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Implicative Almost Distributive Lattices

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Bahir Dar University

College of Science

Department of Mathematics

A Project on
Implicative Almost Distributive Lattices

By

Desalew Gashaye

September, 2024

Bahir Dar, Ethiopia

Bahir Dar University

College of Science

Department of Mathematics

A project on

Implicative Almost Distributive Lattices

A Project submitted to the department of mathematics in partial fulfillment of the requirements for the degree of "Master of Science in Mathematics".

By

Desalew Gashaye Asmare

Advisor: Tilahun Mekonnen (PhD)

September, 2024

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Approval of the Project for Defense

I hereby certify that I have supervised, read and evaluated this project entitled "Implicative Almost Distributive Lattices" by Desalew Gashaye prepared under my guidance. I recommend that the project is submitted for oral defense.

Advisor Name: Tilahun Mekonnen (PhD)

Signature ___________________________

Date _________/__________/__________

Bahir Dar University

College of Science

Department of Mathematics

Approval of the Project for Defense Result

We hereby certify that we have examined this project entitled "Implicative Almost Distributive Lattice" by Desalew Gashaye Asmare. We recommend that Desalew Gashaye Asmare is approved for the degree of "Master of Science in Mathematics".

Board of Examiners

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External examiner:	________________	___________	___/___/_____
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Abstract

The concept of partial order set, lattice, distributive lattice, almost distributive lattice, implicative algebra, and lattice implicative algebra are introduced by different authors.

In this project work our aim is to introduce implicative almost distributive lattices as a generalization of implicative algebras in the class of almost distributive lattices.

The main objective of this project is to understand the concept of implicative almost distributive lattice.

Table of contents

Contents **Page**

Chapter One

1. Introduction and Preliminaries

1.0 Introduction

In chapter one of this project our aim is to understand the concept of partial ordering set, lattice, distributive lattice, almost distributive lattice, lattice implication algebra and implicative algebra. In chapter two of this project our aim is to understand the concept of implicative almost distributive lattice, prove theorems and lemmas of implicative almost distributive lattice and fill gaps.

1.1 Review Literature

The concept of general lattice theory was introduced by Gratzer, G. [3] in 1978.To establish an alternative logic for knowledge representation and reasoning, Xu [8] proposed a logical algebra-lattice implication algebra in 1993 by combining algebraic lattice and implication algebra. Lattice implication algebra is an important non-classical logical algebra, it has been studied by researchers. Lattice-valued logic is an important form of many-valued logic which extends the field of truth-values to lattices. More importantly, lattice-valued logic can represent the uncertainty, especially the incomparable property of people's thinking, judging and decision. In lattice implication algebra, the lattice is defined to describe uncertainties, especially for the incomparability and the implication is designed to describe the ways of human's reasoning. Xu et al. [7] have established the lattice valued propositional logic Lp(X) and lattice valued first ordered logic Lf(X) and the gradual lattice valued propositional logic Lvpl and the gradual lattice valued first order logic Lvfl [9] by taking lattice implication algebra as a truth value field. V. Kulluru and Berhanu Bekele [4] introduced the concept of implicative algebras and obtained certain properties. Further they proved that implicative algebra is equipped with a structure of a bounded lattice and proved that it is a lattice implication algebra. The concept of an almost distributive lattice (ADL) was introduced in 1981 by U.M. Swamy and G.C. Rao [6] as a common abstraction to most of the existing ring theoretic and lattice theoretic generalization of Boolean algebra. [1] B.A. Alaba, M. Alamneh, and T. Mekonnen, introduced the concept of implicative almost distributive lattice (IADLs) as generalization of implicative algebra in the class of ADLs. They proved some properties and equivalent conditions in an implicative almost distributive lattice.

1.2 Preliminaries

First, we recall certain definition and properties of lattice, distributive lattice, almost distributive lattice, lattice implication algebra and implicative algebra that are required in the next chapter.

1.2.1. Lattices, Distributive Lattices and Almost Distributive Lattices

In this section first we recall some concepts of partial ordering set, lattice, distributive lattice and almost distributive lattice that will be useful for our work.

Definition 1.2.1.1.[3]: A binary relation " $\leq$ " on a non-empty set L is called partial ordering relation if it satisfies the following properties, for any x, y, z $\in$L.

1) $x \leq x$ ……………………………………………………………(reflexive)
2) $x \leq y$ and $y \leq x$ implies $x = y$………………………………… (anti-symmetric)
3) $x \leq y$ and $y \leq z$ implies $x \leq z$ ……………………….………. (transitive)

A set equipped with this relation (L, $\leq$) is called partially order set or poset.

Example:1.2.1.2. Let S be a non-empty set. Then $(P(S), \subseteq)$, where $P(S)$ is the power set of S is a Poset.

Definition 1.2.1.3.[2, 5]: A poset (L, $\leq$) is a lattice if sup {x, y} and inf {x, y} exist in L for all x, y $\in$ L.

Example:1.2.1.4: The set of all natural numbers N = {1, 2 . . .} with the usual order of $\leq$ is a poset. By defining sup {x, y} as a bigger of the two elements and inf {x, y} as the smaller of the two elements of N it follows a lattice.

Now we recall an equivalent definition of Lattice.

Definition 1.2.1.5.[3]: An algebra $(L, \vee, \wedge)$ of type (2,2) is called a lattice if for all x, y, z $\in L$ it satisfies the following properties.

1) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ ………………………… [Commutative law]
2) $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$……. [Associative law]
3) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$…………………..…… [Absorption law]
4) $x \vee x = x$ and $x \wedge x = x$ ……………………..….…………. [Idempotent law]

Note: a) $\vee$ and $\wedge$ read as "join" and "meet" respectively and both are binary operations.

b) In (2,2), 2 represent binary operations

c) For x, y $\in$L, $x \vee y$ = sup {x, y} and $x \wedge y$ = inf {x, y}

d) If $(L, \vee, \wedge)$ is a lattice then the element 0 of L is called zero element or least element of L, i.e. $0 \wedge x = 0$ for all $x \in L$ and an element 1 of L is called one element or top element of L, i.e. $x \vee 1 = 1$ for all $x \in L$. If L has 0 and 1 then L is called bounded lattice.

Note that, from the lattice $(L, \leq)$ of Definition 1.2.1.1 (as a poset) we can obtain $(L, \wedge, \vee)$ by defining $x \wedge y = \text{glb}\{x, y\}$ and $x \vee y = \text{lub}\{x, y\}$ (as an algebra).

In other words, in lattice $(L, \wedge, \vee)$, by defining $x \leq y$ if and only if $x \wedge y = x$ or equivalently $x \vee y = y$, we have $(L, \leq)$ is a lattice.

Theorem 1.2.1.6.[5]: In any lattice $(L, \wedge, \vee)$ the following are equivalent:

1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
2) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
3) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
4) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$, for all $x, y, z \in L$.

Definition 1.2.1.7.[5]: A lattice $(L, \wedge, \vee)$ that satisfies one and hence all of the identities in Theorem 1.2.1.4 is called distributive lattice.

Definition 1.2.1.8.[5, 6]: An algebra $(L, \vee, \wedge, 0)$ of type (2,2,0) is called an almost distributive lattice (ADL) with 0 if it satisfies the following axioms:

1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3) $(x \vee y) \wedge y = y$
4) $(x \vee y) \wedge x = x$
5) $x \vee (x \wedge y) = x$
6) $0 \wedge x = 0$, for all $x, y, z \in L$.

Example1.2.1.9. Every distributive lattice $(L, \vee, \wedge)$ is ADL.

Definition 1.2.1.10.[6]: Let L be a non empty set. Fix $x_0 \in L$. For any $x, y \in L$, define $x \vee y = x$, $x \wedge y = y$ if $x \neq x_0$; $x_0 \wedge y = x_0$ and $x_0 \vee y = y$. Then $(L, \vee, \wedge, x_0)$ is called a discrete ADL with x_0 as its 0.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $x, y \in L$, define $x \leq y$ if and only if $x = x \wedge y$ or equivalently $x \vee y = y$, then $\leq$ is a partial ordering on L.

