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Intuitionistic Fuzzy Pms-Subalgebra of A Pms-Alge

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BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS

A PROJECT WORK

ON

INTUITIONISTIC FUZZY PMS-SUBALGEBRA OF A PMS-ALGEBRA

By

Messay Zewdu

June, 2024

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INTUITIONISTIC FUZZY PMS-SUBALGEBRA OF A PMS-ALGEBRA

A Project Submitted to the Department of Mathematics in Partial Fulfilment of the Requirements for the Degree of "Master of Science in Mathematics"

By

Messay Zewdu

Advisor: Yohannes Gedamu (PhD, Asso. Professor)

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APPROVAL OF THE PROJECT

I here by certify that I have supervised, read and evaluated this project entitled **Intuitionistic fuzzy PMS-subalgebra of a PMS algebra**" by **Messay Zewdu** prepared under my guidance. I recommend that the project is submitted for oral defense.

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Advisor Signature Date

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS

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Messay Zewdu

A Project submitted to the Department of Mathematics, College of Science Bahir Dar University in Partial Fulfillment of the Requirements for the Degree of "Master of Science in Mathematics".

Board of Examiners

External examiner	Signature	Date
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Internal Examiner 2:	Signature	Date
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ABSTRACT

In this project, we introduced the notion of intuitionistic fuzzy PMS subalgebra of a PMS-algebra. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra is examined, and some interesting results are obtained. We proved some properties on homomorphism and cartesian product of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra.

LIST OF SYMBOLS AND ABBREVIATIONS

$A \times B$	*Cartesian product of A and B*
μ_A	*Membership function of an intuitionistic fuzzy set A*
ν_A	*Non-membership function of an intuitionistic fuzzy set A*
A^-	*Complement of A*
χ_A	*Characteristic function of A*
$\cup$	*Union*
$\cap$	*Intersection*
$\subseteq$	*Subset*
$\in$	*Belongs to*
$\notin$	*Does not belongs to*
$\Rightarrow$	*Implies*
$\Leftrightarrow$	*If and only if*
$U(\mu_A, t)$	*Upper t-level sets*
$L(\nu_A, t)$	*Lower s-level sets*
$Im(\mu_A)$	*Image of* μ_A
$Im(\nu_A)$	*Image of* ν_A
$f(A)$	*Homomorphic image of A*
$f^{-1}(B)$	*Homomorphic inverse image of B*
inf	*Infimum*
sup	*Supremum*
max	*Maximum*
min	*Minimum*
IF	*Intuitionistic fuzzy*
IFS	*Intuitionistic fuzzy set(subset)*
$\Box A$	*Necessity operator on IFS A*
$\Diamond A$	*Possibility operator on IFS A*

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

BCK and BCI algebras are algebraic structures in abstract algebra that describe fragments of the propositional calculus involving implication known as BCI and BCK logics.
The notion of BCK algebra was introduced by Y. Imai and K. Iseki in 1966 [5]. In, 1980 K. Iseki [6] introduced the notion of a BCI-algebra as a generalization of a BCK-algebra and investigated some of their properties. The main advantage of BCK and BCI algebras is that they can be used to study the properties of logical systems with fewer axioms than Boolean algebra. Since then many researchers have introduced several new algebras as a generalization of BCK/BCI-algebras and have been extensively studied the properties of these generalized algebras. In 2016, Sithar Selvam and Nagalakshmi [14] introduced a new algebraic structure called PMS-algebra.

In 1874, Georg Cantor [3] introduced the concept of set theory (a classical or crisp set) as fundamental theory in mathematics and he defined a set to be a collection of definite and distinguishable objects. Classical set contains elements that satisfy precise properties of membership. In classical set theory, a subset A of a nonempty set X can be defined by its characteristic function

$$\chi_A : X \to \{0,1\} \text{ as } \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

.

the value $'0'$ is used to represents non-membership and the value$'1'$ is used to represents *memberships of the elements of x in X to set A. Many challenges of uncertainty exist in the real physical world, such as the inability to predict the property of "belongingness" of an object with certainty. Uncertainty may emerge as a result of incomplete information regarding the problem, information that is not completely reliable, or information received from more than one source.*

In 1965 L. Zadeh [16] introduced the concept of fuzzy set as a generalization of classical sets to handle uncertainty and vagueness mathematically which Cantor set could not address. Fuzzy sets are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree of freedom. Let X be a non-empty set. A fuzzy set A in X is defined as $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ where the functions $\mu_A : X \to [0,1]$ define the degree of membership of the element x in X to set A. The value ' 0 ' is used to represent complete non-membership, the value ' 1 ' is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. In case, when $\mu_A(x)$ maps X into $[0,1]$, A is a crisp set and $\mu_A(x)$ is identical to the characteristic function of a crisp set. A characteristic function is a special case of a membership function. So that a crisp set is a special case of a fuzzy set.

After the introduction of the concept of fuzzy sets by Zadeh, several researchers studied on fuzzification of important mathematical structures. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971 by Rosenfeld. A. Rosenfeld [10] used fuzzy set in the realm of group theory and introduced fuzzy subgroups of a group, and then he fuzzified the group structures and proved a number of results. Since then many researchers are engaged in extending the concepts/results of abstract algebra to the broader framework of the fuzzy setting. However, not all the results on groups and rings can be fuzzified. In 1982, Liu[7] introduced fuzzy sets in the realm of ring theory. In 1991, O. G. Xi [15] applied the concept of fuzzy sets to BCK algebras. Sithar Selvam and Nagalakshmi in [14] introduced the concept of a fuzzy PMS-subalgebra and fuzzy PMS-ideal of a PMS-algebra and established various properties in detail.

