subseteq \bigcap_{i \in J} I_i$

Now, since $m, n \in \bigcap_{i \in J} I_i$, we have $m, n \in I_i, \forall i \in J$

$\Rightarrow (m @ (n @ w)) @ w \subseteq I_i, \forall i \in J \Rightarrow (m @ (n @ w)) @ w \subseteq \bigcap_{i \in J} I_i$

Hence $\bigcap_{i \in J} I_i$ is a hyper ideal of B (by (i) – (iii) above).

(2) Suppose $I_i, \forall i \in J$ are strong hyper ideals of B:

(i) As $I_i, \forall i \in J$ are strong hyper ideals of B, we have: $0 \in I_i, \forall i \in J \Rightarrow 0 \in \bigcap_{i \in J} I_i$.

(ii) Suppose $n \in \bigcap_{i \in J} I_i$, $m @ n \preceq \bigcap_{i \in J} I_i$. We need to show that $m \in \bigcap_{i \in J} I_i$.

As $n \in I_i, \forall i \in J$, $m @ n \preceq \bigcap_{i \in J} I_i$, we have: $n \in I_i, \forall i \in J$, $m @ n \preceq I_i \forall i \in J$.

$\Rightarrow m \in I_i, \forall i \in J$ and $m \in I_2 \Rightarrow m \in \bigcap_{i \in J} I_i$

(iii) Suppose $m, n, w \in \bigcap_{i \in J} I_i$. We need to show that $(m @ (n @ w)) @ w \preceq \bigcap_{i \in J} I_i$.

Now, since $m, n, w \in \bigcap_{i \in J} I_i$, we have $m, n, w \in I_i, \forall i \in J$ and $m, n, w \in I_2$.

$\Rightarrow (m @ (n @ w)) @ w \preceq I_i, \forall i \in J \Rightarrow (m @ (n @ w)) @ w \preceq \bigcap_{i \in J} I_i$.

Hence $\bigcap_{i \in J} I_i$ is strong hyper ideal of B (by (i) – (iii) above).

(3) Let $I_i, \forall i \in J$ be weak hyper ideals of B.

Then since $\bigcap_{i \in J} I_i$ is (strong) hyper ideal of B as proved above in (1) and (2), then following this

and by Proposition 2.1.4 (3) or by Proposition 2.4.2 (1), $\bigcap_{i \in J} I_i$ is weak hyper ideal of B. \square

Chapter 3

Fuzzy Hyper Structures of BCL–Algebra

3.1. Fuzzy Hyper Subalgebras of Hyper BCL–Algebra

In this section, we introduce the idea of a fuzzy hyper subalgebra in a hyper BCL–algebra. A fuzzy hyper subalgebra uses fuzzy subsets to describe how strongly elements belong to a subalgebra, even when operations give multiple outputs. We define fuzzy weak hyper subalgebra, fuzzy hyper subalgebra, and fuzzy strong hyper subalgebra, and provide examples and theorems that explain their structure.

Definition 3.1.1. *Let $(B; @, 0)$ be a hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called a fuzzy hyper subalgebra of B if, $\forall m, n \in B$:*

- (i) $\eta(0) \geq \eta(m)$
- (ii) $\inf_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}$.

Definition 3.1.2. *Let $(B; @, 0)$ be a hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called a fuzzy weak hyper subalgebra of B if, $\forall m, n \in B$:*

- (i) $\eta(0) \geq \eta(m)$
- (ii) $\sup_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}$.

Definition 3.1.3. *Let $(B; @, 0)$ be a hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called a fuzzy strong hyper subalgebra of B if, $\forall m, n \in B$:*

- (i) $\eta(0) \geq \eta(m)$
- (ii) $\inf_{u \in m @ n} \eta(u) \geq \max\{\eta(m), \eta(n)\}$.

Example 3.1.1. *Let $B = \{0, p, q, r\}$ be a set and $(B; @, 0)$ be a hyper BCL–algebra as defined in Table 2.3 of Example 2.3.1 above. Define fuzzy subsets $\eta_1, \eta_2, \eta_3 : B \rightarrow [0, 1]$ by:*

$$\eta_1(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.8, & \text{if } m = p, \\ 0.6, & \text{if } m = q, r, \end{cases} \quad \eta_2(m) = \begin{cases} 0.9, & \text{if } m = 0, \\ 0.8, & \text{if } m = p, \\ 0.4, & \text{if } m = q, \\ 0.2, & \text{if } m = r, \end{cases} \quad \eta_3(m) = 1, \text{ if } m = 0, p, q, r.$$

Then η_1 is fuzzy (weak) hyper subalgebra but not fuzzy strong hyper subalgebra of B ; η_3 is fuzzy (weak, strong) hyper subalgebra of B and η_2 is fuzzy weak hyper subalgebra but η_2 is neither fuzzy hyper subalgebra nor fuzzy strong hyper subalgebra of B by the following explained reason:

$$\begin{aligned} \inf_{u \in p @ q} \eta_2(u) &= \inf_{u \in \{p, r\}} \eta_2(u) = \min\{\eta_2(p), \eta_2(r)\} = \min\{0.8, 0.2\} = 0.2 \\ &\geq \min\{\eta_2(p), \eta_2(q)\} = \min\{0.8, 0.4\} = 0.4 \text{ is not true} \end{aligned}$$

This assures that it is not fuzzy hyper subalgebra of B .

From the preceding illustration, we have the following for fuzzy strong hyper subalgebra of B :

$$\begin{aligned} \inf_{u \in p @ q} \eta_2(u) &= \inf_{u \in \{p, r\}} \eta_2(u) = \min\{\eta_2(p), \eta_2(r)\} = \min\{0.8, 0.2\} = 0.2 \\ &\geq \max\{\eta_2(p), \eta_2(q)\} = \max\{0.8, 0.4\} = 0.8 \text{ is not true.} \end{aligned}$$

This also assures that η_2 is not fuzzy strong hyper subalgebra of B .

Lemma 3.1.1. Let $\eta : B \rightarrow [0, 1]$ be a fuzzy (weak, strong) hyper subalgebra of a hyper BCL-algebra B . If η is onto then $\eta(0) = 1$.

Proof. If η is onto, $\exists n \in B$ such that $\eta(n) = 1$ and since $\eta(0) \geq \eta(n)$, $\forall n \in B$, then, $\eta(0) = 1$.

Or if η is onto, $\eta(0)$ must attain the maximum value 1 of $[0, 1]$ since $\eta(0) \geq \eta(m)$, $\forall m \in B$. □

Theorem 3.1.1. Let $\eta : B \rightarrow [0, 1]$ be a fuzzy (weak) hyper subalgebra of a hyper BCL-algebra B , then the set $B_\eta = \{m \in B : \eta(m) = \eta(0)\}$ is a (weak) hyper subalgebra of B , respectively.

Proof. Let $B_\eta = \{m \in B : \eta(m) = \eta(0)\}$, where η is a fuzzy (weak,) hyper subalgebra of B .

We need to show that B_η satisfies the conditions of a (weak) hyper subalgebra.

By the first axiom of any one of Definition 3.1.1, 3.1.2, 3.1.3, we have:

$\eta(0) \geq \eta(m)$, $\forall m \in B$. Thus, $\eta(0)$ is the maximum membership value.

Then, $0 \in B_\eta$ since $\eta(0) = \eta(0)$, $0 \in B$ and therefore, $B_\eta \neq \emptyset$.

(1) Let $m, n \in B_\eta$. Then, by definition of B_η (hypothesis), $\eta(m) = \eta(n) = \eta(0)$.

Since η is a fuzzy hyper subalgebra, we have:

$\inf_{u \in m@n} \eta(u) \geq \min \{\eta(m), \eta(n)\}, \forall m, n \in B$, by Definition 3.1.1 (ii)

$\Rightarrow \inf_{u \in m@n} \eta(u) \geq \min \{\eta(0), \eta(0)\} = \eta(0)$. This means that $\forall u \in m@n, \eta(u) \geq \eta(0)$.

But since every $u \in m@n$ satisfies $\eta(u) \geq \eta(0)$ and $\eta(0)$ is the maximum possible value in B_η ,

then it follows that $\eta(u) = \eta(0), \forall u \in m@n$. Thus, $m@n \subseteq B_\eta$.

Therefore, B_η is a hyper subalgebra of B .

(2) Let η be a *fuzzy weak hyper subalgebra*. By Definition 3.1.2, $\forall m, n \in B$, the following hold:

(i) $\eta(0) \geq \eta(m)$,

(ii) $\sup_{u \in m@n} \eta(u) \geq \min \{\eta(m), \eta(n)\}$.

The fact that $B_\eta \neq \emptyset$ is the same as prove in (1) above so that $B_\eta \neq \emptyset$

Now, let $m, n \in B_\eta$ then $\eta(m) = \eta(n) = \eta(0) \Rightarrow \sup_{u \in m@n} \eta(u) \geq \min\{\eta(m), \eta(n)\} = \eta(0)$.

Since $\eta(0)$ is the maximum value in B_η , it follows that $\forall u \in m@n, \eta(u) \geq \eta(0)$, and thus $\eta(u) = \eta(0)$. Hence $m@n \subseteq B_\eta$ and then $m@n \prec B_\eta$, (by Proposition 2.1.4 (3)).

Therefore, B_η is a weak hyper subalgebra of B . □

Remark 3.1.1. As proved in Lemma 2.2.1 (1), S_s is any strong hyper subalgebra of a hyper hyper BCL–algebra B , implies $S_s = \{0\}$, guides us to $B = \{0\}$ is strong hyper subalgebra of B is seen trivial.

Theorem 3.1.2. Let S be a nonempty subset of a hyper BCL–algebra B and $\delta, \sigma \in [0, 1]$ with

$\delta > \sigma$ and let η be a fuzzy subset of B defined by: $\eta(m) = \begin{cases} \delta, & \text{if } m \in S, \\ \sigma, & \text{if } m \notin S. \end{cases}$ Then:

(1) η is a fuzzy (weak) hyper subalgebra of B if and only if S is a (weak) hyper subalgebra of B , respectively.

(2) If η is a fuzzy strong hyper subalgebra of B then S is a strong hyper subalgebra of B but not the converse.

Proof. Suppose $S \subseteq B$ and η is a fuzzy (weak, strong) hyper subalgebra of B . Then we need to prove S is a (weak, strong) hyper subalgebra of B .

(1) (i) To prove S is hyper subalgebra of B if and only if η is fuzzy hyper subalgebra of B :

Suppose S is hyper subalgebra of B .

Let $m, n \in S$. By definition $\eta(m) = \eta(n) = \delta$, and since η is a fuzzy hyper subalgebra,

we have $\inf_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\} = \delta \Rightarrow \eta(u) \geq \delta, \forall u \in m @ n$.

But $\eta(u) \leq \delta, \forall u \in M$ and $\eta(u) \geq \delta, \forall u \in m @ n \Rightarrow \eta(u) = \delta, \forall u \in m @ n, u \in B \Rightarrow m @ n \subseteq M$
 $\Rightarrow S$ is hyper subalgebra of M .

Conversely, suppose S is hyper subalgebra of M .

Then we need to show that η is a fuzzy hyper subalgebra of B .

(a) Since S is hyper subalgebra of B , $0 \in S$, by Lemma 2.2.1

$\Rightarrow \eta(0) = \delta, \forall m \in S$.

(b) For arbitrary $m, n \in B$, we consider three cases as follows:

(i) $m, n \in S \Rightarrow m @ n \subseteq S$, since S is hyper subalgebra of B

$\Rightarrow \eta(m) = \delta = \eta(n) = \eta(m @ n)$, by definition

$\Rightarrow \inf_{u \in m @ n} \eta(u) = \delta, \forall u \in m @ n$ and $\forall m, n \in S$

$\Rightarrow \inf_{u \in m @ n} \eta(u) = \delta \geq \min\{\eta(m), \eta(n)\} = \min\{\delta, \delta\} = \delta$

(ii) $m \in S$ and $n \notin S$ (or $n \in S$ and $m \notin S$) \Rightarrow either $m @ n \subseteq S$ or $m @ n \not\subseteq S$

$\Rightarrow \eta(m) = \delta, \eta(n) = \sigma$ (or $\eta(n) = \delta, \eta(m) = \sigma$) and $\eta(m @ n) \geq \sigma$, by definition

$\Rightarrow \inf_{u \in m @ n} \eta(u) = \sigma, \forall u \in m @ n$ and $\forall m, n \in S$

$\Rightarrow \inf_{u \in m @ n} \eta(u) = \sigma \geq \min\{\eta(m), \eta(n)\} = \min\{\delta, \sigma\} = \sigma$

(iii) $m, n \notin S \Rightarrow$ either $m @ n \subseteq S$ or $m @ n \not\subseteq S$

$\Rightarrow \eta(m) = \sigma = \eta(n)$ and $\eta(m @ n) \geq \sigma$, by definition

$\Rightarrow \inf_{u \in m @ n} \eta(u) = \sigma, \forall u \in m @ n$ and $\forall m, n \in S$

$\Rightarrow \inf_{u \in m @ n} \eta(u) = \sigma \geq \min\{\eta(m), \eta(n)\} = \min\{\sigma, \sigma\} = \sigma$

In either case, $\inf_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}, \forall m, n \in B$.

Thus, η is fuzzy hyper subalgebra of B , by (a) and (b) above.

(ii) To prove S is weak hyper subalgebra of B if and only if η is fuzzy weak hyper subalgebra of B :

Suppose η is weak hyper subalgebra of B , then we show S is weak hyper subalgebra of B .

(a) As shown in (1) above, and by Proposition 2.2.2 (2) that every hyper subalgebra is weak hyper subalgebra, $\eta(0) = \delta$ so that $0 \in S$. By Theorem 3.1.1 (second part) implies the level set $B_\eta = S$ is a weak hyper subalgebra of B . (Or from (1) above, we have shown that:

$m@n \subseteq S \Rightarrow m@n \prec B$, by Proposition 2.1.4 (3)). Thus, S is weak hyper subalgebra of B .

Conversely, suppose S is weak hyper subalgebra of B .

Then we need to show that η is a fuzzy weak hyper subalgebra of B .

But from (1) above, S is hyper subalgebra of $B \Rightarrow S$ is weak hyper subalgebra of B

$$\Rightarrow \inf_{u \in m@n} \eta(u) \geq \min\{\eta(m), \eta(n)\}, \forall m, n \in B$$

$$\Rightarrow \sup_{u \in m@n} \eta(u) \geq \inf_{u \in m@n} \eta(u) \geq \min\{\eta(m), \eta(n)\}, \forall m, n \in B$$

$$\Rightarrow \sup_{u \in m@n} \eta(u) \geq \min\{\eta(m), \eta(n)\}, \forall m, n \in B$$

Thus, η is weak fuzzy hyper subalgebra of B as shown above.

(2) (i) To prove if S is strong hyper subalgebra of B then η is fuzzy strong hyper subalgebra of B :

Suppose η is strong hyper subalgebra of B , then we show S is strong hyper subalgebra of B .

Since $\sup_{u \in m@n} \eta(u) \geq \max\{\eta(m), \eta(n)\}, \forall m, n \in B, \forall m, n \in S$ and by the first axiom of

definition of fuzzy strong hyper subalgebra of B , $\eta(0) \geq \eta(m), \forall m \in S$ and $\eta(m) \leq \delta, \forall m \in B$

$$\Rightarrow \eta(0) \leq \delta \text{ and hence } \eta(0) = \delta \Rightarrow 0 \in S = \{0\} \Rightarrow S \text{ is strong hyper subalgebra of } B.$$

(ii) For "the converse may not be true", suppose S is strong hyper subalgebra of B .

Then we need to justify that η is not a fuzzy strong hyper subalgebra of B .

Let $B = \{0, p, q\}$ as define by Table 2.4 and $S = \{0\}$ is strong hyper subalgebra of B .

Now when $n = 0 \in S$ and $m = p \notin S \Rightarrow p@0 = \{p\} \not\subseteq S \Rightarrow \eta(m) = \delta$, and $\eta(n) = \sigma = \eta(m@n)$

$$\Rightarrow \inf_{u \in m@n} \eta(u) = \sigma \geq \max\{\eta(m), \eta(n)\} = \max\{\delta, \sigma\} = \delta, \text{ which is not true as } \delta > \sigma.$$

Therefore, the converse for Theorem 3.3.3 (2) is not necessarily true.

□

Corollary 3.1.1. Let S be a nonempty subset of a hyper BCL–algebra B and χ_S be the Characteristic

$$\text{function defined by: } \chi_S(m) = \begin{cases} 1, & \text{if } m \in S, \\ 0, & \text{if } m \notin S, \end{cases}$$

Then χ_S is a fuzzy (weak, strong) hyper subalgebra of B if and only if S is a (weak, strong) hyper subalgebra of B , respectively.

Proof. The proof of this corollary is similar to the proof of Theorem 3.1.2 above as η in Theorem 3.1.2 is the generalization of this Characteristic function, χ_S and all the steps are the same except changing δ by 1 and σ by 0 for this case. □

Proposition 3.1.1. Let $\eta : B \rightarrow [0, 1]$ be fuzzy (weak, strong) hyper subalgebra of B . Then, the following hold $\forall m, n \in B$:

$$(1) \{m\} \prec \{n\} \Rightarrow \sup_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\},$$

$$(2) \{m\} \preceq \{n\} \Rightarrow \sup_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}.$$

Proof.

$$(1) \{m\} \prec \{n\} \Rightarrow 0 \in m @ n \Rightarrow \sup_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}; \text{ since } \eta(0) \geq \eta(m), \forall m \in B.$$

$$(2) \{m\} \preceq \{n\} \Rightarrow 0 \in m @ n \Rightarrow \sup_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}; \text{ since } \eta(0) \geq \eta(m), \forall m \in B. \quad \square$$

Theorem 3.1.3. Let $(B; @, 0)$ be a hyper BCL–algebra and η be fuzzy hyper subalgebra of B such that $m, n, w \in B$. Then

$$(1) \{w\} \subseteq m @ n \Rightarrow \eta(w) \geq \min\{\eta(m), \eta(n)\},$$

$$(2) w @ n \subseteq m @ n \Rightarrow \sup_{u \in w @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}.$$

Proof. (1) Let $\{w\} \subseteq m @ n$. From Definition 3.1.1 we have: $\{w\} \subseteq m @ n$ (or $w \in m @ n$)

By definition of fuzzy hyper subalgebra, $\inf_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}$ and

by definition of infimum, $\forall u \in m @ n$, we have $\eta(u) \geq \inf_{v \in m @ n} \eta(v)$.

In Particular, taking $u = w$, $\eta(w) \geq \inf_{u \in m @ n} \eta(u)$.

Combining the two inequalities gives: $\eta(w) \geq \inf_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}$
 $\Rightarrow \eta(w) \geq \min\{\eta(m), \eta(n)\}$

(2) Assume $w @ n \subseteq m @ n$. Then by (1) above, $\forall u \in w @ n$, we have $\eta(u) \geq \min\{\eta(m), \eta(n)\}$

$\Rightarrow \sup_{u \in w @ n} \eta(u) \geq \eta(u) \geq \min\{\eta(m), \eta(n)\} \Rightarrow \sup_{u \in w @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\}$ \square

Theorem 3.1.4. *Let η be a fuzzy (weak, strong) hyper subalgebra of a hyper BCL–algebra B . Then for any $\alpha \in [0, 1]$, the set $B_\eta^\alpha = \{m : \eta(m) \geq \alpha, m \in B\}$ is a (weak, strong) hyper subalgebra of B , respectively.*

Proof. Let $m, n \in B_\eta^\alpha$. Then $\eta(m) \geq \alpha, \eta(n) \geq \alpha$. If η is fuzzy hyper subalgebra, then:

$$\inf_{u \in m @ n} \eta(u) \geq \min\{\eta(m), \eta(n)\} \geq \alpha \Rightarrow \inf_{u \in m @ n} \eta(u) \geq \alpha \Rightarrow \eta(u) \geq \alpha, \forall u \in m @ n$$

$$\Rightarrow \inf_{u \in m @ n} \eta(u) \geq \alpha, \forall u \in B_\eta^\alpha \Rightarrow m @ n \subseteq B_\eta^\alpha \text{ and hence } B_\eta^\alpha \text{ is hyper subalgebra of } B$$

and then the weak and strong cases follow similarly. \square

Remark 3.1.2. *Let $\eta : B \rightarrow [0, 1]$ be a constant function and $c \in [0, 1)$. Then following Theorem 3.1.4 and Corollary 3.1.1, we can generalize the following:*

- (1) *If $\eta(m) = 1, \forall m \in B$, then η is a fuzzy (weak, strong) hyper subalgebra of B .*
- (2) *If $\eta(m) = c, \forall m \in B$ then η is fuzzy (weak) hyper subalgebra of B , but not strong hyper subalgebra of B .*

Theorem 3.1.5. *Let η_1, η_2 be fuzzy (weak, strong) hyper subalgebras of B such that $\forall m \in B, \eta_1(m) \geq \eta_2(m)$, and suppose $B_{\eta_1}^\alpha$ is as defined in Theorem 3.1.4. Then, $B_{\eta_2}^\alpha \subseteq B_{\eta_1}^\alpha$.*

Proof. If $m \in B_{\eta_2}^\alpha$, then $\eta_2(m) \geq \alpha$. Since $\eta_1(m) \geq \eta_2(m) \geq \alpha \Rightarrow \eta_1(m) \geq \alpha \Rightarrow m \in B_{\eta_1}^\alpha$.

$$\Rightarrow B_{\eta_2}^\alpha \subseteq B_{\eta_1}^\alpha \text{ and hence the proof.} \quad \square$$

Proposition 3.1.2. *Let B_η^α be as defined in 3.1.4 and $\eta : B \rightarrow (0, 1]$ be a fuzzy subset. Then η is a fuzzy (weak, strong) hyper subalgebra of B if and only if the set B_η^α is a (weak, strong) hyper subalgebra of $B, \forall \alpha \in [0, 1]$.*

Proof. Follows directly from the proofs of Lemma 3.1.4 (for the sufficient condition) and Theorem 3.1.5 (for the necessary condition). \square

3.2. Fuzzy Hyper Deductive Systems of Hyper BCL–algebra

This section focuses on fuzzy hyper deductive systems in hyper BCL–algebra. These systems combine fuzzy set theory with hyper-operations to represent uncertain logical reasoning. We define fuzzy weak hyper deductive system, fuzzy hyper deductive system, and fuzzy strong hyper deductive system, and explain how fuzzy membership values behave under complex hyper-operations. Through examples and theorems, we show how these systems support generalized logical deductions in uncertain environments. *Here we denote **DS** to represent deductive system.*

Definition 3.2.1. *Let $(B; @, 0)$ be a hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called fuzzy hyper deductive system (DS) of B if the following are satisfied, $\forall m, n, w, u, v \in B$:*

- (i) $\sup_{v \in (m@u)@u} \eta(v) \geq \eta(m)$
- (ii) $\inf_{v \in m@(n@w)} \eta(v) \geq \min\{\eta(m), \eta(n)\}.$

Definition 3.2.2. *Let $(B; @, 0)$ be hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called a fuzzy weak hyper DS of B if the following are satisfied, $\forall m, n, w, u, v \in B$:*

- (i) $\sup_{v \in (m@u)@u} \eta(v) \geq \eta(m)$
- (ii) $\sup_{v \in m@(n@w)} \eta(v) \geq \min\{\eta(m), \eta(n)\}.$

Definition 3.2.3. *Let $(B; @, 0)$ be hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called a strong fuzzy hyper DS of B if the following are satisfied, $\forall m, n, w, u, v \in B$:*

- (i) $\inf_{v \in (m@u)@u} \eta(v) \geq \eta(m)$
- (ii) $\inf_{v \in m@(n@w)} \eta(v) \geq \min\{\eta(m), \eta(n)\}.$

Example 3.2.1. *Let $(B; @, 0)$ be a hyper BCL–algebra as defined in Table 2.3 of Example 2.3.1 and define fuzzy subsets by $\eta_1, \eta_2, \eta_3 : B \rightarrow [0, 1]$ by:*

$$\eta_1(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.5, & \text{if } m = p, r, \\ 0.2, & \text{if } m = q, \end{cases} \quad \eta_2(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.6, & \text{if } m = p, \\ 0.3, & \text{if } m = q, \\ 0.1, & \text{if } m = r, \end{cases} \quad \eta_3(m) = \begin{cases} 0.7, & \text{if } m = 0, \\ 0.4, & \text{if } m = p, q, r. \end{cases}$$

Then η_3 is fuzzy (weak, strong) hyper DS of B . In contrast, η_1 is fuzzy (weak) hyper DS of B but not fuzzy strong hyper DS of B , and η_2 is fuzzy weak hyper DS of B but η_2 is neither fuzzy hyper DS of B nor fuzzy strong hyper DS of B by the the following counter examples:

$$(i) \quad \inf_{v \in 0@(0@r)} \eta_1(v) = \min\{\eta_1(0), \eta_1(r)\} = \min\{1, 0.5\} = 0.5$$

$$\geq \min\{\eta_1(0), \eta_1(0)\} = \eta_1(0) = 1, \text{ is not true confirming that } \eta_1 \text{ is not fuzzy strong hyper DS of } B.$$

$$(ii) \quad \inf_{v \in p@(0@r)} \eta_2(v) = \min\{\eta_2(p), \eta_2(r)\} = \min\{0.6, 0.1\} = 0.1$$

$$\geq \min\{\eta_2(p), \eta_2(0)\} = \min\{0.6, 0.8\} = 0.6, \text{ is not true confirming that } \eta_2 \text{ is not fuzzy hyper DS of } B, \text{ and again}$$

$$\inf_{v \in 0@(0@r)} \eta_2(v) = \min\{\eta_2(0), \eta_2(r)\} = \min\{0.8, 0.1\} = 0.1$$

$$\geq \min\{\eta_2(0), \eta_2(0)\} = \eta_2(0) = 0.8, \text{ is not true confirming that } \eta_2 \text{ is not fuzzy strong hyper DS of } B.$$

Lemma 3.2.1. Let $\eta : B \rightarrow [0, 1]$ be fuzzy subset where η is a fuzzy (weak, strong) hyper DS of B . Then the following hold:

- (1) $\eta(0) \geq \eta(m); \forall m \in B$.
- (2) η is onto $\Rightarrow \eta(0) = 1$.

Proof. (1) Definitions 3.2.1 – 3.2.3, condition (i) requires $\eta(0) \geq \eta(m), \forall m \in B$.

Since $B \neq \emptyset \exists m \in B$ such that $0 \in m@m$. Then by hypothesis pick $m \in B$ such that $(m@m)@m = \{0\}$. Or more clearly:

Since η satisfies axiom (i) of the corresponding fuzzy DS,

$$\sup_{v \in (m@m)@m} \eta(v) \geq \eta(m) \text{ (respectively, } \inf_{v \in (m@u)@u} \eta(v) \geq \eta(m)).$$

But $(m@m)@m = \{0\}$. Then the left-hand side equals $\eta(0)$.

Hence $\eta(0) \geq \eta(m)$. As m is arbitrary, the result follows.

(2) η is onto $\Rightarrow \eta(0) = 1$ (which is immediate), as it is similar to the proof of Lemma 3.1.1.

More clearly, if η is onto, $\exists n \in B$ such that $\eta(n) = 1$ and since $\eta(0) \geq \eta(n), \forall n \in B$, by (1), we have, $\eta(0) = 1$ or if η is onto, $\eta(0)$ must attain the maximum value 1 of $[0, 1]$,

since $\eta(0) \geq \eta(m), \forall m \in B$. □

Theorem 3.2.1. *If $\eta : B \rightarrow [0, 1]$ is a (weak, strong) hyper DS of a hyper BCL–algebra B , then B_η is a (weak, strong) hyper DS of B where the set $B_\eta = \{m \in B : \eta(m) = \eta(0)\}$.*

Proof. For $m, n, u, w \in B_\eta$, we have; $\eta(m) = \eta(n) = \eta(0) = \eta(u) = \eta(w)$

(i) Suppose $m \in B_\eta$. Then using the first axioms of (weak, strong) hyper DS B , we have:

$$\sup_{v \in (m@u)@u} \eta(v) \geq \inf_{v \in (m@u)@u} \eta(v) \geq \eta(m) = \eta(0) \dots\dots\dots (*)$$

$$\Rightarrow \sup_{v \in (m@u)@u} \eta(v) \geq \inf_{v \in (m@u)@u} \eta(v) \geq \eta(0)$$

But $\eta(0) \geq \eta(m)$, $\forall m \in B$ by Lemma 3.2.1 above.

$$\Rightarrow \eta(0) \geq \sup_{v \in (m@u)@u} \eta(v) \geq \inf_{v \in (m@u)@u} \eta(v), \forall v \in (m@u)@u \dots\dots\dots (**)$$

$$\Rightarrow \eta(0) = \sup_{v \in (m@u)@u} \eta(v) = \inf_{v \in (m@u)@u} \eta(v), \forall v \in (m@u)@u, \text{ by } (*) \text{ and } (**) \text{ above.}$$

(ii) using the second axioms of (weak, strong) hyper DS B , we have:

$$\sup_{v \in m@(n@w)} \eta(v) \geq \inf_{v \in m@(n@w)} \eta(v) \geq \min\{\eta(m), \eta(n)\} = \eta(0)$$

$$\Rightarrow \sup_{v \in m@(n@w)} \eta(v) \geq \inf_{v \in m@(n@w)} \eta(v) \geq \eta(0)$$

But $\eta(0) \geq \eta(m)$, $\forall m \in B$ by Lemma 3.2.1 above.

$$\Rightarrow \eta(0) \geq \sup_{v \in m@(n@w)} \eta(v) \geq \inf_{v \in m@(n@w)} \eta(v), \forall v \in m@(n@w)$$

$$\Rightarrow \eta(0) = \sup_{v \in m@(n@w)} \eta(v) = \inf_{v \in m@(n@w)} \eta(v), \forall v \in m@(n@w)$$

Therefore, by (i) and (ii) above, we ratified that B_η is a (weak, strong) hyper DS of B .

□

Theorem 3.2.2. *Let M be a nonempty subset of a hyper BCL–algebra B ; $\delta, \sigma \in [0, 1]$ such*

that $\delta > \sigma$ and let η be a fuzzy subset of B defined by: $\eta(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M. \end{cases}$

Then, η is a fuzzy (weak, strong) hyper DS of B if and only if M is a (weak, strong hyper) hyper DS of B , respectively.

Proof. Suppose $M \subseteq B$ and η is a fuzzy (weak, strong) hyper DS of B . Then we need to prove M is a (weak, strong) hyper DS of B .

(1) To prove M is hyper DS of B if and only if η is fuzzy hyper DS of B :

Suppose M is hyper DS of B .

Let $m, n \in M$. By definition $\eta(m) = \eta(n) = \delta$, and since η is a fuzzy hyper DS, we have:

(i) $\sup_{v \in (m@u)@u} \eta(v) \geq \eta(m) = \delta \Rightarrow \eta(v) \geq \delta, \forall v \in (m@u)@u$. But

$\eta(v) \leq \delta, \forall v \in M$ and $\eta(v) \geq \delta, \forall v \in (m@u)@u \Rightarrow \eta(v) = \delta, \forall v \in (m@u)@u, v \in M$

$\Rightarrow (m@u)@u \subseteq M$, (since $(m@u)@u$ contains 0, when $m = n$).

(ii) $\inf_{v \in m@(n@w)} \eta(v) \geq \min\{\eta(m), \eta(n)\} = \delta \Rightarrow \eta(v) \geq \delta, \forall v \in m@(n@w)$. But

$\eta(v) \leq \delta, \forall v \in B \Rightarrow \eta(v) \geq \delta, \forall v \in m@(n@w) \Rightarrow \eta(v) = \delta, \forall v \in m@(n@w) \Rightarrow v \in B$

$\Rightarrow m@(n@w) \subseteq M$, (since $m@(n@w)$ contains 0, when $m = n$).

$\Rightarrow M$ is hyper DS of B , (by (i) and (ii) above).

Conversely, suppose M is hyper DS of B .

Then we need to show that η is a fuzzy hyper DS of B .

Since M is hyper DS of B , $M = B$ and hence $0 \in B$, by Proposition 2.3.1 (1)

For arbitrary $m, n \in B$, we consider three cases as follows:

(i) $m, n \in M \Rightarrow (m@u)@u \subseteq M$ and $m@(n@w) \subseteq M$, since M is hyper DS of B

$\Rightarrow \eta(m) = \delta = \eta(n) = \eta((m@u)@u) = \eta(m@(n@w))$, by definition

$\Rightarrow \sup_{v \in (m@u)@u} \eta(v) = \delta$ and $\inf_{v \in m@(n@w)} \eta(v) = \delta, \forall v \in (m@u)@u$ and $\forall v \in m@(n@w)$

$\Rightarrow \sup_{v \in (m@u)@u} \eta(v) = \delta \geq \eta(m) = \delta$ and

$\inf_{v \in m@(n@w)} \eta(v) = \delta \geq \min\{\eta(m), \eta(n)\} = \min\{\delta, \delta\} = \delta$

(ii) $m \in M$ and $n \notin M$ (or $n \in M$ and $m \notin M$) \Rightarrow either $(m@u)@u \subseteq M$ or $(m@u)@u \not\subseteq M$;

$m@(n@w) \subseteq M$ or $m@(n@w) \not\subseteq M$

$\Rightarrow \eta(m) = \delta, \eta(n) = \sigma$ (or $\eta(n) = \delta, \eta(m) = \sigma$) and

$\eta(m@u)@u \geq \sigma$ and $\eta(m@(n@w)) \geq \sigma$, by definition

$$\Rightarrow \sup_{v \in (m@n)@u} \eta(v) = \sigma, \quad \inf_{v \in m@(n@w)} \eta(v) = \sigma, \quad \forall v \in (m@u)@u, \quad \forall v \in m@(n@w), \quad \forall m, n \in M$$

$$\Rightarrow \sup_{u \in (m@n)@u} \eta(v) = \sigma \geq \eta(m) = \sigma \quad \text{and}$$

$$\Rightarrow \inf_{v \in m@(n@w)} \eta(v) = \sigma \geq \min\{\eta(m), \eta(n)\} = \min\{\delta, \sigma\} = \sigma$$

(iii) $m, n \neq M \Rightarrow (m@n)@u \subseteq M$ or $(m@n)@u \not\subseteq M$; $m@(n@w) \subseteq M$ or $m@(n@w) \not\subseteq M$

$\Rightarrow \eta(m) = \sigma = \eta(n)$ and $\eta((m@u)@u) \geq \sigma, \eta(m@(n@w)) \geq \sigma$, by definition

$$\Rightarrow \sup_{u \in (m@n)@u} \eta(v) = \sigma \geq \eta(m) = \sigma \quad \text{and}$$

$$\Rightarrow \inf_{v \in m@(n@w)} \eta(v) = \sigma \geq \min\{\eta(m), \eta(n)\} = \min\{\delta, \sigma\} = \sigma$$

In either case, $\inf_{u \in m@n} \eta(u) \geq \min\{\eta(m), \eta(n)\}, \forall m, n \in B$.

Thus, η is fuzzy hyper DS of B , by (a) and (b) above.

(2) To prove M is weak/strong hyper DS of B if and only if η is fuzzy weak/strong hyper DS of B .

We follow similar steps as for (1) above by simply changing "inf" by "sup" or vice-versa which do not violate the persistences.

Therefore, η is weak fuzzy hyper DS of B as shown above. □

Corollary 3.2.1. *Let D be a nonempty subset of a hyper BCL-algebra B and χ be the*

$$\text{Characteristic function defined by: } \chi(m) = \begin{cases} 1, & \text{if } m \in D, \\ 0, & \text{if } m \notin D. \end{cases}$$

Then χ is a fuzzy (weak, strong) hyper DS of B if and only if D is a (weak, strong) hyper DS of B , respectively.

Proof. Following directly from Theorem 3.2.2, by setting $\delta = 1$ and $\sigma = 0$ so that χ is just the special case of η when the membership values are chosen at extreme levels of the interval $[0, 1]$.

Therefore, χ is fuzzy (weak, strong) hyper DS of B if and only if D is (weak, strong) DS of B . □

Corollary 3.2.2. *Let the fuzzy subset $\eta : B \rightarrow [0, 1]$ be a constant function on B such that $\eta(m) = c, \forall m \in B$ where $c \in [0, 1]$, then η is a fuzzy (weak, strong) hyper DS of B .*

Proof. The proof for this is also a particular case of Theorem 3.2.2 such that if $\delta = c$, then the function η is constant on all of B and in such a case, $B_\eta = \{m \in B : \eta(m) = \eta(0) = c\} = B$.

But B itself is always a (weak, strong) hyper DS of a hyper BCL–algebra (since it is closed under all the operations). Therefore, η is a fuzzy (weak, strong) hyper DS of B . □

Theorem 3.2.3. *Let η be a fuzzy (weak, strong) hyper DS of B . Then for each $\alpha \in [0, 1]$, the level set: $B_\eta^\alpha = \{m \in B : \eta(m) \geq \alpha\}$ is a (weak, strong) hyper DS of B .*

Proof. Suppose B is hyper DS and $\alpha \in [0, 1]$

(1) Since $B \neq \emptyset$, $\exists m \in B$ such that $\eta(0) \geq \eta(m)$ (by Lemma 3.2.1 (1)) and combining this with the hypothesis, we obtain, $\eta(0) \geq \eta(m) \geq \alpha \Rightarrow \eta(0) \geq \alpha \Rightarrow 0 \in B_\eta^\alpha$.

(2) Let $m \in B_\eta^\alpha$. Then $\eta(m) \geq \alpha$. By the first axiom of definition of fuzzy strong hyper DS, or

Definition 3.2.3 (1) with the hypothesis: $\inf_{v \in (m@u)@u} \eta(v) \geq \eta(m) \geq \alpha \Rightarrow \inf_{v \in (m@u)@u} \eta(v) \geq \alpha$

Furthermore, from definitions 3.2.1 (1) and 3.2.1 (1) and since $\sup_{v \in (m@u)@u} \eta(v) \geq \inf_{v \in (m@u)@u} \eta(v)$

We obtain, $\sup_{v \in (m@u)@u} \eta(v) \geq \inf_{v \in (m@u)@u} \eta(v) \geq \eta(m) \geq \alpha \Rightarrow v \in B_\eta^\alpha, \forall v \in (m@u)@u$

(3) Let $m, n \in B_\eta^\alpha$. Then $\eta(m) \geq \alpha, \eta(n) \geq \alpha$. By definitions of fuzzy hyper DS Definition

3.2.1 (2) and fuzzy strong hyper DS Definition 3.2.3 (2) with the hypothesis:

$\inf_{v \in (m@n)@w} \eta(v) \geq \min\{\eta(m), \eta(n)\} \geq \min\{\alpha, \alpha\} = \alpha \Rightarrow \inf_{v \in (m@u)@u} \eta(v) \geq \alpha$

Furthermore, from definitions 3.2.1 (2) and since $\sup_{v \in (m@n)@w} \eta(v) \geq \inf_{v \in (m@n)@w} \eta(v) \geq \alpha$

$\sup_{v \in (m@n)@w} \eta(v) \geq \alpha \Rightarrow v \in B_\eta^\alpha, \forall v \in (m@n)@w$

Thus B_η^α is a (weak, strong) hyper DS □

Proposition 3.2.1. *Let $\eta : B \rightarrow [0, 1]$ be fuzzy (strong, weak) hyper DS of B . Then the following hold:*

(1) $\{m\} \prec \{n\} \Rightarrow \inf_{u \in m@n} \eta(u) \geq \eta(m), \forall m, n \in B,$

(2) $\{m\} \preceq \{n\} \Rightarrow \inf_{u \in m@n} \eta(u) \geq \eta(m), \forall m, n \in B.$

Proof. Let $\eta : B \rightarrow [0, 1]$ be a fuzzy (strong, weak) hyper DS of the hyper BCL–algebra B . By the definition of such systems (see Definition 3.2.1–3.2.3), for any $m, n \in B$, we have:

$$\eta(m) \leq \min\{\eta(n), \inf_{u \in m @ n} \eta(u)\} \implies \eta(m) \leq \eta(n); \eta(m) \leq \inf_{u \in m @ n} \eta(u)$$

$\implies \inf_{u \in m @ n} \eta(u) \geq \eta(m)$ always holds under the given condition.

If $\{m\} \preceq \{n\}$ then $\{m\} \prec \{n\}$ by 2.1.4 (9)

$$\implies \{m\} \preceq \{n\} \text{ then } \{m\} \prec \{n\} \implies \eta(m) \leq \inf_{u \in m @ m} \eta(u).$$

Hence, If $\{m\} \preceq \{n\}$ then $\inf_{u \in m @ n} \eta(u) \geq \eta(m)$.

In both cases, $\inf_{u \in m @ n} \eta(u) \geq \eta(m)$ holds which completes the proof. □

Remark 3.2.1. From Proposition 3.2.1 since: $\{m\} \prec \{n\} \implies \inf_{u \in m @ n} \eta(u) \geq \eta(m)$ and $\{m\} \preceq \{n\}$

$$\implies \inf_{u \in m @ n} \eta(u) \geq \eta(m) \text{ and } \sup_{u \in m @ n} \eta(u) \geq \inf_{u \in m @ n} \eta(u) \geq \eta(m) \text{ and } \inf_{u \in m @ n} \eta(u) \geq \eta(m).$$

Then, (1) If $\{m\} \prec \{n\}$, then $\sup_{u \in m @ n} \eta(u) \geq \eta(m)$.

(2) If $\{m\} \preceq \{n\}$, then $\sup_{u \in m @ n} \eta(u) \geq \eta(m)$.

Proposition 3.2.2. Let B be a hyper BCL-algebra and $\eta: B \rightarrow [0, 1]$ be a fuzzy subset such that $\eta(m) \leq \eta(n), \forall m, n \in B$. Then the following hold:

(1) If η is a fuzzy (weak) hyper DS of B , then $\sup_{v \in m @ n} \eta(v) \geq \eta(m)$, and

(2) If η is a fuzzy strong hyper DS of B , then $\inf_{v \in m @ n} \eta(v) \geq \min\{\eta(m), \eta(n)\}$.

Proof. Let $m, n \in B$ and assume that $\eta(m) \leq \eta(n)$.

(1) Since η is a fuzzy (weak) hyper DS B , it satisfies:

$$\sup_{v \in m @ u} \eta(v) \geq \eta(m), \forall m \in B \text{ by Definitions 3.2.1 and 3.2.1, and}$$

Now taking $u = n$, we have: $\sup_{v \in m @ n} \eta(v) \geq \eta(m)$.

(2) Since η is a fuzzy strong hyper DS of B , it satisfies:

$$\inf_{v \in m @ u} \eta(v) \geq \eta(m), \text{ (by Definition 3.2.3) and then } \inf_{v \in m @ n} \eta(v) \geq \eta(m), \text{ when } u = n.$$

Moreover, since $\eta(m) \leq \eta(n)$, it follows that: $\min\{\eta(m), \eta(n)\} = \eta(m)$ and then

$$\inf_{v \in m @ n} \eta(v) \geq \eta(m) = \min\{\eta(m), \eta(n)\} \text{ and therefore, } \inf_{v \in m @ n} \eta(v) \geq \min\{\eta(m), \eta(n)\}.$$

□

Theorem 3.2.4. For a hyper BCL–algebra B and a fuzzy subset $\eta : B \rightarrow [0, 1]$, the following hierarchy holds:

- (1) If η is fuzzy strong hyper DS of B , then η is fuzzy hyper DS of B .
- (2) If η is fuzzy hyper DS of B , then η is fuzzy weak hyper DS of B .
- (3) If η is fuzzy strong hyper DS of B , then η is fuzzy weak hyper DS of B

Proof. Since the axioms for the definitions of fuzzy (weak, strong) hyper DS of B are inter-related by exchanging inf to sup or sup to inf and since the following:

$$\sup_{v \in (m@u)@u} \eta(v) \geq \inf_{v \in (m@u)@u} \eta(v), \forall m \in B, \quad \sup_{v \in m@(n@w)} \eta(v) \geq \inf_{v \in m@(n@w)} \eta(v), \forall m, n \in B,$$

we get the proofs for (1) – (2) become straightforward and (3) can be verified by transitivity from (1) to (2). \square

Remark 3.2.2. The converses of each of the three statements in Theorem 3.2.4 may not be true, demonstrated by the subsequent counter example.

Example 3.2.2. Let $(B: @, 0)$ be a hyper BCL–algebra as defined by Table 2.3 of Example 2.3.1 and $\eta_1, \eta_2 : B \rightarrow [0, 1]$ be fuzzy subsets. Then the converses of the three statements in Theorem 3.2.4 may not be necessarily true which we explain each converse as follows:

$$\text{Suppose, } \eta_1(m) = \begin{cases} 0.9, & \text{if } m = 0, \\ 0.5, & \text{if } m = p, \\ 0.1, & \text{if } m = q, r, \end{cases} \quad \text{and} \quad \eta_2(m) = \begin{cases} 0.9, & \text{if } m = 0, \\ 0.8, & \text{if } m = p, \\ 0.6, & \text{if } m = q, \\ 0.2, & \text{if } m = r. \end{cases}$$

Then, by mechanical calculations, it has been checked that η_1 is fuzzy hyper DS of B and η_2 is fuzzy weak hyper DS of B .

(1) As stated above η_1 is fuzzy hyper DS of B and from Table 2.3, since

$$(p@r)@r = \{p, r\}@r = \{0, p, r\} \quad \text{and} \quad \inf_{v \in (p@r)@r} \eta_1(v) = \inf_{v \in \{0, p, r\}} \eta_1(v) = 0.1 \geq \eta_1(p) = 0.5$$

which is false ratifying that: η_1 is fuzzy strong hyper DS of B if η_1 is fuzzy hyper DS of B is not true.

Therefore, the converse: η_1 is fuzzy strong hyper DS of B if η_1 is fuzzy hyper DS of B is not true.

(2) As stated above η_2 is fuzzy weak hyper DS of B . and from Table 2.3, since:

$$(p@r)@r = \{p, r\}@r = \{0, p, p\} \text{ and } \inf_{v \in (p@r)@r} \eta_2(v) = \inf_{v \in \{0, p, r\}} \eta_2(v) = 0.2 \geq \eta_2(p) = 0.8$$

which is false showing that: η_2 is not fuzzy hyper DS of B .

Therefore, η_2 is fuzzy hyper DS of B if η_2 is fuzzy weak hyper DS of B is not true.

(3) As stated above η_2 is fuzzy weak hyper DS of B . and from Table 2.3, since:

$$p @ (q @ 0) = p @ \{q\} = \{p, r\} \text{ and}$$

$$\inf_{v \in p@(q@0)} \eta_2(v) = \inf_{v \in \{p, r\}} \eta_2(v) = 0.2 \geq \min\{\eta_2(p), \eta_2(q)\} = \min\{0.8, 0.6\} = 0.6 \text{ which is false}$$

showing that η_2 is not fuzzy strong hyper DS of B .

Therefore, η_2 is fuzzy strong hyper DS of B if η_2 is fuzzy weak hyper DS of B is not true.

3.3. Fuzzy Hyper Ideals of Hyper BCL–algebra

Under this section, we explore fuzzy hyper ideals in hyper BCL–algebras, which generalize the classical notion of ideals using fuzzy logic and hyper-operations. Fuzzy hyper ideals measure how well elements fit into an ideal based on their degree of membership. We define fuzzy weak hyper ideal, fuzzy hyper ideal, and fuzzy strong hyper ideals, and their properties are analyzed through theorems and illustrative examples. Their relationships and differences are also discussed in detail.

Definition 3.3.1. Let $(B; @, 0)$ be hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called a fuzzy hyper ideal of B if, $\forall m, n, u, v, w \in B$:

$$(i) \quad \eta(m) \geq \min\{\eta(n), \sup_{u \in m@n} \eta(u)\}$$

$$(ii) \quad \sup_{v \in (m@(n@w))@w} \eta(v) \geq \min\{\eta(m), \eta(n)\}$$

Definition 3.3.2. Let $(B; @, 0)$ be hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called a fuzzy weak hyper ideal of B if, $\forall m, n, u, v, w \in B$:

$$(i) \quad \eta(m) \geq \min\{\eta(n), \inf_{u \in m@n} \eta(u)\}$$

$$(ii) \quad \sup_{v \in (m@(n@w))@w} \eta(v) \geq \min\{\eta(m), \eta(n)\}$$

Definition 3.3.3. Let $(B; @, 0)$ be hyper BCL–algebra. Then a fuzzy subset $\eta : B \rightarrow [0, 1]$ is called a fuzzy strong hyper ideal of B if, $\forall m, n, u, v, w \in B$:

- (i) $\eta(m) \geq \min\{\eta(n), \sup_{u \in m@n} \eta(u)\}$
- (ii) $\inf_{v \in (m@(n@w))@w} \eta(v) \geq \min\{\eta(m), \eta(n)\}$

Example 3.3.1. Let $(B; @, 0)$ be a hyper BCL–algebra as defined in Table 2.3 of Example 2.3.1 and define fuzzy subsets $\eta_1, \eta_2, \eta_3 : B \rightarrow [0, 1]$ by:

$$\eta_1(m) = \begin{cases} 0.9, & \text{if } m = 0, \\ 0.7, & \text{if } m = p, \\ 0.1, & \text{if } m = q, r, \end{cases}, \quad \eta_2(m) = \begin{cases} 0.9, & \text{if } m = 0, \\ 0.5, & \text{if } m = p, \\ 0.4, & \text{if } m = q, \\ 0.2, & \text{if } m = r, \end{cases}, \quad \eta_3(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.3, & \text{if } m = p, q, r. \end{cases}$$

From Example 3.3.1, fuzzy subset η_1 is fuzzy (weak) hyper ideal of B but not fuzzy strong hyper ideal of B . Also, η_2 is fuzzy weak hyper ideal of B but not fuzzy hyper ideal of B nor fuzzy strong hyper ideal of B . Additionally, η_3 is fuzzy (weak, strong) hyper ideal of B with the following explanations, taking some cases and similar steps can follow for all the remaining cases.

(1) First we show for is fuzzy (weak) hyper ideal of B but not fuzzy strong hyper ideal of B by taking the case $m = p, n = 0,$ and $w = q$:

(i) $\eta_1(p) \geq \min\{\eta_1(0), \sup_{u \in p@0} \eta_1(u)\} \Rightarrow \eta_1(p) \geq \min\{\eta_1(0), \eta_1(p)\},$ as $p@0 = \{p\}$

$\Rightarrow 0.7 \geq \min\{0.9, 0.7\} = 0.7,$ (the first axiom for fuzzy hyper ideal holds) and

$\eta_1(p) \geq \min\{\eta_1(0), \inf_{u \in p@0} \eta_1(u)\} \Rightarrow \eta_1(p) \geq \min\{\eta_1(0), \eta_1(p)\},$ as $p@0 = \{p\}$

$\Rightarrow 0.7 \geq \min\{0.9, 0.7\} = 0.7,$ (the first axiom for fuzzy weak hyper ideal also holds)

(ii) $\sup_{v \in (p@(0@q))@q} \eta_1(v) \geq \min\{\eta_1(0), \eta_1(p)\}$

$\Rightarrow \sup_{v \in \{p, r\}} \eta_1(v) \geq \min\{\eta_1(0), \eta_1(p)\},$ as $(p@(0@q))@q = (p@\{0\})@q = \{p\}@q = \{p, r\}$

$\Rightarrow \sup_{v \in \{p, r\}} \eta_1(v) = \max\{\eta_1(p), \eta_1(r)\} = \max\{0.7, 0.1\} = 0.7$

$\geq \min\{\eta_1(0), \eta_1(p)\} = \min\{0.9, 0.7\} = 0.7,$

(the second axioms for fuzzy (weak) hyper ideals hold since both have the same second axiom)

(iii) But taking the second axiom of fuzzy strong hyper ideal of B , we have the following:

$$\inf_{v \in (p @ (0 @ q)) @ q} \eta_1(v) \geq \min\{\eta_1(0), \eta_1(p)\} \text{ and } (p @ (0 @ q)) @ q = (p @ \{0\}) @ q = \{p\} @ q = \{p, r\}$$

$$\Rightarrow \inf_{v \in \{p, r\}} \eta_1(v) \geq \min\{\eta_1(0), \eta_1(p)\}, \text{ and}$$

$$\inf_{v \in \{p, r\}} \eta_1(v) = \min\{\eta_1(p), \eta_1(r)\} = \min\{0.7, 0.1\} = 0.1$$

$$\Rightarrow \sup_{v \in \{p, r\}} \eta_1(v) = \eta_1(r) = 0.1 \geq \min\{\eta_1(0), \eta_1(p)\} = \min\{0.9, 0.7\} = 0.7, \text{ which is false}$$

and hence, η_1 is not fuzzy strong hyper ideal of B because the second axiom does not hold, as shown above.

(2) Second we show for η_2 is fuzzy weak hyper ideal of B but not fuzzy (strong) hyper ideal of B , by taking the case $m = p$, $n = 0$, and $w = r$:

$$(i) \eta_2(p) \geq \min\{\eta_2(p), \inf_{u \in p @ p} \eta_2(u)\} \Rightarrow \eta_2(p) \geq \min\{\eta_2(p), \inf_{u \in \{0, p\}} \eta_2(u)\}, \text{ as } p @ p = \{0, p\}$$

$$\Rightarrow \eta_2(p) \geq \min\{\eta_2(p), \eta_2(p)\}, \text{ as } \inf_{u \in \{0, p\}} \eta_2(u) = \eta_2(p) \text{ since } \eta_2(0) \geq \eta_2(p), \forall p \in B$$

$$\Rightarrow 0.5 \geq \min\{0.5, 0.5\} = 0.5, \text{ (the first axiom for fuzzy weak hyper ideal holds) and}$$

$$(ii) \sup_{v \in (p @ p @ r) @ r} \eta_2(v) \geq \min\{\eta_2(p), \eta_2(p)\} = \eta_2(p)$$

$$\Rightarrow \sup_{v \in \{0, p, r\}} \eta_2(v) \geq \min\{\eta_2(p), \eta_2(p)\} = \eta_2(p), \text{ as } (p @ (p @ r)) @ r = \{0, p, r\}$$

$$\Rightarrow \sup_{v \in \{0, p, r\}} \eta_2(v) = \eta_2(0) = 0.9 \geq \eta_2(p) = 0.5,$$

(the second axiom for fuzzy weak hyper ideal holds)

(iii) But taking the first axioms of fuzzy (strong) hyper ideal of B (as both have the same first axioms), we have the following:

$$\eta_2(p) \geq \min\{\eta_2(0), \sup_{u \in p @ p} \eta_2(u)\} \Rightarrow \eta_2(p) \geq \min\{\eta_2(0), \sup_{u \in \{0, p\}} \eta_2(u)\}, \text{ as } p @ p = \{0, p\}$$

$$\Rightarrow \eta_2(p) \geq \min\{\eta_2(0), \eta_2(0)\} = \eta_2(0), \text{ as } \sup_{u \in \{0, p\}} \eta_2(u) = \eta_2(0) \text{ since } \eta_2(0) \geq \eta_2(p), \forall p \in B$$

$$\Rightarrow 0.5 \geq 0.9, \text{ which is false (the first axioms for fuzzy (strong) hyper ideal do not hold)}$$

and hence, η_2 is not fuzzy (strong) hyper ideal of B because the first axioms do not hold, as shown above.

(3) In a similar fashion, we can justify that η_3 is fuzzy (weak, strong) hyper ideal of B by taking different cases like for (1) and (2) above.

Proposition 3.3.1. For a hyper BCL–algebra B , the following hold:

- (1) Every fuzzy strong hyper ideal of B is fuzzy hyper ideal of B .
- (2) Every fuzzy hyper ideal of B is fuzzy weak hyper ideal of B .
- (3) Every fuzzy strong hyper ideal of B is fuzzy weak hyper ideal of B .
- (4) Let $\eta : B \rightarrow [0, 1]$ be fuzzy (weak) hyper ideal of B . Then $\eta(0) \geq \eta(m), \forall m \in B$.

Proof. (1) Let η be fuzzy strong hyper ideal of B . Then we need to prove η is fuzzy hyper ideal:

Definition 3.3.1 (i) and Definition 3.3.3 (i) are the same, so we need to prove Definition 3.3.3 (ii) and Definition 3.3.1 (ii).

Then Definition 3.3.3 (ii):
$$\inf_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n)\}$$

$$\Rightarrow \sup_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \inf_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n)\}$$

$$\Rightarrow \sup_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n)\}$$

$$\text{since, } \sup_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \inf_{v \in (m @ (n @ w)) @ w} \eta(v), \forall m, n \in B$$

Therefore, by the above justifications, we ratify that every fuzzy strong hyper ideal is fuzzy hyper ideal of a hyper BCL–algebra

- (2) Definition 3.3.1 (ii) and Definition 3.3.2 (ii) are the same so we need to prove Definition 3.3.1 (i) implies Definition 3.3.2 (i) and then:

$$\text{Definition 3.3.1 (i): } \eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\}$$

$$\Rightarrow \eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\} \geq \min\{\eta(n), \inf_{u \in m @ n} \eta(u)\}$$

$$\text{since, } \sup_{u \in m @ n} \eta(u) \geq \inf_{u \in m @ n} \eta(u)$$

$$\text{Thus, } \eta(m) \geq \min\{\eta(n), \inf_{u \in m @ n} \eta(u)\}$$

Therefore, by the above justifications, we ratify that every fuzzy hyper ideal is fuzzy weak hyper ideal of a hyper BCL–algebra

- (3) Definition 3.3.3 (ii) and Definition 3.3.1 (ii) are the same. Thus we only need to show that (i) Definition 3.3.3 (i) implies Definition 3.3.1 (i) so that:

$$\text{Definition 3.3.3 (i): } \eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\}$$

$$\eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\} \geq \min\{\eta(n), \inf_{u \in m @ n} \eta(u)\}$$

$$\text{since, } \sup_{u \in m @ n} \eta(u) \geq \inf_{u \in m @ n} \eta(u)$$

Thus, $\eta(m) \geq \min\{\eta(n), \inf_{u \in m @ n} \eta(u)\}$, Definition 3.3.1 (i)

$$(ii) \text{ Definition 3.3.3 (ii): } \inf_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n)\}$$

$$\sup_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \inf_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n)\},$$

$$\text{since, } \sup_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \inf_{v \in (m @ (n @ w)) @ w} \eta(v)$$

Thus, $\sup_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n)\}$, Definition 3.3.1 (ii).

Therefore, by the above justifications, we ratify that every fuzzy strong hyper ideal is fuzzy weak hyper ideal of a hyper BCL–algebra

(4) Suppose $\eta : B \rightarrow [0, 1]$ is fuzzy (weak) hyper ideal of B .

Then since $B \neq \emptyset$, $\exists m \in B$ such that $0 \in (m @ (m @ m)) @ m$ by Proposition 2.1.2 (iv),

$$\sup_{v \in (m @ (m @ m)) @ m} \eta(v) = \eta(0) \text{ and by Definitions 3.3.1 (ii) and 3.3.2 (ii) we have:}$$

$$\eta(0) = \sup_{v \in (m @ (m @ m)) @ m} \eta(v) \geq \min\{\eta(m), \eta(m)\} = \eta(m) \Rightarrow \eta(0) \geq \eta(m), \forall m \in B.$$

Note: Let η be a fuzzy hyper set in a hyper BCL–algebra. Then, generally;

η is fuzzy strong hyper ideal $\Rightarrow \eta$ is fuzzy hyper ideal $\Rightarrow \eta$ is fuzzy weak hyper ideal of B . \square

Remark 3.3.1. For a hyper BCL–algebra B , the converses in Proposition 3.3.1 (1), (2) and (3) are not necessarily true; in other words the following may not be necessarily true:

(1) Every fuzzy weak hyper ideal of B may not be fuzzy strong hyper ideal of B .

(2) Every fuzzy hyper ideal of B may not be fuzzy weak hyper ideal of B .

(3) Every fuzzy hyper ideal of B may not be fuzzy strong hyper ideal of B .

All of these statements, (1) – (3) above, are demonstrated by the subsequent counter example.

Example 3.3.2. Let the fuzzy subset η on the BCL–algebra B of Example 2.2 Table 2.2 be defined

$$\text{as follows: } \eta(m) = \begin{cases} 0.9, & \text{if } m = 0, \\ 0.7, & \text{if } m = q, \\ 0.4, & \text{if } m = p, r. \end{cases}$$

(1) Then, based on Table 2.2 of Example 2.2.2, η is a fuzzy weak hyper ideal but we have the following for fuzzy strong hyper ideal as axiom (iii) of fuzzy strong hyper ideal fails:

$$\begin{aligned} \text{(iii)} \quad \inf_{u \in (m @ (n @ w)) @ q} \eta(u) &\geq \min\{\eta(m), \eta(n)\}, \text{ for } (q @ (q @ q)) @ q = \{0, p, q, r\}, \text{ and} \\ \sup_{u \in (q @ (q @ q)) @ q} \eta(u) &= \sup\{\eta(0), \eta(p), \eta(q), \eta(r)\} = \sup\{0.9, 0.7, 0.4, 0.4\} = 0.9 \\ &\geq \min\{\eta(m), \eta(n), \eta(w)\} = \min\{\eta(q), \eta(q), \eta(q)\} = \eta(q) = 0.7 \end{aligned}$$

which is true axiom (iii) of fuzzy (weak) hyper ideal

$$\inf_{u \in (q @ (q @ q)) @ q} \eta(u) = \inf\{\eta(0), \eta(p), \eta(q), \eta(r)\} = \inf\{0.9, 0.7, 0.4, 0.4\} = 0.4 \geq 0.7$$

which is false showing that axiom (iii) for fuzzy strong hyper ideal fails. Hence η is a fuzzy (weak) hyper ideal but not fuzzy strong hyper ideal of B , when the hyper operation is as defined in Table 2.2 of Example 2.2.2, above.

(2) Based on Table 2.3 of Example 2.3.1, above, η is fuzzy weak hyper ideal but not fuzzy hyper ideal of B since $p @ q = \{q, r\}$ leads to show η is a fuzzy weak hyper ideal of B but not fuzzy hyper ideal of B as the second axiom of fuzzy weak hyper ideal holds while for fuzzy hyper ideal fails to hold as explained below:

$$\begin{aligned} \text{(ii)} \quad \eta(p) &\geq \min\{\eta(q), \inf_{u \in p @ q} \eta(u)\}; \quad \eta(p) \geq \min\{\eta(q), \sup_{u \in p @ q} \eta(u)\}; \quad p @ q = \{q, r\} \\ \eta(p) = 0.4 &\geq \min\{\eta(q), \inf_{u \in p @ q} \eta(u)\} = \min\{0.7, \min\{0.7, 0.4\}\} = \min\{0.7, 0.4\} = 0.4 \end{aligned}$$

which is true showing that η is weak hyper ideal, however, checking for fuzzy hyper ideal:

$$\eta(p) = 0.4 \geq \min\{\eta(q), \sup_{u \in p @ q} \eta(u)\} = \min\{0.7, \sup\{0.7, 0.4\}\} = \min\{0.7, 0.7\} = 0.7$$

which is false showing that η is not fuzzy hyper ideal of B .

Therefore, η is a fuzzy weak hyper ideal but not a fuzzy hyper ideal of B , when the hyper operation is as defined as in Table 2.3 of Example 2.3.1, above.

(3) η is fuzzy weak hyper ideal of $B \not\Rightarrow \eta$ is fuzzy strong hyper ideal of B and

η is fuzzy hyper ideal of $B \not\Rightarrow \eta$ is fuzzy weak hyper ideal of B .

$\Rightarrow \eta$ is fuzzy hyper ideal of $B \not\Rightarrow \eta$ is fuzzy strong hyper ideal of B , by transitivity

\Rightarrow If η is fuzzy hyper ideal of B then η may not be fuzzy strong hyper ideal of B , by transitivity

\Rightarrow Every fuzzy hyper ideal of B may not be fuzzy strong hyper ideal of B , by transitivity.

Theorem 3.3.1. Let η be a fuzzy weak hyper ideal of B such that η is injective and satisfies:

$$\eta(m) = \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\}, \quad \inf_{v \in (m @ (n @ w)) @ w} \eta(v) = \min\{\eta(m), \eta(n)\}, \quad \forall m, n \in B.$$

Then η is a fuzzy strong hyper ideal of B .

Proof. We are given that η is a fuzzy weak hyper ideal. To prove that η is a fuzzy strong hyper ideal, we must verify the following two axioms (as in Definition 3.3.3):

(i) $\eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\}, \quad \forall m, n \in B.$

(ii) $\inf_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n)\}, \quad \forall m, n, w \in B.$

(i) By assumption, for every $m, n \in B$, we have, $\eta(m) = \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\}.$

This is a stronger condition than what is required in the definition of fuzzy strong hyper ideal because: $\eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\}$ and hence axiom (i) is satisfied.

(ii) The key part is to check that: $\inf_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n), \eta(w)\}.$

We are given that $\forall v \in (m @ (n @ w)) @ w, \quad \eta(v) = \min\{\eta(m), \eta(n)\}.$

Taking the infimum over all such v gives: $\inf_{v \in (m @ (n @ w)) @ w} \eta(v) = \min\{\eta(m), \eta(n)\}.$

Therefore, $\inf_{v \in (m @ (n @ w)) @ w} \eta(v) \geq \min\{\eta(m), \eta(n)\},$ which establishes axiom (ii).

In conclusion, the injectivity of the fuzzy subset η ensures that no two distinct elements in B share the same membership degree, making the equality conditions meaningful and avoiding trivial cases where η is constant. This guarantees that the given equalities in the hypothesis enforce the exact structure required by axioms (i) and (ii). Since the two axioms of fuzzy strong hyper ideals are satisfied, we conclude that η is a fuzzy strong hyper ideal of B . □

Proposition 3.3.2. Let $\eta : B \rightarrow [0, 1]$ be fuzzy (weak, strong) hyper ideal of B and $m, n, w \in B$. Then the following hold:

(1) $\{n\} = m @ n \Rightarrow \eta(m) \geq \eta(n)$

(2) $\{m\} \prec \{n\} \Rightarrow \eta(m) \geq \eta(n);$

(3) $m \preceq n \Rightarrow \eta(m) \geq \eta(n).$

Proof. (1) Let $\{n\} = m @ n \subseteq I \Rightarrow \eta(n) = \sup_{u \in m @ n} \eta(u) = \inf_{u \in m @ n} \eta(u) = \eta(m @ n)$ and by

axiom (i) of either of Definitions 3.3.1, 3.3.2, 3.3.3, we have:

$$\eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\} = \min\{\eta(n), \eta(n)\} = \eta(n) \Rightarrow \eta(m) \geq \eta(n).$$

Therefore, $\eta(m) \geq \eta(n)$

(2) Let $\{m\} < \{n\} \Rightarrow 0 \in m @ n \Rightarrow \eta(0) = \sup_{u \in m @ n} \eta(u)$

By Definitions 3.3.2 (i), 3.3.3 (i) we have; $\eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\}$

$\Rightarrow \eta(m) \geq \min\{\eta(n), \eta(0)\} = \eta(n)$ as $\eta(0) = \eta(n), \forall n \in B$, by Proposition 3.3.1 (4)

$\Rightarrow \eta(m) \geq \eta(n)$

(3) Let $m \preceq n$. Thus $0 \in m @ n$.

Following similar patterns as (2), we obtain: $\eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\} \geq \eta(n)$ □

Lemma 3.3.1. *Let η be a fuzzy (weak, strong) hyper ideal of B . Then if η is onto then $\eta(0) = 1$.*

Proof. If η is onto, $\exists n \in B$ such that $\eta(n) = 1$ and since $\eta(0) \geq \eta(n), \forall n \in B$, implies $\eta(0) = 1$. Then Definitions 3.3.1 - enforces $\eta(0) \geq \eta(m), \forall m \in B$, (similar to the proof of Lemma 3.1.1). □

Theorem 3.3.2. *Let $\emptyset \neq M$ be a subset of a hyper BCL-algebra B and let $\delta, \sigma \in [0, 1]$ with $\delta > \sigma$.*

Let η be a fuzzy subset in B defined by $\eta(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M. \end{cases}$

Then η is a fuzzy (weak, strong) hyper ideal of B if and only if M is a (weak, strong) hyper ideal of B , respectively.

Proof. We prove the theorem by considering both directions (in a similar way as Theorem 3.2.2 except the difference with structures of the axioms so that we want to show some directions to relate these two theorems).

Assume that η is a fuzzy (weak, strong) hyper ideal of B .

We need to show that M is a (weak, strong) hyper ideal of B .

(i) For the defining conditions of fuzzy hyper ideals, if $m, n \in M$, then:

$$\eta(m) = \delta, \quad \eta(n) = \delta.$$

Thus, from the fuzzy hyper ideal axiom we have: $\delta = \eta(m) \geq \min\{\eta(n), \sup_{u \in m @ n} \eta(u)\}$.

Since $\delta > \sigma$, it follows that $\sup_{u \in m@n} \eta(u) = \delta$ so that $u \in M$ for all $u \in m@n$.

Hence, $m@n \subseteq M$.

(ii) Similarly, from axiom (ii) of fuzzy (weak, strong) hyper ideal we have:

$$\sup_{v \in (m@(n@w))@w} \eta(v) \geq \min\{\eta(m), \eta(n)\}.$$

If $m, n \in M$, then the right-hand side equals δ , so that $\sup_{v \in (m@(n@w))@w} \eta(v) = \delta$,

which implies $v \in M, \forall v \in (m@(n@w))@w$.

Therefore, M is a (weak, strong) hyper ideal of B .

Conversely, assume that M is a (weak, strong) hyper ideal of B .

Then, we need to show that η is a fuzzy (weak, strong) hyper ideal of B .

(i) Let $m, n \in B$. If $m \in M$, then $\eta(m) = \delta$.

Since M is a hyper ideal, for all $u \in m@n$ we have $u \in M$, and hence $\eta(u) = \delta$.

Therefore, $\eta(m) = \delta \geq \min\{\eta(n), \sup_{u \in m@n} \eta(u)\}$.

If $m \notin M$, then $\eta(m) = \sigma$. Since $\sigma < \delta$, the inequality still holds because the minimum on the right-hand side is always bounded above by σ .

(ii) For axiom (ii), let $m, n, w \in B$.

If $m, n \in M$, then for every $v \in (m@(n@w))@w$, by closure $v \in M$, so $\eta(v) = \delta$.

Hence, $\sup_{v \in (m@(n@w))@w} \eta(v) = \delta \geq \min\{\eta(m), \eta(n)\}$.

If at least one of m, n is not in M , then the right-hand side is σ , and the inequality holds automatically.

Thus η satisfies the axioms of fuzzy (weak, strong) hyper ideal of B .

Then, we conclude that η is a fuzzy (weak, strong) hyper ideal of B if and only if M is a (weak, strong) hyper ideal of B .

□

Corollary 3.3.1. *Let I be a nonempty subset of a hyper BCL–algebra B and χ be the Characteristic*

$$\text{function defined by: } \chi(m) = \begin{cases} 1, & \text{if } m \in I, \\ 0, & \text{if } m \notin I. \end{cases}$$

Then χ is a fuzzy (weak, strong) hyper ideal of B if and only if I is a (weak, strong) hyper ideal of B , respectively.

Proof. Following directly from Theorem 3.3.2, by setting $\delta = 1$ and $\sigma = 0$ so that χ is just the special case of η when the membership values are chosen at extreme levels of the interval $[0, 1]$. Therefore, χ is fuzzy (weak, strong) hyper DS of B if and only if I is (weak, strong) DS of B . \square

Theorem 3.3.3. *Let η be a fuzzy (weak, strong) hyper ideal of a hyper BCL–algebra B , and let $\alpha \in (0, 1]$. Then the non-empty set: $B_\eta^\alpha = \{m \in B : \eta(m) \geq \alpha\}$ is a (weak, strong) hyper ideal of B .*

Proof. Let B be hyper BCL–algebra and $\eta : B \rightarrow [0, 1]$ be fuzzy (weak, strong) hyper ideal in B such that $B_\eta^\alpha = \{m \in B : \eta(m) \geq \alpha\}$. Then we claim that B_η^α is (weak, strong) hyper ideal of B .

(1) Let $n \in B_\eta^\alpha$ and $m@n \subseteq B_\eta^\alpha$, then $\eta(v) \geq \alpha, \forall v \in m@n$ and $\eta(n) \geq \alpha$

$$\Rightarrow \sup_{v \in m@n} \eta(v) \geq \alpha \text{ and } \eta(n) \geq \alpha.$$

From a fuzzy hyper ideal (Definition 3.3.1 (i)), we have: $\eta(m) \geq \min\{\eta(n), \sup_{u \in m@n} \eta(u)\}$.

$$\Rightarrow \eta(m) \geq \min\{\eta(n), \sup_{u \in m@n} \eta(u)\} \geq \min\{\alpha, \alpha\} \Rightarrow \eta(m) \geq \alpha \Rightarrow m \in B_\eta^\alpha,$$

Thus, the first axiom for hyper ideal holds.

(2) Let $m, n, w \in B_\eta^\alpha \Rightarrow \eta(m) \geq \alpha, \eta(n) \geq \alpha$ and from Definition 3.3.1 (ii):

$$\sup_{v \in (m@(n@w))@w} \eta(v) \geq \min\{\eta(n), \eta(m)\} \geq \min\{\alpha, \alpha\} = \alpha$$

$$\Rightarrow \sup_{v \in (m@(n@w))@w} \eta(v) \geq \alpha \Rightarrow v \in B_\eta^\alpha, \forall v \in (m@(n@w))@w \Rightarrow (m@(n@w))@w \subseteq B_\eta^\alpha$$

Thus, the second axiom for hyper ideal holds.

Therefore, By (1) and (2) above, B_η^α is hyper ideal of B .

(3) Since $0 \in B \Rightarrow \eta(0) \geq \eta(m) \geq \alpha \Rightarrow \eta(0) \geq \alpha \Rightarrow 0 \in B_\eta^\alpha$

Thus, the first axioms for weak hyper ideal strong hyper ideal hold.

(4) Let $n \in B_\eta^\alpha$ and $m@n \prec B_\eta^\alpha \Rightarrow \eta(n) \geq \alpha$ and $(\exists u \in m@n) (\exists v \in B_\eta^\alpha) (0 \in u@v)$

$$\Rightarrow \eta(n) \geq \alpha \text{ and } \sup_{w \in u@v} \eta(w) \geq \alpha \Rightarrow \eta(n) \geq \alpha \text{ and } \sup_{u \in m@n} \eta(u) \geq \alpha$$

From Definition 3.3.1 (i): $\eta(m) \geq \min\{\eta(n), \sup_{u \in m@n} \eta(u)\} \geq \min\{\eta(n), \inf_{u \in m@n} \eta(u)\}$

$$\geq \min\{\alpha, \inf_{u \in m@n} \eta(u)\} \geq \alpha \Rightarrow \eta(m) \geq \alpha \Rightarrow m \in B_\eta^\alpha$$

Therefore, by (2), (3) and (4) above, B_η^α is weak hyper ideal of B .