Theorem 1.2.1.11.[6]: Let L be an ADL such that $x, y, z \in L$. Then the following conditions hold.

1) $x \vee y = x \Leftrightarrow x \wedge y = y$
2) $x \vee y = y \Leftrightarrow x \wedge y = x$
3) $x \wedge y = y \wedge x = x$ whenever $x \leq y$
4) $\wedge$ is associative

5) $x \wedge y \wedge z = y \wedge x \wedge z$
6) $(x \vee y) \wedge z = (y \vee x) \wedge z$
7) $x \wedge y \leq y$ and $x \leq x \vee y$
8) If $x \leq z$ and $y \leq z$, then $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$

Theorem 1.2.1.12.[6]: In an ADL L, the following are equivalent.

1) L is a distributive lattice
2) The poset $(L, \leq)$ is directed above
3) $x \vee y = y \vee x$ for all $x, y \in L$
4) $x \wedge y = y \wedge x$ for all $x, y \in L$
5) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for all $x, y, z \in L$
6) $\theta = \{(x, y) \in L \times L: y \wedge x = x\}$ is anti symmetric

Definition 1.2.1.13: In an ADL L, $(L, \leq)$ be a poset, $S \subseteq L$ and $a \in L$. Then

1) a is called a lower bound of S if $a \leq x$ for all $x \in S$.
2) a is an upper bound of S if $x \leq a$ for all $x \in S$.
3) a is called the greatest lower bound or infimum of S if a is a lower bound of S and for any lower bound b of S, we have $b \leq a$. In this case we write a = glb of S or a = inf S.
4) a is called the least upper bound or supremum of S if a is an upper bound of S and for any upper bound b of S, we have $a \leq b$. In this case we write a = lub of S or a = sup S.

Definition 1.2.1.14.[6]: An element m of an ADL L is called maximal if for any $x \in L$, $m \leq x$ implies $m = x$.

1.2.2. Lattice Implicative Algebra and Implicative Algebra

In this section we define lattice implicative algebra and implicative algebra.

Definition 1.2.2.1.[8]: A bounded lattice $(L, \vee, \wedge, 0, 1)$ with an order reversing involution "$'$" and binary operation "$\rightarrow$" is called a lattice implication algebra if for any $x, y, z \in L$, it satisfying the following axioms.

1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$
2) $x \rightarrow x = 1$
3) $x \rightarrow y = y' \rightarrow x'$
4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$
5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$
6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

Definition 1.2.2.2.[4]: An algebra $(L, \to, ', 0,1)$ of type $(2,1,0,0)$ is called implicative algebra if it satisfies the following conditions.

1) $x \to (y \to z) = y \to (x \to z)$
2) $1 \to x = x$
3) $x \to 1 = 1$
4) $x \to y = y' \to x'$
5) $(x \to y) \to y = (y \to x) \to x$
6) $0' = 1$ for $x, y, z \in L$.

Definition 1.2.2.3.[4]: A relation $\leq$ on an implicative algebra is defined as follows:

$x \leq y \Leftrightarrow x \to y = 1$, for all $x, y \in L$.

Lemma 1.2.2.4.[4]: In an implication algebra L, the following hold: for all $x, y, z \in L$,

1) $x \to x = 1$
2) $1' = 0$
3) $0 \to x = 1$
4) $x \to y = 1 = y \to x \Leftrightarrow x = y$
5) $x \to y = 1$ and $y \to z = 1$, then $x \to z = 1$
6) $x \leq y \Leftrightarrow z \to x \leq z \to y$ and $y \to z \leq x \to z$
7) $((x \to y) \to y) \to y = x \to y$
8) $(x \to y) \to [(y \to z) \to (x \to z)] = 1$
9) $(x')' = x$
10) $x' = x \to 0$, all $x, y, z \in L$.

We define two binary operations $\vee$ and $\wedge$ on an implicative algebra L respectively as follows.

$x \vee y = (x \to y) \to y = (y \to x) \to x$,

$x \wedge y = [(x \to y) \to x']' = [(y \to x) \to y']'$ for all $x, y \in L$.

Theorem 1.2.2.5.[4]: In an implicative algebra L, the following conditions hold:

1) $x \wedge y \leq x, y \leq x \vee y$
2) $x \leq y, x \leq z$ implies $x \leq y \wedge z$
3) $y \leq x, z \leq x$ implies $(y \vee z) \leq x$
4) $(x \vee y) \to z \leq x \to z$ and $(x \vee y) \to z \leq y \to z$
5) $x \to z \leq (x \wedge y) \to z$ and $y \to z \leq (x \wedge y) \to z$
6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$
7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

8) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
9) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

Theorem 1.2.2.6.[4]: Let $(L, \to, ', 0, 1)$ be an implicative algebra. Then $(L, \vee, \wedge, \to, ', 0, 1)$ is a lattice implicative algebra.

Chapter Two

2. Implicative Almost Distributive Lattices

In this chapter we introduce the concepts of implicative almost distributive lattices (IADLs) as a generalization of implicative algebras in the class of ADLs and develop related theory of IADLs. In this section, we define implicative almost distributive lattice and study some properties and equivalent conditions of implicative almost distributive lattices.

Definition 2.1.[1]: Let (L, ∨, ∧, 0, m) be an ADL with 0 and maximal element m. Then an algebra (L, ∨, ∧, →, ′, 0, m) of type (2, 2, 2, 1, 0, 0) is called implicative almost distributive lattice if it satisfies the following conditions:

1) $x \vee y = (x \to y) \to y$
2) $x \wedge y = [(x \to y) \to x'] '$
3) $x \to (y \to z) = y \to (x \to z)$
4) $m \to x = x$
5) $x \to m = m$
6) $x \to y = y' \to x'$
7) $0' = m$, for all x, y, z ∈ L.

Here after the symbol L stands for an IADL (L, ∨, ∧, →, ′, 0, m) unless otherwise specified.

Example 2.2. Let L be a discrete ADL with zero and at least two elements. Fix m (≠ 0) ∈ L to be maximal element and define the binary operation → on L as follows, for any x, y ∈ L, and ′unary operation on L.

$$x \to y = \begin{cases} 0\,, if x \neq 0\,, y = 0 \\ y\ , \quad if\ x = m \\ m, \quad otherwise \end{cases}$$

To show it satisfies IADL we will follow the following:

case 1:x ≠ 0 and y = 0, then $(x \to y) \to y = (x \to 0) \to 0 = 0 \to 0 = m$.

Case 2: x = m and y ≠ 0, then $(x \to y) \to y = (m \to y) \to y = y \to y = m$.

Case3: x ≠ m and y ≠ 0, then $(x \to y) \to y = m \to y = y$.

Thus, we can show the remaining properties by using these three cases. This means that it satisfies all properties or conditions to be implicative almost distributive lattice.

Then the structure (L, ∨, ∧,→, 0, m) is an implicative almost distributive lattice and is called discrete IADL.

Example 2.3. Let L = {0, x, y, z, m} be a set. Define partial ordered relation on L as $0 < x < y < z < m$ and also define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ for all x, y, z ∈ L. Define the unary operation ′ and the binary operation → on L as follows:

a	a′
0	m
x	z
y	y
z	x
m	0

Table 1

→	0	x	y	z	m
0	m	m	m	m	m
x	z	m	m	m	m
y	y	z	m	m	m
z	x	y	z	m	m
m	0	x	y	z	m

Table 2

Proof: From table 1 we observe that: $(x')' = x$ for all x in *L*.

From table 2 for all x, y, z in *L* we observe that:

i. $x \to y = m$ if and only if $x \leq y$

ii. $x \to y = m$ if and only if $y' \to x' = m$.

iii. $m \to x = x$, which is found in the last row

iv. $x \to m = m$, which is found in the last column.