K. T. Atanassov [1] introduced the idea of intuitionistic fuzzy set as a generalization of *fuzzy set by considering both the membership and the nonmembership grades of an object in a set. As a result, it describes an intuitionistic fuzzy set A, by* $\{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ *where* $\mu_A : X \to [0,1]$ *is a membership function and* $\upsilon_A : X \to [0,1]$ *is a nonmembership function. An intuitionistic fuzzy set is more effective than a fuzzy set in dealing with ambiguity and uncertainty since it assigns a membership and nonmembership degree to each element of a set. Since its appearance, mathematicians have applied this fundamental concept to a number of algebraic structures. Biswas [2] studied intuitionistic fuzzy subgroups of a group using the concept of intuitionistic fuzzy sets. Senapati et al. [11] investigated intuitionistic fuzzification of subalgebras and ideals of BG-algebras. In 2011, Mostafa et al.[8] introduced the intuitionistic fuzzy KU-ideals in KU-algebra and investigated some related properties.*

In this project, we understand intuitionstic fuzzy PMS-subalgebra of a PMS algebra and explain the idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra and establish the relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra.. Further we extend some properties on homomorphism and Cartesian product of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra.

This project is divided into 2 chapters in which chapter 1 is an introduction and preliminaries. In this chapter we recall the existing literature namely definitions, example and results concerning PMS algebra, fuzzy set, intuitionistic fuzzy set which is more relevant to the subsequent chapters. Further we recall the definitions, notions, examples and certain properties of homomorphism and Cartesian product. We gave several characterizations for fuzzy subalgebra of PMS algebra.

In Chapter 2 we introduced the notion of intuitionstic fuzzy PMS-subalgebra of a PMS-algebra. We established the relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra and we furnish certain examples of intuitionstic fuzzy PMS-Subalgebra of a PMS Algebra. Further more we prove some properties on homomorphism and cartesian product of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra.

1.2 Preliminaries

Definition 1.2.1. *[14] A nonempty set X with a constant 0 and a binary operation $'\star'$ is called PMS-algebra if it satisfies the following axioms:*

(i). $0 \star x = x$,

(ii). $(y \star x) \star (z \star x) = z \star y$, *for all* $x, y, z \in X$.

Example 1.2.2. *Let* $X = \{0, 1, 2\}$ *be the set with the following table.*

$\star$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Then $(X, \star, 0)$ *is a PMS – algebra.*

Remark 1.2.3. *In* X, *we define a binary relation* $\leq$ *by* $x \leq y$ *if and only if* $x \star y = 0$.

Definition 1.2.4. *[14] Let S be a nonempty subset of a PMS-algebra X, then S is called a PMS-sub algebra of X if $x \star y \in S$, for all $x, y \in S$.*

Example 1.2.5. *Let* $X = \{0, a, b, c\}$ *be the set with the following table.*

$\star$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	c	c	0	b
c	b	b	c	0

Then $(X, \star, 0)$ *is a PMS – algebra and* $S = \{0, a\}$ *is a PMS-subalgebra.*

Example 1.2.6. *Let Z be the set of all integers, and let $\star$ be a binary relation on Z defined by $x \star y = y - x$, for all $x, y \in Z$, where $'-'$ the usual subtraction of integers. Then $(Z, \star, 0)$ is a PMS-algebra since;*

1. $0 \star x = x - 0 = x$,
2. $(y \star x) \star (z \star x) = (z \star x) - (y \star x) = (x - z) - (x - y) = y - z = z \star y$.

Clearly, the set E of all even integers is a PMS-subalgebra of a PMS-algebra Z, since $x \star y = y - x \in E$, for all $x, y \in E$.

Proposition 1.2.7. *In any **PMS**-algebra* $(X, \star, 0)$ *the following properties hold for all* $x, y, z \in X$.

1. $x \star x = 0$,
2. $(y \star x) \star x = y$,
3. $x \star (y \star x) = y \star 0$,
4. $(y \star x) \star z = (z \star x) \star y$,
5. $(x \star y) \star 0 = y \star x = (0 \star y) \star (0 \star x)$.

Definition 1.2.8. *[16] Let* X *be a non-empty set. A fuzzy set* A *in* X *is defined as* $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ *where the functions* $\mu_A : X \to [0, 1]$ *define the degree of membership of the element* x *in* X *to set* A.

Example 1.2.9. *Consider a fuzzy set* A *representing "young people" in a universe of discourse* X *(ages of people from 0 to 100). The membership function might be defined as:*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \leq 20 \\ \dfrac{30 - x}{10}, & \text{if } 20 < x \leq 30 \\ 0, & \text{if } x > 30 \end{cases}$$

This function assigns a membership value of 1 to all people aged 20 or younger, and the membership gradually decreases for ages 21 to 30, becoming 0 for those over 30.

Definition 1.2.10. *[16] Let* A *and* B *be two fuzzy sets in the universe of discourse* X*, then the operations over fuzzy sets are defined as follows:*

1. $A \subseteq B \Rightarrow \mu_A(x) \leq \mu_B(x)$,
2. $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$,
3. $\mu_{A\cap B}(x) = min\{\mu_A(x), \mu_B(x)\}$,
4. $\mu_{A\cup B}(x) = max\{\mu_A(x), \mu_B(x)\}$,
5. $\mu_A^c(x) = 1 - \mu_A(x)$ *for all* $x \in X$.

Example 1.2.11. *Let A and B be fuzzy subsets of $X = \{a, b, c\}$ defind by:*

$$\mu_A(x) = \begin{cases} 0.4, & \text{if } x = a \\ 0.6, & \text{if } x = b \\ 0.3, & \text{if } x = c \end{cases} \quad \text{and} \quad \mu_B(x) = \begin{cases} 0.5, & \text{if } x = a \\ 0.3, & \text{if } x = b \\ 0.4, & \text{if } x = c \end{cases}.$$

Then $A \cap B$ and $A \cup B$ is given by;

$$\mu_{A\cap B}(x) = \begin{cases} 0.4, & \text{if } x = a \\ 0.3, & \text{if } x = b, c \end{cases} \quad \text{and} \quad \mu_{A\cup B}(x) = \begin{cases} 0.5, & \text{if } x = a \\ 0.6, & \text{if } x = b \\ 0.4, & \text{if } x = c \end{cases}.$$

Definition 1.2.12. *[16] Let $\{A_i\} i \in I$ be a collection of fuzzy subsets of X. Then*

1. $\cap_{i\in I}\mu_{A_i}(x) = \inf_{i\in I} \mu_{A_i}(x)$ *for all $x \in X$*

2. $\cup_{i\in I}\mu_{A_i}(x) = \sup_{i\in I} \mu_{A_i}(x)$ *for all $x \in X$*

Definition 1.2.13. *[4] Let A be a fuzzy set with membership function μ_A of a set X. For a fixed $t \in [0, 1]$, the set $U(\mu_A, t) = \{x \in X | \mu_A(x) \geq t\}$ is called an upper t-level subset (upper level subset, upper level cut) of A and the set $L(\mu_A, t) = \{x \in X | \mu_A(x) \leq t\}$ is called a lower t-level subset (lower level subset, lower level cut) of A.*
Note: If $t_1 \leq t_2$, then $U(\mu_A, t_2) \subseteq U(\mu_A, t_1)$ and $L(\mu_A, t_1) \subseteq L(\mu_A, t_2)$.