(5) By (3) for the first axiom, following similar patterns as for the above for the second and third axioms of definition of strong hyper ideal, we can show that B_η^α is strong hyper ideal of B . \square

Lemma 3.3.2. Let $(B; @, 0)$ be a hyper BCL-algebra and η be fuzzy (weak, strong) hyper ideal of B such that $m, n, w \in B$. Then the following hold:

$$(1) m@n \preceq \{w\} \Rightarrow \eta(m) \geq \eta(n);$$

$$(2) m@n \prec \{w\} \Rightarrow \eta(m) \geq \eta(n)$$

Proof. (1) Let $m @ n \preceq \{w\} \Rightarrow u \preceq w, \forall u \in m @ n$

By the fuzzy hyper ideal property, (order-reversing), $\eta(u) \geq \eta(w)$

By Definition 3.3.1 (ii): $\eta(m) \geq \min\{\eta(n), \inf_{u \in m @ n} \eta(u)\}$

$$\Rightarrow \inf_{u \in m @ n} \eta(u) \geq \eta(w) \text{ and } \eta(w) \geq \eta(n) \Rightarrow \eta(m) \geq \eta(n)$$

(2) Let $m @ n \prec \{w\}$. Then $\exists u \in m @ n \ni u \preceq w$.

Then, following similar patterns as (1) above, we get:

$$\eta(m) \geq \min\{\eta(n), \eta(u)\}. n \in I \text{ and } m @ n \preceq \{w\} \Rightarrow \eta(m) \geq \eta(n)$$

□

Remark 3.3.2. In a hyper BCL-algebra, there is no correspondence among fuzzy (weak, strong) hyper subalgebras, fuzzy (weak, strong) hyper deductive systems, and fuzzy (weak, strong) hyper ideals, as for one of the cases that a fuzzy hyper SA may not be fuzzy hyper ideal and the converse does not hold either is demonstrated by the following counterexample, and also similar results apply to the other similar correspondences

Example 3.3.3. Suppose $(B; @, 0)$ is a hyper BCL-algebra where $B = \{0, p, q, r\}$ is as defined in Table 2.3 and define two fuzzy subsets η_1 and η_2 as follows:

$$\eta_1(m) = \begin{cases} 1, & \text{if } m = 0 \\ 0.8, & \text{if } m = q, \\ 0.3, & \text{if } m = p, r \end{cases} \quad \eta_2(m) = \begin{cases} 0.8, & \text{if } m = p, \\ 0.3, & \text{if } m = 0, q, r \end{cases}$$

Then, η_1 is fuzzy hyper SA but not fuzzy hyper ideal and η_2 is fuzzy hyper ideal but not fuzzy hyper SA as justified below:

(1) To show that η_1 is fuzzy hyper SA but not fuzzy hyper ideal of B , we take $m = r, n = q$, as a demonstrating example:

$$(i) \eta_1(0) = 1 \geq \eta_1(m), \forall m \in B$$

$$(ii) \inf_{u \in m @ n} \eta_1(u) = \inf_{u \in r @ q} \eta_1(u) = \inf_{u \in \{0, p\}} \eta_1(u) = 0.3 \geq \min\{\eta_1(r), \eta_1(q)\} = \min\{0.8, 0.3\} = 0.3$$

holds and also it holds for all the others in similar ways and hence η_1 is fuzzy hyper SA of B .

However, since $\eta_1(m) \eta_1(r) = 0.3 \geq \{\eta_1(n), \sup_{u \in (m@n)} \eta_1(u)\} = \{\eta_1(q), \sup_{u \in (r@q)} \eta_1(u)\}$
 $= \{\eta_1(q), \sup_{u \in (0, p)} \eta_1(u)\} = \{0.8, \max\{1, 0.3\}\} = \min\{0.8, 1\} = 0.8$ which is false so that

η_1 is not fuzzy hyper ideal of B .

(2) To show η_2 is fuzzy hyper ideal but not fuzzy hyper SA, we take $m = r$, $n = q$, $w = r$ as a demonstrating example:

$$(i) \eta_2(m) \eta_2(r) = 0.3 \geq \{\eta_2(n), \sup_{u \in (m@n)} \eta_2(u)\} = \{\eta_2(q), \sup_{u \in (r@q)} \eta_2(u)\}$$

$$\inf_{u \in r@q} \eta_2(u) = \inf_{u \in \{0, p\}} \eta_2(u) = \min\{\eta_2(0), \eta_2(p)\} = 0.3 \geq \min\{\eta_2(r), \eta_2(q)\}$$

$$= \min\{0.3, 0.8\} = 0.3 \text{ holds and similarly for the other choices of } m, n \in B.$$

(ii) Since $(m@(n@w))@w = (r@(q@r))@r = (r@{q, r})@r = \{0, p, r\}@r = \{0, p, r\}$

$$\sup_{u \in (m@(n@w))@w} \eta_2(u) = \sup_{u \in (r@(q@r))@r} \eta_2(u) = \sup_{u \in \{0, p, r\}} \eta_2(u) = \max\{\eta_2(0), \eta_2(p), \eta_2(r)\}$$

$$= \max\{0.3, 0.8, 0.3\} = 0.8 \geq \min\{\eta_2(r), \eta_2(q)\} = \min\{0.3, 0.3\} = 0.3 \text{ holds}$$

$$\inf_{u \in r@q} \eta_2(u) = \inf_{u \in \{0, p\}} \eta_2(u) = \min\{\eta_2(0), \eta_2(p)\} = 0.3 \geq \min\{\eta_2(r), \eta_2(q)\}$$

$$= \min\{0.3, 0.8\} = 0.3 \text{ holds and similarly for the other choices of } m, n \in B.$$

Hence, η_2 is fuzzy hyper ideal of B ; however, η_2 is not fuzzy hyper SA of B by the next reason:

$\eta_2(0) = 0.3 \geq \eta_2(p) = 0.8$ is false which tells us that one of the axioms for fuzzy hyper SA does not hold so that η_2 is not fuzzy hyper SA of B .

Chapter 4

Pythagorean Fuzzy Structures of BCL–Algebra

In this chapter, we introduce the notions of a fuzzy subalgebra, fuzzy deductive system, fuzzy ideal and Pythagorean fuzzy subalgebra, Pythagorean fuzzy deductive system, and Pythagorean fuzzy ideal of BCL–algebra. Alongside, we state and prove different theorems, explore the relations among each structures, discuss some properties of the operations like intersection and union of each of the structures, and attempt to investigate some relevant results.

For this section and the subsequent sections; unless otherwise specified, we denote B and B^P for "BCL–Algebra $(B; \otimes, 0)$ " and "Pythagorean fuzzy set (η, τ) ", respectively, where the functions: $\eta : L \rightarrow [0, 1]$ and $\tau : L \rightarrow [0, 1]$ define the degrees of membership and non-membership, respectively, satisfying the condition: $0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1$.

*In this chapter and the subsequent ones, we shall use the abbreviations **SA** and **DS** to denote **subalgebra** and **deductive system**, respectively.*

4.1. Fuzzy Substructures of BCL–Algebra

Under this section, we define fuzzy subalgebra (SA) of BCL–algebra and then discuss some properties and theorems on fuzzy SA of BCL–algebra accompanied by corresponding proofs.

In this section, we newly define SA, fuzzy SA, DS, fuzzy DS, ideal and fuzzy ideal of the BCL–algebra; $(B, \otimes, 0)$ one by one and then following each of these definitions, we explore different relevant properties as theorems, lemmas, propositions with proofs.

Definition 4.1.1. *Let $\eta : B \rightarrow [0, 1]$ be a fuzzy subset on the BCL–algebra, B . Then we call the fuzzy subset $\bar{\bar{\eta}}$ on B is square deviation of η , if and only if $(\bar{\bar{\eta}}(m))^2 = 1 - (\eta(m))^2, \forall m \in B$.*

Remark 4.1.1.

(1) For the square deviation $\bar{\eta}$ on B , $\bar{\eta} : B \rightarrow [-1, 1]$ since $\forall m \in B$, $(\bar{\eta}(m))^2 = 1 - (\eta(m))^2$
 $\Rightarrow \bar{\eta}(m) = \pm \sqrt{1 - (\eta(m))^2} \Rightarrow \bar{\eta} : B \rightarrow [-1, 1]$.

(2) The Square deviation $\bar{\eta}$ could be also defined in any non-empty set R of any algebra like BCI/BCK-algebra as $\bar{\eta} : R \rightarrow [-1, 1]$.

(3) For a fuzzy subset $\eta : R \rightarrow [0, 1]$ and its square deviation $\bar{\eta} : R \rightarrow [0, 1]$ of a non-empty set

$$R \text{ we have the following: } (\eta(m))^2 + (\bar{\eta}(m))^2 = (\eta(m))^2 + (1 - (\eta(m))^2) = 1$$

$$\Rightarrow (\eta(m))^2 + (\bar{\eta}(m))^2 = 1 \Rightarrow 0 \leq (\eta(m))^2 + (\bar{\eta}(m))^2 \leq 1$$

Proposition 4.1.1. Let $(B; \otimes, 0)$ be a BCL-algebra and let $\eta : B \rightarrow [0, 1]$ be a fuzzy subset. For the complement $\bar{\eta}$ and square deviation $\bar{\eta}$ of η , we have the following:

- (1) $(\bar{\eta}(m))^2 \geq \bar{\eta}(m)$, $\forall m \in B$,
- (2) $(\bar{\eta}(m))^2 \geq (\bar{\eta}(m))^2$, $\forall m \in B$.

Proof. (1) Since $\eta(m) \in [0, 1]$, $\forall m \in B \Rightarrow \eta(m) \geq (\eta(m))^2 \Rightarrow -\eta(m) \leq -(\eta(m))^2$
 $\Rightarrow 1 - \eta(m) \leq 1 - (\eta(m))^2 \Rightarrow 1 - \eta(m) = \bar{\eta} \leq 1 - (\eta(m))^2 = (\bar{\eta}(m))^2$
 $\Rightarrow \bar{\eta}(m) \leq (\bar{\eta}(m))^2$ or $(\bar{\eta}(m))^2 \geq \bar{\eta}(m)$

(2) $(\bar{\eta}(m))^2 = (1 - \eta(m))^2 = (1 - \eta(m))(1 - \eta(m)) \leq (1 - \eta(m))(1 + \eta(m))$
 $= 1 - (\eta(m))^2 = (\bar{\eta}(m))^2$
 $\Rightarrow (\bar{\eta}(m))^2 \leq (\bar{\eta}(m))^2$ or $(\bar{\eta}(m))^2 \geq (\bar{\eta}(m))^2$

□

Definition 4.1.2. Let S be a nonempty subset of the BCL-algebra, B . Then S is called subalgebra (SA) of B if and only if $m \otimes n \in S$, $\forall m, n \in S$.

Example 4.1.1. Let $B = \{0, p, q, r\}$ be a set and let \otimes as defined in Table 1.1 of Example 1.1.1 above. Then $(B; \otimes, 0)$ is a BCL-algebra (See [42]), and besides, we have the following:

- (1) $\{0\}$, $\{0, p\}$, $\{0, q\}$, $\{0, r\}$, $\{0, p, r\}$, B are SAs of B ,
- (2) $\{0, p, q\}$, $\{p, q\}$, $\{p, r\}$, $\{q, r\}$, $\{p, q, r\}$ are not SAs of B , for $0 \neq p$, $0 \neq q$, $0 \neq r$.

Exampes 4.1.1 (1) and (2) are explained as follows taking some cases :

(1) $0 \otimes 0 = 0 \in \{0\}$, $0 \otimes 0 = 0 \otimes p = p \otimes p = 0 \in \{0, p\}$, $p \otimes 0 = p \in \{0, p\}$;
 $0 \otimes p = 0 \otimes r = p \otimes p = r \otimes r = r \otimes p = 0 \otimes 0 = 0 \in \{0, p, r\}$,
 $p \otimes 0 = p \otimes r = p = \{0, p, r\}$, and so on, and hence all listed in (1) are SAs of B .

(2) $p \otimes q = r \notin \{0, p, q\}$; $p \otimes q = r \notin \{p, q\}$; $r \otimes p = 0 \notin \{p, r\}$; $r \otimes q = 0 \notin \{q, r\}$;
 $r \otimes p = 0 \notin \{p, q, r\}$ and hence all listed in (2) are not SAs of B .

Lemma 4.1.1. *Let B be a BCL–algebra and S be SA of B , then $0 \in S$.*

Proof. Since $S \neq \emptyset$, $\exists m \in S$ such that $m \otimes m = 0 \in S \Rightarrow 0 \in S$ (by Definition 1.1.8 (i)) □

Proposition 4.1.2. *Let S be SA of a BCL–algebra B . Then $\forall m \in S$, $m \otimes 0 \in S$.*

Proof. By Lemma 4.1.1, we have $0 \in S$ and by definition of SA, for $m \in S$, we have:

$0 \otimes m$, $m \otimes 0 \in S \Rightarrow m \otimes 0 \in S$. □

Theorem 4.1.1. *Let B be a BCL–algebra. Then B and $\{0\}$ are (trivial) SAs of B .*

Proof. Since B is a BCL–algebra, B is closed under \otimes and by definition and $0 \otimes 0 = 0 \in \{0\}$.

Therefore B and $\{0\}$ are (trivial) SA of B . □

Definition 4.1.3. *A non-empty subset D of BCL–algebra $(B; \otimes, 0)$ is called a deductive system (DS) of B if it satisfies, $\forall m, n, u, w \in B$:*

(i) $m \in D \Rightarrow (m \otimes u) \otimes u \in D$,

(ii) $m, n \in D \Rightarrow m \otimes (n \otimes w) \in D$.

Example 4.1.2. *Let \otimes be as defined in Table 1.1 of Example 1.1.1 of $B = \{0, p, q, r\}$.*

Then in Table 1.1, it has been shown that B is a BCL–algebra. Furthermore, by mechanical calculations, we have checked that:

(1) $\{0\}$, $\{0, p, r\}$, $\{0, q, r\}$ and B are DSs of B but

(2) $\{p\}$, $\{q\}$, $\{r\}$, $\{0, p\}$, $\{0, q\}$, $\{p, r\}$, $\{p, r\}$, $\{q, r\}$, $\{0, p, q\}$, $\{p, q, r\}$ are not DSs of B .

We can explain Example 4.1.2 by looking at a few cases to show the pattern for the others in the following ways:

(1) Take $D = \{0, p, r\}$ and $u = w = q$. Then:

(i) $(0 \otimes q) \otimes q = 0 \otimes q = 0 \in D$, $(p \otimes q) \otimes q = r \otimes q = 0 \in D$, $(r \otimes q) \otimes q = 0 \otimes q = 0 \in D$, and so on

(ii) $0 \otimes (p \otimes q) = 0 \otimes r = 0 \in D$, $p \otimes (r \otimes q) = p \otimes 0 = p \in D$, $(r \otimes (p \otimes q)) = r \otimes r = 0 \in D$, and so on

(2) Take $E = \{0, p, q\}$ and $w = r \in B$ and we justifying that in one of the expressions one axiom fails to be satisfied and hence:

$(q \otimes (p \otimes r)) = q \otimes p = r \notin E$, showing that the second axiom for DS of B fails, and so on

Lemma 4.1.2. *If D is a DS of the BCL–algebra, $(B; \otimes, 0)$, then $0 \in D$.*

Proof. Suppose D is DS of B .

$D \neq \emptyset \Rightarrow \exists m \in D$ such that $(m \otimes m) \otimes m \in D \Rightarrow 0 \otimes m = 0 \in D$ (by Definition 1.1.8 (i)). \square

Proposition 4.1.3. *Let D be DS of a BCL–algebra B . If $m \in D$ then $m \otimes 0 \in D$*

Proof. Let D be DS of a BCL–algebra B such that $m \in D$. Also by Lemma 4.1.2

$0 \in D$. Now take $n = 0$ and using Definition 4.1.3 (ii), we have, $m, n \in D \Rightarrow m \otimes (n \otimes w) \in D \Rightarrow m \otimes (0 \otimes w) \in D \Rightarrow m \otimes 0 \in D$, since $0 \otimes w = 0$ by Proposition 2.1.1 (3).

Alternatively, since $m \otimes 0 = m$, $\forall m \in B$ by Proposition 2.1.1 (5), then $m \in D \Rightarrow m \otimes 0 = m \in D$.

Therefore, $m \otimes 0 \in D$, $\forall m \in D$. \square

Definition 4.1.4. *For a non-empty-subset I of B , we say that I is called an ideal of B if and only if the following axioms are satisfied, $\forall m, n, w \in B$*

- (i) $m \otimes n \in I$ and $n \in I \Rightarrow m \in I$, and
- (ii) $m, n \in I \Rightarrow (m \otimes (n \otimes w)) \otimes w \in I$.

Example 4.1.3. *Let the BCL–algebra $(B; \otimes, 0)$ be as defined in Table 1.1 of Example 1.1.1. Thus from the table, the following can be easily obtained:*

- (1) $\{0\}$, $\{0, r\}$, B are ideals of BCL–algebra, B ,
- (2) $\{p\}$, $\{0, p\}$, $\{p, r\}$, $\{q, r\}$, $\{0, p, q\}$ and $\{0, p, r\}$ are not ideals of B .

We can explain Example 4.1.3 by looking at a few cases to show the pattern for the others in the following ways:

(1) Take $I = \{0, r\}$:

(i) $0, r \otimes 0 = r \in I \Rightarrow q \in I$ and $r, 0 \otimes r = 0 \in I \Rightarrow 0 \in I$, hold, and so on.

(ii) Let $w = p$: $(0 \otimes (r \otimes p)) \otimes p = (0 \otimes 0) \otimes p = 0 \otimes p = 0 \in I$ holds,

and $(r \otimes (0 \otimes p)) \otimes p = (r \otimes 0) \otimes p = r \otimes p = 0 \in I$ holds, and so on.

(2) Take $J = \{0, p\}$, $w = q$, then: $(p \otimes (0 \otimes q)) \otimes q = (p \otimes 0) \otimes q = p \otimes q = r \notin J$,

that is one condition for one of the axioms fails, and so on.

Lemma 4.1.3. *If I is an ideal of a BCL–algebra, $(B; \otimes, 0)$, then $0 \in I$.*

Proof. Suppose I is an ideal of a BCL–algebra B .

$I \neq \emptyset \Rightarrow \exists m \in I$ such that $(m \otimes (m \otimes m)) \otimes m \in I$ (by Definition 4.1.4 (ii))

$\Rightarrow (m \otimes 0) \otimes m \in I$, since $m \otimes m = 0$ by Definition 1.1.8 (i)

$\Rightarrow m \otimes m \in I$, since $m \otimes 0 = m$ by Proposition 2.1.1 (4)

$\Rightarrow m \otimes m = 0 \in I$ (since $m \otimes m = 0$ by Definition 1.1.8 (i)). □

Proposition 4.1.4. *Let I be an ideal of a BCL–algebra with $m, n \in B$ and \leq be the binary relation on B . If $m \leq n$ and $n \in I$ then $m \in I$.*

Proof. Suppose I is an ideal of a BCL–algebra B and \leq be the binary relation on B . Then $\forall m, n \in B$, we have $m \otimes n = 0 \in I$. Now $m \otimes n = 0 \in I, n \in I \Rightarrow m \in I$, by definition of ideal in B . □

Corollary 4.1.1. *Let I be an ideal of a BCL–algebra with $m, n, p \in B$ and \leq be the binary relation on B . If $n \in I, p \leq n$ and $m \otimes p \in I$ then $m \in I$.*

Proof. Suppose I is an ideal of a BCL–algebra B , \leq is the binary relation on B and $m, n \in B$, such that $n \in I, p \leq n$ and $m \otimes p \in I$. Then we have $n \in I$ and $p \otimes n = 0 \in I \Rightarrow p \in I$ by Proposition 4.1.4. Again, $p \in I$ and $m \otimes p \in I \Rightarrow m \in I$, by Definition 4.1.4 (i). □

Theorem 4.1.2. *If I is an ideal of a BCL–algebra B then I is a SA of B .*

Proof. Let $(B; \otimes, 0)$ be a BCL–algebra. Suppose I is an ideal of B . We need to show that I is SA of B ; in other words, we claim that I is closed under \otimes .

Now, suppose $m, n \in I \subseteq B \Rightarrow (m \otimes (n \otimes w)) \otimes w \in I, \forall w \in B$.

Since $w \in B$ is arbitrary, choose $w = n \Rightarrow (m \otimes (n \otimes n)) \otimes n \in I$

$\Rightarrow (m \otimes 0) \otimes n \in I$, since $n \otimes n = 0$ by Definition 1.1.8 (i)

$\Rightarrow m \otimes n \in I$, since $m \otimes 0 = m$ by Proposition 2.1.1 (5)

Therefore, $m \otimes n \in I \Rightarrow I$ is SA of B . □

Remark 4.1.2. *For the converses of Theorem 4.1.2, we state the following remarks which are illustrated by the next example:*

(1) *The converse of Theorem 4.1.2 is not necessarily true, .*

(2) *Except Theorem 4.1.2, a SA may not be deductive system, a deductive system may be neither SA nor ideal and ideal may not be deductive system.*

Example 4.1.4. We justify the above remark by the following counter examples:

- (1) From Example 4.1.1, it has been checked and listed that $\{0, p, r\}$ is SA of B ; however, from Example 4.1.3, it has been checked and listed that $\{0, p, r\}$ is not ideal of B .
- (2) For other than (1), we provide the following counter examples from Examples 4.1.1, 4.1.2 and 4.1.3 given above
 - (i) From Example 4.1.1, it has been checked and listed that $\{0, p\}$ is SA of B ; however, from Example 4.1.2, it has been checked and listed that $\{0, p\}$ is not deductive system of B .
 - (ii) From Example 4.1.2, it has been checked and listed that $\{0, p, q\}$ is deductive system of B ; however, from Example 4.1.1, it has been checked that $\{0, p, q\}$ is not SA of B .
 - (iii) From Example 4.1.2, it has been checked and listed that $\{0, p, q\}$ is deductive system of B ; however, from Example 4.1.3, it has been checked and listed that $\{0, p, q\}$ is not ideal of B .
 - (iv) From Example 4.1.3, it has been checked and listed that $\{0, q\}$ is ideal of B ; however, from Example 4.1.2, it has been checked and listed that $\{0, p, r\}$ is not deductive system of B .

4.1.1 Fuzzy Subalgebra of BCL–Algebra

Under this subsection, we introduce and define SA and fuzzy SA of BCL–algebra which had not been defined nor discussed in the prior literatures so far.

Definition 4.1.1.1. Let η be a fuzzy subset in B . Then η is called a fuzzy SA of B if and only if $\eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\}$, $\forall m, n \in B$.

Example 4.1.1.1. Let $B = \{0, p, q, r\}$ and \otimes be a binary operation as defined in Table 1.1 and

$$\text{let the fuzzy subset: } \eta : B \rightarrow [0, 1] \text{ be defined as: } \eta(m) = \begin{cases} 0.7, & \text{if } m = 0, \\ 0.3, & \text{if } m = q, \\ 0.1, & \text{if } m = p, r. \end{cases}$$

Then $(B; \otimes, 0)$ is a BCL–algebra (see Table 1.1), and besides, η is fuzzy SA of B by the following explanations for some cases and possibly, it will be similar to explain the remaining cases:

Take $m = p$, and $n = q \Rightarrow p \otimes q = r$, and then

$$\eta(p \otimes q) = \eta(r) = 0.1 \geq \min\{\eta(p), \eta(q)\} = \min\{0.1, 0.3\} = 0.1 \text{ holds, and so on.}$$

Lemma 4.1.1.1. Let B be a BCL–algebra and $\eta : B \rightarrow [0, 1]$ be a fuzzy subset. If η is a fuzzy SA of B , then $\eta(0) \geq \eta(m)$, $\forall m \in B$.

Proof. For $m \in B$, we have: $m \otimes m = 0$. Then by definition of fuzzy SA of B , we have;

$$\eta(0) = \eta(m \otimes m) \geq \min\{\eta(m), \eta(m)\} = \eta(m) \Rightarrow \eta(0) \geq \eta(m) \quad \square$$

Proposition 4.1.1.1. *Let S be a non-empty subset of a BCL-algebra B and χ_S be the Characteristic function. Then χ_S is fuzzy SA of B if and only if S is SA of B .*

Proof. Suppose $\chi_S : S \rightarrow \{0, 1\}$ is a Characteristic function defined as: $\chi_S(m) = \begin{cases} 1, & \text{if } m \in S, \\ 0, & \text{if } m \notin S. \end{cases}$

Let χ_S be fuzzy SA of B , and let $m, n \in S$. Now, we need to show S is SA of B . Then,

$m \otimes n \in S \Rightarrow \chi_S(m) = \chi_S(n) = 1$ and $\chi_S(m \otimes n) \geq \min\{\chi_S(m), \chi_S(n)\} = \min\{1, 1\} = 1$
 $\Rightarrow \chi_S(m \otimes n) \geq 1$. But $1 \geq \chi_S(m \otimes n)$ and by definition of χ_S , we have:

$$\chi_S(m \otimes n) = 1 \Rightarrow m \otimes n \in S$$

Hence, $m, n \in S \Rightarrow m \otimes n \in S$ and therefore S is a SA of B .

Conversely, suppose S is a SA of B

We need show that $\chi_S(m \otimes n) \geq \min\{\chi_S(m), \chi_S(n)\}, \forall m, n \in B$.

Then following three cases we show: $\chi_S(m \otimes n) \geq \min\{\chi_S(m), \chi_S(n)\}, \forall m, n \in B$.

Case(1) : If $m, n \in S$, then $m \otimes n \in S$ (by the closure property of the binary operation \otimes)
 $\Rightarrow \chi_S(m \otimes n) = 1 \geq \min\{1, 1\} = \min\{\chi_S(m), \chi_S(n)\} \Rightarrow \chi_S(m \otimes n) \geq \min\{\chi_S(m), \chi_S(n)\}$,

Case(2) : If $m \notin S, n \notin S$, (but $m, n \in B$), then, $\chi_S(m) = 0 = \chi_S(n)$, and $\chi_S(m \otimes n) \geq 0$
 $\Rightarrow \chi_S(m \otimes n) \geq 0 = \min\{\chi_S(m), \chi_S(n)\} \Rightarrow \chi_S(m \otimes n) \geq \min\{\chi_S(m), \chi_S(n)\}$

Case(3) : If $m \in S$ and $n \notin S$ (or $n \in S$ and $m \notin S$)

$$\Rightarrow \chi_S(m) = 1 \text{ and } \chi_S(n) = 0 \text{ (or } \chi_S(n) = 1 \text{ and } \chi_S(m) = 0)$$

$$\Rightarrow \min\{\chi_S(m), \chi_S(n)\} = \min\{1, 0\} = 0 \text{ and } \chi_S(m \otimes n) \geq 0$$

$$\Rightarrow \chi_S(m \otimes n) \geq 0 = \min\{\chi_S(m), \chi_S(n)\} \Rightarrow \chi_S(m \otimes n) \geq \min\{\chi_S(m), \chi_S(n)\}$$

Hence, in all the cases (1 – 3), we have checked that:

$$\chi_S(m \otimes n) \geq \min\{\chi_S(m), \chi_S(n)\}, \forall m, n \in B.$$

Thus, χ_S is a membership function of B and hence, χ_S is fuzzy SA of B . □

Theorem 4.1.1.1. *Let $\emptyset \neq S \subseteq B$ and consider the fuzzy subset η of B such that:*

$$\eta(m) = \begin{cases} \delta, & \text{if } m \in S, \\ \gamma, & \text{otherwise,} \end{cases} \quad \forall \gamma, \delta \in [0, 1] \text{ with } \gamma \leq \delta$$

Then, η is fuzzy SA of B if and only if S is SA of B .

Proof. Similar to the proof of Theorem 4.1.1.1, above, by replacing δ for 1 and γ for 0. □

Theorem 4.1.1.2. Let η be a fuzzy subset in a BCL–algebra B and $\bar{\eta}$ be its square deviation such that the sets: $B_\eta = \{m \in B : \eta(m) = \eta(0)\}$ and $B_{\bar{\eta}} = \{m \in B : \bar{\eta}(m) = 1 - \eta(0) = \bar{\eta}(0)\}$.

Then B_η and $B_{\bar{\eta}}$ are SAs of B .

Proof. Let B be a BCL–algebra and $m, n \in B$. We prove that: B_η and $B_{\bar{\eta}}$ are SAs of B .

Let $m, n \in B_\eta$ or $\eta(m) = \eta(0) = \eta(n)$ and $\bar{\eta}(m) = 1 - \eta(0) = \bar{\eta}(n) = \bar{\eta}(m)$

Then $\eta(m \otimes n) = \eta(0) = \eta(m) = \eta(n) \geq \min\{\eta(m), \eta(n)\}$

And $\bar{\eta}(m \otimes n) = 1 - \eta(0)$

$$\geq \min\{1 - \bar{\eta}(0), 1 - \bar{\eta}(0)\} = \min\{\bar{\eta}(m), \bar{\eta}(n)\}$$

Thus, B_η and $B_{\bar{\eta}}$ are SAs of B . □

Theorem 4.1.1.3. Let η be a fuzzy subset in B . Then, for $m, n \in B$, η is fuzzy SA of B if $\eta(m \otimes n) = \eta(0), \forall m, n \in B$.

Proof. Let η be a fuzzy subset in B such that $\eta(m \otimes n) = \eta(0), \forall m, n \in B$ is a fuzzy subset in B .

Then we need to prove that η is fuzzy SA of B .

$$\eta(m \otimes n) = \eta(0) = \min\{\eta(0), \eta(0)\} \geq \min\{\eta(m), \eta(n)\} \Rightarrow (\eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\})$$

Thus η fuzzy a SA of B . □

Remark 4.1.1.1. The converse of the preceding theorem is not true which we confirm it by the example below.

Example 4.1.1.2. Let $B = \{p, q, r, 0\}$ and the binary operation \otimes be as defined in Table 1.1 of Example 1.1.1 above, and define a fuzzy subset η as follows:

$$\eta(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.6, & \text{if } m = p, \\ 0.2, & \text{if } m = q, r. \end{cases}$$

Then from the BCL–algebra $(B; \otimes, 0)$, η is a fuzzy SA of B .

But by taking $m = q$ and $n = p$, we have: $m \otimes n = q \otimes p = r$

$$\eta(m \otimes n) = \eta(q \otimes p) = \eta(r) = 0.2 \neq \eta(0) = 0.8$$

Then η is a fuzzy SA of B does not imply $\eta_B(m \otimes n) = \eta_B(0)$.

Theorem 4.1.1.4. *The intersection of any two fuzzy SAs η_1 and η_2 of B is also a fuzzy SAs of B .*

Proof. Let η_1 and η_2 be any fuzzy SA of B with $m, n, w \in B$.

Then we need to prove that $\eta_1 \cap \eta_2$ is a fuzzy SA of B .

$$\begin{aligned} \eta_1 \cap \eta_2(m \otimes n) &= \min\{(\eta_1(m \otimes n), \eta_2(m \otimes n))\} \\ &\geq \min\{\min\{\eta_1(m), \eta_1(n)\}, \min\{\eta_2(m), \eta_2(n)\}\} \\ &= \min\{\min\{\eta_1(m), \eta_2(m)\}, \min\{\eta_1(n), \eta_2(n)\}\} \\ &= \min\{\eta_1 \cap \eta_2(m), \eta_1 \cap \eta_2(n)\} \text{ and} \end{aligned}$$

Hence, η_1 and η_2 is a fuzzy SA of B . □

The above theorem can also be generalized to any family of fuzzy SAs of B as in the following corollary.

Corollary 4.1.1.1. *The intersection of any family of fuzzy SAs, η_i where $\{\eta_i : i \in I\}$ in B*

is also a fuzzy SA of B , where $\bigcap_{i \in I} \eta_i(m) = \inf_{i \in I} \{\eta_i(m)\}$.

Proof. Let $\eta_i, \forall i \in I$, be family of fuzzy SAs of B with $m, n, w \in B$.

Then we need to prove that $\bigcap_{i \in I} \eta_i$ is a fuzzy SA of B .

$$\begin{aligned} \bigcap_{i \in I} \eta_i(m \otimes n) &= \inf_{i \in I} \{\eta_i(m \otimes n)\} \\ &\geq \inf_{i \in I} \{\min\{\eta_i(m), \eta_i(n)\}\} \\ &\geq \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\ &= \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\ &= \min\{\bigcap_{i \in I} \eta_i(m), \bigcap_{i \in I} \eta_i(n)\} \end{aligned}$$

Hence, $\bigcap_{i \in I} \eta_i$ is a fuzzy SA of B . □

Remark 4.1.1.2. *The union of any two fuzzy SAs of a BCL–algebra B need not be fuzzy SA of B , which we show it by a counter example as follows.*

Example 4.1.1.3. *Let $(B; \otimes, 0)$, where $B = \{p, q, r, 0\}$ be BCL–algebra with binary operation " \otimes " defined by the Table 4.1 below, and define two fuzzy SAs of B , say η_1 and η_2 as follows:*

\otimes	0	p	q	r
0	0	0	0	0
p	p	0	r	p
q	q	r	0	q
r	r	0	p	0

Table 4.1: A Cayley table showing BCL–algebra $(B; \otimes, 0)$ (See [42])

From Table 4.1 above, $(B; \otimes, 0)$ is BCL–algebra and now define η_1 and η_2 as follows:

$$\eta_1(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.4, & \text{if } m = p, \\ 0, & \text{if } m = q, r; \end{cases} \quad \eta_2(m) = \begin{cases} 0.9, & \text{if } m = 0, \\ 0.5, & \text{if } m = q, \\ 0, & \text{if } m = p, r; \end{cases} \quad \Rightarrow (\eta_1 \cup \eta_2)(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.4, & \text{if } m = p, \\ 0.5, & \text{if } m = q, \\ 0, & \text{if } m = r. \end{cases}$$

Take $m = p$ and $n = q \Rightarrow m \otimes n = p \otimes q = r$, then we need to show that:

$$(\eta_1 \cup \eta_2)(p \otimes q) \geq \min\{(\eta_1 \cup \eta_2)(p), (\eta_1 \cup \eta_2)(q)\} \text{ is not true.}$$

$$\begin{aligned} \text{Now, } (\eta_1 \cup \eta_2)(p \otimes q) &= (\eta_1 \cup \eta_2)(r) = 0 \\ &\geq \min\{(\eta_1 \cup \eta_2)(p), (\eta_1 \cup \eta_2)(q)\} \\ &= \min\{0.4, 0.5\} = 0.4 \text{ which is false,} \end{aligned}$$

Therefore, the union of any two fuzzy SAs of BCL–algebra, B is not necessarily a fuzzy SA of B .

4.1.2 Fuzzy Deductive Systems of BCL–Algebra

Under this subsection, we introduce and define deductive system deductive system (DS) and fuzzy DS of BCL–algebra B .

Definition 4.1.2.1. A fuzzy subset η in a BCL–algebra B is called a fuzzy deductive system (DS) of B , if the following axioms are satisfied, $\forall m, n, u, w \in B$:

- (i) $\eta((m \otimes u) \otimes u) \geq \eta(m)$
- (ii) $\eta(m \otimes (n \otimes w)) \geq \min\{\eta(m), \eta(n)\}$

Example 4.1.2.1. Let \otimes be as defined in Table 4.1 of Example 4.1.1.3 of $B = \{0, p, q, r\}$.

$$\text{Define a fuzzy subset } \eta : B \rightarrow [0, 1] \text{ by: } \eta(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.6, & \text{if } m = p, r, \\ 0.2, & \text{if } m = q. \end{cases}$$

Then, η is fuzzy DS of B by the following explanations for some cases and possibly, it will be similar to explain the remaining cases:

Let $m = r$, $n = p$, $w = u = q$, then:

- (i) $\eta((r \otimes q) \otimes q) = \eta(p \otimes q) = \eta(r) = 0.6 \geq \eta(r) = 0.6$ holds,
- (ii) $\eta(r \otimes (p \otimes q)) = \eta(r \otimes r) = \eta(0) = 0.8 \geq \min\{\eta(r), \eta(q)\} = \min\{0.6, 0.2\} = 0.2$ holds,
and so on.

Lemma 4.1.2.1. Let η be a fuzzy subset in B . If η is a fuzzy DS of B , then the following hold:

- (i) $\eta(0) \geq \eta(m)$, $\forall m \in B$
- (ii) $m \otimes n = n \Rightarrow \eta(n) \geq \eta(m)$ and hence $\eta(n) \geq \min\{\eta(m), \eta(n)\}$, $\forall m, n \in B$
- (iii) $m \otimes n = n \otimes m \Rightarrow \eta((m \otimes w) \otimes w) \geq \eta(w)$, $\eta((m \otimes w) \otimes w) \geq \min\{\eta(m), \eta(w)\}$, $\forall m, n, w \in B$

Proof. Let η be fuzzy DS of B . Then $B \neq \emptyset$ by definition of BCL-algebra

- (i) Let $m \in B$, then we have: $m \otimes m = (m \otimes m) \otimes m = 0 \otimes m = 0$, $\forall m \in B$

Then $\eta(0) = \eta((m \otimes m) \otimes m) \geq \eta(m) \Rightarrow \eta(0) \geq \eta(m)$, $\forall m \in B$.

- (ii) Let $m \otimes n = n$, $\forall m \in B$

$\Rightarrow \eta((m \otimes n) \otimes n) \geq \eta(m)$ by Definition 4.1.2.1 (i), where $w = n$, we have:

$(\eta(m \otimes n) \otimes n) = \eta(n) \geq \eta(m)$ and then $\eta(n) \geq \eta(m) \Rightarrow \eta(n) \geq \min\{\eta(m), \eta(n)\}$.

- (iii) Let $m \otimes n = n \otimes m$

$$\begin{aligned} \Rightarrow \eta((m \otimes w) \otimes w) &= \eta((w \otimes (m \otimes w))) = \eta((w \otimes (w \otimes m))) \\ &\geq \min\{\eta(w), \eta(w)\} = \eta(w). \end{aligned}$$

Thus, $\eta((m \otimes w) \otimes w) \geq \eta(w)$.

□

Proposition 4.1.2.1. Let U be a nonempty subset of B such that χ_U is Characteristic function. Then χ_U is fuzzy DS of B if and only if U is DS of B .

Proof. Let $\chi_U: U \rightarrow \{0, 1\}$ be Characteristic function defined as: $\chi_U(m) = \begin{cases} 1, & \text{if } m \in U, \\ 0, & \text{if } m \notin U. \end{cases}$

Suppose χ_U is fuzzy DS of B

$$\Rightarrow \chi_U((m \otimes w) \otimes w) \geq \chi_U(m) \text{ and } \chi_U((m \otimes (n \otimes w)) \geq \min\{\chi_U(m), \chi_U(n)\}$$

Now, we need to prove that U is DS of B .

- (i) Let $m \in U \Rightarrow \chi_U(m) = 1$

Then $\chi_U((m \otimes w) \otimes w) \geq \chi_U(m) = 1$. But $\chi_U((m \otimes w) \otimes w) \leq 1$ by definition of χ_U

$$\Rightarrow \chi_U((m \otimes w) \otimes w) = 1 \Rightarrow (m \otimes w) \otimes w \in U.$$

(ii) Again, let $m, n \in U \Rightarrow \chi_U(m) = 1 = \chi_U(n)$ Then $\chi_U(m \otimes (n \otimes w)) \geq \min\{\chi_U(m), \chi_U(n)\} = 1$.

But $\chi_U(m \otimes (n \otimes w)) \leq 1$ by definition $\Rightarrow \chi_U(m \otimes (n \otimes w)) = 1 \Rightarrow m \otimes (n \otimes w) \in U$

Thus, by (i) and (ii) above, U is DS of B .

Conversely, suppose U is DS of B . We claim that χ_U is fuzzy DS of B , which means we show:

$$(1) \chi_U((m \otimes w) \otimes w) \geq \chi_U(m)$$

$$(2) \chi_U(m \otimes (n \otimes u)) \geq \min\{\chi_U(m), \chi_U(n)\}$$

(1) Now we prove: $\chi_U((m \otimes w) \otimes w) \geq \chi_U(m)$ by taking the following two cases:

Case (i) Let $m \in U \Rightarrow (m \otimes w) \otimes w \in U$ by the hypothesis

$$\Rightarrow \chi_U((m \otimes w) \otimes w) = \chi_U(m) = 1 \Rightarrow \chi_U((m \otimes w) \otimes w) = 1 \geq \chi_U(m)$$

Case (ii) Let $m \notin U$ then (either $(m \otimes w) \otimes w \in U$ or $(m \otimes w) \otimes w \notin U$)

$$\Rightarrow \chi_U(m) = 0 \text{ and either } \chi_U((m \otimes w) \otimes w) = 1 \text{ or } \chi_U((m \otimes w) \otimes w) = 0$$

$$\Rightarrow \chi_U((m \otimes w) \otimes w) \geq 0 = \chi_U(m) \Rightarrow \chi_U((m \otimes w) \otimes w) \geq \chi_U(m)$$

$$\text{By cases (i) – (ii) above, we obtain: } \chi_U((m \otimes w) \otimes w) \geq \chi_U(m)$$

(2) We also prove $\chi_U(m \otimes (n \otimes u)) \geq \min\{\chi_U(m), \chi_U(n)\}$ by following three cases:

Case (i) Let $m, n \in U \Rightarrow m \otimes (n \otimes w) \in U$ by the hypothesis

$$\Rightarrow \chi_U(m \otimes (n \otimes w)) = \chi_U(m) = 1 = \chi_U(n)$$

$$\Rightarrow \chi_U(m \otimes (n \otimes w)) = 1 \geq \min\{\chi_U(m), \chi_U(n)\}$$

$$\Rightarrow \chi_U(m \otimes (n \otimes w)) \geq \min\{\chi_U(m), \chi_U(n)\}$$

Case (ii) Let $m \in U, n \notin U$ (or $m \notin U, n \in U$)

$$\Rightarrow \text{either } m \otimes (n \otimes w) \in U \text{ or } m \otimes (n \otimes w) \notin U \text{ holds}$$

$$\Rightarrow \chi_U(m \otimes (n \otimes w)) \geq 0 \text{ and } \chi_U(m) = 1, \chi_U(n) = 0 \text{ (or } \chi_U(m) = 0, \chi_U(n) = 1)$$

$$\Rightarrow \chi_U(m \otimes (n \otimes w)) \geq 0 = \min\{\chi_U(m), \chi_U(n)\} = \min\{1, 0\}$$

$$\Rightarrow \chi_U(m \otimes (n \otimes w)) \geq \min\{\chi_U(m), \chi_U(n)\}$$

Case (iii) Let $m \notin U$ or $n \notin U$ and (either $m \otimes (n \otimes w) \in U$ or $m \otimes (n \otimes w) \notin U$)

$$\Rightarrow \chi_U(m) = 0 = \chi_U(n) \text{ and either } \chi_U(m \otimes (n \otimes w)) = 1 \text{ or } \chi_U(m \otimes (n \otimes w)) = 0$$

$$\Rightarrow \chi_U(m \otimes (n \otimes w)) \geq 0 = \min\{\chi_U(m), \chi_U(n)\} = \min\{0, 0\}$$

Then by cases (i) – (iii) above, we arrive at the result: $\chi_U(m \otimes (n \otimes u)) \geq \min\{\chi_U(m), \chi_U(n)\}$.

As a whole, by (1) and (2) above, we conclude that χ_U is fuzzy DS of B .

□

Theorem 4.1.2.1. *The intersection, $\eta_1 \cap \eta_2$, of any two fuzzy DSs, η_1 and η_2 , of BCL-algebra B is fuzzy DS of B .*

Proof. Let η_1 and η_2 be any two fuzzy DSs of B .

Then we need to prove that $\eta_1 \cap \eta_2$ is a fuzzy DS of B and let $m, n, w \in B$.

$$(i) (\eta_1 \cap \eta_2)((m \otimes w) \otimes w) = \min\{\eta_1((m \otimes w) \otimes w), \eta_2((m \otimes w) \otimes w)\} \\ \geq \min\{\eta_1(m), \eta_2(m)\} = (\eta_1 \cap \eta_2)(m) \quad \text{and}$$

$$(ii) (\eta_1 \cap \eta_2)(m \otimes (n \otimes w)) = \min\{\eta_1(m \otimes (n \otimes w)), \eta_2(m \otimes (n \otimes w))\} \\ \geq \min\{\min\{\eta_1(m), \eta_1(n)\}, \min\{\eta_2(m), \eta_2(n)\}\} \\ = \min\{\min\{\eta_1(m), \eta_2(m)\}, \min\{\eta_1(n), \eta_2(n)\}\} \\ = \min\{(\eta_1 \cap \eta_2)(m), (\eta_1 \cap \eta_2)(n)\} \quad \text{and}$$

Hence by (i) and (ii) above, $\eta_1 \cap \eta_2$ is a fuzzy DS of B . □

The above theorem can also be generalized to any family of fuzzy DSs as in the following corollary.

Corollary 4.1.2.1. *The intersection of any family $\{\eta_i : i \in I\}$ of fuzzy DS, in B is also a fuzzy DS of B .*

Proof. Let $\eta_i, \forall i \in I$, be family of fuzzy DSs of B . Then we need to prove that $\bigcap_{i \in I} \eta_i$ is a fuzzy DS of B . Let $m, n, w \in B$.

$$(i) \bigcap_{i \in I} \eta_i((m \otimes w) \otimes w) = \inf_{i \in I} \{\eta_i((m \otimes w) \otimes w)\} \\ \geq \inf_{i \in I} \{\eta_i(m)\} \\ \geq \bigcap_{i \in I} \eta_i(m), \quad \forall m, w \in B \quad \text{and}$$

$$(ii) \left(\bigcap_{i \in I} \eta_i\right)(m \otimes (n \otimes w)) = \inf_{i \in I} \{\eta_i(m \otimes (n \otimes w))\} \\ \geq \inf_{i \in I} \{\min\{\eta_i(m), \eta_i(n)\}\} \\ \geq \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\ = \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\ = \min\left\{\bigcap_{i \in I} \eta_i(m), \bigcap_{i \in I} \eta_i(n)\right\}, \quad \forall m, n \in B.$$

Hence by (i) and (ii) above, $\bigcap_{i \in I} \eta_i$ is a fuzzy DS of B . □

□

Remark 4.1.2.1. *The union of any two fuzzy DSs. of B is not necessarily fuzzy DS of B.*

Example 4.1.2.2. *Let $(B; \otimes, 0)$ be a BCL–algebra, where $B = \{0, p, q, r\}$ and " \otimes " be as defined in the Table 1.1 in Example 1.1.1 above, and define two fuzzy DSs., η_1 and η_2 of B as follows:*

$$\eta_1(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.5, & \text{if } m = p, \\ 0.1, & \text{if } m = q, r, \end{cases} \quad \text{and} \quad \eta_2(m) = \begin{cases} 0.6, & \text{if } m = 0, \\ 0.5, & \text{if } m = q, \\ 0.2, & \text{if } m = p, r, \end{cases}$$

$$\text{and then } (\eta_1 \cup \eta_2)(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.5, & \text{if } m = p, q, \\ 0.2, & \text{if } m = r. \end{cases}$$

It is easy to check that η_1 and η_2 are fuzzy DSs of B but to show that their union is not necessarily fuzzy DS of B, we justify it as follows using the above pairs of fuzzy DSs of B which are η_1 and η_2 .

Take $m = q$, $n = p$ and $w = 0 \Rightarrow q \otimes (p \otimes 0) = q \otimes p = r$ Then

$$(i) \quad (\eta_1 \cup \eta_2)(p \otimes (r \otimes q)) = (\eta_1 \cup \eta_2)(r) = 0.2 \geq \min\{(\eta_1 \cup \eta_2)(q), (\eta_1 \cup \eta_2)(p)\} \\ = \min\{0.5, 0.5\} = 0.5 \quad \text{which is not true.}$$

Thus, by the above justifications, the union of any two fuzzy DS. of B is not necessarily a fuzzy DS of B.

Theorem 4.1.2.2. *Let $\eta : B \rightarrow [0, 1]$ be fuzzy subset in B such that for some $b \in B$ $B^b = \{\forall m \in B : \eta(m) \geq \eta(b)\}$. Then, if η is fuzzy DS of B then B^b is DS of B.*

Proof. Suppose η is fuzzy DS of B. We need to show that B^b is DS of B.

$$(1) \quad \text{Let } m \in B^b \Rightarrow \eta(m) \geq \eta(b) \Rightarrow \eta((m \otimes u) \otimes u) \geq \eta(m) \geq \eta(b)$$

$$\Rightarrow \eta((m \otimes u) \otimes u) \geq \eta(b) \Rightarrow ((m \otimes u) \otimes u) \in B^b$$

$$(2) \quad \text{Let } m, n \in B^b \Rightarrow \eta(m) \geq \eta(b); \eta(n) \geq \eta(b)$$

$$\Rightarrow \eta(m \otimes (n \otimes w)) \geq \min\{\eta(m), \eta(n)\} \geq \min\{\eta(b), \eta(b)\} = \eta(b)$$

$$\Rightarrow \eta(m \otimes (n \otimes w)) \geq \eta(b) \Rightarrow ((m \otimes w) \otimes w) \in B^b$$

Therefore, by (1) and (2) above, B^b is DS of B. □

Remark 4.1.2.2. *The converse of Theorem 4.1.2.2 above may not be true, which is justified by the following counter example.*

Example 4.1.2.3. Suppose $\eta : B \rightarrow [0, 1]$ is fuzzy subset in $(B; \otimes, 0)$ discussed in Example 4.1.2 and take the fuzzy DS η of B defined in Example 4.1.2.1.

Now, take $B^b = \{p, r\} = \{\forall m \in B : \eta(m) = \eta(b)\}$, where $b = p, r$. Then, after careful checking, it has been listed that $B^b = \{0, p, r\}$ is not DS of B .

Thus, η is fuzzy DS of B but B^b is not DS of B .

4.1.3 Fuzzy Ideal of BCL–Algebra

Under this subsection, we introduce and define ideal and fuzzy ideal of BCL–algebra B which had not been defined nor discussed in the prior literatures so far.

Definition 4.1.3.1. A fuzzy subset $\eta : B \rightarrow [0, 1]$, where the mapping η define the degrees of membership satisfying the following axioms, $\forall m, n, w \in B$ is called fuzzy ideal of B .

- (i) $\eta(m) \geq \min\{\eta(m \otimes n), \eta(n)\}$,
- (ii) $\eta((m \otimes (n \otimes w)) \otimes w) \geq \min\{\eta(m), \eta(n)\}$

Example 4.1.3.1. Let the set B and the binary operation \otimes be as defined in Table 1.1 of Example 1.1.1, above and define a fuzzy subset $\eta : B \rightarrow [0, 1]$ by

$$\eta_1(m) = \begin{cases} 0.7, & \text{if } m = 0, \\ 0.3, & \text{if } m = p, q, r; \end{cases} \quad \eta_2(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.6, & \text{if } m = p, r, \\ 0.4, & \text{if } m = q; \end{cases} \quad \eta_3(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.7, & \text{if } m = p, \\ 0.4, & \text{if } m = q, r. \end{cases}$$

Then, η_1, η_2 and η_3 are fuzzy ideals of the BCL–algebra B by the following explanations for some cases and possibly, it will be similar to explain the remaining cases:

Let $m = r, n = q, w = p$, then:

$$(i) \quad \eta_1(r) = 0.3 \geq \min\{\eta_1(r \otimes q), \eta_1(q)\} = \min\{\eta_1(0), \eta_1(q)\} = \min\{0.7, 0.3\} = 0.3 \text{ holds;}$$

$$\eta_2(r) = 0.6 \geq \min\{\eta_2(r \otimes q), \eta_2(q)\} = \min\{\eta_2(0), \eta_2(q)\} = \min\{1, 0.4\} = 0.4 \text{ holds;}$$

$$\eta_3(r) = 0.4 \geq \min\{\eta_3(r \otimes q), \eta_3(q)\} = \min\{\eta_3(0), \eta_3(q)\} = \min\{1, 0.4\} = 0.4 \text{ holds;}$$

$$(ii) \quad \eta_1((r \otimes (q \otimes p)) \otimes p) = \eta_1((r \otimes r) \otimes p) = \eta_1(0) = 0.7$$

$$\geq \min\{\eta_1(r \otimes q), \eta_1(q)\} = \min\{\eta_1(r), \eta_1(q)\}$$

$$= \min\{0.3, 0.3\} = 0.3 \text{ holds;}$$

$$\begin{aligned}\eta_2((r \otimes (q \otimes p)) \otimes p) &= \eta_2((r \otimes r) \otimes p) = \eta_2(0) = 1 \\ &\geq \min\{\eta_2(r \otimes q), \eta_2(q)\} = \min\{\eta_2(r), \eta_2(q)\} \\ &= \min\{0.6, 0.4\} = 0.4 \text{ holds;}\end{aligned}$$

$$\begin{aligned}\eta_3((r \otimes (q \otimes p)) \otimes p) &= \eta_3((r \otimes r) \otimes p) = \eta_3(0) = 1 \\ &\geq \min\{\eta_3(r \otimes q), \eta_3(q)\} = \min\{\eta_3(r), \eta_3(q)\} \\ &= \min\{0.4, 0.4\} = 0.4 \text{ holds; and so on.}\end{aligned}$$

Lemma 4.1.3.1. *Let η be fuzzy ideal of BCL–algebra B . Then $\eta(0) \geq \eta(m), \forall m \in B$.*

Proof. Since $B \neq \emptyset$, let $m \in B$, then as proved in the proof of Lemma 4.1.3 we have:

$$\begin{aligned}(m \otimes (m \otimes)) \otimes m &= 0 \\ \Rightarrow \eta((m \otimes m) \otimes m) &= \eta(0) \geq \min\{\eta(m), \eta(m)\} = \eta(m) \\ \Rightarrow \eta(0) &\geq \eta(m)\end{aligned}$$

□

Proposition 4.1.3.1. *Let η be fuzzy ideal of BCL–algebra B such that $m \leq n, \forall m, n \in B$. Then $\eta(m) \geq \eta(n)$, that is, in a fuzzy ideal, η is order reversing, where " \leq " is a binary relation on B defined by Definition 1.1.8, above.*

Proof. Let η be a fuzzy ideal of BCL–algebra B such that $m \leq n, \forall m, n \in B$.

We need to prove: $\eta(m) \geq \eta(n), \forall m, n \in B$.

By the binary relation " \leq " defined on B , we have: $m \leq n$, if and only if $m \otimes n = 0$

$$\text{Now, } \eta(m) \geq \min\{\eta(m \otimes n), \eta(n)\} = \min\{\eta(0), \eta(n)\} = \eta(n)$$

$$\Rightarrow \eta(m) \geq \eta(n) \text{ (since, } \eta(0) \geq \eta(n), \forall n \in B, \text{ by Lemma 4.1.3.1)}$$

That is $m \leq n \Rightarrow \eta(m) \geq \eta(n)$ or η is order reversing.

□

Proposition 4.1.3.2. *Let η be a fuzzy ideal of B , and " \leq " be a binary relation on B . Then the following hold, $\forall m, n, w \in B$:*

- (1) $\eta(m \otimes n) = \eta(0) \Rightarrow \eta(m) \geq \eta(n)$
- (2) $n \leq m \otimes n \Rightarrow \eta(m) \geq \eta(m \otimes n)$ and
- (3) $m \otimes n \leq n \Rightarrow \eta(m) \geq \eta(n)$
- (4) $w \otimes n \leq m \Rightarrow \eta(w) \geq \min\{\eta(m \otimes n), \eta(n)\}$

Proof. Suppose η is fuzzy ideal of BCL–algebra B , then we claim (1), (2), (3), (4), $\forall m, n, w \in B$.

Now, let $m, n, w \in B$.

(1) Suppose $\eta(m \circledast n) = \eta(0)$, then we need to prove: $\eta(m) \geq \eta(n)$

$$\eta(m) \geq \min\{\eta(m \circledast n), \eta(n)\}$$

$$= \min\{\eta(0), \eta(n)\} = \eta(n)$$

$\Rightarrow \eta(m) \geq \eta(n)$, which is the required result.

(2) Suppose $n \leq m \circledast n \Rightarrow \eta(n) \geq \eta(m \circledast n)$, by the order reversing property of the binary relation " \leq " on fuzzy ideal of B so that we need to prove that: $\eta(m) \geq \eta(m \circledast n)$:

$$\eta(m) \geq \min\{\eta(m \circledast n), \eta(n)\} = \eta(m \circledast n)$$

$\Rightarrow \eta(m) \geq \eta(m \circledast n)$, which is the required result.

(3) Suppose $m \circledast n \leq n \Rightarrow \eta(m \circledast n) \geq \eta(n)$, by the the order reversing property of the binary relation " \leq " on fuzzy ideal of B .

Then we need to prove that: $\eta(m) \geq \eta(n)$: $\eta(m) \geq \min\{\eta(m \circledast n), \eta(n)\} = \eta(n)$

$\Rightarrow \eta(m) \geq \eta(n)$, which is the required result.

(4) Suppose $w \circledast n \leq m \Rightarrow \eta(w \circledast n) \geq \eta(m)$, by the the order reversing property of the binary relation " \leq " on fuzzy ideal of B .

Then we need to prove that: $\eta(w) \geq \min\{\eta(m \circledast n), \eta(n)\}$

$$\eta(w) \geq \min\{\eta(w \circledast n), \eta(n)\} \geq \min\{\eta(m), \eta(n)\}$$

$$\geq \min\{\min\{\eta(m \circledast n), \eta(n)\}, \eta(n)\} = \min\{\eta(m \circledast n), \eta(n)\}$$

Thus, $\eta(w) \geq \min\{\eta(m \circledast n), \eta(n)\}$

□

Theorem 4.1.3.1. *The intersection of any two fuzzy ideals, η_1 and η_2 , of the BCL–algebra, B is also a fuzzy ideal of B .*

Proof. Let η_1 and η_2 be any two fuzzy ideals of B . Then we need to prove that $\eta_1 \cap \eta_2$ is a fuzzy ideal of B , $\forall m, n, w \in B$.

$$(i) \quad (\eta_1 \cap \eta_2)(m) = \min\{\eta_1(m), \eta_2(m)\}$$

$$\geq \min\{\min\{\eta_1(n), \eta_1(m \circledast n)\}, \min\{\eta_2(n), \eta_2(m \circledast n)\}\}$$

$$= \min\{\min\{\eta_1(n), \eta_2(n)\}, \min\{\eta_1(m \circledast n), \eta_2(m \circledast n)\}\}$$

$$= \min\{(\eta_1 \cap \eta_2)(n), (\eta_1 \cap \eta_2)(m \circledast n)\} \quad \text{and}$$

$$(ii) \quad (\eta_1 \cap \eta_2)((m \circledast (n \circledast w)) \circledast w) = \min\{(\eta_1((m \circledast (n \circledast w)) \circledast w), (\eta_2((m \circledast (n \circledast w)) \circledast w))\}$$

$$\geq \min\{\min\{\eta_1(m), \eta_1(n)\}, \min\{\eta_2(m), (\eta_2(n))\}\}$$

$$\begin{aligned} &\geq \min\{\min\{\eta_1(m), \eta_2(m)\}, \min\{\eta_1(n), \eta_2(n)\}\} \\ &= \min\{(\eta_1 \cap \eta_2)(m), (\eta_1 \cap \eta_2)(n)\}. \end{aligned}$$

Hence by (i) and (ii) above, $\eta_1 \cap \eta_2$ is a fuzzy ideal of BCL–algebra. □

The above theorem can also be generalized to any family of fuzzy ideals as stated in the corollary following.

Corollary 4.1.3.1. *The intersection $\bigcap_{i \in I} \eta_i$ of any family of fuzzy ideals, $\{\eta_i : i \in I\}$, in B is a fuzzy ideal of BCL–algebra B .*

Proof. Let $\eta_i, \forall i \in I$, be family of fuzzy ideals of B and, $m, n, w \in B$. Then we need to prove that $\bigcap_{i \in I} \eta_i$ is a fuzzy ideal of B .

$$\begin{aligned} \text{(i)} \quad \bigcap_{i \in I} \eta_i(m) &= \inf_{i \in I} \{\eta_i(m)\} \\ &\geq \inf_{i \in I} \{\min\{\eta_i(n), \eta_i(m \otimes n)\}\} \\ &\geq \min\{\inf_{i \in I} \{\eta_i(n), \eta_i(m \otimes n)\}\} \\ &= \{\min\{\inf_{i \in I} \{\eta_i(n)\}, \inf_{i \in I} \{\eta_i(m \otimes n)\}\} \\ &= \min\left\{\bigcap_{i \in I} \eta_i(n), \bigcap_{i \in I} \eta_i(m \otimes n)\right\}, \forall m, n \in B \text{ and} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \bigcap_{i \in I} \eta_i[(m \otimes (n \otimes w)) \otimes w] &= \inf_{i \in I} \{\eta_i[(m \otimes (n \otimes w)) \otimes w]\} \\ &\geq \inf_{i \in I} \{\min\{\eta_i(m), \eta_i(n)\}\} \\ &\geq \min\{\inf_{i \in I} \{\eta_i(m), \eta_i(n)\}\} \\ &= \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\ &= \min\left\{\bigcap_{i \in I} \eta_i(m), \bigcap_{i \in I} \eta_i(n)\right\}, \forall m, n, w \in B. \end{aligned}$$

Hence by (i) and (ii) above, $\eta_1 \cap \eta_2$ is a fuzzy ideal of BCL–algebra. □

Remark 4.1.3.1. *The union of any two fuzzy ideals of a BCL–algebra, B is not necessarily fuzzy ideal of BCL–algebra B . (We justify this remark by the following example.)*

Example 4.1.3.2. Let $(B; \otimes, 0)$ be a BCL–algebra, as defined in Table 1.1 of Example 1.1.1 and take two fuzzy ideals; η_2 and η_3 in Example 4.1.3.1 above:

We have checked that η_2 and η_3 are fuzzy ideals of B but to show that their union is not necessarily a fuzzy ideal of B , we justify it as follows using the pairs of fuzzy ideals of B :

$$(\eta_2 \cup \eta_3)(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.7, & \text{if } m = r, \\ 0.6, & \text{if } m = p, \\ 0.4, & \text{if } m = q. \end{cases}$$

Take $m = r$, $n = 0$, $w = q$

$$\Rightarrow [(m \otimes (n \otimes w)) \otimes w] = [(r \otimes (0 \otimes q)) \otimes q] = [(r \otimes 0) \otimes q] = r \otimes q = p.$$

Then we show that at least one of the axioms for ideals fails as follows:

$$\begin{aligned} (\eta_2 \cup \eta_3) [(r \otimes (0 \otimes q)) \otimes q] &= (\eta_2 \cup \eta_3)(p) = 0.6 \\ &\geq \min\{(\eta_2 \cup \eta_3)(r), (\eta_2 \cup \eta_3)(0)\} \\ &= \min\{0.7, 1\} = 0.7, \text{ which is not true.} \end{aligned}$$

Thus, we deduce that the union of two fuzzy ideals of BCL–algebras B may not be fuzzy ideal of B .

Theorem 4.1.3.2. Let $\eta : B \rightarrow [0, 1]$ be a fuzzy subset in a BCL–algebra $(B; \otimes, 0)$ such that $B^b = \{\forall m \in B : \eta(m) \geq \eta(b), \text{ for some } b \in B\}$. If η is fuzzy ideal of B then B^b is ideal of B .

Proof. Suppose η is fuzzy ideal of B , or $\forall m, n, w \in B$, the following hold:

- (1) $\eta(m) \geq \min\{\eta(m \otimes n), \eta(n)\}$ and
- (2) $\eta((m \otimes (n \otimes w)) \otimes w) \geq \min\{\eta(m), \eta(n)\}$

We need to prove that B^b is ideal of B ; in other words, we need to show the following:

- (a) $m \otimes n, n \in B^b \Rightarrow m \in B^b$,
- (b) $m, n \in B^b \Rightarrow (m \otimes (n \otimes w)) \otimes w \in B^b$

- (a) Let $m \otimes n, n \in B^b$, for some $b \in B \Rightarrow \eta(m \otimes n) \geq \eta(b)$ and $\eta(n) \geq \eta(b)$

$$\text{But by (1): } \eta(m) \geq \min\{\eta(m \otimes n), \eta(n)\} \geq \min\{\eta(b), \eta(b)\} = \eta(b)$$

$$\Rightarrow \eta(m) \geq \eta(b) \Rightarrow m \in B^b, \text{ for some } b \in B.$$

- (b) Let $m, n \in B^b$, for some $b \in B$. Then $\eta(m) \geq \eta(b)$, $\eta(n) \geq \eta(b)$

$$\text{And by (2), } \eta((m \otimes (n \otimes w)) \otimes w) \geq \min\{\eta(m), \eta(n)\}$$

$$\geq \min\{\eta(b), \eta(b)\} = \eta(b)$$

$$\Rightarrow \eta[(m \otimes (n \otimes w)) \otimes w] \geq \eta(b) \Rightarrow (m \otimes (n \otimes w)) \otimes w \in B^b$$

Then, by (a), (b) above, B^b is a fuzzy ideal of B .

Remark 4.1.3.2. *The converse of Theorem 4.1.3.2 above may not be true, which is justified by the following counter example.*

Example 4.1.3.3. *Suppose $\eta : B \rightarrow [0, 1]$ is fuzzy subset in $(B; \otimes, 0)$ discussed in Example 4.1.3 and take the fuzzy subset η defined in Example 4.1.3.1, which has been checked that η is fuzzy DS of B .*

Now, take $B^b = \{p, r\} = \{\forall m \in B : \eta(m) = \eta(b)\}$, where $b = p, r$.

Then, after careful checking, it has been listed that $B^b = \{0, p, r\}$ is not DS of B .

Thus, η is fuzzy DS of B but B^b is not DS of B .

□

Lemma 4.1.3.2. *Let η be fuzzy subset in B . If $\eta(m) = \eta(0), \forall m \in B$, then η is fuzzy ideal of B .*

Proof. Suppose η is fuzzy subset in B such that $\eta(m) = \eta(0)$. We need to prove: η is a fuzzy ideal of BCL–algebra B :

(i) Again from the hypothesis, $\eta(0) = \eta(m), \forall m \in B$

$$\Rightarrow \eta(0) = \eta(m) = \eta(n) = \eta(m \otimes n), \forall m, n \in B \text{ (since, also } m \otimes n \in B)$$

$$\Rightarrow \eta(m) = \eta(0) = \min\{\eta(0), \eta(0)\} = \min\{\eta(m \otimes n), \eta(n)\}$$

$$\Rightarrow \eta(m) = \min\{\eta(m \otimes n), \eta(n)\} \Rightarrow \eta(m) \geq \min\{\eta(m \otimes n), \eta(n)\}$$

(ii) Similar to (i) above, since $\eta(0) = \eta(m), \forall m \in B$

$$\Rightarrow \eta(0) = \eta(m) = \eta(n) = \eta(w) = \eta[(m \otimes (n \otimes w)) \otimes w], \forall m, n, w \in B$$

$$\Rightarrow \eta((m \otimes (n \otimes w)) \otimes w) \geq \min\{\eta(m), \eta(n)\}$$

Thus, by (i) and (ii) above, η is fuzzy ideal of B .

□

Theorem 4.1.3.3. *Let $(B; \otimes, 0)$ be a BCL–algebra. Then the following hold:*

(1) *Every fuzzy deductive system of B is fuzzy SA of B .*

(2) *Every fuzzy ideal of B is fuzzy SA of B .*

Proof. Suppose B is BCL–algebra and $\eta : B \rightarrow [0, 1]$ is fuzzy subset.

- (1) Suppose η is fuzzy DS of B . Then we need to show that η is fuzzy SA of B . Now as η is fuzzy DS of B , then $\forall m, n, w \in B; \eta(m \otimes (n \otimes w)) \geq \min\{\eta(m), \eta(n)\}$ by Definition 4.1.2.1 (ii) (second axiom of definition of fuzzy DS).

Then, since $w \in B$ is arbitrary, put $w = 0$

$$\Rightarrow \eta(m \otimes (n \otimes 0)) \geq \min\{\eta(m), \eta(n)\}$$

$$\Rightarrow \eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\}, \text{ since } n \otimes 0 = n \text{ by Proposition 2.1.1 (5)}$$

$$\Rightarrow \eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\} \text{ that the axiom for fuzzy SA is satisfied.}$$

Therefore the fuzzy DS η of B is fuzzy SA of B .

- (2) Suppose η if fuzzy ideal of B . Then we need to show that η is fuzzy SA of B . Now as η is fuzzy ideal of B , we have, $\forall m, n, w \in B; \eta((m \otimes (n \otimes w)) \otimes w) \geq \min\{\eta(m), \eta(n)\}$ by definition 4.1.3.1 (ii) (third axiom of definition of fuzzy ideal).

Then since $w \in B$ is arbitrary, put $w = n$

$$\Rightarrow \eta((m \otimes (n \otimes n)) \otimes n) \geq \min\{\eta(m), \eta(n)\}$$

$$\Rightarrow \eta(m \otimes 0) \otimes n \geq \min\{\eta(m), \eta(n)\} \text{ since } n \otimes n = 0 \text{ by Definition 1.1.8 (BCL-algebra)}$$

$$\Rightarrow \eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\} \text{ since } m \otimes 0 = m \text{ by Proposition 2.1.1 (5)}$$

$$\Rightarrow \eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\} \text{ that the axiom for fuzzy SA is satisfied.}$$

Therefore the fuzzy ideal η of B is fuzzy SA of B . □

The converses of the two valid statements in Theorem 4.1.3.3 (1) and (2), as well as the other relationships between fuzzy deductive systems and fuzzy ideals, are stated in the following remark and justified by the succeeding counterexample for all being not necessarily true.

Remark 4.1.3.3. *The only general links between the three fuzzy substructures; fuzzy SA, fuzzy DS and fuzzy ideal, in a BCL-algebra involve the statements that any fuzzy DS is a fuzzy SA and any fuzzy ideal is a fuzzy SA as proved in Theorem 4.1.3.3. The converse of either of these two facts, and any other general connection whatsoever between the three structures, is not true in general.*

Example 4.1.3.4. *Apart from the statements proved in Theorem 4.1.3.3, we provide justifications for the remaining statements given in Remark 4.1.3.3 by the following counter examples.*

- (1) *Every fuzzy SA of B may not be fuzzy DS of B (converse of Theorem 4.1.3.3 (1)).*

Suppose η is fuzzy SA of B as defined in Example 4.1.1.1 defined on the BCL-algebra B in

Table 1.1 of Example 1.1.1, and take $m = q$, $n = p$ and $w = 0$ so that,

$$\begin{aligned}\eta((m \otimes n) \otimes w) &= \eta((q \otimes p) \otimes 0) = \eta(r \otimes 0) \text{ (since } q \otimes p = r \text{ from Table 1.1)} \\ &= \eta(r) \text{ (since } r \otimes 0 = r \text{ from Table 1.1)} \\ &= 0.1, \text{ but that}\end{aligned}$$

$$\begin{aligned}\eta((m \otimes n) \otimes w) &= \eta(q \otimes (p \otimes 0)) = 0.1 \geq \eta(q) \text{ by Definition 4.1.2.1 (i) (one axiom of DS)} \\ &= 0.3 \text{ which is false or one axiom of fuzzy DS fails.}\end{aligned}$$

Therefore, fuzzy SA of a BCL–algebra is not necessarily fuzzy DS of B.

(2) Every fuzzy SA of B may not be fuzzy ideal of B (converse of Theorem 4.1.3.3 (2)).

Suppose η is fuzzy SA of B as defined in Example 4.1.1.1 defined on the BCL–algebra B in Table 1.1 of Example 1.1.1, and take $m = r$ and $n = q$

$$\begin{aligned}\eta((m) \otimes n) &= \eta((r) \otimes q) = 0.1 = \min\{\eta(r \otimes q), \eta(q)\} \text{ (Definition 4.1.3.1(ii)) (axiom of fuzzy ideal)} \\ &= \min\{\eta(0), \eta(q)\} \text{ (since } r \otimes q = 0 \text{ from Table 1.1)} \\ &= \min\{0.7, 0.3\} \\ &= 0.3 \text{ which is false and hence one axiom of fuzzy ideal fails.}\end{aligned}$$

Therefore, fuzzy SA of a BCL–algebra is not necessarily fuzzy ideal of B.

(3) Every fuzzy DS of B may not be fuzzy ideal of B.

Suppose η is fuzzy DS of B as defined in Example 4.1.2.1 defined on the BCL–algebra B in Table 4.1 and take $m = q$, $n = p$ which implies $q \otimes p = r$

$$\begin{aligned}\eta(m) \otimes n &= \eta(q) \otimes p = 0.2 \geq \min\{\eta(q \otimes p), \eta(p)\} = \min\{\eta(r), \eta(p)\} = \min\{0.6, 0.6\} \\ &= 0.6 \text{ which is false clarifying that one axiom of fuzzy ideal fails.}\end{aligned}$$

Therefore, fuzzy DS of a BCL–algebra is not necessarily fuzzy ideal of B.

(4) Every fuzzy ideal of B may not be fuzzy DS of B.

Suppose η is fuzzy ideal of B where $\eta = \eta_2$ as defined in Example 4.1.3.1 defined on the BCL–algebra B in Table 1.1 of Example 1.1.1, and take $m = p$, $w = r$ and $n = r$

$$\begin{aligned}\eta((m \otimes w) \otimes n) &= \eta((p \otimes r) \otimes r) = \eta((q \otimes r)) = \eta(q) = 0.4 \\ &\geq \eta(p) \text{ (Definition 4.1.2.1(i)) (first axiom of fuzzy DS)} \\ &= 0.6 \text{ which is false or one axiom of fuzzy ideal fails.}\end{aligned}$$

Therefore, fuzzy SA of a BCL–algebra is not necessarily fuzzy DS of B fails.

4.2. Pythagorean Fuzzy Substructures of BCL–Algebra

In this section, we introduce Pythagorean fuzzy subalgebra, Pythagorean fuzzy deductive System and Pythagorean fuzzy ideal of BCL–algebra, state and prove different properties and theorems for each of the Pythagorean fuzzy subalgebra, Pythagorean fuzzy deductive System and Pythagorean fuzzy Ideal of BCL–algebra.