Now take x, y, z ∈ *L* with $x < y < z$. Then

1) $x \vee y = \max\{x, y\} = y$ and $(x \to y) \to y = m \to y = y$

   It implies that $x \vee y = (x \to y) \to y$ for all x, y ∈ *L*.

2) $x \wedge y = \min\{x, y\} = x$ and $[(x \to y) \to x']' = (m \to x')' = (m \to z)' = (z') = x$ It implies that $x \wedge y = [(x \to y) \to x']'$ for all x, y ∈ *L*.

3) $x \to (y \to z) = x \to m = m$ and $y \to (x \to z) = y \to m = m$.

   It implies that $x \to (y \to z) = y \to (x \to z)$ for all x, y, z ∈ *L*.

4) $m \to x = x$ for all x ∈ *L*.

5) $x \to m = m$ for all x ∈ *L*.

6) $x \to y = m$ and $y' \to x' = m$ it implies that $x \to y = y' \to x'$ for all x, y ∈ *L*.

7) $0' = m$.

Therefor (L, ∨, ∧, →, ′, 0, m) is an implicative almost distributive lattice.

Lemma 2.4: Let L be an IADL. Then for any $x, y \in L$, the following conditions hold.

1) $[(x \to y) \to y] \wedge m = [(y \to x) \to x] \wedge m$
2) $[(x \to y) \to x']' \wedge m = [((y \to x) \to y')'] \wedge m$
3) $x \to x = m$
4) $m' = 0$
5) $(x')' = x$ (is called involution)
6) $x' = x \to 0$

Proof. Let $x, y \in L$. Using definition of IADL, we verify the following:

1) $((x \to y) \to y) \wedge m = (x \vee y) \wedge m = (y \vee x) \wedge m = ((y \to x) \to x) \wedge m$ (by Theorem 1.2.1.12)
2) $((x \to y) \to x')' \wedge m = (x \wedge y) \wedge m = (y \wedge x) \wedge m = ((y \to x) \to y')' \wedge m$ (by Theorem 1.2.1.12)
3) $x \to x = (m \to x) \to x = m \vee x = m$ (by Definition 2.1)
4) $m' = m \to m' = 0' \to m' = m \to 0 = 0$ (by Definition 2.1)
5) $(x')' = m \to (x')' = 0' \to (x')' = x' \to 0 = x' \to m' = m \to x = x$ (by Definition 2.1)
6) $x' = m \to x' = 0' \to x' = x \to 0$.

The following theorem refers to direct product of implicative almost distributive lattices and its proof is direct consequence of definition of IADL.

Theorem 2.5: Let L and H be two IADLs. Then $M = L \times H$ is an IADL with point wise operation.

Proof. Let L and H be two IADLs with maximal element m and $m*$ respectively.

Let $M = L \times H$.

Claim: M is an IADL.

Define point wise operation as follows: for all $(a, b), (c, d) \in M$,

$(a, b) \to (c, d) = (a \to c, b \to d)$, $(a, b) \vee (c, d) = (a \vee c, b \vee d)$

$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d)$, $(a, b)' = (a', b')$.

Now for all $(a, b), (c, d), (e, f) \in M$, using these definitions and definition of IADL we get,

1) $((a, b) \vee (c, d)) = (a \vee c, b \vee d) = ((a \to c) \to c, (b \to d) \to d) = ((a, b) \to (c, d)) \to (c, d)$
2) $((a, b) \wedge (c, d)) = (a \wedge c, b \wedge d) = ([(a \to c) \to a']', [(b \to d) \to b']')$
$= [(a \to c) \to a', (b \to d) \to b']' = [(a \to c, b \to d) \to (a', b')]'$
$= [((a, b) \to (c, d)) \to (a', b')]' = [((a, b) \to (c, d)) \to (a, b)']'$
3) $(a, b) \to ((c, d) \to (e, f)) = (a, b) \to ((c \to e), (d \to f)) = ((a \to (c \to e), b \to (d \to f))$
$= ((c \to (a \to e), d \to (b \to f)) = (c, d) \to ((a, b) \to (e, f))$

4) $(m, m*) \to (a, b) = (a, b)$

5) $(a, b) \to (m, m*) = (m, m*)$

6) $(a, b) \to (c, d) = (a \to c, b \to d) = (c' \to a', d' \to b') = (c', d') \to (a', b') = (c, d)' \to (a, b)'$

7) $(0, 0*)' = (m, m*)$.

Therefore, M is an IADL.

Example 2.6. Every implicative algebra $(L, \to, ', 0, 1)$ is an IADL.

In the following example, we give a method of constructing IADL which is neither implicative algebra nor discrete IADL.

Example 2.7. Let L be an implicative algebra and D be discrete IADL with 0 and at least two elements. Then $M = L \times D$ is an IADL with respect to point wise operation. But M is not implicative algebra since D is not. Also, M is not discrete IADL since L is not.

- $M = L \times D$ is IADL with respect to point wise operation because it respects the distributive lattice structure and implicative algebra operations. M inherits distributive lattice structure from D and L. Hence M is lattice with distributive property.
- M is not implicative algebra because D does not satisfy the necessary implicative algebra property. It fails to satisfy the implication properties needed for M to be implicative algebra. If D does not have required implicative algebra properties M cannot inherit them from D.
- M is not discrete IADL because L does not have the discrete structure required for such a lattice. Particularly every element should be comparable (totally ordered). This failure implies that M inherits this lack of discrete ness from L.

Now we define the following ideas in IADL L:

a) The relation $\leq$ on L is defined as, for any $x, y \in L$, $x \leq y$ if and only if $x \to y = m$.

b) The maximal element m of L is defined as if $m \leq x$ for any $x \in L$, then $m = x$.

c) The principal ideal generated by the maximal element m of L is denoted by (m] and defined as $(m] = \{m \wedge x : x \in L\}$.

Now we have the following remark.

1) For any $x, y \in L$, $x \wedge y = x \Leftrightarrow x \leq y$.

2) The relation $\leq$ on L is a partial ordering and hence $(L, \leq)$ is a poset.

Proof Let $x, y, z \in L$.

1) Let $x, y \in L$. Assume $x \wedge y = x$. Then $x \to y = (x \wedge y) \to y = m$. Therefore, $x \leq y$.

Conversely, assume $x \leq y$ i.e., $x \rightarrow y = m$, then $(x \wedge y) \rightarrow x = m$ implies $x \wedge y \leq x$ and $x \rightarrow (x \wedge y) = (x \rightarrow x) \wedge (x \rightarrow y) = m \rightarrow (x \rightarrow y) = x \rightarrow y = m$ implies $x \leq x \wedge y$. Thus $x \wedge y = x$ (by using Definition 2.1 and properties in ADL).

2) We prove the relation $\leq$ on L is a partial ordering relation.

a) $x \rightarrow x = m$ implies $x \leq x$. (by (3) of Lemma 2.4 and Definition (a) above).
   Therefore, $\leq$ is reflexive.

b) Assume $x \leq y$ and $y \leq x$. Then $x \rightarrow y = m$ and $y \rightarrow x = m$

Claim: $x = y$.

Since $x \leq y \Leftrightarrow x \wedge y = x = (x \vee y) \wedge x = y \wedge x$ (or $x \vee y = y = y \vee x$),

we have $x = m \rightarrow x = (y \rightarrow x) \rightarrow x = y \vee x = x \vee y = (x \rightarrow y) \rightarrow y = m \rightarrow y = y$.

Therefore, $\leq$ is anti-symmetric.

c) Assume $x \leq y$ and $y \leq z$. Then $x \rightarrow y = m$ and $y \rightarrow z = m$ (by using definition (a) above).

Now by using definition of IADL we get,

$x \rightarrow z = x \rightarrow (m \rightarrow z) = x \rightarrow ((y \rightarrow z) \rightarrow z) = x \rightarrow (y \vee z) = x \rightarrow (z \vee y)$ [Since $y \leq z$ implies $y \vee z = z = z \vee y$] $= x \rightarrow ((z \rightarrow y) \rightarrow y) = (z \rightarrow y) \rightarrow (x \rightarrow y) = (z \rightarrow y) \rightarrow m = m$.