Definition 1.2.14. *[14] A fuzzy set A in a PMS-algebra X is called fuzzy PMS-subalgebra of X if $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$, for all $x, y \in X$.*

Example 1.2.15. *Let $X = \{0, a, b, c\}$ be a PMS - algebra with the following Caley table.*

$\star$	0	a	b	c
0	0	a	b	c
a	b	0	a	b
b	a	b	0	c
c	c	c	a	0

Define $\mu_A : X \to [0, 1]$ by $\mu_A(x) = \begin{cases} 0.8, & \text{if } x = 0 \\ 0.5, & \text{if } x = b \\ 0.4, & \text{if } x = a, c \end{cases}$. *Then A is a fuzzy PMS-*

subalgebra of X.

Definition 1.2.16. *[1] An intuitionistic fuzzy set (IFS) A in a nonempty set* X *is an object having the form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$*, where the functions* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *define the degree of membership and the degree of non membership, respectively, satisfying the condition* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$*, for all* $x \in X$*.*
Obviously, each ordinary fuzzy set may be written as $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$*. For the sake of simplicity we write* $A = (\mu_A, \nu_A)$ *for an intuitionistic fuzzy set* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$*.*

Definition 1.2.17. *[1] Let* A *and* B *be two intuitionistic fuzzy subsets of the set* X*, where* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$*, then,*

1. $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$*, for all* $x \in X$
2. $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$*, for all* $x \in X$
3. $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\}$
4. $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\}$
5. $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$
6. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$
7. $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$
8. $\cap_{i\in I} A_i = \left\{\left\langle x, \mu_{\cap_{i\in I} A_i}(x), \nu_{\cup_{i\in I} A_i}(x)\right\rangle \mid x \in X\right\}$*, where* $\cap_{i\in I}\mu_{A_i}(x) = \inf_{i\in I} \mu_{A_i}(x)$ *and* $\cup_{i\in I}\nu_{A_i}(x) = \sup_{i\in I} \nu_{A_i}(x)$
9. $\cup_{i\in I} A_i = \left\{\left\langle x, \mu_{\cup_{i\in I} A_i}(x), \nu_{\cap_{i\in I} A_i}(x)\right\rangle \mid x \in X\right\}$*, where* $\cup_{i\in I}\mu_{A_i}(x) = \sup_{i\in I} \mu_{A_i}(x)$ *and* $\cap_{i\in I}\nu_{A_i}(x) = \inf_{i\in I} \nu_{A_i}(x)$*.*

Definition 1.2.18. *[1] Let* $A = (\mu_A, \nu_A)$ *be an intuitionistic fuzzy set in a nonempty set* X*. For* $t, s \in [0,1]$*, the set* $U(\mu_A, t) = \{x \in X \mid \mu_A(x) \geq t\}$ *is called an upper* t*-level subset of* A *and the set* $L(\nu_A, s) = \{x \in X \mid \nu_A(x) \leq s\}$ *is called the lower* s*-level subset of* A*.*

Let $A = (\mu_A, \nu_A)$ *be an intuitionistic fuzzy set on* X *. For* $s, t \in [0,1]$ *with* $s + t \leq 1$*,*
(i). the set $A^{(t,s)} := \{x \in X \mid t \leq \mu_A(x), \nu_A(x) \leq s\}$ *is called an* (t,s)*-level subset of* A*. The set of all* $(t,s) \in Im(\mu_A) \times Im(\nu_A)$ *such that* $s + t \leq 1$ *is called the image of* $A = (\mu_A, \nu_A)$*.*
(ii). the set $A^{(t,s)} := \{x \in X \mid t < \mu_A(x), \nu_A(x) < s\}$ *is called a strong (t, s)-level subset of* A*.*

Note that:

$$\begin{aligned}A^{(t,s)} &= \{x \in X \mid \mu_A(x) \geq t, \nu_A(x) \leq s\}\\ &= \{x \in X \mid \mu_A(x) \geq t\} \cap \{x \in X \mid \nu_A(x) \leq s\}\\ &= U(\mu_A, t) \cap L(\nu_A, s).\end{aligned}$$

Definition 1.2.19. *[12] Let* X *and* Y *be nonempty sets and $f : X \to Y$ be a mapping. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy subsets of* X *and* Y *respectively. Then the image of A under f is defined as $f(A) = \{\langle y, \mu_{f(A)}(y), \nu_{f(A)}(y)\rangle \mid y \in Y\}$, where*

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu_A(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{f(A)}(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \nu_A(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}.$$

The inverse image of B under f is denoted by $f^{-1}(B)$ and is defined as:

$$f^{-1}(B)(x) = \{\langle x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)\rangle \mid x \in X\},$$

where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)})(x) = \nu_B(f(x))$, for all $x \in X$.

Definition 1.2.20. *[12] An intuitionistic fuzzy subset A in a nonempty set X with the degree of membership $\mu_A : X \to [0, 1]$ and the degree of non membership $\nu_A : X \to [0, 1]$ is said to have sup-inf property, if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that $\mu_A(x_0) = \sup\limits_{t\in T} \mu_A(t)$ and $\nu_A(x_0) = \inf\limits_{t\in T} \nu_A(t)$.*

Definition 1.2.21. *[13] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy subsets of X and Y respectively. Then the Cartesian product of A and B is defined as*

$$A \times B = \{\langle (x, y), \mu_{A\times B}(x, y), \nu_{A\times B}(x, y)\rangle \mid x \in X, y \in Y\},$$

where $\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ and $\nu_{A\times B}(x, y) = \max\{(\nu_A(x), \nu_B(y))\}$ such that $\mu_{A\times B} : X \times Y \to [0, 1]$ and $\nu_{A\times B} : X \times Y \to [0, 1]$, for all $x \in X$ and $y \in Y$.

