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BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

MATHEMATICS DEPARTMENT

A PROJECT

ON

SOLVING ELECTRICAL RLC CIRCUIT PROBLEMS OF FRACTIONAL ORDER

BY

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SEPTEMBER, 2024

BAHIR DAR, ETHIOPI

Solving electrical RLC circuit problems of fractional order

A project submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of "Master of Science in Mathematics"

By

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September, 2024

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DEPARTMENT OF MATHEMATICS

Approval of the project for Oral defense

I hereby certify that I have supervised, read, and evaluated this project entitled "Solving electrical RLC circuit problems of fractional order" by Zelalem Seyoum prepared under my guidance. I recommend that the project be submitted for oral defense.

Eshetu Haile (PhD)______ __________________ __________________

Advisor`s name signature Date

BAHIR DAR UNIVERSITY

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DEPARTMENT OF MATHEMATICS

We hereby certify that we have examined this project entitled "Solving electrical RLC circuit problems of fractional order" by Zelalem Seyoum. We recommend that this project be approved for the degree of "Master of Science in Mathematics".

Board of examiners

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External examiner's name	signature	Date
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Internal examiner's name	signature	Date
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ACKNOWLEDGMENT

All thanks go first to God. I would like to thank Dr. Eshetu Haile for his kindness, support, continuous encouragement, and guidance in the preparation of this project. I would like to thank Goh General Secondary School's main and vice directors for their continuous support in providing computer and some reference materials while I was preparing this project. Furthermore, I would like to extend my thanks to my co-workers and my family for their encouragement and support throughout my project work.

KEY WORDS

Resister: Is the given amount of resistance that controls the amount of current flows through it.

Inductor: It is acoil that stores energy in a magnetic field when an electric current flows through it.

Capacitor: Is a device that stores electrical energy by accumulating electric charges on two closely spaced surfaces that are insulated from each other.

Rate of change: Is the speed at which a variable changes over a specific period of time.

Reactance: is the opposition presented to alternating current by inductance and capacitance.

Impedance: Impedance extends the concept of resistance to alternating current (AC) circuits, and possesses both magnitude and phase, unlike resistance, which has only magnitude. It can be represented as a complex number, with the same units as resistance.

Damping: Is the loss of energy of an oscillating system by dissipation; damping is an influence within or upon an oscillatory system that has the effect of reducing or preventing its oscillation.

Over damping: If $R^2 > \frac{4L}{C}$

Under damping: If $R^2 < \frac{4L}{C}$

Critically damping: If $R^2 = \frac{4L}{C}$

ABSTRACT

The main purpose of this project is to show the application of fractional calculus on solving electrical circuit problems. The basic concepts of fractional calculus on solving electrical RLC circuit problems are justified. Applications of fractional calculus on solving electrical circuit problems have been demonstrated with illustrative examples in the project.

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CHAPTER 1

INTRODUCTION AND PRELIMINARY

1.1. Introduction

Many authors developed different models used to solve an electrical RLC circuited of fractional order so that the investigator tries to develops, state and verify such like models. Indeed, the concept of integration and differentiation is familiar to all who have studied elementary calculus.

We know, for instance, that if $f(x) = x^2$, and then integrating $f(x)$ to the 1^{st} order results in $\int f(x)dx = \frac{1}{3}x^3 + c_1$, and integrating the same function to the 2^{nd} order results in $\int[\int f(x)dx]\, dx = \frac{1}{12}x^4 + c_1x + c_2$. Similarly, $\frac{d}{dx}f(x) = 2x$, and $\frac{d^2}{dx^2}f(x) = 2$. However, what if we want to integrate our function $f(x)$ to the ${}^1\!/_2{}^{th}$ order, or find its ${}^1\!/_2{}^{th}$ order derivative? How could we define our operations? Better still, would our results have a meaning or an application comparable to that of the familiar integer order operations?
In several works, fractional order derivative are used to represent the behavior of an electrical circuits. For instance, fractional differential models serve to design analogue and digital filters of fractional order, and some works concern the fractional order description of magnetically coupled coils or the behavior of circuits and systems with resistors, inductors or capacitors. This project work is to address the study of the described electrical RLC circuit systems: And that answers the question “How is the variation of charge or current at the instant time.

An RLC circuit is an electrical circuit consisting of an inductor (L), resistor (R), and capacitor (C) connected either in series or parallel. However this study depends on a series types of an alternating circuit. As much as possible; the last modified electrical RLC circuit equation of fractional order arises in different real world situations such as, on an electric current line transmission, radio waves e.t.c. For the sake of such like significant we have been choosen this topic as aproject work.

This project work consists of two chapters. The first chapter focuses on the concept of fractional calculus with some functions. In the second chapter, we will see the application of fractional calculus on RLC circuit problems with in the illustrative examples with the analysis of current flows and charge variation through a circuit.

1.2. Some functions

In this section we include the definitions and basic theorems of some basic functions useful for the study of fractional differentiation and integration such as, Gamma function, Mittag-Leffler functions, Error function, Laplace transform necessarily.

1.2.1 Gamma function

Gamma function is one of the most useful and fundamental special functions of a mathematical analysis .It emerged essentially from an attempt by Euler to give a meaning to n!, when $n \in \mathbb{R}^+$who in 1729 undertook the problem of interpolating n, between the positive integer values of n.

Definition 1 [1-11]: For any real or complex number z, Gamma function is defined as;

$$\Gamma(z) = \begin{cases} \int_0^\infty e^{-t}\, t^{z-1} dt \\ \prod_{j=0}^{n-1} (z-j) \end{cases} \text{and, for } \mathbb{R}(z) > 0, n \in \mathbb{N}.$$

Some properties are;

- ✓ $\Gamma(n) = (n-1)!,$ for $n \in \mathbb{N}$.
- ✓ $\Gamma(n+1) = n\,\Gamma(n)$, for $n \in \mathbb{R} \geq 0$.

Example: $a.\,\Gamma(1) = \int_0^\infty e^{-t}\, t^{1-1} dt = \int_0^\infty e^{-t}\, dt = -[e^{-\infty} - e^0] = 0 + 1 = 1.$

$b.\,\Gamma(2) = (2-1)! = 1! = 1.$

$c.\,\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1} dt = \int_0^\infty e^{-t}\, t^{-\frac{1}{2}} dt.$

Let $t = y^2$, then $dt = 2ydy$, and we know $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} (y^2)^{-\frac{1}{2}} 2ydy = 2\int_0^\infty e^{-y^2} dy.$

Equivalently we can write $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx,$ if we multiply together we get:

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy.$$

It is a double integral over the first quadrant, and can be evaluated in polar coordinates to get:

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-(r^2)} drd\theta = \sqrt{\pi}.$$

$\mathrm{d}, \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$

1.2.1. Mittag-Leffler function

The Mittag-Leffler function is the cornerstone of fractional calculus. Several books and excellent papers describe the importance of these types of operators. The concept of Mittag-Leffler function in calculus was introduced and the integral associated to the nonsingular fractional operator with Mittag-Leffler kernel was found by using the Laplace transform.

Mittag-Leffler function is the most fundamental and useful special function used for solving fractional order differential equation. It can be applied in diverse fields both theoretical and practical, including the flow of fluids, electric networks, probability, and the theory of statistical distribution.

We denote $E_\alpha(z)$ the Mittag-Leffler function and by $E_{\alpha,\beta}(z)$ it's generalized version.

Definition 2[9-11]: (Mittag-Leffler function- one parameter):

$$E_\alpha(\mathrm{z}) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \text{for } \alpha \in \mathbb{C}, \mathbb{R}(\alpha) > 0 \text{ and } z \in \mathbb{C}.$$

Definition 3[9-11]: (Generalized Mittag-Leffler functions of two parameter):

$$E_{\alpha,\beta}(\mathrm{z}) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \text{for } \alpha \in \mathbb{C}\ \beta \in \mathbb{C} \text{ and } \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0, z \in \mathbb{C}$$

This generalized form was introduced by [8].

Theorem 1: If $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$, then $E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z\, E_{\alpha,\alpha+\beta}(z)$, for

$\alpha \in \mathbb{C}, \beta \in \mathbb{C}, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0 \ and\ z \in \mathbb{C}$

Proof:

by definition, $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} = \sum_{k=-1}^{\infty} \frac{zz^k}{\Gamma[\alpha k+(\alpha+\beta)]} = \frac{1}{\Gamma(\beta)} + \sum_{k=0}^{\infty} \frac{zz^k}{\Gamma[\alpha k+(\alpha+\beta)]}$

$$= \frac{1}{\Gamma(\beta)} + z\, E_{\alpha,\alpha+\beta}(z), \text{where } \alpha \in \mathbb{C}, \beta \in \mathbb{C}, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0 \text{ and } z \in \mathbb{C}.$$

Theorem 2: If $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$, then $E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha x \frac{d}{dx} E_{\alpha,\beta+1}(z)$, for $\alpha \in \mathbb{C}, \beta \in \mathbb{C}, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0$ and $z \in \mathbb{C}$.

Corollary 1: $\frac{d}{dz} E_{\alpha,\beta+1}(z) = \frac{1}{\alpha z}\left[E_{\alpha,\beta}(z) - \beta E_{\alpha,\beta+1}(z)\right]$, for $\alpha \in \mathbb{C}, \beta \in \mathbb{C}, \mathbb{R}(\alpha) > 0$, $\mathbb{R}(\beta) > 0$ and $z \in \mathbb{C}$.

Corollary 2: $\frac{d}{dz}E_{\alpha,\beta}(z) = \frac{1}{\alpha z}\left[E_{\alpha,\beta-1}(z) - (\beta - 1)E_{\alpha,\beta}(z)\right]$, for $\alpha \in \mathbb{C}, \beta \in \mathbb{C}, \mathbb{R}(\alpha) > 0$, $\mathbb{R}(\beta) > 0$ and $z \in \mathbb{C}$.

The M-L function provides a simple generalization of the exponential function because of the substitution of $n! = (n + 1)n!$ With $(n\,\alpha)! = (n\,\alpha + 1)$. Particular cases of generalized Mittag-Leffler function two parameter, recover elementary functions are recovered.

- ✓ $E_0(z) = \frac{1}{1-z}, |z|$
- ✓ $E_{\alpha,\beta}(0) = 1$
- ✓ $E_{1,1}(z) = E_1(z) = \mathrm{e}^z$
- ✓ $E_{1,2}(z) = \frac{\mathrm{e}^z}{z} - \frac{1}{z}$
- ✓ $E_{1,3}(z) = \frac{\mathrm{e}^z - 1 - z}{z^2}$

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Also we have

- ✓ $E_{1,m}(z) = \frac{1}{z^{m-1}}\{\mathrm{e}^z - \sum_{k=0}^{m-1}\frac{z^k}{k!}\}$,

The hyperbolic sine and cosine are also specific cases of the Mittag-Leffler function.

- ✓ $E_{2,1}(z^2) = \sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k)!} = \frac{\cosh(z)}{z}$.
- ✓ $E_{2,2}(z^2) = \sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(\alpha k+2)} = \sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}$.

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- ✓ $E_{\alpha,1}(z) = E_\alpha(z), \forall\alpha > 0$

And we continue with the hyperbolic function. We look at the order of n

$h_r(z,n) = \sum_{k=0}^{\infty}\frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1}E_{n,r}(z^n)$, which is also a Mittag-Leffler function.

The other one is an error function. We take the general form of the Mittag-Leffler function and put α instead of $\frac{1}{2}$ and put β instead of 1.

$$E_{\frac{1}{2},1}(z) = E_{\alpha,\beta}(z) = \sum_{k=0(}^{\infty} \frac{z^k}{\frac{k}{2}+1)} = e^{-z^2} erfc(-z)$$

The error function is also defined by the following: $\mathrm{Erfc(z)} = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2}\, dt$.