This implies $x \leq z$. Therefore, $\leq$ is transitive.

Thus $\leq$ is a partial ordering on L (by (a), (b) and (c)). Hence, $(L, \leq)$ is a poset.

The following theorem is used to prove useful results in IADL. Now we have some results obtained in IADL.

Theorem 2.8: Let m be a maximal element of L. Then the following conditions hold:

1) $0 \rightarrow x = m$
2) $x \vee m = m$ for all $x \in L$
3) $x \wedge m = x$ for all $x \in L$
4) $(m] = L$
5) $x \wedge y \leq x \rightarrow y$
6) $y \leq x \rightarrow y$
7) $x \wedge z \leq y$ implies $z \leq x \rightarrow y$
8) $x \leq y$ if and only if $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$
9) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$
10) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$
11) $(x \rightarrow z) \rightarrow (x \rightarrow y) = (z \rightarrow x) \rightarrow (z \rightarrow y)$
12) $(x \vee y)' = x' \wedge y'$
13) $(x \wedge y)' = x' \vee y'$

14) $x \leq y$, $x \leq z$ implies $x \leq y \wedge z$

15) $y \leq x$, $z \leq x$ implies $(y \vee z) \leq x$

16) $(x \vee y) \to z \leq x \to z$ and $(x \vee y) \to z \leq y \to z$

17) $x \to z \leq (x \wedge y) \to z$ and $y \to z \leq (x \wedge y) \to z$

18) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

19) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

20) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

21) $x \to (y \vee z) = (x \to y) \vee (x \to z)$, for all $x, y, z \in L$.

Proof. Let m be maximal elements of L and for any $x \in L$. Then by Definition 2.1 and Lemma 2.4 we have,

1) $0 \to x = x' \to 0' = x' \to m = m$.

2) $m = m \to m = (x \to m) \to m = x \vee m$

3) $x \wedge m = ((x \to m) \to x')' = (m \to x')' = (x')' = x$

4) Let $x \in (m]$. Then by definition of principal ideal generated by m we get $m \wedge x = x \in L$.

Therefore, $(m] \subseteq L$..(1).

Conversely, $x \in (m] = \{x = m \wedge x : x \in L\}$ we get $x \in L$ and m is maximal element of L implies $x = m \wedge x \in (m]$. Therefore, $L \subseteq (m]$..(2)

Hence, from (1) and (2) we have $L = (m]$.

5) $(x \wedge y) \leq x \to y = (x \wedge y) \to (x \to y) = x \to ((x \wedge y) \to y) = x \to ([(x \to y) \to x')]' \to y)$

$= x \to (y' \to (x \to y) \to x') = x \to (x \to y) \to (y' \to x') = x \to (x \to y) \to (x \to y)$

$= x \to m = m$ (by Definition 2.1), we have $x \wedge y \leq x \to y$.

6) $y \leq x \to y = y \to (x \to y) = x \to (y \to y)$ (using 3 of Definition 2.1)

$= x \to m = m$. Therefore, $y \leq x \to y$.

7) $x \wedge z \leq y$ implies $z \leq x \to y$. Assume $x \wedge z \leq y$. Then by using Definition 2.1 and (2), we get $(x \wedge z) \to y = [(x \to z) \to x']' \to y = m$

$\Rightarrow y' \to ((x \to z) \to x') = m$

$\Rightarrow (x \to z) \to (y' \to x') = m$

$\Rightarrow (x \to z) \to (x \to y) = m$.

Therefore, $x \to z \leq x \to y$. Thus, $z \leq x \to z \leq x \to y$.

Hence, $x \wedge z \leq y$ implies $z \leq x \to y$.

8) $x \leq y \Leftrightarrow z \to x \leq z \to y$ and $y \to z \leq x \to z$.

Assume $x \leq y$, then $x \to y = m$.

Now, consider $(z \to x) \to (z \to y) = (x' \to z') \to (y' \to z')$

$= y' \to ((x' \to z') \to z')$

$= y' \to (((x' \to z') \to z') \wedge m)$

$= y' \to (((z' \to x') \to x') \wedge m)$

$= y' \to ((z' \to x') \to x')$

$= (z' \to x') \to (y' \to x')$

$= (x \to z) \to (x \to y) = m$

Hence, $x \leq y$ implies $z \to x \leq z \to y$………………………………………………… (1)

Similarly, consider $(y \to z) \to (x \to z) = x \to ((y \to z) \to z) = x \to (((y \to z) \to z) \wedge m)$

$= x \to (((z \to y) \to y) \wedge m) = (z \to y) \to (x \to y) = (z \to y) \to m = m.$

Hence, $x \leq y$ implies $y \to z \leq x \to$ …………………………………………………..... (2)

($\Leftarrow$) suppose $z \to x \leq z \to y$, then $(z \to x) \to (z \to y) = m$.

Now, $x \to y = x \to (m \to y) = x \to ((z \to x) \to (z \to y) \to y)$

$= (z \to x) \to (z \to y) \to (x \to y)$

$= (x' \to z') \to (y' \to z') \to (y' \to x')$

$= y' \to ((x' \to z') \to z') \to (y' \to x')$

$= y' \to ((z' \to x') \to x') \to (y' \to x')$

$= (z' \to x') \to (y' \to x') \to (y' \to x')$

$= (z' \to x') \to m = m.$

Hence, $z \to x \leq z \to y$ implies $x = y$…………………………………………………… (3)

Consider $y \to z \leq x \to z$, then $(y \to z) \to (x \to z) = m$

Now, $x \to y = x \to (m \to y) = x \to ((y \to z) \to (x \to z) \to y)$

$= (y \to z) \to (x \to z) \to (x \to y)$

$= x \to ((y \to z) \to z) \to (x \to y)$

$= x \to ((z \to y) \to y) \to (x \to y)$

$= (z \to y) \to (x \to y) \to (x \to y)$

$= (z \to y) \to m = m.$

Hence $y \to z \leq x \to z$ implies $x = y$………………………………………………….…… (4)

Therefore, from (1), (2), (3) and (4), we have $x \leq y \Leftrightarrow z \to x \leq z \to y$ and $y \to z \leq x \to z$.

9) $((x \to y) \to y) \to y = (y \to (x \to y)) \to (x \to y)$ [since $((x \to y) \to y) = ((y \to x) \to x)$]

$= (x \to (y \to y)) \to (x \to y)$ [by Definition 1.2.2.3(1)]

$= (x \to m) \to (x \to y)$ (since $y \to y = 1$)

$= m \to (x \to y)$ [by Definition 2.1(5)].

$= x \to y$

10) $(x \to y) \leq (y \to z) \to (x \to z) = (x \to y) \to ((y \to z) \to (x \to z))$

$= (x \to y) \to (x \to ((y \to z) \to z))$

$= (x \to y) \to ((x \to ((y \to z) \to z)) \wedge m)$ (by Lemma 2.4)

$= (x \to y) \to (x \to ((z \to y) \to y))$

$= (x \to y) \to ((z \to y) \to (x \to y))$

$= (z \to y) \to ((x \to y) \to (x \to y)) = (z \to y) \to m = m.$

Therefore, $(x \to y) \to ((y \to z) \to (x \to z)) = m$. Hence, $(x \to y) \leq (y \to z) \to (x \to z)$.

11) $(x \to z) \to (x \to y) = (z \to x) \to (z \to y)$. By Theorem 2.8 and Lemma 2.4 we have,

$(x \to z) \to (x \to y) = (z' \to x') \to (y' \to x') = y' \to ((z' \to x') \to x')$

$= y' \to (((z' \to x') \to x') \wedge m) = y' \to ((x' \to z') \to z') = (x' \to z') \to (y' \to z')$

$= (z \to x) \to (z \to y)$. Hence, $(x \to z) \to (x \to y) = (z \to x) \to (z \to y)$.