Remark 1.2.22. *Let X and Y be PMS-algebras, for all $(x, y), (u, v) \in X \times Y$, we define '$\star$' on $X \times Y$ by $(x, y) \star (u, v) = (x \star u, y \star v)$. Clearly $(X \times Y; \star, (0, 0))$ is a PMS-algebra.*

Chapter 2

Intuitionistic fuzzy PMS-subalgebra of a PMS algebra

2.1 Intuitionistic Fuzzy PMS-subalgebras

In this section we introduce the notion of intuitionistic fuzzy PMS-subalgebra and investigated some of its properties.

Definition 2.1.1. *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra X is called an intuitionistic fuzzy PMS-subalgebra of X if:*

1. $\mu_A(x \star y) \geq \min\{\mu_A(x), \mu_A(y)\}$,
2. $\nu_A(x \star y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Example 2.1.2. *Consider $X = \{0, a, b, c\}$ such that $(X, \star, 0)$ is a PMS algebra with table below;*

$\star$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	c	c	0	b
c	b	b	c	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by;

$$\mu_A(x) = \begin{cases} 0.8, & if\ x = 0 \\ 0.5, & if\ x = a, b \\ 0.6, & if\ x = c \end{cases} \qquad and \qquad \nu_A(x) = \begin{cases} 0.2, & if\ x = 0 \\ 0.4, & if\ x = a, b \cdot \\ 0.3, & if\ x = c \end{cases}$$

For intuitionistic fuzzy set A in a PMS-algebra X with membership values $\mu_A(x)$ and non membership values $\nu_A(x)$ as defined above, Definition 2.1.1 is satisfied. Therefore, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS- subalgebra of the PMS-algebra X.

Lemma 2.1.3. *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X, then $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X. Since $x \star x = 0$ for every $x \in X$ by Proposition 1.2.7(1), we have:

$$\begin{aligned} \mu_A(0) &= \mu_A(x \star x) \\ &\geq min\{\mu_A(x), \mu_A(x)\} \\ &= \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} \nu_A(0) &= \nu_A(x \star x) \\ &\leq max\{\nu_A(x), \nu_A(x)\} \\ &= \nu_A(x). \end{aligned}$$

Hence, $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$. □

Lemma 2.1.4. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X, if $x \star y \leq z$, then $\mu_A(x) \geq min\{\mu_A(y), \mu_A(z)\}$ and $\nu_A(x) \leq max\{\nu_A(y), \nu_A(z)\}$.*

Proof. Suppose $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X. Let $x, y, z \in X$ such that $x \star y \leq z$. Then by the binary relation $\leq$ defined in X, we have $(x \star y) \star z = 0$.

Thus, by Definition 1.2.1 and Proposition 1.2.7 (4), we have;

$$\begin{aligned}\mu_A(x) &= \mu_A(0 \star x) \\ &= \mu_A(((x \star y) \star z) \star x) \\ &= \mu_A(((z \star y) \star x) \star x) \\ &= \mu_A((x \star x) \star (z \star y)) \\ &= \mu_A(0 \star (z \star y)) \\ &= \mu_A(z \star y) \geq min\{\mu_A(z), \mu_A(y)\}\end{aligned}$$

Hence, $\mu_A(x) \geq min\{\mu_A(z), \mu_A(y)\}$. Similarly, $\nu_A(x) \leq max\{\nu_A(z), \nu_A(y)\}$. □

Theorem 2.1.5. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X and let $x \in X$, then $\mu_A(x \star y) = \mu_A(y)$ and $\nu_A(x \star y) = \nu_A(y)$ for each $y \in X$ if and only if $\mu_A(x) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0)$, where 0 is a constant in X.*

Proof. Suppose $\mu_A(x \star y) = \mu_A(y)$ and $\nu_A(x \star y) = \nu_A(y)$ for each $y \in X$. Then we need to show that $\mu_A(x) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0)$, where 0 is a constant in X. By Lemma 2.1.3, $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for each $x \in X$. By Proposition 1.2.7 (2) $(x \star 0) \star 0 = x$. Then

$$\begin{aligned}\mu_A(x) &= \mu_A((x \star 0) \star 0) \\ &\geq min\{\mu_A(x \star 0), \mu_A(0)\} \\ &= min\{\mu_A(0), \mu_A(0)\} \\ &= \mu_A(0).\end{aligned}$$

Also,

$$\begin{aligned}\nu_A(x) &= \nu_A((x \star 0) \star 0) \\ &\leq max\{\nu_A(x \star 0), \nu_A(0)\} \\ &= max\{\nu_A(0), \nu_A(0)\} \\ &= \nu_A(0).\end{aligned}$$

Hence, $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$. Therefore, $\mu_A(x) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0)$.
Conversely, Suppose $\mu_A(x) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0)$.
Then we need to prove that $\mu_A(x \star y) = \mu_A(y)$ and $\nu_A(x \star y) = \nu_A(y)$ for each $y \in X$.

By Lemma 2.1.3, $\mu_A(x) \geq \mu_A(y)$ and $\nu_A(x) \leq \nu_A(y)$ for each $y \in X$. Since A is an intuitionistic fuzzy PMS-subalgebra of X.