4.2.1 Pythagorean Fuzzy Subalgebras of BCL–Algebra

Definition 4.2.1.1. Let $(B; \otimes, 0)$ be a BCL–algebra. A Pythagorean fuzzy set $B^P = (\eta, \tau)$ in B , where the fuzzy mappings $\eta : B \rightarrow [0, 1]$ and $\tau : B \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership, respectively in B such that $0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1$ is called Pythagorean fuzzy SA of B if the following two axioms are satisfied; $\forall m, n \in B$:

$$(\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and } (\tau(m \otimes n))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

Example 4.2.1.1. Let \otimes be as defined in Table 4.1 of Example 4.1.1.3 of $B = \{0, p, q, r\}$. Let the fuzzy subsets $\eta : B \rightarrow [0, 1]$ and $\tau : B \rightarrow [0, 1]$ where $(\eta(m))^2 + (\tau(m))^2 \leq 1$ be defined as follows:

$$\eta(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.6, & \text{if } m = p, \\ 0.3, & \text{if } m = q, r, \end{cases} \quad \text{and} \quad \tau(m) = \begin{cases} 0.4, & \text{if } m = 0, \\ 0.7, & \text{if } m = p, \\ 0.9, & \text{if } m = q, r. \end{cases}$$

Then by using basic calculations, we check that $B^P = (\eta, \tau)$ is Pythagorean fuzzy SA of B as follows:

Take $m = q, n = p \Rightarrow m \otimes n = r$, and then

$$(i) (\eta(q \otimes p))^2 = (\eta(r))^2 = 0.09 \geq \{(\eta(q))^2, (\eta(p))^2\} = \min\{0.09, 0.36\} = 0.09 \text{ hold and}$$

$$(ii) (\tau(q \otimes p))^2 = (\tau(r))^2 = 0.81 \leq \{(\tau(q))^2, (\tau(p))^2\} = \max\{0.81, 0.49\} = 0.81 \text{ also hold}$$

and for all the other cases, similarly the two fuzzy subsets hold so that $B^P = (\eta, \tau)$ is a

Pythagorean fuzzy SA of B .

Lemma 4.2.1.1. If $B^P = (\eta, \tau)$ is a Pythagorean fuzzy SA of B , where $\bar{\eta}$ and $\bar{\tau}$ are square deviations of η and τ of B , respectively, then $\forall m \in B$, we have:

$$(\eta(0))^2 \geq (\eta(m))^2 \text{ and } (\tau(0))^2 \leq (\tau(m))^2; (\bar{\tau}(0))^2 \geq (\bar{\tau}(m))^2 \text{ and } (\bar{\eta}(0))^2 \leq (\bar{\eta}(m))^2.$$

Proof. Since $B \neq \emptyset$, for $m \in B$, we have: $m \otimes m = 0$. Then as $m \otimes m = 0$, then:

$$\begin{aligned} (\eta(0))^2 &= (\eta(m \otimes m))^2 \geq \min\{(\eta(m))^2, (\eta(m))^2\}, \text{ since } (\eta, \tau) \text{ is a Pythagorean fuzzy SA of } B \\ &= (\eta(m))^2 \end{aligned}$$

$$\Rightarrow (\eta(0))^2 \geq (\eta(m))^2 \text{ and}$$

$$\begin{aligned} (\tau(0))^2 &= (\tau(m \otimes m))^2 \leq \max\{(\tau(m))^2, (\tau(m))^2\}, \text{ since } (\eta, \tau) \text{ is a Pythagorean fuzzy SA of } B \\ &= (\tau(m))^2 \end{aligned}$$

$$\Rightarrow (\tau(0))^2 \leq (\tau(m))^2,$$

$$\begin{aligned} (\bar{\tau}(0))^2 &= 1 - (\tau(0))^2 \geq 1 - (\tau(m))^2, \text{ since } (\tau(0))^2 \leq (\tau(m))^2 \Rightarrow -(\tau(0))^2 \geq -(\tau(m))^2 \\ &= (\bar{\tau}(m))^2 \end{aligned}$$

$$\Rightarrow (\bar{\tau}(0))^2 \geq (\bar{\tau}(m))^2 \text{ and}$$

$$\begin{aligned} (\bar{\eta}(0))^2 &= 1 - (\eta(0))^2 \leq 1 - (\eta(m))^2, \text{ since } (\eta(0))^2 \geq (\eta(m))^2 \Rightarrow -(\eta(0))^2 \leq -(\eta(m))^2 \\ &= (\bar{\eta}(m))^2 \Rightarrow (\bar{\eta}(0))^2 \leq (\bar{\eta}(m))^2 \end{aligned}$$

□

Proposition 4.2.1.1. *Let S be a non-empty subset of a BCL-algebra B and χ_S be the Characteristic function with square deviation $\bar{\chi}_S$. Then:*

$$(1) \bar{\chi}_S(m) = \bar{\bar{\chi}}_S(m), \text{ and}$$

$$(2) B^P = (\chi_S, \bar{\chi}_S) \text{ is a Pythagorean fuzzy SA of } B \text{ if and only if } S \text{ is SA of } B.$$

Proof. (1) Suppose $\chi_S : S \rightarrow \{0, 1\}$ is a Characteristic function. Then it is defined as:

$$\chi_S(m) = (\chi_S(m))^2 = \begin{cases} 1, & \text{if } m \in S, \\ 0, & \text{if } m \notin S \end{cases} \Rightarrow \bar{\bar{\chi}}_S(m) = \bar{\chi}_S(m) = \begin{cases} 0, & \text{if } m \in S, \\ 1, & \text{if } m \notin S \end{cases} = (\bar{\chi}_S(m))^2$$

(2) Let χ_S be a membership function of B , and let $m, n \in S$,

Now, we show S is SA of $B \Rightarrow m \otimes n \in S \Rightarrow \chi_S$ is a fuzzy SA of B and $m, n \in S$

$$\Rightarrow \chi_S(m) = \chi_S(n) = 1; \chi_S(m \otimes n) \geq \min\{\chi_S(m), \chi_S(n)\} = \min\{1, 1\} = 1 \Rightarrow \chi_S(m \otimes n) \geq 1$$

But $1 \geq \chi_S(m \otimes n) \Rightarrow \chi_S(m \otimes n) = 1 \Rightarrow m \otimes n \in S$. Similarly, $\bar{\chi}_S(m \otimes n) = 0 \Rightarrow m \otimes n \in S$

Hence, $m, n \in S \Rightarrow m \otimes n \in S$ and therefore, S is a SA of B .

Conversely, suppose S is a SA of B .

Thus at first, for the non-membership function $\bar{\bar{\chi}}_S$, let us show that

$$(\bar{\bar{\chi}}_S(m \otimes n))^2 \leq \max\{(\chi_S(m))^2, (\chi_S(n))^2\}, \forall m, n \in B.$$

Then after, following similar steps for the membership function χ_S , one can show that

$$(\chi_S(m \otimes n))^2 \geq \min\{(\chi_S(m))^2, (\chi_S(n))^2\}, \forall m, n \in B \text{ so that we omit the proof for this part.}$$

Case(1) : If $m, n \in S \Rightarrow m \otimes n \in S$ ($(\bar{\bar{\chi}}_S(m))^2 = 0 = (\bar{\bar{\chi}}_S(n))^2 = (\eta(m \otimes n))^2$), then

$$\begin{aligned} m \otimes n \in S &\Rightarrow (\bar{\bar{\chi}}_S(m \otimes n))^2 = 0 \leq \max\{0, 0\} = \max\{(\bar{\bar{\chi}}_S(m))^2, (\bar{\bar{\chi}}_S(n))^2\} \\ &\Rightarrow (\bar{\bar{\chi}}_S(m \otimes n))^2 \leq \max\{(\bar{\bar{\chi}}_S(m))^2, (\bar{\bar{\chi}}_S(n))^2\}, \end{aligned}$$

Case(2) : If $m \notin S, n \notin S$, (but $m, n \in B$)

$$\Rightarrow (\bar{\bar{\chi}}_S(m))^2 = 1 = (\bar{\bar{\chi}}_S(n))^2, \text{ and } (\bar{\bar{\chi}}_S(m \otimes n))^2 \leq 1$$

(since any fuzzy subset is less that or equal to 1)

$$\Rightarrow (\bar{\bar{\chi}}_S(m \otimes n))^2 \leq 1 = \max\{(\bar{\bar{\chi}}_S(m))^2, (\bar{\bar{\chi}}_S(n))^2\}$$

$$\Rightarrow (\bar{\bar{\chi}}_S(m \otimes n))^2 \leq \max\{(\bar{\bar{\chi}}_S(m))^2, (\bar{\bar{\chi}}_S(n))^2\}$$

Case(3) : If $m \in S$ and $n \notin S$ (or $n \in S$ and $m \notin S$)

$$\Rightarrow \bar{\bar{\chi}}_S(m) = 0 \text{ and } \bar{\bar{\chi}}_S(n) = 1 \quad (\text{or } \bar{\bar{\chi}}_S(n) = 0 \text{ and } \bar{\bar{\chi}}_S(m) = 1)$$

$$\Rightarrow \max\{(\bar{\bar{\chi}}_S(m))^2, (\bar{\bar{\chi}}_S(n))^2\} = \max\{1, 0\} = 1 \quad \text{and} \quad (\bar{\bar{\chi}}_S(m \otimes n))^2 \leq 1$$

$$\Rightarrow (\bar{\bar{\chi}}_S(m \otimes n))^2 \leq 1 = \max\{(\bar{\bar{\chi}}_S(m))^2, (\bar{\bar{\chi}}_S(n))^2\}$$

$$\Rightarrow (\bar{\bar{\chi}}_S(m \otimes n))^2 \leq \max\{(\bar{\bar{\chi}}_S(m))^2, (\bar{\bar{\chi}}_S(n))^2\}$$

Hence, in all the cases (1) – (3), we have checked that:

$$(\bar{\bar{\chi}}_S(m \otimes n))^2 \leq \max\{(\bar{\bar{\chi}}_S(m))^2, (\bar{\bar{\chi}}_S(n))^2\}, \forall m, n \in B \text{ (and similarly for } \chi_S).$$

Thus, $\bar{\bar{\chi}}_S$ is non-membership function on B (but χ_S is membership function on B).

Thus $B^P = (\chi_S, \bar{\bar{\chi}}_S)$ is Pythagorean fuzzy SA of B if and only if S is SA of B. □

Theorem 4.2.1.1. Let $\emptyset \neq S \subseteq B$ and consider the fuzzy subset η and its complement in B such that:

$$\eta(m) = \begin{cases} \delta, & \text{if } m \in S \\ \gamma, & \text{otherwise,} \end{cases} \quad \text{and then} \quad \bar{\eta}(m) = \begin{cases} 1 - \delta, & \text{if } m \in S \\ 1 - \gamma, & \text{otherwise,} \end{cases} \quad \forall \gamma, \delta \in [0, 1] \text{ with } \gamma \leq \delta$$

Then $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B if and only if S is SA of B.

Proof. The proof follows similar reasoning as Theorem 4.2.1.1 above, with the substitution of

1 by δ and 0 by σ for η , and 0 by $1-\delta$, 1 for $1-\sigma$ for $\bar{\eta}$

□

Lemma 4.2.1.2. Let $\eta: B \rightarrow [0, 1]$ be a fuzzy subset on a BCL–algebra, B such that $\bar{\eta}$ and $\bar{\bar{\eta}}$ are the complement and the square deviation of η , respectively. Then,

$$(1) (\bar{\eta}(m))^2 = \bar{\bar{\eta}}(m) \Leftrightarrow \eta(m) = 0 \text{ or } \eta(m) = 1$$

$$(2) \bar{\eta}(m) = \bar{\bar{\eta}}(m) \Leftrightarrow \eta(m) = 0 \text{ or } \eta(m) = 1$$

Proof. (1) Suppose $(\bar{\eta}(m))^2 = \bar{\bar{\eta}}(m)$

$$\Leftrightarrow (\bar{\eta}(m))^2 = (1 - \eta(m))^2 = 1 - 2\eta(m) + (\eta(m))^2 = \bar{\bar{\eta}}(m) = 1 - (\eta(m))^2$$

$$\Leftrightarrow 1 - 2\eta(m) + (\eta(m))^2 = 1 - (\eta(m))^2 \Leftrightarrow 1 + 2(\eta(m))^2 - 1 - 2\eta(m) = 0$$

$$\Leftrightarrow (\eta(m))^2 - \eta(m) = 0 \Leftrightarrow \eta(m)(\eta(m) - 1) = 0 \Leftrightarrow \eta(m) = 0 \text{ or } \eta(m) = 1.$$

(2) Suppose $\bar{\eta}(m) = \bar{\bar{\eta}}(m) \Leftrightarrow \bar{\eta}(m) = 1 - \eta(m) = \bar{\bar{\eta}}(m) = 1 - (\eta(m))^2 \Leftrightarrow (\eta(m))^2 - \eta(m) = 0$

$$\Leftrightarrow \eta(m)(1 - \eta(m)) = 0 \Leftrightarrow \eta(m) = 0 \text{ or } \eta(m) = 1.$$

□

Theorem 4.2.1.2. Let η be a fuzzy subset in B and $\bar{\bar{\eta}}$ be its square deviation such that the sets:

$$B_{\eta} = \{m \in B : (\eta(m))^2 = (\eta(0))^2\} \quad \text{and} \quad B_{\bar{\bar{\eta}}} = \{m \in B : (\bar{\bar{\eta}}(m))^2 = (\bar{\bar{\eta}}(0))^2\}.$$

Then B_{η} and $B_{\bar{\bar{\eta}}}$ are SAs of B if and only if $(\eta, \bar{\bar{\eta}})$ is Pythagorean fuzzy SA of B .

Proof. Let B be a BCL–algebra and suppose $m, n \in B$ such that $(\eta, \bar{\bar{\eta}})$ is a Pythagorean fuzzy SA of B , so that $(\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and $(\bar{\bar{\eta}}(m \otimes n))^2 \leq \max\{(\bar{\bar{\eta}}(m))^2, (\bar{\bar{\eta}}(n))^2\}$

So, we need to prove that: B_{η} and $B_{\bar{\bar{\eta}}}$ are SAs of B .

$$\text{Let } m, n \in B_{\eta} \text{ or } (\eta(m))^2 = (\eta(0))^2 = (\eta(n))^2 \quad \text{and} \quad (\bar{\bar{\eta}}(m))^2 = 1 - (\eta(0))^2 = (\bar{\bar{\eta}}(n))^2$$

$$\text{But } (\eta(m \otimes n))^2 = (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$$

$$= \min\{(\eta(0))^2, (\eta(0))^2\} = (\eta(0))^2$$

$$\text{And } (\bar{\bar{\eta}}(m \otimes n))^2 \leq \max\{(\bar{\bar{\eta}}(m))^2, (\bar{\bar{\eta}}(n))^2\} = \max\{(\bar{\bar{\eta}}(0))^2, (\bar{\bar{\eta}}(0))^2\} = (\bar{\bar{\eta}}(0))^2$$

$$\text{Then, } (\eta(m \otimes n))^2 \geq (\eta(0))^2 \quad \text{and} \quad (\bar{\bar{\eta}}(m \otimes n))^2 \leq (\bar{\bar{\eta}}(0))^2$$

\Rightarrow By Lemma 4.2.1.1 above, we have:

$$(\eta(0))^2 \geq (\eta(m \otimes n))^2 \Rightarrow (\bar{\bar{\eta}}(0))^2 \leq (\bar{\bar{\eta}}(m \otimes n))^2, \forall m, n \in B$$

$$\Rightarrow (\eta(m \otimes n))^2 = (\eta(0))^2 \text{ and } (\bar{\eta}(0))^2 = (\bar{\eta}(m \otimes n))^2, \forall m, n \in B$$

$\Rightarrow m \otimes n \in B_\eta$ and $m \otimes n \in B_{\bar{\eta}}$; $\forall m, n \in B$. Thus B_η and $B_{\bar{\eta}}$ are SAs of B .

Conversely, let B_η and $B_{\bar{\eta}}$ be SAs of $B \Rightarrow m \otimes n \in B_\eta$ and $m \otimes n \in B_{\bar{\eta}}$, $\forall m, n \in B$

$$\Rightarrow (\eta(m \otimes n))^2 = (\eta(0))^2 \text{ and } \bar{\eta}(m \otimes n) = \bar{\eta}(0), \forall m, n \in B$$

Then we need to prove that $(\eta, (\bar{\eta})^2)$, is Pythagorean fuzzy SA of B

Let $m, n \in B_\eta$ and $m, n \in B_{\bar{\eta}}$

$$\Rightarrow m \otimes n \in B_\eta, \forall m, n \in B \text{ and then } m, n, m \otimes n \in B_\eta$$

$$(m \otimes n \in B_{\bar{\eta}}, \forall m, n \in B \text{ and then } m, n, m \otimes n \in B_{\bar{\eta}})$$

$$\Rightarrow (\eta(m))^2 = (\eta(0))^2 = (\eta(n))^2 \text{ and } (\bar{\eta}(m))^2 = (\bar{\eta}(0))^2 = (\bar{\eta}(n))^2$$

$$\Rightarrow (\eta(m \otimes n))^2 = (\eta(0))^2 \text{ and } (\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(0))^2$$

$$\Rightarrow (\eta(m \otimes n))^2 = (\eta(0))^2 \geq \min\{(\eta(0))^2, (\eta(0))^2\} = \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(0))^2 \leq \max\{(\bar{\eta}(0))^2, (\bar{\eta}(0))^2\} = \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Therefore, $(\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B . □

Theorem 4.2.1.3. Let η be a fuzzy subset in B . Then, for $m, n \in B$, $(\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B if $(\eta(m \otimes n))^2 = (\eta(0))^2$ and $(\bar{\eta}_s(m \otimes n))^2 = (\bar{\eta}(0))^2$.

Proof. Let η be a fuzzy subset in B such that $(\eta(m \otimes n))^2 = (\eta(0))^2$ and $(\bar{\eta}(m))^2 = (\bar{\eta}(0))^2$,

$\forall m, n \in B$ is a fuzzy subset in B . Then we need to prove: $(\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B .

$$(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(0))^2 = \max\{(\bar{\eta}(0))^2, (\bar{\eta}(0))^2\} \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

$$\Rightarrow (\bar{\eta}(m \otimes n))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\} \text{ (and similarly, } (\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\})$$

Thus $(\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B . □

Remark 4.2.1.1. The converse of the preceding theorem (Theorem 4.2.1.3) is not true which we confirm it by the next example (Example 4.2.1.2 below).

Example 4.2.1.2. Let the fuzzy subsets η and $\bar{\eta}$ in B be defined as follows:

$$\eta(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.7, & \text{if } m = p, \\ 0.3, & \text{if } m = q, r, \end{cases} \quad \Rightarrow \quad (\bar{\eta}(m))^2 = \begin{cases} 0.36, & \text{if } m = 0, \\ 0.51, & \text{if } m = p, \\ 0.91, & \text{if } m = q, r. \end{cases}$$

Now, from Table 4.1 in Example 4.1.1.3 above, $(B; \otimes, 0)$ is BCL–algebra and

$B^P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B . But by taking $m = q$ and $n = p$, we have:

$$(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(q \otimes p))^2 = (\bar{\eta}(r))^2 = 0.91 \neq (\bar{\eta}(0))^2 = 0.36 \text{ and}$$

$$(\eta(m \otimes n))^2 = (\eta(q \otimes p))^2 = (\eta(r))^2 = 0.09 \neq (\eta(0))^2 = 0.64$$

Then $B^P = (\eta_s, \bar{\eta})$ is a Pythagorean fuzzy SA of B does not imply $(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(0))^2$

$$\text{nor } (\eta(m \otimes n))^2 = (\eta(0))^2.$$

Theorem 4.2.1.4. *The intersection of any two Pythagorean fuzzy SAs of a BCL–algebra B is Pythagorean fuzzy SA of B .*

Proof. Let $B_1 = (\eta_1, \tau_1)$ and $B_2 = (\eta_2, \tau_2)$ be any two Pythagorean fuzzy SAs of B . Then we need to prove that $B_1 \cap B_2$ is a Pythagorean fuzzy SA of B . Thus, let $m, n \in B$.

$$\begin{aligned} ((\eta_1 \cap \eta_2)(m \otimes n))^2 &= \min\{(\eta_1(m \otimes n))^2, (\eta_2(m \otimes n))^2\} \\ &\geq \min\{\min\{(\eta_1(m))^2, (\eta_1(n))^2\}, \min\{(\eta_2(m))^2, (\eta_2(n))^2\}\} \\ &= \min\{\min\{(\eta_1(m))^2, (\eta_2(m))^2\}, \min\{(\eta_1(n))^2, (\eta_2(n))^2\}\} \\ &= \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\} \end{aligned}$$

$$((\eta_1 \cap \eta_2)(m \otimes n))^2 \geq \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\} \quad \text{and}$$

$$\begin{aligned} ((\tau_1 \cap \tau_2)(m \otimes n))^2 &= \max\{(\tau_1(m \otimes n))^2, (\tau_2(m \otimes n))^2\} \\ &\leq \max\{\max\{(\tau_1(m))^2, (\tau_1(n))^2\}, \max\{(\tau_2(m))^2, (\tau_2(n))^2\}\} \\ &= \max\{\max\{(\tau_1(m))^2, (\tau_2(m))^2\}, \max\{(\tau_1(n))^2, (\tau_2(n))^2\}\} \\ &= \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2\} \end{aligned}$$

$$((\tau_1 \cap \tau_2)(m \otimes n))^2 \leq \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2\}$$

Hence, the intersection of two Pythagorean fuzzy SAs of a BCL–algebra B is also Pythagorean fuzzy

SA of B as proved above, $B_1 \cap B_2$ is a Pythagorean fuzzy SA of B . □

The above theorem can also be generalized to any family of Pythagorean fuzzy SA of B as in the corollary below.

Corollary 4.2.1.1. *The intersection, $\left(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i\right)$, of any family of Pythagorean fuzzy SAs, $\{(\eta_i, \tau_i) : i \in I\}$, of B is also a Pythagorean fuzzy SA of B .*

Proof. Let $\{(\eta_i, \tau_i) : i \in I\}$, and $B_2 = (\eta_2, \tau_2)$ be family of Pythagorean fuzzy SAs of B . Then we need to prove that $\left(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i\right)$, is a Pythagorean fuzzy SA of B . Thus, let $m, n \in B$.

$$\begin{aligned} \bigcap_{i \in I} (\eta_i(m \otimes n))^2 &= \inf_{i \in I} \{(\eta_i(m \otimes n))^2\} \\ &\geq \inf_{i \in I} \{\min\{(\eta_i(m))^2, (\eta_i(n))^2\}\} \\ &\geq \min\{\inf_{i \in I} \{(\eta_i(m))^2, (\eta_i(n))^2\}\} \\ &= \min\{\inf_{i \in I} \{(\eta_i(m))^2\}, \inf_{i \in I} \{(\eta_i(n))^2\}\} \\ &= \min\left\{\bigcap_{i \in I} (\eta_i(m))^2, \bigcap_{i \in I} (\eta_i(n))^2\right\}, \quad \forall m, n \in B \quad \text{and} \end{aligned}$$

$$\begin{aligned} \bigcap_{i \in I} (\tau_i(m \otimes n))^2 &= \sup_{i \in I} \{(\tau_i(m \otimes n))^2\} \\ &\leq \sup_{i \in I} \{\max\{(\tau_i(m))^2, (\tau_i(n))^2\}\} \\ &\leq \max\{\sup_{i \in I} \{(\tau_i(m))^2, (\tau_i(n))^2\}\} \\ &= \max\{\sup_{i \in I} \{(\tau_i(m))^2\}, \sup_{i \in I} \{(\tau_i(n))^2\}\} \\ &= \max\left\{\bigcap_{i \in I} (\tau_i(m))^2, \bigcap_{i \in I} (\tau_i(n))^2\right\}, \quad \forall m, n \in B \end{aligned}$$

Therefore, $\left(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i\right)$ is a Pythagorean fuzzy SA of B . □

Remark 4.2.1.2. *The union of any two Pythagorean fuzzy SAs of a BCL–algebra B need not be Pythagorean fuzzy SA of B , which we show it by an example as follows.*

Example 4.2.1.3. Let $(B; \otimes, 0)$ be as defined in Table 4.1 of Example 4.1.1.3 and define two membership functions η_1 and η_2 of the Pythagorean fuzzy SAs $B_1 = (\eta_1, \tau_1)$, $B_2 = (\eta_2, \tau_2)$ of B respectively to show that the union fails by using only the membership functions, as follows :

$$\eta_1(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.8, & \text{if } m = p, \\ 0.2, & \text{if } m = q, r, \end{cases} \quad \text{and} \quad \eta_2(m) = \begin{cases} 0.7, & \text{if } m = 0, \\ 0.6, & \text{if } m = q, \\ 0.1, & \text{if } m = p, r, \end{cases}$$

and we get:

$$(\eta_1 \cup \eta_2)(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0.8, & \text{if } m = p, \\ 0.6, & \text{if } m = q, \\ 0.2, & \text{if } m = r, \end{cases} \Rightarrow ((\eta_1 \cup \eta_2)(m))^2 = \begin{cases} 1, & \text{if } m = 0, \\ 0.64, & \text{if } m = p, \\ 0.36, & \text{if } m = q, \\ 0.04, & \text{if } m = r. \end{cases}$$

Take $m = p$ and $n = q \Rightarrow m \otimes n = p \otimes q = r$, then we need to show that

$$((\eta_1 \cup \eta_2)(p \otimes q))^2 \geq \min\{((\eta_1 \cup \eta_2)(p))^2, ((\eta_1 \cup \eta_2)(q))^2\} \text{ is not true.}$$

$$\text{Now, } ((\eta_1 \cup \eta_2)(p \otimes q))^2 = ((\eta_1 \cup \eta_2)(r))^2 = 0.04$$

$$\geq \min\{((\eta_1 \cup \eta_2)(p))^2, ((\eta_1 \cup \eta_2)(q))^2\}$$

$$= \min\{0.64, 0.36\} = 0.36 \text{ which is false,}$$

$$\Rightarrow (\eta \cup \eta)(p \otimes p)^2 = 0.04 \geq \min\{((\eta \cup \eta)(p))^2, ((\eta \cup \eta)(q))^2\} = 0.36 \text{ is false.}$$

Therefore, the union of two Pythagorean fuzzy SAs of BCL–algebra B is not necessarily a Pythagorean fuzzy SA of B .

Proposition 4.2.1.2. Let U be non-empty subset of B such that χ_U is Characteristic function and the square deviation; $(\bar{\chi}_U(m))^2 = 1 - (\chi_U(m))^2$. Then for Pythagorean fuzzy SA, $B^P = (\chi_U, \bar{\chi}_U)$ of B , the accuracy function α_p , the score function s_p and the degree of indeterminacy π_p are respectively given as follows, $\forall m \in B$:

$$(a) \quad \alpha_p(m) = 1, \quad (b) \quad s_p(m) = \begin{cases} 1, & \text{if } m \in U, \\ -1, & \text{if } m \notin U, \end{cases} \quad (c) \quad \pi_p(m) = 0.$$

Proof. Let $\chi_U : U \rightarrow \{0, 1\}$ be Characteristic function and the square deviation $\bar{\chi}_U : U \rightarrow \{0, 1\}$ be such that $(\bar{\chi}_U(m))^2 = 1 - (\chi_U(m))^2$.

$$\text{Then: } \chi_{\mathcal{U}}(m) = \begin{cases} 1, & \text{if } m \in \mathcal{U}, \\ 0, & \text{if } m \notin \mathcal{U}, \end{cases} \quad \text{and then } (\bar{\chi}_{\mathcal{U}}(m))^2 = \begin{cases} 0, & \text{if } m \in \mathcal{U}, \\ 1, & \text{if } m \notin \mathcal{U}. \end{cases}$$

Hence, as proved in Theorem 4.2.1.1, $B^P = (\chi_{\mathcal{U}}, \bar{\chi}_{\mathcal{U}})$ is a Pythagorean fuzzy SA of B, and then:

(a) The accuracy function:

$$\alpha_p(m) = (\chi_{\mathcal{U}}(m))^2 + (\bar{\chi}_{\mathcal{U}}(m))^2 = \begin{cases} 1 + 0, & \text{if } m \in \mathcal{U}, \\ 0 + 1, & \text{if } m \notin \mathcal{U} \end{cases} = 1, \quad \forall m \in B$$

(b) The score function:

$$s_p(m) = (\chi_{\mathcal{U}}(m))^2 - (\bar{\chi}_{\mathcal{U}}(m))^2 = \begin{cases} 1 - 0, & \text{if } m \in \mathcal{U}, \\ 0 - 1, & \text{if } m \notin \mathcal{U} \end{cases} = \begin{cases} 1, & \text{if } m \in \mathcal{U}, \\ -1, & \text{if } m \notin \mathcal{U}, \end{cases}, \quad \forall m \in B,$$

(c) The degree of indeterminacy:

$$\pi_p(m) = \sqrt{1 - ((\chi_{\mathcal{U}}(m))^2 + (\bar{\chi}_{\mathcal{U}}(m))^2)} = \sqrt{1 - \alpha_p(m)} = \sqrt{1 - 1} = 0, \quad \forall m \in B. \quad \square$$

Proposition 4.2.1.3. Let η be a fuzzy subset in a BCL-algebra B such that

$$\eta(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \gamma, & \text{if } m \notin M, \end{cases} \Rightarrow (\bar{\eta}(m))^2 = \begin{cases} 1 - \delta^2, & \text{if } m \in M, \\ 1 - \gamma^2, & \text{if } m \notin M, \end{cases} \quad \text{where } \delta, \gamma \in [0, 1] \text{ and } \delta > \gamma.$$

Then M is SA of B if and only if $B^P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B.

Proof. Let M be SA of B, then we need to prove that $(\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B.

Case(i) : Let $m \in M, n \in M \Rightarrow m \otimes n \in M$, since M is SA of B

$$\text{Then } \eta(m) = \eta(n) = \delta \Rightarrow \eta(m \otimes n) = \delta \Rightarrow \eta(m \otimes n) = \eta(m) = \eta(n) = \delta$$

$$\Rightarrow \eta(m \otimes n) \geq \delta = \min\{\eta(m), \eta(n)\} \text{ and also,}$$

$$(\bar{\eta}(m \otimes n))^2 = 1 - \delta^2 = (\bar{\eta}(m))^2 = (\bar{\eta}(n))^2 \leq 1 - \delta^2 = \min\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Case(ii) : Let $m \in M, n \notin M$ (or similarly let $m \notin M, n \in M$)

(This second part can be completed in a similar way as the first part.)

$$\Rightarrow \eta(m) = \delta, \eta(n) = \gamma \Rightarrow \eta(m \otimes n) \geq \gamma = \min\{\eta(m), \eta(n)\} = \min\{\delta, \gamma\}$$

$$\Rightarrow \eta(m \otimes n) \geq \gamma = \min\{\eta(m), \eta(n)\} \text{ and}$$

$$(\bar{\eta}(m \otimes n))^2 \leq 1 - \gamma^2 = \max\{1 - \delta^2, 1 - \gamma^2\} = \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Case(iii) : Let $m \notin M, n \notin M \Rightarrow \eta(m) = \eta(n) = \gamma$

$\Rightarrow \eta(m \otimes n) \geq \gamma = \eta(m) = \eta(n) \Rightarrow \eta(m \otimes n) \geq \gamma = \min\{\eta(m), \eta(n)\}$ and also,

$$(\bar{\eta}(m \otimes n))^2 \leq 1 - \gamma^2 = (\bar{\eta}(m))^2 = (\bar{\eta}(n))^2 = \min\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Hence $(\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B.

Conversely, suppose $(\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B.

Then we show the M is SA of B, (here we can use either η or $\bar{\eta}$ so that we simply use η).

Let $m, n \in M \Rightarrow \eta(m) = \eta(n) = \delta$ and $\eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\} = \min\{\delta, \delta\} = \delta$

but $(\eta(m \otimes n) \leq \delta$, by the hypothesis $\Rightarrow \eta(m \otimes n) = \delta \Rightarrow m \otimes n \in M$

Therefore, M is SA. of B. □

Theorem 4.2.1.5. *Let η be a fuzzy subset in B such that $\eta(m \otimes n) = \eta(n)$. Then $(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2, \forall m, n \in B$. Furthermore, $B^P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B if and only if η and $\bar{\eta}$ are constants.*

Proof. Suppose $\eta(m \otimes n) = \eta(n) \Rightarrow (\eta(m \otimes n))^2 = (\eta(n))^2$.

Then $(\bar{\eta}(m \otimes n))^2 = 1 - (\eta(m \otimes n))^2 = 1 - (\eta(n))^2 = (\bar{\eta}(n))^2$.

Let $B^P = (\eta^2, \bar{\eta})$ be Pythagorean fuzzy SA of B. Then:

$$(\eta(m \otimes n))^2 = (\eta(n))^2 \text{ and then } (\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2.$$

Now we need to prove that η and then $\bar{\eta}$ are constants, for $\forall m, n \in B$

(or $(\eta(m))^2 = (\eta(n))^2$ and then $(\bar{\eta}(m))^2 = (\bar{\eta}(n))^2, \forall m, n \in B$.)

Since $B^P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B, η is a fuzzy SA of B, and also we have:

$\Rightarrow (\eta(0))^2 = (\eta(0 \otimes m))^2 = (\eta(m))^2, \forall m \in B$, and again

$$(\eta(0))^2 = (\eta(0 \otimes n))^2 = (\eta(n))^2, \forall n \in B \text{ (since } 0 \otimes m = 0 \text{ by Proposition 2.1.1 (3))}$$

$\Rightarrow (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2, \forall m, n \in B$

$\Rightarrow (\eta(m))^2 = (\eta(n))^2, \forall m, n \in B$ and hence, η is constant.

Additionally, $(\bar{\eta}(m))^2 = 1 - (\eta(m))^2 = 1 - (\eta(n))^2 = ((\bar{\eta}(n))^2$ so that $\bar{\eta}$ is constant, too.

Conversely, suppose η and $\bar{\eta}$ are constants, or $\eta(m) = \eta(n)$ and $(\bar{\eta}(m))^2 = (\bar{\eta}(n))^2$

such that, $\forall m, n \in B$: $\eta(m \otimes n) = \eta(n)$ and $(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2$.

Then we need to prove that: $B^P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B .

$$\eta(m) = \eta(n) = \eta(m \otimes n) = \min\{\eta(m), \eta(n)\} \geq \min\{\eta(m), \eta(n)\}$$

$$\Rightarrow (\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ (and then } (\bar{\eta}(m \otimes n))^2 \leq \max\{(\bar{\eta}(m))^2, \bar{\eta}(n)\})$$

Therefore, $B^P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of B . □

Theorem 4.2.1.6. *Let η be a fuzzy subset such that η is a membership function and $\bar{\eta}$ is its square deviation in a BCL-algebra B . Suppose $B_1^b = \{\forall m \in B : (\eta(m))^2 \geq (\eta(b))^2, \text{ for some } b \in B.\}$ and then $B_2^b = \{\forall m \in B : (\bar{\eta}(m))^2 \leq (\bar{\eta}(b))^2, \text{ for some } b \in B.\}$*

If $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B then B_1^b and B_2^b are SAs of B .

Proof. Suppose $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B , or we need to show that:

$$(1) \quad m, n \in B_1^b \Rightarrow (\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$$

$$(2) \quad m, n \in B_1^b \Rightarrow (\eta(m \odot n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$$

$$(3) \quad m, n \in B_2^b \Rightarrow (\bar{\eta}(m \otimes n))^2 \leq \min\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

$$(4) \quad m, n \in B_2^b \Rightarrow (\bar{\eta}(m \odot n))^2 \leq \min\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

We prove all the four as follows:

$$(1) \quad m, n \in B_1^b \Rightarrow (\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$$

$$\geq \min\{(\eta(b))^2, (\eta(b))^2\}$$

$$= (\eta(b))^2 \text{ and}$$

$$\Rightarrow (\eta(m \otimes n))^2 \geq (\eta(b))^2, \text{ for some } b \in B \Rightarrow (m \otimes n) \in B_1^b$$

$$\Rightarrow m, n \in B_1^b \Rightarrow (m \otimes n) \in B_1^b$$

$$(2) \quad m, n \in B_1^b \Rightarrow (\eta(m \odot n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$$

$$\geq \min\{(\eta(b))^2, (\eta(b))^2\}$$

$$= (\eta(b))^2 \text{ and}$$

$$\Rightarrow (\eta((m \otimes n))^2 \geq (\eta(b))^2, \text{ for some } b \in B \Rightarrow (m \odot n) \in B_1^b$$

$$\Rightarrow m, n \in B_1^b \Rightarrow (m \otimes n) \in B_1^b$$

Therefore, from (1) and (2) above, B_1^b is SA of B.

$$\begin{aligned} (3) \quad m, n \in B_2^b &\Rightarrow (\bar{\eta}(m \otimes n))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\} \\ &\leq \max\{(\bar{\eta}(b))^2, (\bar{\eta}(b))^2\} \\ &= (\bar{\eta}(b))^2 \text{ and} \end{aligned}$$

$$\Rightarrow (\bar{\eta}((m \otimes n))^2 \leq (\bar{\eta}(b))^2, \text{ for some } b \in B \Rightarrow (m \otimes n) \in B_2^b$$

$$\Rightarrow m, n \in B_2^b \Rightarrow (m \otimes n) \in B_2^b$$

$$\begin{aligned} (4) \quad m, n \in B_2^b &\Rightarrow (\bar{\eta}(m \odot n))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\} \\ &\leq \max\{(\bar{\eta}(b))^2, (\bar{\eta}(b))^2\} \\ &= (\bar{\eta}(b))^2 \text{ and} \end{aligned}$$

$$\Rightarrow (\eta((m \otimes n))^2 \leq (\eta(b))^2, \text{ for some } b \in B \Rightarrow (m \odot n) \in B_2^b$$

$$\Rightarrow m, n \in B_2^b \Rightarrow (m \otimes n) \in B_2^b$$

Therefore, from (3) and (4) above, B_2^b is SA of B. □

Corollary 4.2.1.2. *Let η be a fuzzy subset and $\bar{\eta}$ be its square deviation in B. Then $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B if and only if η is fuzzy SA of B. Furthermore, the accuracy function α_p , the score function s_p and the degree of indeterminacy π_p are respectively given as follows, $\forall m \in B$:*

$$(i) \quad \alpha_p(m) = 1, \quad (ii) \quad s_p(m) = 2(\eta(m))^2 - 1 = 1 - 2(\bar{\eta}(m))^2, \quad (iii) \quad \pi_p(m) = 0.$$

Proof. Suppose $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B. Then we prove that η is fuzzy SA of B.

$(\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B implies

$$(\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and } (\bar{\eta}(m \otimes n))^2 \geq \min\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

From these two inequalities: $(\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ is the axiom for the definition of fuzzy SA η of B so that η is fuzzy SA of B.

Conversely, suppose η is fuzzy SA of B and then we show $(\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B.

As η is fuzzy SA of B , we have: $(\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$

$$\Rightarrow -(\eta(m \otimes n))^2 \leq -\min\{(\eta(m))^2, (\eta(n))^2\} \Rightarrow -(\eta(m \otimes n))^2 \leq \max\{-(\eta(m))^2, -(\eta(n))^2\}$$

$$\Rightarrow 1 - (\eta(m \otimes n))^2 \leq \max\{1 - (\eta(m))^2, 1 - (\eta(n))^2\} \Rightarrow (\bar{\eta}(m \otimes n))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Hence, $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B .

Furthermore, the values for α_p , s_p and π_p are justified as follows:

$$(i) \quad \alpha_p(m) = (\eta(m))^2 + (\bar{\eta}(m))^2 = (\eta(m))^2 + (1 - \eta(m))^2 = 1$$

$$(ii) \quad s_p(m) = (\eta(m))^2 - (\bar{\eta}(m))^2 = (\eta(m))^2 - (1 - \eta(m))^2 = 2(\eta(m))^2 - 1 \\ = 2(1 - \bar{\eta}(m))^2 - 1 = 1 - 2(\bar{\eta}(m))^2$$

$$(iii) \quad \pi_p(m) = \sqrt{1 - \alpha_p(m)} = \sqrt{1 - 1} = 0.$$

□

Remark 4.2.1.3. *The converse of Theorem 4.2.1.6 is not necessarily true, which is justified by the next counter example.*

Example 4.2.1.4. *Suppose $\eta : B \rightarrow [0, 1]$ is fuzzy subset in $(B; \otimes, 0)$ discussed in Example 4.1.1 and take the fuzzy subset η defined in Example 4.2.1.1, which has been checked that η is fuzzy SA of B which implies $(\eta, \bar{\eta})$ is Pythagorean fuzzy SA of B , by Corollary 4.2.1.2 above.*

Now, take $B^b = \{p, r\} = \{\forall m \in B : \eta(m) = \eta(b)\}$, where $b = p, r$. Then, after careful checking, it has been listed that $B^b = \{p, r\}$ is not SA of B .

Thus, η is fuzzy SA of B but B^b is not SA of B .

Theorem 4.2.1.7. *Let $B^P = (\eta, \tau)$ be a Pythagorean fuzzy SA of B and $m, n \in B$.*

If $\eta(m \otimes n) = \eta(n)$, and $\tau(m \otimes n) = \tau(n)$, $\forall m, y \in B$, then the following hold:

$$(1) \quad \text{The accuracy function: } \alpha_p(m) \leq 2 - [(\eta(0))^2 + (\tau(0))^2],$$

$$(2) \quad \text{The score function: } s_p(m) \leq 1 - 2(\tau(0))^2,$$

$$(3) \quad \text{The degree of indeterminacy: } \pi_p(m) \text{ is such that } (\pi_p(m))^2 \geq (\eta(m))^2 - (\eta_B(0))^2.$$

Proof. Let $B^P = (\eta, \tau)$ be a Pythagorean fuzzy SA of B ; $\eta(m \otimes n) = \eta(n)$, $\tau(m \otimes n) = \tau(n)$

Since, $B \neq \emptyset$, $\exists m \in B$ such that $0 = 0 \otimes m \Rightarrow \eta(0) = \eta(0 \otimes m) = \eta(m)$, $\forall m \in B$

$$\Rightarrow \eta(0) = \eta(m) = \eta(n) = \eta(w), \forall m, n, w \in B$$

And similarly, $\tau(0) = \tau(m) = \tau(n) = \tau(w), \forall m, n, w \in B$

$$\text{But } 0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1 \Rightarrow 0 \leq (\eta(m))^2 + (\tau(0))^2 \leq 1$$

$$\Rightarrow (\eta(m))^2 \leq 1 - (\tau(0))^2 \dots\dots\dots (*)$$

$$0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1 \Rightarrow 0 \leq (\eta(0))^2 + (\tau(m))^2 \leq 1 \Rightarrow (\tau(m))^2 \leq 1 - (\eta(0))^2 \dots\dots (**)$$

Then, $\forall m \in B$, we have the following:

(1) The accuracy function:

$$\begin{aligned} a_p(m) &= (\eta(m))^2 + (\tau(m))^2 \leq (1 - (\eta(0))^2) + (1 - (\tau(0))^2) \dots\dots\dots (\text{From } (*) \text{ and } (**)) \\ &= 2 - (\eta(0))^2 - (\tau(0))^2 \end{aligned}$$

(2) The score function:

$$s_p(m) = (\eta(m))^2 - (\tau(m))^2 \leq (1 - (\eta(0))^2) - (\tau(0))^2 = 1 - (\eta(0))^2 - (\tau(0))^2$$

(3) The degree of indeterminacy:

$$\begin{aligned} (\pi_p(m))^2 &= 1 - (\eta(m))^2 - (\tau(m))^2 \geq 1 - (\eta(m))^2 - (1 - (\eta(0))^2) = (\eta(0))^2 - (\eta(m))^2 \\ \Rightarrow (\pi_p(m))^2 &= (\eta(0))^2 - (\eta(m))^2. \end{aligned} \quad \square$$

Theorem 4.2.1.8. *If $B^P = (\eta, \tau)$ is Pythagorean fuzzy SA of B such that $\bar{\eta}$ and $\bar{\tau}$ are square deviations and complements of η , respectively and $\bar{\bar{\eta}}$ and $\bar{\bar{\tau}}$ are square deviations and complements of τ , respectively, then the following ordered pairs are each Pythagorean fuzzy SAs of B:*

- | | | | |
|--------------------------|--------------------------------|--------------------------------------|--------------------------------------------|
| (1) $(\eta, \bar{\eta})$ | (2) $(\eta, \bar{\bar{\eta}})$ | (3) $(\bar{\tau}, \bar{\eta})$ | (4) $(\bar{\tau}, \bar{\bar{\eta}})$ |
| (5) $(\bar{\tau}, \tau)$ | (6) $(\bar{\bar{\tau}}, \tau)$ | (7) $(\bar{\bar{\tau}}, \bar{\eta})$ | (8) $(\bar{\bar{\tau}}, \bar{\bar{\eta}})$ |

Proof. $B^P = (\eta, \tau)$ is Pythagorean fuzzy SA of B means η is membership function and τ is non-membership function of B. As explained by Corollary 4.2.1.2 for $\bar{\eta}$ (which is similar for $\bar{\eta}$) and for the non-membership map τ : $(\tau(m \otimes n))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$

$$\begin{aligned} \Rightarrow -(\tau(m \otimes n))^2 &\geq -\min\{(\tau(m))^2, (\tau(n))^2\} \Rightarrow -(\tau(m \otimes n))^2 \geq \min\{-(\tau(m))^2, -(\tau(n))^2\} \\ \Rightarrow 1 - (\tau(m \otimes n))^2 &\geq \min\{1 - (\tau(m))^2, 1 - (\tau(n))^2\} \Rightarrow (\bar{\bar{\tau}}(m \otimes n))^2 \geq \min\{(\bar{\bar{\tau}}(m))^2, (\bar{\bar{\tau}}(n))^2\} \end{aligned}$$

Then we deduce that $\bar{\bar{\tau}}$ is membership map (and similarly the complement $\bar{\tau}$ is also membership map) for the Pythagorean fuzzy SA (η, τ) of B.

This implies the square deviations; $\bar{\tau}$ is membership function and $\bar{\eta}$ is non-membership function of the Pythagorean fuzzy SA of B.

Hence η , $\bar{\tau}$ and $\bar{\tau}$ are membership functions, and τ , $\bar{\eta}$ and $\bar{\eta}$ are non-membership functions of B. Then pairing each membership function with any of the non-membership function of B, we obtain Pythagorean fuzzy SAs of B and hence the proofs of the above and other similar pairs are assured as proved previously for different forms. □

4.2.2 Pythagorean Fuzzy Deductive Systems of BCL–Algebra

In this subsection, we introduce and define Pythagorean fuzzy DS of BCL–algebra, state and prove some important and novel properties of the Pythagorean fuzzy DS of BCL–algebra.

Definition 4.2.2.1. A Pythagorean fuzzy set $B^P = (\eta, \tau)$, in a BCL–algebra B is called a Pythagorean fuzzy deductive system of B where the functions $\eta : B \rightarrow [0, 1]$ and $\tau : B \rightarrow [0, 1]$ define the degree of membership and degree of non-membership, respectively, if and only if the following axioms are satisfied, $\forall m, n, w \in B$:

- (i) $(\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2$ and $(\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2$
- (ii) $(\eta(m \otimes (n \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and $(\tau(m \otimes (n \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$.

Example 4.2.2.1. Let B and the binary operation \otimes be as defined as in Table 1.1 of Example 1.1.1 above and let the fuzzy subsets $\eta : B \rightarrow [0, 1]$ and $\tau : B \rightarrow [0, 1]$ be defined as follows:

$$\eta(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.5, & \text{if } m = r, \\ 0.1 & \text{if } m = p, q, \end{cases} \quad \text{and} \quad \tau(m) = \begin{cases} 0.2, & \text{if } m = 0, \\ 0.4, & \text{if } m = r, \\ 0.7, & \text{if } m = p, q. \end{cases}$$

By manipulating basic calculations, we check that $B^P = (\eta, \tau)$ is Pythagorean fuzzy DS of B as follows:

Take $m = r$, $n = p$, $u = 0$, $w = q$, and then $(r \otimes 0) \otimes 0 = r \otimes 0 = r$ and $(r \otimes (p \otimes q)) = r \otimes r = 0$

(i) $(\eta(r \otimes 0) \otimes 0)^2 = (\eta(r))^2 = 0.25 \geq (\eta(r))^2 = 0.25$ holds and

$(\tau(r \otimes 0) \otimes 0)^2 = (\tau(r))^2 = 0.16 \leq (\tau(r))^2 = 0.16$ holds

- (ii) $(\eta(r \otimes (p \otimes q)))^2 = (\eta(0))^2 = 0.64 \geq \min\{(\eta(r))^2, (\eta(p))^2\} = \min\{0.64, 0.01\} = 0.01$ holds
 $(\tau(r \otimes (p \otimes q)))^2 = (\tau(0))^2 = 0.04 \leq \max\{(\tau(r))^2, (\tau(p))^2\} = \max\{0.16, 0.49\} = 0.49$ holds

The other cases, also hold similarly so that $B^P = (\eta, \tau)$ is Pythagorean fuzzy DS of B .

Theorem 4.2.2.1. Let $B^P = (\eta, \tau)$ be Pythagorean fuzzy set in B . If B^P is Pythagorean fuzzy DS of B , then the following hold, $\forall m, n, u \in B$:

- (i) $(\eta(0))^2 \geq (\eta(m))^2$ and $(\tau(0))^2 \leq (\tau(m))^2$
(ii) $m \otimes n = n \Rightarrow (\eta(n))^2 \geq (\eta(m))^2$ and $(\tau(n))^2 \leq (\tau(m))^2$;
 $(\eta(m \otimes (n \otimes w)))^2 = (\eta(w))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and
 $(\tau(m \otimes (n \otimes w)))^2 = (\tau(w))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$
(iii) $m \otimes n = n \otimes m \Rightarrow (\eta(m \otimes w \otimes m))^2 \geq (\eta(m))^2$ and $(\eta(m \otimes w \otimes m))^2 \geq (\eta(w))^2$;
 $(\tau((m \otimes w) \otimes m))^2 \leq (\tau(m))^2$ and $(\tau((m \otimes w) \otimes m))^2 \leq (\eta(w))^2$.

Proof. Let $B^P = (\eta, \tau)$ be Pythagorean fuzzy DS of

- (i) From the first axiom of the Pythagorean fuzzy DS; (η, τ) : $(\eta(m \otimes u) \otimes u)^2 \geq \eta(m)$, since u is arbitrary element of B , put $u = m$. Then, $(m \otimes u) \otimes u = (m \otimes m) \otimes m = 0 \otimes m = 0$ and then $(\eta(m \otimes u) \otimes u)^2 \geq (\eta(m))^2 \Rightarrow (\eta(0))^2 \geq (\eta(m))^2$

- (ii) Let $m \otimes n = n$, then from the first axiom of Pythagorean fuzzy DS of B , we have:

$$\Rightarrow (\eta(m \otimes n) \otimes n)^2 \geq (\eta(m))^2 \text{ and } (\tau(m \otimes n) \otimes n)^2 \leq (\tau(m))^2$$

$$\Rightarrow (\eta(m \otimes n) \otimes n)^2 = (\eta(n))^2 \geq (\eta(m))^2 \text{ and}$$

$$(\tau(m \otimes n) \otimes n)^2 = (\tau(n))^2 \leq (\tau(m))^2 \text{ and also from the second axiom:}$$

$$(\eta(m \otimes (n \otimes w)))^2 = (\eta(n \otimes w))^2 = (\eta(w))^2 = (\eta(w))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau(m \otimes (n \otimes w)))^2 = (\tau(n \otimes w))^2 = (\tau(w))^2 = (\tau(w))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

- (iii) Let $m \otimes n = n \otimes m$ and by the axioms of Pythagorean fuzzy DS of B , we have:

$$\Rightarrow (\eta(m \otimes w \otimes m))^2 = (\eta(m \otimes (m \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(m))^2\} = (\eta(m))^2 \text{ and}$$

$$\Rightarrow (\eta(m \otimes w \otimes m))^2 \geq (\eta(m))^2 \text{ (by the second axiom) and}$$

$$\begin{aligned}
&\Rightarrow (\eta(m \otimes w) \otimes m))^2 = (\eta(w \otimes m) \otimes m))^2 \geq (\eta(w))^2 \text{ and} \\
&\Rightarrow (\eta(m \otimes w) \otimes m))^2 \geq (\eta(w))^2 \text{ (by the first axiom) and for } \tau: \\
&\Rightarrow (\tau(m \otimes w) \otimes m))^2 = (\tau(m \otimes (m \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(m))^2\} = (\tau(m))^2 \text{ and} \\
&\Rightarrow (\tau(m \otimes w) \otimes m))^2 \leq (\tau(m))^2 \text{ (by the second axiom) and} \\
&\Rightarrow (\tau(m \otimes w) \otimes m))^2 = (\tau(w \otimes m) \otimes m))^2 \leq (\tau(w))^2 \text{ and} \\
&\Rightarrow (\tau(m \otimes w) \otimes m))^2 \leq (\tau(w))^2 \text{ (by the first axiom)}
\end{aligned}$$

□

Theorem 4.2.2.2. *Let η and its square deviation $\bar{\eta}$ be fuzzy subsets in BCL–algebra B such that $(\eta(m \otimes n))^2 = (\eta(n))^2$. Then $(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2, \forall m, n \in B$. Additionally, $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy DS of B if and only if η and then $\bar{\eta}$ are constants. Furthermore, the accuracy function $\alpha_D(m)$, the score function $s_D(m)$ and the degree of indeterminacy $\pi_D(m)$ are respectively given as: $\forall m \in B$:*

$$(a) \alpha_D(m) = 1, \quad (b) s_D(m) = 2(\eta(m))^2 - 1, \quad (c) \pi_D(m) = 0.$$

Proof. Suppose $(\eta(m \otimes n))^2 = (\eta(n))^2$

$$\text{Then } (\bar{\eta}(m \otimes n))^2 = 1 - (\eta(m \otimes n))^2 = 1 - (\eta(n))^2 = (\bar{\eta}(n))^2 \text{ and}$$

Let $B^P = (\eta, \bar{\eta})$ be Pythagorean fuzzy ideal of B , (and then $(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2$),

Now, we claim to verify that η and $\bar{\eta}$ are constants,

$$\text{(or } (\eta(m))^2 = (\eta(n))^2 \text{ and then } (\bar{\eta}(m))^2 = (\bar{\eta}(n))^2, \forall m, n \in B.)$$

As $B^P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy DS of B , η is a fuzzy DS of B , and hence

we have: $\eta(0) = \eta(0 \otimes m), \forall m \in B$, (since $0 = 0 \otimes m, \forall m \in B$ by Proposition 2.1.1)

$$\Rightarrow (\eta(0))^2 = (\eta((0 \otimes m) \otimes m))^2 = (\eta(m))^2, \forall m \in B, \text{ and again}$$

$$(\eta(0))^2 = (\eta((0 \otimes n) \otimes n))^2 = (\eta(n))^2, \forall m \in B$$

$$\Rightarrow (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2 \Rightarrow \eta(0)^2 = \eta(m)^2 = \eta(n)^2, \forall m, n \in B,$$

Or $(\eta(m))^2 = (\eta(n))^2, \forall m, n \in B$ and hence, η is constant, and analogously, $\bar{\eta}$ is, too.

Conversely, let η and $\bar{\eta}$ be constants, or: $\eta(m) = \eta(n)$ and $\bar{\eta}(m) = \bar{\eta}(n), \forall m, n \in B$

$$\Rightarrow (\eta(m \otimes n))^2 = (\eta(n))^2 \text{ and } (\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2.$$

Now, we prove: $B^P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy DS of BCL-algebra B, as follows:

- (i) $(\eta(m))^2 = (\eta(n))^2 = (\eta((m \otimes n) \otimes n))^2$
 $\Rightarrow (\eta((m \otimes n) \otimes n))^2 \geq (\eta(m))^2$, (and then $\{(\bar{\eta}((m \otimes n) \otimes n))^2 \leq (\bar{\eta}(m))^2\}$)
- (ii) $(\eta(m \otimes (n \otimes w)))^2 = (\eta(m))^2 = (\eta(n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and then
 $(\bar{\eta}(m \otimes (n \otimes w)))^2 = (\bar{\eta}(m))^2 = (\bar{\eta}(n))^2 \leq \min\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$

Therefore, by (i) and (ii) above, $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy DS of B. Furthermore:

- (a) $\alpha_p(m) = (\eta(m))^2 + (\bar{\eta}(m))^2 = (\eta(m))^2 + (1 - (\eta(m))^2) = 1$
(b) $s_p(m) = (\eta(m))^2 - (\bar{\eta}(m))^2 = (\eta(m))^2 - (1 - (\eta(m))^2) = 2(\eta(m))^2 - 1$
(c) $\pi_p(m) = \sqrt{1 - \alpha_p(m)} = 0.$

□

Proposition 4.2.2.1. Let U be a non-empty subset of B such that χ_U is Characteristic function and $\bar{\chi}_U$ is its square deviation. Then $B^P = (\chi_U, \bar{\chi}_U)$ is Pythagorean fuzzy DS of B if and only if U is DS of B .

Proof. Let $\chi_U: U \rightarrow \{0, 1\}$ be Characteristic function and the square deviation $\bar{\chi}_U: U \rightarrow \{0, 1\}$ be

$$\text{defined as: } \chi_U(m) = \begin{cases} 1, & \text{if } m \in U, \\ 0, & \text{if } m \notin U, \end{cases} \text{ and then } (\bar{\chi}_U(m))^2 = \begin{cases} 0, & \text{if } m \in U, \\ 1, & \text{if } m \notin U. \end{cases}$$

Under such particular case where $1^2 = 1$ and $0^2 = 0$, we omit squaring the membership and non-membership expressions. Now, suppose $B^P = (\chi_U, \bar{\chi}_U)$ is Pythagorean fuzzy DS of B . Then,

- (1) $\chi_U((m \otimes u) \otimes u) \geq \chi_U(m)$ and $\bar{\chi}_U((m \otimes u) \otimes u) \leq \bar{\chi}_U(m)$
(2) $\chi_U(m \otimes (n \otimes w)) \geq \min\{\chi_U(m), \chi_U(n)\}$ and $\bar{\chi}_U(m \otimes (n \otimes w)) \leq \max\{\bar{\chi}_U(m), \bar{\chi}_U(n)\}.$

Then we claim U is DS of B or $m, n \in U \Rightarrow (m \otimes u) \otimes u \in U$ and

$$m, n \in U \Rightarrow m \otimes (n \otimes w) \in U, \forall u, w \in B.$$

Let $m, n \in U \Rightarrow \chi_U(m) = 1 = \chi_U(n)$ and $\bar{\chi}_U(m) = 0 = \bar{\chi}_U(n)$

Then $\chi_U((m \otimes u) \otimes u) \geq \chi_U(m) = 1$; $\bar{\chi}_U((m \otimes u) \otimes u) \leq \bar{\chi}_U(m) = 0$ and

$\chi_U((m \otimes u) \otimes u) \geq 1$; $\bar{\chi}_U((m \otimes u) \otimes u) \leq 0$ and $\chi_U(m \otimes (n \otimes w)) \geq 1$; $\bar{\chi}_U(m \otimes (n \otimes w)) \leq 0$

But by the definitions of χ_U and $\bar{\chi}_U$, we also have:

$$\chi_u((m \otimes u) \otimes u) \leq 1; \bar{\chi}_u((m \otimes u) \otimes u) \geq 0 \text{ and } \chi_u(m \otimes (n \otimes w)) \leq 1; \bar{\chi}_u(m \otimes (n \otimes w)) \geq 0$$

Therefore, we deduce the following:

$$\chi_u((m \otimes u) \otimes u) = 1; \bar{\chi}_u((m \otimes u) \otimes u) = 0 \text{ and } \chi_u(m \otimes (n \otimes w)) = 1; \bar{\chi}_u(m \otimes (n \otimes w)) = 0$$

$(m \otimes u) \otimes u \in U$ and $(m \otimes (n \otimes w)) \in U$ and hence U is DS of B .

Therefore, U is DS of B .

Conversely, suppose U is DS of B .

We claim that $B^P = (\chi_u, \bar{\chi}_u)$ is Pythagorean fuzzy DS of B , or the following:

$$(1) \chi_u((m \otimes u) \otimes u) \geq \chi_u(m) \text{ and } \bar{\chi}_u((m \otimes u) \otimes u) \leq \bar{\chi}_u(m)$$

$$(2) \chi_u(m \otimes (n \otimes w)) \geq \min\{\chi_u(m), \chi_u(n)\} \text{ and } \bar{\chi}_u(m \otimes (n \otimes w)) \leq \min\{\bar{\chi}_u(m), \bar{\chi}_u(n)\}$$

To prove each, we follow different cases as below:

Claim (1): $\chi_u((m \otimes u) \otimes u) \geq \chi_u(m)$ and $\bar{\chi}_u((m \otimes u) \otimes u) \leq \bar{\chi}_u(m), \forall u \in U$

Case (1.1) Let $m \in U$ and $u \in B \Rightarrow ((m \otimes u) \otimes u) \in U$ (by hypothesis, since U is DS of B .)

$$\Rightarrow \chi_u((m \otimes u) \otimes u) = \chi_u(m) = 1 \Rightarrow \chi_u((m \otimes u) \otimes u) \geq \chi_u(m) \text{ and}$$

$$\Rightarrow \bar{\chi}_u((m \otimes u) \otimes u) = \bar{\chi}_u(m) = 0 \Rightarrow \bar{\chi}_u((m \otimes u) \otimes u) \leq \bar{\chi}_u(m)$$

Case (1.2) Let $m \notin U$ and $(m \otimes n) \otimes n \in U, u \in B$

$$\Rightarrow \chi_u((m \otimes n) \otimes n) = 1 \text{ and } \chi_u(m) = 0 \Rightarrow \chi_u((m \otimes n) \otimes n) \geq \chi_u(m) \text{ and}$$

$$\bar{\chi}_u((m \otimes n) \otimes n) = 0 \text{ and } \bar{\chi}_u(m) = 1 \Rightarrow \bar{\chi}_u((m \otimes n) \otimes n) \leq \bar{\chi}_u(m)$$

Case (1.3) Let $m \notin U$ and $((m \otimes n) \otimes n) \notin U, u \in B$

$$\Rightarrow \chi_u((m \otimes n) \otimes n) = \chi_u(m) = 0 \Rightarrow \chi_u((m \otimes n) \otimes n) \geq \chi_u(m) \text{ and}$$

$$\Rightarrow \bar{\chi}_u((m \otimes n) \otimes n) = \bar{\chi}_u(m) = 1 \Rightarrow \bar{\chi}_u((m \otimes n) \otimes n) \leq \bar{\chi}_u(m).$$

Claim (2): $\chi_u(m \otimes (n \otimes w)) \geq \min\{\chi_u(m), \chi_u(n)\}$ and $\bar{\chi}_u(m \otimes (n \otimes w)) \leq \max\{\bar{\chi}_u(m), \bar{\chi}_u(n)\}$

Case (2.1) Let $m, n \in U$ and $w \in B \Rightarrow (m \otimes (n \otimes w)) \in U$ by hypothesis, since U is DS of B .

$$\Rightarrow \chi_u(m \otimes (n \otimes w)) = 1 = \chi_u(m) = \chi_u(n) \Rightarrow \chi_u(m \otimes (n \otimes w)) \geq \min\{\chi_u(m), \chi_u(n)\} \text{ and}$$

$$\bar{\chi}_u(m \otimes (n \otimes w)) = 0 = \bar{\chi}_u(m) = \bar{\chi}_u(n) \Rightarrow \bar{\chi}_u(m \otimes (n \otimes w)) \leq \max\{\bar{\chi}_u(m), \bar{\chi}_u(n)\}$$

Case (2.2) Let $m \notin U$ and $n \in U$. But either $(m \otimes n) \otimes n \in U$ or $(m \otimes n) \otimes n \notin U$

$$\Rightarrow \chi_u(m \otimes (n \otimes w)) \geq 0; \chi_u(m) = 0, \chi_u(n) = 1 \Rightarrow \chi_u(m \otimes (n \otimes w)) \geq \min\{\chi_u(m), \chi_u(n)\} = 0;$$

$$\bar{\chi}_u((m \otimes (n \otimes w)) \leq 1, \bar{\chi}_u(m) = 1, \bar{\chi}_u(n) = 0 \Rightarrow \bar{\chi}_u((m \otimes n) \otimes n) \leq \max\{\bar{\chi}_u(m), \bar{\chi}_u(n)\}$$

(And the same results hold when $m \notin U$ and $n \in U$)

Case (2.3) Let $m \notin U$ and $n \notin U$. But either $(m \otimes n) \otimes n \in U$ or $(m \otimes n) \otimes n \notin U$

$$\Rightarrow \chi_u((m \otimes (n \otimes w)) \geq 0; \chi_u(m) = 0 = \chi_u(n) \Rightarrow \chi_u(m \otimes (n \otimes w)) \geq \min\{\chi_u(m), \chi_u(n)\} = 0;$$

$$\bar{\chi}_u((m \otimes (n \otimes w)) \leq 1, \bar{\chi}_u(m) = 1 = \bar{\chi}_u(n) \Rightarrow \bar{\chi}_u((m \otimes n) \otimes n) \leq \max\{\bar{\chi}_u(m), \bar{\chi}_u(n)\}$$

Therefore, by the three corresponding cases of the two claims above, $B^P = (\chi_u, \bar{\chi}_u)$

is Pythagorean fuzzy DS of B. □

Theorem 4.2.2.3. *The intersection of any two Pythagorean fuzzy DSs, D_1 and D_2 , of B is also a Pythagorean fuzzy DS of B.*

Proof. Let $D_1 = (\eta_1, \tau_1)$ and $D_2 = (\eta_2, \tau_2)$ be any two Pythagorean fuzzy DSs of B.

Then we need to prove that $D_1 \cap D_2$ is a Pythagorean fuzzy DS of B. Let $m, n, u, w \in B$.

$$(i) (\eta_1 \cap \eta_2)((m \otimes u) \otimes u)^2 = \min\{(\eta_1((m \otimes u) \otimes u))^2, (\eta_2((m \otimes u) \otimes u))^2\} \\ \geq \min\{(\eta_1(m))^2, (\eta_2(m))^2\} = (\eta_1 \cap \eta_2(m))^2 \quad \text{and}$$

$$(\tau_1 \cap \tau_2((m \otimes u) \otimes u))^2 = \max\{(\tau_1((m \otimes u) \otimes u))^2, (\tau_2((m \otimes u) \otimes u))^2\} \\ \leq \max\{(\tau_1(m))^2, (\tau_2(m))^2\} = (\tau_1 \cap \tau_2(m))^2, \quad \forall m, u \in B$$

$$(ii) (\tau_1 \cap \tau_2(m \otimes (n \otimes w)))^2 = \max\{(\tau_1(m \otimes (n \otimes w)))^2, (\tau_2(m \otimes (n \otimes w)))^2\} \\ \leq \max\{\max\{(\tau_1(m))^2, (\tau_1(n))^2\}, \max\{(\tau_2(m))^2, (\tau_2(n))^2\}\} \\ = \max\{\max\{(\tau_1(m))^2, (\tau_2(m))^2\}, \max\{(\tau_1(n))^2, (\tau_2(n))^2\}\} \\ = \max\{(\tau_1 \cap \tau_2(m))^2, (\tau_1 \cap \tau_2(n))^2\}, \quad \forall m, n, w \in B \quad \text{and similarly:}$$

$$(\eta_1 \cap \eta_2(m \otimes (n \otimes w)))^2 \geq \min\{(\eta_1 \cap \eta_2(m))^2, (\eta_1 \cap \eta_2(n))^2\}, \quad \forall m, n, w \in B.$$

(also proved in fuzzy DS of B in Theorem 4.1.2.1 (ii) above). Hence by (i) and (ii) above,

$D_1 \cap D_2$ is a Pythagorean fuzzy DS of B. □

Corollary 4.2.2.1. *The intersection, $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, of any family of Pythagorean fuzzy DSs, $\{(\eta_i, \tau_i) : i \in I\}$, of B is also a Pythagorean fuzzy DS of B.*

Proof. Let (η_i, τ_i) be family of Pythagorean fuzzy DSs of B. Then we need to prove that

$(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, is a Pythagorean fuzzy DS of B. Let $m, n, w \in B$.

$$\begin{aligned} \text{(i)} \quad \bigcap_{i \in I} (\eta_i((m \otimes u) \otimes u))^2 &= \inf_{i \in I} \left\{ (\eta_i((m \otimes u) \otimes u))^2 \right\} \\ &\geq \inf_{i \in I} \left\{ (\eta_i(m))^2 \right\} \\ &= \bigcap_{i \in I} (\eta_i(m))^2, \quad \forall m, u \in B \quad \text{and} \end{aligned}$$

$$\begin{aligned} \bigcap_{i \in I} (\tau_i((m \otimes u) \otimes u))^2 &= \sup_{i \in I} \left\{ (\tau_i((m \otimes u) \otimes u))^2 \right\} \\ &\leq \sup_{i \in I} \left\{ (\tau_i(m))^2 \right\} \\ &= \bigcap_{i \in I} (\tau_i(m))^2, \quad \forall m, u \in B \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \bigcap_{i \in I} (\tau_i(m \otimes (n \otimes w)))^2 &= \sup_{i \in I} \left\{ (\tau_i(m \otimes (n \otimes w)))^2 \right\} \\ &\leq \sup_{i \in I} \left\{ \max \{ (\tau_i(m))^2, (\tau_i(n))^2 \} \right\} \\ &\leq \max \left\{ \left(\sup_{i \in I} \{ (\tau_i(m))^2 \} \right), \left(\sup_{i \in I} \{ (\tau_i(n))^2 \} \right) \right\} \\ &= \max \left\{ \left(\sup_{i \in I} \{ (\tau_i(m))^2 \} \right), \left(\sup_{i \in I} \{ (\tau_i(n))^2 \} \right) \right\} \\ &= \max \left\{ \bigcap_{i \in I} (\tau_i(m))^2, \bigcap_{i \in I} (\tau_i(n))^2 \right\} \quad \text{and} \end{aligned}$$

$$\begin{aligned} \bigcap_{i \in I} (\eta_i(m \otimes (n \otimes w)))^2 &= \inf_{i \in I} \left\{ (\eta_i(m \otimes (n \otimes w)))^2 \right\} \\ &\geq \inf_{i \in I} \left\{ \min \{ (\eta_i(m))^2, (\eta_i(n))^2 \} \right\} \\ &\geq \min \left\{ \inf_{i \in I} \{ (\eta_i(m))^2 \}, \inf_{i \in I} \{ (\eta_i(n))^2 \} \right\} \\ &= \min \left\{ \inf_{i \in I} \{ (\eta_i(m))^2 \}, \inf_{i \in I} \{ (\eta_i(n))^2 \} \right\} \\ &= \min \left\{ \bigcap_{i \in I} (\eta_i(m))^2, \bigcap_{i \in I} (\eta_i(n))^2 \right\}, \quad \forall m, n, w \in B. \end{aligned}$$

Therefore, by (i) and (ii) above, $\left(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i\right)$, is a Pythagorean fuzzy DS of B . \square

Remark 4.2.2.1. *The union of any two Pythagorean fuzzy DSs. of B is not necessarily Pythagorean fuzzy DS. of B , justified by the next counter example (Example 4.2.2.2, below).*

Example 4.2.2.2. *Let $(B; \otimes, 0)$ be a BCL–algebra, where $B = \{a, b, c, 0\}$ and let " \otimes " be as defined in the Table 1.1 of Example 1.1.1 above, and define two Pythagorean fuzzy DSs*

$D_1 = (\eta_1, \tau_1)$ and $D_2 = (\eta_2, \tau_2)$ as follows:

$$\begin{aligned} (\eta_1(m))^2 &= \begin{cases} 0.8, & \text{if } m = 0, \\ 0.5, & \text{if } m = p, \\ 0.1, & \text{if } m = q, r, \end{cases} & \text{and} & (\tau_1(m))^2 = \begin{cases} 0.2, & \text{if } m = 0, \\ 0.4, & \text{if } m = p, r, \\ 0.7, & \text{if } m = q, \end{cases} \\ (\eta_2(m))^2 &= \begin{cases} 0.5, & \text{if } m = 0, \\ 0.4, & \text{if } m = q, \\ 0.2, & \text{if } m = p, r, \end{cases} & \text{and} & (\tau_2(m))^2 = \begin{cases} 0.1, & \text{if } m = 0, \\ 0.2, & \text{if } m = p, \\ 0.6, & \text{if } m = q, r. \end{cases} \end{aligned}$$

It is easy to check that D_1 and D_2 are Pythagorean fuzzy DSs. of B ; however, to show that their union, $D_1 \cup D_2$, is not necessarily a Pythagorean fuzzy DS. of B , we justify it as follows using the above pairs of Pythagorean fuzzy DSs of B , where $(B; \otimes, 0)$ is as defined in Table 1.1, above:

$$\begin{aligned} ((\eta_1 \cup \eta_2)(m))^2 &= \begin{cases} 0.8, & \text{if } m = 0, \\ 0.5, & \text{if } m = p, \\ 0.4, & \text{if } m = q, \\ 0.2, & \text{if } m = r, \end{cases} & \text{and} & ((\tau_1 \cup \tau_2)(m))^2 = \begin{cases} 0.1, & \text{if } m = 0, \\ 0.2, & \text{if } m = p, \\ 0.4, & \text{if } m = r, \\ 0.6, & \text{if } m = q. \end{cases} \end{aligned}$$

Now, take $m = p$, $n = q$ and $u = 0 \Rightarrow (m \otimes (n \otimes u)) = (p \otimes (q \otimes 0)) = (p \otimes q) = r$. Then

$$\begin{aligned} ((\eta_1 \cup \eta_2)(p \otimes (q \otimes 0)))^2 &= (\eta_1 \cup \eta_2)(r)^2 = 0.2 \\ &\geq \min\{((\eta_1 \cup \eta_2))(p)^2, ((\eta_1 \cup \eta_2))(q)^2\} \\ &= \min\{0.5, 0.4\} = 0.4, \text{ which is not true.} \end{aligned}$$

which is not true. Thus, by the above justifications, the union of any two Pythagorean fuzzy DSs. of B is not necessarily a Pythagorean fuzzy DS. of B .