Therefore; $E_{\frac{1}{2},1}(z) = E_{\alpha,\beta}(z) = \sum_{k=0(}^{\infty} \frac{z^k}{\frac{k}{2}+1)} = e^{z^2} \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} d_t = e^{z^2} \left[1 + \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} d_{t\tau}\right]$

1.3. Laplace Transform

Definition 4 [12-16]**:** Let $f(t)$ is a given function defined for all positive value of t, then the Laplace transform of real valued function $f(t)$ is defined by an improper integral $\int_0^{\infty} f(t) e^{-st}\, dt$ if it exists, where s is the parameter. We denote the Laplace transform $f(t)$ by $\mathcal{L}[f(t)]$ or $F(s)$ and is given by $F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$.

- ✓ Inverse**:** Let $\mathcal{L}[f(t)] = F(s)$, then $f(t)$is called an inverse Laplace transform of $F(s)$ and we write it as: $\mathcal{L}^{-1}[F(s)] = f(t)$
- ✓ First shifting property: If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[e^{at} f(t)] = F(s-a)$, $t$hen
- ✓ Second shifting property: Let $g(t) = \begin{cases} f(t-a), if t > a \\ 0, if\ 0 \le t < a \end{cases}$, then

  $$\mathcal{L}[g(t)] = e^{-as} F(s),\ \text{where } \mathcal{L}[f(t)] = F(s).$$
- ✓ The function $u_\alpha(t) = u(t-\alpha)$, for each $\alpha > 0$ defined on $[0,\infty]$ and is given by: $u_\alpha(t) = \begin{cases} 0, if\ 0 \le t < \alpha \\ 1, if\ t > \alpha \end{cases}$ and is called a unit step function since, $u_\alpha(t)$ is piecewise continuous and bounded on $[0,\infty]$. Its Laplace transform exists for all $s > 0, \mathcal{L}[u_\alpha(t)] = \frac{e^{-\alpha s}}{s}$.
- ✓ Derivatives: $\mathcal{L}\left[\frac{d^n}{dx^n} f(t)\right] = S^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f(0)$, where $n \in \mathbb{Z}$.
- ✓ Generally, for $n \in \mathbb{Z}, If\ \mathcal{L}[\mathrm{f(t)}] = F(s)$, then $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$.

1.4. Fractional Calculus

The origins of fractional calculus can be traced back to the end of 17th century, and it has been around almost as long as calculus [1-8]. However, application cases of fractional calculus are not officially reported until 1960s [1-3]. During this time, many fractional studies were called other names. For example, the 1889' Curie's law' proposed a capacitive universal model, which was later recognized in the late 20th century as the fractional capacitance model or constant phase element [17-33]; Abel's integral equation in 1845 was actually a fractional differential equation [3], and so on. In other words, because classical calculus is a special case of fractional calculus, quite a few anomalous phenomena can be seen as normal ones under the wider scheme of fractional order dynamics. That is to say, the dynamic response of fractional order RLC circuit model provides a universal way to analyse the dynamics of real RLC circuits. While fractional calculus is widely used in electronics and control theory, it is also applied to other fields like bioengineering [12], biology [12-21], and electrochemical [17-31], where various fractional order equivalent circuit models are introduced. Mean-while, partial differential equations also appear in the modeling of these phenomena [19].

In recent years, constant phase element greatly describes some electronically phenomena and have better accuracy comparing with the classical capacitor model and has been widely used in equivalent circuit modeling [16, 17] and impedance analysis [18, 19]. Fractional-order equivalent circuit models with constant phase element and fractional order inductance L concern RL, RC, LC and RLC in recent researches. The RC and RL models are analysed in [17-33]. The current, voltage and power analysis of fractional order RC and RL elements are provided in [17-29]. By applying the Laplace transform of the Caputo fractional derivative, analysis of the time and frequency properties of fractional order RC electrical circuit is reported in [29]. The impedance characteristic, the sensitivity analysis and pure imaginary impedance condition for the fractional order RC and RL circuits in the frequency domain is provided in [30]. Analysis of the fractional order LC circuit in the steady state regime is discussed in [33]. The transient regime analysis of the fractional order series RLC circuit is provided in [21, 30, 31] and the parallel one is analysed in [32]. A method for electric circuit's responses determination under different voltage source signals of the fractional order RLC circuit is reported in [30, 31]. It seems that fractional order RLC circuit is more universal, but only the physical meaning of fractional capacitance is clear.

In addition, the analysis of fractional order RLC circuit is either based on the assumption or the analysis is carried out with complex Mittag-Leffler function term series, so the physical meaning is unclear and the algorithm is easy to scatter. It should be noted that the fractional order RLC circuit is usually a non-commensurate fractional order model. The dynamic response analysis of it remains open [23]. The time domain oscillation and frequency domain resonance of RLC circuit are fundamental to many real-world applications. In frequency domain, the steady state regime of fractional order RLC circuit is discussed in [30, 31]. The resonance, quality factor and stability analysis of fractional order RLC circuit are provided in [29]; the resonance condition and frequency characteristics are reported in [30]. However, the time domain analysis depends on the series of Mittag-Leffler function terms, the characteristic analysis is difficult to start, the numerical calculation is easy to scatter, and i.e. the classical series method takes the risk of divergence in time domain and large error in frequency domain. This is the bottleneck that affects the popularization and application of fractional circuit model. Fractional order oscillator equation is analyzed in [21-34]. In the field of electronics, oscillators have important applications in signal generation.

In recent years, more and more theory and design of fractional order oscillators is provided. The topology of the Wien bridge oscillator family is analyzed in [29], while [24-27] provides four practical sinusoidal oscillators. The fractional-order differential equations design of sinusoidal oscillators is reported in [31]. The Barhkausen conditions for fractional order oscillate systems and fractional generalization of some famous integer-order sinusoidal oscillators is shown in [26]. Analysis of the fractional order operational transform resistance amplifiers based oscillator is provided in [18] and some fractional order sinusoidal oscillators is designed in [20-34].

Fractional Calculus is an extension of the integer-order calculus that considers derivatives of any real or complex order. Fractional calculus was born in 1695 with a letter that L'Hopital wrote to Leibniz, where the derivative of order 1/2 is suggested. Since then, many mathematicians, like Laplace, Riemann, Liouville, Abel, among others, contributed to the development of this subject. One of the first applications of fractional calculus was due to Abel in his solution to the tautochrone problem [1-3].

According to many authors, the rule of fractional derivative has no unique definition till now, and there exist several definitions, including Grunewald-Letnikov, Riemann-Louisville, Weyl, Riesz, and Caputo representation. In the Caputo sense, the derivative of a constant is zero and we can properly define the initial conditions for the fractional differential equations, So that they can be handled analogously to the classical integer case. Caputo fractional derivative implicate a memory effect by means of convolution between the integer order derivative and a power of time [4-8].

Despite the long history of fractional calculus in the field of mathematics a large amount of real world applications of this field has appeared mainly during the last decades. This type of calculus has become so wider that almost no a branch of science and engineering can be found without fractional calculus, and a lot of books have been written in these regards. Increasing the use of fractional calculations has increased the variety of questions and resulted in various basic definitions for fractional integral and derivative.

We recall that the Riemann-Liouville definition entails physically unacceptable initial conditions; conversely for the Liouville - Caputo fractional derivative, the initial conditions are expressed in terms of integer order derivatives having direct physical significance [3-11].

Nonlinear differential equations with integer or fractional order have played a very important role in various fields of science and engineering, such as mechanics, electricity, chemistry, biology, control theory, signal processing and image processing $[16, 17]$. In all these scientific fields, it is important to obtain exact or approximate solutions of nonlinear fractional differential equations.

There exists a vast literature on different definitions of fractional derivatives and integrals. The most popular ones are the Riemann–Liouville and the Caputo derivatives. However in this project work applies the Liouville-Caputo derivative sense [3-9].

The differentiation and integration of a function f(x) with differential operator D and integral operator ∫ are defined as:

$$\frac{d}{dx} f(x) = f'(x) = D\, f(x)$$

$$\frac{d^2}{dx^2} f(x) = f'' = D^2\, f(x)$$

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$$\frac{d^n}{dx^n} f(x) = f^n(x) = D^n\, f(x), for\ n\ =\ 0, 1, 2, 3 \ldots$$

For instance, if $f(x) = x^m$, where $m \in \mathbb{N}$, then $\frac{d^n}{dx^n} f(x) = \frac{m!}{(m-n)!} (f(x))^{m-n}$

$$= \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m-n+1)} (x^{m-n}), if\ n < m + 1 \\ m!, \qquad\qquad if\ m = n \end{cases}.$$

Let $f(x) = x^m$, where $m \in \mathbb{R}$, then

$$\frac{d^n}{dx^n} f(x) = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m-n+1)} (x^{m-n}), if\ (m-n) < 1 \\ m!, \qquad\qquad if \quad n = m \end{cases} \ and\ n = 0,1,2,3 \ldots$$

Now; $\frac{d}{dt} f(x) = f'(x) = lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$\frac{d^2}{dx^2}f(x) = f''(x) = \lim_{h\to 0}\frac{f'(x+h) - f'(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\{\frac{f((x+h)+h) - f(x+h)}{h} - \frac{f(x+h) - f(x)}{h}\}$$
$$= \lim_{h\to 0}\frac{1}{h}\{\frac{f(x+2h) - 2f(x+h) + f(x)}{h}\}$$
$$= \lim_{h\to 0}\frac{1}{h^2}\{f(x+2h) - 2f(x+h) + f(x)\}.$$

Looking closely at those equations, we notice a recurring pattern in their computations.

In general, the n^{th} order derivative of the function $f(x)$ is described as follows:

$$\frac{d^n}{dx^n}f(x) = f^{(n)}(x) = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+kh),$$

where, $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ and $\binom{-n}{k} = (-1)^k\binom{n}{k}, n = 0,1,2,3,$

By following the same analogue, we will get great simplified formula/method/ developed by Grunewald Letnikov, I. Petras.

We know already:

$D^1e^{\lambda x} = \lambda e^{\lambda x}, D^2e^{\lambda x} = \lambda^2 e^{\lambda x}, \ldots, D^n e^{\lambda x} = \lambda^n e^{\lambda x},$ when n is an integer. Why not to replace n by $\frac{1}{2}$ and write, $D^{\frac{1}{2}}e^{\lambda x} = \lambda^{\frac{1}{2}}e^{\lambda x}$? Why not to go further and put $\sqrt{2}$ instead of n? Let us write

$D^\alpha e^{\lambda x} = \lambda^\alpha e^{\lambda x}$, for any values of α is an integer, rational, irrational, or complex.

We naturally want $e^{\lambda x} = D^1[D^{-1}e^{\lambda x}]$, since $e^{\lambda x} = D^1[\frac{1}{\lambda}e^{\lambda x}]$, we have;

$$D^{-1}[e^{\lambda x}] = \frac{1}{\lambda}e^{\lambda x} = \int e^{\lambda x}dx.$$

Definition 5 $[1,2]$**:** if the function $f(x)$ is continuous on $(0,\infty)$, then the Riemann-Liouville fractional derivative of order $\alpha \geq 0$ is defined as:

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}f(t)dt, \text{for } \alpha \in (1,\infty),\ n-1 < \alpha \leq n, \text{for } n \in \mathbb{N}.$$

Similarly,

$D^{-2}[e^{\lambda x}] = \int\int e^{\lambda x}dxdx = \lambda^{-2}e^{\lambda x} = \frac{1}{\lambda^2}e^{\lambda x}$. So it is reasonable to interpret D^α when α is a negative integer and $-n$ as the n^{th} iterated integral.

D^α Represents α order derivative if α is positive number and integral if α is a negative number.