Then, $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\} = \mu_A(y)$ and $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\} = \nu_A(y)$. Thus, $\mu_A(x \star y) \geq \mu_A(y)$ and $\nu_A(x \star y) \leq \nu_A(y)$ for each $y \in X$. But, using Proposition 1.2.7 (2) and 1.2.7 (5) it follws that

$$\begin{aligned}
\mu_A(y) &= \mu_A((y \star x) \star x) \\
&\geq min\{\mu_A(y \star x), \mu_A(x)\} \\
&= min\{\mu_A((x \star y) \star 0), \mu_A(x)\} \\
&\geq min\{min\{\mu_A(x \star y), \mu_A(0)\}, \mu_A(x)\} \\
&= min\{\mu_A(x \star y), \mu_A(x)\} = \mu_A(x \star y)
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(y) &= \nu_A((y \star x) \star x) \\
&\leq max\{\nu_A(y \star x), \nu_A(x)\} \\
&= max\{\nu_A((x \star y) \star 0), \nu_A(x)\} \\
&\leq max\{max\{\nu_A(x \star y), \nu_A(0)\}, \nu_A(x)\} \\
&= max\{\nu_A(x \star y), \nu_A(x)\}
\end{aligned}$$

$= \nu_A(x \star y)$. Hence, $\mu_A(x \star y) = \mu_A(y)$ and $\nu_A(x \star y) = \nu_A(y)$ for each $y \in X$. □

Theorem 2.1.6. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X. If $\mu_A(x \star y) = \mu_A(0)$ and $\nu_A(x \star y) = \nu_A(0)$ for all $x, y \in X$, then $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.*

Proof. Let $x, y \in X$ such that $\mu_A(x \star y) = \mu_A(0)$ and $\nu_A(x \star y) = \nu_A(0)$.

Claim: $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$. Now;

$$\begin{aligned}
\mu_A(x) &= \mu_A((y \star y) \star x) \\
&= \mu_A((x \star y) \star y) \\
&\geq min\{\mu_A(x \star y), \mu_A(y)\} \\
&= min\{\mu_A(0), \mu_A(y)\} \\
&= \mu_A(y)
\end{aligned}$$

Conversely,

$$\begin{aligned}
\mu_A(y) &= \mu_A((x \star x) \star y) \\
&= \mu_A((y \star x) \star x) \\
&\geq min\{\mu_A(y \star x), \mu_A(x)\} \\
&= min\{\mu_A((x \star y) \star 0), \mu_A(x)\} \\
&\geq min\{min\{\mu_A(x \star y), \mu_A(0)\}, \mu_A(x)\} \\
&= min\{\mu_A(0), \mu_A(x)\} \\
&= \mu_A(x)
\end{aligned}$$

Thus, $\mu_A(x) = \mu_A(y)$. By similar argument we have $\nu_A(x) = \nu_A(y)$. □

Theorem 2.1.7. *The intersection of any two intuitionistic fuzzy PMS-subalgebras of X is also an intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra X.

Claim: $A \cap B$ is an intuitionistic fuzzy PMS-subalgebra of X.Then for $x, y \in X$, we have

$$\begin{aligned}
\mu_{(A\cap B)}(x \star y) &= min\{\mu_A(x \star y), \mu_B(x \star y)\} \\
&\geq min\{min\{\mu_A(x), \mu_A(y)\}, min\{\mu_B(x), \mu_B(y)\}\} \\
&= min\{min\{\mu_A(x), \mu_B(x)\}, min\{\mu_A(y), \mu_B(y)\}\} \\
&= min\{\mu_{(A\cap B)}(x), \mu_{(A\cap B)}(y)\}.
\end{aligned}$$

and

$$\begin{aligned}
\nu_{(A\cap B)}(x \star y) &= max\{\nu_A(x \star y), \nu_B(x \star y)\} \\
&\leq max\{max\{\nu_A(x), \nu_A(y)\}, max\{\nu_B(x), \nu_B(y)\}\} \\
&= max\{max\{\nu_A(x), \nu_B(x)\}, max\{\nu_A(y), \nu_B(y)\}\} \\
&= max\{\nu_{(A\cap B)}(x), \nu_{(A\cap B)}(y)\}.
\end{aligned}$$

Hence, $A \cap B$ is an intuitionistic fuzzy PMS-subalgebra of X. □

Corollary 2.1.8. *If $\{A_i \mid i \in I\}$ be a family of intuitionistic fuzzy PMS-subalgebra of X. Then $\cap_{i\in I} A_i$ is also an intuitionistic fuzzy PMS-subalgebra of X, where $\cap_{i\in I}\mu_{A_i}(x) = \inf_{i\in I} \mu_{A_i}(x)$ and $\cap_{i\in I}\nu_{A_i}(x) = \sup_{i\in I} \nu_{A_i}(x)$.*

Remark 2.1.9. *The union of any two intutionistic fuzzy PMS-subalgebras of a PMS-algebra X is not necassarily an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X.*

Example 2.1.10. *Let $X = \{0, 1, 2, 3\}$ be a set with the table below is a PMS-algebra.*

$\star$	0	1	2	3
0	0	1	2	3
1	2	0	1	2
2	1	2	0	1
3	3	1	2	0

and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x = 1, 2 \\ 0, & \text{if } x = 3 \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.4, & \text{if } x = 1, 2 \\ 1, & \text{if } x = 3 \end{cases}.$$

and let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy set in X defined by

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.6, & \text{if } x = 1, 3 \\ 0, & \text{if } x = 2 \end{cases} \quad \text{and} \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.2, & \text{if } x = 1, 3 \\ 1, & \text{if } x = 2 \end{cases}.$$

Now,

$$\begin{aligned} \mu_{(A \cup B)}(1 \star 0) &= \mu_{(A \cup B)}(2) \\ &= max\{\mu_A(2), \mu_B(2)\} \\ &= max\{0.5, 0\} \\ &= 0.5........(1), \end{aligned}$$

$$\begin{aligned}\mu_{(A\cup B)}(1\star 0) &= min\{\mu_{(A\cup B)}(1), \mu_{(A\cup B)}(0)\}\\ &= min\{max\{\mu_A(1), \mu_B(1)\}, max\{\mu_A(0), \mu_B(0)\}\}\\ &= min\{max\{0.5, 0.6\}, max\{1, 1\}\}\\ &= min\{0.6, 1\}\\ &= 0.6.......(2),\end{aligned}$$

$$\begin{aligned}\nu_{(A\cup B)}(1\star 0) &= \nu_{(A\cup B)}(2)\\ &= min\{\mu_A(2), \mu_B(2)\}\\ &= min\{0.4, 1\}\\ &= 0.4........(3),\end{aligned}$$

$$\begin{aligned}\nu_{(A\cup B)}(1\star 0) &= max\{\nu_{(A\cup B)}(1), \nu_{(A\cup B)}(0)\}\\ &= max\{min\{\nu_A(1), \nu_B(1)\}, min\{\mu_A(0), \nu_B(0)\}\}\\ &= max\{min\{0.4, 0.2\}, min\{0, 0\}\}\\ &= max\{0.2, 0\}\\ &= 0.2.......(4).\end{aligned}$$