Theorem 4.2.2.4. *Let η be a fuzzy subset such that η is a membership function and $\bar{\eta}$ is its square deviation in B . Suppose $\eta^b = \{\forall m \in B: (\eta(m))^2 \geq (\eta(b))^2, \text{ for some } b \in B\}$ and then $\bar{\eta}^b = \{\forall m \in B:$*

$(\overline{\eta}(m))^2 \leq (\overline{\eta}(b))^2$, for some $b \in B$. Then $B^P = (\eta, \overline{\eta})$ is Pythagorean fuzzy DS of B if and only if $B^b = (\eta^b, \overline{\eta}^b)$ is Pythagorean fuzzy DS of B .

Proof. Suppose $B^P = (\eta, \overline{\eta})$ is Pythagorean fuzzy DS of B , or $\forall m, n, u, \in B$:

We need to show that $B^b = (\eta^b, \overline{\eta}^b)$ is Pythagorean fuzzy DS of B .

(1) For $m \in \eta^b$; $(\eta((m \otimes n) \otimes n))^2 \geq (\eta(m))^2 \geq (\eta(b))^2$, for some $b \in B$ and

$$(\overline{\eta}((m \otimes n) \otimes n))^2 \leq (\overline{\eta}(m))^2 \leq (\overline{\eta}(b))^2; \text{ for some } b \in B$$

(2) For $m, n \in \eta^b$, $\exists b \in B$; $(\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \geq (\eta(b))^2$, and

$$(\overline{\eta}(m \otimes (n \otimes u)))^2 \leq \max\{(\overline{\eta}(m))^2, (\overline{\eta}(n))^2\} \leq (\eta(b))^2; \text{ some } b \in B.$$

Therefore, $B^b = (\eta^b, \overline{\eta}^b)$ is a Pythagorean fuzzy DS of B .

Conversely, suppose $\eta^b = \{m \in B: (\eta(m))^2 \geq (\eta(b))^2, \text{ some } b \in B\}$ and

$\overline{\eta}^b = \{m \in B: (\overline{\eta}(m))^2 \leq (\overline{\eta}(b))^2, \text{ some } b \in B\}$ such that $B^b = (\eta^b, \overline{\eta}^b)$ is Pythagorean

fuzzy DS of B . We need to prove that $B^P = (\eta, \overline{\eta})$ is Pythagorean fuzzy DS of B . By the

hypothesis we have the following:

(1) $m \in \eta^b \Rightarrow ((m \otimes n) \otimes n) \in \eta^b$ and $m \in \overline{\eta}^b \Rightarrow ((m \otimes n) \otimes n) \in \overline{\eta}^b$:

$$\Rightarrow (\eta((m \otimes n) \otimes n))^2 \geq (\eta(b))^2, (\overline{\eta}((m \otimes n) \otimes n))^2 \leq (\overline{\eta}(b))^2; \text{ some } b \in B$$

(2) $m, n \in \eta^b \Rightarrow m \otimes (n \otimes u) \in \eta^b$ and $m, n \in \overline{\eta}^b \Rightarrow m \otimes (n \otimes u) \in \overline{\eta}^b$

$$\Rightarrow (\eta^b(m \otimes (n \otimes u)))^2 \geq (\eta(b))^2, \text{ and } (\overline{\eta}^b(m \otimes (n \otimes u)))^2 \leq (\overline{\eta}(b))^2, \text{ some } b \in B.$$

Thus, by (1) and (2), $B^b = (\eta^b, \overline{\eta}^b)$ is Pythagorean fuzzy DS of B .

□

4.2.3 Pythagorean Fuzzy Ideals of BCL–Algebra

Under this subsection, we introduce Pythagorean fuzzy ideal of BCL–algebra and following this we explore different properties as theorem and justify then with corresponding proofs.

Definition 4.2.3.1. A Pythagorean fuzzy set $P = (\eta, \tau)$, where the mappings: $\eta : B \rightarrow [0, 1]$ and $\tau : B \rightarrow [0, 1]$ define the degrees of membership and non-membership, respectively in B is called a Pythagorean fuzzy ideal of BCL–algebra B , if the following axioms are satisfied, $\forall m, n, w \in B$:

- (i) $(\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\}$ and $(\tau(m))^2 \leq \max\{(\tau(m \otimes n))^2, (\tau(n))^2\}$,
- (ii) $(\eta((m \otimes (n \otimes w)) \otimes w))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and
- $$(\tau((m \otimes (n \otimes w)) \otimes w))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

Example 4.2.3.1. Let B be a BCL–algebra and the binary operation \otimes be as defined in Table 4.1 above and let the fuzzy subsets $\eta : B \rightarrow [0, 1]$ and $\tau : B \rightarrow [0, 1]$ be defined as follows:

$$\eta(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.5, & \text{if } m = p, q, r, \end{cases} \quad \text{and} \quad \tau(m) = \begin{cases} 0.5, & \text{if } m = 0, \\ 0.7, & \text{if } m = p, q, r. \end{cases}$$

Then by manipulating basic calculations, we check that $P = (\eta, \tau)$ is a Pythagorean fuzzy ideal of BCL–algebra B as follows:

Take $m = q$, $n = r$, $w = p$, $q \otimes r = q$, $(q \otimes (r \otimes p)) \otimes p = (q \otimes 0) \otimes p = q \otimes p = r$ and

$$(i) \quad (\eta(q))^2 \geq \{(\eta(q \otimes r))^2, (\eta(r))^2\}$$

$$\Rightarrow (\eta(q))^2 = 0.25 \geq \min\{(\eta(q))^2, (\eta(r))^2\} = \min\{0.25, 0.25\} = 0.25 \text{ holds and}$$

$$(\tau(q))^2 \leq \max\{(\tau(q \otimes r))^2, (\tau(r))^2\}$$

$$\Rightarrow (\tau(q))^2 = 0.25 \leq \max\{(\tau(q))^2, (\tau(r))^2\} = \max\{0.25, 0.25\} = 0.25 \text{ also holds}$$

$$(ii) \quad (\eta(q \otimes (r \otimes p)) \otimes p)^2 \geq \min\{(\eta(q))^2, (\eta(r))^2\}$$

$$\Rightarrow (\eta(r))^2 = 0.25 \geq \min\{(\eta(q))^2, (\eta(r))^2\} = \min\{0.25, 0.25\} = 0.25 \text{ holds and}$$

$$(\tau(q \otimes (r \otimes p)) \otimes p)^2 \leq \max\{(\tau(q))^2, (\tau(r))^2\}$$

$$\Rightarrow (\eta(r))^2 = 0.25 \leq \max\{(\eta(q))^2, (\eta(r))^2\} = \max\{0.25, 0.25\} = 0.25 \text{ holds}$$

And also, the other cases similarly hold and then $P = (\eta, \tau)$ is Pythagorean fuzzy ideal of B .

Lemma 4.2.3.1. Let $P = (\eta, \tau)$ be Pythagorean fuzzy ideal of BCL–algebra B . Then, $\forall m \in B$
 $(\eta(0))^2 \geq (\eta(m))^2$ and $(\tau(0))^2 \leq (\tau(m))^2$

Proof. Let B be a BCL–algebra. $B \neq \emptyset$ and hence $\exists m \in B$ such that:

$$(m \otimes (m \otimes m)) \otimes m = (m \otimes 0) \otimes m = m \otimes m = 0, \text{ (as proved in Lemma 4.1.3).}$$

$$\text{Thus } (\eta((m \otimes (m \otimes m)) \otimes m))^2 = (\eta(0))^2 \geq \min\{(\eta(m))^2, (\eta(m))^2\} = (\eta(m))^2$$

$$\Rightarrow (\eta(0))^2 \geq (\eta(m))^2 \text{ and then it is analogous to show } (\tau(0))^2 \leq (\tau(m))^2$$

□

Proposition 4.2.3.1. Let $P = (\eta, \tau)$ be Pythagorean fuzzy ideal of BCL–algebra B such that $m \leq n, \forall m, n \in B$. Then $(\eta(m))^2 \geq (\eta(n))^2$ and $(\tau(m))^2 \leq (\tau(n))^2$, that is, in a Pythagorean fuzzy ideal of BCL–algebra $P = (\eta, \tau)$, the fuzzy subsets η and τ are order reversing and order preserving, respectively.

Proof. Let (η, τ) be a Pythagorean fuzzy ideal of BCL–algebra B such that $m \leq n, \forall m, n \in B$.

We need to prove: $(\eta(m))^2 \geq (\eta(n))^2$ and $(\tau(m))^2 \leq (\tau(n))^2, \forall m, n \in B$.

Then by the binary relation " \leq " defined on B , we have: $m \leq n$, if $m \otimes n = 0$

$$\Rightarrow (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} = \min\{(\eta(0))^2, (\eta(n))^2\} = (\eta(n))^2$$

Thus, $m \leq n \Rightarrow (\eta(m))^2 \geq (\eta(n))^2 \Rightarrow \eta$ is order reversing, and

$$(\tau(m))^2 \leq \max\{(\tau(m \otimes n))^2, (\tau(n))^2\} = \max\{(\tau(0))^2, (\tau(n))^2\} = (\tau(n))^2$$

$$\Rightarrow m \leq n \Rightarrow (\tau(m))^2 \leq (\tau(n))^2 \Rightarrow \tau \text{ is order preserving.}$$

□

Proposition 4.2.3.2. Let $P = (\eta, \tau)$ be a Pythagorean fuzzy ideal of BCL–algebra B . Then the following hold true, $\forall m, n, w \in B$:

$$(1) (\eta(m \otimes n))^2 = (\eta(0))^2 \Rightarrow (\eta(m))^2 \geq (\eta(n))^2 \text{ and } (\tau(m))^2 \leq (\tau(n))^2$$

$$(2) n \leq m \otimes n \Rightarrow (\eta(m))^2 \geq (\eta(m \otimes n))^2 \text{ and } (\tau(m))^2 \leq (\tau(m \otimes n))^2$$

$$(3) m \otimes n \leq n \Rightarrow (\eta(m))^2 \geq (\eta(n))^2 \text{ and } (\tau(m))^2 \leq (\tau(n))^2$$

$$(4) w \otimes n \leq m \Rightarrow (\eta(w))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \text{ and } (\tau(w))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

Proof. Suppose (η, τ) is a Pythagorean fuzzy ideal of BCL–algebra B ,

(1) Suppose $(\eta(m \otimes n))^2 = (\eta(0))^2$, and then we need to prove:

$$(\eta(m))^2 \geq (\eta(n))^2 \text{ and } (\tau(m))^2 \leq (\tau(n))^2$$

$$\text{Now, } (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} = \min\{(\eta(0))^2, (\eta(n))^2\} = (\eta(n))^2$$

$$\Rightarrow (\eta(m))^2 \geq (\eta(n))^2 \text{ and}$$

$$(\tau(m))^2 \leq \max\{(\tau(m \otimes n))^2, (\tau(n))^2\} = \max\{(\tau(0))^2, (\tau(n))^2\} = (\tau(n))^2$$

$$\Rightarrow (\tau(m))^2 \leq (\tau(n))^2.$$

Thus, $(\eta(m))^2 \geq (\eta(n))^2$ and $(\tau(m))^2 \leq (\tau(n))^2$

(2) Suppose $n \leq m \circledast n \Rightarrow (\eta(n))^2 \geq (\eta(m \circledast n))^2$ and $(\tau(n))^2 \leq (\tau(m))^2$, by Proposition 4.2.3.1

so that we need to prove that: $(\eta(m))^2 \geq (\eta(m \circledast n))^2$ and $(\tau(m))^2 \leq (\tau(m \circledast n))^2$:

$$(\eta(m))^2 \geq \min\{(\eta(m \circledast n))^2, (\eta(n))^2\} = (\eta(m \circledast n))^2 \Rightarrow (\eta(m))^2 \geq (\eta(m \circledast n))^2, \text{ and}$$

$$(\tau(m))^2 \leq \max\{(\tau(m \circledast n))^2, (\tau(n))^2\} = (\tau(m \circledast n))^2 \Rightarrow (\tau(m))^2 \leq (\tau(m \circledast n))^2,$$

Thus, $(\eta(m))^2 \geq (\eta(m \circledast n))^2$ and $(\tau(m))^2 \leq (\tau(m \circledast n))^2$

(3) Let $m \circledast n \leq n \Rightarrow (\eta(n))^2 \leq (\eta(m \circledast n))^2$ and $(\tau(n))^2 \geq (\tau(m \circledast n))^2$, by Proposition 4.2.3.1

then we prove that $(\eta(m))^2 \geq (\eta(n))^2$ and $(\tau(m))^2 \leq (\tau(n))^2$:

$$\Rightarrow (\eta(m))^2 \geq \min\{(\eta(m \circledast n))^2, (\eta(n))^2\} = (\eta(n))^2 \Rightarrow (\eta(m))^2 \geq (\eta(n))^2 \text{ and}$$

$$\Rightarrow (\tau(m))^2 \leq \max\{(\tau(m \circledast n))^2, (\tau(n))^2\} = (\tau(n))^2 \Rightarrow (\tau(m \circledast n))^2 \leq (\tau(n))^2$$

Thus, $(\eta(m))^2 \geq (\eta(n))^2$ and $(\tau(m))^2 \leq (\tau(n))^2$

(4) Let $w \circledast n \leq m \Rightarrow (\eta(m))^2 \leq (\eta(w \circledast n))^2$ and $(\tau(m))^2 \geq (\tau(w \circledast n))^2$,

by Proposition 4.2.3.1.

Then we need to prove that $(\eta(w))^2 \geq \min\{(\eta(m \circledast n))^2, (\eta(n))^2\}$

$$\Rightarrow (\eta(w))^2 \geq \min\{(\eta(w \circledast n))^2, (\eta(n))^2\} \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$\geq \min\{\min\{(\eta(m \circledast n))^2, (\eta(n))^2\}, (\eta(n))^2\} = \min\{(\eta(m \circledast n))^2, (\eta(n))^2\}$$

Thus, $(\eta(w))^2 \geq \min\{(\eta(m \circledast n))^2, (\eta(n))^2\}$ and

$$(\tau(w))^2 \leq \max\{(\tau(w \circledast n))^2, (\tau(n))^2\} \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

$$\leq \max\{\max\{(\tau(m))^2, (\tau(n))^2\}, (\tau(n))^2\} = \max\{(\tau(m))^2, (\tau(n))^2\}.$$

Thus, $(\tau(w))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$

□

Proposition 4.2.3.3. *Let U be a non-empty subset of B such that χ_U is Characteristic function in B with the square deviation such that $(\overline{\chi_U}(m))^2 = 1 - (\chi_U(m))^2$. Then $P = (\chi_U, \overline{\chi_U})$ is Pythagorean fuzzy ideal of BCL-algebra B if and only if U is ideal of B . Furthermore, the accuracy function α , the score function s and the degree of indeterminacy π are, respectively given as follows, $\forall m \in B$:*

$$(a) \quad a(m) = 1, \quad (b) \quad s(m) = \begin{cases} 1, & \text{if } m \in U, \\ -1, & \text{if } m \notin U. \end{cases} \quad (c) \quad \pi(m) = 0.$$

Proof. Let $\chi_U : U \rightarrow \{0, 1\}$ be a Characteristic function and the square deviation $\bar{\chi}_U : U \rightarrow \{0, 1\}$

$$\text{be defined as: } \chi_U(m) = \begin{cases} 1, & \text{if } m \in U, \\ 0, & \text{if } m \notin U, \end{cases} \quad \text{and then } (\bar{\chi}_U(m))^2 = \begin{cases} 0, & \text{if } m \in U, \\ 1, & \text{if } m \notin U. \end{cases}$$

Let $P = (\chi_U, \bar{\chi}_U)$ be a Pythagorean fuzzy ideal of BCL–algebra B ,

Now we need to prove that U is ideal of BCL–algebra B :

$$(i) \quad \text{Let } n, m \otimes n \in U \Rightarrow (\chi_U(n))^2 = (\chi_U(m \otimes n))^2 = 1 \text{ and } (\bar{\chi}_U(n))^2 = (\bar{\chi}_U(m \otimes n))^2 = 0$$

$$\Rightarrow (\chi_U(m))^2 \geq \min\{(\chi_U(n))^2, (\chi_U(m \otimes n))^2\} = \min\{1, 1\} = 1 \text{ and}$$

$$(\bar{\chi}_U(m))^2 \leq \max\{(\bar{\chi}_U(n))^2, (\bar{\chi}_U(m \otimes n))^2\} = \min\{0, 0\} = 0,$$

since $(\chi, \bar{\chi})$ is Pythagorean fuzz idea

But $(\chi_U(m))^2 \leq 1$ and $(\bar{\chi}_U(m))^2 \geq 0, \forall m \in B$, by the definitions.

Hence $(\chi_U(m))^2 = 1$ and $(\bar{\chi}_U(m))^2 = 0 \Rightarrow m \in U$

$$(ii) \quad \text{Let } m, n \in U \Rightarrow (\chi_U(m))^2 = (\chi_U(n))^2 = 1 \text{ and } (\bar{\chi}_U(m))^2 = (\bar{\chi}_U(n))^2 = 0$$

$$\Rightarrow (\chi_U((m \otimes (n \otimes w)) \otimes w))^2 \geq \min\{(\chi_U(m))^2, (\chi_U(n))^2\} = \min\{1, 1\} = 1 \text{ and}$$

$$(\bar{\chi}_U((m \otimes (n \otimes w)) \otimes w))^2 \leq \max\{(\bar{\chi}_U(m))^2, (\bar{\chi}_U(n))^2\} = \min\{0, 0\} = 0$$

But $(\chi_U((m \otimes (n \otimes w)) \otimes w))^2 \leq 1$ and $(\bar{\chi}_U((m \otimes (n \otimes w)) \otimes w))^2 \geq 0, \forall m, n, w \in B$, by

the definitions of χ_U and its square deviation $\bar{\chi}$ of the BCL–algebra.

Hence $(\chi_U((m \otimes (n \otimes w)) \otimes w))^2 = 1$ and $(\bar{\chi}_U((m \otimes (n \otimes w)) \otimes w))^2 = 0$

Then, $(m \otimes (n \otimes w)) \otimes w \in U$, which means U is ideal of B , by the above (i) and (ii).

Conversely, for an ideal U of B , we need to prove that P is a Pythagorean fuzzy ideal of B :

Part (A): First, we need to show that:

$$(\chi_U(m))^2 \geq \min\{(\chi_U(n))^2, (\chi_U(m \otimes n))^2\} \text{ and}$$

$$(\chi_U(m))^2 \leq \max\{(\bar{\chi}_U(n))^2, (\bar{\chi}_U(m \otimes n))^2\}, \forall m, n \in U$$

We show this proof by considering the following four cases:

Case(1) : If $n, m \otimes n \in U$, then $m \in U$

$$\Rightarrow (\chi_u(m))^2 = 1 \geq \min\{1, 1\} = \min\{(\chi_u(n))^2, (\chi_u(m \otimes n))^2\} \text{ and}$$

$$(\bar{\chi}_u(m))^2 = 0 \leq \max\{0, 0\} = \max\{(\bar{\chi}_u(n))^2, (\bar{\chi}_u(m \otimes n))^2\}$$

$$\Rightarrow (\chi_u(m))^2 \geq \min\{(\chi_u(n))^2, (\chi_u(m \otimes n))^2\} \text{ and}$$

$$(\bar{\chi}_u(m))^2 \leq \max\{(\bar{\chi}_u(n))^2, (\bar{\chi}_u(m \otimes n))^2\}.$$

Case(2) : If $n \notin U, m \otimes m \notin U$

$$\Rightarrow (\chi_u(n))^2 = 0 = (\chi_u(m \otimes n))^2, (\chi_u(m))^2 \geq 0, \forall m \in B \text{ by definition of } \chi_u \text{ and}$$

$$(\bar{\chi}_u(n))^2 = 1 = (\bar{\chi}_u(m \otimes n))^2, (\chi_u(m))^2 \leq 1, \forall m \in B$$

$$\Rightarrow (\chi_u(m))^2 \geq 0 = \min\{\chi(n), \chi(m \otimes n)\} \text{ and } (\chi_u(m))^2 \leq 1 = \max\{\bar{\chi}(n), \bar{\chi}(m \otimes n)\}$$

$$\Rightarrow (\chi_u(m))^2 \geq \min\{(\eta(n))^2, (\eta(m \otimes n))^2\} \text{ and } (\bar{\chi}_u(m))^2 \leq \max\{\bar{\chi}(n), \bar{\chi}(m \otimes n)\}$$

Case(3) : If $n \in U$ and $m \otimes n \notin U$

$$\Rightarrow (\chi_u(n))^2 = 1, (\chi_u(m \otimes n))^2 = 0 \text{ and } (\bar{\chi}_u(n))^2 = 0, (\bar{\chi}_u(m \otimes n))^2 = 1$$

$$\Rightarrow (\chi_u(m))^2 \geq 0 = \min\{(\chi_u(n))^2, (\chi_u(m \otimes n))^2\} = \min\{1, 0\} = 0 \text{ and}$$

$$\Rightarrow (\bar{\chi}_u(m))^2 \leq 1 = \max\{(\bar{\chi}(n))^2, (\bar{\chi}(m \otimes n))^2\} = \max\{1, 0\} = 1$$

$$\text{Hence, } (\chi_u(m))^2 \geq \min\{(\chi_u(n))^2, (\chi_u(m \otimes n))^2\} \text{ and}$$

$$(\chi_u(m))^2 \leq \max\{(\bar{\chi}_u(n))^2, (\bar{\chi}_u(m \otimes n))^2\}, \forall m, n \in U.$$

Case(4) : If $n \notin U$ and $m \otimes n \in U$

$$\Rightarrow (\chi_u(n))^2 = 0, (\chi_u(m \otimes n))^2 = 1 \text{ and } (\bar{\chi}_u(n))^2 = 1, (\bar{\chi}_u(m \otimes n))^2 = 0$$

$$\Rightarrow (\chi_u(m))^2 \geq 0 = \min\{(\chi_u(n))^2, (\chi_u(m \otimes n))^2\} = \min\{0, 1\} = 0 \text{ and}$$

$$\Rightarrow (\bar{\chi}_u(m \otimes n))^2 \leq 1 = \max\{\bar{\chi}(n), \bar{\chi}(m)\} = \max\{1, 0\} = 1$$

$$\text{Hence, } (\chi_u(m))^2 \geq \min\{(\chi_u(n))^2, (\chi_u(m \otimes n))^2\} \text{ and}$$

$$(\bar{\chi}_u(m))^2 \leq \max\{(\bar{\chi}_u(n))^2, (\bar{\chi}_u(m \otimes n))^2\}, \forall m, n \in U$$

Therefore, in all the cases (1-4) above, we see that:

$$(\chi_u(m))^2 \geq \min\{(\chi_u(n))^2, (\chi_u(m \otimes n))^2\} \text{ and}$$

$$(\bar{\chi}_u(m))^2 \leq \max\{(\bar{\chi}_u(n))^2, (\bar{\chi}_u(m \otimes n))^2\}, \forall m, n \in U.$$

Part (B): Second, we can simply show that

$$(\chi_u[(m \otimes (n \otimes w) \otimes w)])^2 \geq \min\{(\chi_u(m))^2, (\chi_u(n))^2\} \text{ and}$$

$$(\bar{\chi}_u[(m \otimes (n \otimes w) \otimes w)])^2 \leq \max\{(\bar{\chi}_u(m))^2, (\bar{\chi}_u(n))^2\}$$

The proof of this part is analogous to the proof of Part A above.

Therefore, $(\chi_u, \bar{\chi}_u)$ is Pythagorean fuzzy ideal of B, by (A) and (B) above.

(a) The accuracy function:

$$\alpha(m) = (\chi_u(m))^2 + (\bar{\chi}_u(m))^2 = \begin{cases} 1 + 0, & \text{if } m \in U, \\ 0 + 1, & \text{if } m \notin U \end{cases} = 1, \forall m \in B$$

(b) The score function:

$$s(m) = (\chi_u(m))^2 - (\bar{\chi}_u(m))^2 = \begin{cases} 1 - 0, & \text{if } m \in U, \\ 0 - 1, & \text{if } m \notin U \end{cases} = \begin{cases} 1, & \text{if } m \in U, \\ -1, & \text{if } m \notin U. \end{cases}$$

(c) The degree of indeterminacy:

$$\pi(m) = \sqrt{1 - [(\chi_u(m))^2 + (\bar{\chi}_u(m))^2]} = \begin{cases} \sqrt{1 - (1 + 0)}, & \text{if } m \in U, \\ \sqrt{1 - (0 + 1)}, & \text{if } m \notin U \end{cases} = 0.$$

□

Corollary 4.2.3.1. Let U be a non-empty subset of B such that χ_u is Characteristic function with the complement $\bar{\chi}_u$ such that $\bar{\chi}_u(m) = 1 - \chi_u(m)$, $\forall m \in U$. Then $P = (\chi_u, \bar{\chi}_u)$ is Pythagorean fuzzy ideal of BCL-algebra B if and only if U is ideal of B . Furthermore, the accuracy function α , the score function s and the degree of indeterminacy π respectively have the same Corresponding values as in Proposition 4.2.3.3 above.

Proof. The proof for $P = (\chi_u, \bar{\chi}_u)$ is Pythagorean fuzzy ideal of BCL-algebra B if and only if U is ideal of B and deriving functional values are analogous to the proof and corresponding functional values of Proposition 4.2.3.3 above. □

Theorem 4.2.3.1. Let η be a fuzzy subset with the square deviation $\bar{\eta}$ in a BCL-algebra B such that $(\eta(m \otimes n))^2 = (\eta(n))^2$ and then $(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2$, $\forall m, n \in B$.

Then $P = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of BCL-algebra B if and only if η and then $\bar{\eta}$ are constants. Furthermore, the accuracy function $\alpha(m)$, the score function, $s(m)$ and the

degree of indeterminacy $\pi(m)$ are respectively given as follows, $\forall m \in B$:

$$(a) \quad \alpha(m) = 1, \quad (b) \quad s(m) = 2(\eta(m)) - 1 \quad (c) \quad \pi(m) = 0.$$

Proof. Suppose $(\eta(m \otimes n))^2 = (\eta(n))^2$.

$$\text{Then } (\bar{\eta}(m \otimes n))^2 = 1 - (\eta(m \otimes n))^2 = 1 - (\eta(n))^2 = (\bar{\eta}_u(n))^2.$$

Now, let $P = (\eta, \bar{\eta})$ be a Pythagorean fuzzy ideal of BCL–algebra B and

$$(\eta(m \otimes n))^2 = (\eta(n))^2 \quad (\text{and then } (\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2)$$

We prove that: η and then $\bar{\eta}$ are constants, or $\forall m, n \in B$, $(\eta(m))^2 = (\eta(n))^2$

$$\text{and then } (\bar{\eta}(m))^2 = (\bar{\eta}(n))^2.$$

Since $P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy ideal of BCL–algebra B , η is a fuzzy ideal of

BCL–algebra B , and hence by one of its axioms, we have:

$$(\eta(0))^2 = (\eta(m \otimes m))^2, \forall m \in B, \Rightarrow (\eta(0))^2 = (\eta(m \otimes m))^2 = (\eta(m))^2, \forall m \in B.$$

$$\text{Again, } (\eta(0))^2 = (\eta(0 \otimes n))^2 = (\eta(n))^2, \forall n \in B \Rightarrow (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2, \forall m, n \in B$$

Or $\eta(0) = \eta(m) = \eta(n)$, $\forall m, n \in B$ and similarly, $\bar{\eta}(0) = \bar{\eta}(m) = \bar{\eta}(n)$, $\forall m, n \in B$ and hence,

η is a constant, and $\bar{\eta}$ is, too.

Conversely, suppose η and $\bar{\eta}$ are constants, or $(\eta(m))^2 = (\eta(n))^2$ and $(\bar{\eta}(m))^2 = (\bar{\eta}_u(n))^2$

such that $(\eta(m \otimes n))^2 = (\eta(n))^2$ and $(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}_u(n))^2$, $\forall m, n \in B$.

We need to prove that $P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy ideal of BCL–algebra B , or simply η is a fuzzy ideal of BCL–algebra B .

$$(i) \quad (\eta(0))^2 = (\eta(m \otimes m))^2 = (\eta(m))^2 \Rightarrow (\eta(0))^2 \geq (\eta(m))^2 \text{ and then } (\bar{\eta}(0))^2 \leq (\bar{\eta}(m))^2$$

$$(ii) \quad (\eta(m))^2 = (\eta(n))^2 = (\eta(m \otimes n))^2 = \min\{(\eta(m \otimes n))^2, (\eta(n))^2\}$$

$$\geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\}$$

$$\Rightarrow (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \text{ and } (\bar{\eta}(m))^2 \leq \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\}$$

(iii) η is constant, and then

$$(\eta[(m \otimes (n \otimes w)) \otimes w])^2 = (\eta(m))^2 = (\eta(n))^2, \forall m, n, w \in B.$$

$$\Rightarrow (\eta[(m \otimes (n \otimes w) \otimes w)])^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$$

$$(\text{and then } (\bar{\eta}[(m \otimes (n \otimes w) \otimes w)])^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}.$$

Therefore, by (i), (ii) and (iii) above, η is a fuzzy ideal of BCL–algebra B and $\bar{\eta}$ is the complement of η so that $P = (\eta, (\bar{\eta}))$ is a Pythagorean fuzzy ideal of BCL–algebra B .

$$(a) \quad \alpha(m) = (\eta(m))^2 + (\bar{\eta}(m))^2 = (\eta(m))^2 + 1 - (\eta(m))^2 = 1$$

$$(b) \quad s(m) = (\eta(m))^2 - (\bar{\eta}(m))^2 = (\eta(m))^2 - (1 - (\eta(m))^2) = 2(\eta(m))^2 - 1$$

$$(c) \quad \pi(m) = \sqrt{1 - \alpha(m)} = \sqrt{1 - 1} = 0$$

□

Theorem 4.2.3.2. *The intersection of any two Pythagorean fuzzy ideals of the BCL–algebra B of B_1 and B_2 , is also a Pythagorean fuzzy ideal of B .*

Proof. Let $B_1 = (\eta_1, \tau_1)$ and $B_2 = (\eta_2, \tau_2)$ be any two Pythagorean fuzzy ideals of B . Then we need to prove that $B_1 \cap B_2$ is a Pythagorean fuzzy ideal of B and let $m, n, w \in B$.

$$(i) \quad (\eta_1 \cap \eta_2(m))^2 = \min\{(\eta_1(m))^2, (\eta_2(m))^2\}$$

$$\geq \min\{\min\{(\eta_1(n))^2, (\eta_1(m \otimes n))^2\}, \min\{(\eta_2(n))^2, (\eta_2(m \otimes n))^2\}\}$$

$$= \min\{\min\{(\eta_1(n))^2, (\eta_2(n))^2\}, \min\{(\eta_1(m \otimes n))^2, (\eta_2(m \otimes n))^2\}\}$$

$$= \min\{(\eta_1 \cap \eta_2(n))^2, (\eta_1 \cap \eta_2(m \otimes n))^2\} \quad \text{and}$$

$$(\tau_1 \cap \tau_2(m))^2 = \max\{(\tau_1(m))^2, (\tau_2(m))^2\}$$

$$\leq \max\{\max\{(\tau_1(n))^2, (\tau_1(m \otimes n))^2\}, \max\{(\tau_2(n))^2, (\tau_2(m \otimes n))^2\}\}$$

$$= \max\{\max\{(\tau_1(n))^2, (\tau_2(n))^2\}, \max\{(\tau_1(m \otimes n))^2, (\tau_2(m \otimes n))^2\}\}$$

$$= \max\{(\eta_1 \cap \eta_2(n))^2, (\eta_1 \cap \eta_2(m \otimes n))^2\} \quad \text{and}$$

$$(ii) \quad (\eta_1 \cap \eta_2((m \otimes (n \otimes w) \otimes w)))^2 = \min\{(\eta_1((m \otimes (n \otimes w) \otimes w)))^2, (\eta_2((m \otimes (n \otimes w) \otimes w)))^2\}$$

$$\geq \min\{\min\{(\eta_1(m))^2, (\eta_1(n))^2\}, \min\{(\eta_2(m))^2, (\eta_2(n))^2\}\}$$

$$= \min\{(\eta_1 \cap \eta_2(m))^2, (\eta_1 \cap \eta_2(n))^2\} \quad \text{and}$$

$$(\tau_1 \cap \tau_2((m \otimes (n \otimes w) \otimes w)))^2 = \max\{(\tau_1((m \otimes (n \otimes w) \otimes w)))^2, (\tau_2((m \otimes (n \otimes w) \otimes w)))^2\}$$

$$\geq \max\{\max\{(\tau_1(m))^2, (\tau_1(n))^2\}, \max\{(\tau_2(m))^2, (\tau_2(n))^2\}\}$$

$$\begin{aligned} &\geq \max\{\max\{(\tau_1(m))^2, (\tau_2(m))^2\}, \max\{(\tau_1(n))^2, (\tau_2(n))^2\}\} \\ &= \max\{(\tau_1 \cap \tau_2(m))^2, (\tau_1 \cap \tau_2(n))^2\} \end{aligned}$$

Hence by (i) and (ii) above, $B_1 \cap B_2$ is a Pythagorean fuzzy ideal of BCL–algebra B. \square

The above theorem can also be generalized to any family of Pythagorean fuzzy ideals as in the following corollary.

Corollary 4.2.3.2. *The intersection, $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, of any family of Pythagorean fuzzy ideals, $\{(\eta_i, \tau_i) : i \in I\}$, of B is also a Pythagorean fuzzy ideal of B.*

Proof. Let $\{(\eta_i, \tau_i) : i \in I\}$, be family of Pythagorean fuzzy ideals of B. Then we need to prove that $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, is a Pythagorean fuzzy ideal of B and let $m, n, w \in B$.

$$\begin{aligned} \text{(i)} \quad \bigcap_{i \in I} (\eta_i(m))^2 &= \inf_{i \in I} \{(\eta_i(m))^2\} \\ &\geq \inf_{i \in I} \{\min\{(\eta_i(n))^2, (\eta_i(m \otimes n))^2\}\} \\ &\geq \min\{\inf_{i \in I} \{(\eta_i(n))^2, (\eta_i(m \otimes n))^2\}\} \\ &= \min\{\inf_{i \in I} \{(\eta_i(n))^2\}, \inf_{i \in I} \{(\eta_i(m \otimes n))^2\}\} \\ &= \min\left\{\bigcap_{i \in I} (\eta_i(n))^2, \bigcap_{i \in I} (\eta_i(m \otimes n))^2\right\} \quad \text{and} \end{aligned}$$

$$\begin{aligned} \bigcap_{i \in I} (\tau_i(m))^2 &= \sup_{i \in I} \{(\tau_i(m))^2\} \\ &\leq \sup_{i \in I} \{\max\{(\tau_i(n))^2, (\tau_i(m \otimes n))^2\}\} \\ &\leq \max\{\sup_{i \in I} \{(\tau_i(n))^2, (\tau_i(m \otimes n))^2\}\} \\ &= \max\{\sup_{i \in I} \{(\tau_i(n))^2\}, \sup_{i \in I} \{(\tau_i(m \otimes n))^2\}\} \\ &= \max\left\{\bigcap_{i \in I} (\tau_i(n))^2, \bigcap_{i \in I} (\tau_i(m \otimes n))^2\right\}, \quad \forall m, n \in B. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \bigcap_{i \in I} (\eta_i((m \otimes (n \otimes w)) \otimes w))^2 &= \inf_{i \in I} \left\{(\eta_i((m \otimes (n \otimes w)) \otimes w))^2\right\} \\ &\geq \inf_{i \in I} \{\min\{(\eta_i(m))^2, (\eta_i(n))^2\}\} \end{aligned}$$

$$\begin{aligned}
&\geq \min\left\{\inf_{i \in I}\{(\eta_i(m))^2, (\eta_i(n))^2\}\right\} \\
&= \min\left\{\inf_{i \in I}\{(\eta_i(m))^2\}, \inf_{i \in I}\{(\eta_i(n))^2\}\right\} \\
&= \min\left\{\bigcap_{i \in I}(\eta_i(m))^2, \bigcap_{i \in I}(\eta_i(n))^2\right\} \quad \text{and} \\
\bigcap_{i \in I}(\tau_i((m \otimes (n \otimes w)) \otimes w))^2 &= \sup_{i \in I}\left\{\left(\tau_i((m \otimes (n \otimes w)) \otimes w)\right)^2\right\} \\
&\leq \sup_{i \in I}\left\{\max\{(\tau_i(m))^2, (\tau_i(n))^2\}\right\} \\
&\leq \max\left\{\sup_{i \in I}\{(\tau_i(m))^2\}, \sup_{i \in I}\{(\tau_i(n))^2\}\right\} \\
&= \max\left\{\sup_{i \in I}\{(\tau_i(m))^2\}, \sup_{i \in I}\{(\tau_i(n))^2\}\right\} \\
&= \max\left\{\bigcap_{i \in I}(\tau_i(m))^2, \bigcap_{i \in I}(\tau_i(n))^2\right\}, \quad \forall m, n \in B.
\end{aligned}$$

Hence by (i) and (ii) above, $\left(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i\right)$ is a Pythagorean fuzzy ideal of B . □

Remark 4.2.3.1. *The union of two Pythagorean fuzzy ideals of a BCL–algebra B is not necessarily a Pythagorean fuzzy ideal of BCL–algebra B which we justify it as follows.*

Example 4.2.3.2. *Let $(B; \otimes, 0)$ be a BCL–algebra, where $B = \{0, p, q, r\}$ and let " \otimes " be as defined in the Table 4.1 of Example 4.1.1.3, above and define the next two Pythagorean fuzzy ideals:*

$B_1 = (\eta_1, \tau_1)$ and $B_2 = (\eta_2, \tau_{B_2})$ as follows:

$$\begin{aligned}
(\eta_1(m))^2 &= \begin{cases} 0.6, & \text{if } m = 0, \\ 0.5, & \text{if } m = r, \\ 0.1, & \text{if } m = p, q, \end{cases} \quad \text{and} \quad (\tau_1(m))^2 = \begin{cases} 0.2, & \text{if } m = 0, \\ 0.4, & \text{if } m = p, r, \\ 0.7, & \text{if } m = q, \end{cases} \\
(\eta_2(m))^2 &= \begin{cases} 0.5, & \text{if } m = 0, \\ 0.4, & \text{if } m = p, r, \\ 0.2, & \text{if } m = q, \end{cases} \quad \text{and} \quad (\tau_2(m))^2 = \begin{cases} 0.1, & \text{if } m = 0, \\ 0.2, & \text{if } m = r, \\ 0.6, & \text{if } m = p, q. \end{cases}
\end{aligned}$$

It is easy to check that B_1 and B_2 are Pythagorean fuzzy ideals of B but to show that their union is not necessarily a Pythagorean fuzzy ideal of B , we justify it as follows using the above pairs of Pythagorean fuzzy ideals of B :

$$((\eta_1 \cup \eta_2)(m))^2 = \begin{cases} 0.6, & \text{if } m = 0, \\ 0.5, & \text{if } m = r, \\ 0.4, & \text{if } m = p, \\ 0.2, & \text{if } m = q, \end{cases} \quad \text{and} \quad ((\tau_1 \cup \tau_2)(m))^2 = \begin{cases} 0.1, & \text{if } m = 0, \\ 0.2, & \text{if } m = r, \\ 0.4, & \text{if } m = p, \\ 0.6, & \text{if } m = q. \end{cases}$$

Take $m = r, n = 0, w = q \Rightarrow (m \otimes (n \otimes w)) \otimes w = (r \otimes (0 \otimes q)) \otimes q = (r \otimes 0) \otimes q = r \otimes q = p$.

Then $(\eta_1 \cup \eta_2 [(r \otimes (0 \otimes q)) \otimes q])^2 = (\eta_1 \cup \eta_2(p))^2 = 0.4$

$$\geq \min\{((\eta_1 \cup \eta_2)(r))^2, ((\eta_1 \cup \eta_2)(0))^2\}$$

$$= \min\{0.5, 0.6\} = 0.5 \text{ is not true.}$$

Thus, the union of two Pythagorean fuzzy ideals of BCL–algebras of B is not necessarily a Pythagorean fuzzy ideal of B .

Theorem 4.2.3.3. Let η be a fuzzy subset such that η is a membership function and $\bar{\eta}$ is its square deviation in a BCL–algebra B . Suppose $\eta^b = \{\forall m \in B: (\eta(m))^2 \geq (\eta(b))^2\}$, for some $b \in B$ and then $\bar{\eta}^b = \{\forall m \in B: (\bar{\eta}(m))^2 \leq (\bar{\eta}(b))^2\}$, for some $b \in B$. Then $B^P = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of B if and only if $B^b = (\eta^b, \bar{\eta}^b)$ is Pythagorean fuzzy ideal of B .

Proof. Suppose $P = (\eta^b, \bar{\eta}^b)$ is Pythagorean fuzzy ideal of BCL–algebra B , or $\forall m, n, w, \in B$:

$$(1) (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \quad \text{and} \quad (\bar{\eta}(m))^2 \leq \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\}$$

$$(2) (\eta[((m \otimes n) \otimes w) \otimes w])^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \quad \text{and}$$

$$(\bar{\eta}[\((m \otimes n) \otimes w) \otimes w])^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

We need to prove that $B^b = (\eta^b, \bar{\eta}^b)$ is a Pythagorean fuzzy ideal of BCL–algebra B .

(i) Let $m \otimes n, n \in \eta^b$, for some $b \in B \Rightarrow (\eta(m \otimes n))^2 \geq (\eta(b))^2$ and $(\eta(n))^2 \geq (\eta(b))^2$

$$\text{But by (2), } (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \geq \min\{(\eta(b))^2, (\eta(b))^2\} = ((\eta(b))^2)$$

$$\Rightarrow (\eta(m))^2 \geq ((\eta(b))^2) \Rightarrow m \in \eta^b, \text{ for some } b \in B \Rightarrow m \otimes n, n \in \eta^b \Rightarrow m \in \eta^b \quad \text{and}$$

$$m \otimes n, n \in \bar{\eta}^b, \text{ for some } b \in B \Rightarrow (\bar{\eta}(m \otimes n))^2 \leq (\bar{\eta}(b))^2 \quad \text{and} \quad (\bar{\eta}(n))^2 \leq (\bar{\eta}(b))^2$$

$$\text{But by (2), } (\bar{\eta}(m))^2 \leq \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\} \leq \max\{(\bar{\eta}(b))^2, (\bar{\eta}(b))^2\} = \bar{\eta}(b))^2$$

$$\Rightarrow (\bar{\eta}(m))^2 \leq (\bar{\eta}(b))^2 \Rightarrow m \in \bar{\eta}^b, \text{ for some } b \in B.$$

Which means $m, n \in B \Rightarrow m \otimes n, n \in \bar{\eta}^b \Rightarrow m \in \bar{\eta}^b$

(ii) Let $m, n, w \in \eta^b$, for some $b \in B \Rightarrow (\eta(m))^2 \geq (\eta(b))^2, (\eta(n))^2 \geq (\eta(b))^2$

And by (2), $(\eta(((m \otimes n) \otimes w) \otimes w))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \geq \min\{(\eta(b))^2, (\eta(b))^2\} = (\eta(b))^2$

$\Rightarrow (\eta(((m \otimes n) \otimes w) \otimes w))^2 \geq (\eta(b))^2 \Rightarrow ((m \otimes n) \otimes w) \otimes w \in \eta^b$ and

For $m, n \in \bar{\eta}^b$, for some $b \in B: \Rightarrow (\bar{\eta}(m))^2 \leq (\bar{\eta}(b))^2, (\bar{\eta}(n))^2 \leq (\bar{\eta}(b))^2$

And by (2), $(\bar{\eta}(((m \otimes n) \otimes w) \otimes w))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\} \leq \max\{(\bar{\eta}(b))^2, (\bar{\eta}(b))^2\} = (\bar{\eta}(b))^2$

$\Rightarrow (\bar{\eta}(((m \otimes n) \otimes w) \otimes w))^2 \leq (\bar{\eta}(b))^2 \Rightarrow ((m \otimes n) \otimes w) \otimes w \in \bar{\eta}^b$

Thus, by (i), (ii) and (iii) above, $P = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of BCL–algebra B

Then $B^b = (\eta^b, \bar{\eta}^b)$ is a Pythagorean fuzzy ideal of BCL–algebra B.

Conversely, suppose $\eta^b = \{\forall m \in B : (\eta(m))^2 \geq (\eta(b))^2, \text{ for some } b \in B\}$ and

$\bar{\eta}^b = \{\forall m \in B : (\bar{\eta}(m))^2 \leq (\bar{\eta}(b))^2, \text{ for some } b \in B\}$ are such that $B^b = (\eta^b, \bar{\eta}^b)$ is a Pythagorean

fuzzy ideal of BCL–algebra B. Then we need to prove that $P = ((\eta, \bar{\eta}))$ is Pythagorean fuzzy ideal of BCL–algebra B. By the hypothesis we have the following:

(1) $m \otimes n, n \in \eta^b \Rightarrow m \in \eta^b$ and $m \otimes n, n \in \bar{\eta}^b \Rightarrow m \in \bar{\eta}^b$:

$\Rightarrow (\eta(m \otimes n))^2 \geq (\eta(b))^2, (\eta(n))^2 \geq (\eta(b))^2 \Rightarrow (\eta(m))^2 \geq (\eta(b))^2, \text{ for some } b \in B$

$\Rightarrow (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \geq ((\eta(b))^2, \text{ for some } b \in B$

$\Rightarrow (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\}$ and similarly

$\Rightarrow (\bar{\eta}(m))^2 \leq \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\}$

(2) $m, n \in \eta^b \Rightarrow (m \otimes (n \otimes w)) \otimes w \in \eta^b$ and $m, n \in \bar{\eta}^b \Rightarrow (m \otimes (n \otimes w)) \otimes w \in (\bar{\eta}^b)^2$

$\Rightarrow (\eta(m))^2 \geq (\eta(b))^2, (\eta(n))^2 \geq (\eta(b))^2$

$\Rightarrow (\eta((m \otimes (n \otimes w))))^2 \geq (\eta(b))^2, \text{ for some } b \in B$

$\Rightarrow (\eta((m \otimes (n \otimes w))))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \geq ((\eta(b))^2, \text{ for some } b \in B$ and similarly,

$\Rightarrow (\eta((m \otimes (n \otimes w))))^2 \geq ((\eta(b))^2, \text{ for some } b \in B$ and similarly,

$(\bar{\eta}((m \otimes (n \otimes w))))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\} \leq (\bar{\eta}(b))^2, \text{ for some } b \in B$

$$\Rightarrow (\bar{\eta}[(m \otimes (n \otimes w))])^2 \leq (\bar{\eta}(b))^2, \text{ for some } b \in B$$

Thus, by (1) and (2) above, $P = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of BCL–algebra B. □

Lemma 4.2.3.2. *Let η be a fuzzy subset and $\bar{\eta}$ be its square deviation in a BCL–algebra B.*

If $(\eta(m))^2 = (\eta(0))^2$ and $(\bar{\eta}(m))^2 = (\bar{\eta}(0))^2, \forall m \in B$, then $P = (\eta, \bar{\eta})$ is a Pythagorean fuzzy ideal of B.

Proof. Suppose η and $\bar{\eta}$ are fuzzy subsets in B such that $(\eta(m))^2 = (\eta(0))^2$ and $(\bar{\eta}(m))^2 = (\bar{\eta}(0))^2$.

Then we prove: $P = ((\eta, \bar{\eta}))$ is a Pythagorean fuzzy ideal of B:

(i) Again from the hypothesis, $(\eta(0))^2 = (\eta(m))^2, \forall m \in B$

$$\Rightarrow (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2 = (\eta(m \otimes n))^2, \forall m, n \in B \text{ (since, also } m \otimes n \in B)$$

$$\Rightarrow (\eta(m))^2 = (\eta(0))^2 = \min\{(\eta(0))^2, (\eta(0))^2\} = \min\{(\eta(m \otimes n))^2, (\eta(n))^2\}$$

$$\Rightarrow (\eta(m))^2 = \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \Rightarrow (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\}$$

$$\text{and } (\bar{\eta}(0))^2 = (\bar{\eta}(m))^2, \forall m \in B \text{ (by the hypothesis)}$$

$$\Rightarrow (\bar{\eta}(0))^2 = (\bar{\eta}(m))^2 = (\bar{\eta}(n))^2 = (\bar{\eta}(m \otimes n))^2, \forall m, n \in B \text{ (since, also } m \otimes n \in B)$$

$$\Rightarrow (\bar{\eta}(m))^2 = (\bar{\eta}(0))^2 = \max\{(\bar{\eta}(0))^2, (\bar{\eta}(0))^2\} = \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\}$$

$$\Rightarrow (\bar{\eta}(m))^2 = \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\} \Rightarrow ((\bar{\eta}(m))^2 \leq \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\})$$

(ii) Similarly as (i) above, since $(\eta(0))^2 = (\eta(m))^2, \forall m \in B$

$$\Rightarrow (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2 = (\eta[(m \otimes (n \otimes w)) \otimes z])^2, \forall m, n, w \in B$$

$$(\eta[(m \otimes (n \otimes w)) \otimes z])^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$$

$$\text{and similarly, for } \bar{\eta}: (\bar{\eta}[(m \otimes (n \otimes w)) \otimes z])^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Thus, by (i) and (ii) above, $P = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of BCL–algebra B. □

Theorem 4.2.3.4. *Let $P = (\eta, \tau)$ be Pythagorean fuzzy ideal of B and $m, n \in B. \forall m \in B$, if $(\eta(m \otimes n))^2 = (\eta(m))^2$ and $(\tau(m \otimes n))^2 = (\tau(m))^2$ then, $\forall m, n \in B$, the following hold:*

(1) *The accuracy function:* $a(m) \leq 2 - [(\eta(0))^2 + (\tau(0))^2],$

(2) *The score function:* $s(m) \leq 1 - \eta(0)^2 - \tau(0)^2,$

(3) *The degree of indeterminacy:* $\pi(m)$ is such that $(\pi(m))^2 \geq (\eta(0))^2 + (\tau(0))^2 - 1.$

Proof. Let $P = (\eta, \tau)$ be a Pythagorean fuzzy ideal of BCL–algebra B such that;

$$(\eta(m \otimes n))^2 = (\eta(m))^2 \quad \text{and} \quad (\tau(m \otimes n))^2 = (\tau(m))^2$$

$$\begin{aligned} \text{Since, } \forall m \in B, 0 = m \otimes m &\Rightarrow (\eta(0))^2 = (\eta(m \otimes m))^2 = (\eta(m))^2 \Rightarrow (\eta(0))^2 = (\eta(m))^2, \\ &\Rightarrow (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2, \forall m, n \in B \end{aligned}$$

And similarly, $(\tau(0))^2 = (\tau(m))^2 = (\tau(n))^2, \forall m, n \in B$, But, $\forall m, n \in B$, we have:

$$0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1 \Rightarrow 0 \leq (\eta(m))^2 + (\tau(0))^2 \leq 1 \Rightarrow (\eta(m))^2 \leq 1 - (\tau(0))^2.$$

$$\text{And, } 0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1 \Rightarrow 0 \leq (\eta(0))^2 + (\tau(m))^2 \leq 1 \Rightarrow (\tau(m))^2 \leq 1 - (\eta(0))^2$$

(1) The accuracy function:

$$\begin{aligned} a(m) &= (\eta(m))^2 + (\tau(m))^2 \leq (1 - (\eta(0))^2) + (1 - (\tau(0))^2) = 2 - (\eta(0))^2 - (\tau(0))^2 \\ &\Rightarrow a(m) \leq 2 - (\eta(0))^2 - (\tau(0))^2 = 2 - [(\eta(0))^2 + (\tau(0))^2], \forall m, n \in B \end{aligned}$$

(2) The score function:

$$\begin{aligned} s(m) &= (\eta(m))^2 - (\tau(m))^2 \leq (1 - (\eta(0))^2) - (\tau(m))^2 \\ &= (1 - (\eta(0))^2) - (\tau(0))^2, \text{ as } \tau(m) = \tau(0) \\ &= 1 - (\eta(0))^2 - (\tau(0))^2 \\ &\Rightarrow s(m) \leq 1 - (\eta(0))^2 - (\tau(0))^2, \forall m, n \in B \end{aligned}$$

(3) The degree of indeterminacy:

$$\begin{aligned} (\pi(m))^2 &= 1 - (\eta(m))^2 - (\tau(m))^2 = 1 - a(m) \geq 1 - (2 - ((\eta(0))^2 - (\tau(0))^2)) \\ &\quad (\text{since, } a(m) \leq 2 - (\eta(0))^2 - (\tau(0))^2 \Rightarrow -a(m) \geq -(2 - (\eta(0))^2 - (\tau(0))^2)) \\ &\Rightarrow (\pi(m))^2 \geq (\eta(0))^2 + (\tau(0))^2 - 1, \forall m \in B. \end{aligned}$$

□

Remark 4.2.3.2. *As the relationships whether the fuzzy SA, fuzzy DS, and fuzzy ideal of BCL–algebra B discussed in Theorem 4.1.3.3 and Remark 4.1.3.3, as justified by Example 4.1.3.4, then the same results also hold for the relationships among Pythagorean fuzzy substructures of B (Pythagorean fuzzy SA, Pythagorean fuzzy DS and Pythagorean fuzzy ideal).*

Chapter 5

Pythagorean Fuzzy Structures of Liu^{B} –Algebra

Under this chapter, we introduce the notions of fuzzy substructures (fuzzy subalgebra, fuzzy deductive system and fuzzy ideal) and Pythagorean fuzzy substructures (Pythagorean fuzzy subalgebra, Pythagorean fuzzy deductive system and Pythagorean fuzzy ideal) of Liu^{B} –algebra by using the concept of the newly introduced Liu^{B} –algebra under which we have stated and justified different properties. The related concepts to Pythagorean fuzzy sets of Liu^{B} –algebra like accuracy function, score function, degree of indeterminacy and square deviation are also discussed.

Under this section and the subsequent subsections, we abbreviate "LBA" for the " Liu^{B} –algebra". We also denote "L" for the set in the Liu^{B} –algebra $(L; \otimes, \odot, 0)$ and " L^{B} " for the the "Pythagorean fuzzy substructure $\text{L}^{\text{B}} = (\eta, \tau)$ of Liu^{B} –algebra L", unless otherwise specified.

5.1. Basic Concepts of Liu^{B} –Algebra and Its Substructures

In this subsection, we introduce new definitions and examples of Liu^{B} –algebra under which subalgebra, deductive system and ideal are explored where Liu^{B} –algebra $(L; \otimes, \odot, 0)$ is newly defined based on the BCL–algebra $(B; \otimes, 0)$, (not on BCL⁺–algebra $(B; \otimes, 1)$, as previously defined in Definition 1.1.12, for Liu–algebra [46]), and the substructures of Liu^{B} –algebra are originally defined correspondingly.

Definition 5.1.1. *An algebra $(L; \otimes, \odot, 0)$ in which L is a non-empty set; \otimes and \odot are two binary operations defined on L ; 0 is constant in L , where for every element $m, n, w \in L$, the following axioms are satisfied is called Liu^{B} –algebra (LBA):*

- (1) $(L; \otimes, 0)$ is BCL–algebra;
- (2) $(L; \odot)$ is a semi-group (\odot is associative in L);
- (3) \odot is both right and left distributive over \otimes

$$\text{or } \begin{cases} m \odot (n \otimes w) = (m \odot n) \otimes (m \odot w), \\ (n \otimes w) \odot m = (n \odot m) \otimes (w \odot m), \end{cases} \quad \forall m, n, w \in L.$$

Example 5.1.1. Let $L = \{0, m, n, w\}$ be a set and let two binary operations \otimes and \odot on L be defined by the tables below:

\otimes	0	m	n	w
0	0	0	0	0
m	m	0	w	m
n	n	w	0	w
w	w	0	m	0

\odot	0	m	n	w
0	0	0	0	0
m	0	m	n	w
n	0	n	n	0
w	0	w	0	w

Table 5.1: Tables that define binary operations on LBA , $(L, \otimes, \odot, 0)$ (See [42] for \otimes)

From the table above, all algebraic calculations are checked that $(L, \otimes, \odot, 0)$ is LBA and for one case let us show for some cases as follows:

- (1) $(L; \otimes, 0)$ is BCL–algebra ((See [42])
- (2) $(m \odot n) \odot w = n \odot w = 0 = m \odot (n \odot w) = m \odot 0 \Rightarrow \odot$ is associative,
- (3) $(m \otimes n) \odot w = w \odot w = w = (m \odot w) \otimes (n \odot w) = w \otimes 0$ and
 $w \odot (m \otimes n) = w \odot w = w = (w \odot m) \otimes (w \odot n) = w \otimes 0$
 $\Rightarrow \odot$ is both right and left distributive over \otimes

Most of the above justifications are for some particular cases, similarly, one can check that all the axioms hold for every element of L , the above is to show the way as there is no need to show every step unlike proof of a theorem..

Remark 5.1.1.

- (1) The properties of the binary operation " \otimes " and the binary order relation " \leq " discussed in the BCL–algebra $(B; \otimes, 0)$ in the preceding sections also hold for this section.
- (2) The Liu^B–algebra defined in Definition 5.1.1 above is denoted by Liu^B–algebra for it is defined based on BCL–algebra just to make the distinction from Liu–algebra which was previously defined based on BCL⁺–algebra by Y. H. Liu in 2017 [46] since the two notions are quite different.

Example 5.1.2. Let $L_1 = \{-1, 0, 1\}$ be a set $\forall m, n \in L_1$ and define:

$$m \otimes n = \begin{cases} 0, & \text{if } m = n, \\ m, & \text{if } m \neq n, \\ 0, & \text{if } m = 0, \end{cases} \quad \text{and} \quad m \odot n = \begin{cases} m, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

Then $(L_1; \otimes, \odot, 0)$ is Liu^B-algebra (LBA), by the following, particular cases as examples, for $m = 1, n = 0$ and $w = -1$:

(i) $m \otimes m = 0$ hold by the particular definition of \otimes ,

(ii) $m \otimes n = 0$ and $n \otimes m = 0 \Rightarrow m = n$ also hold by the particular definition of \otimes ,

$$\begin{aligned} \text{(iii)} \quad & [((m \otimes n) \otimes w) \otimes ((m \otimes w \otimes n))] \otimes ((w \otimes n) \otimes m) \\ &= [((1 \otimes 0) \otimes (-1)) \otimes ((1 \otimes (-1)) \otimes 0)] \otimes (((-1) \otimes 0) \otimes 1) \\ &= [(1 \otimes (-1)) \otimes (1 \otimes 0)] \otimes (-1 \otimes 1) = (1 \otimes 1) \otimes (-1) = 0 \otimes (-1) = 0 \text{ hold.} \end{aligned}$$

$$(2) \quad (1 \odot 0) \odot (-1) = 0 \odot (-1) = 0 = 1 \odot (0 \odot (-1)) = 1 \odot 0 \Rightarrow \odot \text{ is associative,}$$

$$(3) \quad (1 \otimes 0) \odot (-1) = 1 \odot (-1) = 0 = (1 \odot (-1)) \otimes (0 \odot (-1)) = 0 \otimes 0 \text{ and}$$

$$(-1) \odot (1 \otimes 0) = (-1) \odot 1 = 0 = ((-1) \odot 1) \otimes ((-1) \odot 0) = 0 \otimes 0$$

$\Rightarrow \odot$ is both right and left distributive over \otimes

Most of the above justifications are for some particular cases, similarly, one can check that all the axioms hold for every element of L , the above is to show the way as there is no need to show every step unlike proof of a theorem.

Example 5.1.3. Let $L_2 = \{-i, 0, i\}$; where $i = \sqrt{-1}$ and $\forall m, n \in L_2$ define:

$$m \otimes n = \begin{cases} 0, & \text{if } m = n, \\ m, & \text{if } m \neq n \text{ and } m \neq 0, \\ 0, & \text{if } m = 0, \end{cases} \quad \text{and} \quad m \odot n = \begin{cases} m, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(L_2; \otimes, \odot, 0)$ is LBA which could be justified in a similar way as Example 5.1.2 above with the difference that we replace 1 by i and -1 by $-i$ for this case.

Proposition 5.1.1. Let $(L; \otimes, \odot, 0)$ be LBA Then the statements hereunder hold $\forall m, n \in L$:

$$(a) \quad 0 \odot 0 = 0$$

$$(b) \quad 0 \odot m = 0 = m \odot 0 = 0$$

$$(c) \quad m \odot (n \otimes n) = (n \otimes n) \odot m = 0$$

Proof. Suppose $(L; \otimes, \odot, 0)$ is a Liu^B-algebra and $m \in L$:

$$(a) \quad 0 = (0 \odot m) \otimes (0 \odot m) = 0 \odot (m \otimes m), \text{ since } k \otimes k = 0 \text{ by Definition 1.1.8 (1)}$$

$$= 0 \odot 0, \forall m \in L, \text{ since } \odot \text{ is distributive over } \otimes \text{ by Definition 5.1.1 (3)}$$

$$(b) \quad 0 = (0 \odot m) \otimes (0 \odot m) = (0 \otimes 0) \odot m = 0 \odot m, \text{ similarly as (a) above}$$

$$= (m \odot 0) \otimes (m \odot 0) = m \odot (0 \otimes 0) = m \odot 0, \forall m \in L$$

$$(c) \quad \text{Holds true by (b) as } n \otimes n = 0$$

Generally, $0 \odot 0 = 0 \odot m = m \odot 0 = 0$, by (a) and (b) above

$$= m \odot (n \otimes n) = (n \otimes n) \odot m, \forall m, n \in L, \text{ as } n \otimes n = 0, \forall n \in L. \quad \square$$

Theorem 5.1.1. *Let $(L; \otimes, \odot, 0)$ be LBA and \leq be a binary relation on the BCL-algebra $(L; \otimes, 0)$. Then the statements hereunder hold, $\forall m, n, w \in L$:*

$$(i) \quad m \leq n \Rightarrow w \odot m \leq w \odot n$$

$$(ii) \quad m \leq n \Rightarrow m \odot w \leq n \odot w$$

Proof. Let $(L; \otimes, \odot, 0)$ be a Liu^B-algebra and let $m, n, w \in L$:

$$(i) \quad m \leq n \Rightarrow m \otimes n = 0 \text{ and } (w \odot m) \otimes (w \odot n) = 0 = w \odot (m \otimes n) = w \odot 0 = 0,$$

by Proposition 5.1.1

$$\Rightarrow (w \odot m) \otimes (w \odot n) = 0 \Rightarrow w \odot m \leq w \odot n, \text{ by definition of } \leq.$$

$$(ii) \quad (m \odot w) \otimes (n \odot w) = (m \otimes n) \odot w = 0 \odot w = 0 \text{ by Proposition 5.1.1 (b)}$$

$$\Rightarrow m \odot w \leq n \odot w \text{ by definition of } \leq. \quad \square$$

Remark 5.1.2. *The definition for square deviation $\bar{\eta}$ in Definition 4.1.1 could be extended to the LBA $(L; \otimes, \odot, 0)$ such that $\bar{\eta}(m) \in [-1, 1], \forall m \in L$ is a fuzzy subset and the properties discussed in Proposition 4.1.1 and Remark 4.1.1 also hold in LBA L*

Definition 5.1.2. *Let $(L; \otimes, \odot, 0)$ be LBA and suppose S is a non-empty subset of L . Thus S is known as subalgebra (SA) of L if $m \otimes n \in S$ and $m \odot n \in S, \forall m, n \in S$.*

Example 5.1.4. *Let $L = \{0, m, n, w\}$ be a set and let two binary operations \otimes and \odot on L be defined by the tables below:*

\otimes	0	m	n	w
0	0	0	0	0
m	m	0	w	m
n	n	w	0	w
w	w	0	m	0

\odot	0	m	n	w
0	0	0	0	0
m	0	m	n	w
n	0	0	n	0
w	0	w	0	w

Table 5.2: Tables that define binary operations on LBA , $(L, \otimes, \odot, 0)$ (See [42] for \otimes)

From Table 5.2 above, by checking algebraic calculations, we have checked that $(L, \otimes, \odot, 0)$ is LBA and we have checked the following as well:

- (1) $L, \{0\}, \{0, m\}, \{0, n\}, \{0, w\}, \{0, m, w\}$ are SAs of L ,
- (2) $\{m\}, \{n\}, \{w\}, \{m, n\}, \{m, w\}, \{n, w\}, \{0, m, n\}, \{0, n, w\}, \{m, n, w\}$ are not SAs of L , for $m \neq 0, n \neq 0, w \neq 0$.

In order to explain the above two, we take one set for each and show as follows:

(1) Take $S = \{0, m, w\}$. We show that S is SA of L , so that for the given operations, we have:

$$0 \otimes m = 0 \otimes w = m \otimes m = w \otimes w = w \otimes m = 0 \in S, m \otimes w = m \otimes 0 = m \in S, w \otimes 0 = w \in S.$$

$$\text{Also, } 0 \odot m = m \odot 0 = 0 \odot w = w \odot 0 = 0 \in S, m \odot w = w \odot m = w \odot m = w \in S, m \odot m = m \in S.$$

Therefore, all results of the operations remain in S , and satisfies the axioms of SA. Furthermore, for the other sets listed in (1) above, the results of both binary operations also stay within S and satisfy the axioms of SA.

(2) Take $N = \{0, n, w\}$ and we show that N is not SA of L

$$w, n \in N \text{ but } w \otimes n = m \notin N \text{ so that } N \text{ is not SA of } L.$$

Similarly, the other sets listed in (2) above are also not SAs of L .

Example 5.1.5. Let $L = \{0, -1, 1\}$ be a set and let the two binary operations \otimes and \odot on L be as defined as in Example 5.1.2 above. Then, we have checked that $(L, \otimes, \odot, 0)$ is LBA and we have also checked the following as well:

- (1) $L, \{0\}, \{0, -1\}, \{0, 1\}$ are SAs of L ,
 - (2) $\{-1\}, \{1\}, \{-1, 1\}$ are not SAs of L .
- which could be explained as follows:

(A) First we explain how $(L; \otimes, \odot, 0)$ is LBA:

(1) $(L; \otimes, 0)$ is BCL–algebra since:

- (i) $1 \otimes 1 = (-1) \otimes (-1) = 0 \otimes 0 = 0$ by Definition of \otimes in Example 5.1.2,
- (ii) From (i) above, we generalize $m \otimes n = 0$ and $n \otimes m = 0 \Rightarrow m = n$, from Example 5.1.2
- (iii) To show that the following axiom is satisfied, we take some cases among several as follows:

(a) Taking $(-1), 1, 0$ at a time: $((((-1) \otimes 1) \otimes 0) \otimes (((-1) \otimes 0) \otimes 1)) \otimes ((0 \otimes 1) \otimes (-1)))$
 $= ((-1) \otimes 0) \otimes ((-1) \otimes 1) \otimes (0 \otimes (-1)) = ((-1) \otimes (-1)) \otimes 0 = 0 \otimes 0 = 0$

(b) Taking $1, 1, (-1)$ at a time: $((1 \otimes 1) \otimes (-1)) \otimes ((1 \otimes (-1)) \otimes 1) \otimes (((-1) \otimes 1) \otimes 1)$
 $= ((0 \otimes (-1)) \otimes (1 \otimes 1)) \otimes ((-1) \otimes 1) = (0 \otimes 0) \otimes (-1) = 0 \otimes (-1) = 0$ and so on

Hence $(L; \otimes, 0)$ is a BCL–algebra.

(2) $(L; \odot)$ is a semi-group (\odot is associative):

(a) Taking $(-1), 1, 0$ at a time: $(-1) \odot (1 \odot 0) = (-1) \odot 0 = 0 = ((-1) \odot 1) \odot 0 = 0 \odot 0 = 0$

(b) Taking $1, 1, (-1)$: $1 \odot (1 \odot (-1)) = 1 \odot 0 = 0 = (1 \odot 1) \odot (-1) = 1 \odot (-1) = 0$ and so on

(3) \odot is distributive over \otimes :

(a) Taking $(-1), 1$: $(-1) \odot (1 \otimes 0) = (-1) \odot 1 = 0 = ((-1) \odot 1) \otimes ((-1) \odot 0) = 0 \otimes 0 = 0$

(b) Taking $1, 1, (-1)$: $1 \odot (1 \otimes (-1)) = 1 \odot 1 = 1 = (1 \odot 1) \otimes (1 \odot (-1)) = 1 \otimes 0 = 1$ and so on

We showed some random cases hold for each axioms so that $(L; \otimes, \odot, 0)$ is LBA.

(B) We show some of $L, \{0, 1\}, \{0, -1\}$ are SAs of L :

(1) $S = \{0, 1\}$: $0 \otimes 0 = 0 \otimes 1 = 1 \otimes 1 = 0 \odot 0 = 0 \odot 1 = 1 \odot 0 = 0 \in S, 1 \otimes 0 = 1 \odot 1 = 1 \in S$

(2) $S = \{0, -1\}$: $0 \otimes 0 = 0 \otimes -1 = -1 \otimes -1 = 0 \odot 0 = 0 \odot -1 = -1 \odot 0 = 0 \in S,$

$$-1 \otimes 0 = -1 \odot -1 = -1 \in S \text{ and so on.}$$

(C) We show $\{1\}, \{-1\}, \{1, -1\}$ are not SAs of L :

(1) $1 \otimes 1 = 0 \neq \{1\}, -1 \otimes -1 = 0 \neq \{-1\}, 1 \odot -1 = 0 \neq \{1, -1\},$

$$\otimes 1 = 1 \otimes 1 = 0 \odot 0 = 0 \odot 1 = 1 \odot 0 = 0 \in S, 1 \otimes 0 = 1 \odot 1 = 1 \in S$$

(2) $S = \{0, -1\}$: $0 \otimes 0 = 0 \otimes -1 = -1 \otimes -1 = 0 \odot 0 = 0 \odot -1 = -1 \odot 0 = 0 \in S,$

$$-1 \otimes 0 = -1 \odot -1 = -1 \in S \text{ and so on.}$$

Lemma 5.1.1. Suppose $(L, \otimes, \odot, 0)$ is LBA, and S is SA of L . Then $0 \in S$.

Proof. Since $S \neq \emptyset$, let $m \in S \Rightarrow m \in L \Rightarrow m \otimes m = 0 \in S,$

since $(L; \otimes, 0)$ is BCL–algebra by Definition 5.1.1.

□

Definition 5.1.3. Suppose D is a non-empty subset of L of LBA, $(L; \otimes, \odot, 0)$. Thus D is known as deductive system (DS) of L if and only if the following are satisfied, $\forall m, n, w \in L$:

- (1) $m, n \in D \Rightarrow m \otimes (n \otimes w) \in D$,
- (2) $m \in D \Rightarrow m \odot w \odot w \in D$.

Example 5.1.6. Let $L = \{0, m, n, w\}$ be a set and let two binary operations \otimes and \odot on the LBA L be as defined by Tables 5.1 of Example 5.1.1 above. Then, $(L, \otimes, \odot, 0)$ is LBA and we have checked that we also have the following:

- (1) $L, \{0\}, \{0, n\}$ are DSs of L ,
- (2) $\{0, m\}, \{0, w\}, \{m, n\}, \{m, n, w\}, \{0, n, w\}$ are not DSs of L for $m \neq 0, n \neq 0, w \neq 0$.

Example 5.1.7. Let $L = \{0, -1, 1\}$ be a set and let two binary operations \otimes and \odot on L be as defined in Example 5.1.2 above. Then, $(L, \otimes, \odot, 0)$ is LBA and we have the following as well:

- (1) $L, \{0\}, \{0, -1\}, \{0, 1\}$ are DSs of L ,
 - (2) $\{-1\}, \{1\}, \{-1, 1\}$ are not DSs of L .
- (A) As $(L, \otimes, \odot, 0)$ is LBA is explained in Example 5.1.5 above, we only need to explain (1) and (2) as follows:

(B) We show some of $L, \{0, 1\}, \{0, -1\}$ are DSs of L :

- (1) $D = \{0, 1\}, w = -1$: $(0 \otimes 1) \otimes -1 = 0 \otimes -1 = 0 \in D, (1 \otimes 1) \otimes -1 = 0 \otimes -1 = 0 \in D,$
 $(0 \otimes 0) \otimes -1 = 0 \otimes -1 = 0 \in D, (1 \otimes 0) \otimes -1 = 1 \otimes -1 = 1 \in D$ and
 $0 \odot -1 \odot -1 = 0 \odot -1 = 0 \in D, 1 \odot -1 \odot -1 = 0 \odot -1 = 0 \in D,$ and so on.

- (2) $D = \{-1, 1\}, w = 0$: $1 \odot 0 \odot 0 = 0 \odot 0 = 0 \notin D$ which suffices to conform that $\{-1, 1\}$ is not DS of L and neither $\{1\}$ is DS of L , and so on.

Lemma 5.1.2. Suppose $(L; \otimes, \odot, 0)$ is LBA, and D is DS of L . Then $0 \in D$.

Proof. Since $D \neq \emptyset$, let $m \in D \Rightarrow m \in L \Rightarrow m \odot 0 \odot 0 = m \odot 0 = 0 \odot 0 = 0 \in D$, since $0 \in L$, by Definition 5.1.3 (2) and hence $0 \in D$. □

Definition 5.1.4. Let I be a non-empty subset of LBA L . Thus I is called ideal of LBA $(L; \otimes, \odot, 0)$ if the following axioms are satisfied, $\forall m, n, w \in L$:

- (1) $m \otimes n \in I$, and $n \in I \Rightarrow m \in I$,
- (2) $m, n \in I \Rightarrow (m \otimes n) \odot (n \otimes w) \in I$.

Example 5.1.8. Let $L = \{0, m, n, w\}$ and two binary operations \otimes and \odot on L be as defined as the following table:

\otimes	0	m	n	w
0	0	0	0	0
m	m	0	m	m
n	n	n	0	n
w	w	w	w	0

\odot	0	m	n	w
0	0	0	0	0
m	0	m	0	0
n	0	0	n	0
w	0	0	0	w

Table 5.3: Tables for the LBA $(L, \otimes, \odot, 0)$ (See [42] for \otimes)

Then, by checking algebraic calculations, we have checked that $(L, \otimes, \odot, 0)$ is LBA and we have checked the following as well:

- (1) $L, \{0\}, \{0, m\}, \{0, n\}, \{0, w\}, \{0, m, n\}, \{0, m, w\}, \{0, n, w\}$ are ideals of L .
- (2) $\{m\}, \{n, w\}, \{m, n\}, \{m, n, w\}$ are not ideals of L , for $m \neq 0, n \neq 0, w \neq 0$.