Mean that, $D^{-1} f(x) = \int_0^x f(t)\, dt$.

$$D^{-2} f(x) = \int_0^x \int_0^{t_2} f(t_1)\, dt_1 dt_2 = \int_0^x \int_{t_1}^x f(t_1)\, dt_2 dt_1 = \int_0^x f(t_1) \int_{t_1}^x dt_2 dt_1$$
$$= \int_0^x f(t_1)(x - t_1) dt_1.$$

Using the same procedure, we have that:

$$D^{-3} f(x) = \frac{1}{2}\int_0^x f(t)(x-t)^2 dt \text{ and } D^{-4} f(x) = \frac{1}{2\times 3}\int_0^x f(t)(x-t)^3 dt.$$

Generally: $D^{-n} f(x) = \frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1} dt$ from the method of Dirichlet, 1908. $D^{-\alpha} f(x) = \frac{1}{(\alpha-1)!}\int_0^x f(t)(x-t)^{\alpha-1} dt$, $D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1} dt$, (Riemann), for $0 < \alpha < 1$, and $n = 1,2,3,\ldots$.

For any positive real number α: $Lim_{h\to 0} \frac{1}{h^\alpha}\sum_{k=0}^{n}(-1)^k \binom{\alpha}{k} f(x - kh)$.

Definition 6 [3, 4]**:** The Riemann-Liouville fractional integral of order $\alpha \geq 0$, for a function $f(x)$ which is continuous on $(0, \infty)$ is defined as follows.

$$I^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t) dt, \text{if } \mathbb{R}(\alpha) > 0 \text{ and } \alpha \in \mathbb{C} \\ f(x), \quad \text{if} \quad \alpha = 0 \end{cases}.$$

Definition 7[**5**, **6**]: The Liouville-Caputo differential operator (C) with fractional order $\alpha > 0$ if the function $f(x)$ is continuous on $(0, \infty)$ is defined as: ${}_0^cD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x - t)^{n-\alpha-1} f^{(n)}(t) dt$.

The Laplace transform of this equation with the operator $\mathcal{L}$ is defined as follows.

$\mathcal{L}\,[{}_0^cD_x^\alpha f(x)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$, where $n = R(\alpha) + 1$.

From this expression we have two cases:

$\mathcal{L}\,[{}_0^cD_x^\alpha f(x)] = s^\alpha F(s) - s^{\alpha-1} f(0)$, if $0 < \alpha \leq 1$.

$\mathcal{L}[{}_0^cD_x^\alpha f(x)] = s^\alpha F(s) - s^{\alpha-1} f(s) - s^{\alpha-2} f'(0)$, if $1 < \alpha \leq 2$.

Definition 8[6, 7]**:** The Caputo-Fabrizio fractional differential operator (CF) if $f(x)$ is continuous on $(0, \infty)$ is defined as: ${}_0^{cf}D_x^\alpha f(x) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^x f'(\theta)\, e^{(-\alpha\frac{(x-\theta)}{1-\alpha})} d\theta$, where M is a normalization function such that $M(0) = M(1) = 1$, and $M(\alpha) = \frac{2}{2-\alpha}$.

If $n \geq 1$ *and* $\alpha \in [0, 1]$, CF operator of order $(n + \alpha)$ is defined by:

${}_0^{cf}D_x^{(\alpha+n)} f(x) = {}_0^{cf}D_x^\alpha[{}_0^{cf}D_x^n f(x)]$.

The Laplace transform of this equation is defined as follows:

$\mathcal{L}[{}^{cf}_{0}D_x^{(\alpha+n)} f(x)] = \frac{s^{n+1}\uparrow\mathcal{L}[f(x)-s^n f(0)-s^{n-1}f'(0)\ldots-f^{(n)}(0)}{s+\alpha(1-s)}$. From this expression we have:

$\mathcal{L}[{}^{cf}_{0}D_x^{\alpha} f(x)] = \frac{s\mathcal{L}[f(x)-f(0)}{s+\alpha(1-s)}, n = 0$ *and* $\mathcal{L}\ [{}^{cf}_{0}D_x^{\alpha} f(x)] = \frac{s^2\mathcal{L}[f(x)-sf(0)-f'(0)}{s+\alpha(1-s)}, n = 1.$

Definition 9[21]**:** The Atangana-Baleanu fractional operator (ABC) in Liouville-Caputo sense if f is continuous on $(0,\infty)$ is defined as follows. ${}^{ABC}_{0}D_x^{\alpha} f(x)] = \frac{B(\alpha)}{1-\alpha}\int_\alpha^x f'(\theta)E_\alpha\left[-\alpha\frac{(x-\theta)^\alpha}{1-\alpha}\right]d\theta$, where B is a normalization function such that $B(0) = B(1) = 1$.

The Laplace transform of this equation is defined as follows.

$\mathcal{L}[{}^{ABC}_{0}D_x^{\alpha} f(x)](s) = \frac{B(\alpha)}{1-\alpha}\mathcal{L}[\int_\alpha^x f'(\theta)\, E_\alpha\left[-\alpha\frac{(x-\theta)^\alpha}{1-\alpha}\right]d\theta = \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha\mathcal{L}[f(x)](s)-s^{\alpha-1}f(0)}{s^\alpha+\frac{\alpha}{1-\alpha}}.$

Definition 10[21]: The Atangana-Baleanu also suggests another fractional derivative operator (ABR) in Riemann-Liouville sense if $f(x)$ is continuous on $(0,\infty)$ is defined as follows:

${}^{ABR}_{0}D_x^{\alpha} f(x)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{dx}\int_\alpha^x f(\theta)E_\alpha[-\alpha\frac{(x-\theta)^\alpha}{1-\alpha}]d\theta$, where B is a normalization function such that $B(0) = M(0) = M(1) = 1$.

In general: $M\ (\alpha) = \frac{2}{2-\alpha}$ [9].

The Laplace transform of this equation is defined as follows.

$\mathcal{L}[{}^{ABR}_{0}D_x^{\alpha} f(x)](s) = \frac{B(\alpha)}{1-\alpha}\mathcal{L}[\frac{d}{dx}\int_\alpha^x f(\theta)E_\alpha(-\alpha\frac{(x-\theta)^\alpha}{1-\alpha})d\theta] = \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha\mathcal{L}[f(x)](s)}{s^\alpha+\frac{\alpha}{1-\alpha}}.$

For instance, $f(x) = x$, then find, $D_x^{\frac{1}{2}} f(x)$ with respect to x, then

Solution: $D_x^{\frac{1}{2}} f(x) = \frac{\Gamma(1+1)}{\Gamma\left(1-\frac{1}{2}+1\right)}x^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma\left(\frac{1}{2}+1\right)}\sqrt{x} = \frac{1\Gamma(1)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\sqrt{x} = \frac{\sqrt{x}}{\frac{1}{2}\sqrt{\pi}} = \frac{2\sqrt{x}}{\sqrt{\pi}}, since\ \Gamma\ \left(\frac{1}{2}\right) = \sqrt{\pi}$

CHAPTER 2

APPLICATIONS OF FRACTIONAL CALCULUS ON ELECTRICAL RLC CIRCUIT

2.1. Introduction

Classical electrical circuits consist of resistors, inductors and capacitors. However, these electrical components have a non-conservative feature that involve irreversible dissipative effects, such as ohmic friction or internal friction, thermal memory and nonlinearities due to the effects of the electric and magnetic fields, these dissipative effects are not considered in the standard theoretical calculations [19-33]. These dissipative effects originate non-conservative systems and equations to describe the behavior of these systems must be non-local differential equations in time; with this purpose, in the last decades the Fractional Calculus (FC) allows the investigation of the nonlocal response of multiple phenomena [4-10], the fractional derivatives are memory operators which usually represent dissipative effects or damage. Fractional order calculus considers the history and non-local distributed effects of any physical system, particularly in electrical circuits, the use of fractional order operators allows us to generalize the propagation of electrical signals in devices, circuits and networks [23-34], as well, the modeling of electrical components (capacitors, coils, memristors, domino ladders, tree structures), see [21-28]. [29] Has suggested a fractional differential equation that combines the simple harmonic oscillations of a LC circuit with the discharging of RC circuit. In [33] the simple current source-wire circuit has been studied fractionally using direct and alternating current source. It was shown that the wire acquires an inducting behavior as the current is initiated in it and gradually recovers its resisting behavior, recently, the authors of [21-34] considered theoretically and experimentally the charging and discharging processes of different capacitors in electrical RC circuits, the measured experimental results could be exactly obtained within the fractional calculus approach.

We know that, differentiation is the time rate of change of quantities; the integral rates of change of quantities are known in classical calculus. However, it cannot determine the properties of these variations in a fraction of time. Further, it is important to know the rate of change of quantities in a fractional time for the sake of different real world events. So that we

will investigate a model in fractional order used to determine the solution of electrical RLC circuit problems in a fraction of time like 0.5, 1.5, 3.4, etc. This project work describes the rate of change of current, voltage and charge variations at the instant time.
Current is defined as the motion of charges. The charges that move in a current are called charge carriers. The charge carriers in metals are electron. A current is a motion of charges sustained by an internal electric field that creates a current [24].

An electric field in the wire produces a force that pushes all the electrons and this electric field accelerates all the electrons in the same direction called current. The current is equal to the amount of charge that passes out the end of the wire per unit time. The rate of flow of electric charge – that is, the electric current - is measured in amperes (A). If a charge flows at a rate of one ampere, and continuous to flow like that for a second, then the total amount of charge that has passed is one coulomb(C). $\mathbf{q = it}$, where q stands for the quantity of charge which passes when current (i) flows for a time (t) [24].

Conductivity is a way of measuring a materials ability to allow an electric current to flow. Its unit is Siemens per meter (s/m). The inverse of conductivity is resistivity. It is a measure of how much a material resists a flow of an electric current. And it is a material property depends on the type of materials and temperature.
We know that when a voltage is connected across apiece of wire, it pushes the free electrons so that they flow through the wire and produce an electric current. The electrons start to flow instantaneously because the free electrons are already spread through the wire.
As soon as the voltage is applied, there is an electromotive force on all the electrons, which gets them moving. Electrical circuits transfer energy from batteries to the other components. When the circuit is complete the energy from the battery pushes the current around the circuit and transfers the energy to the component, which can then work. The energy or push that the battery gives to the circuit is called the voltage or electromotive force of the battery. It is measured in volts (v).

The higher the voltage is the greater the amount of energy that can be transferred. Voltage is a measure of the difference in electrical energy between two parts of a circuit. Because the energy is transferred by the component there must be more energy entering the component than there is leaving the component. Voltage is sometimes called potential difference. Potential difference measures the difference in the amount of energy the current is carrying either side of the component. This voltage drop across the component tells us how much energy the component is transferring. Potential difference is defined as energy per unit charge. The unit of potential difference is the volt. Using the definition, we can define the volt as Joule per coulomb. One Joule is the energy exerted by a force of one newton acting to move an object through a distance of one meter.

- Kirchhoff's loop rule stated as in any closed loop in a circuit the sum of the electromotive force is equal to the sum of potential difference.
- Ohm's law defines the relation of current and voltage as $V = iR$.

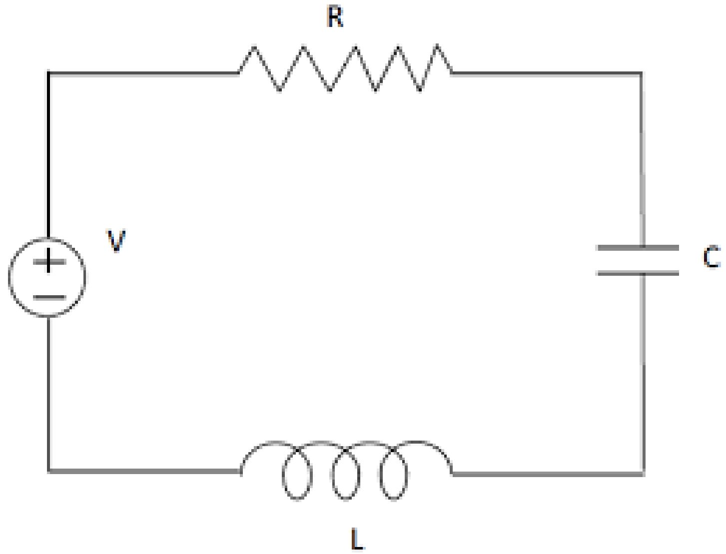
The principle of conservation of energy means that, at any time t, the rate at which energy is supplied by the supply potential difference must equal the sum of the rate at which it is stored in the capacitive and inductive elements and dissipated by the resistive elements (here we assume that ideal capacitors and inductors have no internal resistance) [18-26]. We know that supply potential difference (V) is given by:

The voltage drop across resistor $V_R(t) = Ri(t) = R\frac{d}{dt}q(t)$.