12) $(x \vee y)' = x' \wedge y'$

Consider $(x \vee y)' \to x' \wedge y' = ((x \to y) \to y)' \to ((y' \to x') \to y'')'$

$= ((x \to y) \to y)' \to ((y' \to x') \to y)'$

$= ((y' \to x') \to y) \to ((x \to y) \to y)$

$= ((x \to y) \to y) \to ((x \to y) \to y) = m$ (By Definition 2.1)

Hence, $(x \vee y)' \leq x' \wedge y'$ …………………………………………………………………….…. (1)

And also, $x' \wedge y' \to (x \vee y)' = ((x' \to y') \to x'')' \to ((x \to y) \to y)'$

$= ((y' \to x') \to y'')' \to ((x \to y) \to y)'$

$= ((x \to y) \to y) \to ((y' \to x') \to y'')$

$= ((x \to y) \to y) \to ((y' \to x') \to y)$

$= ((x \to y) \to y) \to ((x \to y) \to y) = m$ (Using Definition 2.1)

Then, $x' \wedge y' \leq (x \vee y)'$ ………………………………….……………………….… (2)

Therefore, from (1) and (2), we have $(x \vee y)' = x' \wedge y'$.

13) $(x \wedge y)' = x' \vee y'$

Consider $(x \wedge y)' \to x' \vee y' = (((y \to x) \to y')')' \to ((x' \to y') \to y')$

$= ((y \to x) \to y') \to ((x' \to y') \to y')$

$= ((y \to x) \to y') \to ((y \to x) \to y') = m$

Hence, $(x \wedge y)' \leq x' \vee y'$ …………….…………………………………………...........(1)

And also, $x' \vee y' \to (x \wedge y)' = ((x' \to y') \to y') \to (((y \to x) \to y')')'$

$= ((x' \to y') \to y') \to ((y \to x) \to y')$

$= ((x' \to y') \to y') \to ((x' \to y') \to y') = m$

Hence, $x' \vee y' \leq (x \wedge y)'$ ………………………………………………….……………... (2).

Therefore, from (1) and (2), we have $(x \wedge y)' = x' \vee y'$.

14) $x \leq y$, $x \leq z$ implies $x \leq y \wedge z$

Assume $x \leq y$ and $x \leq z$, then $x \to y = m$ and $x \to z = m$ respectively.

Then $x \to (y \wedge z) = x \to [(y \to z) \to y']'$

$= [(y \to z) \to y'] \to x'$

$= [(y \to z) \to y'] \to (m \to x')$

$= [(y \to z) \to y'] \to ((x \to z) \to x')$

$= [(z' \to y') \to y'] \wedge m \to ((z' \to x') \to x') \wedge m$

$= [(y' \to z') \to z'] \to ((x' \to z') \to z')$

$= (x' \to z') \to ([(y' \to z') \to z'] \to z')$

$= (x' \to z') \to (y' \to z')$ (Since $([(y' \to z') \to z'] \to z') = (y' \to z')$)

$= y' \to ((x' \to z') \to z') \wedge m)$

$= y' \to ((z' \to x') \to x'))$

$= (z' \to x') \to (y' \to x')$

$= (x \to z) \to (x \to y) = m$ (by using Lemma 2.4). Therefore, $x \leq y \wedge z$.

15) Assume $y \leq x$ and $z \leq x$, then $y \to x = m$ and $z \to x = m$ respectively.

Then $(y \vee z) \to x = ((y \to z) \to z) \to (m \to x)$

$= ((y \to z) \to z) \wedge m) \to (m \to x)$

$= ((z \to y) \to y) \to ((y \to x) \to x)$

$= ((z \to y) \to y) \to ((y \to x) \to x) \wedge m)$

$= ((z \to y) \to y) \to ((x \to y) \to y)$

$= (x \to y) \to (((z \to y) \to y) \to y)$

$= (x \to y) \to (z \to y)$

$= z \to ((x \to y) \to y)$

$= z \to ((y \to x) \to x)$

$= (y \to x) \to (z \to x) = m$. Hence, $y \vee z \leq x$.

16) Consider $((x \vee y) \to z) \leq (x \to z) = ((x \vee y) \to z) \to (x \to z)$

$= (((x \to y) \to y) \to z) \to (x \to z)$

$= ((((x \to y) \to y) \wedge m) \to (z \wedge m)) \to (x \to z)$

$= (((y \to x) \to x) \to z) \to (x \to z)$

$= z \to ((y \to x) \to x) \to (x \to z)$

$= (x \to z) \to (y \to x) \to (x \to z)$

$= (y \to x) \to (x \to z) \to (x \to z)$

$= (y \to x) \to m = m$ (since $(x \to z) \to (x \to z) = m$)

Similarly, we have $((x \vee y) \to z) \leq (y \to z) = ((x \vee y) \to z) \to (y \to z)$

$= (((x \to y) \to y) \to z) \to (y \to z)$

$= z \to ((x \to y) \to y) \to (y \to z)$

$= (y \to z) \to (x \to y) \to (y \to z)$

$= (x \to y) \to (y \to z) \to (y \to z)$

$= (x \to y) \to m = m.$

Therefore, $((x \vee y) \to z) \leq x \to z$ and $((x \vee y) \to z) \leq y \to z$ hold.

17) Consider $(x \to z) \to ((x \wedge y) \to z) = (x \wedge y) \to ((x \to z) \to z)$

$= (x \wedge y) \to (((x \to z) \to z) \wedge m)$

$= (x \wedge y) \to ((z \to x) \to x)$

$= (z \to x) \to ((x \wedge y) \to x)$

$= (z \to x) \to (x' \to (x' \vee y'))$

$= (z \to x) \to (x' \to ((x' \to y') \to y'))$

$= (z \to x) \to ((x' \to y') \to (x' \to y'))$

$= (z \to x) \to ((y \to x) \to (y \to x))$

$= (z \to x) \to m = m$ (by using Lemma 2.4).

Therefore, $(x \to z) \to ((x \wedge y) \to z) = m$. Hence, $x \to z \leq (x \wedge y) \to z$.

Similarly, we have $(y \to z) \to ((x \wedge y) \to z) = (x \wedge y) \to ((y \to z) \to z)$

$= (x \wedge y) \to (((y \to z) \to z) \wedge m)$

$= (x \wedge y) \to ((z \to y) \to y)$

$= (z \to y) \to ((x \wedge y) \to y)$

$= (z \to y) \to (y' \to (x \wedge y)')$

$= (z \to y) \to (y' \to (x' \vee y'))$

$= (z \to y) \to (y' \to (y' \vee x'))$

$= (z \to y) \to (y' \to (y' \to x') \to x')$

$= (z \to y) \to (y' \to x') \to (y' \to x')$

$= (z \to y) \to m = m.$ (by using Lemma 2.4).

Therefore, $y \to z \leq (x \wedge y) \to z$.

18) From the above Theorem 1.2.2.5 (2) and Theorem 1.2.2.5 (4), we have

$(x \vee y) \to z \leq (x \to z) \wedge (y \to z)$ ………………………………………………………… (1)

And it remains to show that the other way $(x \to z) \wedge (y \to z) \leq (x \vee y) \to z$.