In order to explain the above two, we take one set for each and show as follows:

Take $I = \{0, w\}$. We show that I is ideal of L , so that we have:

$$(1) 0 \otimes w = 0 \in I \text{ and } w \in I \Rightarrow 0 \in I, \quad w \otimes 0 = w \in I \text{ and } 0 \in I \Rightarrow w \in I,$$

$$0 \otimes 0 = 0 \in I \text{ and } 0 \in I \Rightarrow 0 \in I, \quad w \otimes w = 0 \in I \text{ and } w \in I \Rightarrow w \in I, \text{ and so on.}$$

$$(2) 0, w \in I, n \in L \Rightarrow (0 \otimes w) \odot (w \otimes n) = 0 \odot w = 0 \in I,$$

$$0, w \in I, n \in L \Rightarrow (w \otimes 0) \odot (0 \otimes n) = w \odot 0 = 0 \in I,$$

$$w \in I, n \in L \Rightarrow (w \otimes w) \odot (w \otimes n) = 0 \odot w = 0 \in I, \text{ and so on.}$$

Therefore, all results of the operations satisfy the two axioms of ideal, I . Furthermore, for the other sets listed in (1) above, the results of both axioms also stay within I and satisfy the axioms of ideal.

(2) Take $N = \{0, n, w\}$ and we show that N is not SA of L

$$w, n \in N \text{ but } w \otimes n = m \notin N \text{ so that } N \text{ is not SA of } L.$$

Similarly, the other sets listed in (2) above are also not SAs of L .

Example 5.1.9. Let $L = \{-i, -1, 0, i, 1\}$ be a set, where $i = \sqrt{-1}$ and two binary operations \otimes and \odot on L be as defined as follows: Then, $(L, \otimes, \odot, 0)$ is LBA and we have the following as well:

$$m \otimes n = \begin{cases} 0, & \text{if } m = n, \\ m, & \text{if } m \neq n \text{ and } m \neq 0, \\ 0, & \text{if } m = 0, \end{cases} \quad \text{and} \quad m \odot n = \begin{cases} m, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have the following as well:

- (1) $\{0\}, \{0, -i\}, \{0, i\}, \{0, -1\}, \{0, 1\}, \{0, -i, -1, 1\}, \{0, -1, i\}, L$ are ideals of L ,

as for example, if $I = \{0, -i, -1, 1\}$, and $m = -i$, and $n = -1$, then by the first axiom of I : $m \otimes n = (-i) \otimes (-1) = (-i) \in I$, $n = -1 \in I \Rightarrow m = (-i) \in I$ hold and for the others hold, too; and $m = -i, n = -1 \in I \Rightarrow (m \otimes n) \odot (n \otimes w) = ((-i) \otimes (-1)) \odot ((-i) \otimes 1) = (-i) \odot (-i) = (-i) \in I$, when $w = 1$ also hold, and for the other listed elements, the axioms for ideals in L hold, too.

- (2) $\{i\}, \{-1\}, \{-1, 1\}, \{(-i), -1\}, \{i, -1, 1\}, L - \{0\}$ are not ideals of L ;

since for example as discussed in (1) above, the first axiom of I fails for most of the sets listed in (2) when $m = -i$ and $n = -1$.

Example 5.1.10. Suppose $L = \{0, m, n\}$ is a set and two binary operations \otimes and \odot on L are given by the table below:

\otimes	0	m	n
0	0	0	0
m	m	0	0
n	n	n	0

\odot	0	m	n
0	0	0	0
m	0	m	0
n	0	0	n

Table 5.4: Tables for the LBA $(L, \otimes, \odot, 0)$ (See [42] for \otimes)

From the above tables, we have checked that $(L, \otimes, \odot, 0)$ is LBA and we have the following as well:

- (1) $\{0\}, \{0, m\}, \{0, n\}, L$ are ideals of L .

as for example, if $I = \{0, m\}$, then by the first axiom of I : $m \otimes 0 = m \in I$, $0 \in I \Rightarrow m \in I$ hold and for the others hold, too; and $m, 0 \in I \Rightarrow (m \otimes 0) \odot (0 \otimes n) = (m \otimes 0) \odot (0 \otimes n) = m \odot 0 = m \in I$, also hold, and for the other listed elements, the axioms for ideals in L hold, too.

- (2) $\{m\}, \{n\}, \{m, n\}$ are not ideals of L for $m \neq 0, n \neq 0$.

since for example as discussed in (1) above, the first axiom of I fails for most of the sets listed in (2) when $I = \{m, n\}$, we have: $m, n \in I \Rightarrow (n \otimes m) \odot (m \otimes 0) = (n \odot m) = 0 \notin I = \{m, n\}$.

Lemma 5.1.3. Suppose $(L; \otimes, \odot, 0)$ is LBA Then if I is ideal of L , then $0 \in I$.

Proof. Since $I \neq \emptyset$, $\exists n \in I \subseteq L$ such that $(n \otimes n) \odot (n \otimes n) = 0 \odot 0 = 0 \in I$ by Definition 5.1.4 (2)

Thus $0 \in I$.

□

Proposition 5.1.2. *Suppose I is a non-empty subset of LBA $(L; \otimes, \odot, 0)$ and \leq is a binary relation on the BCL–algebra $(L; \otimes, 0)$. Then:*

(1) *I is an ideal of LBA L if the following axioms are satisfied, for the binary relation \leq in a BCL–algebra B and $\forall m, n, w \in L$:*

(i) $m \leq n$ and $n \in I \Rightarrow m \in I$,

(ii) $m, n \in I \Rightarrow (m \otimes n) \odot (n \otimes w) \in I$.

(2) *If I is an ideal of L such that $w \in I$ and $n \leq w$ then $n \in I$.*

Proof. Suppose L is LBA and $I \subseteq L$.

(i) $m \leq n$, and $n \in I \Rightarrow m \in I \Rightarrow m \otimes n = 0$ and $n \in I$ with $0 \in I$ by Lemma 5.1.3

Then $0 = m \otimes n \in I$ and $n \in I \Rightarrow m \in I$, here the first axiom for ideal is satisfied.

(ii) $m, n \in I \Rightarrow (m \otimes n) \odot (n \otimes w) \in I$ is the second axiom of an ideal.

Thus, since (i) and (ii) describe the two axioms for ideal being satisfied, I is an ideal of L .

(2) Let $w \in I$ and $n \leq w \Rightarrow w \in I$ and $n \otimes w = 0 \Rightarrow w \in I$ and $0 = n \otimes w \in I$

Thus $w \in I$ and $n \otimes w \in I \Rightarrow n \in I$ by Definition 5.1.4 (1).

□

5.2. Fuzzy Substructures of Liu^B–Algebra

Under this section, we explore three fundamental fuzzy substructures within Liu^B–algebras (LBAs.): fuzzy subalgebras, fuzzy deductive systems, and fuzzy ideals. We introduce precise definitions for each, supported by illustrative examples that clarify their structures and behavior. Key properties are established through rigorous justifications, including the dominance of the zero element’s membership grade across all substructures, the preservation of substructure properties under intersection (with counter examples demonstrating failure under union), equivalence conditions linking crisp substructures to their fuzzy counterparts via characteristic functions.

Theorems, lemmas, and propositions validate fundamental results, while examples and remarks highlight critical distinctions between each pair of substructures. This systematic treatment lays the groundwork for understanding hierarchical relationships and operational compatibility within fuzzy LBA substructures.

5.2.1 Fuzzy Subalgebra of Liu^B–Algebra

Under this subsection, we introduce definitions and give examples of fuzzy subalgebra of LBA , state and prove the properties and theorems of fuzzy subalgebra of LBA

Definition 5.2.1.1. A fuzzy subset η in $(L, \otimes, \odot, 0)$ in a non-empty set L where $\eta : L \rightarrow [0, 1]$ define the membership function is known as a fuzzy subalgebra of L if the two pairs of statements hereunder are satisfied; $\forall m, n \in L$:

- (i) $\eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\}$
- (ii) $\eta(m \odot n) \geq \min\{\eta(m), \eta(n)\}$.

Example 5.2.1.1. Let $L = \{0, m, n, w\}$ be a set and two binary operations \otimes and \odot on L be given as in Table 5.2 of Example 5.1.4 above and the fuzzy subset $\eta : L \rightarrow [0, 1]$ given by:

$$\eta(k) = \begin{cases} 0.9, & \text{if } k = 0, \\ 0.7, & \text{if } k = m, w, \\ 0.4, & \text{if } k = n. \end{cases}$$

Thus, applying mathematical manipulations, like:

$$\eta(m \otimes n) = \eta(w) = 0.7 \geq \min\{\eta(m), \eta(n)\} = \min\{0.7, 0.4\} = 0.4 \text{ and}$$

$\eta(m \odot n) = \eta(n) = 0.4 \geq \min\{\eta(m), \eta(n)\} = \min\{0.7, 0.4\} = 0.4$, both hold and similarly all the others hold which could be verified for each of the elements in L in a similar fashion.

Lemma 5.2.1.1. Let $(L, \otimes, \odot, 0)$ be LBA If η is a fuzzy SA of L , then $\eta(0) \geq \eta(m)$; $\forall m \in L$.

Proof. Let $m \in L$, then we have: $m \otimes m = 0$, by Definition 1.1.8 (1).

Hence, from the first axiom of fuzzy SA of L , we have

$$\eta(0) = \eta(m \otimes m) \geq \min\{\eta(m), \eta(m)\} = \eta(m) \Rightarrow \eta(0) \geq \eta(m), \forall m \in L.$$

□

Theorem 5.2.1.1. Let $(L; \otimes, \odot, 0)$ be LBA, M be a non-empty subset of L and η be a fuzzy

subset in L such that $\eta(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M, \end{cases}$ where $\delta, \sigma \in [0, 1]$, $\delta > \sigma$.

Then M is SA of L if and only if η is a fuzzy SA of L .

Proof. Let M be SA of L , we claim that η is a fuzzy SA of L .

(1) First we prove: $\eta(m \odot n) \geq \min\{\eta(m), \eta(n)\}, \forall m, n \in L$.

Case(i) : Let $m, n \in M \Rightarrow m \odot n \in M$ (since M is SA of L by hypothesis)

Then $\eta(m) = \eta(n) = \delta \Rightarrow \eta(m \odot n) = \delta$

$\Rightarrow \eta(m \odot n) \geq \min\{\eta(m), \eta(n)\} = \min\{\delta, \delta\} = \delta, \forall m, n \in L$.

Case(ii) : Let $n \in M, m \notin M$ (or $m \in M, n \notin M$)

Then $\eta(m) = \sigma, \eta(n) = \delta$ (or $\eta(m) = \delta, \eta(n) = \sigma$)

And again, $m \odot n \notin M$ or $m \odot n \in M \Rightarrow \eta(m \odot n) = \sigma, \eta(m \odot n) = \delta$

$\Rightarrow \eta(m \odot n) \geq \sigma = \min\{\eta(m), \eta(n)\} = \min\{\sigma, \delta\} = \sigma$.

Case(iii) : Let $m \notin M, n \notin M$. Then $\eta(m) = \sigma = \eta(n)$

\Rightarrow Either $m \odot n \notin M$ or $m \odot n \in M \Rightarrow \eta(m \odot n) = \sigma$ or $\eta(m \odot n) = \delta$.

$\Rightarrow \eta(m \odot n) \geq \sigma = \min\{\eta(m), \eta(n)\} = \min\{\sigma, \sigma\} = \sigma$

By Cases (i), (ii) and (iii), above we deduce that: $\eta(m \odot n) \geq \min\{\eta(m), \eta(n)\}, \forall m, n \in L$.

(2) Similar steps can be followed to show that: $\eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\}, \forall m, n \in L$.

Hence η is fuzzy SA of L .

Conversely, suppose η is a fuzzy SA of L , then we claim that M is SA of L .

Let $n, m \in M \Rightarrow \eta(m) = \eta(n) = \delta$ and

$\eta(m \otimes n) \geq \min\{\eta(m), \eta(n)\} = \min\{\delta, \delta\} = \delta \Rightarrow \eta(m \otimes n) \geq \delta$

and since $\delta, \sigma \in [0, 1], \delta > \sigma$, we have $\eta(m \odot n) \leq \delta$ (by definition of η), hypothesis

$\Rightarrow \eta(m \odot n) = \delta \Rightarrow m \odot n \in M$

Again, for $n, m \in M$ it is easy to show that, $m \otimes n \in M$, following the preceding similar steps.

Therefore, M is SA of L . □

Corollary 5.2.1.1. *Let $(L; \otimes, \odot, 0)$ be LBA, M be a non-empty subset of L and χ_M be the Characteristic function on M . Then χ_M is a fuzzy SA of L if and only if M is SA of L .*

Proof. Suppose the Characteristic function $\chi_M : M \rightarrow \{0, 1\}$ is defined as: $\chi_M(m) = \begin{cases} 1, & \text{if } m \in M, \\ 0, & \text{if } m \notin M. \end{cases}$

This proof can be completed by following the steps for Theorem 5.2.1.1 above by simply replacing δ by 1 and σ by 0. □

Theorem 5.2.1.2. *The intersection of any two fuzzy SAs, η_1 and η_2 of LBA L is also a fuzzy SA of L .*

Proof. Let η_1 and η_2 be any two fuzzy SAs of L . Then we claim; $\eta_1 \cap \eta_2$ is a fuzzy

SA of L . Let $m, n \in L$, then, we have:

$$\begin{aligned}
 (1) \quad (\eta_1 \cap \eta_2)(m \otimes n) &= \min\{(\eta_1 \cap \eta_2)(m \otimes n), (\eta_1 \cap \eta_2)(m \otimes n)\} \\
 &\geq \min\{\min\{\eta_1(m), \eta_1(n)\}, \min\{\eta_2(m), \eta_2(n)\}\} \\
 &= \min\{\min\{\eta_1(m), \eta_2(m)\}, \min\{\eta_1(n), \eta_2(n)\}\} \\
 &= \min\{(\eta_1 \cap \eta_2)(m), (\eta_1 \cap \eta_2)(n)\} \\
 \Rightarrow (\eta_1 \cap \eta_2)(m \otimes n) &\geq \min\{(\eta_1 \cap \eta_2)(m), (\eta_1 \cap \eta_2)(n)\}, \forall m, n \in L.
 \end{aligned}$$

(2) Similarly, it is possible to verify that $(\eta_1 \cap \eta_2)(m \odot n) \geq \min\{(\eta_1 \cap \eta_2)(m), (\eta_1 \cap \eta_2)(n)\}$ of fuzzy SAs of L is fuzzy SA of L . □

Corollary 5.2.1.2. *The intersection, $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, of any family of fuzzy SAs, $\{(\eta_i, \tau_i) : i \in I\}$, of LBA L is also a fuzzy SA of L .*

Proof. Let $\{(\eta_i, \tau_i) : i \in I\}$, be family of fuzzy SAs of L . Then we claim; $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, is a fuzzy SA of L and let $m, n \in L$, then, we have:

$$\begin{aligned}
 (1) \quad \bigcap_{i \in I} \eta_i(m \otimes n) &= \inf_{i \in I} \{\eta_i(m \otimes n)\} \\
 &\geq \inf_{i \in I} \{\min\{\eta_i(m), \eta_i(n)\}\} \\
 &\geq \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\
 &= \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\
 &= \min\left\{\bigcap_{i \in I} \eta_i(m), \bigcap_{i \in I} \eta_i(n)\right\}, \forall m, n \in B.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \bigcap_{i \in I} \eta_i(m \odot n) &= \inf_{i \in I} \{\eta_i(m \odot n)\} \\
 &\geq \inf_{i \in I} \{\min\{\eta_i(m), \eta_i(n)\}\} \\
 &\geq \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\
 &= \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\
 &= \min\left\{\bigcap_{i \in I} \eta_i(m), \bigcap_{i \in I} \eta_i(n)\right\}, \forall m, n \in B.
 \end{aligned}$$

Thus, by (1) and (2) above, intersection of fuzzy SAs of L is fuzzy SA of L . □

Remark 5.2.1.1. *The union of two fuzzy SAs of LBA L may not be fuzzy SA of L , justified by the following counter example.*

Example 5.2.1.2. *Let $(L, \otimes, \odot, 0)$ where $L = \{0, m, n, w\}$ and two binary operations \otimes and \odot on L be as defined in Table 5.2 of Example 5.1.4 above and define two fuzzy SAs η_1 and η_2 of L as in Table 5.6 below:*

Elements of L	η_1	η_2	$\eta_1 \cup \eta_2$
0	0.9	0.8	0.9
m	0.3	0.7	0.7
n	0.5	0.3	0.5
w	0.3	0.3	0.3

Table 5.5: A table that defines fuzzy SAs and their unions

Hence, since $m \otimes n = w$: then we have:

$$\begin{aligned} (\eta_1 \cup \eta_2)(m \otimes n) &= (\eta_1 \cup \eta_2)(w) = 0.3 \geq \min\{(\eta_1 \cup \eta_2)(m), (\eta_1 \cup \eta_2)(n)\} \\ &= \min\{0.7, 0.5\} = 0.5, \text{ which is false, and this shows that the union} \\ &\text{of two fuzzy SA of } L \text{ is not necessarily fuzzy SA of } L. \end{aligned}$$

5.2.2 Fuzzy Deductive System of Liu^B—Algebra

Under this subsection, we introduce definitions and give examples of fuzzy deductive systems of LBA, state and prove some properties and theorems of fuzzy deductive systems of LBA

Definition 5.2.2.1. *A fuzzy subset η in L , where $(L; \otimes, \odot, 0)$ is LBA, is known as a fuzzy DS of L if the statements hereunder are satisfied $\forall m, n, w \in L$:*

- (i) $\eta(m \otimes (n \otimes w)) \geq \min\{\eta(m), \eta(n)\}$
- (ii) $\eta(m \odot w \odot w) \geq \eta(m)$.

Example 5.2.2.1. *Let $L = \{0, m, n, w\}$ and two binary operations \otimes and \odot on LBA L be given as in Table 5.2 of Example 5.1.4 above and the fuzzy subset $\eta : L \rightarrow [0, 1]$ given by:*

$$\eta(k) = \begin{cases} 0.8, & \text{if } k = 0, \\ 0.3, & \text{if } k = m, n, w. \end{cases}$$

Thus, it has been checked that η is fuzzy DS of L explained for some cases as follows taking random elements $w, 0, n \in L$ (as one case):

$$(i) \quad \eta(w \circledast (0 \circledast n)) = \eta(w \circledast 0) = \eta(w) = 0.3 \geq \min\{\eta(w), \eta(0)\} = \{0.3, 0.3\} = 0.3 \text{ holds}$$

and so do other cases,

$$(ii) \quad \eta(w \odot n \odot n) = \eta(0) = 0.8 = \geq \eta(w) \text{ 0.3 holds and so do other cases,}$$

Lemma 5.2.2.1. *Let $(L; \circledast, \odot, 0)$ be LBA If η is a fuzzy DS of L , then $\eta(0) \geq \eta(m), \forall m \in L$.*

Proof. For $m \in L$, we have: $m \odot 0 \odot 0 = 0$. Then

$$\Rightarrow \eta(m \odot 0 \odot 0) \geq \eta(m) \Rightarrow \eta(m \odot 0 \odot 0) = \eta(0) \geq \eta(m), \forall m \in L,$$

Hence, $\eta(0) \geq \eta(m), \forall m \in L$. □

Proposition 5.2.2.1. *Let $(L; \circledast, \odot, 0)$ be LBA and M be a non-empty subset of L , then the Characteristic function χ_M on M is a fuzzy DS of L if and only if M is DS of L .*

Proof. Suppose the Characteristic function, χ_M , is fuzzy DS of L where $\chi_M : M \rightarrow \{0, 1\}$ is

$$\text{defined as: } \chi_M(k) = \begin{cases} 1, & \text{if } k \in M, \\ 0, & \text{if } k \notin M. \end{cases}$$

We need to verify that M is DS of L , that is;

$$(1) \quad m \in M \Rightarrow m \odot w \odot w \in M \quad \text{and}$$

$$(2) \quad m, n \in M \Rightarrow (m \circledast (n \circledast w)) \in M.$$

$$(1) \quad \text{Let } m \in M \Rightarrow \chi_M(m) = 1. \text{ Then } \chi_M(m \odot w \odot w) \geq \chi_M(m) = 1 \Rightarrow \chi_M(m \odot w \odot w) \geq 1$$

But $\chi_M(m \odot w \odot w) \leq 1$, by definition of Characteristic function.

$$\Rightarrow \chi_M(m \odot w \odot w) = 1 \Rightarrow m \odot w \odot w \in M.$$

$$(2) \quad \text{Let } m, n \in M \Rightarrow \chi_M(m) = \chi_M(n) = 1$$

$$\text{Then } \chi_M(m \circledast (n \circledast w)) \geq \min\{\chi_M(m), \chi_M(n)\} = \min\{1, 1\} = 1$$

But $\chi_M(m \circledast (n \circledast w)) \leq 1$, by definition of Characteristic function.

$$\Rightarrow \chi_M(m \circledast (n \circledast w)) = 1 \Rightarrow m \circledast (n \circledast w) \in M$$

Therefore, M is a DS of L , by (1) and (2) above.

Conversely, assume M is DS of L . We need to assert that the Characteristic function χ_M is a fuzzy DS of L , claiming to verify the two defining axioms. Then we prove this part by verifying the two

defining axioms where we consider three cases for one and two case for the other as follows:

Axiom (1): $\chi_M(m \otimes (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\}; \forall m, n, w \in L$, (by 3 Cases).

Claim (1): Asserting axiom 1 by considering 3 cases as follows:

Case(i) : Suppose $m, n \in M$ and $w \in L \Rightarrow m \otimes (n \otimes w) \in M$; since M is DS of L

$$\Rightarrow \chi_M(m) = 1 = \chi_M(n) = \chi_M(m \otimes (n \otimes w))$$

$$\Rightarrow \chi_M(m \otimes (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\} = \min\{1, 1\} = 1$$

Case(ii) : Suppose $m \in M, n \notin M$ (or similarly $m \notin M, n \in M$)

$$\Rightarrow \chi_M(m) = 1, \chi_M(n) = 0 \text{ (or similarly, } \chi_M(m) = 0, \chi_M(n) = 1\text{), and}$$

$$\text{either } \chi_M(m \otimes (n \otimes w)) = 1 \text{ or } \chi_M(m \otimes (n \otimes w)) = 0 \Rightarrow \chi_M(m \otimes (n \otimes w)) \geq 0$$

$$\Rightarrow \chi_M(m \otimes (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\} = \min\{1, 0\} = 0$$

Case(iii) : Suppose $m \notin M, n \notin M \Rightarrow \chi_M(m) = 0 = \chi_M(n)$, and

$$\text{either } \chi_M(m \otimes (n \otimes w)) = 1 \text{ or } \chi_M(m \otimes (n \otimes w)) = 0 \Rightarrow \chi_M(m \otimes (n \otimes w)) \geq 0$$

$$\Rightarrow \chi_M(m \otimes (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\} = \min\{0, 0\} = 0$$

Then by Cases (i) – (iii) above, $\chi_M(m \otimes (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\}, \forall m, w \in L$.

Axiom (2): $\chi_M(m \odot w \odot w) \geq \chi_M(m); \forall m \in M, \forall w \in L$

Claim (2): Asserting axiom 2 by considering 2 cases as follows:

Case(i) : Suppose $m \in M \Rightarrow (m \odot w) \odot w \in M$; since M is DS of L

$$\Rightarrow \chi_M(m) = 1 = \chi_M(m \odot w \odot w) \Rightarrow \chi_M(m \odot w \odot w) \geq \chi_M(m) = 1$$

Case(ii) : Let $m \notin M$. Then $\chi_M(m) = 0$, and either $\chi_M(m \odot w \odot w) = 1$ or $\chi_M(m \odot w \odot w) = 0$

$$\Rightarrow \chi_M(m \odot w \odot w) \geq 0 \Rightarrow \chi_M(m \odot w \odot w) \geq \chi_M(m) = 0$$

By Cases (i) – (ii) above, we deduce that $\chi_M(m \odot w \odot w) \geq \chi_M(m), \forall m, w \in L$.

Therefore, by axioms (1) and (2) (Conclusions of claims (1) and (2)) above, we conclude that χ_M is fuzzy DS of L . □

Corollary 5.2.2.1. Let $(L; \otimes, \odot, 0)$ be LBA, $\emptyset \neq M \subseteq L$ and $\eta_M : M \rightarrow [0, 1]$ be a

fuzzy subset in L such that $\delta, \sigma \in [0, 1]$, and $\delta > \sigma$, where $\eta_M(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M. \end{cases}$

Then M is DS of L if and only if η_M is a fuzzy DS of L .

Proof. Similar to the proof of Proposition 5.2.2.1 above with the exception that 1 is replaced by δ and 0 is replaced by σ . \square

Theorem 5.2.2.1. *Intersection of any two fuzzy DSs, η_1 and η_2 of LBA L is fuzzy DS of L.*

Proof. Let η_1 and η_2 be any two fuzzy DSs of L. Then we need to prove:

$\eta_1 \cap \eta_2$ is a fuzzy DS of L, which is justified as follows:

- (i) $(\eta_1 \cap \eta_2)(m \odot w \odot w) = \min\{\eta_1(m \odot w \odot w), \eta_2(m \odot w \odot w)\} \geq \min\{\eta_1(m), \eta_2(m)\}$
 $\Rightarrow (\eta_1 \cap \eta_2)(m \odot w \odot w) \geq \eta_1(m), \forall m \in L,$
- (ii) $(\eta_1 \cap \eta_2)(m \otimes (n \otimes w)) = \min\{\eta_1(m \otimes (n \otimes w)), \eta_2(m \otimes (n \otimes w))\}$
 $\geq \min\{\min\{\eta_1(m), \eta_1(n)\}, \min\{\eta_2(m), \eta_2(n)\}\}$
 $= \min\{\min\{\eta_1(m), \eta_2(m)\}, \min\{\eta_1(n), \eta_2(n)\}\},$
 $= \min\{(\eta_1 \cap \eta_2)(m), (\eta_1 \cap \eta_2)(n)\}$

Thus, intersection of two fuzzy DSs of L is fuzzy DS of L. \square

The above theorem can also be generalized to any family of fuzzy DSs of L as in the corollary below.

Corollary 5.2.2.2. *The intersection, $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, of any family of fuzzy DSs, $\{(\eta_i, \tau_i) : i \in I\}$, of LBA L is also a fuzzy DS of L.*

Proof. Let $\{(\eta_i, \tau_i) : i \in I\}$, be family of fuzzy DSs of L and let $m, n, w \in L$. Then we need

to prove that: $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, is a fuzzy DS of L, which is justified as follows:

- (i) $(\bigcap_{i \in I} \eta_i)(m \odot w \odot w) = \inf_{i \in I} \{\eta_i(m \odot w \odot w)\}$
 $\geq \inf_{i \in I} \{\eta_i(m)\}$
 $= \bigcap_{i \in I} \eta_i(m), \forall m, w \in B.$
- (ii) $(\bigcap_{i \in I} \eta_i)(m \otimes (n \otimes u)) = \inf_{i \in I} \{\eta_i(m \otimes (n \otimes u))\}$
 $\geq \inf_{i \in I} \{\min\{\eta_i(m), \eta_i(n)\}\}$
 $\geq \min\{\inf_{i \in I} \{\eta_i(m), \eta_i(n)\}\}$
 $= \min\{\inf_{i \in I} \{\eta_i(m)\}, \inf_{i \in I} \{\eta_i(n)\}\}$

$$= \min \left\{ \bigcap_{i \in I} \eta_i(m), \bigcap_{i \in I} \eta_i(n) \right\}, \forall m, n, u \in B.$$

□

Thus, by (i) and (ii) above, intersection of two fuzzy DSs of L is fuzzy DS of L.

Remark 5.2.2.1. *The union of two fuzzy DSs of LBA L may not be fuzzy DS of L, justified by the following counter example.*

Example 5.2.2.2. *Let $L = \{0, m, n, w\}$ and the two binary operations \otimes and \odot on L be as defined in Table 5.2 of Example 5.1.4 above and define two fuzzy DSs η_1 and η_2 of L as follows:*

Elements of L	$\eta_1(k)$	$\eta_2(k)$	$(\eta_1 \cup \eta_2)(k)$
0	0.8	1	1
m	0.4	0.6	0.6
n	0.4	0.7	0.7
w	0.8	0.7	0.8

Table 5.6: A table that defines fuzzy SAs and their unions

Then, from Table 5.2 of Example 5.1.4 above, we have that: $w \otimes (n \otimes 0) = w \otimes n = m$.

Also, from Table 5.6 of Example 5.2.2.2 above, we have:

$$\Rightarrow (\eta_1 \cup \eta_2)(w \otimes (n \otimes 0)) = (\eta_1 \cup \eta_2)(m) = 0.6$$

$$\Rightarrow (\eta_1 \cup \eta_2)(w \otimes (n \otimes 0)) = 0.6 \geq \min\{\eta_1 \cup \eta_2(w), \eta_1 \cup \eta_2(n)\}$$

$$= \min\{0.8, 0.7\} = 0.7, \text{ which is false confirming that}$$

the union of two fuzzy DSs of LBA L is not necessarily fuzzy DS of L.

5.2.3 Fuzzy Ideal of Liu^B–Algebra

Under this subsection, we introduce definition and give examples of fuzzy ideal of LBA, state and prove some properties and theorems on fuzzy ideal of LBA

Definition 5.2.3.1. *A fuzzy subset η in LBA $(L; \otimes, \odot, 0)$ is called a fuzzy ideal of LBA if the following statements are satisfied $\forall m, n \in L$:*

$$(i) \quad \eta(m) \geq \min\{\eta(m \otimes n), \eta(n)\}$$

$$(ii) \quad \eta((m \otimes n) \odot (n \otimes w)) \geq \min\{\eta(m), \eta(n)\}.$$

Example 5.2.3.1. Let $L_1 = \{0, m, n, w\}$, $L_2 = \{0, m, n\}$ be sets and binary operations \otimes and \odot on both L_1 and L_2 be as defined in Table 5.3 and Table 5.4 above, respectively and let the fuzzy subsets η_1 in L_1 and η_2 in L_2 be defined as follows, and let $\eta_1 : L_1 \rightarrow [0, 1]$, $\eta_2 : L_2 \rightarrow [0, 1]$ be

$$\text{given by: } \eta_1(k) = \begin{cases} 0.9, & \text{if } k = 0, \\ 0.4, & \text{if } k = m, w, \\ 0.2, & \text{if } k = n, \end{cases} \quad \text{and} \quad \eta_2(k) = \begin{cases} 0.8, & \text{if } k = 0, \\ 0.5, & \text{if } k = m, n. \end{cases}$$

η_1 is fuzzy ideal of L_1 and η_2 is fuzzy ideal of L_2 , where we explain these examples as follows for some cases taking some randomly selected elements of L_1 and L_2 like for instance; $n, 0, w \in L_1$ and $m, n, 0 \in L_2$.

$$(1) (i) \eta_1(n) = 0.2 \geq \min\{\eta_1(n \otimes 0), \eta_1(0)\} = \min\{\eta_1(n), \eta_1(0)\} = \min\{0.2, 0.9\} = 0.2 \text{ holds}$$

and so do the others for any arbitrary choices of elements of L_1 .

$$(ii) \eta_1(n \otimes 0) \odot (0 \otimes m) = \eta_1(n \odot 0) = \eta_1(0) = 0.9$$

$$\geq \min\{\eta_1(n), \eta_1(0)\} = \min\{0.2, 0.9\} = 0.2 \text{ holds and so do the}$$

others for any arbitrary choices of elements of L_1 .

$$(2) (i) \eta_2(m) = 0.5 \geq \min\{\eta_2(m \otimes n), \eta_2(n)\} = \min\{\eta_2(m), \eta_2(n)\} = \min\{0.5, 0.5\} = 0.5$$

holds and so do the others for any arbitrary choices of elements of L_2 .

$$(ii) \eta_2(m \otimes n) \odot (n \otimes 0) = \eta_2(m \odot n) = \eta_2(0) = 0.8$$

$$\geq \min\{\eta_2(m), \eta_2(n)\} = \min\{0.5, 0.5\} = 0.5 \text{ holds and so do the}$$

others for any arbitrary choices of elements of L_2 .

Lemma 5.2.3.1. Let $(L; \otimes, \odot, 0)$ be LBA If η is fuzzy ideal of L , then $\eta(0) \geq \eta(m), \forall m \in L$.

Proof. As $L \neq \emptyset$, $\exists m \in L$ such that $\eta((m \otimes m) \odot (m \otimes m)) \geq \min\{\eta(m), \eta(m)\} = \eta(m)$

$$\Rightarrow \eta((m \otimes m) \odot (m \otimes m)) = \eta(0 \odot 0) = \eta(0), \forall m \in L, \text{ as } m \otimes m = 0 \text{ and } 0 \otimes 0 = 0, \forall m \in L$$

$$\text{and } \eta((m \otimes m) \odot (m \otimes m)) = \eta(0) \geq \min\{\eta(m), \eta(m)\} = \eta(m), \forall m \in L$$

Therefore, $\eta(0) \geq \eta(m), \forall m \in L$

□

Proposition 5.2.3.1. Let M be a non-empty subset of L and χ_M be the Characteristic function, then χ_M is a fuzzy ideal of L if and only if M is ideal of L .

Proof. Suppose $\chi_M : M \rightarrow \{0, 1\}$ is a Characteristic function, then: $\chi_M(m) = \begin{cases} 1, & \text{if } m \in M, \\ 0, & \text{if } m \notin M. \end{cases}$

Suppose χ_M is a fuzzy ideal of L. We need to verify that M is an ideal of L.

(1) Let $m \otimes n, n \in M \Rightarrow \chi_M(m \otimes n) = \chi_M(n) = 1$, and then we claim that $m \in M$

$$\text{But } \chi_M(m) \geq \min\{\chi_M(m \otimes n), \chi_M(n)\} = \min\{1, 1\} = 1$$

$$\Rightarrow \chi_M(m) \geq 1 \text{ but } \chi_M(m) \leq 1, \forall m \in M, \text{ by definition of } \chi_M$$

$$\Rightarrow \chi_M(m) = 1 \Rightarrow m \in M$$

Hence, $m \otimes n, n \in M \Rightarrow m \in M$

(2) Again, let $m, n \in M \Rightarrow \chi_M(m) = \chi_M(n) = 1$.

Here also we claim that $(m \otimes n) \odot (n \otimes w) \in M, \forall m, n, w \in L$

$$\text{Now, } \chi_M((m \otimes n) \odot (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\} = \min\{1, 1\} = 1$$

$$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) \geq 1, \forall m, n \in M$$

but $\chi_M((m \otimes n) \odot (n \otimes w)) \leq 1, \forall m, n \in M$, by definition of χ_M

$$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) = 1 \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$$

Thus, $m, n \in M \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$ and therefore, M is a ideal of L if χ_M is fuzzy ideal of L.

Conversely, suppose M is ideal of L. We need to prove χ_M is a fuzzy ideal of L, $\forall m, n, w \in L$.

That means we need to show that:

$$(1) \chi_M(m) \geq \min\{\chi_M(m \otimes n), \chi_M(n)\}, \forall m, n \in L$$

$$(2) \chi_M((m \otimes n) \odot (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\}, \forall m, n, w \in L$$

(1) To show $\chi_M(m) \geq \min\{\chi_M(m \otimes n), \chi_M(n)\}$, we follow the following three cases:

Case(i) : Let $m \otimes n, n \in M \Rightarrow m \in M$, since M is ideal of L.

$$\text{But from the hypothesis, } \chi_M(m \otimes n) = \chi_M(n) = 1 \Rightarrow \chi_M(m) = 1$$

$$\Rightarrow \chi_M(m) = 1 \geq \min\{\chi_M(m \otimes n), \chi_M(n)\}$$

Case(ii) : Let $m \otimes n \in M, n \notin M$ (or $m \otimes n \notin M, n \in M$), proving one of these is similar to proving the other and so we prove the first part.

$$\Rightarrow \chi_M(m \otimes n) = 1 \text{ and } \chi_M(n) = 0 \text{ but either } m \in M \text{ or } m \notin M \Rightarrow \chi_M(m) = 1 \text{ or } \chi_M(m) = 0$$

$$\Rightarrow \chi_M(m) \geq 0 = \min\{\chi_M(m \otimes n), \chi_M(n)\} = \min\{1, 0\} \Rightarrow \chi_M(m) \geq \min\{\chi_M(m \otimes n), \chi_M(n)\}$$

Case(iii) : Let $m \otimes n \notin M, n \notin M \Rightarrow \chi_M(m \otimes n) = 0 = \chi_M(n)$.

But either $m \in M$ or $m \notin M \Rightarrow \chi_M(m) = 1$ or $\chi_M(m) = 0$

$$\Rightarrow \chi_M(m) \geq 0 = \min\{\chi_M(m \otimes n), \chi_M(n)\} = \min\{0, 0\}$$

Thus, by all the Cases (i) – (iii), above, we obtain $\chi_M(m) \geq \min\{\chi_M(m \otimes n), \chi_M(n)\}$.

(2) To show $\chi_M((m \otimes n) \odot (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\}$, we follow the following

the three cases as similarly as (1) above

Case(i) : Let $m, n \in M \Rightarrow (m \otimes n) \odot (n \otimes w) \in M, \forall w \in L$, since M is ideal of L .

But from the hypothesis, $\chi_M(m) = \chi_M(n) = 1 \Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) = 1$

$$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) = 1 \geq \min\{\chi_M(m), \chi_M(n)\}$$

Case(ii) : Let $m \in M, n \notin M$ (or $m \notin M, n \in M$), proving one of these is similar to proving

the other and so we prove the first part.

$\Rightarrow \chi_M(m) = 1$ and $\chi_M(n) = 0$ but either $(m \otimes n) \odot (n \otimes w) \in M$ or $(m \otimes n) \odot (n \otimes w) \notin M$

$$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) = 1 \text{ or } \chi_M((m \otimes n) \odot (n \otimes w)) = 0$$

$$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) \geq 0 = \min\{\chi_M(m), \chi_M(n)\} = \min\{1, 0\}$$

$$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\}$$

Case(iii) : Let $m \notin M, n \notin M \Rightarrow \chi_M(m) = 0 = \chi_M(n)$.

But either $(m \otimes n) \odot (n \otimes w) \in M$ or $(m \otimes n) \odot (n \otimes w) \notin M$

$$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) = 1 \text{ or } \chi_M((m \otimes n) \odot (n \otimes w)) = 0$$

$$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w)) \geq 0 = \min\{\chi_M(m), \chi_M(n)\} = \min\{0, 0\}$$

Thus, by the Cases (i) – (iii), above we obtain: $\chi_M((m \otimes n) \odot (n \otimes w)) \geq \min\{\chi_M(m), \chi_M(n)\}$.

In general, by (1) and (2) above, we have ratified that the Characteristic function, χ_M is fuzzy ideal of L if M is ideal of L . □

Theorem 5.2.3.1. *The intersection of any two fuzzy ideals, η_1 and η_2 of LBA L is a fuzzy ideal of L .*

Proof. Let η_1 and η_2 be any two fuzzy ideals of L . Then we need to prove:

(i) $(\eta_1 \cap \eta_2)(m) \geq \min\{(\eta_1 \cap \eta_2)(m \otimes n), (\eta_1 \cap \eta_2)(n)\}$ and then:

$$(\eta_1 \cap \eta_2)(m) = \min\{\eta_1(m), \eta_2(m)\}$$

$$\geq \min\{\min\{(\eta_1(m \otimes n), \eta_1(n)), \min\{(\eta_2(m \otimes n), \eta_2(n))\}\}$$

$$\begin{aligned}
&= \min\{\min\{\eta_1(m \otimes n), \eta_2(m \otimes n)\}, \min\{\eta_1(n), \eta_2(n)\}\} \\
&= \min\{(\eta_1 \cap \eta_2)(m \otimes n), (\eta_1 \cap \eta_2)(n)\}
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad (\eta_1 \cap \eta_2)((m \otimes n) \odot (n \otimes w)) &= \min\{\eta_1((m \otimes n) \odot (n \otimes w)), \eta_2((m \otimes n) \odot (n \otimes w))\} \\
&\geq \min\{\min\{(\eta_1(m), (\eta_1)(n))\}, \min\{(\eta_2(m), (\eta_2)(n))\}\} \\
&= \min\{\min\{(\eta_1(m), (\eta_2)(m))\}, \min\{(\eta_1(n), (\eta_2)(n))\}\} \\
&= \min\{(\eta_1 \cap \eta_2)(m), (\eta_1 \cap \eta_2)(n)\}
\end{aligned}$$

Thus by (i) and (ii) above, we ratified that $\eta_1 \cap \eta_2$ is a fuzzy ideal of L .

Therefore, intersection of any two fuzzy ideals of L is also fuzzy ideal of L . \square

The above theorem can also be generalized to any family of ideals of L as in the corollary below.

Corollary 5.2.3.1. *The intersection, $\bigcap_{i \in I} \eta_i$, of any family of fuzzy ideals, $\{(\eta_i, \tau_i) : i \in I\}$, of LBA L is also a fuzzy ideal of L .*

Proof. Let $\{(\eta_i, \tau_i) : i \in I\}$, be family of fuzzy ideals of L . Then we need to prove $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, is fuzzy ideal of L , and then:

$$\begin{aligned}
\text{(i)} \quad (\bigcap_{i \in I} \eta_i)(m) &= \inf_{i \in I} \{\eta_i(m)\} \\
&\geq \inf_{i \in I} \{\min\{\eta_i(m \otimes n), \eta_i(n)\}\} \\
&\geq \min\{\inf_{i \in I} \{\eta_i(m \otimes n), \eta_i(n)\}\} \\
&= \min\{\inf_{i \in I} \{\eta_i(m \otimes n)\}, \inf_{i \in I} \{\eta_i(n)\}\} \\
&= \min\{\bigcap_{i \in I} \eta_i(m \otimes n), \bigcap_{i \in I} \eta_i(n)\}, \forall m, n \in L.
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad \bigcap_{i \in I} \eta_i((m \otimes n) \odot (n \otimes w)) &= \inf_{i \in I} \{\eta_i((m \otimes n) \odot (n \otimes w))\} \\
&\geq \inf_{i \in I} \{\min\{(\eta_i(m), (\eta_i)(n))\}\} \\
&\geq \min\{\inf_{i \in I} \{(\eta_i(m), (\eta_i)(n))\}\}
\end{aligned}$$

$$\begin{aligned}
&= \min\left\{\inf_{i \in I}\{(\eta_i)(m)\}, \inf_{i \in I}\{(\eta_i)(n)\}\right\} \\
&= \min\left\{\bigcap_{i \in I}(\eta_i)(m), \bigcap_{i \in I}(\eta_i)(n)\right\}, \forall m, n, w \in L.
\end{aligned}$$

Thus by (i) and (ii) above, we ratified that $\bigcap_{i \in I} \eta_i$ is a fuzzy ideal of L .

Therefore, intersection of any family fuzzy ideals of LBA L is also fuzzy ideal of L . □

Remark 5.2.3.1. *The union of two fuzzy ideals of L may not be necessarily fuzzy ideal of L which is illustrated by Example 5.2.3.2 below.*

Example 5.2.3.2. *Let $(L, \otimes, \odot, 0)$ where $L = \{0, m, n, w\}$ and two binary operations \otimes and \odot on L be as defined in Table 5.7 below, and the two fuzzy ideals η_1 and η_2 of L as define below:*

$$\eta_1(k) = \begin{cases} 0.9, & \text{if } k = 0, \\ 0.6, & \text{if } k = n, \\ 0.4, & \text{if } k = m, w, \end{cases} \quad \text{and } \eta_2(k) = \begin{cases} 0.8, & \text{if } k = 0, \\ 0.7, & \text{if } k = m, \\ 0.5, & \text{if } k = n, w, \end{cases} \quad \Rightarrow (\eta_1 \cup \eta_2)(k) = \begin{cases} 0.9, & \text{if } k = 0, \\ 0.7, & \text{if } k = m, \\ 0.6, & \text{if } k = n, \\ 0.5, & \text{if } k = w. \end{cases}$$

\otimes	0	m	n	w
0	0	0	0	0
m	m	0	w	m
n	n	w	0	n
w	w	w	n	0

\odot	0	m	n	w
0	0	0	0	0
m	0	m	w	n
n	0	w	n	m
w	0	n	m	w

Table 5.7: Tables for the LBA $(L, \otimes, \odot, 0)$ (See [42] for \otimes)

From Table 5.7 above, it has been shown that $(L, \otimes, \odot, 0)$ is LBA and we have:

Since w is an arbitrary element of L , choose, $m = w$ so that:

$$(m \otimes n) \odot (n \otimes m) = w \odot w = w, \text{ and then}$$

$$\begin{aligned}
(\eta_1 \cup \eta_2)((m \otimes n) \odot (n \otimes m)) &= (\eta_1 \cup \eta_2)(w) = 0.5 \\
&\geq \min\{(\eta_1 \cup \eta_2)(m), (\eta_1 \cup \eta_2)(n)\} \\
&= \min\{0.7, 0.6\} = 0.6
\end{aligned}$$

which is false, confirming that union of two fuzzy ideals of L is not necessarily fuzzy ideal of L .

Proposition 5.2.3.2. *Let η be a fuzzy subset in LBA L such that $\eta(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M. \end{cases}$*

where $\delta, \sigma \in [0, 1]$, and $\delta > \sigma$. Then M is ideal of L if and only if η is a fuzzy ideal of L .

Proof. The proof this proposition is Similar to the proof of Proposition 5.2.3.1 above with the exception that 1 is replaced by δ and 0 is replaced by σ . □

Remark 5.2.3.2. *In a Liu^B-algebra (LBA), the classes of fuzzy subalgebras, fuzzy DS, and fuzzy ideals are mutually non-equivalent; in other words, a fuzzy subset satisfying the conditions of one of these structures does not necessarily satisfy the axioms of the others, and hence none of the three notions implies (or is implied by) the others.*

Example 5.2.3.3. *To justify some of the relations of any two, let us take some examples as follows and the rest could be elaborated in similar ways. Let $\eta : L \rightarrow [0, 1]$ be a fuzzy subset in LBA L where the binary operations are defined as in Table 5.7 of Example 5.2.3.2.*

(1) *Let η be fuzzy ideal of L such that $\eta(0) = 0.9$, $\eta(m) = 0.8 = \eta(n)$, $\eta(m \otimes n) = \eta(w) = 0.3$, but one of the axioms for fuzzy subalgebra of L fails as:*

$\eta(m \otimes n) = \eta(w) = 0.3 \geq \min\{\eta(m), \eta(n)\} = \min\{0.8, 0.8\} = 0.8$ *which is false showing fuzzy ideal of L may not be fuzzy subalgebra of L .*

(2) *On the other hand, let η be fuzzy subalgebra of L where $\eta(0) = 0.9 = \eta(w)$, $\eta(m) = 0.6$, $\eta(n) = 0.8$, and $\eta(m \otimes n) = \eta(w) = 0.9$ implies:*

$\eta(m) = 0.6 \geq \min\{\eta(m \otimes n), \eta(n)\} = \min\{\eta(w), \eta(n)\} = \min\{0.9, 0.8\} = 0.8$, *showing one axiom for fuzzy ideal of L fails meaning every fuzzy deductive system of L may not be fuzzy ideal of L . The other justifications could be explained in similar ways as the above.*

5.3. Pythagorean Fuzzy Substructures of Liu^B-Algebra

Under this section, we investigate Pythagorean fuzzy substructures in Liu^B-algebra (LBA), introducing Pythagorean fuzzy subalgebra, Pythagorean fuzzy deductive system and Pythagorean fuzzy ideal of Liu^B-algebra. We examine properties like membership deviations, accuracy functions, and their interrelations through theorems and examples, establishing foundational results for Pythagorean fuzzy Liu^B-algebraic frameworks.

For the whole of this subsection and subsequent subsections, we denote L^B for the Pythagorean fuzzy substructure $L^B = (\eta, \tau)$ of Liu^B-algebra $(L; \otimes, \odot, 0)$, unless otherwise specified.

Before proceeding to this subsection, we briefly recall some concepts in the form of a remark. We then develop the basic subsection to establish the foundational notions and properties of Pythagorean fuzzy LBA and their substructures. In particular, we examine the fundamental behavior of membership and non-membership functions, introduce the concept of square deviation for fuzzy subsets, and characterize Pythagorean fuzzy substructures through key theorems.

Remark 5.3.1.

For the fuzzy subset $\eta : L \rightarrow [0, 1]$ in the Liu^B-algebra $(L; \otimes, \odot, 0)$, $\forall m \in L$:

(1) $\eta(m) \geq (\eta(m))^2$ and $\tau(m) \geq (\tau(m))^2$, since $\eta(m) \in [0, 1]$ and $\tau(m) \in [0, 1]$

$$\Rightarrow \eta(m) - (\eta(m))^2 \geq 0 \text{ and } \tau(m) - (\tau(m))^2 \geq 0$$

$$\Rightarrow \eta(m)(1 - \eta(m)) \geq 0 \text{ and } \tau(m)(1 - \tau(m)) \geq 0$$

$$\Rightarrow \eta(m)(\bar{\eta}(m)) \geq 0 \text{ and } \tau(m)(\bar{\tau}(m)) \geq 0.$$

(2) As proved in Proposition 4.1.1 for the BCL-algebra $(B; \otimes, 0)$, the expressions stated below also hold for the Liu^B-algebra $(L; \otimes, \odot, 0)$ for the fuzzy subset $\eta : L \rightarrow [0, 1]$, the complement $\bar{\eta} : L \rightarrow [0, 1]$ of η and square deviation $\bar{\bar{\eta}} : L \rightarrow [-1, 1]$ of η as defined by Definition 4.1.1, and also we have the following:

(i) $(\bar{\bar{\eta}}(m))^2 \geq \bar{\eta}(m)$, $\forall m \in B$,

(ii) $(\bar{\bar{\eta}}(m))^2 \geq (\bar{\eta}(m))^2$, $\forall m \in B$.

5.3.1 Pythagorean Fuzzy Subalgebra of Liu^B-Algebra

Under this subsection, we introduce definition and give example of Pythagorean fuzzy subalgebra (SA) of Liu^B-algebra (LBA), state and prove some properties and theorems of Pythagorean fuzzy SA of LBA

Definition 5.3.1.1. A Pythagorean fuzzy set $L^B = (\eta, \tau)$ in LBA $(L; \otimes, \odot, 0)$ in which the functions $\eta : L \rightarrow [0, 1]$ and $\tau : L \rightarrow [0, 1]$ define the membership and the non-membership degrees respectively in L is called a Pythagorean fuzzy SA of L if the following two pairs of statements are satisfied; $\forall m, n \in L$:

(i) $(\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and $(\tau(m \otimes n))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$

(ii) $(\eta(m \odot n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and $(\tau(m \odot n))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$.

Example 5.3.1.1. Suppose $L = \{0, m, n, w\}$ and two binary operations \otimes and \odot on L are given by the tables 5.2 of Example 5.1.4 above. Let the Pythagorean fuzzy set $L^B = (\eta, \tau)$ such that $\eta : L \rightarrow [0, 1]$ and $\tau : L \rightarrow [0, 1]$ are given by

$$\eta(k) = \begin{cases} 0.9, & \text{if } k = 0, \\ 0.4, & \text{if } k = m, n, w, \end{cases} \quad \text{and} \quad \tau(k) = \begin{cases} 0.3, & \text{if } k = 0, \\ 0.7, & \text{if } k = m, n, w. \end{cases}$$

In Example 5.3.1.1, we explain how the membership function η and non-membership function τ represent the Pythagorean fuzzy SA of $(L; \otimes, \odot, 0)$, where $L = \{0, m, n, w\}$, as follows:

$$\begin{aligned} (\eta(m \otimes n))^2 &= (\eta(w))^2 = 0.16 \geq \min\{(\eta(m))^2, (\eta(n))^2\} = \min\{0.16, 0.16\} = 0.16, \text{ and} \\ (\tau(m \otimes n))^2 &= (\tau(w))^2 = 0.49 \leq \min\{(\tau(m))^2, (\tau(n))^2\} = \max\{0.49, 0.49\} = 0.49, \\ (\eta(m \odot n))^2 &= (\eta(n))^2 = 0.16 \geq \min\{(\eta(m))^2, (\eta(n))^2\} = \min\{0.16, 0.16\} = 0.16 \text{ and} \\ (\tau(m \odot n))^2 &= (\tau(n))^2 = 0.49 \leq \min\{(\tau(m))^2, (\tau(n))^2\} = \max\{0.49, 0.49\} = 0.49, \text{ and so on.} \end{aligned}$$

Lemma 5.3.1.1. If $L^B = (\eta, \tau)$ is a Pythagorean fuzzy SA of L , then $\forall m \in L$:

$$(\eta(0))^2 \geq (\eta(m))^2 \text{ and } (\tau(0))^2 \leq (\tau(m))^2.$$

Proof. Since $L \neq \emptyset$, let $m \in L$, we have: $m \otimes m = 0$, by Definition 1.1.8 (1). Then

$$\begin{aligned} (\eta(0))^2 &= (\eta(m \otimes m))^2 \geq \min\{(\eta(m))^2, (\eta(m))^2\} = \eta(m), \forall m \in L, \text{ and} \\ (\tau(0))^2 &= (\tau(m \otimes m))^2 \leq \max\{(\tau(m))^2, (\tau(m))^2\} = (\tau(m))^2, \forall m \in L \\ \Rightarrow (\eta(0))^2 &\geq (\eta(m))^2, \text{ and } (\tau(0))^2 \leq (\tau(m))^2, \forall m \in L. \end{aligned}$$

□

Proposition 5.3.1.1. Let M be a non-empty subset of a Liu^B-algebra L and $L^P = (\chi_M, \bar{\chi}_M)$, where χ_M is the Characteristic function and $\bar{\chi}_M$ is the square deviation of χ_M . Then $(\chi_M, \bar{\chi}_M)$ is a Pythagorean fuzzy SA of L if and only if M is SA of L .

Proof. Suppose $\chi_M : M \rightarrow \{0, 1\}$ is a Characteristic function defined as:

$$\chi_M(m) = \begin{cases} 1, & \text{if } m \in M, \\ 0, & \text{if } m \notin M. \end{cases} \text{ and hence } \bar{\chi}_M(m) = (\bar{\chi}_M(m))^2 = \begin{cases} 0, & \text{if } m \in M, \\ 1, & \text{if } m \notin M. \end{cases}$$

Now, suppose $L^P = (\chi_M, \bar{\chi}_M)$ is Pythagorean fuzzy SA of L and $m, n \in L$ which implies the Characteristic function χ_M is a fuzzy SA of L , by Proposition 4.2.1.1.

We need to verify that M is a SA of L .

Here, we need to justify this claim by using the non-membership part as follows since using the membership part can be justified similarly :

Now, we show M is SA of $L \Rightarrow m \otimes n \in M$ and $m \odot n \in M, \forall m, n \in L$

Let $m, n \in L \Rightarrow (\bar{\chi}_M(m))^2 = (\bar{\chi}_M(n))^2 = 0$ and

$$\chi_M(m \otimes n) \leq \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\} = \max\{0, 0\} = 0$$

$$\Rightarrow (\bar{\chi}_M(m \otimes n))^2 \leq 0$$

$$\text{But } 0 \leq (\bar{\chi}_M(m \otimes n))^2 \Rightarrow (\bar{\chi}_M(m \otimes n))^2 = 0 \Rightarrow m \otimes n \in S.$$

$$\text{Similarly, } (\bar{\chi}_M(m \odot n))^2 = 0 \Rightarrow m \otimes n \in M$$

Hence, $m, n \in M \Rightarrow m \otimes n \in M$; $m \odot n \in M$ and therefore M is a SA of L .

Conversely, suppose M is a SA of L

Thus at first, for the non-membership function $\bar{\chi}_M$, let us show that

$$(\chi_M(m \otimes n))^2 \geq \max\{(\chi_M(m))^2, (\chi_M(n))^2\}, \forall m, n \in L.$$

Then after, following similar steps for the membership function χ_M , one can show that

$$(\bar{\chi}_M(m \otimes n))^2 \leq \min\{(\chi_M(m))^2, (\chi_M(n))^2\}, \forall m, n \in L \text{ so that, understanding this,}$$

we omit the proof for this part.

Case(1) : If $m, n \in M \Rightarrow m \otimes n \in M \Rightarrow (\bar{\chi}_M(m))^2 = 0 = (\bar{\chi}_M(n))^2 = (\bar{\chi}_M(m \otimes n))^2$, then

$$m \otimes n \in M \Rightarrow (\bar{\chi}_M(m \otimes n))^2 = 0 \leq \max\{0, 0\} = \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\}$$

$$\Rightarrow (\bar{\chi}_M(m \otimes n))^2 \leq \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\},$$

Case(2) : If $m \notin S, n \notin S$, (but $m, n \in L$)

$$\Rightarrow (\bar{\chi}_M(m))^2 = 1 = (\bar{\chi}_M(n))^2, \text{ and } (\bar{\chi}_M(m \otimes n))^2 \leq 1$$

(since any fuzzy subset is less than or equal to 1)

$$\Rightarrow (\bar{\chi}_M(m \otimes n))^2 \leq 1 = \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\} = \max\{1, 1\}$$

$$\Rightarrow (\bar{\chi}_M(m \otimes n))^2 \leq \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\}$$

Case(3) : If $m \in M$ and $n \notin M$ (or $n \in M$ and $m \notin S$)

$$\Rightarrow (\bar{\chi}_M(m))^2 = 0 \text{ and } (\bar{\chi}_M(n))^2 = 1 \quad (\text{or } (\bar{\chi}_M(n))^2 = 0 \text{ and } (\bar{\chi}_M(m))^2 = 1)$$

$$\Rightarrow \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\} = \max\{1, 0\} = 1 \quad \text{and } (\bar{\chi}_M(m \otimes n))^2 \leq 1$$

$$\Rightarrow (\bar{\chi}_M(m \otimes n))^2 \leq 1 = \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\}$$

$$\Rightarrow (\bar{\chi}_M(m \otimes n))^2 \leq \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\}$$

Hence, in all the cases (1) – (3), we have checked that:

$$(\overline{\chi}_M(m \otimes n))^2 \leq \max\{(\overline{\chi}_M(m))^2, (\overline{\chi}_M(n))^2\}, \forall m, n \in L \text{ and similarly for } \chi_M:$$

$$(\overline{\chi}_M(m \odot n))^2 \leq \max\{(\overline{\chi}_M(m))^2, (\overline{\chi}_M(n))^2\}, \forall m, n \in L.$$

Thus $L^P = (\chi_M, \overline{\chi}_M)$ is Pythagorean fuzzy SA of L if and only if M is SA of L . \square

Theorem 5.3.1.1. *The intersection of any two Pythagorean fuzzy SAs, $L_T^B = (\eta_1 \cap \eta_2, \tau_1 \cap \tau_2)$, of LBA L is also a Pythagorean fuzzy SA of L .*

Proof. Let L_1^B and L_2^B be any two Pythagorean fuzzy SA of L . Then we need to prove:

$L_T^B = (\eta_1 \cap \eta_2, \tau_1 \cap \tau_2)$ is a Pythagorean fuzzy SA of L . For $m, n \in L$, we have:

$$\begin{aligned} \text{(i)} \quad ((\eta_1 \cap \eta_2)(m \otimes n))^2 &= \min\{(\eta_1(m \otimes n))^2, (\eta_2(m \otimes n))^2\} \\ &\geq \min\{\min\{(\eta_1(m))^2, (\eta_1(n))^2\}, \min\{(\eta_2(m))^2, (\eta_2(n))^2\}\} \\ &= \min\{\min\{(\eta_1(m))^2, (\eta_2(m))^2\}, \min\{(\eta_1(n))^2, (\eta_2(n))^2\}\} \\ &= \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\} \\ &= \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\}, \forall m, n \in L. \end{aligned}$$

Hence, $((\eta_1 \cap \eta_2)(m \otimes n))^2 \geq \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\}, \forall m, n \in L$.

Similarly, $((\tau_1 \cap \tau_2)(m \otimes n))^2 \leq \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2\}, \forall m, n \in L$.

$$\begin{aligned} \text{(ii)} \quad ((\eta_1 \cap \eta_2)(m \odot n))^2 &= \min\{(\eta_1(m \odot n))^2, (\eta_2(m \odot n))^2\} \\ &\geq \min\{\min\{(\eta_1(m))^2, (\eta_1(n))^2\}, \min\{(\eta_2(m))^2, (\eta_2(n))^2\}\} \\ &= \min\{\min\{(\eta_1(m))^2, (\eta_2(m))^2\}, \min\{(\eta_1(n))^2, (\eta_2(n))^2\}\} \\ &= \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\} \\ &= \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\}, \forall m, n \in L. \end{aligned}$$

Hence, $((\eta_1 \cap \eta_2)(m \odot n))^2 \geq \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\}, \forall m, n \in L$.

Similarly, $((\tau_1 \cap \tau_2)(m \odot n))^2 \leq \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2\}, \forall m, n \in L$.

Thus, intersection of any two Pythagorean fuzzy SAs of LBA L is Pythagorean fuzzy SA of L . \square

Corollary 5.3.1.1. *The intersection, $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, of any family of Pythagorean fuzzy SAs, $\{(\eta_i, \tau_i) : i \in I\}$, of LBA L is also a Pythagorean fuzzy SA of L .*

Proof. Let $\{(\eta_i, \tau_i) : i \in I\}$, be family of Pythagorean fuzzy SAs of L. Then we need to prove:

$(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, is a Pythagorean fuzzy SA of L. Then for $m, n \in L$, we have:

$$\begin{aligned}
 \text{(i) } \bigcap_{i \in I} (\eta_i(m \otimes n))^2 &= \inf_{i \in I} (\eta_i(m \otimes n))^2 \\
 &\geq \inf_{i \in I} \{\min\{(\eta_i(m))^2, (\eta_i(n))^2\}\} \\
 &\geq \min\left\{\inf_{i \in I} \{(\eta_i(m))^2, (\eta_i(n))^2\}\right\} \\
 &= \min\left\{\inf_{i \in I} \{(\eta_i(m))^2\}, \inf_{i \in I} \{(\eta_i(n))^2\}\right\} \\
 &= \min\left\{\bigcap_{i \in I} (\eta_i(m))^2, \bigcap_{i \in I} (\eta_i(n))^2\right\} \quad \text{and} \\
 \bigcap_{i \in I} (\tau_i(m \otimes n))^2 &= \sup_{i \in I} (\tau_i(m \otimes n))^2 \\
 &\leq \sup_{i \in I} \{\max\{(\tau_i(m))^2, (\tau_i(n))^2\}\} \\
 &\leq \max\left\{\sup_{i \in I} \{(\tau_i(m))^2, (\tau_i(n))^2\}\right\} \\
 &= \max\left\{\sup_{i \in I} \{(\tau_i(m))^2\}, \sup_{i \in I} \{(\tau_i(n))^2\}\right\} \\
 &= \max\left\{\bigcap_{i \in I} (\tau_i(m))^2, \bigcap_{i \in I} (\tau_i(n))^2\right\}, \forall m, n \in L.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } \bigcap_{i \in I} (\eta_i(m \odot n))^2 &= \inf_{i \in I} (\eta_i(m \odot n))^2 \\
 &\geq \inf_{i \in I} \{\min\{(\eta_i(m))^2, (\eta_i(n))^2\}\} \\
 &\geq \min\left\{\inf_{i \in I} \{(\eta_i(m))^2, (\eta_i(n))^2\}\right\} \\
 &= \min\left\{\inf_{i \in I} \{(\eta_i(m))^2\}, \inf_{i \in I} \{(\eta_i(n))^2\}\right\} \\
 &= \min\left\{\bigcap_{i \in I} (\eta_i(m))^2, \bigcap_{i \in I} (\eta_i(n))^2\right\} \quad \text{and}
 \end{aligned}$$

$$\begin{aligned}
 \bigcap_{i \in I} (\tau_i(m \odot n))^2 &= \sup_{i \in I} (\tau_i(m \odot n))^2 \\
 &\leq \sup_{i \in I} \{\max\{(\tau_i(m))^2, (\tau_i(n))^2\}\} \\
 &\leq \max\left\{\sup_{i \in I} \{(\tau_i(m))^2, (\tau_i(n))^2\}\right\}
 \end{aligned}$$

$$\begin{aligned}
&= \max \left\{ \sup_{i \in I} \{(\tau_i(m))^2\}, \sup_{i \in I} \{(\tau_i(n))^2\} \right\} \\
&= \max \left\{ \bigcap_{i \in I} (\tau_i(m))^2, \bigcap_{i \in I} (\tau_i(n))^2 \right\}, \forall m, n \in L.
\end{aligned}$$

Thus, intersection of Pythagorean fuzzy SAs of L is Pythagorean fuzzy SA of L . \square

Remark 5.3.1.1. *The union of two Pythagorean fuzzy SA of a Liu^B-algebra L may not be Pythagorean fuzzy SA of L which is justified by Example 5.3.1.2 below.*

Example 5.3.1.2. *Assume $L = \{0, m, n\}$ and two binary operations \otimes and \odot on L as defined in Table 5.2 of Example 5.1.4 above. Then it has been shown that $(L, \otimes, \odot, 0)$ is a Liu^B-algebra. Define two Pythagorean fuzzy SA of LBA $L_1^B = (\eta_1, \tau_1)$ and $L_2^B = (\eta_2, \tau_2)$, as follows:*

$$\begin{aligned}
\eta_1(k) &= \begin{cases} 0.9, & \text{if } k = 0, \\ 0.6, & \text{if } k = m, \\ 0.4, & \text{if } k = n, w, \end{cases} & \text{and} & \tau_1(k) = \begin{cases} 0, & \text{if } k = 0, \\ 0.5, & \text{if } k = n, \\ 0.3, & \text{if } k = m, w, \end{cases} & \text{and} \\
\eta_2(k) &= \begin{cases} 0.8, & \text{if } k = 0, \\ 0.7, & \text{if } k = m, n, \\ 0.5, & \text{if } k = w, \end{cases} & \text{and} & \tau_2(k) = \begin{cases} 0.2, & \text{if } k = 0, \\ 0.4, & \text{if } k = m, \\ 0.6, & \text{if } k = n, w. \end{cases} & \text{then} \\
(\eta_1 \cup \eta_2)(k) &= \begin{cases} 0.9, & \text{if } k = 0, \\ 0.7, & \text{if } k = m, n, \\ 0.5, & \text{if } k = w, \end{cases} & \text{and} & (\tau_1 \cup \tau_2)(k) = \begin{cases} 0, & \text{if } k = 0, \\ 0.3, & \text{if } k = m, w, \\ 0.5, & \text{if } k = n. \end{cases}
\end{aligned}$$

From the table of \otimes in Table 5.2, as $m \otimes n = w$: we have:

$$\begin{aligned}
(\eta_1 \cup \eta_2)(m \otimes n)^2 &= (\eta_1 \cup \eta_2)(w)^2 = (0.5)^2 = 0.25 \\
&\geq \min\{((\eta_1 \cup \eta_2)(m))^2, ((\eta_1 \cup \eta_2)(n))^2\} \\
&= \min\{0.49, 0.49\} \\
&= 0.49, \text{ which is false.}
\end{aligned}$$

This tells us that the union of two Pythagorean fuzzy SA of LBA is not necessarily Pythagorean fuzzy SA of LBA

Proposition 5.3.1.2. *Let M be a non-empty subset of LBA L and η be a fuzzy subset in LBA L*

$$\text{such that } (\eta(m))^2 = \begin{cases} \delta^2, & \text{if } m \in M, \\ \sigma^2, & \text{if } m \notin M, \end{cases} \text{ and then } (\bar{\eta}(m))^2 = \begin{cases} 1 - \delta^2, & \text{if } m \in M, \\ 1 - \sigma^2, & \text{if } m \notin M, \end{cases}$$

where $\delta, \sigma \in [0, 1]$, $\delta > \sigma \Rightarrow \delta^2, \sigma^2 \in [0, 1]$, $\delta^2 > \sigma^2$

Then M is SA of L if and only if $(\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of L .

Proof. Suppose M is SA of L , we prove that $(\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of L .

Case(i) : Let $m, n \in M \Rightarrow m \otimes n \in M$

Then $(\eta(m))^2 = (\eta(n))^2 = \delta^2 \Rightarrow (\eta(m \otimes n))^2 = \delta^2$ and

$$(\bar{\eta}(m))^2 = (\bar{\eta}(n))^2 = 1 - \delta^2 \Rightarrow (\bar{\eta}(m \otimes n))^2 = 1 - \delta^2$$

$\Rightarrow (\eta(m \otimes n))^2 \geq \delta^2 = \min\{(\eta(m))^2, (\eta(n))^2\}$ and also,

$$(\bar{\eta}(m \otimes n))^2 \leq 1 - \delta^2 = \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Similarly, $m, n \in M \Rightarrow m \odot n \in M \Rightarrow (\eta(m))^2 = (\eta(n))^2 = \delta^2 \Rightarrow (\eta(m \odot n))^2 = \delta^2$ and

$$(\bar{\eta}(m))^2 = (\bar{\eta}(n))^2 = 1 - \delta^2 \Rightarrow (\bar{\eta}(m \odot n))^2 = 1 - \delta^2$$

$\Rightarrow (\eta(m \odot n))^2 \geq \delta^2 = \min\{(\eta(m))^2, (\eta(n))^2\}$, and similarly

$$\Rightarrow (\bar{\eta}(m \odot n))^2 \leq 1 - \delta^2 = \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Case(ii) : Let $n \in M, m \notin M$ (or $m \in M, n \notin M$) $\Rightarrow m \otimes n \notin M$ or $m \otimes n \in M$

$\Rightarrow (\eta(m))^2 = \delta^2, (\eta(m))^2 = \sigma^2 \Rightarrow (\eta(m \otimes n))^2 = \sigma^2$ or $(\eta(m \otimes n))^2 = \delta^2$ (for first case)

$\Rightarrow (\eta(m \otimes n))^2 \geq \sigma^2 = \min\{(\eta(m))^2, (\eta(n))^2\} = \min\{\sigma^2, \delta^2\} = \sigma^2$ and

$$(\bar{\eta}(m \otimes n))^2 \leq \sigma^2 = \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\} = \max\{\delta^2, \sigma^2\} = \sigma^2$$

Similarly, for $m \in M, n \notin M$, or $m, n \notin M$, we obtain similar deductions.

Thus, $(\eta, \bar{\eta})$ is Pythagorean SA of L .

Conversely, suppose $(\eta, \bar{\eta})$ is a Pythagorean fuzzy SA of L .

Let $n, m \in M \Rightarrow (\eta(m))^2 = (\eta(n))^2 = \delta^2$ and

$(\eta(m \otimes n))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} = \min\{\delta^2, \delta^2\} = \delta^2 \Rightarrow (\eta(m \otimes n))^2 \geq \delta^2$ and since

$\delta^2, \sigma^2 \in [0, 1]$, $\delta^2 > \sigma^2$, we have $(\eta(m \otimes n))^2 \leq \delta^2 \Rightarrow (\eta(m \otimes n))^2 = \delta^2 \Rightarrow m \otimes n \in M$.

Following similar steps as above for $n, m \in M$, we can show that, $m \odot n \in M$.

Therefore, M is SA of L . □

5.3.2 Pythagorean Fuzzy Deductive System of Liu^B–Algebra

Under this subsection, we introduce definitions and give examples of deductive system (DS), fuzzy DS and Pythagorean fuzzy DS of LBA, state and prove some properties and theorems on Pythagorean fuzzy DS of LBA, which are our wide range focuses.

Definition 5.3.2.1. A Pythagorean fuzzy set $L^B = (\eta, \tau)$ in LBA $(L; \otimes, \odot, 0)$ in which the functions $\eta : L \rightarrow [0, 1]$ and $\tau : L \rightarrow [0, 1]$ define the membership and the non-membership degrees, respectively, in L is known as a Pythagorean fuzzy DS of L if the two pairs of statements hereunder are satisfied; $\forall m, n \in L$:

- (i) $(\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and $(\tau(m \otimes (n \otimes u)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$
- (ii) $(\eta(m \odot w \odot w))^2 \geq (\eta(m))^2$ and $(\tau(m \odot w \odot w))^2 \leq (\tau(m))^2$

Example 5.3.2.1. Let $(L; \otimes, \odot, 0)$ be LBA, where $L = \{0, m, n, w\}$ and two binary operations \otimes and \odot on L be as given by Table 5.4 of Example 5.1.10, above.

Let the fuzzy subset $\eta : L \rightarrow [0, 1]$ and $\tau : L \rightarrow [0, 1]$ be defined by:

$$\eta(k) = \begin{cases} 0.8, & \text{if } k = 0, \\ 0.3, & \text{if } k = m, n, \end{cases} \quad \text{and} \quad \tau(k) = \begin{cases} 0.2, & \text{if } k = 0, \\ 0.8, & \text{if } k = m, n. \end{cases}$$

In Example 5.3.2.1, we explain how the membership function η and non-membership function τ represent the Pythagorean fuzzy DS of $(L; \otimes, \odot, 0)$, where $L = \{0, m, n\}$, as follows:

$$(\eta(m \otimes (n \otimes 0)))^2 = (\eta(0))^2 = 0.64 \geq \min\{(\eta(m))^2, (\eta(n))^2\} = \min\{0.09, 0.09\} = 0.09, \quad \text{and}$$

$$(\tau(m \otimes (n \otimes 0)))^2 = (\tau(0))^2 = 0.04 \leq \max\{(\tau(m))^2, (\tau(n))^2\} = \max\{0.64, 0.64\} = 0.64,$$

$$(\eta(m \odot n \odot 0))^2 = (\eta(0))^2 = 0.64 \geq (\eta(m))^2 = 0.09 \quad \text{and}$$

$$(\tau(m \odot n \odot 0))^2 = (\tau(0))^2 = 0.04 \leq (\tau(m))^2 = 0.64, \quad \text{and so on.}$$

Theorem 5.3.2.1. Let $\eta : L \rightarrow [0, 1]$ be a fuzzy subset in LBA L and $\bar{\eta}$ be its square deviation. Then η is fuzzy DS of L if and only if $(\eta, \bar{\eta})$ is Pythagorean fuzzy DS of L .