The voltage drop across inductor $V_L(t) = L\frac{d}{dt}i(t) = L\frac{d^2}{dt^2}q(t)$.

The voltage drop across capacitor $V_C(t) = \int_0^t i(x)d_x = \frac{1}{C}q(t)$.

On using Kirchhoff's voltage law around any loop in a circuit, the voltage raised must equal to the voltage drops. Therefore we will get the following equation for electrical RLC circuit represented in the figure 1



$$E\ (t) = V_r + V_L + V_C = L\frac{d^2}{dt^2}q(t) + R\frac{d}{dt}q(t) + \frac{1}{C}q(t).$$

In the form of current the equation of electrical RLC circuit represented in the figure 1 is given as; $E\ (t) = Ri(t) + L\frac{d}{dt}i(t) + \frac{1}{c}\int_0^t i(t) + V\ (0) = V_P\,sin(\omega t)$, where V_p is the value of peak potential difference for the circuit.

We also know that power is $\frac{work\ done}{time} = \frac{\Delta W}{\Delta t}$.

In the case of electrical energy, we also know that Power = potential difference×current.

In an alternating circuit, we know that the current is given by: $\mathrm{i} = \mathrm{i_p}\sin(\omega t - \theta)$, where $\mathrm{i_p}$ is the peak value for the current and θ is the phase angle for the circuit. By differentiating from both sides with respect to time t, we get; $\mathrm{R}\frac{d}{dt}\mathrm{i(t)} + \mathrm{L}\frac{d^2}{dt^2}\mathrm{i(t)} + \frac{1}{c}\mathrm{i(t)} = \mathrm{V_P}\omega t\cos(\omega t)$

Consider: $\mathrm{E\ (t)} = \mathrm{V_P}\omega t\cos(\omega t)$, then $\mathrm{E\ (t)} = \mathrm{L}\frac{d^2}{dt^2}\mathrm{i(t)} + \mathrm{R}\frac{d}{dt}\mathrm{i(t)} + \frac{1}{c}\mathrm{i(t)}$, dividing both sides by L we get: $\frac{\mathrm{E(t)}}{\mathrm{L}} = \frac{d^2}{dt^2}\mathrm{i(t)} + \frac{\mathrm{R}}{\mathrm{L}}\frac{d}{dt}\mathrm{i(t)} + \frac{1}{\mathrm{LC}}\mathrm{i(t)}$.

The amount of current (i) flow through the wire in the circuit $\mathrm{i(t)} = \frac{\mathrm{V}}{\mathrm{R}}$

The amount of charge (q) in the circuit; $\mathrm{q(t)} = \frac{\mathrm{V_l}}{\mathrm{V}}$

The voltage through impedance $\mathrm{V_l} = \mathrm{Vq} = \frac{\mathrm{V}}{\mathrm{R}}\omega_0\mathrm{L}$

An oscillating circuit in series, in general, is an electrical circuit consisting of three kinds of circuit elements: a resistor with a resistance R measured in ohms, an inductor with an inductance L measured in henneries, and a capacitor with capacitance C measured in farads. The change with respect to time of the electric charge $q(t)$ in the shell of the capacitor is described by the second order homogeneous differential equation.

$$\mathrm{L}\frac{d^2}{dt}\mathrm{q(t)} + \mathrm{R}\frac{d}{dt}\mathrm{q(t)} + \frac{\mathrm{q(t)}}{\mathrm{C}} = 0$$

The term $\mathrm{q(t)}$ and C are very important because its lack implies that we have not an oscillating circuit. The main goal of this work is the study of the above two second order differential equations (one in terms of $i(t)$ and the other in terms of $\mathrm{q(t)}$ from the point of view of the fractional calculus. Unlike other works, in which the pass from ordinary derivative to fractional one is direct, a systematic way to construct fractional differential equations for the physical systems has been proposed.

Consider an RLC-circuit, as shown in the above Figure 1. To create an electromotive force $E_0 f$(t) V (volts), a resistor with resistance R Ω (ohms), an inductor with inductance L H (henrys), and a capacitor with capacitance C F (farads) are connected in series as shown in the above Figure 1. According to Kirchhoff's voltage law (KVL), the impressed voltage $E_0 f$(t) equals the sum of the voltage drops across the three loop components R, L and C in the circuit shown in the above Figure 1. Current I passing through a resistor, inductor or capacitor create a voltage drop (voltage difference measured in volts) at both ends. As per experiments; these drops are:

- For a resistor of resistance R ohm, the voltage drop is Ri (Ohm's law);
- For an inductor with inductance L henry(H), the voltage drop is $L\frac{di}{dt}$
- For a capacitor of capacitance C farads(F), the voltage drop is $\frac{q}{C}$
- The charge on the capacitor, q coulombs, is linked to the current by

$$i(t) = \frac{dq}{dt} \Rightarrow q(t) = \int i(t)d_t$$

According to Kirchhoff's voltage law (KVL), we get the "integro-differential equation" in Figure 1 above for an RLC-circuit with electromotive force $E_0 f(t)$ (E_0 constant) as a model.

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i(t)d_t = E_0 f\ (t).$$

To remove the integral we differentiate, with respect to t,

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i\ = E_0 f'(t).$$

This demonstrates that the current in an RLC-circuit may be determined as the solution of a nonhomogeneous second order ordinary differential equation with constant coefficients [22_33].

2.2. Electrical RLC Circuit Problems

The three basic elements used in electronic circuits are the resister, capacitor and inductor. They each play an important role in how an electronic circuit behaves. They also have their own standard symbols and units of measurement.

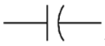
Resistors (R): A resistor represents a given amount of resistance in a circuit. Resistance is a measure of how the flow of electric current is opposed or "resisted." It is defined by Ohm's law which says the resistance equals the voltage divided by the current.

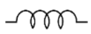
$$\text{Resistance} = \frac{\text{voltage}}{\text{current}}$$

or

$$\text{R} = \frac{\text{V}}{\text{i}}$$

Resistance is measured in Ohms. The Ohm is often represented by the omega symbol Ω. The symbol for resistance is a zigzag line like this  .

Capacitors (C): A capacitor represents the amount of capacitance in a circuit. The capacitance is the ability of a component to store an electrical charge. You can think of it as the "capacity" to store a charge. The capacitance is defined by the equation, $C = \frac{q}{V}$, where q is the charge in coulombs and V is the voltage. In a DC circuit, a capacitor becomes an open circuit blocking any DC current from passing the capacitor. Only AC current will pass through a capacitor. Capacitance is measured in Farads. The symbol for capacitance is two parallel lines like this .

Inductors (L): An inductor represents the amount of inductance in a circuit. The inductance is the ability of a component to generate electromotive force due to a change in the flow of current. A simple inductor is made by looping a wire into a coil. Inductors are used in electronic circuits to reduce or oppose the change in electric current. In a DC circuit, an inductor looks like a wire. It has no affect when the current is constant. Inductance only has an effect when the current is changing as in an AC circuit. It is measured in Henrys. The symbol for inductance is a series of coils as like this 
Combinations of capacitors, inductors, and resistors are used to build passive filters that will only allow electronic signals of certain frequencies to pass through.

Impedance (Z): Is a measure of an opposition to electrical flow. It is measured in ohms. For DC systems, impedance and resistance are the same. If the circuit elements are a Resister(R), a Capacitor(C) & an Inductor(L), then the $\text{Impedance}(Z) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ & phase angle, $\theta < 0$, if $\omega L > \frac{1}{\omega C}$, or $\theta > 0$, if $\omega L < \frac{1}{\omega C}$.

The combination of inductor (L) & capacitor(C) is called reactance. $S = \omega L - \frac{1}{\omega C}$.

The amplitude $(I_0) = \frac{E_0}{\sqrt{R^2+S^2}}$ & phase $\text{lag}(\theta) = \frac{S}{R}$.

Impedance $(Z) = \sqrt{R^2 + S^2}$ & $\frac{E}{i} = R$ (Ohm's law) it is also known as apparent reactance.

$$\text{Curent amplitude} = \frac{E_0}{R}.$$

$$\text{Capacitive reactance} = \frac{1}{\omega C}.$$

Inductive reactance $= \omega L$.

Z is minimum if $\omega L = \frac{1}{\omega C}$ this is a resonance.

If $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$, then the system of the circuit will be over damped response.

If $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$, then the system of the circuit will be critically damped response.

If $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$, then the system of the circuit will be under damped response.

$\sqrt{(\frac{1}{\sqrt{LC}})^2 - (\frac{R}{2L})^2}$ is called the damped resonance frequency or the damped natural frequency.

Kirchhoff's junction rule states that the total current flowing into a point is equal to the total current flowing out of that point.

Ohm's law can be summarized using the equation $V = iR$. This can be used to analyze circuits and solve circuit problems involving potential difference, current and resistance. The power dissipated in an electrical component is $P = iV = i^2 R$.

So our solutions of electrical RLC circuit problems analyzed based on these the above concepts; hence we will state and describe these solutions according to dampness with respect to reactance.

According to Kirchhoff's laws of the equation of RLC electrical circuit is:

$$V(t) = V_r + V_L + V_C$$

As the time varying voltage across RLC is a function on which has sinusoidal (oscillating curve) given the nature of radio wave (as sine or cosine time graph). So that the total voltage through the RLC circuit is defined as: $V(t) = V_0 \sin(\omega t)$, where ωt is angular frequency.

Hence; $V(t) = V_r + V_L + V_C = Ri(t) + L\frac{di}{dt} + \frac{1}{c}\int_0^t i(t)dt + V(0) = V_0 \sin(\omega t)$.

By differentiating from both sides with respect to time (t), we get:

$$L\frac{d^2}{dt^2}i + R\frac{di}{dt} + \frac{1}{c}i(t) = V_0 \omega \cos(\omega t).$$

Consider $E'(t) = V_0 \omega \cos(\omega t)$, then $E'(t) = L\frac{d^2}{dt^2}i(t) + R\frac{d}{dt}i(t) + \frac{1}{c}i(t)$, dividing by L from both sides we get; $\frac{E'(t)}{L} = \frac{d^2}{dt^2}i(t) + \frac{R}{L}\frac{d}{dt}i(t) + \frac{1}{LC}i(t)$.

It is called Kirchhoff's laws of second order differential electrical RLC circuit.

The fractional order calculus has been applied in the circuit theory for many years [22-24]. Reviews of numerous formulations of fractional order derivatives can be found in classical monographs [19-21].

The main reason for applying fractional order derivatives to the modeling of electrical-circuit elements stems from its nonlocal properties when compared to the classical integral order definition.

Unlike integral order derivatives, fractional order derivatives include a memory of all previous states of the considered circuit element (i.e., time-domain history) in calculations. Furthermore, with regard to the design process, an additional parameter, namely the order of fractional order derivative appears; which allows for design flexibility and optimization freedom [17, 18]. In this project, we will review the circuit-level modeling of transmission lines based on fractional order derivatives rather than analysis. In fractional order models of transmission lines, the fractional order inductance can be useful for modeling of the skin effect whil it is the fractional order capacitance is able to model various non-idealities of characteristics of dielectric media (e.g., accumulation of electric charge along the line and memory effects in dielectric polarization). As demonstrated in many experimental works [16_24], the fractional order transmission-line model allows for more compact and accurate analytical modeling over a wide-frequency band when compared to the traditional integral order modeling.

In this paper, we will find the analytic solution of a fractional differential equation associated with electrical RLC circuit. The fractional order differential equation of the above equation can be written as follow.