Now $[(x \to z) \wedge (y \to z)] \to (x \vee y) \to z$

$= [((x \to z) \to (y \to z)) \to (x \to z)']' \to [((x \to y) \to y) \to z]$

$= [((x \to y) \to y) \to z]' \to [((x \to z) \to (y \to z)) \to (x \to z)']$

$= [(x \to z) \to (y \to z)] \to [(((x \to y) \to y) \to z' \to (x \to z)']$

$= [(x \to z) \to (y \to z)] \to [(x \to z) \to (((x \to y) \to y) \to z)]$

$= [y \to ((x \to z) \to z)] \to [((x \to y) \to y) \to ((x \to z) \to z)]$

$= ((x \to y) \to y) \to [(y \to ((x \to z) \to z) \to ((x \to z) \to z))]$

$= ((x \to y) \to y) \to [((((x \to z) \to z) \to y) \to y)]$

$= ((x \to z) \to z) \to y \to [(((x \to y) \to y) \to y)]$

$= (((x \to z) \to z) \to y) \to (x \to y)$

$\geq x \to ((x \to z) \to z) = (x \to z) \to (x \to z) = m.$

Hence $(x \to z) \wedge (y \to z) \to (x \vee y) \to z = 1$, then $(x \to z) \wedge (y \to z) \leq (x \vee y) \to z$... (2)

Therefore, from (1) and (2) we have $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

19) Clearly from the above Theorem 1.2.2.5 (3) and Theorem 1.2.2.5 (5), we have

$(x \to z) \vee (y \to z) \leq (x \wedge y) \to z$.…………………………………………………….… (1)

And it remains to show that the other way $(x \wedge y) \to z \leq (x \to z) \vee (y \to z)$.

Now $(x \wedge y) \to z \to (x \to z) \vee (y \to z)$

$= [((y \to x) \to y')' \to z] \to [((y \to z) \to (x \to z)) \to (x \to z]$

$= [z' \to ((y \to x) \to y')] \to [((y \to z) \to (x \to z)) \to (x \to z)]$

$= [(y \to x) \to (z' \to y')] \to [((y \to z) \to (x \to z)) \to (x \to z)]$

$= [(y \to x) \to (y \to z)] \to [((y \to z) \to (x \to z)) \to (x \to z)]$

$= (y \to z) \to (x \to z) \to [((y \to x) \to (y \to z)) \to (x \to z)]$

$= x \to ((y \to z) \to z) \to [((y \to x) \to (y \to z)) \to (x \to z)]$

$= x \to ((z \to y) \to y) \to [((y \to x) \to (y \to z)) \to (x \to z)]$

$= (z \to y) \to (x \to y) \to [((y \to x) \to (y \to z)) \to (x \to z)]$

$= (y \to x) \to (y \to z) \to [((z \to y) \to (x \to y)) \to (x \to z)]$

$= (y \to x) \to (y \to z) \to [x \to ((z \to y) \to y) \to (x \to z)]$

$= (y \to x) \to (y \to z) \to [x \to ((y \to z) \to z) \to (x \to z)]$

$= (y \to x) \to (y \to z) \to [(y \to z) \to (x \to z) \to (x \to z)]$

$= (y \to x) \to (y \to z) \to [(y \to z) \to m]$

$= (y \to x) \to (y \to z) \to m = m.$

Hence $(x \wedge y) \to z \to (x \to z) \vee (y \to z) = m$, then $(x \wedge y) \to z \to (x \to z) \vee (y \to z)$ .. (2)

Therefore, from (1) and (2), we have $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

20) $x \to (y \wedge z) = x \to ((y \to z) \to y')'$ (by Definition 2.1(2)).

$= ((y \to z) \to y') \to x'$ (by 6 Definition 2.1).

$= ((z' \to y') \to y') \to x'$ (by 6 Definition 2.1).

$= (z' \vee y') \to x'$ (by 1 of Definition 1.2.1.1)

$= (y' \vee z') \to x'$ (commutativity of join)

$= (y' \to x') \wedge (z' \to x')$ (by 6 Definition 1.2.2.1)

$= (x \to y) \wedge (x \to z)$

21) $x \to (y \vee z) = (y \vee z)' \to x'$ (by 6 of Definition 2.1)

$= (y' \wedge z') \to x'$

$= (y' \to x') \vee (z' \to x')$ (by 7 of Definition1.2.2.1)

$= (x \to y) \vee (x \to z)$ (by Definition 2.1).

Remark 2.9: By 8 of Theorem 2.8, $x \leq y$, implies $y' \leq x'$ for any $x, y \in L$. Such condition is called order reversing.

Theorem 2.10. In L, the following conditions are equivalent for all $x, y \in L$.

1) L is implicative algebra
2) $x \wedge y \leq x, y \leq x \vee y$
3) $x \vee y$ is the least upper bound of $\{x, y\}$
4) $x \wedge y$ is the greatest lower bound for $\{x, y\}$

Proof: Let $x, y, z, t \in L$.

(1) ⇒ (2). Assume L is implicative algebra.

Then $(x \wedge y) \to x = [(x \to y) \to x']' \to x = x' \to [(x \to y) \to x'] = (x \to y) \to (x' \to x')$

$= (x \to y) \to m = m$. Therefore, $x \wedge y \leq x$.

Similarly, $x \wedge y \leq y$ implies $x \wedge y \to y = ([(x \to y) \to x']' \to y) = y' \to [(x \to y) \to x']$

$= (x \to y) \to (y' \to x') = (x \to y) \to (x \to y) = m$.

And $x \leq x \vee y = x \to (x \vee y) = x \to ((x \to y) \to y = (x \to y) \to (x \to y) = m$.

Therefore, $x \leq x \vee y$.

Similarly, $y \leq (x \vee y) = y \to (x \vee y) = y \to ((x \to y) \to y) = y \to (((x \to y) \to y) \wedge m)$

$= y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = m$. Therefore, $y \leq x \vee y$.

(2) ⇒ (3). Assume (2). From 2, we get $x \leq x \vee y$ and $y \leq x \vee y$. Thus $x \vee y$ is an upper bound of $\{x, y\}$. Let t be any other upper bound of $\{x, y\}$. Then $x \leq t$ and $y \leq t$. By Theorem 2.8 and our assumptions $(x \vee y) \to t = (x \to t) \wedge (y \to t) = m \wedge m = m$. Thus, $x \vee y \leq t$. Hence, $x \vee y$ is least upper bound of $\{x, y\}$

(3) ⇒ (2). Since $x \vee y$ is least upper bound of $\{x, y\}$, we have $x, y \leq x \vee y$ and then $x \wedge (x \vee y) = x$, $y \wedge (x \vee y) = y$. Now $x \wedge y = x \wedge (y \wedge (x \vee y)) = y \wedge x \wedge (x \vee y) = y \wedge x$.

This implies $\wedge$ is commutative and hence by Theorem 1.2.1.10 L is distributive lattice so that $x \wedge y \leq x, y$.

(2) ⇒ (4). Assume (2). From 2, $x \wedge y$ is lower bound of $\{x, y\}$. Let t be any other lower bound of $\{x, y\}$. Then $t \to x = m$ and $t \to y = m$. Now from conditions of Definition 2.1, Lemma 2.4, and Theorem 2.8, it follows that

$$\begin{aligned} t \to (x \wedge y) &= (x' \vee y') \to t' = (x' \vee y') \to (m \to t') = (x' \vee y') \to (t \to y) \to t' \\ &= (x' \vee y') \to ((y' \to t') \to t') = (x' \vee y') \to ((t' \to y') \to y') = (t' \to y') \to ((x' \vee y') \to y') \\ &= (t' \to y') \to (((x' \to y') \to y') \to y') = (t' \to y') \to ((x' \to y') = ((x' \to (y' \to t') \to t') \\ &= (y' \to t') \to (x' \to t') = (t \to y) \to (t \to x) = m. \end{aligned}$$

That is, $t \leq x \wedge y$. Hence, $x \wedge y$ is the greatest lower bound of $\{x, y\}$.

(4) ⇒ (1). Assume 4 holds. From 4, it follows $x \wedge y \leq x$ and $x \wedge y \leq y$.

1) $x \wedge x \leq x$ implies $x \leq x$. Therefore, $\leq$ is reflexive.

2) Let $x \leq y$ and $y \leq x$. Then $x \to y = y \to x = m$. We have $x = y$. Therefore, $\leq$ is anti symmetric.

3) Let $x \leq y$ and $y \leq z$. Since $\wedge$ associative in L and $x \wedge y = x$ and $y \wedge z = y$.

We have, $x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$. Therefore, $\leq$ is transitive.

Thus, $(L, \leq)$ is a poset. And also, $x \wedge y \leq x$ implies $x \wedge y = (x \wedge y) \wedge x = y \wedge x$. By Theorem 1.2.1.13, L is distributive lattice. Hence, L is an implicative algebra.