Proof. Let η be a fuzzy subset in LBA L and $\bar{\eta}$ be its square deviation such that η is fuzzy DS of L :

- (i) $\eta(m \otimes (n \otimes u)) \geq \min\{\eta(m), \eta(n)\}$
- (ii) $\eta(m \odot w \odot w) \geq \eta(m)$

Then we claim that $(\eta, \bar{\eta})$ is Pythagorean fuzzy DS of L:

- (i) $\eta(m \otimes (n \otimes u)) \geq \min\{\eta(m), \eta(n)\} \Rightarrow (\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and
 $\bar{\eta}(m \otimes (n \otimes u)) \leq \max\{\bar{\eta}(m), \bar{\eta}(n)\} \Rightarrow (\bar{\eta}(m \otimes (n \otimes u)))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$
- (ii) $\eta(m \odot w \odot w) \geq \eta(m) \Rightarrow (\eta(m \odot w \odot w))^2 \geq (\eta(m))^2$ and
 $(\bar{\eta}(m \odot w \odot w))^2 \leq (\bar{\eta}(m))^2$

Therefore, $(\eta, \bar{\eta})$ is Pythagorean fuzzy DS of L.

Conversely, suppose $(\eta, \bar{\eta})$ is Pythagorean fuzzy DS of L, then we claim that η is fuzzy DS of L and hence:

- (i) $(\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \Rightarrow \eta(m \otimes (n \otimes u)) \geq \min\{\eta(m), \eta(n)\}$
- (ii) $(\eta(m \odot w \odot w))^2 \geq (\eta(m))^2 \Rightarrow \eta(m \odot w \odot w) \geq \eta(m)$.

Hence, η is fuzzy DS of L. □

Proposition 5.3.2.1. Let M be a non-empty subset of L and $L^B = (\chi_M, \bar{\chi}_M)$, where χ_M is the Characteristic function and $\bar{\chi}_M$ is its complement, then L^B is a Pythagorean fuzzy DS of L if and only if M is DS of L .

Proof. Suppose $\chi_M : M \rightarrow \{0, 1\}$ is a Characteristic function defined as:

$$\chi_M(k) = \begin{cases} 1, & \text{if } k \in M, \\ 0, & \text{if } k \notin M, \end{cases} \quad \text{and then } \bar{\chi}_M(k) = \begin{cases} 0, & \text{if } k \in M, \\ 1, & \text{if } k \notin M. \end{cases}$$

$$\Rightarrow (\bar{\chi}_M(k))^2 = \begin{cases} 0, & \text{if } k \in M, \\ 1, & \text{if } k \notin M. \end{cases} = (\bar{\bar{\chi}}_M(k))^2 = \begin{cases} 0, & \text{if } k \in M, \\ 1, & \text{if } k \notin M. \end{cases}$$

and as proved in Theorem 5.3.2.1, the Characteristic function χ_M is a fuzzy DS of L and this implies $(\chi_M, \bar{\chi}_M)$ is Pythagorean fuzzy DS of L , where $\bar{\bar{\chi}}_M$ is the square deviation of $\bar{\chi}_M$ and then we only need to show the inequalities for the non-membership part, the complement, as for the membership part is justified by the proof of Theorem 5.3.2.1 and using the fact that $(\bar{\chi}(m))^2 = (\bar{\bar{\chi}}(m))^2, \forall m \in L$, we proceed as follows::

- (i) $(\bar{\chi}(m \otimes (n \otimes u)))^2 = (\bar{\bar{\chi}}(m \otimes (n \otimes u)))^2 \leq \max\{(\bar{\chi}(m))^2, (\bar{\chi}(n))^2\}$
 $= \max\{(\bar{\bar{\chi}}(m))^2, (\bar{\bar{\chi}}(n))^2\}$
 $\Rightarrow (\bar{\chi}(m \otimes w \otimes w))^2 \leq \max\{(\bar{\chi}(m))^2, (\bar{\chi}(n))^2\}$ and
- (ii) $(\bar{\chi}(m \odot w \odot w))^2 = (\bar{\bar{\chi}}(m \odot w \odot w))^2 \leq (\bar{\bar{\chi}}(m))^2 = (\bar{\chi}(m))^2$
 $\Rightarrow (\bar{\chi}(m \odot (n \odot u)))^2 \leq \max\{(\bar{\chi}(m))^2, (\bar{\chi}(n))^2\}$

Thus, $(\chi_M, \bar{\chi}_M)$ is a Pythagorean fuzzy DS of L . □

Theorem 5.3.2.2. *The intersection of any two Pythagorean fuzzy DS of L is also a Pythagorean fuzzy DS of L.*

Proof. Let L_1^B and L_2^B be any two Pythagorean fuzzy DS of L. Then we need to prove:

$L_{\cap}^B = (\eta_1 \cap \eta_2, \tau_1 \cap \tau_2)$ is a Pythagorean fuzzy DS of L.

$$\begin{aligned} \text{(i)} \quad ((\eta_1 \cap \eta_2)(m \otimes (n \otimes w)))^2 &= \min\{(\eta_1(m \otimes (n \otimes w)))^2, (\eta_2(m \otimes (n \otimes w)))^2\} \\ &\geq \min\{\min\{(\eta_1(m))^2, (\eta_1(n))^2\}, \min\{(\eta_2(m))^2, (\eta_2(n))^2\}\} \\ &= \min\{\min\{(\eta_1(m))^2, (\eta_2(m))^2\}, \min\{(\eta_1(n))^2, (\eta_2(n))^2\}\} \\ &= \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\}, \forall m, n \in L, \text{ and} \end{aligned}$$

$$\begin{aligned} ((\tau_1 \cap \tau_2)(m \otimes (n \otimes w)))^2 &= \max\{(\tau_1(m \otimes (n \otimes w)))^2, (\tau_2(m \otimes (n \otimes w)))^2\} \\ &\leq \max\{\max\{(\tau_1(m))^2, (\tau_1(n))^2\}, \max\{(\tau_2(m))^2, (\tau_2(n))^2\}\} \\ &= \max\{\max\{(\tau_1(m))^2, (\tau_2(m))^2\}, \max\{(\tau_1(n))^2, (\tau_2(n))^2\}\} \\ &= \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2\}, \forall m, n \in L. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad ((\eta_1 \cap \eta_2)(m \odot w \odot w))^2 &= \min\{(\eta_1(m \odot w \odot w))^2, (\eta_2(m \odot w \odot w))^2\} \\ &\geq \min\{(\eta_1(m))^2, (\eta_2(m))^2\} \\ &= ((\eta_1 \cap \eta_2)(m))^2, \forall m, n \in L, \text{ and} \end{aligned}$$

$$\begin{aligned} ((\tau_1 \cap \tau_2)(m \odot w \odot w))^2 &= \max\{(\tau_1(m \odot w \odot w))^2, (\tau_2(m \odot w \odot w))^2\} \\ &\leq \max\{(\tau_1(m))^2, (\tau_2(m))^2\} \\ &= ((\tau_1 \cap \tau_2)(m))^2, \forall m, n \in L. \end{aligned}$$

Thus, intersection of any two Pythagorean fuzzy DSs of L is Pythagorean fuzzy DS of L. \square

The preceding theorem can also be generalized to any family of DSs of L as in the corollary below.

Corollary 5.3.2.1. *The intersection, $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, of any family of Pythagorean fuzzy DSs, $\{(\eta_i, \tau_i) : i \in I\}$, of LBA L is also a Pythagorean fuzzy DS of L.*

Proof. Let L_1^B and L_2^B be any two Pythagorean fuzzy DS of L. Then we need to prove:

$(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$ is a Pythagorean fuzzy DS of L, we follow the following steps:

$$\begin{aligned}
\text{(i)} \quad \left(\bigcap_{i \in I} (\eta_i(m \otimes (n \otimes w))) \right)^2 &= \inf_{i \in I} \{ (\eta_i(m \otimes (n \otimes w)))^2 \} \\
&\geq \inf_{i \in I} \{ \min \{ (\eta_1(m))^2, (\eta_2(n))^2 \} \} \\
&\geq \min \{ \inf_{i \in I} \{ (\eta_1(m))^2 \}, (\eta_2(n))^2 \} \\
&\geq \min \{ \inf_{i \in I} \{ (\eta_1(m))^2 \}, \inf_{i \in I} \{ (\eta_2(n))^2 \} \} \\
&\geq \min \left\{ \bigcap_{i \in I} (\eta_1(m))^2, \bigcap_{i \in I} (\eta_2(n))^2 \right\}, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\left(\bigcap_{i \in I} (\tau_i(m \otimes (n \otimes w))) \right)^2 &= \sup_{i \in I} \{ (\tau_i(m \otimes (n \otimes w)))^2 \} \\
&\leq \sup_{i \in I} \{ \max \{ (\tau_1(m))^2, (\tau_2(n))^2 \} \} \\
&\leq \max \{ \sup_{i \in I} \{ (\tau_1(m))^2 \}, (\tau_2(n))^2 \} \\
&= \max \{ \sup_{i \in I} \{ (\tau_1(m))^2 \}, \sup_{i \in I} \{ (\tau_2(n))^2 \} \} \\
&= \max \left\{ \bigcap_{i \in I} (\tau_1(m))^2, \bigcap_{i \in I} (\tau_2(n))^2 \right\}, \forall m, n, w \in L.
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad \bigcap_{i \in I} (\eta_i(m \odot w \odot w))^2 &= \inf_{i \in I} \{ (\eta_i(m \odot w \odot w))^2 \} \\
&\geq \inf_{i \in I} \{ (\eta_i(m))^2 \} \\
&= \bigcap_{i \in I} (\eta_i(m))^2, \quad \forall m, w \in L \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\bigcap_{i \in I} (\tau_i(m \odot w \odot w))^2 &= \sup_{i \in I} \{ (\tau_i(m \odot w \odot w))^2 \} \\
&\leq \sup_{i \in I} \{ (\tau_i(m))^2 \} \\
&= \bigcap_{i \in I} (\tau_i(m))^2, \quad \forall m, w \in L.
\end{aligned}$$

Thus, intersection of any family of Pythagorean fuzzy DSs of L is Pythagorean fuzzy DS of L. \square

Remark 5.3.2.1. Union of two Pythagorean fuzzy DS of LBA L may not be necessarily Pythagorean fuzzy DS of L illustrated by the following counter example (Example 5.3.2.2 below).

Example 5.3.2.2. Let $L = \{0, m, n, w\}$ be a set and the two binary operations \otimes and \odot on L be as defined in Table 5.2 of Example 5.1.4 above and define two fuzzy DSs η_{D_1} and η_{D_2} of L as follows:

k	$\eta_1(k)$	$\eta_2(k)$	$(\eta_1 \cup \eta_2)(k)$	$\tau_1(k)$	$\tau_2(k)$	$(\tau_1 \cup \tau_2)(k)$
0	0.8	1	1	0.2	0.3	0.2
m	0.4	0.6	0.6	0.6	0.6	0.6
n	0.4	0.7	0.7	0.6	0.4	0.4
w	0.8	0.7	0.8	0.2	0.4	0.2

Table 5.8: A table that defines fuzzy SAs and their unions

Then, from Table 5.2 of Example 5.1.4 above, we have that: $w \otimes (n \otimes 0) = w \otimes n = m$.

Also, from Table 5.8 of Example 5.3.2.2 above, we have:

$$\begin{aligned} \Rightarrow ((\eta_1 \cup \eta_2)(w \otimes (n \otimes 0)))^2 &= ((\eta_1 \cup \eta_2)(m))^2 = 0.36 \\ \Rightarrow ((\eta_1 \cup \eta_2)(w \otimes (n \otimes 0)))^2 &= 0.36 \geq \min\{((\eta_1 \cup \eta_2)(w))^2, ((\eta_1 \cup \eta_2)(n))^2\} \\ &= \min\{0.64, 0.49\} = 0.49, \text{ which is false} \end{aligned}$$

confirming that the union of two fuzzy DSs of LBA L is not necessarily fuzzy DS of L .

Theorem 5.3.2.3. Let η be a fuzzy subset in LBA L and $M \subseteq L$ such that for $\delta, \sigma \in [0, 1]$, and $\delta > \sigma$

$$\eta(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M, \end{cases} \quad \text{then the square deviation: } (\bar{\eta}(m))^2 = \begin{cases} 1 - \delta^2, & \text{if } m \in M, \\ 1 - \sigma^2, & \text{if } m \notin M. \end{cases}$$

Then M is DS of L if $L^B = (\eta, \bar{\eta})$ is a Pythagorean fuzzy DS of L .

Proof. (1) It is simple mathematical computation that $0 \leq (\eta(m))^2 + (\bar{\eta}(m))^2 \leq 1, \forall m \in M$, since

$$(\eta(m))^2 + (\bar{\eta}(m))^2 = (\eta(m))^2 + (1 - (\eta(m))^2) = 1$$

(2) To check that the axioms of Pythagorean fuzzy DS are satisfied, we proceed as follows:

$$\eta(m) \in [\sigma, \delta] \subseteq [0, 1] \text{ and } (\bar{\eta}(m))^2 \in [1 - \delta^2, 1 - \sigma^2] \subseteq [0, 1]$$

Suppose M is DS of L , then we prove that $L^B = (\eta, \bar{\eta})$ is a Pythagorean fuzzy DS of L with the following two pairs of Claim:

$$(i) \quad (\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\bar{\eta}(m \otimes (n \otimes u)))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

$$(ii) \quad (\eta(m \odot w \odot w))^2 \geq (\eta(m))^2 \text{ and } (\bar{\eta}(m \odot w \odot w))^2 \leq (\bar{\eta}(m))^2$$

Then we prove this claims by following three cases as follows:

Case(i) : Let $m, n \in M \Rightarrow m \otimes (n \otimes u) \in M$

$$\text{Then } (\eta(m))^2 = (\eta(n))^2 = \delta^2 \Rightarrow (\eta(m \otimes (n \otimes u)))^2 = \delta^2 \text{ and}$$

$$(\bar{\eta}(m))^2 = (\bar{\eta}(n))^2 = 1 - \delta^2 \Rightarrow (\bar{\eta}(m \otimes (n \otimes u)))^2 = 1 - \delta^2$$

$$\Rightarrow (\eta(m \otimes (n \otimes u)))^2 \geq \delta^2 = \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\bar{\eta}(m \otimes (n \otimes u)))^2 \leq 1 - \delta^2 = \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

$$\Rightarrow (\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\bar{\eta}(m \otimes (n \otimes u)))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Case(ii) : Let $m \in M, n \notin M$

(or the case for $m \notin M, n \in M$, is similar and hence left)

$$\Rightarrow (\eta(m))^2 = \delta^2 \text{ and } (\eta(n))^2 = \sigma^2$$

but $(\eta(m \otimes (n \otimes u)))^2 \geq \sigma^2$ and $\bar{\eta}(m \otimes (n \otimes u))^2 \leq 1 - \delta^2$, by the hypothesis

$$\Rightarrow (\eta(m \otimes (n \otimes u)))^2 \geq \sigma^2 = \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$\bar{\eta}(m \otimes (n \otimes u))^2 \leq 1 - \delta^2 = \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

$$\Rightarrow (\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$\bar{\eta}(m \otimes (n \otimes u))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

Case(iii) : Let $m \notin M, n \notin M \Rightarrow (\eta(m))^2 = \sigma^2 = (\eta(n))^2$

but $(\eta(m \otimes (n \otimes u)))^2 \geq \sigma^2$ and $\bar{\eta}(m \otimes (n \otimes u))^2 \leq 1 - \delta^2$, by the hypothesis

$$\Rightarrow (\eta(m \otimes (n \otimes u)))^2 \geq \sigma^2 = \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$\bar{\eta}(m \otimes (n \otimes u))^2 \leq 1 - \delta^2 = \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

$$\Rightarrow (\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$\bar{\eta}(m \otimes (n \otimes u))^2 \leq \max\{(\bar{\eta}(m))^2, \bar{\eta}(n))^2\}$$

Therefore, by the cases (i) – (iii) above, we have shown that $(\eta, \bar{\eta})$ is Pythagorean fuzzy DS of L provided M is DS of L.

Conversely, suppose $(\eta, \bar{\eta})$ is Pythagorean fuzzy DS of L so that we claim that M is DS of L.

Now, $(\eta, \bar{\eta})$ is Pythagorean fuzzy DS of L is followed by:

$$(i) (\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and } (\bar{\eta}(m \otimes (n \otimes u)))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}$$

$$(ii) (\eta(m \odot w \odot w))^2 \geq (\eta(m))^2 \text{ and } (\bar{\eta}(m \odot w \odot w))^2 \leq (\bar{\eta}(m))^2$$

and we use only one of the inequalities from each pair of axiom (the membership for one and the non-membership for the other) as follows:

$$(i) (\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and let } m, n \in M$$

$$\Rightarrow (\eta(m))^2 = (\eta(m))^2 = \delta^2 \text{ and } (\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \\ = \min\{\delta^2, \delta^2\} = \delta^2$$

$$\Rightarrow (\eta(m \otimes (n \otimes u)))^2 \geq \delta^2 \Rightarrow m \otimes (n \otimes u) \in M$$

$$(ii) (\bar{\eta}(m \odot w \odot w))^2 \leq (\bar{\eta}(m))^2 = 1 - \delta^2$$

$$\Rightarrow (\bar{\eta}(m \odot w \odot w))^2 \leq 1 - \delta^2 \Rightarrow m \odot w \odot w \in M.$$

Therefore, by (i) and (ii) above, M is DS of L provided $(\eta, \bar{\eta})$ Pythagorean fuzzy DS of L. \square

Corollary 5.3.2.2. Let M be a non-empty subset of LBA L and $L^B = (\chi_M, \bar{\chi}_M)$, where χ_M is the Characteristic function and $\bar{\chi}_M$ is its square deviation, then L^B is a Pythagorean fuzzy DS of L if and only if M is DS of L.

Proof. The proof of this corollary is similar to the proof of Theorem 5.3.2.3 above with the execution that δ^2 is replaced by 1, σ^2 is replaced by 0 for χ_M (membership function) and $1 - \delta^2$ is replaced by 0 and $1 - \sigma^2$ is replaced by 1 for $\bar{\chi}_M$ (non-membership function). \square

Theorem 5.3.2.4. Let $(L; \otimes, \odot, 0)$ be Liu^B-algebra and $L^B = (\eta, \tau)$ be a Pythagorean fuzzy DS of L such that $\bar{\eta}, \bar{\tau}$ are square deviations and complements of η and τ , respectively. Then all the following are also Pythagorean fuzzy DSs of L:

- | | | | |
|--------------------------|--------------------------|--------------------------------|--------------------------------|
| (1) $(\eta, \bar{\eta})$ | (2) $(\eta, \bar{\tau})$ | (3) $(\bar{\tau}, \bar{\eta})$ | (4) $(\bar{\tau}, \bar{\eta})$ |
| (5) $(\bar{\tau}, \tau)$ | (6) $(\bar{\tau}, \tau)$ | (7) $(\bar{\tau}, \eta)$ | (8) $(\bar{\tau}, \eta)$ |

Proof. Since the pairs: (η, τ) and $(\eta, \bar{\eta})$ are Pythagorean fuzzy DSs by the hypothesis and as proved in Theorem 5.3.2.3, respectively, it follows that η is a membership function and hence DS of L. Here $\bar{\eta}$ and $\bar{\bar{\eta}}$ serve as the complement and square deviation functioning of η acting as non-membership components. Furthermore, τ represents the non-membership part of the Pythagorean fuzzy DS, (η, τ) , where $\bar{\tau}$ and $\bar{\bar{\tau}}$ are its complement and square deviation, respectively, acting as a membership functions) where both of these also form DSs on L.

Therefore, each pairing of a membership function with a non-membership function or square deviation yields a Pythagorean fuzzy DS For clarification, consider the pair (η, τ) as a Pythagorean fuzzy DS on L. In this case, η and τ represent the membership and non-membership functions respectively), and their corresponding complements form the ordered pair $(\bar{\tau}, \bar{\eta})$. Now we claim to show that $(\bar{\tau}, \bar{\eta})$ is Pythagorean fuzzy DS of L and then the others could be proved similarly. Then, suppose (η, τ) is Pythagorean fuzzy DS of L so that $(\bar{\eta}, \bar{\tau})$ is ordered pair of corresponding complements. Then:

$$\begin{aligned} & (\eta(m \otimes (n \otimes u))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and } (\eta(m \odot w \odot w))^2 \geq (\eta(m))^2 \\ & \Rightarrow -(\eta(m \otimes (n \otimes u))^2 \leq -\min\{(\eta(m))^2, (\eta(n))^2\} \text{ and } -(\eta(m \odot w \odot w))^2 \leq -(\eta(m))^2 \\ & \Rightarrow 1-(\eta(m \otimes (n \otimes u))^2 \leq 1-\min\{(\eta(m))^2, (\eta(n))^2\} \text{ and } 1-(\eta(m \odot w \odot w))^2 \leq 1-(\eta(m))^2 \\ & \Rightarrow 1-(\eta(m \otimes (n \otimes u))^2 \leq \max\{1-(\eta(m))^2, 1-(\eta(n))^2\} \text{ and} \\ & \quad 1-(\eta(m \odot w \odot w))^2 \leq 1-(\eta(m))^2 \\ & \Rightarrow (\bar{\eta}(m \otimes (n \otimes u))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\} \text{ and } (\bar{\eta}(m \odot w \odot w))^2 \leq (\bar{\eta}(m))^2 \end{aligned}$$

This shows that $\bar{\eta}$ is non-membership function which satisfies the inequality for the non-membership part of Pythagorean fuzzy DS.

Similarly, applying some convenient mathematical arithmetic, we have:

$$(\bar{\eta}(m))^2 \leq \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\}, \text{ and } (\bar{\eta}(m \odot w \odot w))^2 \leq (\bar{\eta}(m))^2$$

And also, we have the following corresponding to the non-membership function τ yielding membership functions as:

$$\begin{aligned} & (\tau(m \otimes (n \otimes u))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\} \text{ and } (\tau(m \odot w \odot w))^2 \geq (\tau(m))^2 \\ & \Rightarrow -(\tau(m \otimes (n \otimes u)))^2 \geq -\max\{(\tau(m))^2, (\tau(n))^2\} \text{ and } -(\tau(m \odot w \odot w))^2 \geq -(\tau(m))^2 \\ & \Rightarrow 1-(\tau(m \otimes (n \otimes u)))^2 \geq 1-\max\{(\tau(m))^2, (\tau(n))^2\} \text{ and } 1-(\tau(m \odot w \odot w))^2 \geq 1-(\tau(m))^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow 1 - (\tau(m \otimes (n \otimes u)))^2 &\geq \min\{1 - (\tau(m))^2, 1 - (\tau(n))^2\} \text{ and} \\ 1 - (\tau(m \odot w \odot w))^2 &\geq 1 - (\tau(m))^2 \\ \Rightarrow (\bar{\tau}(m \otimes (n \otimes u)))^2 &\geq \min\{(\bar{\tau}(m))^2, (\bar{\tau}(n))^2\} \text{ and } (\bar{\tau}(m \odot w \odot w))^2 \geq (\bar{\tau}(m))^2 \end{aligned}$$

This shows that $\bar{\tau}$ is membership function which satisfies the inequality for the membership part of Pythagorean fuzzy DS. Applying some convenient mathematical arithmetic, we have:

$$(\bar{\tau}(m))^2 \geq \min\{(\bar{\tau}(m \otimes n))^2, (\bar{\tau}(n))^2\}, \text{ and } (\bar{\tau}(m \odot w \odot w))^2 \leq (\bar{\tau}(m))^2$$

That means τ is non-membership function which guides us to $\bar{\tau}$ as membership function and hence Pairing any membership function with any of the non-membership function as $(\bar{\eta}, \tau)$ and $(\bar{\eta}, \tau)$ are Pythagorean fuzzy DSs of L and so form Pythagorean fuzzy DS of L . Therefore, (η, τ) is Pythagorean fuzzy DS of L implies $(\bar{\tau}, \bar{\eta})$ is also Pythagorean fuzzy DS of L and hence all the claims in this theorem also hold true which can be justified in similar ways.

□

5.3.3 Pythagorean Fuzzy Ideal of Liu^B–Algebra

Under this subsection, we introduce definition and give examples of Pythagorean fuzzy ideal of LBA, state and prove some properties and theorems on Pythagorean fuzzy ideal of LBA

Definition 5.3.3.1. A Pythagorean fuzzy set $L^B = (\eta, \tau)$ in $(L; \otimes, \odot, 0)$ in which the functions $\eta : L \rightarrow [0, 1]$ and $\tau : L \rightarrow [0, 1]$ define the membership and the non-membership degrees respectively in L is known as a Pythagorean fuzzy ideal of L if the two pair of statements hereunder are satisfied; $\forall m, n \in L$:

$$\begin{aligned} (i) \quad (\eta(m))^2 &\geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \text{ and } (\tau(m))^2 \leq \max\{(\tau(m \otimes n))^2, (\tau(n))^2\} \\ (ii) \quad (\eta((m \otimes n) \odot (n \otimes w)))^2 &\geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and} \\ (\tau((m \otimes n) \odot (n \otimes w)))^2 &\leq \max\{(\tau(m))^2, (\tau(n))^2\}. \end{aligned}$$

Example 5.3.3.1. Suppose $L = \{0, m, n\}$ and two binary operations \otimes and \odot on L are as given by the Table 5.4 of Example 5.1.10 above and let the Pythagorean fuzzy set $L^B = (\eta, \tau)$ be such that: $\eta : L \rightarrow [0, 1]$ and $\tau : L \rightarrow [0, 1]$ are given by:

$$\eta(k) = \begin{cases} 0.9, & \text{if } k = 0, \\ 0.4, & \text{if } k = m, n, \end{cases} \quad \text{and} \quad \tau(k) = \begin{cases} 0.3, & \text{if } k = 0, \\ 0.7, & \text{if } k = m, n. \end{cases}$$

In Example 5.3.3.1, we explain how the Pythagorean fuzzy set (η, τ) represent the Pythagorean fuzzy ideal of $(L; \otimes, \odot, 0)$, where $L = \{0, m, n\}$, as follows:

$$\begin{aligned}
(\eta(\mathfrak{m}))^2 &= 0.16 \\
&\geq \min\{(\eta(\mathfrak{m} \otimes \mathfrak{n}))^2, (\eta(\mathfrak{n}))^2\} = \min\{(\eta(0))^2, (\eta(\mathfrak{n}))^2\} \\
&= \min\{0.81, 0.16\} \\
&= 0.16, \text{ and}
\end{aligned}$$

$$\begin{aligned}
(\tau(\mathfrak{m}))^2 &= 0.49 \\
&\leq \max\{(\tau(\mathfrak{m} \otimes \mathfrak{n}))^2, (\tau(\mathfrak{n}))^2\} \\
&= \max\{(\tau(0))^2, (\tau(\mathfrak{n}))^2\} \\
&= \max\{0.09, 0.49\} \\
&= 0.49,
\end{aligned}$$

$$\begin{aligned}
(\eta((\mathfrak{m} \otimes \mathfrak{n}) \odot (\mathfrak{n} \otimes 0)))^2 &= (\eta((0) \odot (\mathfrak{n})))^2 = (\eta(0))^2 = 0.81 \\
&\geq \min\{(\eta(\mathfrak{m}))^2, (\eta(\mathfrak{n}))^2\} \\
&= \min\{0.16, 0.16\} \\
&= 0.16
\end{aligned}$$

$$\begin{aligned}
(\tau((\mathfrak{m} \otimes \mathfrak{n}) \odot (\mathfrak{n} \otimes 0)))^2 &= (\tau((0) \odot (\mathfrak{n})))^2 = (\tau(0))^2 = 0.09 \\
&\leq \max\{(\tau(\mathfrak{m}))^2, (\tau(\mathfrak{n}))^2\} \\
&= \max\{0.49, 0.49\} \\
&= 0.49 \text{ and so on.}
\end{aligned}$$

Lemma 5.3.3.1. *If $L^B = (\eta, \tau)$ is a Pythagorean fuzzy ideal of L , then $\forall \mathfrak{m} \in B$;*

$$(\eta(0))^2 \geq (\eta(\mathfrak{m}))^2 \text{ and } (\tau(0))^2 \leq (\tau(\mathfrak{m}))^2.$$

Proof. Since $L \neq \emptyset$, $\exists \mathfrak{m} \in L$ such that $\mathfrak{m} \otimes \mathfrak{m} = 0$ and $(\mathfrak{m} \otimes \mathfrak{m}) \odot (\mathfrak{m} \otimes \mathfrak{m}) = 0 \odot 0 = 0$ and then,

$$(\eta((\mathfrak{m} \otimes \mathfrak{m}) \odot (\mathfrak{m} \otimes \mathfrak{m})))^2 = (\eta(0))^2 \geq \min\{(\eta(\mathfrak{m}))^2, (\eta(\mathfrak{m}))^2\} = \{(\eta(\mathfrak{m}))^2\}$$

$$\Rightarrow (\eta(0))^2 \geq (\eta(\mathfrak{m}))^2, \forall \mathfrak{m} \in L, \text{ and similarly,}$$

$$(\tau((\mathfrak{m} \otimes \mathfrak{m}) \odot (\mathfrak{m} \otimes \mathfrak{m})))^2 = (\tau(0))^2 \leq \max\{(\tau(\mathfrak{m}))^2, (\tau(\mathfrak{m}))^2\} = \{(\tau(\mathfrak{m}))^2\}$$

$$\Rightarrow (\tau(0))^2 \leq (\tau(\mathfrak{m}))^2, \forall \mathfrak{m} \in L,$$

Therefore, $(\eta(0))^2 \geq (\eta(\mathfrak{m}))^2$, and $(\tau(0))^2 \leq (\tau(\mathfrak{m}))^2, \forall \mathfrak{m} \in L$.

□

Proposition 5.3.3.1. Let M be a non-empty subset of L and $L^B = (\chi_M, \bar{\chi}_M)$, where χ_M is the Characteristic function and $\bar{\chi}_M$ is its complement, then L^B is a Pythagorean fuzzy ideal of L if and only if M is ideal of L . Furthermore, the accuracy function α_M , the score function s_M and the degree of indeterminacy π_M are respectively given hereunder, $\forall m \in L$:

$$(a) \alpha_M(m) = I, \quad (b) s_M(m) = \begin{cases} 1, & \text{if } m \in M, \\ -1, & \text{if } m \notin M. \end{cases} \quad (c) \pi_M(m) = 0.$$

Proof. Suppose $\chi_M : M \rightarrow \{0, 1\}$ is a Characteristic function defined as:

$$\chi_M(k) = (\chi_M(k))^2 = \begin{cases} 1, & \text{if } k \in M, \\ 0, & \text{if } k \notin M, \end{cases} \quad \text{and then } \bar{\chi}_M(k) = (\bar{\chi}_M(k))^2 = \begin{cases} 0, & \text{if } k \in M, \\ 1, & \text{if } k \notin M. \end{cases}$$

and suppose $L^B = (\chi_M, \bar{\chi}_M)$ is a Pythagorean fuzzy ideal of L . We verify M is ideal of L .

Clearly, $(\chi_M(m))^2 = \chi_M(m)$ and $(\bar{\chi}_M(m))^2 = \bar{\chi}_M(m)$

(1) Let $m \otimes n, n \in M \Rightarrow (\chi_M(m \otimes n))^2 = (\chi_M(n))^2 = 1$, and then we claim that $m \in M$

But $(\chi_M(m))^2 \geq \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\} = \min\{1, 1\} = 1$

$\Rightarrow (\chi_M(m))^2 \geq 1$ but $(\chi_M(m))^2 \leq 1, \forall m \in M$, by definition of χ_M

$\Rightarrow (\chi_M(m))^2 = 1 \Rightarrow m \in M$

And similarly, $(\bar{\chi}_M(m))^2 = 0 \Rightarrow m \in M$ and hence, $m \otimes n, n \in M \Rightarrow m \in M$

(2) Again, let $m, n \in M \Rightarrow (\chi_M(m))^2 = (\chi_M(n))^2 = 1$.

Here also we claim that $(m \otimes n) \odot (n \otimes w) \in M, \forall w \in L$ as χ_M is fuzzy ideal of L .

Now, $(\chi_M((m \otimes n) \odot (n \otimes w)))^2 \geq \min\{(\chi_M(m))^2, (\chi_M(n))^2\} = \min\{1, 1\} = 1$

$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w))^2 \geq 1, \forall m, n \in M$

but $\chi_M((m \otimes n) \odot (n \otimes w))^2 \leq 1, \forall m, n \in M$, by definition of χ_M

$\Rightarrow \chi_M((m \otimes n) \odot (n \otimes w))^2 = 1 \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$

Again, following similar steps, we could show that:

$\Rightarrow (\bar{\chi}_M((m \otimes n) \odot (n \otimes w)))^2 = 0 \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$

Thus $m, n \in M \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$

Therefore, M is a ideal of L if $L^B = (\chi_M, \bar{\chi}_M)$ is Pythagorean fuzzy ideal of L .

Conversely, suppose M is an ideal of L . We need to prove that $L^B = (\chi_M, \bar{\chi}_M)$ is a Pythagorean fuzzy ideal of L , $\forall m, n \in L$, that means we show the following two pairs of statements hold:

$$(1) (\chi_M(m))^2 \geq \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\} \quad \text{and}$$

$$(\bar{\chi}_M(m))^2 \leq \max\{(\bar{\chi}_M(m \otimes n))^2, (\bar{\chi}_M(n))^2\}$$

$$(2) (\chi_M((m \otimes n) \odot (n \otimes w)))^2 \geq \min\{(\chi_M(m))^2, (\chi_M(n))^2\} \quad \text{and}$$

$$(\bar{\chi}_M((m \otimes n) \odot (n \otimes w)))^2 \leq \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\}$$

(1) To show $(\chi_M(m))^2 \geq \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\}$, we follow three cases:

Case(i): Let $m \otimes n, n \in M \Rightarrow m \in M$, since M is ideal of L .

$$\text{But from the hypothesis, } (\chi_M(m \otimes n))^2 = (\chi_M(n))^2 = 1 \Rightarrow (\chi_M(m))^2 = 1$$

$$\Rightarrow (\chi_M(m))^2 = 1 \geq \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\} = \min\{1, 1\}$$

$$\Rightarrow (\chi_M(m))^2 \geq \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\}$$

Case(ii): Let $m \otimes n \in M, n \notin M$ (or $m \otimes n \notin M, n \in M$), and we prove one as the other is similar.

$$\Rightarrow (\chi_M(m \otimes n))^2 = 1 \text{ and } (\chi_M(n))^2 = 0.$$

$$\text{But either } m \in M \text{ or } m \notin M \Rightarrow (\chi_M(m))^2 = 1 \text{ or } (\chi_M(m))^2 = 0$$

$$\Rightarrow (\chi_M(m))^2 \geq 0 = \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\} = \min\{1, 0\}$$

$$\Rightarrow (\chi_M(m))^2 \geq \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\}$$

Case(iii): Let $m \otimes n \notin M, n \notin M \Rightarrow (\chi_M(m \otimes n))^2 = 0 = (\chi_M(n))^2$.

$$\text{But either } m \in M \text{ or } m \notin M \Rightarrow (\chi_M(m))^2 = 1 \text{ or } (\chi_M(m))^2 = 0$$

$$\Rightarrow (\chi_M(m))^2 \geq 0 = \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\} = \min\{0, 0\}$$

$$\Rightarrow (\chi_M(m))^2 \geq \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2\}$$

Thus, by all the Cases (i) – (iii), above we obtain $(\chi_M(m))^2 \geq \min\{(\chi_M(m \otimes n), \chi_M(n))^2\}$.

Following each similar step of the membership part for the non-membership part, we have:

$$(\bar{\chi}_M(m))^2 \leq \max\{(\bar{\chi}_M(m \otimes n), \bar{\chi}_M(n))^2\}.$$

(2) To show $(\chi_M((m \otimes n) \odot (n \otimes w)))^2 \geq \min\{(\chi_M(m))^2, (\chi_M(n))^2\}$, similar to (1)

above, we follow the following three cases:

Case(i) : Let $m, n \in M \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$, since M is ideal of L .

$$\begin{aligned} \text{But from the hypothesis, } (\chi_M(m))^2 &= (\chi_M(n))^2 = 1 \Rightarrow (\chi_M((m \otimes n) \odot (n \otimes w)))^2 = 1 \\ \Rightarrow (\chi_M((m \otimes n) \odot (n \otimes w)))^2 &= 1 \geq \min\{(\chi_M(m))^2, (\chi_M(n))^2\} \end{aligned}$$

Case(ii) : Let $m \in M, n \notin M$ (or $m \notin M, n \in M$), and we prove one as the other similar.

$$\Rightarrow (\chi_M(m))^2 = 1 \text{ and } (\chi_M(n))^2 = 0.$$

But either $(m \otimes n) \odot (n \otimes w) \in M$ or $(m \otimes n) \odot (n \otimes w) \notin M$

$$\Rightarrow (\chi_M((m \otimes n) \odot (n \otimes w)))^2 = 1 \text{ or } (\chi_M((m \otimes n) \odot (n \otimes w)))^2 = 0$$

$$\Rightarrow (\chi_M((m \otimes n) \odot (n \otimes w)))^2 \geq 0 = \min\{(\chi_M(m))^2, (\chi_M(n))^2\} = \min\{1, 0\}$$

$$\Rightarrow (\chi_M((m \otimes n) \odot (n \otimes w)))^2 \geq \min\{(\chi_M(m))^2, (\chi_M(n))^2\}$$

Case(iii) : Let $m \notin M, n \notin M$

$$\Rightarrow (\chi_M(m))^2 = 0 = (\chi_M(n))^2.$$

But either $(m \otimes n) \odot (n \otimes w) \in M$ or $(m \otimes n) \odot (n \otimes w) \notin M$

$$\Rightarrow (\chi_M((m \otimes n) \odot (n \otimes w)))^2 = 1 \text{ or } (\chi_M((m \otimes n) \odot (n \otimes w)))^2 = 0$$

$$\Rightarrow (\chi_M((m \otimes n) \odot (n \otimes w)))^2 \geq 0 = \min\{(\chi_M(m))^2, (\chi_M(n))^2\} = \min\{0, 0\}$$

Thus, by all the Cases (i) – (iii), above we obtain:

$$(\chi_M((m \otimes n) \odot (n \otimes w)))^2 \geq \min\{(\chi_M(m))^2, (\chi_M(n))^2\}.$$

Following each step for the non-membership part, we can also obtain the result:

$$(\bar{\chi}_M((m \otimes n) \odot (n \otimes w)))^2 \leq \max\{(\bar{\chi}_M(m))^2, (\bar{\chi}_M(n))^2\}.$$

In general, by (1) and (2) above, we have assured that the Characteristic function along with its complement, $L^B = (\chi_M, \bar{\chi}_M)$ is Pythagorean fuzzy ideal of L if M is ideal of L . Furthermore, we have the next justifications for α_M , s_M and π_M :

(a) The accuracy function:

$$\alpha_M(m) = (\chi_M(m))^2 + (\bar{\chi}_M(m))^2 = \begin{cases} 1 + 0, & \text{if } m \in M, \\ 0 + 1, & \text{if } m \notin M \end{cases} = 1, \forall m \in L$$

(b) The score function:

$$s_M(m) = (\chi_M(m))^2 - (\bar{\chi}_M(m))^2 = \begin{cases} 1 - 0, & \text{if } m \in M, \\ 0 - 1, & \text{if } m \notin M, \end{cases} = \begin{cases} 1, & \text{if } m \in M, \\ -1, & \text{if } m \notin M. \end{cases}$$

(c) The degree of indeterminacy:

$$\pi_M(m) = \sqrt{1 - \left[(\chi_M(m))^2 + (\bar{\chi}_M(m))^2 \right]} = \begin{cases} \sqrt{0 - (1+0)}, & \text{if } m \in M, \\ \sqrt{0 - (0+1)}, & \text{if } m \notin M \end{cases} = 0, \forall m \in L. \quad \square$$

Corollary 5.3.3.1. Let η be a fuzzy subset in LBA L and M be a non-empty subset of L such that

$$\eta_M(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M, \end{cases} \quad \text{and its complement } \bar{\eta}_M(m) = \begin{cases} 1 - \delta, & \text{if } m \in M, \\ 1 - \sigma, & \text{if } m \notin M. \end{cases}$$

where $\delta, \sigma \in [0, 1]$, $\delta > \sigma$. Then M is ideal of L if and only if $(\eta_M, \bar{\eta}_M)$ is Pythagorean fuzzy ideal of L .

Proof. Analogous to the proof of Theorem 5.3.3.1 above. □

Remark 5.3.3.1. In LBA, as the membership parts of the classes of Pythagorean Structures, which are fuzzy subalgebras, fuzzy deductive systems, and fuzzy ideals are mutually non-equivalent as discussed in Remark 5.2.3.2 and demonstrated by Example 5.2.3.2, the same non-equivalence also hold among the classes of Pythagorean fuzzy subalgebras, Pythagorean fuzzy deductive systems, and Pythagorean fuzzy ideals.

Theorem 5.3.3.1. Let $(L; \otimes, \odot, 0)$ be LBA and the fuzzy subset $\eta : L \rightarrow [0, 1]$ be fuzzy ideal of L and $\bar{\eta} : L \rightarrow [0, 1]$ such that $\bar{\eta}(m) = 1 - \eta(m)$, $\forall m \in L$ be the complement of η on L . Then $L^B = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of L .

Proof. From the hypothesis that η is fuzzy ideal of L , we have:

$$\begin{aligned} \eta(m) &\geq \min\{\eta(m \otimes n), \eta(n)\} \quad \text{and} \quad \eta((m \otimes n) \odot (n \otimes w)) \geq \min\{\eta(m), \eta(n)\} \\ \Rightarrow (\eta(m))^2 &\geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \quad \text{and} \\ (\eta((m \otimes n) \odot (n \otimes w)))^2 &\geq \min\{(\eta(m))^2, (\eta(n))^2\}, \quad \text{and again by Remark 5.3.1 (2) (ii):} \\ (\bar{\eta}(m))^2 &\leq (\bar{\eta}(m))^2 \leq 1 - (\eta(m))^2 \leq 1 - \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \\ &= \max\{1 - (\eta(m \otimes n))^2, 1 - (\eta(n))^2\} \quad \text{and} \\ &= \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\} \quad \text{which means} \\ (\bar{\eta}(m))^2 &\leq \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\} \quad \text{and similarly,} \\ (\bar{\eta}((m \otimes n) \odot (n \otimes w)))^2 &\leq \min\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}. \end{aligned}$$

Therefore, $L^B = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of L . □

Theorem 5.3.3.2. Let L be LBA and $\eta : L \rightarrow [0, 1]$ be fuzzy subset and $\bar{\bar{\eta}}$ be the square deviation of η in L . Then, $L^B = (\eta, \bar{\bar{\eta}})$ is a Pythagorean fuzzy ideal of L if and only if η is fuzzy ideal of L .

Furthermore, the accuracy function α_L , the score function s_L and the degree of indeterminacy π_L are respectively given as follows, $\forall m \in L$:

$$(a) \alpha_L(m) = 1, \quad (b) s_L(m) = 2(\eta(m))^2 - 1, \quad (c) \pi_L(m) = 0.$$

Proof. The forward implication that: if $L^B = (\eta, \bar{\eta})$ is a Pythagorean fuzzy ideal of L , then η is fuzzy ideal of L is straightforward. Thus it suffices to prove the Converse that:

$L^B = (\eta, \bar{\eta})$ is a Pythagorean fuzzy ideal of L if η is fuzzy ideal of L and hence the following.

Suppose η is fuzzy ideal of L and let $m \in L$.

$$\Rightarrow \eta(m) \geq \min\{\eta(m \otimes n), \eta(n)\} \text{ and } \eta((m \otimes n) \odot (n \otimes w)) \geq \min\{\eta(m), \eta(n)\}$$

$$\Rightarrow -\eta(m) \leq -\min\{\eta(m \otimes n), \eta(n)\} \text{ and } -\eta((m \otimes n) \odot (n \otimes w)) \leq -\min\{\eta(m), \eta(n)\}$$

$$\Rightarrow 1 - (\eta(m))^2 \leq 1 - \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \text{ and}$$

$$1 - (\eta((m \otimes n) \odot (m \otimes n)))^2 \leq 1 - \min\{(\eta(m))^2, (\eta(n))^2\}, \text{ and}$$

$$\Rightarrow (\bar{\eta}(m))^2 \leq \max\{1 - (\eta(m \otimes n))^2, 1 - (\eta(n))^2\} \text{ and}$$

$$(\bar{\eta}((m \otimes n) \odot (m \otimes n)))^2 \leq \max\{(1 - \eta(m))^2, 1 - (\eta(n))^2\}, \text{ and}$$

$$\Rightarrow (\bar{\eta}(m))^2 \leq \min\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\}, \forall m \in L \text{ and}$$

$$(\bar{\eta}((m \otimes n) \odot (m \otimes n)))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}, \forall m \in L \text{ and}$$

Therefore, $L^B = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of L . Furthermore, we have the following:

$$(a) \alpha_L(m) = (\eta(m))^2 + (\bar{\eta}(m))^2 = (\eta(m))^2 + (1 - (\eta(m))^2) = 1$$

$$(b) s_L(m) = (\eta(m))^2 - (\bar{\eta}(m))^2 = (\eta(m))^2 - (1 - (\eta(m))^2) = 2(\eta(m))^2 - 1$$

$$(c) \pi_L(m) = \sqrt{1 - \alpha_L(m)} = 0$$

□

Remark 5.3.3.2. Let L be LBA By Theorem 5.3.3.1, since $L \subseteq L$ and because $\bar{\chi}_L(m) = \bar{\bar{\chi}}_L(m)$, $\forall m \in L$, where χ_L is the Characteristic function on L ; $\bar{\chi}_L$ and $\bar{\bar{\chi}}_L$ are its complement and square deviation, respectively, it follows that $L_1^B = (\chi_L, \bar{\chi}_L)$ is a Pythagorean fuzzy ideal of L .

Similarly $L_2^B = (\chi_L, \bar{\bar{\chi}}_L)$ is also a Pythagorean fuzzy ideal of L .

Theorem 5.3.3.3. Let L be LBA, $\emptyset \neq M \subseteq L$ and $\eta_M : L \rightarrow [0, 1]$ be a fuzzy subset in LBA L such

$$\text{that } \eta_M(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M, \end{cases} \text{ and the square deviation } (\bar{\eta}_M(m))^2 = \begin{cases} 1 - \delta^2, & \text{if } m \in M, \\ 1 - \sigma^2, & \text{if } m \notin M. \end{cases}$$

where $\delta, \sigma \in [0, 1]$, $\delta > \sigma \Rightarrow 1 - \delta^2 \leq 1 - \sigma^2$.

Then M is ideal of L if and only if $L^B = (\eta_M, \bar{\eta}_M)$ is a Pythagorean fuzzy ideal of L .

Proof. η_M is fuzzy ideal of L if and only if M is ideal of L as stated in Corollary 5.3.3.1 followed by Theorem 5.3.3.1 above. Then we only need to show that: M is ideal of L if and only if the following two axioms hold for the non-membership part as follows:

- (1) $(\bar{\eta}_M(m))^2 \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\}$ and
- (2) $(\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2 \leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2\}$

Let M be ideal of L , then we need to prove:

Case(i) : Let $n \in M$, $m \otimes n \in M \Rightarrow m \in M$

$$\begin{aligned} \text{Then } (\bar{\eta}_M(m \otimes n))^2 &= (\bar{\eta}_M(m))^2 = (\bar{\eta}_M(n))^2 = 1 - \delta^2 \Rightarrow (\bar{\eta}_M(m))^2 = 1 - \delta^2 \\ \Rightarrow (\bar{\eta}_M(m))^2 &\leq 1 - \delta^2 = \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\} \\ \Rightarrow (\bar{\eta}_M(m))^2 &\leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\} \text{ and similarly,} \\ \Rightarrow (\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2 &\leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2\}. \end{aligned}$$

Case(ii) : Let $n \in M$, $m \otimes n \notin M \Rightarrow m \notin M$ or $m \in M$

$$\begin{aligned} \Rightarrow (\bar{\eta}_M(n))^2 &= 1 - \delta^2, (\bar{\eta}_M(m \otimes n))^2 = 1 - \sigma^2 \text{ but } (\bar{\eta}_M(m))^2 = 1 - \sigma^2 \text{ or } (\bar{\eta}_M(m))^2 = 1 - \delta^2 \\ \Rightarrow (\bar{\eta}_M(m))^2 &\leq 1 - \delta^2 = \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\} = \max\{1 - \sigma^2, 1 - \delta^2\} \\ \Rightarrow (\bar{\eta}_M(m))^2 &\leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\} \end{aligned}$$

$$\text{Similarly, } (\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2 \leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2\}$$

Now we arrive at similar deductions for **Case(iii)** : when $n \notin M$, $m \otimes n \in M \Rightarrow m \in M$

or $m \notin M$ and **Case(iv)** : when $n \notin M$, $m \otimes n \notin M \Rightarrow m \in M$ or $m \notin M$

Thus, based on all the above cases, we conclude that both conditions (1) and (2) are satisfied.

Therefore, $(\eta_M, \bar{\eta}_M)$ is Pythagorean fuzzy ideal of LBA L .

Conversely, suppose (1) $(\bar{\eta}_M(m))^2 \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\}$ and

(2) $(\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2 \leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2\}$ hold.

$B^P = (\eta, \tau)$ is a Pythagorean fuzzy SA of B

Then, we need to show that M is ideal of L , which means:

(a) $n, m \otimes n \in M \Rightarrow m \in M$ and

(b) $m, n \in M \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$ Then:

(a) Let $n, m \otimes n \in M \Rightarrow (\bar{\eta}_M(n \otimes m))^2 = (\bar{\eta}_M(n))^2 = 1 - \delta^2$ and

$$(\bar{\eta}_M(m))^2 \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\} = \max\{1 - \delta^2, 1 - \delta^2\} = 1 - \delta^2$$

$\Rightarrow (\bar{\eta}_M(m))^2 \leq 1 - \delta^2$ and since $\delta, \sigma \in [0, 1], \delta > \sigma \Rightarrow 1 - \sigma^2 \geq 1 - \delta^2$, we have:

$$(\bar{\eta}_M(m))^2 \geq 1 - \delta^2 \Rightarrow (\bar{\eta}_M(m))^2 = 1 - \delta^2 \Rightarrow m \in M$$

(b) Let $n, m \in M$, then we need to show that $(m \otimes n) \odot (n \otimes w) \in M$.

Now $(\bar{\eta}_M(m))^2 = (\bar{\eta}_M(n))^2 = 1 - \delta^2$ and

$$(\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2 \leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2\} = \max\{1 - \delta^2, 1 - \delta^2\} = 1 - \delta^2$$

$\Rightarrow (\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2 \leq 1 - \delta^2$ and by similar implications as (a) above, we have:

$$(\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2 \geq 1 - \delta^2 \text{ by definition of } \bar{\eta}_M$$

$$\Rightarrow (\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2 = 1 - \delta^2 \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$$

Therefore, based on cases (a) and (b) shown above, we have confirmed that M is ideal of L . \square

Theorem 5.3.3.4. Let L be LBA $\eta : L \rightarrow [0, 1]$ be a fuzzy subset and $\bar{\eta} : L \rightarrow [0, 1]$ be the square deviation of η in L such that $(\eta(m \otimes n))^2 = (\eta(n))^2$ and $(\eta(m \odot n))^2 = (\eta(n))^2$. Then $(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2; (\bar{\eta}(m \odot n))^2 = (\bar{\eta}(n))^2, \forall m, n \in L$, and η and $\bar{\eta}$ are constants.

Proof. Suppose $(\eta(m \otimes n))^2 = (\eta(n))^2$ and $(\eta(m \odot n))^2 = (\eta(n))^2$.

Then $(\bar{\eta}(m \otimes n))^2 = 1 - (\eta(m \otimes n))^2 = 1 - (\eta(n))^2 = (\bar{\eta}(n))^2$ and

$$(\bar{\eta}(m \odot n))^2 = 1 - (\eta(m \odot n))^2 = 1 - (\eta(n))^2 = (\bar{\eta}(n))^2.$$

As proved in Theorem 5.3.3.2 above, $L = (\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of L ,

$\Rightarrow (\eta(m \otimes n))^2 = (\eta(n))^2$ and $(\eta(m \odot n))^2 = (\eta(n))^2$ and then

$$(\bar{\eta}(m \otimes n))^2 = (\bar{\eta}(n))^2 \text{ and } (\bar{\eta}(m \odot n))^2 = (\bar{\eta}(n))^2$$

Now we need to verify that η and $\bar{\eta}$ are constants,

$$((\eta(m))^2 = (\eta(n))^2 \text{ and then } (\overline{\eta}(m))^2 = (\overline{\eta}(n))^2, \forall m, n \in L.)$$

Since $L^B = (\eta, \overline{\eta})$ is a Pythagorean fuzzy ideal of L , η is a fuzzy ideal of L , and hence by Proposition 2.1.1 (3) and Proposition 5.1.1 (b), we have:

$$0 = (0 \otimes m) = (0 \odot m), \forall m \in L \text{ and then } \eta(0) = \eta(0 \otimes m) = \eta(0 \odot m), \forall m \in L,$$

$$\Rightarrow (\eta(0))^2 = (\eta(0 \otimes m))^2 = (\eta(m))^2 = (\eta(0 \odot m))^2, \forall m \in L, \text{ and again}$$

$$(\eta(0))^2 = (\eta(0 \otimes n))^2 = (\eta(0 \odot n))^2 = (\eta(n))^2, \forall n \in L$$

$$\Rightarrow (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2, \forall m, n \in L,$$

Or $(\eta(m))^2 = (\eta(n))^2, \forall m, n \in L$ and hence, η is constant, and analogously, $\overline{\eta}$ is, too. \square

Theorem 5.3.3.5. Let $L^B = (\eta, \tau)$ be Pythagorean fuzzy ideal of LBA L and $m, n \in L$.

If $\eta(m \otimes n) = \eta(n), \tau(m \otimes n) = \tau(n), \eta(m \odot n) = \eta(n), \tau(m \odot n) = \tau(n)$ then,

$\forall m, n \in L$, the following hold:

$$(1) \text{ The accuracy function: } a_L(m) \leq 2 - [(\eta(0))^2 + (\tau(0))^2],$$

$$(2) \text{ The score function: } s_L(m) \leq 1 - [(\eta(0))^2 + \tau(0))^2],$$

$$(3) \text{ The degree of indeterminacy: } \pi_L(m) \text{ is such that: } (\pi_L(m))^2 \leq (\eta(0))^2 + (\tau(0))^2 - 1.$$

Proof. Let $L^B = (\eta, \tau)$ be a Pythagorean fuzzy ideal of L ;

$$(\eta(m \otimes n))^2 = (\eta(n))^2, (\tau(m \otimes n))^2 = (\tau(n))^2 \text{ and}$$

$$(\eta(m \odot n))^2 = (\eta(n))^2, (\tau(m \odot n))^2 = (\tau(n))^2$$

Since, $\forall m \in L, 0 = 0 \otimes m = 0 \odot m$ (by Proposition 2.1.1 (3) and Proposition 5.1.1 (b))

$$\Rightarrow (\eta(0))^2 = (\eta(0 \otimes m))^2 = (\eta(0 \odot m))^2 = (\eta(m))^2 = (\eta(0 \otimes n))^2 = (\eta(0 \odot n))^2 = (\eta(n))^2$$

$$\Rightarrow (\eta(0))^2 = (\eta(m))^2 = (\eta(n))^2, \forall m, n \in L$$

$$\text{Similarly, } (\tau(0))^2 = (\tau(m))^2 = (\tau(n))^2, \forall m, n \in L$$

$$\text{But } 0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1 \Rightarrow 0 \leq (\eta(m))^2 + (\tau(0))^2 \leq 1$$

$$\Rightarrow (\eta(m))^2 \leq 1 - (\tau(0))^2. \text{ And, } 0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1$$

$$\Rightarrow 0 \leq (\eta(0))^2 + (\tau(m))^2 \leq 1 \Rightarrow (\tau(m))^2 \leq 1 - (\eta(0))^2$$

Then $\forall m \in L$, we have the following:

(1) The accuracy function:

$$a_L(m) = (\eta(m))^2 + (\tau(m))^2 \leq (1 - (\eta(0))^2) + (1 - (\tau(0))^2) = 2 - (\eta(0))^2 - (\tau(0))^2$$

(2) The score function:

$$s_L(m) = (\eta(m))^2 - (\tau(m))^2 \leq (1 - (\eta(0))^2) - (\tau(0))^2 = 1 - (\eta(0))^2 - (\tau(0))^2$$

(3) The degree of indeterminacy:

$$\begin{aligned} (\pi_L(m))^2 &= 1 - (\eta(m))^2 - (\tau(m))^2 = 1 - a_L(m) \leq 1 - [2 - (\eta(0))^2 - (\tau(0))^2] \\ &= (\eta(0))^2 + (\tau(0))^2 - 1. \end{aligned}$$

□

Theorem 5.3.3.6. *Let $L = (\eta, \tau)$ be a Pythagorean fuzzy ideal of LBA L , $\bar{\eta}$, $\bar{\tau}$ are complements and $\bar{\bar{\eta}}$, $\bar{\bar{\tau}}$ are square deviations of η , τ , respectively. Then ordered pairs hereunder are each Pythagorean fuzzy ideals of L :*

- | | | | |
|--------------------------|--------------------------------|--------------------------------------|--------------------------------------------|
| (1) $(\eta, \bar{\eta})$ | (2) $(\eta, \bar{\bar{\eta}})$ | (3) $(\bar{\tau}, \bar{\eta})$ | (4) $(\bar{\tau}, \bar{\bar{\eta}})$ |
| (5) $(\bar{\tau}, \tau)$ | (6) $(\bar{\bar{\tau}}, \tau)$ | (7) $(\bar{\bar{\tau}}, \bar{\eta})$ | (8) $(\bar{\bar{\tau}}, \bar{\bar{\eta}})$ |

Proof. Since the pairs: (η, τ) , $(\eta, \bar{\eta})$ and $(\eta, \bar{\bar{\eta}})$ are Pythagorean fuzzy ideals as established in Corollary 5.3.3.1, Theorem 5.3.3.1 and Theorem 5.3.3.2, respectively, it follows that η is a membership function and hence ideal of L . Here $\bar{\eta}$ and $\bar{\bar{\eta}}$ serve as its complement and square deviation functioning as non-membership components. Furthermore, τ represents the non-membership part of the Pythagorean fuzzy ideal, (η, τ) , where $\bar{\tau}$ and $\bar{\bar{\tau}}$ are its complement and square deviation acting as membership functions) where both of these also form ideals on L .

Therefore, each pairing of a membership function with a non-membership function or square deviation yields a Pythagorean fuzzy ideal. For clarification, consider the pair (η, τ) as a Pythagorean fuzzy ideal on L . In this case, η and τ represent the membership and non-membership functions respectively, and their corresponding complements form the ordered pair $(\bar{\tau}, \bar{\eta})$. Now we claim to show that $(\bar{\tau}, \bar{\eta})$ is Pythagorean fuzzy ideal of L and then the others could be proved similarly. Suppose (η, τ) is Pythagorean fuzzy ideal of L so that $(\bar{\eta}, \bar{\tau})$ is ordered pair of corresponding complements. Then:

$$\begin{aligned} (\eta(m))^2 &\geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\} \text{ and } (\eta(m \otimes n) \odot (n \otimes w))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \\ &\Rightarrow -(\eta(m))^2 \leq -\min\{(\eta(m \otimes n))^2, (\eta(n))^2\}, \text{ and} \end{aligned}$$

$$\begin{aligned}
& -(\eta((m \otimes n) \odot (n \otimes w)))^2 \leq -\min\{(\eta(m))^2, (\eta(n))^2\} \\
\Rightarrow & 1 - (\eta(m))^2 \leq 1 - \min\{(\eta(m \otimes n))^2, (\eta(n))^2\}, \text{ and} \\
& 1 - (\eta((m \otimes n) \odot (n \otimes w)))^2 \leq 1 - \min\{(\eta(m))^2, (\eta(n))^2\} \\
\Rightarrow & (\bar{\eta}(m))^2 \leq \max\{(\bar{\eta}(m \otimes n))^2, (\bar{\eta}(n))^2\}, \text{ and} \\
& (\bar{\eta}((m \otimes n) \odot (n \otimes w)))^2 \leq \max\{(\bar{\eta}(m))^2, (\bar{\eta}(n))^2\}
\end{aligned}$$

That means η is membership function which guides us to $\bar{\eta}$ as non-membership function and hence $(\eta, \bar{\eta})$ is Pythagorean fuzzy ideal of L . Again, going back from the last to the first of the preceding steps for the non-membership function τ to get the membership function $\bar{\tau}$, we have got $(\bar{\tau}, \tau)$ is Pythagorean fuzzy ideal of L . Therefore, (η, τ) is Pythagorean fuzzy ideal of L implies $(\bar{\tau}, \bar{\eta})$ is also Pythagorean fuzzy ideal of L and hence all the claims in this theorem also hold which can be justified in similar ways. \square

Theorem 5.3.3.7. *The intersection of any two Pythagorean fuzzy ideals of LBA L is also a Pythagorean fuzzy ideal of L .*

Proof. Let $L_1^B = (\eta_1, \tau_1)$ and $L_2^B = (\eta_2, \tau_2)$ be any two Pythagorean fuzzy ideal of L and let us denote $L_{\cap}^B = (\eta_1 \cap \eta_2, \tau_1 \cap \tau_2)$. Then we need to prove: $L_{\cap}^B = (\eta_1 \cap \eta_2, \tau_1 \cap \tau_2)$ is Pythagorean fuzzy ideal of L . Now, let $m, n \in L$. Then,

$$\begin{aligned}
\text{(i)} \quad & ((\eta_1 \cap \eta_2)(m))^2 = \min\{(\eta_1(m))^2, (\eta_2(m))^2\} \\
& \geq \min\{\min\{(\eta_1(m \otimes n))^2, (\eta_1(n))^2\}, \min\{(\eta_2(m \otimes n))^2, (\eta_2(n))^2\}\} \\
& = \min\{\min\{(\eta_1(m \otimes n))^2, (\eta_2(m \otimes n))^2, \min\{(\eta_1(n))^2, (\eta_2(n))^2\}\}\} \\
& = \min\{((\eta_1 \cap \eta_2)(m \otimes n))^2, ((\eta_1 \cap \eta_2)(n))^2\} \quad \text{and} \\
& ((\tau_1 \cap \tau_2)(m))^2 = \max\{(\tau_1(m))^2, (\tau_2(m))^2\} \\
& \leq \max\{\max\{(\tau_1(m \otimes n))^2, (\tau_1(n))^2\}, \max\{(\tau_2(m \otimes n))^2, (\tau_2(n))^2\}\} \\
& = \max\{\max\{(\tau_1(m \otimes n))^2, (\tau_2(m \otimes n))^2, \max\{(\tau_1(n))^2, (\tau_2(n))^2\}\}\} \\
& = \max\{((\tau_1 \cap \tau_2)(m \otimes n))^2, ((\tau_1 \cap \tau_2)(n))^2\} \\
\text{(ii)} \quad & ((\eta_1 \cap \eta_2)((m \otimes n) \odot (n \otimes w)))^2 \\
& = \min\{(\eta_1((m \otimes n) \odot (n \otimes w)))^2, (\eta_2((m \otimes n) \odot (n \otimes w)))^2\} \\
& \geq \min\{\min\{(\eta_1(m))^2, (\eta_1(n))^2\}, \min\{(\eta_2(m))^2, (\eta_2(n))^2\}\}
\end{aligned}$$

$$\begin{aligned}
&= \min\{\min\{(\eta_1(m))^2, (\eta_2(m))^2, \min\{(\eta_1(n))^2, (\eta_2(n))^2\}\}\} \\
&= \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\} \quad \text{and} \\
&(\tau_1 \cap \tau_2)((m \otimes n) \odot ((n \otimes w)))^2 \\
&= \max\{(\tau_1((m \otimes n) \odot ((n \otimes w))))^2, (\tau_2((m \otimes n) \odot ((n \otimes w))))^2\} \\
&\leq \max\{\max\{(\tau_1(m))^2, (\tau_1(n))^2\}, \max\{(\tau_2(m))^2, (\tau_2(n))^2\}\} \\
&= \max\{\max\{(\tau_1(m))^2, (\tau_2(m))^2, \max\{(\tau_1(n))^2, (\tau_2(n))^2\}\} \\
&= \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2\}
\end{aligned}$$

Thus, by (i) and (ii) above, intersection of any two Pythagorean fuzzy ideals of LBA L is Pythagorean fuzzy ideal of L. \square

Corollary 5.3.3.2. *The intersection, $(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i)$, of any family of Pythagorean fuzzy ideals, $\{(\eta_i, \tau_i) : i \in I\}$, of LBA L is also a Pythagorean fuzzy ideal of L.*

Proof. Let $\{(\eta_i, \tau_i) : i \in I\}$, be family of Pythagorean fuzzy ideals of L and let $m, n \in L$. Then,

$$\begin{aligned}
\text{(i)} \quad \bigcap_{i \in I} (\eta_i(m))^2 &= \inf_{i \in I} \{(\eta_i(m))^2\} \\
&\geq \inf_{i \in I} \{\min\{(\eta_i(m \otimes n))^2, (\eta_i(n))^2\}\} \\
&\geq \min\{\inf_{i \in I} \{(\eta_i(m \otimes n))^2, (\eta_i(n))^2\}\} \\
&= \min\{\inf_{i \in I} \{(\eta_i(m \otimes n))^2\}, \inf_{i \in I} \{(\eta_i(n))^2\}\} \\
&= \min\left\{\bigcap_{i \in I} (\eta_i(m \otimes n))^2, \bigcap_{i \in I} (\eta_i(n))^2\right\} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\bigcap_{i \in I} (\tau_i(m))^2 &= \sup_{i \in I} \{(\tau_i(m))^2\} \\
&\leq \sup_{i \in I} \{\max\{(\tau_i(m \otimes n))^2, (\tau_i(n))^2\}\} \\
&\leq \max\{\sup_{i \in I} \{(\tau_i(m \otimes n))^2, (\tau_i(n))^2\}\} \\
&= \max\{\sup_{i \in I} \{(\tau_i(m \otimes n))^2\}, \sup_{i \in I} \{(\tau_i(n))^2\}\} \\
&= \max\left\{\bigcap_{i \in I} (\tau_i(m \otimes n))^2, \bigcap_{i \in I} (\tau_i(n))^2\right\}, \forall m, n \in L.
\end{aligned}$$

$$\begin{aligned}
(ii) \quad \bigcap_{i \in I} (\eta_i((m \otimes n)) \odot ((n \otimes w)))^2 &= \inf_{i \in I} \{(\eta_i((m \otimes n)) \odot ((n \otimes w)))^2\} \\
&\geq \inf_{i \in I} \{\min\{(\eta_i(m))^2, (\eta_i(n))^2\}\} \\
&\geq \min\{\inf_{i \in I} \{(\eta_i(m))^2, (\eta_i(n))^2\}\} \\
&= \min\{\inf_{i \in I} \{(\eta_i(m))^2\}, \inf_{i \in I} \{(\eta_i(n))^2\}\} \\
&= \min\left\{\bigcap_{i \in I} (\eta_i(m))^2, \bigcap_{i \in I} (\eta_i(n))^2\right\} \quad \text{and} \\
\bigcap_{i \in I} (\tau_i((m \otimes n)) \odot ((n \otimes w)))^2 &= \sup_{i \in I} \{(\tau_i((m \otimes n)) \odot ((n \otimes w)))^2\} \\
&\leq \sup_{i \in I} \{\max\{(\tau_i(m))^2, (\tau_i(n))^2\}\} \\
&\leq \max\{\sup_{i \in I} \{(\tau_i(m))^2, (\tau_i(n))^2\}\} \\
&= \max\{\sup_{i \in I} \{(\tau_i(m))^2\}, \sup_{i \in I} \{(\tau_i(n))^2\}\} \\
&= \max\left\{\bigcap_{i \in I} (\tau_i(m))^2, \bigcap_{i \in I} (\tau_i(n))^2\right\}, \forall m, n, w \in L.
\end{aligned}$$

Thus, intersection of Pythagorean fuzzy ideal of L is Pythagorean fuzzy ideal of L . \square

Remark 5.3.3.3. Union of two Pythagorean fuzzy ideals of LBA L may not be Pythagorean fuzzy ideal of L illustrated by Example 5.3.3.2 below.

Example 5.3.3.2. It suffices to show that one part of an axiom fails so that we show the failure of the membership part of the first axiom as follows:

Then let $(L; \otimes, \odot, 0)$ where $L = \{0, m, n, w\}$ and two binary operations \otimes and \odot on L be as defined in Table 5.7 of Example 5.2.3.2 above and define two fuzzy ideals η_1 and η_2 of L as follows:

k	$\eta_1(k)$	$\eta_2(k)$	$(\eta_1 \cup \eta_2)(k)$	$\tau_1(k)$	$\tau_2(k)$	$(\tau_1 \cup \tau_2)(k)$
0	0.8	0.9	0.9	0.2	0.3	0.2
m	0.4	0.7	0.7	0.7	0.4	0.4
n	0.7	0.5	0.7	0.5	0.6	0.5
w	0.4	0.5	0.5	0.7	0.6	0.6

Table 5.9: A table that defines fuzzy ideal and their unions

Hence, from Table 5.7 above, as $(n \otimes m) \odot (m \otimes n) = w \odot w = w$, we have:

$$\begin{aligned} ((\eta_1 \cup \eta_2)((n \otimes m) \odot (m \otimes n)))^2 &= ((\eta_1 \cup \eta_2)(w))^2 = (0.5)^2 = 0.25 \\ &\geq \min\{(\eta_1 \cup \eta_2)(n), (\eta_1 \cup \eta_2)(m)\} \\ &= \min\{(0.7)^2, (0.7)^2\} = \min\{0.49, 0.49\} = 0.49 \end{aligned}$$

which is false, ratifying that union of two fuzzy ideals of L is not necessarily fuzzy ideal of L .

Remark 5.3.3.4. Let $P_1 = (\eta_1, \tau_1)$, $P_2 = (\eta_2, \tau_2)$, and $P_3 = (\eta_3, \tau_3)$ represent a Pythagorean fuzzy SA, a Pythagorean fuzzy DS, and a Pythagorean fuzzy ideal of the Liu^B-algebra L , respectively. The connections, whether one implies another or not, among these three depend on the corresponding relationships between fuzzy subalgebra, fuzzy DS, and fuzzy ideal of L as discussed earlier in Remark 5.2.3.2 and shown in Example 5.2.3.3. In short, the same kind of result also applies to these Pythagorean fuzzy substructures; that is, there is no implication or inclusion relation between any two of P_1 , P_2 , and P_3 .

Chapter 6

Fundamental Structures in (Pythagorean) Fuzzy Composite Structures

In this chapter, we introduce the notions and explore different properties of "Cartesian product" on fuzzy SA of BCL–algebra, "level sets" on Pythagorean fuzzy DS of BCL–algebra, "homomorphisms" on Pythagorean fuzzy DS of Liu^B–algebra, "Pythagorean (β, θ) –fuzzy ideal" on Pythagorean fuzzy Liu^B–algebra, The fundamental structures: "homomorphisms", "level sets", "Cartesian product" and the parameters (β, θ) applied to other substructures are well introduced with essential guidance. However, certain extended or overlapping concepts are intentionally omitted to avoid redundancy and remain open for systematic exploration in subsequent research endeavor or are reserved to be explored in our future works.

6.1. Cartesian Products in Fuzzy Subalgebra of BCL–Algebra

Under this section, we discuss the notion of Cartesian product on fuzzy SAs of BCL–algebras. We demonstrate that the Cartesian product of two such fuzzy SAs is itself a fuzzy SA of BCL–algebra, and we investigate several other related results.

Remark 6.1.1. *Definitions 1.5.4 of Chapter 1 above about the Cartesian product of fuzzy subsets η_1 and η_2 on an algebra R can be summarized as:*

$$\begin{aligned}
 (\eta_1 \times \eta_2)(m, n) &= \min\{\eta_1(m), \eta_2(n)\}, \forall m, n \in R \text{ and} \\
 (\eta_1 \times \eta_2)((m_1, n_1) \otimes (m_2, n_2)) &= (\eta_1 \times \eta_2)(m_1 \otimes m_2, n_1 \otimes n_2) \\
 &= \min\{\eta_1(m_1 \otimes m_2), \eta_2(n_1 \otimes n_2)\}, \forall m_1, m_2, n_1, n_2 \in R.
 \end{aligned}$$

Example 6.1.1. *Let the fuzzy subsets η_1 and η_2 be as defined in Table 5.6 of Example 5.2.1.2, and let the binary operation \otimes be as defined in Table 5.1 of Example 5.1.1 above, and take:*

$m_1 = m, m_2 = n, n_1 = w, n_2 = m.$ Then:

$$(m_1, n_1) \otimes (m_2, n_2) = (m, w) \otimes (n, m) = (m \otimes n, w \otimes m) = (w, 0) \text{ and then}$$

$$\begin{aligned}\eta_1 \times \eta_2((m_1, n_1) \otimes (m_2, n_2)) &= \eta_1 \times \eta_2(w, 0) = \min\{\eta_1(w), \eta_2(0)\} \\ &= \min\{0.8, 0.8\} = 0.8 \text{ and}\end{aligned}$$

$$\eta_1 \times \eta_1(m_1, m_2) = \min\{\eta_1(m_1), \eta_1(m_2)\} = \min\{\eta_1(m), \eta_1(n)\} = \min\{0.4, 0.4\} = 0.4$$

Theorem 6.1.1. *Let η_1 and η_2 be two fuzzy SAs of a BCL–algebra B . Then, $(\eta_1 \times \eta_2)$ is also fuzzy SAs of $B \times B$, that is to say the Cartesian product of two fuzzy SAs of B is also fuzzy SA of $B \times B$.*

Proof. Suppose the fuzzy subsets η_1 and η_2 are fuzzy SAs of B . We claim that $(\eta_1 \times \eta_2)$ is also fuzzy SA of $B \times B$, in other words we need to show that:

$$\eta_1 \times \eta_1((m_1, n_1) \otimes (m_2, n_2)) = \min\{\eta_1 \times \eta_2(m_1, n_1), (\eta_1 \times \eta_2)(m_2, n_2)\} \text{ and then}$$

$$\begin{aligned}\eta_1 \times \eta_2((m_1, n_1) \otimes (m_2, n_2)) &= \eta_1 \times \eta_2((m_1 \otimes m_2, n_1 \otimes n_2)) \\ &= \min\{\eta_1(m_1 \otimes m_2), \eta_2(n_1 \otimes n_2)\} \\ &\geq \min\{\min\{\eta_1(m_1), \eta_1(m_2)\}, \min\{\eta_2(n_1), \eta_2(n_2)\}\} \\ &= \min\{\min\{\eta_1(m_1), \eta_2(n_1)\}, \min\{\eta_1(m_2), \eta_2(n_2)\}\} \\ &= \min\{(\eta_1 \times \eta_2)(m_1, n_1), (\eta_1 \times \eta_2)(m_2, n_2)\} \quad \square\end{aligned}$$

Corollary 6.1.1. *Let $\{\eta_i : i \in I\}$ be family of fuzzy SAs of a BCL–algebra B . Then, the Cartesian product, $\prod_{i \in I} \eta_i$ is also SA of $\prod_{i \in I} B$.*

Proof. Let the fuzzy subsets $\{\eta_i : i \in I\}$ be family of fuzzy SAs of B . We claim that $\prod_{i \in I} \eta_i$ is also fuzzy SA of $\prod_{i \in I} B$, in other words we need to show that:

$$\prod_{i \in I} \eta_i((m_i, n_i) \otimes (u_i, w_i)) = \min\left\{\prod_{i \in I} \eta_i(m_i, n_i), \prod_{i \in I} \eta_i(u_i, w_i)\right\}.$$

$$\begin{aligned}\prod_{i \in I} \eta_i((m_i, n_i) \otimes (u_i, w_i)) &= \prod_{i \in I} \eta_i((m_i \otimes u_i, n_i \otimes w_i)) \\ &= \min\left\{\prod_{i \in I} \eta_i(m_i \otimes u_i), \prod_{i \in I} \eta_i(n_i \otimes w_i)\right\} \\ &= \min\left\{\min\left\{\prod_{i \in I} (m_i), \prod_{i \in I} (u_i)\right\}, \min\left\{\prod_{i \in I} (n_i), \prod_{i \in I} (w_i)\right\}\right\} \\ &= \min\left\{\prod_{i \in I} \eta_i(m_i, n_i), \prod_{i \in I} \eta_i(u_i, w_i)\right\}.\end{aligned} \quad \square$$

Remark 6.1.2. The converse of Theorem 6.3.1 may not be necessarily true, that is if $\eta_i \times \eta_2$ is fuzzy SA of BCL–algebra B , then η_1, η_2 may not be SA(s) of B .