$L\frac{d^\alpha}{dt^\alpha}\, i(t) + \; R\frac{d^\beta}{dt^\beta} i(t) + \; \frac{1}{c} i(t) \; = \; E'(t)$, where $1 < \; \alpha \; \leq \; 2, 0 < \beta \; \leq 1$.

$\frac{d^\alpha}{dt^\alpha} i(t) + \; \frac{R}{L}\frac{d^\beta}{dt^\beta} i(t) + \; \frac{1}{LC} i(t) \; = E_0\omega\, cos\, \omega t$, and $(\alpha - \beta) > 0$.

where $lim_{\alpha \to 2}\frac{d^\alpha}{dt^\alpha} i(t) = \frac{d^2}{dt^2} i(t)$ & $lim_{\beta \to 1}\frac{d^\beta}{dt^\beta} i(t) = \frac{d}{dt} i(t)$.

By taking the Laplace transform from both sides,

$$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha} i(t) + \frac{R}{L}\frac{d^\beta}{dt^\beta} i(t) + \frac{1}{LC} i(t)\right] = \mathcal{L}\left[\frac{E_0\omega}{L} cos\, \omega t\right]$$

$$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha}\, i(t)\right] + \mathcal{L}\left[\frac{R}{L}\frac{d^\beta}{dt^\beta} i(t)\right] + \mathcal{L}\left[\frac{1}{LC} i(t)\right] = \mathcal{L}\left[\frac{E_0\omega}{L} cos\, \omega t\right]$$

$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha} i(t)\right] + \; \frac{R}{L}\mathcal{L}[\frac{d^\beta}{dt^\beta} i(t)\Big] + \frac{1}{LC}\mathcal{L}\Big[i(t)\Big] = E_0\omega\mathcal{L}[cos\, \omega t]$, by properties of Laplace

Using the Laplace transform of the Caputo differential operator is:

$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha} i(t)\right] = s^\alpha\mathcal{L}[i(t)] - s^{\alpha-1} i(0) - s^{\alpha-2} i'(0)$, $\mathcal{L}\left[\frac{d^\beta}{dt^\beta} i(t)\right] = s^\beta\mathcal{L}[i(t)] - s^{\beta-1} i(0)$.

And $E_0\omega\mathcal{L}[cos\, \omega t] = E_0\omega[\frac{s}{s^2+\omega^2}]$, where $\mathcal{L}[i(t)] = i(s), E_0\omega = \mathbb{R}$, s is the parameter (may be real or complex).

Now; $s^{\alpha}\mathcal{L}[i(t)] - s^{\alpha-1}i(0) - s^{\alpha-2}i'(0) + \frac{R}{L}\left[s^{\beta}\mathcal{L}[i(t)] - s^{\beta-1}i(0)\right] + \frac{1}{LC}\mathcal{L}[i(t)]$

$$= \frac{E_0\omega}{L}\left[\frac{s}{s^2+\omega^2}\right].$$

$\Rightarrow s^{\alpha}\mathcal{L}[i(t)] + \frac{R}{L}\left[s^{\beta}\mathcal{L}[i(t)]\right] + \frac{1}{LC}\mathcal{L}[i(t)] = \frac{E_0\omega}{L}\left[\frac{s}{s^2+\omega^2}\right] + s^{\alpha-1}i(0) + s^{\alpha-2}i'(0) + \frac{R}{L}s^{\beta-1}i(0).$

$\Rightarrow \mathcal{L}[i(t)]\left(s^{\alpha} + \frac{R}{L}s^{\beta} + \frac{1}{LC}\right) = \frac{E_0\omega}{L}\left(\frac{s}{s^2+\omega^2}\right) + s^{\alpha-1}i(0) + s^{\alpha-2}i'(0) + \frac{R}{L}s^{\beta-1}i(0).$

By assuming the initial condition $i(0) = A \; and \; i'(0) = B$ and further using equation.

$I(s) = \frac{\frac{E_0\omega}{L}\left(\frac{s}{s^2+\omega^2}\right) + s^{\alpha-1}A + s^{\alpha-2}B + \frac{R}{L}s^{\beta-1}A}{s^{\alpha} + \frac{R}{L}s^{\beta} + \frac{1}{LC}}$, where, $\mathcal{L}[i(t)] = I(s)$ applying inverse Laplace transform we have: $i(t) = \mathcal{L}^{-1}\left[\frac{\frac{E_0\omega}{L}\left(\frac{s}{s^2+\omega^2}\right) + s^{\alpha-1}A + s^{\alpha-2}B + \frac{R}{L}s^{\beta-1}A}{s^{\alpha} + \frac{R}{L}s^{\beta} + \frac{1}{LC}}\right]$

$$= \mathcal{L}^{-1}\left[\frac{\frac{E_0\omega}{L}\left(\frac{s}{s^2+\omega^2}\right)}{s^{\alpha} + \frac{R}{L}s^{\beta} + \frac{1}{LC}}\right] + \mathcal{L}^{-1}\left[\frac{s^{\alpha-1}A}{s^{\alpha} + \frac{R}{L}s^{\beta} + \frac{1}{LC}}\right] + \mathcal{L}^{-1}\left[\frac{s^{\alpha-2}B}{s^{\alpha} + \frac{R}{L}s^{\beta} + \frac{1}{LC}}\right]$$

$$+ \mathcal{L}^{-1}\left[\frac{\frac{R}{L}s^{\beta-1}A}{s^{\alpha} + \frac{R}{L}s^{\beta} + \frac{1}{LC}}\right], by \; sum \; rule.$$

where $1 < \alpha \leq 2$ and $0 < \beta \leq 1$.

In this study the researcher will obtain the analytic solution of the Liouville-Caputo equation sense of electrical RLC circuit of the above equation involving the fractional operator with Mittag-Leffler kernel, by considering different source terms E(t) as;

- A unit step source (constants),
- an exponential source (powers), and
- Periodic source (trigonometry's)

Special cases

CaseI: When constant electromotive force is applied, i.e. $E(t) = E_0$, then

$L\frac{d^{\alpha}}{dt^{\alpha}}i(t) + R\frac{d^{\beta}}{dt^{\beta}}i(t) + \frac{1}{c}i(t) = E'(t)$, reduces into, $\frac{d^{\alpha}}{dt^{\alpha}}i(t) + \frac{R}{L}\frac{d^{\beta}}{dt^{\beta}}i(t) + \frac{1}{LC}i(t) = E_0$.

The technique we have applied to find out the analytic solution the same technique will apply to find out the solution and the analytic solution will be:

$L\frac{d^{\alpha}}{dt^{\alpha}}i(t) + R\frac{d^{\beta}}{dt^{\beta}}i(t) + \frac{1}{c}i(t) = 0$, since E_0 is constant, then ${E_0}' = 0$.

By applying the Laplace transform from both sides, we have;

$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha} i(t)\right] + \mathcal{L}\left[\frac{R}{L}\frac{d^\beta}{dt^\beta} i(t)\right] + \mathcal{L}\left[\frac{1}{LC} i(t)\right] = \mathcal{L}[0].$

$\Rightarrow s^\alpha \mathcal{L}[i(t)] - s^{\alpha-1} i(0) - s^{\alpha-2} i'(0) + \frac{R}{L}\left[s^\beta \mathcal{L}[i(t)] - s^{\beta-1} i(0)\right] + \mathcal{L}\left[\frac{1}{LC} i(t)\right] = 0,$

$since\ \mathcal{L}[k] = \frac{k}{s}, for\ k = constant.$

$\Rightarrow \mathcal{L}[i(t)]\left(s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}\right) = s^{\alpha-1} i(0) + s^{\alpha-2} i(0) + \frac{R}{L}s^{\beta-1} i(0).$

$\Rightarrow I(t) = \frac{s^{\alpha-1} I(0) + s^{\alpha-2} I'(0) + \frac{R}{L}s^{\beta-1} I(0)}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}$. By assuming the initial condition $i(0) = A$ $and\ i'(0) = B$.

$I(t) = \frac{s^{\alpha-1}A + s^{\alpha-2}B + \frac{R}{L}s^{\beta-1}A}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}$. By applying inverse Laplace transform, we have:

$$i(t) = \mathcal{L}^{-1}\left[\frac{s^{\alpha-1}A + s^{\alpha-2}B + \frac{R}{L}s^{\beta-1}A}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}\right].$$

Case II: When an exponential electromotive force is applied, i.e. $E(t) = e^t$, then $L\frac{d^\alpha}{dt^\alpha} i(t) + R\frac{d^\beta}{dt^\beta} i(t) + \frac{1}{c} i(t) = E'(t)$, reduces into; $\frac{d^\alpha}{dt^\alpha} i(t) + \frac{R}{L}\frac{d^\beta}{dt^\beta} i(t) + \frac{1}{LC} i(t) = e^t$. The technique we have applied to find out the analytic solution the same technique will apply to find out the solution and the analytic solution will be:

$L\frac{d^\alpha}{dt^\alpha} i(t) + R\frac{d^\beta}{dt^\beta} i(t) + \frac{1}{c} i(t) = e^t, since\ \frac{de^t}{dx} = e^t$. By applying the Laplace transform from both sides, we have: $\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha} i(t)\right] + \mathcal{L}\left[\frac{R}{L}\frac{d^\beta}{dt^\beta} i(t)\right] + \mathcal{L}\left[\frac{1}{LC} i(t)\right] = \mathcal{L}[e^t]$

$\Rightarrow s^\alpha \mathcal{L}[i(t)] - s^{\alpha-1} i(0) - s^{\alpha-2} i'(0) + \frac{R}{L}\left[s^\beta \mathcal{L}[i(t)] - s^{\beta-1} i(0)\right] + \mathcal{L}\left[\frac{1}{LC} i(t)\right]$

$$= \frac{1}{s-1}, since\mathcal{L}[e^{at}] = \frac{1}{s-1}$$

$$\Rightarrow \mathcal{L}[i(t)]\left(s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}\right) = \frac{1}{s-1} + s^{\alpha-1} i(0) + s^{\alpha-2} i'(0) + \frac{R}{L}s^{\beta-1} i(0)$$

$$\Rightarrow \mathcal{L}[i(t)] = \frac{\frac{1}{s-1} + s^{\alpha-1} i(0) + s^{\alpha-2} i'(0) + \frac{R}{L}s^{\beta-1} i(0)}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}.$$

By assuming the initial conditions $i(0) = A$ and $i'(0) = B$, and $\mathcal{L}[i(t)] = I(t)$, then

$I(t) = \frac{\frac{1}{s-1} + s^{\alpha-1}A + s^{\alpha-2}B + \frac{R}{L}s^{\beta-1}A}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}$. By applying inverse Laplace transform, we have:

$$i(t) = \mathcal{L}^{-1}\left[\frac{\frac{1}{s-1} + s^{\alpha-1}A + s^{\alpha-2}B + \frac{R}{L}s^{\beta-1}A}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}\right].$$

Case III: When a periodic source (trigonometric function) electromotive force is applied, i.e.