From (1) and (2) we see that $\mu_{(A\cup B)}(1\star 0) = 0.5 < 0.6 = min\{\mu_{(A\cup B)}(1), \mu_{(A\cup B)}(0)\}$ *and from (3) and (4) we see that* $\nu_{(A\cup B)}(1\star 0) = 0.4 > 0.2 = max\{\nu_{(A\cup B)}(1), \nu_{(A\cup B)}(0)\}$ *which is a contradiction. This shows that the union of any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra X may not be an intutionistic fuzzy PMS-subalgebra.*

Lemma 2.1.11. *Let* $A = (\mu_A, \nu_A)$ *be an intutionistic fuzzy set in* X*. Then the following statements hold for any* $x, y \in X$*.*

1. $1 - max\{\mu_A(x), \mu_A(y)\} = min\{1 - \mu_A(x), 1 - \mu_A(y)\}$,
2. $1 - min\{\mu_A(x), \mu_A(y)\} = max\{1 - \mu_A(x), 1 - \mu_A(y)\}$,
3. $1 - max\{\nu_A(x), \nu_A(y)\} = min\{1 - \nu_A(x), 1 - \nu_A(y)\}$,
4. $1 - min\{\nu_A(x), \nu_A(y)\} = max\{1 - \nu_A(x), 1 - \nu_A(y)\}$.

Theorem 2.1.12. *An intuitionistic fuzzy subset* $A = (\mu_A, \nu_A)$ *of a PMS-algebra X is an intuitionistic fuzzy PMS-subalgebra of X if and only if the fuzzy subsets* μ_A *and* ν_A^- *are fuzzy subalgebras of X.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X.
Claim: The fuzzy subsets μ_A and ν_A^- are fuzzy subalgebras of X. Clearly, μ_A is a fuzzy PMS-subalgebra of X directly follows from the fact that $A = (\mu_A, \nu_A)$ an intuitionistic fuzzy PMS-subalgebra of X. Now for all $x, y \in X$,

$$\begin{aligned}\nu_A^-(x \star y) &= 1 - \nu_A(x \star y)\\ &\geq 1 - max\{\nu_A(x), \nu_A(y)\}\\ &= min\{1 - \nu_A(x), 1 - \nu_A(y)\} \text{ (by Lemma 2.1.11(3))}\\ &= min\{\nu_A^-(x), \nu_A^-(y)\}.\end{aligned}$$

Therefore, ν_A^- is a fuzzy PMS -subalgebra of X.
Conversely, Suppose μ_A and ν_A^- are fuzzy PMS-subalgebras of X. So, we need to show that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X. Since μ_A and ν_A^- are fuzzy PMS-subalgebras of X, we have that $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A^-(x \star y) \geq min\{\nu_A^-(x), \nu_A^-(y)\}$, for all $x, y \in X$. Now it suffices to show that $\nu(x \star y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.

$$\begin{aligned}1 - \nu_A(x \star y) &= \nu_A^-(x \star y)\\ &\geq min\{\nu_A^-(x), \nu_A^-(y)\}\\ &= min\{1 - \nu_A(x), 1 - \nu_A(y)\}\\ &= 1 - max\{\nu_A(x), \nu_A(y)\}. \text{ (by Lemma 2.1.11(3))}\end{aligned}$$

$\Rightarrow \nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\}$, for all $x, y \in X$.
Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X. □

Corollary 2.1.13. *If μ_A is a fuzzy PMS-subalgebra of X, then $A = (\mu_A, \mu_A^-)$ is an intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Suppose μ_A is a fuzzy PMS-subalgebra of X.
Then we want to show that $A = (\mu_A, \mu_A^-)$ is an intuitionistic fuzzy PMS-subalgebra of X. Since μ_A is a fuzzy PMS-subalgebra of X, it follows that $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$.

Then it suffices to show that $\mu_A^-(x \star y) \leq max\{\mu_A^-(x), \mu_A^-(y)\}$.

$$\begin{aligned}
\mu_A^-(x \star y) &= 1 - \mu_A(x \star y) \\
&\leq 1 - min\{\mu_A(x), \mu_A(y)\} \\
&= max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\
&= max\{\mu_A^-(x), \mu_A^-)(y)\}
\end{aligned}$$

$\Rightarrow \mu_A^-(x \star y) \leq max\{\mu_A^-(x), \mu_A^-(y)\}$.
Hence, $A = (\mu_A, \mu_A^-)$ is an intuitionistic fuzzy PMS-subalgebra of X. $\square$

Corollary 2.1.14. *If ν_A^- is a fuzzy PMS-subalgebra of X, then $A = (\nu_A^-, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X.*

Theorem 2.1.15. *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra X is an intuitionistic fuzzy PMS-subalgebra of X if and only if $\square A = (\mu_A, \mu_A^-)$ and $\lozenge A = (\nu_A^-, \nu_A)$ are intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Assume that an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X is an intuitionistic fuzzy PMS-subalgebra of X, then $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\}$. Claim: $\square A = (\mu_A, \mu_A^-)$ and $\lozenge A = (\nu_A^-, \nu_A)$ are intuitionistic fuzzy PMS-subalgebras of X.
(i) To show that $\square A$ is an intuitionistic fuzzy PMS-subalgebra of X, it suffices to show that $\mu_A^-(x \star y) \leq max\{\mu_A^-(x), \mu_A^-(y)\}$,for all $x, y \in X$. Let $x, y \in X$, then:

$$\begin{aligned}
\mu_A^-(x \star y) &= 1 - \mu_A(x \star y) \\
&\leq 1 - min\{\mu_A(x), \mu_A(y)\} \\
&= max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\
&= max\{\mu_A^-(x), \mu_A^-(y)\}
\end{aligned}$$

$\Rightarrow \mu_A^-(x \star y) \leq max\{\mu_A^-(x), \mu_A^-(y)\}$, $\forall x, y \in X$.
Hence, $\square A$ is an intuitionistic fuzzy PMS-subalgebra of X.
(ii) to show that $\lozenge A$ is an intuitionistic fuzzy PMS-subalgebra of X, it suffices to show that
$\nu_A^-(x \star y) \geq min\{\nu_A^-(x), \nu_A^-(y)\}$, for all $x, y \in X$.