Theorem 2.11: For any $x, y, z \in L$, the following are equivalent.

1) L is implicative algebra
2) The poset $(L, \leq)$ is directed above
3) $(L, \vee, \wedge)$ is a distributive lattice
4) $\vee$ is commutative
5) $\wedge$ is commutative
6) $\vee$ is right distributive over $\wedge$
7) The relation $\theta := \{(x, y) \in L \times L : y \wedge x = x\}$ is anti symmetric.

Proof. Let $x, y, z \in L$.

(1) ⇒ (2): Assume L is an implicative algebra. Then L is a lattice and for all $x, y \in L$, there exists $1 \in L$ such that $x \leq 1$ and $y \leq 1$. This implies the poset $(L, \leq)$ is directed above.

(2) ⇒ (3): Assume the poset $(L, \leq)$ is directed above. For every pair $x, y \in L$, there exists a least upper bound $x \vee y$, and since L is lattice, there is also greatest lower bound $x \wedge y$. For to be distributive, we need to show that for all $x, y, z \in L$. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. This implies

$(L, \vee, \wedge)$ is a distributive lattice. (3) ⇒ (4), Assume $(L, \vee, \wedge)$ is a distributive lattice, in distributive lattice both, both $\vee$ (join) and $\wedge$ (meet) area commutative by definition. For any elements $x, y \in L$, we have $x \vee y = y \vee x$.

(4) ⇒ (5) if $\vee$ is commutative, then the lattice properties ensure that $\wedge$ is also commutative. This follows directly from the duality between operations $\vee$ and $\wedge$.

(5) ⇔ (6) are clear from theorem 1.2.1.10. (6) ⇔ (7) Assume (6), the relation θ defines a partial order if it is anti symmetric, that is if $x \wedge y = x$ and $x \wedge y = y$, then $x = y$. Hene θ anti symmetric. We finish the theorem by establishing (7) ⇒ (1): Assume (7) and suppose $x, y \in L$. Then $(x \wedge y) \wedge (y \wedge x) = y \wedge x$ and $(y \wedge x) \wedge (x \wedge y) = x \wedge y$ so that the elements $(x \wedge y, y \wedge x)$, $(y \wedge x, x \wedge y)$ belongs to θ, and hence $x \wedge y = y \wedge x$. Now, by theorem 2.10, we have $(L, \leq)$ is a poset in which, for any $x, y \in L$, $x \wedge y$ is greatest lower bound (glb) of x and y and $x \vee y$ is the least upper bound (lub) of x and y so that L is a lattice and hence a distributive lattice. This implies $(L, \leq)$ is directed above by (1). Hence, L is an implicative algebra.

The following is also a characterization of an IADL. If L is an ADL with 0, then for any $a \in L$, the interval $[0, a]$ is a bounded distributive lattice. Hence, we can extend many concepts existing in the class of distributive lattices to the class of ADLs. The following theorem justifies the definition of IADL given in Definition 2.1.

Theorem 2.12: Let L be an ADL with 0 and a maximal element m. Then the following are equivalent.

1) L is IADL
2) $[0, a]$ is implicative algebra for all $a \in L$
3) $[0, m]$ is implicative algebra.

Proof. Let L be an ADL with 0 and maximal element m and $a \in L$.

(1) ⇒ (2). Assume that L is an IADL and $a \in L$. We know that $[0, a]$ is a bounded distributive lattice.

Now, define a binary operation $\rightarrow^a$ on $[0, a]$ by $x \rightarrow^a y = (x \rightarrow y) \wedge a$ for any $x, y \in [0, a]$. Suppose $c = d$ and $e = f$, then $c \rightarrow^a e = (c \rightarrow e) \wedge a = (d \rightarrow f) \wedge a = d \rightarrow^a f$ for all $c, d, e, f \in [0, a]$. Thus $\rightarrow^a$ on $[0, a]$ is well defined. Let $x, y, z \in [0, a]$:

1) Since $x = x \wedge a$, and $y \wedge a = y$, we have,

$$\begin{aligned} x \rightarrow^a (y \rightarrow^a z) &= [x \rightarrow (y \rightarrow^a z)] \wedge a \\ &= [x \rightarrow ((y \rightarrow z) \wedge a] \wedge a \\ &= [x \rightarrow (y \rightarrow z)] \wedge a \\ &= [y \rightarrow (x \rightarrow z)] \wedge a \end{aligned}$$

$= [y \to ((x \to z) \wedge a)] \wedge a$

$= [y \to (x \to^{a} z)] \wedge a$

$= y \to^{a} (x \to^{a} z)$.

2) $a \to^{a} x = (a \to x) \wedge a = x \wedge a = x$.
3) $x \to^{a} a = (x \to a) \wedge a = a \wedge a = a$.
4) $x \to^{a} y = (x \to y) \wedge a = (y' \to x') \wedge a = y' \to^{a} x'$.
5) $(x \to^{a} y) \to^{a} y = ((x \to^{a} y) \to y) \wedge a = [((x \to y) \wedge a) \to y] \wedge a = [(x \to y) \to y] \wedge a$
$= [(y \to x) \to x] \wedge a = (y \to^{a} x) \to^{a} x$.
6) $0' = 0 \to^{a} 0 = (0 \to 0) \wedge a = a \wedge a = a$.

Therefore, [0, a] is an implicative algebra for all $a \in L$.

(2) ⇒ (3). Assume [0, a] is implicative algebra, since $a = m$ is an element of L, and we are given that [0, a] is implicative for all $a \in L$, it follows trivially that [0, m] is implicative. The condition applies to all $a \in L$, and m is specific element in L. Therefore [0, m] is implicative.

(3) ⇒ (1). Assume that [0, m] is an implicative algebra in which the binary operation (→) is denoted by $\to^{m}$. Define $x \to y = x \wedge m \to^{m} y \wedge m$ for any $x, y \in L$. Let $x, y, z \in L$,

1) $x \vee y = [x \wedge m \to^{m} y \wedge m] \wedge m \to^{m} y \wedge m = (x \to y) \to y$.
2) $x \wedge y = [(x \wedge m \to^{m} y \wedge m) \wedge m \to^{m} x' \wedge m]' = [(x \to y) \to x']'$.
3) $x \to (y \to z) = x \wedge m \to^{m} (y \wedge m \to^{m} z \wedge m) \wedge m = x \wedge m \to^{m} (y \wedge m \to^{m} z \wedge m)$
$= y \wedge m \to^{m} (x \wedge m \to^{m} z \wedge m) = y \wedge m \to^{m} (x \wedge m \to^{m} z \wedge m) \wedge m = y \to (x \to z)$.
4) $x \to y = x \wedge m \to^{m} y \wedge m = (y \wedge m)' \to^{m} (x \wedge m)') = y' \to x'$.
5) $m \to x = m \wedge m \to^{m} x \wedge m = x$.
6) $x \to m = x \wedge m \to^{m} m \wedge m = m$.
7) $0' = 0 \to 0 = 0 \wedge m \to^{m} 0 \wedge m = m$.

Therefore, L is IADL

Lemma 2.13: Let $x, y \in L$. Then the following are equivalent.

1) $(x] \subseteq (y]$
2) $y \wedge x = x$
3) $x \wedge t \leq y \wedge t$ all $t \in L$.

Proof. Let $x, y, z \in L$. We can easily prove using Lemma 2.4 and Theorem 2.8 as follows,

(1) ⇒ (2). Since $x \in (x] \subseteq (y]$, then $x \in (y]$ so that $x = y \wedge t$ for some $t \in L$.

Thus, $y \wedge x = y \wedge (y \wedge t) = (y \wedge y) \wedge t = y \wedge t = x$ (since ∧ is associative).