$E(t) = sin\,\omega t$, then $L\frac{d^\alpha}{dt^\alpha}i(t) + R\frac{d^\beta}{dt^\beta}i(t) + \frac{1}{c}i(t) = E'(t)$ reduces into;

$\frac{d^\alpha}{dt^\alpha}i(t) + \frac{R}{L}\frac{d^\beta}{dt^\beta}i(t) + \frac{1}{LC}i(t) = sin\,\omega t.$

$\frac{d^\alpha}{dt^\alpha}i(t) + \frac{R}{L}\frac{d^\beta}{dt^\beta}i(t) + \frac{1}{LC}i(t) = \frac{\omega}{L}cos\,\omega t$, since $\frac{d}{dt}sin\,\omega t = \omega\, cos\,\omega t.$

The technique we have applied to find out the analytic solution the same technique will apply to find out the solution and the analytic solution will be:

$$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha}i(t) + \frac{R}{L}\frac{d^\beta}{dt^\beta}i(t) + \frac{1}{LC}i(t)\right] = \mathcal{L}\left[\frac{E_0\omega}{L}cos\,\omega t\right]$$

$$\Rightarrow s^\alpha\mathcal{L}[i(t)] - s^{\alpha-1}i(0) - s^{\alpha-2}i'(0) + \frac{R}{L}\left[s^\beta\mathcal{L}[i(t)] - s^{\beta-1}i(0)\right] + \frac{1}{LC}\mathcal{L}[i(t)]$$

$$= \frac{E_0\omega}{L}\left[\frac{s}{s^2+\omega^2}\right], \text{where } \mathcal{L}[cos\,\omega t] = \frac{s}{s^2+\omega^2}.$$

$$\Rightarrow s^\alpha\mathcal{L}[i(t)] + \frac{R}{L}\left[s^\beta\mathcal{L}[i(t)]\right] + \frac{1}{LC}\mathcal{L}[i(t)]$$

$$= \frac{E_0\omega}{L}\left[\frac{s}{s^2+\omega^2}\right] + s^{\alpha-1}i(0) + s^{\alpha-2}i'(0) + \frac{R}{L}s^{\beta-1}i(0)$$

$$\Rightarrow \mathcal{L}[i(t)]\left(s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}\right) = \frac{E_0\omega}{L}\left(\frac{s}{s^2+\omega^2}\right) + s^{\alpha-1}i(0) + s^{\alpha-2}i'(0) + \frac{R}{L}s^{\beta-1}i(0).$$

By assuming the initial condition, $i(0) = A \text{ and } i'(0) = B$, then

$I(t) = \frac{\frac{E_0\omega}{L}(\frac{s}{s^2+\omega^2}) + s^{\alpha-1}A + s^{\alpha-2}B + \frac{R}{L}s^{\beta-1}A}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}$. Applying inverse Laplace transform we have:

$$i(t) = \mathcal{L}^{-1}\left[\frac{\frac{E_0\omega}{L}\left(\frac{s}{s^2+\omega^2}\right) + s^{\alpha-1}A + s^{\alpha-2}B + \frac{R}{L}s^{\beta-1}A}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{\frac{E_0\omega}{L}\left(\frac{s}{s^2+\omega^2}\right)}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}\right] + \mathcal{L}^{-1}\left[\frac{s^{\alpha-1}A}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}\right] + \mathcal{L}^{-1}\left[\frac{s^{\alpha-2}B}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}\right] + \mathcal{L}^{-1}\left[\frac{\frac{R}{L}s^{\beta-1}A}{s^\alpha + \frac{R}{L}s^\beta + \frac{1}{LC}}\right]$$

Case IV: When a periodic source (trigonometric function) electromotive force is applied, i.e. $E(t) = cos\,\omega t$, then $L\frac{d^\alpha}{dt^\alpha}i(t) + R\frac{d^\beta}{dt^\beta}i(t) + \frac{1}{c}i(t) = E'(t)$ reduces into;

$\frac{d^\alpha}{dt^\alpha}i(t) + \frac{R}{L}\frac{d^\beta}{dt^\beta}i(t) + \frac{1}{LC}i(t) = cos\,\omega t$

$$\frac{d^\alpha}{dt^\alpha}i(t) + \frac{R}{L}\frac{d^\beta}{dt^\beta}i(t) + \frac{1}{LC}i(t) = -\frac{\omega}{L}sin\,\omega t\text{ , since } \frac{d\,cos\,\omega t}{dt} = -\omega\,sin\,\omega t$$

The technique we have applied to find out the analytic solution the same technique we will apply to find out the solution and the analytic solution will be:

$$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha}i(t) + \frac{R}{L}\frac{d^\beta}{dt^\beta}i(t) + \frac{1}{LC}i(t)\right] = \mathcal{L}\left[-\frac{E_0\omega}{L}sin\,\omega t\right], \mathcal{L}[sin\,\omega t] = \frac{\omega}{s^2+a^2}$$

$$\Rightarrow s^\alpha\mathcal{L}[i(t)] - s^{\alpha-1}i(0) - s^{\alpha-2}i'(0) + \frac{R}{L}\left[s^\beta\mathcal{L}[i(t)] - s^{\beta-1}Ii\right] + \frac{1}{LC}\mathcal{L}[i(t)]$$

$$= -\frac{E_0\omega}{L}\left[\frac{\omega}{s^2+\omega^2}\right], \text{where } \mathcal{L}[sin\,\omega t] = \frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow\ s^{\alpha}\mathcal{L}[i(t)] + \frac{R}{L}\left[s^{\beta}\mathcal{L}[i(t)]\right] + \frac{1}{LC}\mathcal{L}[i(t)]$$

$$= -\frac{E_0\omega}{L}\left[\frac{\omega}{s^2+\omega^2}\right] + s^{\alpha-1}i(0) + s^{\alpha-2}i'(0) + \frac{R}{L}s^{\beta-1}i(0)$$

$$\Rightarrow \mathcal{L}[i(t)]\left(s^{\alpha} + \frac{R}{L}s^{\beta} + \frac{1}{LC}\right) = -\frac{E_0\omega}{L}\left(\frac{\omega}{s^2+\omega^2}\right) + s^{\alpha-1}i(0) + s^{\alpha-2}i'(0) + \frac{R}{L}s^{\beta-1}i(0).$$

By assuming the initial conditions $i(0) = A\ and\ i'(0) = B$, then we have: $I(t) = \frac{-\frac{E_0\omega}{L}(\frac{\omega}{s^2+\omega^2})+s^{\alpha-1}A+s^{\alpha-2}B+\frac{R}{L}s^{\beta-1}A}{s^{\alpha}+\frac{R}{L}s^{\beta}+\frac{1}{LC}}$. Applying inverse Laplace transform we have:

$$i(t) = \mathcal{L}^{-1}\left[\frac{-\frac{E_0\omega}{L}\left(\frac{\omega}{s^2+\omega^2}\right)+s^{\alpha-1}A+s^{\alpha-2}B+\frac{R}{L}s^{\beta-1}A}{s^{\alpha}+\frac{R}{L}s^{\beta}+\frac{1}{LC}}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{-\frac{E_0\omega}{L}\left(\frac{\omega}{s^2+\omega^2}\right)}{s^{\alpha}+\frac{R}{L}s^{\beta}+\frac{1}{LC}}\right] + \mathcal{L}^{-1}\left[\frac{s^{\alpha-1}A}{s^{\alpha}+\frac{R}{L}s^{\beta}+\frac{1}{LC}}\right] + \mathcal{L}^{-1}\left[\frac{s^{\alpha-2}B}{s^{\alpha}+\frac{R}{L}s^{\beta}+\frac{1}{LC}}\right] + \mathcal{L}^{-1}\left[\frac{\frac{R}{L}s^{\beta-1}A}{s^{\alpha}+\frac{R}{L}s^{\beta}+\frac{1}{LC}}\right]$$

Case V: When we take $\beta = 1$, equation $L\frac{d^{\alpha}}{dt^{\alpha}}i(t) + R\frac{d^{\beta}}{dt^{\beta}}i(t) + \frac{1}{c}i(t) = E'(t)$ reduces to the form; $\frac{d^{\alpha}}{dt^{\alpha}}i(t) + \frac{R}{L}\frac{d}{dt}i(t) + \frac{1}{LC}i(t) = E_0$, where $1 < \alpha \le 2$.

$$\frac{d^2}{dt^2}i(t) + \frac{R}{L}\frac{d}{dt}i(t) + \frac{1}{LC}i(t) = E_0, \text{where } 0 < \beta < \alpha \le 2.$$

Case VI: when we take $\alpha = 2$, then equation $L\frac{d^{\alpha}}{dt^{\alpha}}i(t) + R\frac{d^{\beta}}{dt^{\beta}}i(t) + \frac{1}{c}i(t) = E'(t)$ reduces to the form; $\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{d^{\beta}i(t)}{dt^{\beta}} + \frac{i(t)}{LC} = E_0$, where $0 < \beta \le 1$ and, then we can rewrite as follows. $\frac{d^2}{dt^2}i(t) + \frac{R}{L}\frac{d}{dt}i(t) + \frac{1}{LC}i(t) = E_0$, where $1 < 2\beta \le 2$.

2.3. Illustrative examples

Example 1: A capacitor 0.02 Farads, a resister 20 ohms and an inductor 2 henry are connected in series with an electromotive force of E volts. At $t = 0$ the charge on the capacitor and the current in the circuit are zero. Find the charge and current at any time $t > 0$. If a. $E(t) = 200$ volts b. $E(t) = 100\sin 3t$ c. $E(t) = 10\cos 2t$ d. $E(t) = e^t$

Solution: According to the given conditions of the problem has a critically damped response, since $\frac{R}{2L} = \frac{20}{4} = 5$ and $\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.02\times2}} = 5$, so that $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$.

Let q and i be the instantaneous charge and current respectively at time t. By Kirchhoff's law; $L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1q(t)}{C} = E(t)$. On putting the given values,

$$2\frac{d^2q(t)}{dt^2} + 20\frac{dq(t)}{dt} + \frac{1q(t)}{0.02} = E(t).$$

a. $2\frac{d^2q(t)}{dt^2} + 20\frac{dq(t)}{dt} + \frac{1q(t)}{0.02} = 200\text{volts}$, particularly for $\alpha = 2$ & $\beta = 1$

$\Rightarrow \frac{d^2q(t)}{dt^2} + 10\frac{dq(t)}{dt} + 25q(t) = 150$

$\Rightarrow \mathcal{L}\left[\frac{d^2q(t)}{dt^2} + 10\frac{dq(t)}{dt} + 25q(t)\right] = \mathcal{L}[150]$

$\Rightarrow s^2\mathcal{L}[q(t)] - sq(0) - q'(0) + 10[\mathcal{L}[q(t)] - q(0)] + 25\mathcal{L}[q(t)] = \frac{150}{s}.$

Applying initial conditions at $t = 0, q(t) = 0, \& \, q'(t) = 0$, we have: $s^2\mathcal{L}[q(t)] + 10[s\mathcal{L}[q(t)]] + 25\mathcal{L}[q(t)] = \frac{150}{s}$, then $\mathcal{L}[q(t)][s^2 + 10s + 25] = \frac{150}{s}$

$\therefore \mathcal{L}[q(t)] = \frac{150}{s[s^2+10s+25]}$

By using fractional decomposition as follows; we get the next one.

$$\frac{150}{s[s^2+10s+25]} = \frac{A}{s} + \frac{B}{(s+5)} + \frac{C}{(s+5)^2}.$$

$A[s^2 + 10s + 25] + Bs(s + 5) + Cs = 150$

$A + b = 0, 10A + 5B + C = 0 \; and \; 25A = 150, then$

$A = 6, B = -6 \; and \; C = -30$

$\Rightarrow Q(t) = \frac{150}{s[(s+5)^2]} = \frac{6}{s} - \frac{6}{(s+5)} - \frac{30}{(s+5)^2}.$

$\Rightarrow Q(t) = \frac{6}{s} - \frac{6}{(s+5)} - \frac{30}{(s+5)^2}$, since $\mathcal{L}[q(t)] = Q(t)$.

By taking the inverse Laplace transform from both sides, we get:

Fore $\mathcal{L}^{-1}[F(s-a)] = \mathcal{L}^{-1}\{F(s)|s \to s-a\} = e^{at}f(t)$, then Q(t) will be:

$$q(t) = 6 - 6e^{-5t} - 30e^{-5t}t$$

Therefore the charge variation through the circuit of the given problem determined by

$$\mathbf{q(t) = 6 - 6e^{-5t} - 30e^{-5t}t, for\ t > 0}$$

$$q(t) = 6 - 6E_{1,1}(-5t) - 30E_{1,2}(-5t),$$

since, $e^{-5t} = E_{1,1}(-5t) \; and \; te^{-5t} = tE_{1,2}(-5t)$.

Moreover the current in RLC circuit is also determined by the equation based on the relation; q = it, i.e. current is the time rate of change of an electric charge.