Example 6.1.2. Let $(B; \otimes, 0)$ be a BCL–algebra defined by Table 1.1 of Example 1.1.1 and define two fuzzy subsets η_1 and η_2 of B as follows:

$$\eta_1(m) = \begin{cases} 0.9, & \text{if } m = 0, \\ 0.7, & \text{if } m = p, q, \\ 0.2, & \text{if } m = r, \end{cases} \quad \text{and} \quad \eta_2(m) = \begin{cases} 0.8, & \text{if } m = 0, \\ 0.6, & \text{if } m = p, r, \\ 0.5, & \text{if } m = q. \end{cases}$$

$$\text{Then } (\eta_1 \times \eta_2)(m, n) = \begin{cases} 0.8, & \text{if } m = 0; & n = 0, \\ 0.7, & \text{if } m = p, q; & n = 0, \\ 0.6, & \text{if } m = 0, p, q; & n = p, r, \\ 0.5, & \text{if } m = 0, p, q; & n = q, \\ 0.2, & \text{if } m = r; & n = 0, p, q, r. \end{cases}$$

Now, by following algebraic calculation, we have checked that $\eta_1 \times \eta_2$ is fuzzy SA of $\prod_{i \in I} B$. However, η_1 is not SAs of B by the following justification:

$\eta_1(p \otimes q) = \eta_1(r) = 0.2 \geq \min\{\eta_1(p), \eta_1(q)\} = \min\{0.7, 0.7\} = 0.7$ which is false showing that η_1 is not SA of B .

Proposition 6.1.1. Let η be fuzzy SA of the BCL–algebra $(B; \otimes, 0)$, then the following hold $\forall m, n, u, w \in B$:

- (1) $(\eta \times \eta)((m, n) \otimes (m, n)) = \eta(0)$, and then $(\eta \times \eta)(0) = \eta(0)$
- (2) $(\eta \times \eta)((m, m) \otimes (n, n)) = \eta(m \otimes n)$
- (3) $(\eta \times \eta)((m, n) \times (u, n)) = \eta(m \otimes u)$,
- (4) $(\eta \times \eta)((m, n) \times (m, w)) = \eta(n \otimes w)$.

Proof. Suppose η is fuzzy SA of the BCL–algebra B and $m, n, u, w \in B$. Then since

$\eta(0) \geq \eta(m), \forall m \in B$ by Lemma 4.1.1.1, the following hold:

- (1) $(\eta \times \eta)((m, n) \otimes (m, n)) = (\eta \times \eta)(m \otimes m, n \otimes n) = (\eta \times \eta)(0, 0) = \min\{\eta(0), \eta(0)\} = \eta(0)$,
and then $(\eta \times \eta)(0) = (\eta \times \eta)((m, n) \otimes (m, n)) = \eta(0)$
- (2) $(\eta \times \eta)((m, m) \otimes (n, n)) = (\eta \times \eta)(m \otimes n, m \otimes n) = \min\{\eta(m \otimes n), \eta(m \otimes n)\} = \eta(m \otimes n)$
- (3) $(\eta \times \eta)((m, n) \times (u, n)) = (\eta \times \eta)(m \otimes u, n \otimes n) = \min\{\eta(m \otimes u), \eta(n \otimes n)\}$

$$= \min\{\eta(\mathfrak{m} \otimes \mathfrak{u}), \eta(0)\} = \eta(\mathfrak{m} \otimes \mathfrak{u}),$$

$$(4) \quad (\eta \times \eta)((\mathfrak{m}, \mathfrak{n}) \times (\mathfrak{m}, \mathfrak{w})) = (\eta \times \eta)(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{n} \otimes \mathfrak{w}) = \min\{\eta(0), \eta(\mathfrak{n} \otimes \mathfrak{w})\}$$

$$= \min\{\eta(0), \eta(\mathfrak{n} \otimes \mathfrak{w})\} = \eta(\mathfrak{n} \otimes \mathfrak{w}).$$

□

Lemma 6.1.1. *Let η_1 and η_2 be fuzzy SAs of B , then $(\eta_1 \times \eta_2)(0, 0) \geq (\eta_1 \times \eta_2)(\mathfrak{m}, \mathfrak{n})$, $\forall \mathfrak{m}, \mathfrak{n} \in B$.*

Proof. Let η_1 and η_2 be fuzzy SAs B . Then

$$(\eta_1 \times \eta_2)(0, 0) = \min\{\eta_1(0), \eta_2(0)\} \geq \min\{\eta_1(\mathfrak{m}), \eta_2(\mathfrak{n})\}, \text{ since}$$

$$\eta_1(0) \geq \eta_1(\mathfrak{m}) \text{ and } \eta_2(0) \geq \eta_2(\mathfrak{n}), \forall \mathfrak{m}, \mathfrak{n} \in B, \text{ by Lemma 4.1.1.1}$$

$$\text{Hence, } (\eta_1 \times \eta_2)(0, 0) \geq (\eta_1 \times \eta_2)(\mathfrak{m}, \mathfrak{n}), \forall \mathfrak{m}, \mathfrak{n} \in B.$$

□

Corollary 6.1.2. *Let η_1 and η_2 be two fuzzy SAs of a BCL–algebra B . Then, $(\eta_1 \times \eta_2)((0, 0) \otimes (\mathfrak{m}, \mathfrak{n})) \geq (\eta_1 \times \eta_2)(\mathfrak{m}, \mathfrak{n})$.*

Proof. Suppose η_1 and η_2 are fuzzy SAs of a BCL–algebra B . Then, $\forall \mathfrak{m}, \mathfrak{n} \in B$:

$$\begin{aligned} (\eta_1 \times \eta_2)((0, 0) \otimes (\mathfrak{m}, \mathfrak{n})) &= \eta_1 \times \eta_2((0 \otimes \mathfrak{m}), (0 \otimes \mathfrak{n})) \\ &= \eta_1 \times \eta_2(0, 0), \text{ since } 0 \otimes \mathfrak{m} = 0, \forall \mathfrak{m} \in B \text{ by Proposition 2.1.1 (3)} \\ &= \min\{\eta_1(0), \eta_2(0)\} \\ &\geq \min\{\eta_1(\mathfrak{m}), \eta_2(\mathfrak{n})\}, \text{ as } \eta_1(0) \geq \eta_1(\mathfrak{m}), \eta_2(0) \geq \eta_2(\mathfrak{n}), \forall \mathfrak{m}, \mathfrak{n} \in B \\ &= (\eta_1 \times \eta_2)(\mathfrak{m}, \mathfrak{n}). \end{aligned}$$

$$\text{Therefore, } (\eta_1 \times \eta_2)((0, 0) \otimes (\mathfrak{m}, \mathfrak{n})) \geq (\eta_1 \times \eta_2)(\mathfrak{m}, \mathfrak{n}).$$

□

Corollary 6.1.3. *Let $\{\eta_i : i \in I\}$ be family of fuzzy SAs of a BCL–algebra B . Then, the following hold:*

$$(1) \quad \prod_{i \in I} \eta_i(0_i) \geq \prod_{i \in I} \eta_i(\mathfrak{m}_i), \quad \forall \mathfrak{m}_i \in B; i \in I$$

$$(2) \quad \prod_{i \in I} \eta_i((0_i) \times (\mathfrak{m}_i)) \geq \prod_{i \in I} \eta_i(\mathfrak{m}_i), \quad \forall \mathfrak{m}_i \in B; i \in I$$

Proof. (1) Let $\{\eta_i : i \in I\}$ be family of fuzzy SAs of B . Then

$$\prod_{i \in I} \eta_i(0_i) = \min \left\{ \prod_{i \in I} \eta_i(0) \right\}$$

$$\geq \min \left\{ \prod_{i \in I} \eta_i(m_i) \right\}, \text{ since } \eta_i(0) \geq \eta_i(m), \forall m_i \in B, \text{ and } \forall i \in I$$

by Lemma 4.1.1.1 and Lemma 6.1.1

$$\text{Hence, } \prod_{i \in I} \eta_i(0_i) = \min \left\{ \prod_{i \in I} \eta_i(0) \right\}, \forall m_i \in B \text{ and } \forall i \in I.$$

(2) Suppose $\{\eta_i : i \in I\}$ are fuzzy SAs of a BCL–algebra B . Then, $\forall m_i \in B$ and $\forall i \in I$:

$$\begin{aligned} \prod_{i \in I} \eta_i((0_i) \times (m_i)) &= \prod_{i \in I} \eta_i((0_i \otimes m_i)) \\ &= \prod_{i \in I} \eta_i(0), \text{ since } 0 \otimes m = 0, \forall m \in B \text{ and } \forall i \in I \text{ by Proposition 2.1.1 (3)} \\ &= \prod_{i \in I} \eta_i(0_i) \\ &\geq \prod_{i \in I} \eta_i(m_i), \forall m_i \in B \text{ and } \forall i \in I. \end{aligned}$$

□

Theorem 6.1.2. Let η_1 and η_2 be the fuzzy SAs of the BCL–algebra $(B; \otimes, 0)$. Then the Cartesian product $(\eta_1 \times \eta_2) : B \times B \rightarrow [0, 1]$ has the property: $(\eta_1 \times \eta_2)(m, n) = \eta_2 \times \eta_1(n, m)$

Proof. Suppose η_1 and η_2 are the fuzzy SAs of the BCL–algebra B . Then, $\forall m, n \in B$:

$$(\eta_1 \times \eta_2)(m, n) = \min\{\eta_1(m), \eta_2(n)\} = \min\{\eta_2(n), \eta_1(m)\} = \eta_2 \times \eta_1(n, m)$$

□

Proposition 6.1.2. Let η_1 and η_2 be two fuzzy subsets in B . If $(\eta_1 \times \eta_2)$ is a fuzzy SA of $B \times B$, then, the following hold:

- (1) Either $\eta_1(0) \geq \eta_1(m)$ or $\eta_2(0) \geq \eta_2(m)$, $\forall m \in B$.
- (2) If $\eta_1(0) \geq \eta_1(m)$, $\forall m \in B$, then either $\eta_2(0) \geq \eta_1(m)$ or $\eta_2(0) \geq \eta_2(m)$.
- (3) If $\eta_2(0) \geq \eta_2(m)$, $\forall m \in B$, then either $\eta_1(0) \geq \eta_2(m)$ or $\eta_1(0) \geq \eta_1(m)$.
- (4) Either η_1 or η_2 is fuzzy SA of B .

Proof. We prove (1), (2), and (3) by contradiction as in the following ways:

(1) Suppose that $\eta_1(0) < \eta_1(m)$ and $\eta_2(0) < \eta_2(m)$, for some $m, n \in B$, then:

$$((\eta_1 \times \eta_2))(m, n) = \min\{\eta_1(m), \eta_1(n)\} > \min\{\eta_1(0), \eta_2(0)\} = ((\eta_1 \times \eta_2))(0, 0), \text{ a contradiction}$$

to $\eta_1 \times \eta_2$ is a fuzzy SA of B . Hence $\eta_1(0) \geq \eta_1(m)$ or $\eta_2(0) \geq \eta_2(m)$, $\forall m \in B$.

(2) Suppose that $\eta_2(0) < \eta_1(m)$ and $\eta_2(0) < \eta_2(n)$, $\forall m, n \in B$, then

$$\begin{aligned} ((\eta_1 \times \eta_2))(m, n) &= \min\{\eta_1(m), \eta_2(n)\} > \min\{\eta_2(0), \eta_2(0)\} = \eta_2(0) = \min\{\eta_1(0), \eta_2(0)\} \\ &= ((\eta_1 \times \eta_2))(0, 0), \end{aligned}$$

which is a contradiction to $(\eta_1 \times \eta_2)$ is a fuzzy SA of B.

Hence either $\eta_2(0) \geq \eta_1(m)$ or $\eta_2(0) \geq \eta_2(m)$.

(3) Similar to Proof (2).

(4) By (1) if $\eta_1(0) \geq \eta_1(m)$, $\forall m \in B$ then by (2), we have the following:

$$\eta_2(0) \geq \eta_1(m) \text{ or } \eta_2(0) \geq \eta_2(m).$$

$$\text{Now } \eta_1(m) = \min\{\eta_1(m), \eta_1(m)\} = \min\{\eta_1(m), \eta_2(0)\} = ((\eta_1 \times \eta_2))(m, 0).$$

Hence, for $m, w \in B$, we have:

$$\begin{aligned} \eta_1(m+w) &= \min\{\eta_1(m+w), \eta_2(0)\} = ((\eta_1 \times \eta_2))(m+w, 0) = ((\eta_1 \times \eta_2))(m+w, 0+0) \\ &= ((\eta_1 \times \eta_2))((m, 0) + (w, 0)) = \min\{((\eta_1 \times \eta_2))(m, 0), ((\eta_1 \times \eta_2))(w, 0)\} \\ &= \min\{\min\{\eta_1(m), \eta_2(0)\}, \min\{\eta_1(w), \eta_2(0)\}\} \\ &= \min\{\eta_1(m), \eta_1(w)\}. \eta_1(m) = \min\{\eta_1(m), \eta_2(0)\} \\ &= ((\eta_1 \times \eta_2))(m, 0) \geq \min\{((\eta_1 \times \eta_2))(m, 0) - (w, 0), ((\eta_1 \times \eta_2))(w, 0)\} \\ &= \min\{((\eta_1 \times \eta_2))(m-w, 0), ((\eta_1 \times \eta_2))(w, 0)\} \\ &= \min\{\min\{\eta_1(m-w), \eta_2(0)\}, \min\{\eta_1(w), \eta_2(0)\}\} \\ &= \min\{\eta_1(m-w), \eta_1(w)\}. \end{aligned}$$

Hence η_1 is a fuzzy SA of B, and similarly, we can prove that η_2 is a fuzzy SA of B. □

6.2. Level Sets in Pythagorean Fuzzy Deductive System of BCL–Algebra

In this section, we study the notion of level sets of fuzzy DSs of BCL–algebra. Characterizations of fuzzy DSs of BCL–algebra in terms of its level subsets are also given.

Definition 6.2.1. Let $B^D = (\eta, \tau)$ be Pythagorean fuzzy DS of a BCL–algebra B. Then the set:

$$B_{(\delta, \sigma)}^D = \{m \in B : (\eta(m))^2 \geq \delta^2, (\tau(m))^2 \leq \sigma^2\} \text{ is called Pythagorean fuzzy level DS of B.}$$

Definition 6.2.2. Let $B^D = (\eta, \tau)$ be Pythagorean fuzzy DS of a BCL–algebra B such that $t, s \in [0, 1]$. Then $DS: U(\eta, t) = \{m \in B : (\eta(m))^2 \geq t^2\}$ is called an upper t -level DS of B and $DS: L(\tau, s) = \{m \in B : (\tau(m))^2 \leq s^2\}$ is called a lower s -level DS of B .

Remark 6.2.1. Let $B^D = (\eta, \tau)$ be Pythagorean fuzzy DS of BCL–algebra B such that $t, s \in [0, 1]$. If $s \leq t$ then $U(\eta, t) \subseteq U(\eta, s)$ and $L(\tau, s) \subseteq L(\tau, t)$.

(There are order–preserving correspondence between \leq and \subseteq under lower level DS of B and also order-reversing correspondence between \leq and \subseteq under upper level DS of B).

Theorem 6.2.1. If $B^D = (\eta, \tau)$ is a Pythagorean fuzzy DS of BCL–algebra B , then the sets $B_\eta = \{m \in B : (\eta(m))^2 = (\eta(0))^2\}$ and $B_\tau = \{m \in B : (\tau(m))^2 = (\tau(0))^2\}$ are DSs of BCL–algebra of B .

Proof. Assume that $B^D = (\eta, \tau)$ is a Pythagorean fuzzy DS of BCL–algebra of B , then we claim to show B_η and B_τ are DS of B (fulfilling the axioms defined in Definition 4.1.3).

Clearly, both B_η and B_τ contain the element zero.

(i) Let $m \in B_\eta, u \in B$. Then $(\eta(m))^2 = (\eta(0))^2$.

$$\text{So, } (\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2 = \min\{(\eta(0))^2, (\eta(0))^2\} = (\eta(0))^2.$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq (\eta(0))^2 \text{ but } (\eta((m \otimes u) \otimes u))^2 \leq (\eta(0))^2, \text{ as } \eta \text{ is a fuzzy subset.}$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 = (\eta(0))^2$$

$$\Rightarrow (m \otimes u) \otimes u \in B_\eta, \forall u \in B, \text{ by the hypothesis.}$$

$$\text{Similarly, } m, n \in B_\eta \Rightarrow m \otimes (n \otimes w) \in B_\eta, \forall w \in B.$$

(ii) Let $m, n \in B_\tau \Rightarrow (\tau(m))^2 = (\tau(0))^2 = (\tau(n))^2$

$$\Rightarrow (\tau(m \otimes (n \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\} = \max\{(\tau(0))^2, (\tau(0))^2\} = (\tau(0))^2.$$

$$\Rightarrow (\tau(m \otimes (n \otimes w)))^2 \leq (\tau(0))^2 \text{ but } (\tau(m \otimes (n \otimes w)))^2 \geq (\tau(0))^2, \text{ since } \tau \text{ is fuzzy subset}$$

$$\Rightarrow (\tau(m \otimes (n \otimes w)))^2 = (\tau(0))^2$$

$$\Rightarrow m \otimes (n \otimes w) \in B_\tau, \forall w \in B, \text{ by the hypothesis.}$$

$$\text{Similarly, } m \in B_\tau \Rightarrow (m \otimes u) \otimes u \in B_\tau, \forall u \in B$$

Hence, by (i) and (ii), the sets B_η and B_τ are DS of BCL–algebra of B . □

Theorem 6.2.2. A Pythagorean fuzzy set $B^D = (\eta, \tau)$ of a BCL–algebra B is a Pythagorean fuzzy DS of B if and only if the level subsets $U(\eta, t)$ and $L(\tau, s)$ are DSs of B , $\forall t \in \text{Im}(\eta)$, $\forall s \in \text{Im}(\tau)$ with $t^2 + s^2 \leq 1$, where $\text{Im}(\eta)$ and $\text{Im}(\tau)$ represent images of the level subsets η and τ , respectively.

Proof. Assume that $B^D = (\eta, \tau)$ is a Pythagorean fuzzy DS of B .

(i) Let $m, u \in B$ such that $m \in U(\eta, t)$. Then, $(\eta(m))^2 \geq t^2$. Since $B^D = (\eta, \tau)$

is a Pythagorean fuzzy DS of B , we have: $(\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2 \geq t^2$

$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq t^2$. Then we justified that $m \in U(\eta, t) \Rightarrow (m \otimes u) \otimes u \in U(\eta, t)$.

Similarly, $m, n \in U(\eta, t) \Rightarrow m \otimes (n \otimes w) \in U(\eta, t)$, $\forall w \in B$

Hence, $U(\eta, t)$ is a DS of B .

(ii) Also, let $m, u \in B$ such that $m \in L(\tau, s)$. Then, $(\tau(m))^2 \leq s^2$.

Since $B^D = (\eta, \tau)$ is a Pythagorean fuzzy DS of B , we have:

$(\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2 \leq s^2 \Rightarrow (\tau((m \otimes u) \otimes u))^2 \leq s^2$

$\Rightarrow (m \otimes u) \otimes u \in L(\tau, s)$, $\forall u \in B$.

Similarly, $m, n \in L(\tau, s) \Rightarrow m \otimes (n \otimes w) \in L(\tau, s)$, $\forall w \in B$

Hence, $L(\tau, s)$ is a DS of B .

Conversely, assume that, the level subsets $U(\eta, t)$ and $L(\tau, s)$ are DSs of B

for any $t \in \text{Im}(\eta)$ and $s \in \text{Im}(\tau)$ with $t^2 + s^2 \leq 1$.

(1) Let $m, u \in B$ and let $t \in \text{Im}(\eta)$ such that $t^2 = (\eta(m))^2$. Then, $(\eta(m))^2 \geq t^2$

$\Rightarrow m \in U(\eta, t)$. As $U(\eta, t)$ is a DS of B , it follows: $(m \otimes u) \otimes u \in U(\eta, t)$

$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq t^2 = (\eta(m))^2 \Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2$.

Also, let $m, u \in B$ and let $s \in \text{Im}(\tau)$ such that $s^2 = (\tau(m))^2$.

$\Rightarrow (\tau(m))^2 \leq s^2 \Rightarrow m \in L(\tau, s) \Rightarrow$ Since $L(\tau, s)$ is a DS of B , we have:

$(m \otimes u) \otimes u \in L(\tau, s) \Rightarrow (\tau((m \otimes u) \otimes u))^2 \leq s^2 = (\tau(m))^2$

$\Rightarrow (\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2$.

(2) Let $m, n, w \in B$, $t \in \text{Im}(\eta)$ such that $t^2 = \min\{(\eta(m))^2, (\eta(n))^2\}$

$$\Rightarrow \min\{(\eta(m))^2, (\eta(n))^2\} \geq t^2 \Rightarrow (\eta(m))^2 \geq t^2, (\eta(n))^2 \geq t^2 \Rightarrow m, n \in U(\eta, t).$$

Now, since $U(\eta, t)$ is a DS of B , we have: $m \otimes (n \otimes w) \in U(\eta, t)$

$$\Rightarrow (\eta(m \otimes (n \otimes w)))^2 \geq t^2 = (\eta(m))^2 \Rightarrow (\eta(m \otimes (n \otimes w)))^2 \geq (\eta(m))^2.$$

Again, let $m, n, w \in B$ and let $s \in \text{Im}(\tau)$ such that $s^2 = \max\{(\tau(m))^2, (\tau(n))^2\}$.

Then, we have $(\tau(m))^2 \leq s^2, (\tau(n))^2 \leq s^2 \Rightarrow m, n \in L(\tau, s)$

Also, since $U(\tau, s)$ is a DS of B , we have:

$$m \otimes (n \otimes w) \in U(\tau, s). \Rightarrow (\tau(m \otimes (n \otimes w)))^2 \leq s^2 = \leq \max\{(\tau(m))^2, (\tau(n))^2\}.$$

Then, $(\tau(m \otimes (n \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$. Hence, by (1) and (2) above,

we deduce that: $B^D = (\eta, \tau)$ is a Pythagorean fuzzy DS of B .

The following theorem shows that every DS of B can be characterized as level DS of a Pythagorean fuzzy DS of the BCL–algebra, B . □

Theorem 6.2.3. *Every DS of a BCL–algebra B is a level DS of a Pythagorean fuzzy DS $B^D = (\eta, \tau)$ of B .*

Proof. Let M be a DS of B and $B^D = (\eta, \tau)$ be a Pythagorean fuzzy DS in B defined by:

$$\eta(m) = \begin{cases} t, & \text{if } m \in M, \\ 0, & \text{if } m \notin M, \end{cases} \quad \text{and} \quad \tau(m) = \begin{cases} s, & \text{if } m \in M, \\ 1, & \text{if } m \notin M, \end{cases} \quad \forall t, s \in [0, 1], t^2 + s^2 \leq 1$$

Clearly, $U(\eta, t) = M = L(\tau, s)$. And since, $0 \in M$, we have: $(\eta(0))^2 = t^2$ and $(\tau(0))^2 = s^2$

$$\Rightarrow (\eta(0))^2 \geq (\eta(m))^2 \text{ and } (\tau(0))^2 \leq (\tau(m))^2, \forall m \in B.$$

Then we need to prove the following two pairs of statements, $\forall m, n, u, w \in B$:

$$(1) (\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2 \text{ and } (\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2$$

$$(2) (\eta(m \otimes (n \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau(m \otimes (n \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

(1) **Claim:-** $(\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2$ and $(\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2$, and by two cases:

Case(i) : Let $m \in M$, then we have: $(m \otimes u) \otimes u \in M$, since M is a DS of B .

$$\Rightarrow (\eta(m))^2 = t^2 = (\eta((m \otimes u) \otimes u))^2 \text{ and } (\tau(m))^2 = s^2 = (\tau((m \otimes u) \otimes u))^2$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2 \text{ and } (\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2$$

Case(ii) : Let $m \notin M$; however, we have either: $(m \otimes u) \otimes u \in M$ or $(m \otimes u) \otimes u \notin M$.

$$\Rightarrow (\eta(m))^2 = 0; \text{ and either } ((\eta((m \otimes u) \otimes u))^2 = t^2 \text{ or } (\eta((m \otimes u) \otimes u))^2 = 0),$$

$$\text{and also, } (\tau(m))^2 = 1; \text{ but either } ((\tau((m \otimes u) \otimes u))^2 = s^2 \text{ or } (\tau((m \otimes u) \otimes u))^2 = 1)$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq 0 = (\eta(m))^2, \text{ and } (\tau((m \otimes u) \otimes u))^2 \leq 1 = (\tau(m))^2$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2, \text{ and } (\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2$$

(2) $(\eta(m \otimes n \otimes w))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and

$$(\tau((m \otimes u) \otimes u))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\} \text{ could be proved following three cases:}$$

Case(i) : Let $m, n \in M$, then we have: $m \otimes (n \otimes w) \in M$, since M is a DS of B .

$$\Rightarrow (\eta(m))^2 = t^2 = (\eta(n))^2 = (\eta(m \otimes (n \otimes w)))^2 \text{ and } (\tau(m))^2 = s^2 = (\tau(n))^2 = (\tau(m \otimes (n \otimes w)))^2$$

$$\Rightarrow (\eta(m \otimes (n \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and } (\tau(m \otimes (n \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

Case(ii) : Let $m \in M$ and $n \notin M$ (or $n \in M$ and $m \notin M$), but we have either:

$$(m \otimes u) \otimes u \in M \text{ or } (m \otimes u) \otimes u \notin M \Rightarrow (\eta(m))^2 = t^2, (\eta(n))^2 = 0 \text{ (or } (\eta(m))^2 = 0, (\eta(n))^2 = t^2)$$

$$\text{but } (\eta((m \otimes u) \otimes u))^2 = t^2 \text{ or } (\eta((m \otimes u) \otimes u))^2 = 0, \text{ and}$$

$$(\tau(m))^2 = s^2, (\tau(n))^2 = 1 \text{ (or } (\tau(m))^2 = 1, (\tau(n))^2 = s^2)$$

$$\text{but } (\tau((m \otimes u) \otimes u))^2 = s^2 \text{ (or } (\tau((m \otimes u) \otimes u))^2 = 1)$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq 0 = \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau((m \otimes u) \otimes u))^2 \leq 1 = \max\{(\tau(m))^2, (\tau(n))^2\}$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau((m \otimes u) \otimes u))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

Case(iii) : Let $m \notin M, n \notin M$, but we have either $(m \otimes u) \otimes u \in M$ or $(m \otimes u) \otimes u \notin M$.

$$\Rightarrow (\eta(m))^2 = 0 = (\eta(n))^2 \text{ and } (\eta((m \otimes u) \otimes u))^2 = t^2 \text{ or } (\eta((m \otimes u) \otimes u))^2 = 0, \text{ and}$$

$$(\tau(m))^2 = 1 = (\tau(n))^2 \text{ and } (\tau((m \otimes u) \otimes u))^2 = s^2 \text{ or } (\tau((m \otimes u) \otimes u))^2 = 1$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq 0 = \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau((m \otimes u) \otimes u))^2 \leq 1 = \max\{(\tau(m))^2, (\tau(n))^2\}$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau((m \otimes u) \otimes u))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

Therefore, $B^D = (\eta, \tau)$ is a Pythagorean fuzzy DS in B , and hence, M is a level DS of B corresponding to Pythagorean fuzzy DS $B^D = (\eta, \tau)$ of B .

We can also prove the following theorem as a generalization of Theorem 6.2.3, above. \square

Theorem 6.2.4. *Let M be any nonempty subset of a BCL-algebra B and $B^D = (\eta_M, \tau_M)$ be Pythagorean fuzzy set in B defined by:*

$$\eta_M(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M \end{cases} \quad \text{and} \quad \tau_M(m) = \begin{cases} \alpha, & \text{if } m \in M, \\ \gamma, & \text{if } m \notin M \end{cases}$$

$$\forall \delta, \sigma, \alpha, \gamma \in [0, 1] \text{ with } \delta \geq \sigma, \alpha \leq \gamma \text{ and } \delta^2 + \alpha^2 \leq 1, \sigma^2 + \gamma^2 \leq 1.$$

Then B^D is a Pythagorean fuzzy DS of B if and only if M is DS of B .

Proof. Let B^D be a Pythagorean fuzzy DS of B then we need to prove that M is DS of B .

Then $(\eta(0))^2 \geq (\eta(m))^2 = \delta^2$ and $(\tau(0))^2 \leq (\tau(m))^2 = \alpha^2$. Hence $0 \in M$.

$$(1) m \in M \Rightarrow (\eta_M(m \otimes u) \otimes u))^2 \geq (\eta_M(m))^2 = \delta^2 \text{ and } (\tau_M(m \otimes u) \otimes u))^2 \leq (\tau_M(m))^2 = \alpha^2$$

Hence $(m \otimes u) \otimes u \in M$.

$$(2) m, n \in M \Rightarrow (\eta_M(m \otimes (n \otimes w)))^2 \geq (\eta_M(m))^2 = \delta^2 \text{ and}$$

$$(\tau_M(m \otimes (n \otimes w)))^2 \leq (\tau_M(m))^2 = \alpha^2$$

Hence, $m \otimes (n \otimes w) \in M$. Thus, by (1) and (2) above, we deduce that M is DS of B .

Conversely, let M be DS of B . So we prove: $B^D = (\eta_M, \tau_M)$ is Pythagorean fuzzy DS of B .

(a) (i) Let $m \in B \Rightarrow (m \otimes u) \otimes u \in M$ and, we have:

$$(\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2 = \delta^2 \text{ and } (\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2 = \alpha^2$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2 \text{ and } (\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2, \forall m \in M.$$

(ii) Let $m \notin B \Rightarrow (m \otimes u) \otimes u \in M$ or $(m \otimes u) \otimes u \notin M$ and, we have:

$$(\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2 = \sigma^2 \text{ and } (\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2 = \gamma^2$$

$$\Rightarrow (\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2 \text{ and } (\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2, \forall m \in M.$$

(b) To prove: $(\eta(m \otimes (n \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and

$(\tau(m \otimes (n \otimes u)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}, \forall m, n \in B$; we follow three cases:

(i) Let $m, n \in B \Rightarrow (m \otimes (n \otimes w)) \in M$ and, we have:

$$(\eta(m \otimes (n \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} = \delta^2 \text{ and}$$

$$(\tau(m \otimes (n \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\} = \gamma^2$$

(ii) Let $m \in B, n \notin B$ (or $m \notin B, n \in B$) then, we have:

$$(\eta(m \otimes (n \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} = \min\{\delta^2, \sigma^2\} = \sigma^2 \text{ and}$$

$$(\tau(m \otimes u) \otimes u)^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\} = \max\{\alpha^2, \gamma^2\} = \gamma^2$$

(iii) Let $m \notin B, n \notin B$ then, we have:

$$(\eta(m \otimes (n \otimes w)))^2 \geq \sigma^2 = \min\{(\eta(m))^2, (\eta(n))^2\} = \min\{\sigma^2, \sigma^2\} \text{ and}$$

$$(\tau(m \otimes u) \otimes u)^2 \leq \gamma^2 = \max\{(\tau(m))^2, (\tau(n))^2\} = \max\{\gamma^2, \gamma^2\}$$

Thus, in all the three cases (i), (ii) and (iii) above, we deduce that:

$$(\eta((m \otimes (n \otimes w))))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau((m \otimes u) \otimes u))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}, \forall m, n \in B.$$

Thus, by cases (a) and (b) above, we deduce that $B^D = (\eta_M, \tau_M)$ is Pythagorean fuzzy DS of B. \square

Theorem 6.2.5. Let $\{M_i : i = 0, 1, 2, \dots, n\}$ be any of a BCL-algebra B such that $M_0 \subset M_1 \subset M_2 \subset \dots \subset M_n = B$, then there exists a Pythagorean fuzzy DS $B^D = (\eta, \tau)$ of B whose level DSs of B are exactly the DSs $\{M_i\}$ of B.

Proof. Consider the following set of numbers:

$$\delta_0 > \sigma_1 > \delta_2 > \dots > \delta_{n-1} > \delta_n \text{ and } \sigma_0 < \sigma_1 < \sigma_2 < \dots < \sigma_{n-1} < \delta_n, \text{ where each } \delta_i, \sigma_i \in [0, 1]$$

$$\text{with } (\delta_i)^2 + (\sigma_i)^2 \leq 1.$$

Now, suppose $B^D = (\eta, \tau)$ is Pythagorean fuzzy set defined by:

$$\eta(m) = \begin{cases} \delta_0, & \text{if } m \in M_0, \\ \delta_i, & \text{if } m \in M_i - M_{i-1}, \end{cases} \quad \text{and } \tau(m) = \begin{cases} \sigma_0, & \text{if } m \in M_0, \\ \sigma_i, & \text{if } m \in M_i - M_{i-1}, \end{cases} \quad 0 < i \leq n.$$

Clearly, $0 \in M_i$, for $i = 0, 1, 2, \dots, n$, since each M_i is a DS of B.

Then, $(\eta(0))^2 = (\delta_0)^2 = (\eta(m))^2$ and $(\tau(0))^2 = (\sigma_0)^2 = (\tau(m))^2$, if $m \in M_0$, and

$$(\eta(0))^2 = (\delta_i)^2 = (\eta(m))^2 \text{ and } (\tau(0))^2 = (\delta_i)^2 = (\tau(m))^2, \text{ if } m \in M_i - M_{i-1}.$$

In any case, $(\eta(0))^2 \geq (\eta(m))^2$ and $(\tau(0))^2 \leq (\tau(m))^2, \forall m \in B$.

Claims: (1) $(\eta((m \otimes u) \otimes u))^2 \geq (\eta(m))^2$ and $(\tau((m \otimes u) \otimes u))^2 \leq (\tau(m))^2,$

(2) $(\eta(m \otimes (n \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and

$$(\tau(m \otimes (n \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

Letting $\mathbb{W}_n = \{0, 1, 2, \dots, n\}$, we prove (1) by taking two cases and (2) by taking three cases:

(1) **Case(i)**: Let $m \in M_i - M_{i-1}$, for $i \in \mathbb{W}_n$. Then $(\eta(m))^2 = (\delta_i)^2$ and $(\tau(m))^2 = (\sigma_i)^2$

As M_i is DS of B , it follows that: $(m \otimes u) \otimes u \in M_i$. So, either $(m \otimes u) \otimes u \in M_i - M_{i-1}$ or

$(m \otimes u) \otimes u \in M_{i-1}$. In any case, we can deduce that:

$$(\eta((m \otimes u) \otimes u))^2 \geq (\delta_i)^2 = \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau((m \otimes u) \otimes u))^2 \leq (\sigma_i)^2 = \max\{(\tau(m))^2, (\tau(n))^2\}$$

Case(ii): Let $m, n \in M_i - M_{i-1}$, for $i \in \mathbb{W}_n$. Then

$$(\eta(m))^2 = (\delta_i)^2 = (\eta(n))^2 \text{ and } (\tau(m))^2 = (\sigma_i)^2 = (\tau(n))^2.$$

Then since M_i is DS of B , it follows that: $m \otimes (n \otimes w) \in M_i$.

So, either $m \otimes (n \otimes w) \in M_i - M_{i-1}$ or $m \otimes (n \otimes w) \in M_{i-1}$.

In any case, we can deduce that:

$$(\eta(m \otimes (n \otimes w)))^2 \geq (\delta_i)^2 = \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$(\tau(m \otimes (n \otimes w)))^2 \leq (\sigma_i)^2 = \max\{(\tau(m))^2, (\tau(n))^2\}$$

Case(iii): Let $m \in M_i - M_{i-1}$ and $n \in M_j - M_{j-1}$, for $i, j \in \mathbb{W}_n$ and $i > j$

Then $(\eta(m))^2 = (\delta_i)^2, (\eta(n))^2 = (\delta_j)^2$ and $(\tau(m))^2 = (\sigma_i)^2, (\tau(n))^2 = (\sigma_j)^2$.

AS M_i is DS of B : $m \otimes (n \otimes w) \in M_i$ but $m \otimes (n \otimes w) \in M_i - M_{i-1}$ or $m \otimes (n \otimes w) \in M_{i-1}$

Then, we can deduce that: $(\eta(m \otimes (n \otimes w)))^2 \geq (\delta_i)^2 = \min\{(\eta(m))^2, (\eta(n))^2\}$ and

$$(\tau(m \otimes (n \otimes w)))^2 \leq (\sigma_i)^2 = \max\{(\tau(m))^2, (\tau(n))^2\} \Rightarrow m \in M_i \text{ and } n \in M_j.$$

Then since $M_j \subseteq M_i$, we have $n \in M_i$ and hence $m \otimes (n \otimes w) \in M_i$ as M_i is DS of B .

Thus, $(\eta(m \otimes (n \otimes w)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and

$$(\tau(m \otimes (n \otimes w)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}$$

Then, $B^D = (\eta, \tau)$ is Pythagorean fuzzy DS of B . Again, from the definition of

$B^D = (\eta, \tau)$, we have: $\text{Im}(\eta) = \{\delta_0, \delta_1, \delta_2, \dots, \delta_n\}$ and $\text{Im}(\tau) = \{\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_n\}$.

Then the level DS of $B^D = (\eta, \tau)$ are given by the chain of DSs:

$$U(\eta, \delta_0) \subset U(\eta, \delta_1) \subset U(\eta, \delta_2) \subset \dots \subset U(\eta, \delta_n) = B$$

and $L(\tau, \sigma_0) \subset L(\tau, \sigma_1) \subset L(\tau, \sigma_2) \subset \dots \subset L(\tau, \sigma_n) = B$. Now

$$U(\eta, \delta_0) = \{m \in B : (\eta(m))^2 \geq (\delta_0)^2\} = M_0 = \{m \in B : (\tau(m))^2 \leq (\sigma_0)^2\} = L(\tau, \sigma_0).$$

Finally, we need to prove that: $U(\eta, \delta_i) = M_i = L(\tau, \sigma_i)$, for $0 < i \leq n$.

$$\text{Let } m \in M_i \Rightarrow (\eta(m))^2 = (\delta_i)^2 \text{ and } (\tau(m))^2 = (\sigma_i)^2$$

$$\Rightarrow m \in U(\eta, \delta_i) \text{ and } m \in L(\tau, \sigma_i)$$

$$\Rightarrow M_i \subseteq U(\eta, \delta_i) \text{ and } M_i \subseteq L(\tau, \sigma_i)$$

If $m \in U(\eta, \delta_i)$ and $m \in L(\tau, \sigma_i)$ then, $(\eta(m))^2 \geq (\delta_i)^2$ and $(\tau(m))^2 \leq (\sigma_i)^2$

$\Rightarrow m \notin M_j$, for $j > i$. Otherwise, $m \in M_j$, for $j > i \Rightarrow (\eta(m))^2 \geq (\delta_j)^2$ and $(\tau(m))^2 \leq (\sigma_j)^2$,

$\Rightarrow (\delta_i)^2 \geq (\eta(m))^2 \geq (\delta_j)^2$ and $(\delta_i)^2 \leq (\tau(m))^2 \leq (\sigma_j)^2$, which is a contradiction to the

assumption that: $m \in U(\eta, \delta_i)$ and $m \in L(\tau, \sigma_i)$. Hence, $\eta(m) \in \{\delta_0, \delta_1, \delta_2, \dots, \delta_n\}$ and

$\tau(m) \in \{\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_n\}$. Thus, $m \in M_k$, for some $k \leq i$. As $M_k \subseteq M_i \Rightarrow m \in M_i$

$\Rightarrow U(\eta, \delta_i) \subseteq M_i$ and $L(\tau, \sigma_i) \subseteq M_i$. Thus, $U(\eta, \delta_i) = M_i = L(\tau, \sigma_i)$

□

Remark 6.2.2. *If B is a finite BCL–algebra, the number of DSs of B is finite. Similarly, the number of level DSs, related to a fuzzy DS M also appears finite. However, since every level DS of M is inherently a DS of B , not all such level DSs are necessarily distinct. The conditions governing this relationship are detailed in the following theorem.*

Theorem 6.2.6. *Let $B^D = (\eta, \tau)$ be a Pythagorean fuzzy DS of B , then:*

(a) *The upper level DSs $U(\eta, \delta_1)$ and $U(\eta, \delta_2)$ with $\delta_1 < \delta_2$ of a Pythagorean fuzzy DS M of B are equal if there is no $m \in B$ such that $(\delta_1)^2 \leq (\eta(m))^2 < (\delta_2)^2$.*

(b) *The lower level DSs $L(\tau, \sigma_1)$ and $L(\tau, \sigma_2)$ with $\sigma_1 > \sigma_2$ of a Pythagorean fuzzy DS M of B are equal if there is no $m \in B$ such that $(\sigma_1)^2 \geq (\tau(m))^2 > (\sigma_2)^2$.*

Proof. (a) Let $B^D = (\eta, \tau)$ be Pythagorean fuzzy DS of B , $U(\eta, \delta_1) = U(\eta, \delta_2)$, for some $\delta_1 < \delta_2$.

(i) Suppose there exist $m \in B$ such that $(\delta_1)^2 \leq (\eta(m))^2 < (\delta_2)^2$.

This implies $m \in (\eta, \delta_1)$ but $m \notin U(\eta, \delta_2)$, which is a contradiction to the assumption.

$U(\eta, \delta_1) = U(\eta, \delta_2)$. Hence there is no $m \in B$ such that $(\delta_1)^2 \leq (\eta(m))^2 < (\delta_2)^2$.

Conversely, suppose there is no $m \in B$ such that $\delta_1 \leq \eta(m) < \delta_2$. Since $\delta_1 < \delta_2$, we get:

$$U(\eta, \delta_2) \subseteq (\eta, \delta_1)$$

(ii) Now, $m \in (\eta, \delta_1) \Rightarrow (\eta(m))^2 \geq (\delta_1)^2$. So $(\eta(m))^2 \geq (\delta_2)^2$, because $(\eta(m))^2$ does not lie between $(\delta_1)^2$ and $(\delta_2)^2$. Hence, $m \in U(\eta, \delta_2)$ and thus, $(\eta, \delta_1) \subseteq U(\eta, \delta_2)$. Thus, by (i) and (ii) above, $U(\eta, \delta_1) = U(\eta, \delta_2)$

(b) The proof of (b) is similar to (a) above.

(iii) Assume that there exist $m \in B$ such that $(\sigma_1)^2 \geq (\tau(m))^2 > (\delta_2)^2$. By definition, $m \in L(\tau, \sigma_1)$ as $(\sigma_1)^2 \geq (\tau(m))^2$. But as $(\tau(m))^2 > (\sigma_2)^2$, $m \notin L(\tau, \sigma_2)$

which is a contradiction to $L(\tau, \sigma_1) = L(\tau, \sigma_2)$, so that no such m exists.

Conversely, assume there is no $m \in B$ such that $(\sigma_1)^2 \geq (\tau(m))^2 > (\sigma_2)^2$.

Then we show, $L(\tau, \sigma_1) = L(\tau, \sigma_2)$ As $\sigma_1 > \sigma_2$, we get, $L(\tau, \sigma_1) \subseteq L(\tau, \sigma_2)$

(iv) Again, let $m \in L(\tau, \sigma_2) \Rightarrow (\tau(m))^2 \leq (\sigma_2)^2$, and as $\sigma_2 < \sigma_1$ and no m satisfies

$(\sigma_1)^2 \geq (\tau(m))^2 > (\sigma_2)^2$, we get that, $(\tau(m))^2 \leq (\sigma_2)^2 \leq (\sigma_1)^2$. Thus,

$m \in L(\tau, \sigma_1) \Rightarrow L(\tau, \sigma_2) \subseteq L(\tau, \sigma_1)$ Thus, by (iii) and (iv), $L(\tau, \sigma_2) = L(\tau, \sigma_1)$.

□

Theorem 6.2.7. Let $B^D = (\eta, \tau)$ be a Pythagorean fuzzy DS of B with finite images. If $U(\eta, \delta_1) = U(\eta, \delta_2)$ and $L(\tau, \sigma_1) = L(\tau, \sigma_2)$, for any $\delta_1, \delta_2 \in \text{Im}(\eta)$ and $\sigma_1, \sigma_2 \in \text{Im}(\tau)$, then $\delta_1 = \delta_2$ and $\sigma_1 = \sigma_2$.

Proof. Let $B^D = (\eta, \tau)$ be a Pythagorean fuzzy DS of B with finite images such that $U(\eta, \delta_1) = U(\eta, \delta_2)$ and $L(\tau, \sigma_1) = L(\tau, \sigma_2)$, for some $\delta_1, \delta_2 \in \text{Im}(\eta)$ and $\sigma_1, \sigma_2 \in \text{Im}(\tau)$.

So we need to prove that $\delta_1 = \delta_2$ and $\sigma_1 = \sigma_2$ and so assume that $\delta_1 \neq \delta_2$ and $\sigma_1 \neq \sigma_2$.

Without loss of generality, assume that $\delta_1 < \delta_2$ and $\sigma_1 > \sigma_2$.

Let $m \in U(\eta, \delta_2)$, then $(\eta(m))^2 \geq (\delta_2)^2 > (\delta_1)^2 \Rightarrow (\eta(m))^2 > (\delta_1)^2$.

Hence, $m \in U(\eta, \delta_1)$, and so let $m \in B$ such that $(\delta_1)^2 < (\eta(m))^2 < (\delta_2)^2$.

Then, $m \in U(\eta, \delta_1)$ but $m \notin U(\eta, \delta_2) \Rightarrow U(\eta, \delta_2) \subset U(\eta, \delta_1) \Rightarrow U(\eta, \delta_1) \neq U(\eta, \delta_2)$

which contradicts the hypo dissertation that: $U(\eta, \delta_1) = U(\eta, \delta_2)$. Therefore, $\delta_1 = \delta_2$. □

Theorem 6.2.8. Let $B^D = (\eta, \tau)$ be a Pythagorean fuzzy DS of B and let $m \in B$. Then:

(i) $(\eta(m))^2 = (\delta_1)^2$ if and only if $m \in U(\eta, \delta_1)$ but $m \notin U(\eta, \delta_2)$, $\forall \delta_2 > \delta_1$.

(ii) $(\tau(m))^2 = (\sigma_1)^2$ if and only if $m \in L(\tau, \sigma_1)$ but $m \notin L(\tau, \sigma_2)$, $\forall \sigma_2 < \sigma_1$.

Proof. Let $B^D = (\eta, \tau)$ be a Pythagorean fuzzy DS of B and let $m \in B$.

(i) Assume $(\eta)^2(m) = (\delta_1)^2$ so that $m \in U(\eta, \delta_1)$ and let $m \in U(\eta, \delta_2)$, for $\delta_2 > \delta_1$.

Then $(\eta(m))^2 \geq (\delta_2)^2 > (\delta_1)^2$ which contradicts the fact that $(\eta(m))^2 = (\delta_1)^2$.

Hence $m \in U(\eta, \delta_1)$ but $m \notin U(\eta, \delta_2)$, $\forall \delta_2 > \delta_1$.

Conversely, let $m \in U(\eta, \delta_1)$ but $m \notin U(\eta, \delta_2)$, $\forall \delta_2 > \delta_1$, then $m \in U(\eta, \delta_1)$

$\Rightarrow (\eta(m))^2 \geq (\delta_1)^2$, since $m \notin U(\eta, \delta_2)$, $\forall \delta_2 > \delta_1$, we have $(\eta(m))^2 = (\delta_1)^2$.

(ii) Assume $(\tau(m))^2 = (\sigma_1)^2$ so that $m \in L(\tau, \sigma_1)$. Let $m \in L(\tau, \sigma_2)$ for $\sigma_2 < \sigma_1$.

Then $(\tau(m))^2 \leq (\sigma_2)^2 < (\sigma_1)^2$, which contradicts the fact that $(\tau(m))^2 = (\sigma_1)^2$.

Hence $m \in L(\tau, \sigma_1)$ but $m \notin L(\tau, \sigma_2)$, $\forall \sigma_2 < \sigma_1$.

Conversely, let $m \in L(\tau, \sigma_1)$ but $m \notin L(\tau, \sigma_2)$, $\forall \sigma_2 < \sigma_1$, then $m \in L(\tau, \sigma_1)$

$\Rightarrow (\tau(m))^2 \leq (\sigma_1)^2$, since $m \notin L(\tau, \sigma_2)$, $\forall \sigma_2 < \sigma_1$. □

6.3. Homomorphism of Pythagorean Fuzzy Deductive System of Liu^B-algebra.

Under this section, we explore how Pythagorean fuzzy DSs in LBA behave under homomorphisms. It introduces fuzzy subset transformations through algebraic homomorphisms and establishes preservation theorems, including direct, inverse, and epimorphic mappings, highlighting their role in maintaining deductive structure across related algebras.

Definition 6.3.1. Let $(L; \otimes_L, \odot_L, 0_L)$ and $(M; \otimes_M, \odot_M, 0_M)$ be Liu^B-algebras. Let $f: L \rightarrow M$ be a homomorphism on LBA. Then for any fuzzy subset η in M , we define a new fuzzy subset η^f in L by: $(\eta^f(m))^2 = (\eta(f(m)))^2$, $\forall m \in L$.

Definition 6.3.2. A Pythagorean fuzzy set L^B in a non-empty set L with the degrees of membership $\eta : L \rightarrow [0,1]$ and non-membership $\tau : L \rightarrow [0,1]$ is said to have the sup–inf property, if for any non-empty subset $M \subseteq L$, $\exists m_0 \in M$ such that: $(\eta(m_0))^2 = \sup_{t \in M} (\eta(t))^2$ and $(\tau(m_0))^2 = \inf_{t \in M} (\tau(t))^2$.

Definition 6.3.3. Let $f : L \rightarrow M$ be a homomorphism of LBAs. Then for any Pythagorean fuzzy set $L^B = (\eta, \tau)$ in M , we define a Pythagorean fuzzy set $L^f = (\eta^f, \tau^f)$ in L by:

$$(\eta^f(m))^2 = (\eta(f(m)))^2 \quad \text{and} \quad (\tau^f(m))^2 = (\tau(f(m)))^2, \quad \forall m \in L.$$

Lemma 6.3.1. Let $f : L \rightarrow M$ be a homomorphic mapping of LBAs. Then $f(0) = 0$.

Proof. Let L be LBA and $m \in L$ (as $L \neq \emptyset$). Then:

$$f(0) = f(m \odot m) = f(m) \odot f(m) = 0, \quad \text{as } f \text{ is homomorphic mapping of LBAs.} \quad \square$$

Theorem 6.3.1. Let $f : L \rightarrow M$ be an epimorphism of LBA. If $L^B = (\eta, \tau)$ is a Pythagorean fuzzy DS of L with the sup–inf property, then the image $f(L^B)$ of L^B is a Pythagorean fuzzy DS of M .

Proof. Suppose $L^B = (\eta, \tau)$ is a Pythagorean fuzzy DS of L with the sup – inf property.

Let $m, n \in M$ with $m_0 \in f^{-1}(m)$, $n_0 \in f^{-1}(n)$ such that:

$$(\eta(m_0))^2 = \sup_{t \in f^{-1}(m)} (\eta(t))^2, \quad (\tau(m_0))^2 = \inf_{t \in f^{-1}(m)} (\tau(t))^2$$

(i) For the first axiom of definition of Pythagorean fuzzy DS of L , we have:

$$(\eta^f(m \otimes (n \otimes u)))^2 = \sup_{t \in f^{-1}(m \otimes (n \otimes u))} (\eta(t))^2 \geq (\eta^f(m))^2, \quad \text{and again,}$$

$$(\tau^f(m \otimes (n \otimes u)))^2 = \inf_{t \in f^{-1}(m \otimes (n \otimes u))} (\tau(t))^2 \leq \{(\tau^f(m))^2\}.$$

(ii) For the second condition of Definition 4.2.2.1, we have:

$$(\eta^f(m \odot w \odot w))^2 = \sup_{t \in f^{-1}(m \odot w \odot w)} (\eta(t))^2 \geq \min\{(\eta^f(m))^2, (\eta^f w)^2\},$$

$$\Rightarrow (\eta^f(m \odot w \odot w))^2 \geq \min\{(\eta^f(m))^2, (\eta^f w)^2\}, \quad \text{and similarly,}$$

$$(\tau^f(m \odot w \odot w))^2 = \inf_{t \in f^{-1}(m \odot w \odot w)} (\tau(t))^2 \leq \max\{(\tau^f(m))^2, (\tau^f w)^2\}.$$

$$(\tau^f(m \odot w \odot w))^2 \leq \max\{(\tau^f(m))^2, (\tau^f w)^2\}.$$

Hence, $f(L^B)$ is a Pythagorean fuzzy DS of M . □

Theorem 6.3.2. *Let $f: L \rightarrow M$ be a homomorphism of LBA. If the Pythagorean fuzzy set $L^B = (\eta, \tau)$ is a Pythagorean fuzzy DS of M , then the Pythagorean fuzzy set $L^f = (\eta^f, \tau^f)$ in L is a Pythagorean fuzzy DS of L .*

Proof. Let f be a homomorphism of LBA and $L^B = (\eta, \tau)$ be a Pythagorean fuzzy DS of M . By Definition 6.3.1 and Lemma 6.3.1, since $f(0) = 0$, we have:

(i) For all $m, n, u, w \in L$, since f is a homomorphism,

$$\begin{aligned} (\eta^f(m \otimes (n \otimes u)))^2 &= (\eta(f(m \otimes (n \otimes u))))^2 \\ &= (\eta(f(m) \otimes (f(n) \otimes f(u))))^2 \\ &\geq \min\{(\eta(f(m)))^2, (\eta(f(n)))^2\}, \text{ which gives} \end{aligned}$$

$$(\eta^f(m \otimes (n \otimes u)))^2 \geq \min\{(\eta^f(m))^2, (\eta^f(n))^2\}. \text{ and}$$

$$\begin{aligned} (\tau^f(m \otimes (n \otimes u)))^2 &= (\tau(f(m \otimes (n \otimes u))))^2 \\ &= (\tau(f(m) \otimes (f(n) \otimes f(u))))^2 \\ &\leq \max\{(\tau(f(m)))^2, (\tau(f(n)))^2\}, \text{ which gives} \end{aligned}$$

$$(\tau^f(m \otimes (n \otimes u)))^2 \geq \min\{(\tau^f(m))^2, (\tau^f(n))^2\}.$$

$$\begin{aligned} \text{(ii) } (\eta^f(m \odot w \odot w))^2 &= (\eta(f(m \odot w \odot w)))^2 \\ &= (\eta(f(m) \odot (f(w) \odot f(w))))^2 \\ &\geq (\eta(f(m)))^2 \text{ which gives} \end{aligned}$$

$$(\eta^f(m \odot w \odot w))^2 = (\eta(f(m \odot w \odot w)))^2 \geq (\eta(f(m)))^2 \text{ and}$$

$$\begin{aligned} (\tau^f(m \odot w \odot w))^2 &= (\tau(f(m \odot w \odot w)))^2 \\ &= (\tau(f(m) \otimes (f(w) \otimes f(w))))^2 \\ &\leq m(\tau(f(m)))^2, \end{aligned}$$

$$\Rightarrow (\tau^f(m \odot w \odot w))^2 \leq (\tau(f(m)))^2,$$

Hence, by (i) and (ii), L^f is a Pythagorean fuzzy DS of L . □

Remark 6.3.1. *The Converse of Theorem 6.3.2 above may not be true as justified by Example 6.3.1*

Example 6.3.1. Let $L = \{0, p\}$, $M = \{0, p, q\}$, and define a map

$$f: L \rightarrow M \text{ by } f(k) = \begin{cases} 0, & \text{if } k=0, \\ a, & \text{if } k=p. \end{cases}$$

We equip L and M with binary operations \otimes and \odot as follows (all other entries omitted are determined by the displayed rules or by using the element 0 as the designated zero):

\otimes_L	0	p
0	0	0
p	p	p

\otimes_M	0	a	b
0	0	0	0
a	a	a	a
b	a	a	a

Table 6.1: Tables of two binary operations \otimes_L and \odot_M

Now, $\forall m, n \in L$, $f(u \otimes_L v) = f(u) \otimes_M f(v)$ by the tables above, Like (as a caese):

$$(i) f(p \otimes_L 0) = f(p \otimes_L p) = f(p) = a = f(p) \otimes_M f(0) = a \otimes_M 0 = a = f(p) \otimes_M f(p) = a \otimes_M a$$

$$\text{and } f(0 \otimes_L 0) = f(0 \otimes_L p) = f(0) = 0 = f(0) \otimes_M f(0) = 0 \otimes_M 0 = f(0) \otimes_M f(p) = 0 \otimes_M a = 0$$

(And similarly for \odot).

Also, f is not surjective (its image is $\{0, a\}$).

Define η and τ on M as follows:

$$\eta(k) = \begin{cases} 1, & \text{if } k=0, \\ 0.9, & \text{if } k=b, \\ 0.7, & \text{if } k=a, \end{cases} \quad \text{and } \tau(k) = \begin{cases} 0, & \text{if } k=0, \\ 0.5, & \text{if } k=a, \\ 0.3, & \text{if } k=b. \end{cases}$$

Each pair satisfies $0 \leq \eta(p)^2 + \tau(p)^2 \leq 1$:

$$1 + 0 \leq 1, \quad 0.49 + 0.25 = 0.74 \leq 1, \quad 0.81 + 0.09 = 0.90 \leq 1.$$

By Definition 6.3.2 the induced Pythagorean fuzzy set $L^f = (\eta^f, \tau^f)$ on L is:

$(\eta^f(m))^2 = (\eta(f(m)))^2$, $(\tau^f(m))^2 = (\tau(f(m)))^2$, and define η^f and τ^f on L as follows:

$$\eta^f(k) = \begin{cases} 1, & \text{if } k=0, \\ 0.7, & \text{if } k=p, \end{cases} \quad \text{and } \tau^f(k) = \begin{cases} 0, & \text{if } k=0, \\ 0.5, & \text{if } k=p, \end{cases}$$

(A) L^f is a Pythagorean fuzzy DS of L .

We verify the two Pythagorean fuzzy DS axioms (Definition 5.3.2.1) on L .

(i) For arbitrary $m, n, u \in L$, $m = n = u = p$, we have: $p \otimes_L (p \otimes_L p) = p \otimes_L p = p$.

So, $(\eta^f(p \otimes_L (p \otimes_L p)))^2 = \eta^f(p)^2 = 0.49 \geq \min\{0.49, 0.49\}$.

Likewise for τ^f we have: $\tau^f(p \otimes_L (p \otimes_L p))^2 = 0.25 \leq \max\{0.25, 0.25\}$.

All other choices involving 0 are: $\eta^f(0) = 1$ and $\tau^f(0) = 0$ give the required inequalities.

(ii) Similar results hold for \odot_L .

Therefore, L^f is a Pythagorean fuzzy DS of L .

$L^B = (\eta, \tau)$ is not a Pythagorean fuzzy DS of M .

We check Pythagorean fuzzy DS axiom (i) in M and find a direct failure.

Take $m = n = u = b \in M$. Using the table for \otimes_M ,

$$b \otimes_M (b \otimes_M b) = b \otimes_M a = a.$$

$$\text{Therefore, } (\eta(b \otimes_M (b \otimes_M b)))^2 = (\eta(a))^2 = 0.49,$$

$$\text{while } \min\{(\eta(b))^2, (\tau(b))^2\} = \min\{0.81, 0.81\} = 0.81.$$

Thus, $0.49 \not\geq 0.81$, and then Pythagorean fuzzy DS axiom (i) fails for $m = n = u = b$.

Hence L^B is not a Pythagorean fuzzy DS of M .

Thus, L^f is a Pythagorean fuzzy DS of L does not imply that L^f is a Pythagorean fuzzy DS of L .

Theorem 6.3.3. Let $f: L \rightarrow M$ be a homomorphism of LBAs. If $L^B = (\eta, \tau)$ is a Pythagorean fuzzy DS of M , then $f^{-1}(L^B) = (\eta^{f^{-1}}, \tau^{f^{-1}})$ defined by:

$$(\eta^{f^{-1}}(m))^2 = (\eta(f(m)))^2, (\tau^{f^{-1}}(m))^2 = (\tau(f(m)))^2, \forall m \in L, \text{ is a Pythagorean fuzzy DS of } L.$$

Proof. We verify the two Pythagorean fuzzy DS axioms (i) and (ii) of Definition 5.3.2.1 for $(\eta^{f^{-1}}, \tau^{f^{-1}})$.

(i) Let $m, n, u \in L$. Using that f is a homomorphism,

$$f(m \otimes (n \otimes u)) = f(m) \otimes (f(n) \otimes f(u)).$$

$$\Rightarrow (\eta^{f^{-1}}(m \otimes (n \otimes u)))^2 = (\eta(f(m \otimes (n \otimes u))))^2 = (\eta(f(m) \otimes (f(n) \otimes f(u))))^2.$$

Since, (η, τ) is a Pythagorean fuzzy DS of M , applying Pythagorean fuzzy DS condition (i) in M :

$$(\eta(f(m) \otimes (f(n) \otimes f(u))))^2 \geq \min\{(\eta(f(m)))^2, (\eta(f(n)))^2\}.$$

But $(\eta(f(m)))^2 = (\eta^{f^{-1}}(m))^2$ and likewise for n , hence

$$(\eta^{f^{-1}}(m \otimes (n \otimes u)))^2 \geq \min\{(\eta^{f^{-1}}(m))^2, (\eta^{f^{-1}}(n))^2\}.$$

The analogous chain of equalities/inequalities for τ (using the Pythagorean fuzzy DS upper-bound for τ in M) gives: $(\tau^{f^{-1}}(m \circledast (n \circledast u)))^2 \leq \max\{(\tau^{f^{-1}}(m))^2, (\tau^{f^{-1}}(n))^2\}$.

(ii) For arbitrary $m, w \in L$, $f(m \odot w \odot w) = f(m) \odot f(w) \odot f(w)$, so

$$(\eta^{f^{-1}}(m \odot w \odot w))^2 = (\eta(f(m) \odot f(w) \odot f(w)))^2.$$

By Pythagorean fuzzy DS condition (ii) in M ,

$$(\eta(f(m) \odot f(w) \odot f(w)))^2 \geq (\eta(f(m)))^2 = (\eta^{f^{-1}}(m))^2, \text{ and similarly}$$

$$(\tau^{f^{-1}}(m \odot w \odot w))^2 \leq (\tau^{f^{-1}}(m))^2.$$

Thus both Pythagorean fuzzy DS conditions hold for $(\eta^{f^{-1}}, \tau^{f^{-1}})$, so $f^{-1}(L^B)$ is a Pythagorean fuzzy DS of L . □

Corollary 6.3.1. *Let $f : L \rightarrow M$ be an epimorphism of LBAs. Then $L^f = (\eta^f, \tau^f)$ is a Pythagorean fuzzy DS of L if and only if $L^B = (\eta, \tau)$ is a Pythagorean fuzzy DS of M , where (η^f, τ^f) is given by: $(\eta^f(x))^2 = (\eta(f(x)))^2$, $(\tau^f(x))^2 = (\tau(f(x)))^2$.*

Proof. Suppose L^f is a Pythagorean fuzzy DS of L and f is an epimorphism (surjective).

Take arbitrary $m, n, u \in M$; choose $a, b, c \in L$ with $f(a) = m, f(b) = n, f(c) = u$.

Then, by Pythagorean fuzzy DS (i) in L for L^f , $(\eta^f(a \circledast (b \circledast c)))^2 \geq \min\{(\eta^f(a))^2, (\eta^f(b))^2\}$.

Replace η^f by $\eta \circ f$ (Definition 6.3.2) to obtain: $(\eta(m \circledast (n \circledast u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$,

and similarly for τ . Also using Lemma 6.3.1 and Pythagorean fuzzy DS (ii) in L yields

the corresponding zero/double- w inequalities in M . Hence L^B is a Pythagorean fuzzy DS of M .

Conversely, assume L^B is a Pythagorean fuzzy DS of M . Define η^f, τ^f on L by composing with f as above. Then for any $x, y, z \in L$ we have, by homomorphism property,

$f(x \circledast (y \circledast z)) = f(x) \circledast (f(y) \circledast f(z))$, and applying Pythagorean fuzzy DS (i) in M gives

$$(\eta^f(x \circledast (y \circledast z)))^2 = (\eta(f(x \circledast (y \circledast z))))^2 \geq \min\{(\eta(f(x)))^2, (\eta(f(y)))^2\} \text{ which is}$$

$$(\eta^f(x \circledast (y \circledast z)))^2 \geq \min\{(\eta^f(x))^2, (\eta^f(y))^2\}.$$

Analogous reasoning for (ii) and for τ^f shows L^f is a Pythagorean fuzzy DS of L . □

Theorem 6.3.4. *Let $f: L_1 \rightarrow L_2$ be an epimorphism of LBAs. If $B_1^P = (\eta_1, \tau_1)$ is a Pythagorean fuzzy DS in L_1 having the sup–inf property, then the image $B_1^P = (\eta_2, \tau_2)$ defined for $n \in L_2$ by: $(\eta_2(n))^2 = \sup\{(\eta_1(m))^2 : f(m) = n\}$, $(\tau_2(n))^2 = \inf\{(\tau_1(m))^2 : f(m) = n\}$, is a Pythagorean fuzzy DS in L_2 .*

Proof. We need check Pythagorean fuzzy DS axioms (i) and (ii) for (η_2, τ_2) .

(i) Let $n_1, n_2, u \in L_2$. For each $n \in \{n_1, n_2, u\}$ pick (since f is surjective) nonempty sets

$F_n = \{m \in L_1 : f(m) = n\}$. By the sup–inf property $\exists m_1 \in F_{n_1}, m_2 \in F_{n_2}, m_u \in F_u$ achieving the sup–inf values for η_2 and τ_2 .

Consider; $m_1 \otimes (m_2 \otimes m_u) \in L_1$. Because f is a homomorphism,

$$f(m_1 \otimes (m_2 \otimes m_u)) = n_1 \otimes (n_2 \otimes u).$$

Applying Pythagorean fuzzy DS axiom (i) in L_1 to (η_1, τ_1) gives

$$(\eta_1(m_1 \otimes (m_2 \otimes m_u)))^2 \geq \min\{(\eta_1(m_1))^2, (\eta_1(m_2))^2\}.$$

Taking suprema over all preimages on the left-hand side (and using that chosen m_1, m_2 attain the suprema) yields $(\eta_2(n_1 \otimes (n_2 \otimes u)))^2 \geq \min\{(\eta_2(n_1))^2, (\eta_2(n_2))^2\}$.

The τ_2 inequality follows similarly using infima and Pythagorean fuzzy DS (i) for τ_1 .

(ii) For $n \in L_2$ and any $w \in L_2$, choose $m \in L_1$ and $t \in L_1$ with $f(m) = n$, $f(t) = w$

and assume the sup–inf are realized at m (for η_2) and at some t (for τ_2). In L_1 Pythagorean fuzzy DS (ii) gives: $(\eta_1(m \odot t \odot t))^2 \geq (\eta_1(m))^2$.

Applying f and passing to sup/inf over fibres yields

$$(\eta_2(n \odot w \odot w))^2 \geq (\eta_2(n))^2, \quad (\tau_2(n \odot w \odot w))^2 \leq (\tau_2(n))^2.$$

Thus (η_2, τ_2) satisfies both Pythagorean fuzzy DS conditions in L_2 . □

Theorem 6.3.5. *Let L_1 and L_2 be Liu^B–algebras and let $f: L_1 \rightarrow L_2$ be a monomorphism of LBAs. If $B_2^P = (\eta_2, \tau_2)$ is a Pythagorean fuzzy DS in L_2 , then the preimage $f^{-1}(B_2^P) = (\eta_1, \tau_1)$ with $(\eta_1(m))^2 = (\eta_2(f(m)))^2$, $(\tau_1(m))^2 = (\tau_2(f(m)))^2$, $\forall m \in L_1$, is a Pythagorean fuzzy DS in L_1 .*

Proof. We must verify the two Pythagorean fuzzy DS conditions (i) and (ii) from Definition 5.3.2.1

for the pair (η_1, τ_1) on L_1 . By hypothesis we define, for every $m \in L_1$,

$$(\eta_1(m))^2 = (\eta_2(f(m)))^2, \quad (\tau_1(m))^2 = (\tau_2(f(m)))^2.$$

These equalities will be used repeatedly to translate statements about η_1, τ_1 on L_1 to statements about η_2, τ_2 on L_2 .

Take arbitrary elements $m, n, u \in L_1$. We must prove

$$(\eta_1(m \otimes (n \otimes u)))^2 \geq \min\{(\eta_1(m))^2, (\eta_1(n))^2\}, \quad \text{and}$$

$$(\tau_1(m \otimes (n \otimes u)))^2 \leq \max\{(\tau_1(m))^2, (\tau_1(n))^2\}.$$

$$(\eta_1(m \otimes (n \otimes u)))^2 = (\eta_2(f(m \otimes (n \otimes u))))^2 \quad \text{by, definition of } \eta_1.$$

Then we use the homomorphism property of f :

$$f(m \otimes (n \otimes u)) = f(m) \otimes (f(n) \otimes f(u)), \quad \text{and hence,}$$

$$(\eta_1(m \otimes (n \otimes u)))^2 = (\eta_2(f(m) \otimes (f(n) \otimes f(u))))^2.$$

Apply Pythagorean fuzzy DS condition (i) in L_2 – Because (η_2, τ_2) is a Pythagorean fuzzy DS in L_2 , Definition 5.3.2.1(i) gives:

$$(\eta_2(f(m) \otimes (f(n) \otimes f(u))))^2 \geq \min\{(\eta_2(f(m)))^2, (\eta_2(f(n)))^2\}.$$

Returning to η_1 notation – $(\eta_2(f(m)))^2 = (\eta_1(m))^2$ and $(\eta_2(f(n)))^2 = (\eta_1(n))^2$. Thus

$$(\eta_1(m \otimes (n \otimes u)))^2 \geq \min\{(\eta_1(m))^2, (\eta_1(n))^2\},$$

which is exactly the required η -inequality of Pythagorean fuzzy DS (i) on L_1 , and again

$$(\tau_1(m \otimes (n \otimes u)))^2 = (\tau_2(f(m) \otimes (f(n) \otimes f(u))))^2 \leq \max\{(\tau_2(f(m)))^2,$$

$$(\tau_2(f(n)))^2\} = \max\{(\tau_1(m))^2, (\tau_1(n))^2\},$$

Thus Pythagorean fuzzy DS condition (i) holds for (η_1, τ_1) .

Verifying Pythagorean fuzzy DS condition (ii) on L_1) and then take arbitrary $m, w \in L_1$.

$$\text{We must show: } (\eta_1(m \odot w \odot w))^2 \geq (\eta_1(m))^2, \quad (\tau_1(m \odot w \odot w))^2 \leq (\tau_1(m))^2.$$

$$\text{Translating the left-hand side for } \eta_1, \text{ we obtain: } (\eta_1(m \odot w \odot w))^2 = (\eta_2(f(m \odot w \odot w)))^2.$$

Using homomorphism property: $f(m \odot w \odot w) = f(m) \odot f(w) \odot f(w)$, so that

$$(\eta_1(m \odot w \odot w))^2 = (\eta_2(f(m) \odot f(w) \odot f(w)))^2.$$

Applying Pythagorean fuzzy DS condition (ii) in L_2 – we have:

$$(\eta_2(f(m) \odot f(w) \odot f(w)))^2 \geq (\eta_2(f(m)))^2.$$

Returning to η_1 notatio: $(\eta_2(f(m)))^2 = (\eta_1(m))^2 \Rightarrow (\eta_1(m \odot w \odot w))^2 \geq (\eta_1(m))^2$,

which is the required η -inequality of Pythagorean fuzzy DS (ii) on L_1 .

Again: $(\tau_1(m \odot w \odot w))^2 = (\tau_2(f(m) \odot f(w) \odot f(w)))^2 \leq (\tau_2(f(m)))^2 = (\tau_1(m))^2$,

and also by Pythagorean fuzzy DS (ii) in L_2 .

Both Pythagorean fuzzy DS conditions (i) and (ii) hold for (η_1, τ_1) on L_1 .

Hence $f^{-1}(B_2^P) = (\eta_1, \tau_1)$ is a Pythagorean fuzzy DS of L_1 .