(2) ⇒ (3). If $y \wedge x = x$, then for any $t \in L$, $(x \wedge t) \to (y \wedge t) = (y \wedge x \wedge t) \to (y \wedge t) =$
$(x \wedge y \wedge t) \to (y \wedge t) = (x \to (y \wedge t)) \vee ((y \wedge t \to (y \wedge t))$ (By 7 of Definition 1.2.2.1)

$= (x \to (y \wedge t)) \vee m = m$. Thus, $x \wedge t \leq y \wedge t$.

(3) ⇒ (1). Let $x \in (x]$ such that $x \wedge t \leq y \wedge t$ all $t \in L$. Put $t = x$, $x \wedge x \leq y \wedge x \leq x$ so that $x = y \wedge x$. Thus, $x \in (y]$. Therefore, $(x] \subseteq (y]$.

Theorem 2.14: Let L be an ADL with zero and maximal element m. Then L is an IADL if and only if the set of principal ideal P I(L) is an implicative algebra.

Proof. Let L be an ADL with zero and maximal element m. Suppose L is an IADL, then since (0] and (m] are least and greatest element of P I(L), (P I(L), ∨, ∧) is bounded. Clearly (P I(L), ⊆) is a poset under set inclusion ⊆. For $(x], (y] \in P\, I(L)$, $(x] \wedge (y] = (x \wedge y] = \inf\{(x], (y]\}$ and $(x] \vee (y] = (x \vee y] = \sup\{(x], (y]\}$. Therefore, (P I(L), ∨, ∧) is a bounded lattice. Now we define $(x] \to (y] = (x \to y]$ for any $x, y \in L$. Using Lemma 2.13 we prove that the binary operation → on P I(L) is well defined. If $(a] = (b]$ and $(c] = (d]$, then $a \wedge b = b$, $b \wedge a = a$, $c \wedge d = d$, $d \wedge c = c$.

Now $a \to c = a \to (d \wedge c) = (a \to d) \wedge (a \to c) = ((a \vee b) \to d) \wedge (a \to c)$

$= (a \to d) \wedge (b \to d) \wedge (a \to c) \leq (b \to d) \wedge (a \to c) \leq a \to c$.

Hence $(b \to d) \wedge (a \to c) = (a \to c)$. This implies that $(a \to c] \subseteq (b \to d]$ and similarly we get $(b \to d] \subseteq (a \to c]$. Therefore, $(a \to c] = (b \to d]$.

Thus, the binary operation → on P I(L) is well defined. Now we can routinely verify that P I(L) is an implicative algebra. Let $(x], (y], (z] \in P\, I(L)$.

1) $(x] \to ((y] \to (z]) = ((x \to (y \to z)]) = ((y \to (x \to z)]) = (y] \to ((x] \to (z])$.
2) $(m] \to (x] = (x]$
3) $(x] \to (m] = (m]$
4) $(x] \to (y] = (y'] \to (x']$
5) $((x] \to (y]) \to (y] = ((y] \to (x]) \to (x]$
6) $(0'] = (m]$.

Conversely, Let L be an ADL with zero and maximal element m. Suppose P I(L) is an implicative algebra. For all $x, y \in L$, define $x \to y = z \wedge m$ where $(x] \to (y] = (z]$ for some $z \in L$. Let $(s] = (t]$, for some $s, t \in L$. Then $s \wedge t = t$ and $t \wedge s = s$.

Now $s \wedge m = t \wedge s \wedge m = s \wedge t \wedge m = t \wedge m$. Thus, the binary operation → on L is well defined. Let $x, y, z \in L$.

1) $(x] \vee (y] = ((x] \to (y]) \to (y] = (r] \to (y]$ so that $x \to y = r \wedge m$ for some $r \in L$, where $(x] \to (y] = (r]$ and $r \to y = v \wedge m$ for some $v \in L$, where $(r] \to (y] = (v]$. Therefore, $(x \to y) \to y = v \wedge m$ where $((x] \to (y]) \to (y] = (v]$.
2) $(x] \wedge (y] = (((x] \to (y]) \to (x'])' = ((r] \to (x'])'$ where $(x] \to (y] = (r]$ for some $r \in L$ and

$((r] \to (x'])' = (v]'$ for some $v \in L$. Therefore, $[(x \to y) \to x']' = v' \wedge m$.

3) Let $(x], (y], (z] \in PI(L)$ and $PI(L)$ is an implicative algebra.

   $(x] \to ((y] \to (z]) = (y] \to ((x] \to (z])$ such that $x \to r = v \wedge m$ where $(x] \to (r] = (v]$ and $y \to z = r \wedge m$, where $(y] \to (z] = (r]$ for some $r, v \in L$. This implies $x \to (y \to z) = v \wedge m$ where $(y] \to ((x] \to (z]) = (x] \to ((y] \to (z]) = (v]$ for some $v \in L$. This also implies $y \to (x \to z) = v \wedge m$ where $(y] \to ((x] \to (z]) = (v]$ for some $v \in L$.

   Therefore, $x \to (y \to z) = y \to (x \to z)$.

4) $(x] \to (y] = (r]$ implies $x \to y = r \wedge m$ for some $r \in L$,

   Now $y' \to x' = ((y \to 0) \to (x \to 0)) = x \to ((y \to 0) \to 0) = x \to y = r \wedge m$ where $(y'] \to (x'] = (r]$. Therefore, $x \to y = y' \to x'$.

5) $(m] \to (x] = (x]$ implies $m \to x = x \wedge m = x$.

6) $(x] \to (m] = (m]$ implies $x \to m = m \wedge m = m$.

7) $(0'] = (m]$ implies $0 \to 0 = m \wedge m = m$. Therefore, L is an IADL.

Remark 2.15: In L with $0 \neq m$, $\to$ can never be associative.

Proof. Suppose $\to$ on L is associative and $0 \neq m$.

Let $a = b = c \in L$. Then, by condition 3 of Definition 2.1, we have

$\Rightarrow (a \to a) \to a = a \to (a \to a)$

$\Rightarrow m \to a = a \to m$

$\Rightarrow a = m$ (This implies $m \to a = a \to m$ and thus $a = m$),

Hence, $0 = m$, leads to a contradiction.

Therefore, $\to$ can never be associative

Conclusion

In this project we have discussed about the concept of implicative almost distributive lattice and studied some of their properties. As a result, we have identified gaps between IAs and ADLs, we have proved theorems which was not proved, give additional examples regarding to our project. Therefore, we are motivated to extend IAs to implicative ADLs in the class of ADLs. We have introduced implicative ADLs (IADLs) as a generalization of implicative algebras in the class of ADLs.

In this project, the theory of implicative almost distributive lattices (IADLs) is introduced and we have developed different results related with IADLs. We discussed about necessary and sufficient condition to IADLs

References

[1] B.A. Alaba, M. Alamneh, and T. Mekonnen, "Implicative almost distributive lattice," International Journal of computing science and Applied Mathematics, vol. 4, no.19-22,2018

[2] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Springer-Verlag, New York, 1981

[3] G. Gratzer, General lattice theory, Acadamic press, Newyork, Sanfransisco (1978)

[4] V. Kolluru and B. Bekele, "Implicative algebras," Momona Ethiopian Journal of Science, vol. 4, no. 1, pp. 90–101, 2012.

[5] G. C. Rao, "almost distributive lattice," PhD. thesis, Department of mathematics, Andhra university (1980)

[6] U. Swamy and G. Rao, "Almost distributive lattices," Journal of the Australian Mathematical Society, vol.31, no. 1, pp. 77–91, 1981gebras, Momona Ethiopian Journal of Science, 4 (2012), no. 1, 90-101.

[7] Y. Xu and K. Qin, "implication algebras and lattice implication algebra classes, j," Hebei Mining and Civil Engineering Institute, vol. 3, pp. 139– 143, 1992.

[8] X. Yang, "Lattice implication algebras [j]," Journal of Southwest Jiaotong University, vol. 1, 1993.

[9] X. Yang, L. Jun, S. Zhenming, and Q. Keyun, "On semantics of l-valued first-order logic lvft," International Journal of General System, vol. 29, no. 1, pp. 53–79, 2000.