$\therefore i(t) = \frac{dq}{dt} = \frac{d}{dt}[6 - 6e^{-5t} - 30e^{-5t}t] = 30e^{-5t} + 150e^{-5t}t - 30e^{-5t}$, for $t > 0$.

b. $2\frac{d^2q(t)}{dt^2} + 20\frac{dq(t)}{dt} + \frac{1q(t)}{0.02} = 100\,sin\,3t$, particularly for $\alpha = \frac{3}{2}$, then $\beta = \frac{\alpha}{2} = \frac{3}{4}$

$\frac{d^{\frac{3}{2}}q(t)}{dt^{\frac{3}{2}}} + 10\frac{d^{\frac{3}{4}}q(t)}{dt^{\frac{3}{4}}} + 25q(t) = 50\,sin\,3t$. Apply Laplace transform from both sides of the equation, we get: $\mathcal{L}\left[\frac{d^{\frac{3}{2}}q(t)}{dt^{\frac{3}{2}}} + 10\frac{d^{\frac{3}{4}}q(t)}{dt^{\frac{3}{4}}} + 25q(t)\right] = \mathcal{L}[50\,sin\,3t]$, then

$$\mathrm{s}^{\frac{3}{2}}\mathcal{L}[\mathrm{q(t)}] - \mathrm{s}^{\frac{3}{2}-1}\mathrm{q}(0) - \mathrm{s}^{\frac{3}{2}-2}\mathrm{q}'(0) + 10\left[\mathrm{s}^{\frac{3}{4}}\mathcal{L}[\mathrm{q(t)}] - \mathrm{s}^{\frac{3}{4}-1}\mathrm{q}(0)\right] + \frac{1}{\mathrm{LC}}\mathcal{L}[\mathrm{q(t)}] = \mathcal{L}[50\sin 3t]$$

Then $\mathrm{s}^{\frac{3}{2}}\mathcal{L}[\mathrm{q(t)}] - \mathrm{s}^{\frac{1}{2}}\mathrm{q}(0) - s^{\frac{-1}{2}}q'(0) + 10\left[\mathrm{s}^{\frac{3}{4}}\mathcal{L}[\mathrm{q(t)}] - \mathrm{s}^{\frac{-1}{4}}\mathrm{q}(0)\right] + 25\mathcal{L}[\mathrm{q(t)}] = 50\left[\frac{3}{s^2+9}\right]$.

By applying the initial conditions we have: $\mathcal{L}[q(t)]\left[s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25\right] = 50\left[\frac{3}{s^2+9}\right]$

$$\mathrm{Q(t)} = \frac{50(\frac{3}{s^2+9})}{(\mathrm{s}^{\frac{3}{2}} + 10\mathrm{s}^{\frac{3}{4}} + 25)}, \text{then } \mathrm{q(t)} = \mathcal{L}^{-1}\left[\frac{150}{(s^2+9)\left(\mathrm{s}^{\frac{3}{2}} + 10\mathrm{s}^{\frac{3}{4}} + 25\right)}\right]$$

$\therefore \mathrm{q(t)} = \mathcal{L}^{-1}\left[\frac{150}{(s^2+9)\left(\mathrm{s}^{\frac{3}{2}}+10\mathrm{s}^{\frac{3}{4}}+25\right)}\right]$.

$\mathrm{s}^{\frac{3}{2}} + 10\mathrm{s}^{\frac{3}{4}} + 25$, is more complex. We need to proceed by recognizing the general form of $\mathcal{L}^{-1}\left[\frac{1}{(s+a)^{\alpha}}\right] = \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{at}$, for $\alpha > 0$. The expression in terms of Mittag-Leffler function can be some-what approximated as: $q(t) = \frac{150}{9}E_{1,2}(-9t^2) + c_1\sin 3t$, where c_1 a constant determined from the initial conditions and $E_{1,2}(-9t^2)$ reflects the integral order Laplace transform involving $\mathbf{s}^{\frac{3}{2}}$ in the denominator. However our expression has mixed orders. Using inverse Laplace transform of the entire fraction would require transforming each part.

In a similar manner the current flow in the circuit of the above given RLC problem is also determined by following the same technique based on the relation $q = it$, i.e. current is the time rate of charge.

$$i(t) = \frac{dq}{dt} = \frac{d}{dt}\left(\frac{150}{9}E_{1,2}(-9t^2) + c_1\sin 3t\right).$$

$\therefore i(t) = \frac{150}{9}\frac{d}{dt}E_{1,2}(-9t^2) + 3c_1\sin 3t$.

c. $2\frac{d^2q(t)}{dt^2} + 20\frac{dq(t)}{dt} + \frac{1q(t)}{0.02} = 10\,cos\,2t$ volts, particularly for $\alpha = \frac{3}{2}$, then $\beta = {}^{\alpha}/_{2} = \frac{3}{4}$.

$\frac{D^{\frac{3}{2}}q(t)}{dt^{\frac{3}{2}}} + 10\frac{d^{\frac{3}{4}}q(t)}{dt^{\frac{3}{4}}} + 25q(t) = 5\,cos\,2t$. Apply Laplace transform from both sides of the equation to get; $\mathcal{L}\left[\frac{d^{\frac{3}{2}}q(t)}{dt^{\frac{3}{2}}} + 10\frac{d^{\frac{3}{4}}q(t)}{dt^{\frac{3}{4}}} + 25q(t)\right] = \mathcal{L}[5\,cos\,2t]$, then

$$s^{\frac{3}{2}}\mathcal{L}[q(t)] - s^{\frac{3}{2}-1}q(0) - s^{\frac{3}{2}-2}q'(0) + 10\left[s^{\frac{3}{4}}\mathcal{L}[q(t)] - s^{\frac{3}{4}-1}q(0)\right] + \frac{1}{LC}\mathcal{L}[q(t)] = \mathcal{L}[5\,cos\,2t].$$

$$s^{\frac{3}{2}}\mathcal{L}[q(t)] - s^{\frac{1}{2}}q(0) - s^{\frac{-1}{2}}q'(0) + 10\left[s^{\frac{3}{4}}\mathcal{L}[q(t)] - s^{\frac{-1}{4}}q(0)\right] + 25\mathcal{L}[q(t)] = 5\frac{s^2}{(s^2+4)}$$

By applying the initial conditions we have: $\mathcal{L}[q(t)]\left[s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25\right] = 5\left[\frac{s^2}{s^2+4}\right]$

$$Q(t) = \frac{5s^2}{(s^2+4)(s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25)}, \text{then } q(t) = \mathcal{L}^{-1}\left[\frac{5s^2}{(s^2+4)\left(s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25\right)}\right]$$

$$\therefore q(t) = \mathcal{L}^{-1}\left[\frac{5s^2}{(s^2+4)\left(s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25\right)}\right]$$

Using the linearity property of Laplace transform we have:

$$q(t) = \mathcal{L}^{-1}\left[\frac{5s^2}{(s^2+4)\left(s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25\right)}\right] = 5\mathcal{L}^{-1}\left[\frac{s^2}{s^2+4} + \frac{1}{s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25}\right]$$

Using the fact $\mathcal{L}^{-1}\left[\frac{s^2}{s^2+4}\right] = cos\,2t$ and $\mathcal{L}^{-1}\left[\frac{1}{s^{\frac{3}{2}}+10s^{\frac{3}{4}}+25}\right] = k_1E_\alpha(-\beta t^2)$, where k_1 is a constant that can be determined from the system parameters, α and β depend on how the roots factor into the original expression.

Using the convolution theorem, the inverse Laplace transform can be expressed as a convolution of the two individual transforms. $q(t) = \left[cos\,2t * \left[\frac{1}{s^{\frac{3}{2}}+10s^{\frac{3}{4}}+25}\right]\right]$

This yield: $q(t) = 5[cos\,2t * k_1E_\alpha(-\beta t^2)]$ putting it all together, we get:

$q(t) = 5\cos 2t * k_1E_\alpha(-\beta t^2) + c,$ Where c is a constant determined by initial conditions. By following a similar analogue the current flow in the above given RLC circuit problem is also determined based on the relation $\mathrm{q} = \mathrm{it}$, i.e. current is the time rate of charge.

$$i(t) = \frac{dq}{dt} = 5\frac{d}{dt}[cos\,2t * k_1E_\alpha(-\beta t^2)]$$

$$= 5\frac{d}{dt}[cos\,2t * k_1E_\alpha(-\beta t^2) + c]$$

$$\therefore i(t) = 10k_1\left[\beta t\, cos\,2t\frac{d}{dt}E_\alpha(-\beta t^2) + E_\alpha(-\beta t^2)\sin 2t\right].$$

d. $2\frac{d^2q(t)}{dt^2} + 20\frac{dq(t)}{dt} + \frac{1q(t)}{0.02} = e^t$volts, particularly for $\alpha = \frac{3}{2}$, then $\beta = {}^{\alpha}/_{2} = \frac{3}{4}$.

$\frac{d^{\frac{3}{2}}q(t)}{dt^{\frac{3}{2}}} + 10\frac{d^{\frac{3}{4}}q(t)}{dt^{\frac{3}{4}}} + 25q(t) = e^t$. Apply Laplace transform from both sides of the equation;

$$\mathcal{L}\left[\frac{d^{\frac{3}{2}}q(t)}{dt^{\frac{3}{2}}} + 10\frac{d^{\frac{3}{4}}q(t)}{dt^{\frac{3}{4}}} + 25q(t)\right] = \mathcal{L}[e^t], \text{then}$$

$$s^{\frac{3}{2}}\mathcal{L}[q(t)] - s^{\frac{3}{2}-1}q(0) - s^{\frac{3}{2}-2}q'(0) + 10\left[s^{\frac{3}{4}}\mathcal{L}[q(t)] - s^{\frac{3}{4}-1}q(0)\right] + \frac{1}{LC}\mathcal{L}[q(t)] = \mathcal{L}[e^t]$$

$$s^{\frac{3}{2}}\mathcal{L}[q(t)] - s^{\frac{1}{2}}q(0) - s^{\frac{-1}{2}}q'(0) + 10\left[s^{\frac{3}{4}}\mathcal{L}[q(t)] - s^{\frac{-1}{4}}q(0)\right] + 25\mathcal{L}[q(t)] = \frac{1}{(s-1)}$$

By applying the initial conditions we have: $\mathcal{L}[q(t)]\left[s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25\right] = 5\left[\frac{1}{(s-1)}\right]$

$$Q(t) = \frac{1}{(s-1)(s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25)}, \text{then } q(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)\left(s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25\right)}\right]$$

$$\therefore q(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)\left(s^{\frac{3}{2}} + 10s^{\frac{3}{4}} + 25\right)}\right]$$

$\mathcal{L}^{-1}\left[\frac{1}{(s-1)\left(s^{\frac{3}{2}}+10s^{\frac{3}{4}}+25\right)}\right] \sim e^t\left[\frac{1}{s^{\frac{3}{2}}+10s^{\frac{3}{4}}+25}\right]$. Since $\mathcal{L}^{-1}\left[\frac{1}{(s^{\alpha}+\lambda s^{\beta}+\cdots)}\right] = t^{\alpha-1}E_{\alpha,\beta}(-\lambda t^{\alpha})$

$$\therefore q(t) \sim e^t t^{\frac{3}{2}-1}E_{\frac{3}{2},\beta}(-\lambda t^{\frac{3}{2}})$$

Example 2: A capacitor 0.001mF, a resister 10 ohms and an inductor 0.25henry are connected in series with an electromotive force is 0 volts. At $t = 0$ the charge on the capacitor and the current in the circuit are zero. Find the charge and current at any time $t > 0$. Where $q_0 = 2C$. the circuit is under-damped response, because discriminator $R^2 < \frac{4L}{c}$, then the circuit is an under-damped response. For different values of α we have:

Case 1: $0.25\frac{d^{\frac{1}{4}}q(t)}{dt^{\frac{1}{4}}} + 40\frac{d^{\frac{1}{4}}q(t)}{dt^{\frac{1}{8}}} + \frac{1q(t)}{25\times10^{-8}} = 0volts,$

$$\Rightarrow \mathcal{L}\left[0.25\frac{d^{\frac{1}{4}}q(t)}{dt^{\frac{1}{4}}} + 40\frac{d^{\frac{1}{4}}q(t)}{dt^{\frac{1}{8}}} + \frac{1q(t)}{25\times10^{-8}}\right] = \mathcal{L}[0].$$

$$\Rightarrow s^{\frac{1}{8}}\mathcal{L}[q(t)] - sq(0) + 160[\mathcal{L}[q(t)] - q(0)] + 4\times10^6\mathcal{L}[q(t)] = 0.$$

Applying initial conditions at $t = 0, q_0 = 2C$ and, & $q_0'(t) = 0$, we have:

$$s^{\frac{1}{4}}\mathcal{L}[q(t)] + 160[s\mathcal{L}[q(t)] + 2C] + 4\times10^6\mathcal{L}[q(t) + 2C] = 0,$$

Then by following the same manner as the above example 1, we get:

$$q(t) = 2 - \left(\frac{25}{3\sqrt{15}}\right)\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} + \left(\frac{25}{9\sqrt[4]{\sqrt{15}}}\right)\frac{t^{\frac{3}{24}}}{\Gamma\left(\frac{7}{4}\right)} + \frac{25}{18}t + \cdots.$$

By applying the same techiniques we have as follows:

Case 2: If $\alpha = 1$, then $q(t) = 2 - 8000\frac{t^2}{2!} + 320000\ \frac{t^3}{3!}\ + \cdots$.

$$\therefore\ q(t) = 2e^{-20t}\ \left(cos60t + \frac{1}{3}\ sin60t\right)$$

$$= 2E_{1,1}(-20t)\left[cos60t + \frac{1}{3}\ sin60t\right]$$

$$= 2E_{1,1}(-20t)\left[\frac{e^{i60t} + e^{-i60t}}{2} + \frac{e^{i60t} - e^{-i60t}}{2i}\right].$$

But, $i(t) = \frac{dq(t)}{dt} = \frac{d}{dt}\left[2e^{-20t}\ \left(cos60t + \frac{1}{3}\ sin60t\right)\right]$, by applaying product and chain rule we get i(t) as:

$$\therefore i(t) = -\frac{400}{3}e^{-20t}\, sin\, 60t.$$

Case 3: If $\alpha = \frac{1}{2}$, then, $\mathrm{q(t)} = 2 - \frac{200}{3}\mathrm{t} + \left(\frac{400}{3\sqrt{30}}\right)\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} + \left(\frac{400}{3}\right)\frac{t^2}{\Gamma(3)} + \cdots$.

$$\therefore q(t) = E_{\frac{3}{2},1}(-\lambda t).$$

But $i(t) = \frac{dq(t)}{dt}$,then the current flow in acircuit will be:

$$i(t) = \frac{d}{dt}\left[2 - \frac{200}{3}\mathrm{t} + \left(\frac{400}{3\sqrt{30}}\right)\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} + \left(\frac{400}{3}\right)\frac{t^2}{\Gamma(3)} + \cdots\right]$$

$$\therefore i(t) = -\frac{200}{3} + \frac{100}{\sqrt{30}}\frac{\sqrt{t}}{\Gamma\left(\frac{5}{2}\right)} + \frac{400}{3}t + \cdots$$

Case 4: if $\alpha = \frac{3}{4}$, then, $\mathrm{q(t)} = 2 -\ \frac{2000}{\sqrt{5}}\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} + \left(\frac{40000}{5^{\frac{3}{4}}}\right)\frac{t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} + 240000\frac{t^3}{\Gamma(4)} + \cdots$

$$= 2E_{\alpha,1}(-\lambda t), \text{where } \alpha \text{ is the fractional exsponents of t.}$$

The current flow will be given by $i(t) = \frac{dq(t)}{dt}$

$$\Rightarrow i(t) = \frac{d}{dt}\left[2 -\ \frac{2000}{\sqrt{5}}\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} + \left(\frac{40000}{5^{\frac{3}{4}}}\right)\frac{t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} + 240000\frac{t^3}{\Gamma(4)} + \cdots\right]$$

$$\therefore i(t) = \frac{300}{\sqrt{5}}\frac{t^{\frac{1}{2}}}{\Gamma(\frac{5}{2})} + \frac{90000}{5^{\frac{3}{4}}}\frac{t^{\frac{5}{4}}}{\Gamma(\frac{13}{4})} + 120000t^2 + \cdots$$

Example 3: A capacitor $C = \frac{1}{300}mF$, a resister $R = 20\Omega$ and an inductor $L = 0.25H$ are connected in series with an electromotive force $E(t) = 100volts$. At $\mathrm{t} = 0$ the charge on the capacitor and the current in the circuit are zero. Find the charge and current at any time $t > 0$. Where $q_0 = 4\text{C}$ and $q_0{}' = i(o) = 0\ A$, then the discriminator $R^2 > \frac{4L}{c}$, then the circuit is an over-damped response. In different values of α we have:

Case 1: If $\alpha = \frac{1}{2}$, then, $q(t) = \text{q(t)} = 4 - 110\text{t} + \left(\frac{4400}{\sqrt{10}}\right)\frac{\text{t}^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} - 14300\ \frac{\text{t}^2}{\Gamma(3)}\ + \cdots$

$$= 4E_{\frac{3}{2},1}(-\lambda t).$$

and again $i(t) = \frac{dq(t)}{dt}$, will give as:

$$i(t) = \frac{\text{d}}{\text{dt}}\left[4 - 110\text{t} + \left(\frac{4400}{\sqrt{10}}\right)\frac{\text{t}^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} - 14300\ \frac{\text{t}^2}{\Gamma(3)}\ + \cdots\right]$$

$$\therefore \text{i(t)} = 110 + \frac{6600}{\sqrt{10}\Gamma\left(\frac{5}{2}\right)}t^{\frac{1}{2}} - 14300t.$$

Case 2: If $\alpha = \frac{1}{4}$, then $q(t) = 4 - \left(\frac{4400}{\sqrt{(80)^3}}\right)\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} + \left(\frac{4400}{\sqrt[4]{(80)^5}}\right)\frac{t^{\frac{3}{4}}}{\Gamma\left(\frac{7}{4}\right)} - \left(\frac{715}{16}\right)t + \cdots$

$$= 4E_{\frac{3}{2},1}(-\lambda t).$$

Then the current flows through the circuit will be $i(t) = \frac{dq(t)}{dt}$,

$$i(t) = \frac{d}{dt}\left[4 - \left(\frac{4400}{\sqrt{(80)^3}}\right)\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} + \left(\frac{4400}{\sqrt[4]{(80)^5}}\right)\frac{t^{\frac{3}{4}}}{\Gamma\left(\frac{7}{4}\right)} - \left(\frac{715}{16}\right)t + \cdots\right]$$

$$\therefore i(t) = -\frac{220}{\sqrt{(80)^3}\Gamma\left(\frac{3}{2}\right)}t^{-\frac{1}{2}} + \frac{3300}{\sqrt[4]{80^5}\Gamma\left(\frac{7}{4}\right)}t^{-\frac{1}{4}} - \frac{715}{16}.$$

Case 3: If $\alpha = \frac{3}{4}$, then

$$q(t) = 4 - 4400 \times \sqrt{\frac{3}{80}}\frac{\text{t}^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} + 352000 \times \left(\frac{3}{80}\right)^{\frac{3}{4}}\frac{t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} + 858000\frac{t^3}{\Gamma(4)} + \cdots$$

$$= 4E_{\frac{3}{2},1}(-\lambda t).$$

$$\text{Then } i(t) = \frac{dq(t)}{dt}$$

Hence $i(t) = \frac{d}{dt}\left[4 - 4400 \times \sqrt{\frac{3}{80}}\frac{\text{t}^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} + 352000 \times \left(\frac{3}{80}\right)^{\frac{3}{4}}\frac{t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} + 858000\frac{t^3}{\Gamma(4)} + \cdots\right]$

$$\therefore i(t) = -6600\sqrt{\frac{3}{80\Gamma\left(\frac{3}{2}\right)}}t^{\frac{1}{2}} + 198000\left(\frac{3}{80}\right)^{\frac{3}{4}}\frac{t^{\frac{5}{4}}}{\Gamma\left(\frac{13}{4}\right)} + 429000t^2.$$

Case 4: If $\alpha = 1$, then $q(t) = 4 - 4400 \times \frac{t^2}{\Gamma(3)} + 352000 \times \frac{t^3}{\Gamma(4)} - 22880000\frac{t^4}{\Gamma(5)} + \cdots$

$$= 4E_{\frac{3}{2},1}(-\lambda t).$$

This solution can be represented as closed form as, $q(t) = \frac{33}{6}e^{-20t} - \frac{11}{6}e^{-60t} + \frac{1}{3}$.

$$i(t) = \frac{d}{dt}\left[4 - 4400 \times \frac{t^2}{\Gamma(3)} + 352000 \times \frac{t^3}{\Gamma(4)} - 22880000\frac{t^4}{\Gamma(5)} + \cdots\right]$$

$$\Rightarrow i(t) = -4400t + 176000t^2 + \frac{11440000}{3}t^3 + \cdots$$

It is the same as $\frac{d}{dt}\left[\frac{33}{6}e^{-20t} - \frac{11}{6}e^{-60t} + \frac{1}{3}\right]$

$$\therefore i(t) = -110e^{-20t} + 110e^{-60t}.$$

2.4. Analysis of examples

In the present paper, analytical solutions of the electrical RLC circuit using the Liouville–Caputo fractional operators were presented. The solutions obtained preserve the dimensionality of the studied system for any value of the exponent of the fractional derivative. We can conclude that the decreasing value of the time fractional order of derivative provides an attenuation of the amplitudes of the oscillations, the system increases its “damping capacity” and the current changes due to the order derivative (causing irreversible dissipative effects such as ohmic friction), and the response of the system evolves from an under-damped behavior into an over-damped behavior.

The fractional differentiation with respect to the time represents a non-local effect of dissipation of energy (internal friction) represented by the fractional order α. The electrical circuit RLC exhibits fractality in time to different scales and shows the existence of heterogeneities in the electrical components (resistance, capacitance, and inductance).Due to the physical process involved (i.e., magnetic hysteresis), these components can present signs of non-linear phenomena and non-locality in time, it is clear that the approximate solutions continuously depend on the time-fractional derivative α.

In realistic manner, the resistance of the cables connecting the circuit elements to each other varies with the temperature. This unstable resistance causes some dissipative effects which make the behavior of RLC electrical circuit is non-linear and non-local in time. Since the temperature of RLC circuit increases with time, an unstable ohmic friction occurs during the process. Also, in a real capacitor series resistance is not zero and in the parallel resistance is not infinite. The series resistance in an inductor is not zero. Hence, the current transmitted to circuit (and also the charge variation of the capacitor) per each small time interval (Δt) exhibits a difference in each time interval step. Hereby, time reversal symmetry for this process is not valid. From a different perspective, to indicate the non-locality in time, the transmitted current can be assumed being unchanged per different time intervals and, the flow period of different time intervals can be assumed being variable depending on the temperature. Consequently, it could be said that using the time fractional derivative instead of standard one is a very useful way to describe the realistic feature of RLC electrical circuit that is non-local in time.

3. Conclusion

We have obtained the analytic solution of the second order fractional differential equation associated with a RLC electrical circuit in the time domain using Caputo derivative in terms of Mittag-Leffler type function which can be implemented for computational study of behavior of current and charge variation. In view of the generality of Mittag-Leffler functions, on specializing the various parameters, we can obtained from our main result, several result containing remarkably wide variety of useful functions and their various RLC electrical circuits. Thus the main result presented in this project would at once yield a large number of results containing a large variety of simpler special functions occurring scientific and technological fields. The results we obtained are closely related with the classical results in RLC circuit. Moreover, our method can be extended to solve another physical problems. In the future, we will use the fractional differential techniques to study another engineering mathematics problems.

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