□

Theorem 6.3.6. *Let $f: L_1 \rightarrow L_2$ be an isomorphism of LBAs. Define $B_2^P = (\eta_2, \tau_2)$ from $B_1^P = (\eta_1, \tau_1)$ by $(\eta_2(f(m)))^2 = (\eta_1(m))^2$, $(\tau_2(f(m)))^2 = (\tau_1(m))^2$, $(\forall m \in L_1)$. Equivalently, for each $n \in L_2$ (writing $m = f^{-1}(n)$), $(\eta_2(n))^2 = (\eta_1(f^{-1}(n)))^2$, $(\tau_2(n))^2 = (\tau_1(f^{-1}(n)))^2$. Then B_1^P is a Pythagorean fuzzy DS in L_1 if and only if B_2^P is a Pythagorean fuzzy DS in L_2 .*

Proof. We prove both directions. Throughout we use Definition 5.3.2.1 for PFDS:

(i) For all m, n, u in an LBA: $(\eta(m \otimes (n \otimes u)))^2 \geq \min\{(\eta(m))^2, (\eta(n))^2\}$ and

$$(\tau(m \otimes (n \otimes u)))^2 \leq \max\{(\tau(m))^2, (\tau(n))^2\}.$$

(ii) $\forall m, w$: $(\eta(m \odot w \odot w))^2 \geq (\eta(m))^2$ and $(\tau(m \odot w \odot w))^2 \leq (\tau(m))^2$.

(A) Assume B_1^P is a Pythagorean fuzzy DS in L_1 . Show B_2^P is a Pythagorean fuzzy DS in L_2 .

Since f is bijective, for all $n \in L_2$ the element $m = f^{-1}(n) \in L_1$ is unique, so the assignments:

$$(\eta_2(n))^2 = (\eta_1(f^{-1}(n)))^2, \quad (\tau_2(n))^2 = (\tau_1(f^{-1}(n)))^2 \quad \text{are well defined.}$$

(a) Pythagorean fuzzy DS condition (i) in L_2 . Take arbitrary $n_1, n_2, u \in L_2$.

Put $m_1 = f^{-1}(n_1)$, $m_2 = f^{-1}(n_2)$, $w = f^{-1}(u)$ (uniquely defined since f is bijective).

$$\text{Then } f(m_1 \otimes (m_2 \otimes w)) = f(m_1) \otimes (f(m_2) \otimes f(w)) = n_1 \otimes (n_2 \otimes u),$$

where we used that f is a homomorphism. Now computing:

$$(\eta_2(n_1 \otimes (n_2 \otimes u)))^2 = (\eta_1(f^{-1}(n_1 \otimes (n_2 \otimes u))))^2 \quad (\text{definition of } \eta_2)$$

$$\begin{aligned}
&= (\eta_1(m_1 \otimes (m_2 \otimes w)))^2 \quad (\text{since } f^{-1}(n_i) = m_i, f^{-1}(u) = w) \\
&\geq \min\{(\eta_1(m_1))^2, (\eta_1(m_2))^2\} \quad (\text{i) for } B_1^P \text{ in } L_1) \\
&= \min\{(\eta_2(n_1))^2, (\eta_2(n_2))^2\}, \quad \text{where the last equality uses } (\eta_2(n_i))^2 = (\eta_1(m_i))^2.
\end{aligned}$$

Thus the η -inequality of Pythagorean fuzzy DS axiom (i) holds in L_2 .

Similarly, for τ :

$$\begin{aligned}
(\tau_2(n_1 \otimes (n_2 \otimes u)))^2 &= (\tau_1(m_1 \otimes (m_2 \otimes w)))^2 \leq \max\{(\tau_1(m_1))^2, (\tau_1(m_2))^2\} \\
&= \max\{(\tau_2(n_1))^2, (\tau_2(n_2))^2\}.
\end{aligned}$$

Hence Pythagorean fuzzy DS condition (i) holds for B_2^P in L_2 .

(b) To show Pythagorean fuzzy DS axiom (ii) in L_2 .

Take arbitrary $n \in L_2$ and $v \in L_2$, put $m = f^{-1}(n)$ and $t = f^{-1}(v)$. Then

$$f(m \odot t \odot t) = f(m) \odot f(t) \odot f(t) = n \odot v \odot v.$$

$$\Rightarrow (\eta_2(n \odot v \odot v))^2 = (\eta_1(f^{-1}(n \odot v \odot v)))^2 = (\eta_1(m \odot t \odot t))^2 \geq (\eta_1(m))^2 = (\eta_2(n))^2.$$

$$\text{And similarly, } (\tau_2(n \odot v \odot v))^2 = (\tau_1(m \odot t \odot t))^2 \leq (\tau_1(m))^2 = (\tau_2(n))^2.$$

Hence Pythagorean fuzzy DS axiom (ii) holds for B_2^P in L_2 .

From Steps (a) and (b), we conclude B_2^P is a Pythagorean fuzzy DS in L_2 .

(B) Conversely, assume B_2^P is a Pythagorean fuzzy DS in L_2 .

We need to show B_1^P is a Pythagorean fuzzy DS in L_1

Since f is an isomorphism, $f^{-1} : L_2 \rightarrow L_1$ exists and is a homomorphism.

We may therefore transport B_2^P back to L_1 via f^{-1} . Concretely $\forall x \in L_1$,

$$(\eta_1(x))^2 = (\eta_2(f(x)))^2, \quad (\tau_1(x))^2 = (\tau_2(f(x)))^2, \quad \text{by the defining relation of the transported pair.}$$

The argument below mirrors Part (A) with roles of L_1 and L_2 swapped.

(i) Pythagorean fuzzy DS axiom (i) in L_1 .

Take arbitrary $m_1, m_2, u \in L_1$. Put $n_i = f(m_i)$ and $v = f(u)$. Then

$$f(m_1 \otimes (m_2 \otimes u)) = n_1 \otimes (n_2 \otimes v).$$

Using Pythagorean fuzzy DS (i) in L_2 for B_2^P ,

$$(\eta_2(n_1 \otimes (n_2 \otimes v)))^2 \geq \min\{(\eta_2(n_1))^2, (\eta_2(n_2))^2\}.$$

Translating back via $(\eta_1(x))^2 = (\eta_2(f(x)))^2$ gives

$$(\eta_1(x_1 \otimes (x_2 \otimes u)))^2 \geq \min\{(\eta_1(x_1))^2, (\eta_1(x_2))^2\},$$

and the analogous τ -inequality holds. Thus Pythagorean fuzzy DS(i) holds for B_1^P .

(ii) Pythagorean fuzzy DS(ii) in L_1 . Take arbitrary $m, w \in L_1$ and put $n = f(m), v = f(w)$.

$$\text{Then } f(m \odot w \odot w) = n \odot v \odot v,$$

$$\text{and Pythagorean fuzzy DS(ii) in } L_2 \text{ gives } (\eta_2(n \odot v \odot v))^2 \geq (\eta_2(n))^2.$$

Translate back to L_1 using $(\eta_1(m))^2 = (\eta_2(f(m)))^2$ to obtain

$$(\eta_1(m \odot w \odot w))^2 \geq (\eta_1(m))^2, \text{ and similarly for } \tau.$$

Hence Pythagorean fuzzy DS axiom (ii) holds for B_1^P .

Combining Steps B1 and B2 we conclude B_1^P is a Pythagorean fuzzy DS in L_1 . □

6.4. Pythagorean (β, θ) -Fuzzy Ideal of Liu^B-Algebra

Under this section, we discuss the notion of (β, θ) -fuzzy ideals on fuzzy Liu^B-algebra under which we demonstrate that the (β, θ) -fuzzy ideals structure of two such fuzzy ideals is itself a fuzzy ideal of Liu^B-algebra, and we investigate several other related results.

Definition 6.4.1. Let $L^B = (\eta, \tau)$ be the Pythagorean fuzzy set in the non-empty set L . Then:

$L^B = (\eta, \tau)$ is called Pythagorean (β, θ) -fuzzy ideal of the LBA L if the following

two pairs of statements are satisfied; $\forall m, n, w \in L$ and $0 \leq \beta < \theta \leq 1$:

$$(i) \max\{(\eta(m))^2, \beta\} \geq \min\{\min\{(\eta(m \otimes n))^2, (\eta(n))^2\}, \theta\} \text{ and}$$

$$\min\{(\tau(m))^2, \theta\} \leq \max\{\max\{(\tau(m \otimes n))^2, (\tau(n))^2\}, \beta\}$$

$$(ii) \max\{(\eta((m \otimes n) \odot (n \otimes w)))^2, \beta\} \geq \min\{\min\{(\eta(m))^2, (\eta(n))^2\}, \theta\} \text{ and}$$

$$\min\{(\tau((m \otimes n) \odot (n \otimes w)))^2, \theta\} \leq \max\{\max\{(\tau(m))^2, (\tau(n))^2\}, \beta\}.$$

Remark 6.4.1. Let $(L, \otimes, \odot, 0)$ be LBA and $0 \leq \beta < \theta \leq 1$, then the following hold, $\forall m, n \in L$:

$$(1) \min\{\min\{m, n\}, \theta\} = \min\{m, n, \theta\}$$

$$(2) \max\{\max\{m, n\}, \beta\} = \max\{m, n, \beta\}$$

(3) Taking into account Remarks (1) and (2) above, Definition 6.4.1 can be redefined as:

$$(i) \max\{(\eta(m))^2, \beta\} \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2, \theta\} \text{ and} \\ \min\{(\tau(m))^2, \theta\} \leq \max\{(\tau(m \otimes n))^2, (\tau(n))^2, \beta\}$$

$$(ii) \max\{(\eta((m \otimes n) \odot (n \otimes w)))^2, \beta\} \geq \min\{(\eta(m))^2, (\eta(n))^2, \theta\} \text{ and} \\ \min\{(\tau((m \otimes n) \odot (n \otimes w)))^2, \theta\} \leq \max\{(\tau(m))^2, (\tau(n))^2, \beta\}.$$

Example 6.4.1. Suppose $L = \{0, m, n\}$ and two binary operations \otimes and \odot on L are as given by Table 5.4 of Example 5.1.10 above and the Pythagorean fuzzy set $L^B = (\eta, \tau)$ be defined as:

$$\eta(k) = \begin{cases} 0.9, & \text{if } k = 0, \\ 0.4, & \text{if } k = m, n, \end{cases} \quad \text{and} \quad \tau(k) = \begin{cases} 0.3, & \text{if } k = 0, \\ 0.7, & \text{if } k = m, n. \end{cases}$$

Now, take $\beta = 0.1$ and $\theta = 0.2$, where $w = m$; so that:

$$(i) \max\{(\eta(m))^2, \beta\} = \max\{(\eta(m))^2, 0.1\} = \max\{0.16, 0.1\} = 0.16 \text{ and}$$

$$\min\{(\eta(m \otimes n))^2, (\eta(n))^2, 0.2\} = \min\{(\eta(0))^2, (\eta(n))^2, 0.2\}$$

$$= \min\{0.81, 0.16, 0.2\} = 0.16, \text{ and thus}$$

$$\max\{(\eta(m))^2, 0.1\} = 0.16 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2, 0.2\} = 0.16 \text{ is true, and}$$

$$\min\{(\tau(m))^2, \theta\} = \min\{(\tau(m))^2, 0.2\} = \min\{0.49, 0.2\} = 0.2$$

$$\leq \max\{(\tau(m \otimes n))^2, (\tau(n))^2, \beta\} = \max\{(\tau(0))^2, (\tau(n))^2, 0.1\}$$

$$= \max\{0.09, 0.49, 0.1\} = 0.49, \text{ is true.}$$

$$(ii) \max\{(\eta((m \otimes n) \odot (n \otimes w)))^2, \beta\} = \max\{(\eta(0 \odot n))^2, 0.1\} = \min\{(\eta(0))^2, 0.1\}$$

$$= \max\{0.81, 0.1\} = 0.81 \text{ and}$$

$$\min\{(\eta(m))^2, (\eta(n))^2, \theta\} = \min\{0.16, 0.16, 0.2\} = 0.16, \text{ and thus}$$

$$\max\{(\eta((m \otimes n) \odot (n \otimes w)))^2, \beta\} = 0.81 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2, \theta\} = 0.16. \text{ and}$$

$$\min\{(\tau((m \otimes n) \odot (n \otimes w)))^2, \theta\} = \min\{(\tau(0 \odot n))^2, 0.2\} = \min\{(\tau(0))^2, 0.2\}$$

$$= \min\{0.09, 0.2\} = 0.09 \text{ and}$$

$$\max\{(\tau(m))^2, (\tau(n))^2, \beta\} = \min\{0.49, 0.49, 0.1\} = 0.1 \text{ and thus}$$

$$\min\{(\tau(m \otimes n) \odot (n \otimes w))^2, \theta\} = 0.09 \leq \max\{(\tau(m \otimes n))^2, (\tau(n))^2, \beta\} = 0.1.$$

Therefore, $L^B = (\eta, \tau)$ a Pythagorean (β, θ) -fuzzy ideal of L , for particular β and θ where $\beta = 0.1$ and $\theta = 0.2$.

Similarly, for any $0 \leq \beta < \theta \leq 1$, $\max\{(\eta(m))^2, \beta\} \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2, \theta\}$;
 $\min\{(\tau(m))^2, \theta\} \geq \max\{(\tau(m \otimes n))^2, (\tau(n))^2, \beta\}$ and therefore,

L^B is Pythagorean (β, θ) -fuzzy ideal of L , for these particular η and τ of L .

Lemma 6.4.1. If $L^B = (\eta, \tau)$ is a Pythagorean (β, θ) -fuzzy ideal of L , then $\forall m \in B$; and for any β, θ such that $0 \leq \beta < \theta \leq 1$, $\max\{(\eta(0))^2, \beta\} \geq \min\{(\eta(m))^2, \theta\}$ and

$$\min\{(\tau(0))^2, \theta\} \leq \max\{(\tau(m))^2, \beta\}.$$

Proof. From Lemma 5.3.3.1 above we have: $(\eta(0))^2 \geq (\eta(m))^2$ and $(\tau(0))^2 \leq (\tau(m))^2, \forall m \in L$,

Now, let $0 \leq \beta < \theta \leq 1$, and then

$$\max\{(\eta(0))^2, \beta\} \geq (\eta(0))^2 \geq (\eta(m))^2 \geq \min\{(\eta(m))^2, \theta\}, \forall \beta, \theta \in [0, 1] \text{ such that } \beta \leq \theta$$

$$\Rightarrow \max\{(\eta(0))^2, \beta\} \geq \min\{(\eta(m))^2, \theta\}, \text{ for } 0 \leq \beta < \theta \leq 1, \forall m \in L \text{ and again}$$

$$\min\{(\tau(0))^2, \theta\} \leq \tau(0)^2 \leq \tau(m)^2, \forall \beta, \theta \in [0, 1] \text{ such that } 0 \leq \beta < \theta \leq 1$$

$$\leq \max\{(\tau(m))^2, \beta\}, \text{ since } \tau(0)^2 \leq \tau(m)^2, \forall m \in L$$

$$\Rightarrow \min\{(\tau(0))^2, \theta\} \leq \max\{(\tau(m))^2, \beta\}, \text{ for } 0 \leq \beta < \theta \leq 1, \forall m \in L$$

Thus, $\max\{(\eta(0))^2, \beta\} \geq \min\{(\eta(m))^2, \theta\}$ and

$$\min\{(\tau(0))^2, \theta\} \leq \max\{(\tau(m))^2, \beta\}, \text{ for } 0 \leq \beta < \theta \leq 1, \forall m \in L.$$

□

Proposition 6.4.1. Let M be an ideal of L and $L^B = (\chi_M, \bar{\chi}_M)$, where χ_M is the Characteristic function and $\bar{\chi}_M$ is its complement, then $L^B = (\chi_M, \bar{\chi}_M)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

Proof. Suppose $0 \leq \beta < \theta \leq 1$ and $\chi_M : M \rightarrow \{0, 1\}$ is a Characteristic function defined as:

$$\chi_M(m) = \begin{cases} 1, & \text{if } m \in M, \\ 0, & \text{if } m \notin M. \end{cases} \quad \text{and then} \quad \bar{\chi}_M(m) = \begin{cases} 0, & \text{if } m \in M, \\ 1, & \text{if } m \notin M. \end{cases}$$

$$\Rightarrow \max\{(\chi_M(m))^2, \beta\} = \begin{cases} \max\{1, \beta\}, & \text{if } m \in M, \\ \max\{0, \beta\}, & \text{if } m \notin M. \end{cases} \quad \text{and}$$

$$\min\{(\bar{\chi}_M(m))^2, \theta\} = \begin{cases} \min\{0, \theta\}, & \text{if } m \in M, \\ \min\{1, \theta\}, & \text{if } m \notin M, \end{cases}$$

$$\Rightarrow \max\{(\chi_M(m))^2, \beta\} = \begin{cases} 1, & \text{if } m \in M, \\ \beta, & \text{if } m \notin M. \end{cases} \quad \text{and} \quad \min\{(\bar{\chi}_M(m))^2, \theta\} = \begin{cases} 0, & \text{if } m \in M, \\ \theta, & \text{if } m \notin M, \end{cases}$$

$$\Rightarrow \max\{(\chi_M(m))^2, \beta\} = 1, \text{ and } \min\{(\bar{\chi}_M(m))^2, \theta\} = 0, \forall m \in M$$

$$\Rightarrow \max\{(\chi_M(m))^2, \beta\} = 1 \geq \min\{(\chi_M(m))^2, \theta\}; \text{ since } \theta \leq 1; (\chi_M(m))^2 \leq 1, \forall m \in M$$

$$\geq \min\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2, \theta\}; \text{ as } (\chi_M(0))^2 \geq (\chi_M(m))^2$$

$$\min\{(\bar{\chi}_M(m))^2, \theta\} = 0 \leq \max\{(\bar{\chi}_M(m))^2, \beta\}; \text{ since } \beta \geq 0; (\bar{\chi}_M(m))^2 \geq 0, \forall m \in M$$

$$\leq \max\{(\bar{\chi}_M(m \otimes n))^2, (\bar{\chi}_M(n))^2, \beta\}; \text{ as } (\bar{\chi}_M(0))^2 \leq (\bar{\chi}_M(m))^2$$

$$\Rightarrow \max\{(\chi_M(m))^2, \beta\} \geq \max\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2, \theta\} \text{ and}$$

$$\min\{(\bar{\chi}_M(m))^2, \theta\} \leq \max\{(\bar{\chi}_M(m \otimes n))^2, (\bar{\chi}_M(n))^2, \beta\}.$$

Following similar steps as the above, it can be shown that:

$$\max\{(\chi_M(m \otimes n) \odot (n \otimes w))^2, \beta\} \geq \max\{(\chi_M(m \otimes n))^2, (\chi_M(n))^2, \theta\} \text{ and}$$

$$\min\{(\bar{\chi}_M((m \otimes n) \odot (n \otimes w)))^2, \theta\} \leq \max\{(\bar{\chi}_M(m \otimes n))^2, (\bar{\chi}_M(n))^2, \beta\}.$$

Thus, $L^B = (\chi_M, \bar{\chi}_M)$ is a Pythagorean (β, θ) -fuzzy ideal of L . □

Theorem 6.4.1. *Every Pythagorean fuzzy ideal $L^B = (\eta, \tau)$ of Liu^B-algebra L is also a Pythagorean (β, θ) -fuzzy ideal of L .*

Proof. Let $L^B = (\eta, \tau)$ be any Pythagorean fuzzy ideal of LBA L . Then we have:

$$(i) \quad (\eta(m))^2 \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2\}; \{(\tau(m))^2\} \leq \max\{(\tau(m \otimes n))^2, (\tau(n))^2\}$$

$$(ii) \quad \{(\eta((m \otimes n) \odot (n \otimes w)))^2\} \geq \min\{(\eta(m))^2, (\eta(n))^2\} \text{ and}$$

$$\{(\tau((m \otimes n) \odot (n \otimes w)))^2\} \leq \max\{(\tau(m))^2, (\tau(n))^2\}.$$

Now, we claim that:

$$(i) \quad \max\{(\eta(m))^2, \beta\} \geq \min\{(\eta(m \otimes n))^2, (\eta(n))^2, \theta\} \text{ and}$$

$$\min\{(\tau(\mathfrak{m}))^2, \theta\} \leq \max\{(\tau(\mathfrak{m} \circledast \mathfrak{n}))^2, (\tau(\mathfrak{n}))^2, \beta\}$$

$$(ii) \max\{(\eta(\mathfrak{m}))^2, \beta\} \geq \min\{(\eta(\mathfrak{m} \odot \mathfrak{n}))^2, (\eta(\mathfrak{n} \odot \mathfrak{m}))^2, (\eta(\mathfrak{n}))^2, \theta\} \text{ and}$$

$$\min\{(\tau(\mathfrak{m}))^2, \theta\} \leq \max\{(\tau(\mathfrak{m} \odot \mathfrak{n}))^2, (\tau(\mathfrak{n} \odot \mathfrak{m}))^2, (\tau(\mathfrak{n}))^2, \beta\}.$$

Now, the proof proceeds as follows, for every β , and θ such that $0 \leq \beta < \theta \leq 1$:

$$(i) \max\{(\eta(\mathfrak{m}))^2, \beta\} \geq (\eta(\mathfrak{m}))^2 \geq \min\{(\eta(\mathfrak{m} \circledast \mathfrak{n}))^2, (\eta(\mathfrak{n}))^2\}$$

$$\geq \min\{(\eta(\mathfrak{m} \circledast \mathfrak{n}))^2, (\eta(\mathfrak{n}))^2, \theta\}$$

$$\text{Thus } \max\{(\eta(\mathfrak{m}))^2, \beta\} \geq \min\{(\eta(\mathfrak{m} \circledast \mathfrak{n}))^2, (\eta(\mathfrak{n}))^2, \theta\} \text{ and}$$

$$\min\{(\tau(\mathfrak{m}))^2, \theta\} \leq \{(\tau(\mathfrak{m}))^2\} \leq \max\{(\tau(\mathfrak{m} \circledast \mathfrak{n}))^2, (\tau(\mathfrak{n}))^2\}$$

$$\leq \max\{(\tau(\mathfrak{m} \circledast \mathfrak{n}))^2, (\tau(\mathfrak{n}))^2, \beta\}$$

$$\text{Thus } \min\{(\tau(\mathfrak{m}))^2, \theta\} \leq \max\{(\tau(\mathfrak{m} \circledast \mathfrak{n}))^2, (\tau(\mathfrak{n}))^2, \beta\}$$

$$(ii) \max\{(\eta((\mathfrak{m} \circledast \mathfrak{n}) \odot (\mathfrak{m} \circledast \mathfrak{n})))^2, \beta\} \geq \{(\eta((\mathfrak{m} \circledast \mathfrak{n}) \odot (\mathfrak{m} \circledast \mathfrak{n})))^2\}$$

$$\geq \min\{(\eta(\mathfrak{m}))^2, (\eta(\mathfrak{n}))^2\}$$

$$\geq \min\{(\eta(\mathfrak{m}))^2, (\eta(\mathfrak{n}))^2, \theta\}$$

$$\text{Thus, } \max\{(\eta((\mathfrak{m} \circledast \mathfrak{n}) \odot (\mathfrak{n} \circledast \mathfrak{w})))^2, \beta\} \geq \min\{(\eta(\mathfrak{m}))^2, (\eta(\mathfrak{n}))^2, \theta\} \text{ and}$$

$$\min\{(\tau((\mathfrak{m} \circledast \mathfrak{n}) \odot (\mathfrak{m} \circledast \mathfrak{n})))^2, \theta\} \leq \{(\tau((\mathfrak{m} \circledast \mathfrak{n}) \odot (\mathfrak{m} \circledast \mathfrak{n})))^2\}$$

$$\leq \max\{(\tau(\mathfrak{m}))^2, (\eta(\mathfrak{n}))^2\}$$

$$\leq \max\{(\tau(\mathfrak{m}))^2, (\eta(\mathfrak{n}))^2, \beta\}$$

$$\text{Thus, } \min\{(\tau((\mathfrak{m} \circledast \mathfrak{n}) \odot (\mathfrak{m} \circledast \mathfrak{n})))^2, \theta\} \leq \max\{(\tau(\mathfrak{m}))^2, (\tau(\mathfrak{n}))^2, \beta\}$$

Then $L_1^{\mathbb{B}} = (\eta, \tau)$ is Pythagorean (β, θ) -fuzzy ideal of LBA L .

□

Remark 6.4.2. *The converse of Theorem 6.4.1 is not true which we ratify it by the following counter example as illustrated by the next counter (Example below 6.4.2).*

Example 6.4.2. Here we give example for, "A Pythagorean (β, θ) -fuzzy ideal of LBA L may not be Pythagorean fuzzy ideal of L , for some β, θ such that $0 \leq \beta < \theta \leq 1$."

Let the LBA $(L, \otimes, \odot, 0)$ be as defined in Tables 5.7 of Example 5.2.3.2 above and define membership and non-membership functions $\eta : L \rightarrow [0, 1]$ and $\tau : L \rightarrow [0, 1]$, respectively, as follows:

$$\eta(k) = \begin{cases} 0.6, & \text{if } k = 0, \\ 0.4, & \text{if } k = m, n, \\ 0.3, & \text{if } k = w, \end{cases} \quad \text{and} \quad \tau(k) = \begin{cases} 0.2, & \text{if } k = 0, \\ 0.5, & \text{if } k = m, n, \\ 0.9, & \text{if } k = w. \end{cases}$$

Now, we show that (η, τ) is not Pythagorean fuzzy ideal of L , as from Table 5.7, since the following:

$$\begin{aligned} w \otimes n = n; \text{ and } (\eta(w))^2 &= (0.3)^2 = 0.09 \geq \min\{(\eta(w \otimes n))^2, (\eta(n))^2\} \\ &= \min\{\eta(n), \eta(n)\} = \min\{(0.4)^2, (0.4)^2\} = 0.16 \end{aligned}$$

which is false so that (η, τ) is not Pythagorean fuzzy ideal of L , as shown by this particular case, above.

However, taking $0.8 = \beta < \theta = 0.9$, then we have:

$$\begin{aligned} \max\{(\eta(w))^2, \beta\} &= \max\{0.09, 0.8\} = 0.8 \geq \min\{(\eta(w \otimes n))^2, (\eta(n))^2, \theta\} \\ &= \min\{(\eta(m))^2, (\eta(n))^2, \theta\} \\ &= \min\{0.16, 0.16, 0.9\} = 0.16, \text{ which is true.} \end{aligned}$$

Furthermore:

$$\begin{aligned} \min\{\tau(w)^2, \theta\} &= \min\{0.09, 0.9\} = 0.09 \leq \max\{(\tau(w \otimes n))^2, (\tau(n))^2, \beta\} \\ &= \max\{(\tau(m))^2, (\tau(n))^2, \beta\} \\ &= \max\{0.16, 0.16, 0.8\} = 0.8 \text{ which is also true,} \end{aligned}$$

Similarly, the second axiom is true, too, as the first axiom is justified as above.

Therefore, for $0.8 = \beta < \theta \leq 0.9$ (η, τ) defined above is Pythagorean (β, θ) -fuzzy ideal of L but not Pythagorean fuzzy ideal of L .

Of course, the above Example holds for any β, θ , where $0 \leq \beta < \theta \leq 1$.

Theorem 6.4.2. The intersection of any two Pythagorean (β, θ) -fuzzy ideals of LBA L is also a Pythagorean (β, θ) -fuzzy ideal of L .

Proof. Let $L_1^B = (\eta_1, \tau_1)$ and $L_2^B = (\eta_2, \tau_2)$ be any two Pythagorean (β, θ) -fuzzy ideals of L .

As proved in Theorem 5.3.3.7 above, we have shown that $L_1^B = (\eta_1, \tau_1)$ and $L_2^B = (\eta_2, \tau_2)$

of L are Pythagorean fuzzy ideals of LBA L implies their intersection $L_{\cap}^B = (\eta_1 \cap \eta_2, \tau_1 \cap \tau_2)$

is a Pythagorean fuzzy ideal of L .

Now, we need to prove that $L_{\cap}^B = (\eta_1 \cap \eta_2, \tau_1 \cap \tau_2)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

Then for any $m, n, w \in L$ and any β, θ where $0 \leq \beta < \theta \leq 1$; we have the following:

$$(i) \max\{((\eta_1 \cap \eta_2)(m))^2, \beta\} \geq \{((\eta_1 \cap \eta_2)(m))^2\}$$

$$\geq \min\{((\eta_1 \cap \eta_2)(m \otimes n))^2, ((\eta_1 \cap \eta_2)(n))^2\}$$

$$\geq \min((\eta_1 \cap \eta_2)(m \otimes n))^2, ((\eta_1 \cap \eta_2)(n))^2, \theta\}$$

Thus, $\max\{((\eta_1 \cap \eta_2)(m))^2, \beta\} \geq \min((\eta_1 \cap \eta_2)(m \otimes n))^2, ((\eta_1 \cap \eta_2)(n))^2, \theta\}$ and

$$\min\{((\tau_1 \cap \tau_2)(m))^2, \theta\} \leq \{((\tau_1 \cap \tau_2)(m))^2\}$$

$$\leq \max\{((\tau_1 \cap \tau_2)(m \otimes n))^2, ((\tau_1 \cap \tau_2)(n))^2\}$$

$$\leq \max\{((\tau_1 \cap \tau_2)(m \otimes n))^2, ((\tau_1 \cap \tau_2)(n))^2, \beta\}$$

Thus, $\min\{((\tau_1 \cap \tau_2)(m))^2, \theta\} \leq \max\{((\tau_1 \cap \tau_2)(m \otimes n))^2, ((\tau_1 \cap \tau_2)(n))^2, \beta\}$

$$(ii) \max\{((\eta_1 \cap \eta_2)((m \otimes n) \odot (n \otimes w)))^2, \beta\} \geq \{((\eta_1 \cap \eta_2)((m \otimes n) \odot (n \otimes w)))^2\}$$

$$\geq \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2\}$$

$$\geq \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2, \theta\}$$

$$\Rightarrow \max\{((\eta_1 \cap \eta_2)((m \otimes n) \odot (n \otimes w)))^2, \beta\} \geq \min\{((\eta_1 \cap \eta_2)(m))^2, ((\eta_1 \cap \eta_2)(n))^2, \theta\}$$

and

$$\min\{((\tau_1 \cap \tau_2)((m \otimes n) \odot (n \otimes w)))^2, \theta\} \leq \{((\tau_1 \cap \tau_2)((m \otimes n) \odot (n \otimes w)))^2\}$$

$$\leq \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2\}$$

$$\leq \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2, \beta\}$$

$$\Rightarrow \min\{((\tau_1 \cap \tau_2)((m \otimes n) \odot (n \otimes w)))^2, \theta\} \leq \max\{((\tau_1 \cap \tau_2)(m))^2, ((\tau_1 \cap \tau_2)(n))^2, \beta\}$$

Therefore, by (i) and (ii) above, we asserted that the intersection of any two Pythagorean (β, θ) -fuzzy ideals of LBA L is Pythagorean (β, θ) -fuzzy ideal of L . \square

The above theorem can also be generalized to any family of Pythagorean (β, θ) -fuzzy ideals of L as in the corollary below.

Corollary 6.4.1. *The intersection of a any family of Pythagorean (β, θ) -fuzzy ideals of L is also a Pythagorean (β, θ) -fuzzy ideal of $LBA L$.*

Proof. Let $\{(\eta_i, \tau_i) : i \in I\}$ be a family of Pythagorean (β, θ) -fuzzy ideals of L .

Then, we need to prove that $\left(\bigcap_{i \in I} \eta_i, \bigcap_{i \in I} \tau_i\right)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

Then for any $m, n, w \in L$ and any β, θ where $0 \leq \beta < \theta \leq 1$; we have the following:

$$\begin{aligned}
 \text{(i) } \max\left\{\bigcap_{i \in I} (\eta_i(m))^2, \beta\right\} &\geq \bigcap_{i \in I} (\eta_i(m))^2 \\
 &= \inf_{i \in I} \{(\eta_i(m))^2\} \\
 &\geq \inf_{i \in I} \{\min\{(\eta_i(m \otimes n))^2, (\eta_i(n))^2\}\} \\
 &\geq \min\left\{\inf_{i \in I} \{(\eta_i(m \otimes n))^2, (\eta_i(n))^2\}\right\} \\
 &= \min\left\{\inf_{i \in I} \{(\eta_i(m \otimes n))^2\}, \inf_{i \in I} \{(\eta_i(n))^2\}\right\} \\
 &= \min\left\{\bigcap_{i \in I} (\eta_i(m \otimes n))^2, \bigcap_{i \in I} (\eta_i(n))^2\right\} \\
 &= \min\left\{\bigcap_{i \in I} (\eta_i(m \otimes n))^2, \bigcap_{i \in I} (\eta_i(n))^2, \theta\right\} \quad \text{and}
 \end{aligned}$$

$$\begin{aligned}
 \min\left\{\bigcap_{i \in I} (\tau_i(m))^2, \theta\right\} &\leq \bigcap_{i \in I} (\tau_i(m))^2 \\
 &= \sup_{i \in I} \{(\tau_i(m))^2\} \\
 &\leq \sup_{i \in I} \{\max\{(\tau_i(m \otimes n))^2, (\tau_i(n))^2\}\} \\
 &\leq \max\left\{\sup_{i \in I} \{(\tau_i(m \otimes n))^2, (\tau_i(n))^2\}\right\} \\
 &= \max\left\{\sup_{i \in I} \{(\tau_i(m \otimes n))^2\}, \sup_{i \in I} \{(\tau_i(n))^2\}\right\} \\
 &= \max\left\{\bigcap_{i \in I} (\tau_i(m \otimes n))^2, \bigcap_{i \in I} (\tau_i(n))^2\right\}
 \end{aligned}$$

$$= \max \left\{ \bigcap_{i \in I} (\tau_i(m \otimes n))^2, \bigcap_{i \in I} (\tau_i(n))^2, \beta \right\}, \forall m, n \in L.$$

$$\begin{aligned} \text{(ii) } \max \left\{ \bigcap_{i \in I} (\eta_i((m \otimes n) \odot (n \otimes w)))^2, \beta \right\} &\geq \bigcap_{i \in I} (\eta_i((m \otimes n) \odot (n \otimes w)))^2 \\ &= \inf_{i \in I} \{ (\eta_i((m \otimes n) \odot (n \otimes w)))^2 \} \\ &\geq \inf_{i \in I} \{ \min \{ (\eta_i(m))^2, (\eta_i(n))^2 \} \} \\ &\geq \min \left\{ \inf_{i \in I} \{ (\eta_i(m))^2, (\eta_i(n))^2 \} \right\} \\ &= \min \left\{ \inf_{i \in I} \{ (\eta_i(m))^2 \}, \inf_{i \in I} \{ (\eta_i(n))^2 \} \right\} \\ &= \min \left\{ \bigcap_{i \in I} (\eta_i(m))^2, \bigcap_{i \in I} (\eta_i(n))^2 \right\} \\ &= \min \left\{ \bigcap_{i \in I} (\eta_i(m))^2, \bigcap_{i \in I} (\eta_i(n))^2, \theta \right\} \quad \text{and} \end{aligned}$$

$$\begin{aligned} \min \left\{ \bigcap_{i \in I} \{ (\tau_i((m \otimes n) \odot (n \otimes w)))^2, \theta \} \right\} &\leq \bigcap_{i \in I} (\tau_i((m \otimes n) \odot (n \otimes w)))^2 \\ &= \sup_{i \in I} \{ (\tau_i((m \otimes n) \odot (n \otimes w)))^2 \} \\ &\geq \sup_{i \in I} \{ \max \{ (\tau_i(m))^2, (\tau_i(n))^2 \} \} \\ &\leq \max \left\{ \sup_{i \in I} \{ (\tau_i(m))^2, (\tau_i(n))^2 \} \right\} \\ &= \max \left\{ \sup_{i \in I} \{ (\tau_i(m))^2 \}, \sup_{i \in I} \{ (\tau_i(n))^2 \} \right\} \\ &= \max \left\{ \bigcap_{i \in I} (\tau_i(m))^2, \bigcap_{i \in I} (\tau_i(n))^2 \right\} \\ &= \max \left\{ \bigcap_{i \in I} (\tau_i(m))^2, \bigcap_{i \in I} (\tau_i(n))^2, \theta \right\}, \forall m, n, w \in L. \end{aligned}$$

Therefore, by (i) and (ii) above, we asserted that the intersection of two Pythagorean (β, θ) -fuzzy ideals of LBA L is Pythagorean (β, θ) -fuzzy ideal of L . \square

Remark 6.4.3. *The union of two Pythagorean (β, θ) -fuzzy ideals of LBA L may not be Pythagorean (β, θ) -fuzzy ideal of L which is illustrated by Example 6.4.3 below.*

Example 6.4.3. It suffices to show for either union of the membership or the non-membership of one of the axioms fails and then we show the membership of the first axiom fails as shown below:

Let $(L, \otimes, \odot, 0)$ be LBA where $L = \{0, m, n\}$ and two binary operations \otimes and \odot on L are as defined in Table 5.4 of Example 5.1.10 above and define two pairs of fuzzy ideals η_1 and η_2 ; τ_1 and τ_2 of L as follows:

k	$\eta_1(k)$	$\eta_2(k)$	$(\eta_1 \cup \eta_2)(k)$	$\tau_1(k)$	$\tau_2(k)$	$(\tau_1 \cup \tau_2)(k)$
m	0.4	0.7	0.7	0.5	0.3	0.3
n	0.6	0.3	0.6	0.2	0.5	0.2
0	0.8	0.9	0.9	0.1	0.2	0.1

Table 6.2: A table that defines fuzzy ideals and their unions

Hence, from Table 6.2 above, as $m \otimes 0 = m$, and taking $\beta = 0.1$, $\theta = 0.2$; we have:

$$\begin{aligned} \max\{((\eta_1 \cup \eta_2)(m))^2, \beta\} &= \max\{0.16, 0.1\} = 0.16 \geq \min\{((\eta_1 \cup \eta_2)(m \otimes 0))^2, ((\eta_1 \cup \eta_2)(0))^2, \theta\} \\ &= \min\{((\eta_1 \cup \eta_2)(m))^2, ((\eta_1 \cup \eta_2)(0))^2, \theta\} \\ &= \min\{0.49, 0.81, 0.2\} = 0.2, \end{aligned}$$

which is false, describing that the union of two Pythagorean (β, θ) -fuzzy ideals of L is not necessarily Pythagorean (β, θ) -fuzzy ideal of L .

Proposition 6.4.2. Let η_M be a fuzzy subset in LBA L , where M is non-empty subset of L such that

$$\eta_M(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M, \end{cases} \quad \text{and the square deviation } \bar{\eta}_M(m) = \begin{cases} 1 - \delta^2, & \text{if } m \in M, \\ 1 - \sigma^2, & \text{if } m \notin M. \end{cases}$$

where $\delta, \sigma \in [0, 1]$, $\delta > \sigma$. Then M is ideal of L if and only if $L^B = (\eta_M, \bar{\eta}_M)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

Proof. Let M be ideal of L , and $0 \leq \beta < \theta \leq 1$. Then we need to prove that $L^B = (\eta_M, \bar{\eta}_M)$ is

a Pythagorean (β, θ) -fuzzy ideal of L , that is:

$$(1) \max\{(\eta_M(m))^2, \beta\} \geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \text{ and}$$

$$\min\{(\bar{\eta}_M(m))^2, \theta\} \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2, \beta\}$$

$$(2) \max\{(\eta_M((m \otimes n) \odot (n \otimes w)))^2, \beta\} \geq \min\{(\eta_M(m))^2, (\eta_M(n))^2, \theta\} \text{ and}$$

$$\min\{(\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2, \theta\} \leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2, \beta\}.$$

(1) Now, we prove these pairs of axioms following the three cases below:

Case(i) : Let $n \in M$, $m \otimes n \in M \Rightarrow m \in M$

$$\begin{aligned} \text{Then } (\eta_M(m \otimes n))^2 &= (\eta_M(n))^2 = \delta^2 \Rightarrow (\eta_M(m))^2 = \delta^2 \text{ and} \\ (\bar{\eta}_M(m \otimes n))^2 &= (\bar{\eta}_M(n))^2 = 1 - \delta^2 \Rightarrow (\bar{\eta}_M(m))^2 = 1 - \delta^2 \\ \Rightarrow \max\{(\eta_M(m))^2, \beta\} &\geq (\eta_M(m))^2 \geq \delta^2 \geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2\} \\ &\geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} = \min\{\delta^2, \delta^2, \theta\} \\ \Rightarrow \max\{(\eta_M(m))^2, \beta\} &\geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \text{ and also} \\ \min\{(\bar{\eta}_M(m))^2, \theta\} &\leq (\bar{\eta}_M(m))^2 \leq 1 - \delta^2 \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\} \\ &\leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2, \beta\} = \max\{\delta^2, \delta^2, \beta\} \end{aligned}$$

Case(ii) : Let $n \in M$, $m \otimes n \notin M$ (or $n \notin M$, $m \otimes n \in M$) \Rightarrow either $m \in M$ or $m \notin M$

$$\begin{aligned} \Rightarrow (\eta_M(m \otimes n))^2 &= \delta^2, (\eta_M(n))^2 = 1 - \delta^2 \text{ (or } (\eta_M(m \otimes n))^2 = 1 - \delta^2, (\eta_M(n))^2 = \delta^2) \\ \Rightarrow \text{either } (\eta_M(m))^2 &= \delta^2 \text{ or } (\eta_M(m))^2 = 1 - \delta^2 \end{aligned}$$

$$\begin{aligned} \text{Now, } \max\{(\eta_M(m))^2, \beta\} &\geq (\eta_M(m))^2 \geq 1 - \delta^2 = \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2\} \\ &\geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \end{aligned}$$

$$\Rightarrow \max\{(\eta_M(m))^2, \beta\} \geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\}$$

and following similar steps as the preceding steps, we obtain:

$$\min\{(\bar{\eta}_M(m))^2, \theta\} \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2, \beta\}$$

Case(iii) : Let $n \notin M$, $m \otimes n \notin M \Rightarrow$ either $m \in M$ or $m \notin M$

$$\Rightarrow (\eta_M(m \otimes n))^2 = 1 - \delta^2 = (\eta_M(n))^2 \Rightarrow \text{either } (\eta_M(m))^2 = \delta^2 \text{ or } (\eta_M(m))^2 = 1 - \delta^2$$

$$\begin{aligned} \text{Now, } \max\{(\eta_M(m))^2, \beta\} &\geq (\eta_M(m))^2 \geq 1 - \delta^2 = \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2\} \\ &\geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \end{aligned}$$

$$\Rightarrow \max\{(\eta_M(m))^2, \beta\} \geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\}$$

and following similar steps as the preceding steps, we obtain:

$$\min\{(\bar{\eta}_M(m))^2, \theta\} \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2, \beta\}$$

(2) In a similar fashion, we can simply show that:

$$\begin{aligned} \max\{(\eta_M((m \otimes n) \odot (n \otimes w)))^2, \beta\} &\geq \min\{(\eta_M(m))^2, (\eta_M(n))^2, \theta\} \text{ and} \\ \min\{(\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2, \theta\} &\leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2, \beta\}. \end{aligned}$$

Therefore, $(\eta_M, \bar{\eta}_M)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

Conversely, suppose $(\eta_M, \bar{\eta}_M)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

Then we need to show M is ideal of L ; in other words:

- (i) $m \otimes n \in M$ and $n \in M \Rightarrow m \in M$;
- (ii) $m, n \in M \Rightarrow ((m \otimes n) \odot (n \otimes w)) \in M$.

Now, we show the second axiom and similarly it becomes simple to prove axiom (i):

Then we, for $m, n \in M \subseteq L$, we need to show $(m \otimes n) \odot (n \otimes w) \in M, \forall m, n, w \in L$:

$$\begin{aligned} \text{Let } m, n \in M &\Rightarrow (\eta_M(m))^2 = (\eta_M(n))^2 = \delta^2 \\ (\eta_M((m \otimes n) \odot (n \otimes w)))^2 &\geq \min\{(\eta_M(m))^2, (\eta_M(n))^2\} \\ \Rightarrow (\eta_M((m \otimes n) \odot (n \otimes w)))^2 &\geq \min\{(\eta_M(m))^2, (\eta_M(n))^2\} = \min\{\delta^2, \delta^2\} = \delta^2 \\ \Rightarrow (\eta_M((m \otimes n) \odot (n \otimes w)))^2 &\geq \delta^2 \text{ and by definition } (\eta_M((m \otimes n) \odot (n \otimes w)))^2 \leq \delta^2 \\ \Rightarrow (\eta_M((m \otimes n) \odot (n \otimes w)))^2 &= \delta^2 \Rightarrow (m \otimes n) \odot (n \otimes w) \in M \end{aligned}$$

Similarly, it is simple to show that $m \otimes n \in M, n \in M \Rightarrow m \in M$.

Therefore, M is ideal of L . □

Corollary 6.4.2. Let η be a fuzzy subset and $\bar{\eta}$ be its complement in LBA L and $\emptyset \neq M \subseteq L$ such that

$$\eta_M(m) = \begin{cases} \delta, & \text{if } m \in M, \\ \sigma, & \text{if } m \notin M, \end{cases} \quad \text{and } \bar{\eta}_M(m) = \begin{cases} 1 - \delta, & \text{if } m \in M, \\ 1 - \sigma, & \text{if } m \notin M, \end{cases} \quad \text{where } \delta, \sigma \in [0, 1], \delta > \sigma.$$

Then M is ideal of L if and only if $(\eta_M, \bar{\eta}_M)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

Proof. Let M be ideal of L , and $0 \leq \beta < \theta \leq 1$. Then we need to prove that $L^B = (\eta_M, \bar{\eta}_M)$ is

a Pythagorean (β, θ) -fuzzy ideal of L , that is:

$$(1) \max\{(\eta_M(m))^2, \beta\} \geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \text{ and}$$

$$\min\{(\bar{\eta}_M(m))^2, \theta\} \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2, \beta\}$$

$$(2) \max\{(\eta_M((m \otimes n) \odot (n \otimes w)))^2, \beta\} \geq \min\{(\eta_M(m))^2, (\eta_M(n))^2, \theta\} \text{ and}$$

$$\min\{(\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2, \theta\} \leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2, \beta\}.$$

(1) Now, we prove these pairs of axioms following the three cases below:

$$\text{(From Proposition 4.1.1, } 1 - \delta^2 = (\bar{\eta}_M(m))^2 \geq (\eta_M(m))^2 = (1 - \delta)^2 = \delta^2 + 1 - 2\delta\text{):}$$

Case(i) : Let $n \in M, m \otimes n \in M \Rightarrow m \in M$

$$\text{Then } (\eta_M(m \otimes n))^2 = (\eta_M(n))^2 = \delta^2 \Rightarrow (\eta_M(m))^2 = \delta^2 \text{ and}$$

$$(\bar{\eta}_M(m \otimes n))^2 = (\bar{\eta}_M(n))^2 = \delta^2 + 1 - 2\delta \Rightarrow (\bar{\eta}_M(m))^2 = \delta^2 + 1 - 2\delta$$

$$\begin{aligned} \Rightarrow \max\{(\eta_M(m))^2, \beta\} &\geq (\eta_M(m))^2 \geq \delta^2 = \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2\} \\ &\geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \end{aligned}$$

$$\Rightarrow \max\{(\eta_M(m))^2, \beta\} \geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \text{ and also}$$

$$\begin{aligned} \min\{(\bar{\eta}_M(m))^2, \theta\} &\leq (\bar{\eta}_M(m))^2 \leq \delta^2 + 1 - 2\delta = \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2\} \\ &\leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2, \beta\} \end{aligned}$$

Case(ii) : Let $n \in M, m \otimes n \notin M$ (or $n \notin M, m \otimes n \in M$) \Rightarrow either $m \in M$ or $m \notin M$

$$\Rightarrow (\eta_M(m \otimes n))^2 = \sigma^2, (\eta_M(n))^2 = \delta^2$$

$$\text{(or } (\eta_M(m \otimes n))^2 = \delta^2, (\eta_M(n))^2 = \sigma^2 \text{)}$$

$$\Rightarrow \text{either } (\eta_M(m))^2 = \delta^2 \text{ or } (\eta_M(m))^2 = \sigma^2$$

$$\begin{aligned} \text{Now, } \max\{(\eta_M(m))^2, \beta\} &\geq (\eta_M(m))^2 \geq \sigma^2 = \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2\} \\ &\geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \end{aligned}$$

$$\Rightarrow \max\{(\eta_M(m))^2, \beta\} \geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\}$$

and following similar steps as the preceding steps, we obtain:

$$\min\{(\bar{\eta}_M(m))^2, \theta\} \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2, \beta\}$$

Case(iii) : Let $n \notin M, m \otimes n \notin M \Rightarrow$ either $m \in M$ or $m \notin M$

$$\Rightarrow (\eta_M(m \otimes n))^2 = \sigma^2 = (\eta_M(n))^2$$

$$\Rightarrow \text{either } (\eta_M(m))^2 = \sigma^2 \text{ or } (\eta_M(m))^2 = \delta^2$$

$$\begin{aligned} \text{Now, } \max\{(\eta_M(m))^2, \beta\} &\geq (\eta_M(m))^2 \geq \sigma^2 = \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2\} \\ &\geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\} \end{aligned}$$

$$\Rightarrow \max\{(\eta_M(m))^2, \beta\} \geq \min\{(\eta_M(m \otimes n))^2, (\eta_M(n))^2, \theta\}$$

and following similar steps as the preceding steps, we obtain:

$$\min\{(\bar{\eta}_M(m))^2, \theta\} \leq \max\{(\bar{\eta}_M(m \otimes n))^2, (\bar{\eta}_M(n))^2, \beta\}$$

(2) In a similar fashion, we can show that:

$$\begin{aligned} \max\{(\eta_M((m \otimes n) \odot (n \otimes w)))^2, \beta\} &\geq \min\{(\eta_M(m))^2, (\eta_M(n))^2, \theta\} \text{ and} \\ \min\{(\bar{\eta}_M((m \otimes n) \odot (n \otimes w)))^2, \theta\} &\leq \max\{(\bar{\eta}_M(m))^2, (\bar{\eta}_M(n))^2, \beta\}, \forall m, n, e \in L. \end{aligned}$$

Therefore, $(\eta_M, \bar{\eta}_M)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

Conversely, suppose $(\eta_M, \bar{\eta}_M)$ is a Pythagorean (β, θ) -fuzzy ideal of L .

We claim that M is ideal of M ; in other words:

$$(i) \quad m \otimes n, n \in M \Rightarrow m \in M;$$

$$(ii) \quad m, n \in M \Rightarrow ((m \otimes n) \odot (n \otimes w)) \in M.$$

Now, we show the second axiom and similarly it becomes simple to prove axiom (i):

$$\text{Let } m, n \in M \Rightarrow (\eta_M(m))^2 = (\eta_M(n))^2 = \delta^2$$

$$(\eta_M(m \otimes n) \odot (n \otimes w))^2 \geq \min\{(\eta_M(m))^2, (\eta_M(n))^2\} = \min\{\delta^2, \delta^2\} = \delta^2$$

$$\Rightarrow (\eta_M(m \otimes n) \odot (n \otimes w))^2 \geq \delta^2 \text{ but } (\eta_M(m \otimes n) \odot (n \otimes w))^2 \leq \delta^2, \text{ by the hypothesis}$$

$$\Rightarrow (\eta_M(m \otimes n) \odot (n \otimes w))^2 = \delta^2 \Rightarrow (m \otimes n) \odot (n \otimes w) \in M, \forall m, n, w \in L.$$

$$(\eta_M(n))^2\} = \min\{\delta^2 + 1 - 2\delta, \delta^2 + 1 - 2\delta\} = \delta^2 + 1 - 2\delta$$

$$\Rightarrow (\eta_M(m \otimes n) \odot (n \otimes w))^2 \geq \delta^2 + 1 - 2\delta \text{ and}$$

$$\text{by definition } (\eta_M(m \otimes n) \odot (n \otimes w))^2 \leq \delta^2 + 1 - 2\delta$$

$$\Rightarrow (\eta_M(m \otimes n) \odot (n \otimes w))^2 = \delta^2 + 1 - 2\delta \Rightarrow (m \otimes n) \odot (n \otimes w) \in M$$

Similarly, it is simple to show that $m \otimes n \in M, n \in M \Rightarrow m \in M$.

Therefore, M is ideal of L . □

Conclusion and Future Works

This section gives a summary of the main ideas and findings of our dissertation. It also suggests possible future research directions that can extend and improve the concepts discussed, focusing on algebraic structures under uncertainty and their practical applications.

I. Conclusion

This dissertation attempts to explore comprehensive study of the advanced structures in algebra, particularly focusing on BCL–algebras and Pythagorean fuzzy set, through the integration of hyper–structures, fuzzy subsets and Pythagorean fuzzy sets. The main purpose of this research has been to extend classical logical algebraic structures by incorporating new mathematical concepts to better model uncertainty, non–determinism and hesitation.

The research began by revisiting the foundational structures of algebra, especially BCL–algebras, which are logical algebras characterized by implication like operations. These algebras serve as an excellent framework for analyzing logical statements, deductive reasoning and structure preserving transformations. Although the classical theory of BCL–algebra is well–developed by Y. Liu in 2011 [42], the limitations of binary operations and crisp set membership in dealing with complex or vague situations are significant. To address this, three major generalizations were explored and combined which are hyper structures, fuzzy subsets and Pythagorean fuzzy sets. The foundational concepts, hyper structures, fuzzy subsets and Pythagorean fuzzy sets were originally introduced by F. Marty in 1934 [49], L. A. Zadeh in 1965 [81] and R. Yager in 2013 [78], respectively. Each of these frameworks offers additional flexibility and expressiveness in dealing with uncertainty and ambiguity within algebraic systems.

The first major advancement involved the introduction and formal development of Hyper BCL–algebras. These are generalizations of classical BCL–algebras where binary operations are replaced by hyper operations which are set–valued mappings that reflect the possibility of multiple outcomes rather than a single one. In this setting, beginning with introducing hyper BCL–algebra starting by defining the concepts of (weak, strong) hyper subalgebras, (weak, strong) hyper deductive systems and (weak, strong) hyper ideals were rigorously defined and studied. The properties of these struc-

tures were analyzed in detail, including how they relate to their classical counterparts and how they behave under basic algebraic operations.

After establishing hyper BCL–algebra and its hyper substructures, the research moved into the realm of fuzzy logic. Fuzzy set theory, introduced and developed by Zadeh in 1965 [81, 82] allows elements to belong to sets with varying degrees of membership, providing a better framework for handling uncertainty than classical (crisp) sets. Building upon the hyper BCL–algebra framework, the dissertation introduced fuzzy (weak, strong) hyper substructures such as fuzzy (weak, strong) hyper subalgebras, fuzzy (weak, strong) hyper deductive systems and fuzzy (weak, strong) hyper ideals. These are sophisticated constructs that allow for a graded understanding of logical implications within systems that already accommodate multiple outcomes. Several characterizations, lemmas and theorems were proven to provide structural insights and support deeper classification of these fuzzy hyper constructions.

However, even fuzzy subsets have limitations because they rely on a single membership function, this shortcoming was addressed by moving to Pythagorean fuzzy sets, a significant innovation in this work. Pythagorean fuzzy sets, as introduced and developed by R. Yager (2013) [78, 79], allow each element to be assigned two independent degrees which are the pair of membership and non–membership mappings, with the condition that the square sum of these two values does not exceed one, symbolical expressed as $0 \leq (\eta(m))^2 + (\tau(m))^2 \leq 1$, for each element of the set under discussion. This condition offers a more flexible and realistic model of hesitation or indeterminacy, which is often present in practical decision–making and reasoning processes. Thus, the Pythagorean fuzzy setting captures more types of uncertainty compared to ordinary fuzzy or intuitionistic fuzzy subsets.

Pythagorean fuzzy substructures of BCL–algebras are systematically developed including Pythagorean fuzzy subalgebra, Pythagorean fuzzy deductive system and Pythagorean fuzzy ideal. In each case, these structures were rigorously defined, their necessary and sufficient conditions were derived and their properties were studied. Various theorems and propositions helped to understand how these structures relate to each other and to their classical or fuzzy analogues.

The Pythagorean fuzzy extension to Pythagorean fuzzy Liu^B–algebras (LBAs.) is a particularly novel contribution of the dissertation. Pythagorean fuzzy Liu^B–algebras are relatively new algebraic structures that combine features of BCL–algebras and semi–groups. This dissertation is one of the first works to rigorously define and analyze fuzzy and Pythagorean fuzzy substructures within this new Pythagorean fuzzy Liu^B–algebras context. Definitions and properties were provided for Pythagorean fuzzy Liu^B–subalgebras, Pythagorean fuzzy deductive systems and Pythagorean fuzzy ideals. A strong emphasis was placed on analyzing how these structures behave under algebraic operations, such as intersection, union and implication–like operations defined in Pythagorean fuzzy Liu^B–algebras.

The final chapter of the dissertation introduces and investigates more fundamental constructions and advanced relationships between the developed structures. One such concept is the homomorphism of Pythagorean fuzzy deductive systems, which involves studying how structure-preserving maps (homomorphisms) affect the behavior of fuzzy and Pythagorean fuzzy substructures. This analysis includes results on homomorphic images and pre-images, which are essential in understanding how algebraic properties are maintained under mappings between different algebras.

In addition, the level set representation of Pythagorean fuzzy structures was developed. This involves slicing the fuzzy structure at a specific degree of membership or non-membership to obtain classical substructures. These level sets bridge the gap between fuzzy logic and classical algebra, offering a visual and analytical tool for converting uncertain data into more manageable crisp subsets.

Another important contribution is the study of Cartesian products of fuzzy subalgebras whose construction is used to combine two or more fuzzy structures to analyze their joint behavior. The interaction between the components of the Cartesian product was analyzed and results were presented showing how membership values propagate in the combined system.

The dissertation also introduced and studied a specialized type of fuzzy ideal called the (β, θ) -Pythagorean fuzzy ideal defined using refined conditions involving thresholds (β, θ) . These ideals provide sharper classification dissertation and enable finer control over the uncertainty sense, especially in practical settings where certain bounds must be respected.

Along sides of introducing all these new ideas, this dissertation also explains important related concepts such as the complement of fuzzy subsets, the square deviation in fuzzy comparisons and special functions like the accuracy function, score function and the degree of indeterminacy. These concepts are discussed in detail under the appropriate sections and subsections of the dissertation.

The results presented in this dissertation are supported by numerous examples, counterexamples and diagrams, which illustrate the definitions and make the abstract concepts more accessible. Where applicable, comparative studies between fuzzy, hyper and Pythagorean fuzzy substructures were conducted to highlight the strengths and weaknesses of each approach.

This dissertation successfully integrates hyper operations, fuzzy subsets and Pythagorean fuzzy sets into the existing theory of BCL-algebra and newly introduced Pythagorean fuzzy sets. It introduces new algebraic structures, proves important theoretical results and develops a robust foundation for future research.

II. Future Works

This dissertation has presented some new directions for further studies, especially in the area of algebraic structures under uncertainty. One of the important future works is to develop the theory of *Pythagorean fuzzy hyper structures* of Hyper BCL–algebra and hyper Liu^B–algebra. Since we have already seen the usefulness of hyper structures in BCL–algebras, it would be natural and interesting to define and study *hyper subalgebras, hyper deductive systems and hyper ideals in Pythagorean fuzzy set*. This would help us to model systems where the operation may have multiple possible outcomes, which is common in uncertain or multi–agent environments.

In addition, future studies may focus on *bipolar fuzzy subsets applied to both BCL–algebras and Pythagorean fuzzy set*. Bipolar fuzzy subsets use both positive and negative membership functions, which are useful to represent situations involving both satisfaction and dissatisfaction. This can expand our understanding of decision-making models and logical frameworks where both acceptance and rejection play a role.

Another important direction is to develop *interval-valued fuzzy structures in BCL–algebra and Liu^B–algebra*. These allow each element to be associated not with a single number, but with an interval that shows the range of possible membership values.

Moreover, studying *fuzzy congruence relations* on these algebras can give us better classification and decomposition dissertation. Along with this, building topological fuzzy structures could help to connect fuzzy algebra with topology, which is useful in logic, analysis and computer science.

More advanced mathematical dissertation such as *category theory, graph theory and soft and fuzzy soft set theory* could be used to explore the relationships between all these algebraic structures.

Finally, **the potential applications of fuzzy hyper BCL–algebras** such as Decision-making and knowledge representation, Artificial intelligence and soft computing, Knowledge and database systems, Control systems and decision logic, Algebraic logic and computer science, Information fusion and pattern recognition, and **the real life application of the Pythagorean fuzzy BCL–algebra and Liu^B–algebra** such as Medical diagnosis systems, Multi-agent and expert systems, Robotics and autonomous systems, Image processing and pattern recognition, Economics and social systems, Cybersecurity and risk analysis could be addressed as critically as possible.

These future directions will continue to build on the strong foundation developed in this dissertation and help researchers in Mathematics contribute more to global knowledge in algebra and fuzzy logic.

Bibliography

- [1] A. Ahmed and E. A. Amhed (2020), Fuzzy BCK-algebras, *Journal of Applied Mathematics and Physics*, **8**(5), 927–932.
- [2] B. Ahmad (1993), Fuzzy BCI-algebras, *J. Fuzzy Math.*, **1**(2), 445–452.
- [3] D. Al-Kadi and R. Hosny (2013), On BCL-algebra, *Journal of Advances in Mathematics*, **3**(2).
- [4] A. Al-Masarwah and A. G. Ahmad (2019), m-polar (α, β) -fuzzy ideals in BCK/BCI-algebras, *Symmetry*, **11**(1), 44.
- [5] B. A. Alaba, B. L. Derseh and Y. G. Wondifraw (2023), On Intuitionistic Fuzzy PMS-Ideals of a PMS-Algebra Under Homomorphism and Cartesian Product, *International Journal of Computational Intelligence Systems*, **16**(1), 68.
- [6] S. H. Ali, F. F. Kareem and A. T.Hameed (2021, February). Hyper Fuzzy AT-ideals of AT-algebra, In *Journal of Physics: Conference Series* **1804**(1), p. 012067), IOP Publishing.
- [7] K. T. Atanassov (1999), Intuitionistic fuzzy sets, In *Intuitionistic fuzzy sets: theory and applications* (pp. 1-137), Heidelberg: Physica-Verlag HD. Atanassov (2012), *On intuitionistic fuzzy sets theory* **283**, Springer.
- [8] P. Bhattacharya and N. P. Mukherjee (1985), Fuzzy relations and fuzzy groups, *Information Sciences*, **36**(3), 267–282.
- [9] R. Borzooei, A. B. Saeid, R. Ameri and A. Rezaei (2014), Involutory BE-algebras, *J. Math. Appl*, **37**, 13-26.
- [10] A. Borzooei, X. L. Xin, E. H. Roh and Y. B. Jun (2019), Int-soft Implicative hyper BCK-ideals in hyper BCK-algebras, *Missouri Journal of Mathematical Sciences*, **31**(2), 152–163.
- [11] A. Borzooei, M. M. Zahed, A. B. Saeid, Y. B. Jun, P. Corsini, V. L. Fotea, ... and S. H. Mayerov, *Journal of Algebraic Hyperstructures and Logical Algebras*.
- [12] D. Busneag (1985), A note on deductive systems of a Hilbert algebra, *Kobe J. Math.*, **2**, 29–35.
- [13] G. Cantor (1874), On a property of the class of all real algebraic numbers, *Crelle's Journal for Mathematics*, **77**, 258–262.
- [14] I. Chajda, R. Halaš and Y. B. Jun (2002), Annihilators and deductive systems in commutative Hilbert algebras, *Commentationes Mathematicae Universitatis Carolinae*, **43**(3), 407–417.

- [15] P. S. Das (1981), Fuzzy groups and level subgroups, *J. MATH. ANALY. AND APPLIC.*, **84**(1), 264-269.
- [16] B. Davvaz (1999), Fuzzy HV-groups, *Fuzzy sets and systems*, **101**(1), 191-195.
- [17] A. K. Deena and H. Rodyna (2013), On BCL-Algebra, *Journal of Advances in Mathematics*, **3**(2), 184–190.
- [18] A. Diego (1966), Sur les algebras de Hilbert, Ed. Herman, *Collection de Logique Math. Serie A*, 21.
- [19] V. N. Dixit, R. Kumar and N. Ajmal (1990), Level subgroups and union of fuzzy subgroups, *Fuzzy Sets and Systems*, **37** (3), 359–371.
- [20] V. N. Dixit, R. Kumar and N. Ajmal (1992), On fuzzy rings. *Fuzzy Sets and Systems*, **49** (2), 205–213.
- [21] W. A. Dudek (1990), On BCC-algebras, *Logique et Analyse*, **33**(129/130), 103–111.
- [22] Y. Feng and P. Corsini (2013), (λ, μ) -fuzzy interior ideals of ordered Γ -semigroups, *Algebra and Discrete Mathematics*, **6**(1), 61–70.
- [23] M. Gulistan, M. Shahzad and S. Ahmed (2015), On (α, β) -fuzzy KU-ideals of KU-algebras, *Afrika Matematika*, 26, 651-661.
- [24] Q. P. Hu and X. Li (1983), *On BCH - algebras*, *Math. Seminar Notes*, **11** (2)313 - 320.
- [25] A. Iampan (2017), *A new branch of the logical algebra: UP-algebras*, *Journal of Algebra and Related Topics*, **5**(1), 35-54.
- [26] K. Iséki (1978), An introduction to the theory of BCK-algebras, *Math. Japon.*, **23**(1), 1–26.
- [27] K. Iseki (1980), On BCI-algebras, In *Math. Seminar Notes* **8**., 235–236).
- [28] K. Iseki and S. Tanaka (1978), *An introduction to theory of BCK algebras*, *Math Japan* **23** , 1–26.
- [29] Z. Jianming and M. Xueling (2004), Fuzzy Deductive Systems of HS-algebras, *Sci. Math. Jpn.*, 155–162.
- [30] Y. B. Jun (1998), On BH-algebras, *Scientiae Mathematicae*, **1**(3), 347–354.
- [31] Y. B. Jun (1999), Fuzzy ideals and fuzzy subalgebras of BCK-algebras, *J. Fuzzy Math.*, **7**(2), 411–418.
- [32] Y. B. Jun (2007), Fuzzy subalgebras with thresholds in BCK/BCI-algebras, *Communications of the Korean Mathematical Society*, **22**(2), 173–181.
- [33] Y. B. Jun and R. Bandaru (2022), Deductive systems of GE-algebras, *Algebraic Structures and Their Applications*, **9**(1), 53–67.

- [34] Y. B. Jun, M. S. Kang and H. S. Kim (2010), Fuzzy structures of hyper-MV-deductive systems in hyper-MV-algebras, *Computers and Mathematics with Applications*, **59**(8), 2982–2989.
- [35] Y. B. Jun, M. S. Kang and H. S. Kim. (2010), Hyper MV-deductive systems of hyper MV-algebras, *Communications of the Korean Mathematical Society*, **25**(4), 537–545.
- [36] M. S.Khan, K. Rahman, S. Abdullah and F. Hasain (2019), On Hyper BCH-Algebra, *Ital J Pure Appl*, 41, 85–96.
- [37] K. H. Kim (2010), A note on BE-algebras, *Sci. Math. Japon.*, **72**(2), 127–132.
- [38] H. S. Kim and Y. H. Kim (2007), On BE-algebras, *Scientiae Mathematicae Japonicae*, **66**(1), 113-116.
- [39] B. Lamesgin (2023), Intuitionistic Fuzzy Structures on PMS-algebras.
- [40] C. Lele, C. Wu, P. Weke, T. Mamadou and G. E. Njock (2001), Fuzzy ideals and weak ideals in BCK-algebras.
- [41] J. Liu (1982), Fuzzy invariant subgroups and fuzzy ideals, *Fuzzy sets and Systems*, **8** (2), 133-139.
- [42] Y. Liu (2011), A new branch of the pure algebra: BCL-algebras, *Advances in Pure Mathematics*, **1**(5), 297–299.
- [43] Y. Liu (2012), On BCL-algebras, *Advances in Pure Mathematics*, 2, 59–61.
- [44] Y. Liu (2013), Partial Orders in BCL^+ -Algebra, *Journal: Journal of Advances in Mathematics*, **5**(1).
- [45] Y. Liu (2015), Filtrations and deductive systems in BCL^+ -algebras, *British Journal of Mathematics and Computer Science*, 8, 274–285.
- [46] Y. Liu (2017), On Liu algebras: a new composite structure of the BCL^+ algebras and the semigroups, *J. Semigroup Theory Appl.*, 2017, Article-ID.
- [47] A. Malik and M. Touqeer (2015), Fuzzy Hyper p-ideals of Hyper BCK-algebras, *Filomat*, **29**(8), 1769–1780.
- [48] S. Mashchenko (2018), Intersections and unions of fuzzy sets of operands, *Fuzzy sets and systems*, 352, 12-25.
- [49] F. Marty (1934), Sur une generalization de la notion de groups, In 8th congress Math. Scandinaves, Stockholm.
- [50] A. A. Mechdesro, B. A. Alaba, T. M. Munie and T. G. Alemayehu (2023), Fuzzy pseudo-UP ideal of pseudo-UP algebra, *Research in Mathematics*, **10**(1), 2234217.
- [51] K. Megalai, & A. Tamilarasi (2011), Fuzzy subalgebras and fuzzy T-ideals in TM-algebra, *Journal of Mathematics and Statistics*, **7**(2), 107-111.

- [52] J. Meng and X. E. Guo (2005), On fuzzy ideals in BCK/BCI-algebras, *Fuzzy sets and Systems*, **149**(3), 509-525.
- [53] S. M. Mostafa, M. A. Abd-Elnaby & M. M. Yousef (2011), Fuzzy ideals of KU-algebras, In *Int. Math. Forum* **6**(63), 3139-3149).
- [54] G. Muhiuddin (2014), Bipolar fuzzy KU-subalgebras/ideals of KU-algebras, *Ann. Fuzzy Math. Inform*, **8**(3), 409-418.
- [55] T. M. Nirmala, D. Jayalakshmi and G. Subbiah (2024), Application of Pythagorean fuzzy sub-algebra of BCK/BCI-Algebra, *African Journal of Biological Science*, **6**(12).
- [56] F. Nisar, R. S. Tariq and S. A. Bhatti (2012), Fuzzy ideals in hyper BCI-algebras, *World Applied Sciences Journal*, **16**(12), 1771–1777.
- [57] J. Peng (2012), Intuitionistic Fuzzy B-algebras, *Research Journal of Applied Sciences, Engineering and Technology*, **4**(21), 4200–4205.
- [58] X. Peng & J. Dai (2017), Approaches to Pythagorean fuzzy stochastic multi-criteria decision making based on prospect theory and regret theory with new distance measure and score function, *International Journal of Intelligent Systems*, **32**(11), 1187-1214.
- [59] X. Peng & J. Dai (2018), Approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function, *Neural Computing and Applications*, **29**(10), 939-954.
- [60] C. Prabpayak and U. Leerawat (2009), On ideals and congruences in KU-algebras, *Sci. Magna*, **5**(1), 54-57.
- [61] T. Priya & T. Ramachandran (2014), Homomorphism and Cartesian product of fuzzy PS-algebra, *Applied Mathematical Sciences*, **8**(67), 3321-3330.
- [62] A. Radfar, A. Rezaei and A. B. Saeid (2014), Hyper BE-algebras, *Novi Sad J. Math.*, **44**(2), 137–147.
- [63] S. B. Rayeni & A. Rezaei (2022), On hyper CI-algebras: as a generalization of hyper BE-algebras.
- [64] A. Romano (2018), A note on ideals and filters in BCC-algebra, *Journal of Universal Mathematics*, **1**(2), 190–194.
- [65] D. A. Romano (2020), Pseudo-UP algebras, An introduction, *Bull. Int. Math. Virtual Inst.*, **10**(2), 349–355.
- [66] A. Rosenfeld (1971), Fuzzy groups, *Journal of Mathematical Analysis and Applications*, **35**(3), 512–517.
- [67] P. S. Selvam and K. T. Nagalakshmi (2016), Role of homomorphism and Cartesian product over fuzzy PMS-algebras, *International Journal of Fuzzy Mathematical Archive*, **11**(1), 1622-1628.

- [68] P. K. Sharma (2011), Homomorphism of Intuitionistic fuzzy groups, In International Mathematical Forum **6**(64), pp. 3169-3178).
- [69] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw and A. Iampan (2016), *Fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform, **12**(6), 739-756.
- [70] V. S. Subha and P. Dhanalakshmi (2021), New type of fuzzy ideals in BCK/BCI algebras, World Scientific News, **153**(2), 80–92.
- [71] A. Tamilarasi & K. Megalai (2011), Fuzzy Subalgebras and fuzzy T-ideals in TM-algebras, Journal of Mathematics and Statistics, **7**(2), 107-111.
- [72] M. Touqeer and M. A. Malik (2016), Fuzzy hyper BCK-implicative ideals of hyper BCK-algebras, Scientiae Mathematicae Japonicae, **79**(2), 93–112.
- [73] F. Wang (2019, October), Intuitionistic (λ, μ) -fuzzy subalgebras in CI-algebras, In Journal of Physics: Conference Series **1324**(1), p. 012015), IOP Publishing.
- [74] Y. G. Wondifraw, B. A. Alaba & B. L. Derseh (2022), Intuitionistic Fuzzy PMS-ideals in a PMS-algebra, Thai Journal of Mathematics, **20**(4), 1535-1548.
- [75] X. L. Xiao (2006), Hyper BCI-Algebras, Discuss. Math., 26, 5–19.
- [76] F. Xiao and W. Ding (2019), Divergence measure of Pythagorean fuzzy sets and its application in medical diagnosis, Applied Soft Computing, 79, 254–267.
- [77] X. L. Xin, W. Ji and X. J. Hua (2011), Fuzzy Filter Spectrum of a BCK Algebra, International Journal of Mathematics and Mathematical Sciences, **211**(1), 795934.
- [78] R. R. Yager (2013), Pythagorean membership grades in multicriteria decision making, IEEE Transactions on Fuzzy Systems, **22**(4), 958–965.
- [79] R. R. Yager (2013, June), Pythagorean fuzzy sets, In 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), 57–61. IEEE.
- [80] M. A. Yousef, and A. Khalaf (2022), Topological pseudo-UP algebras, Journal of Mathematics and Computer Science, **26**(1), 80–89.
- [81] L.A. Zadeh (1965), Fuzzy sets, Information and Control, **8**(3), 338–353.
- [82] L. A. Zadeh (1975), The concept of a linguistic variable and its application to approximate reasoning—I, Information sciences, **8**(3), 199-249.
- [83] M. M. Zahedi, R. A. Borzoei, Y. B. Jun, and A. Hasankhani (2000), Some results on hyper K-algebras, Scientiae Mathematicae, **3**(1), 53–59.
- [84] X. Zhang and Z. Xu (2014), Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, International Journal of Intelligent Systems, **29**(12), 1061–1078.