Let $x, y \in X$, then;

$$\begin{aligned}\nu_A^-(x \star y) &= 1 - \nu_A(x \star y) \\ &\geq 1 - max\{\nu_A^-(x), \nu_A^-(y)\} \\ &= min\{1 - \nu_A(x), 1 - \nu_A(y)\} \\ &= min\{\nu_A^-(x), \mu_A^-(y)\}\end{aligned}$$

$\Rightarrow \nu_A^-(x \star y) \geq min\{\nu_A^-(x), \nu_A^-(y)\}, \forall x, y \in X$.

Hence, $\Diamond A$ is an intuitionistic fuzzy PMS-subalgebra of X.

The proof of the converse of this theorem is trivial. □

2.2 Level Subsets of IF PMS-subalgebras

In this section, the idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. Characterizations of level subsets of a fuzzy PMS-subalgebra of a PMS-algebra are given.

Theorem 2.2.1. *If $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X, then the sets $X_{\mu_A} = \{x \in X \mid \mu_A(x) = \mu_A(0)\}$ and $X_{\nu_A} = \{x \in X \mid \nu_A(x) = \nu_A(0)\}$ are PMS -subalgebra of X.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X and let $x, y \in X_{\mu_A}$. Then $\mu_A(x) = \mu_A(0) = \mu_A(y)$. So,

$$\begin{aligned}\mu_A(x \star y) &\geq min\{\mu_A(x), \mu_A(y)\} \\ &= min\{\mu_A(0), \mu_A(0)\} \\ &= \mu_A(0).\end{aligned}$$

$$\Rightarrow \mu_A(x \star y) \geq \mu_A(0).$$

By Lemma 2.1.3, we get that $\mu_A(x \star y) = \mu_A(0)$ which imply that $x \star y \in X_{\mu_A}$

Also, Let $x, y \in X_{\nu_A}$. Then $\nu_A(x) = \nu_A(0) = \nu_A(y)$ and so,

$$\begin{aligned}\nu_A(x \star y) &\leq max\{\nu_A(x), \nu_A(y)\} \\ &= max\{\nu_A(0), \nu_A(0)\} \\ &= \nu_A(0).\end{aligned}$$

$$\Rightarrow \nu_A(x\star y) \leq \nu_A(0).$$

By Lemma 2.1.3, we get that $\nu_A(x\star y) = \nu_A(0)$ which imply that $x\star y \in X_{\nu_A}$.

Hence, the sets X_{μ_A} and X_{ν_A} are PMS-subalgebras of X. □

Theorem 2.2.2. *Let S be a nonempty subset of a PMS-algebra X and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by:*

$$\mu_A(x) = \begin{cases} p, & \text{if } x \in S \\ q, & \text{if } x \notin S \end{cases} \qquad \text{and} \qquad \nu_A(x) = \begin{cases} r, & \text{if } x \in S \\ s, & \text{if } x \notin S \end{cases}$$

for all $p, q, r, s \in [0,1]$ with $p \geq q, r \leq s$ and $0 \leq p+r \leq 1, 0 \leq q+s \leq 1$.

Then A is an intuitionistic fuzzy PMS-subalgebra of X if and only if S is a PMS-subalgebra of X. Furthermore, in this situation, $X_{\mu_A} = S = X_{\nu_A}$.

Proof. Let A be an intuitionistic fuzzy PMS-subalgebra of X. Then we want to show that S is a PMS-subalgebra of X.

Let $x, y \in X$ such that $x, y \in S$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X, we have: $\mu_A(x\star y) \geq min\{\mu_A(x), \mu_A(y)\} = min\{p,p\} = p$ and $\nu_A(x\star y) \leq max\{\nu_A(x), \nu_A(y)\} = max\{r,r\} = r$.

Hence $x\star y \in S$. So, S is a PMS-subalgebra of X.

Conversely, suppose that S is a PMS-subalgebra of X. We claim to show that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X.

Let $x, y \in X$. Now consider the following cases:

Case (i). If $x, y \in S$ then $x\star y \in S$ since S is a PMS-subalgebra of X.

Thus, $\mu_A(x\star y) = p = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x\star y) = r = max\{\nu_A(x), \nu_A(y)\}$.

Case (ii). If $x \in S, y \notin S$, then $\mu_A(x) = p, \mu_A(y) = q$ and $\nu_A(x) = r, \nu_A(y) = s$. Thus, $\mu_A(x\star y) \geq q = min\{p,q\} = min\{\mu_A(x), \mu_A(y)\} \Rightarrow \mu_A(x\star y) \geq min\{\mu_A(x), \mu_A(y)\}$.

and $\nu_A(x\star y) \leq s = max\{r,s\} = max\{\nu_A(x), \nu_A(y)\}$ implies $\nu_A(x\star y) \leq max\{\nu_A(x), \mu_A(y)\}$.

Case (iii). If $x \notin S, y \in S$, then interchanging the roles of x and y in Case (ii), yields similar results $\mu_A(x\star y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x\star y) \geq max\{\nu_A(x), \mu_A(y)\}$.

Case (iv). If $x, y \notin S$, then $\mu_A(x) = q = \mu_A(y)$ and $\nu_A(x) = s = \nu_A(y)$, this implies that $\mu_A(x\star y) \geq q = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x\star y) \leq s = max\{\nu_A(x), \nu_A(y)\}$.

Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X. Furthermore, we have

$X_{\mu_A} = \{x \in X \mid \mu_A(x) = \mu_A(0)\} = \{x \in X \mid \mu_A(x) = p\} = S$ and $X_{\nu_A} = \{x \in X \mid \nu_A(x) = \nu_A(0)\} = \{x \in X \mid \nu_A(x) = r\} = S$. Hence, $X_{\mu_A} = S = X_{\nu_A}$. $\square$

Theorem 2.2.3. *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra X is an intuitionistic fuzzy PMS-subalgebra of X if and only if the the nonempty level subsets $U(\mu_A, t)$ and $L(\nu_A, s)$ of A are PMS-subalgebras of X for all $t, s \in [0, 1]$ with $0 \leq t+s \leq 1$.*

Proof. Assume that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X such that $U(\mu_A, t) \neq \phi$ and $L(\nu_A, s) \neq \phi$.
Now we claim that $U(\mu_A, t)$ and $L(\nu_A, t)$ are PMS-subalgebras of X for all $t, s \in [0, 1]$ with $0 \leq t + s \leq 1$. Let $x, y \in U(\mu_A, t)$, then we have $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$. Thus, $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\} \geq min\{t, t\} = t$.

$\Rightarrow x \star y \in U(\mu_A, t)$.

Hence, $U(\mu_A, t)$ is a PMS-subalgebra of X.
Also, let $x, y \in L(\nu_A, s)$, then $\nu_A(x) \leq s$ and $\nu_A(y) \leq s$.
So, $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\} \leq max\{s, s\} = s$.

$\Rightarrow x \star y \in L(\nu_A, s)$.

Hence, $L(\nu_A, s)$ is a PMS-subalgebra of X.
Conversely, Suppose that $U(\mu_A, t)$ and $L(\nu_A, s)$ are PMS-subalgebra of X for all $t, s \in [0, 1]$ with $0 \leq t + s \leq 1$.
Claim: A is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X. Let $x, y \in X$ such that $\mu_A(x) = t_1$ and $\mu_A(y) = t_2$ for $t_1, t_1 \in [0, 1]$.
Then $x \in U(\mu_A, t_1)$ and $y \in U(\mu_A, t_2)$.
Choose $t = min\{t_1, t_2\}$, then $t \leq t_1$ and $t \leq t_2$.

$\Rightarrow U(\mu_A, t_1) \subseteq U(\mu_A, t)$ and $U(\mu_A, t_2) \subseteq U(\mu_A, t)$.

$\Rightarrow x, y \in U(\mu_A, t)$.

Since, $U(\mu_A, t)$ is a PMS-Subalgebra of X, it follows that $x \star y \in U(\mu_A, t)$.
Thus, $\mu_A(x \star y) \geq t = min\{t_1, t_2\} = min\{\mu_A(x), \mu_A(y)\}$.
Hence, $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.
And also, let $x, y \in X$ such that $\nu_A(x) = s_1$ and $\nu_A(y) = s_2$ for $s_1, s_2 \in [0, 1]$.
Then, $x \in L(\nu_A, s_1)$ and $y \in L(\nu_A, s_1)$.
Choose $s = max\{s_1, s_2\}$, then $s_1 \leq s$ and $s_2 \leq s$.

$\Rightarrow L(\nu_A, s_1) \subseteq L(\nu_A, s)$ and $L(\nu_A, s_2) \subseteq L(\nu_A, s)$.

$\Rightarrow x, y \in L(\nu_A, s)$,

Since, $L(\nu_A, s)$ is a PMS-subalgebra of X, it follows that $x \star y \in L(\nu_A, s)$.
Thus, $\nu_A(x \star y) \leq s = max\{s_1, s_2\} = max\{\nu_A(x), \nu_A(y)\}$.
Hence, $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.
Hence, A is an intuitionistic fuzzy PMS-subalgebra of a PMS -algebra X. □

Remark 2.2.4. *The PMS-subalgebras $U(\mu_A, t)$ and $L(\nu, s)$ of X for all $t, s \in [0, 1]$ obtained in the above theorem are called level PMS-subalgebras of X.*

Corollary 2.2.5. *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra X is an intuitionistic fuzzy PMS-subalgebra of X if and only if the level subsets $U(\mu_A, t)$ and $L(\nu_A, s)$ of A are PMS-subalgebras of X for all $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $0 \leq t + s \leq 1$.*

Theorem 2.2.6. *Let S be a subset of X and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by:*

$$\mu_A(x) = \begin{cases} t, & \text{if } x \in S \\ 0, & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s, & \text{if } x \in S \\ 1, & \text{if } x \notin S \end{cases}$$

for all $t, s \in [0, 1]$ such that $0 \leq t + s \leq 1$. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X, then S is a level PMS-subalgebra of X.

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X. Then we need to show that S is a level PMS-subalgebra of X. Let $x, y \in S$, then $\mu_A(x) = t = \mu_A(y)$ and $\nu_A(x) = s = \nu_A(y)$. So, $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\} = min\{t, t\} = t$ and $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\} = max\{s, s\} = s$. Which implies that $x \star y \in S$.
Hence, S is a PMS-subalgebra of X. Also, by Theorem 2.2.3, $U(\mu_A, t)$ is a level subalgebra of X, and $U(\mu_A, t) = \{x \in X : \mu_A(x) \geq t\} = S = \{x \in X : \nu_A(x) \leq s\}$. Thus, S is a level PMS-Subalgebra of X corresponding to the intuitionistic fuzzy PMS-subalgebra $A = (\mu_A, \nu_A)$ of X. □

Theorem 2.2.7. *If S is any PMS-subalgebra of X, then there exists an intuitionistic fuzzy PMS-subalgebra A of X, in which S satisfies both the upper level and lower level PMS-subalgebra of A in X.*

Proof. Let S be a PMS-subalgebra of a PMS-algebra X and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by:

$$\mu_A(x) = \begin{cases} t, & \text{if } x \in S \\ 0, & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s, & \text{if } x \in S \\ 1, & \text{if } x \notin S \end{cases}$$

for all $t, s \in [0, 1]$ such that $0 \leq t + s \leq 1$.

Clearly, $U(\mu_A, t) = \{x \in X : \mu_A(x) \geq t\} = S$. Let $x, y \in X$. To prove that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X, we consider the following cases:

Case(i). If $x, y \in S$, then $x \star y \in S$. Since S is a PMS-subalgebra of a PMS-algebra X.
$\mu_A(x) = \mu_A(y) = \mu_A(x \star y) = t$ and $\nu_A(x) = \nu_A(y) = \nu_A(x \star y) = s$.
Therefore, $\mu_A(x \star y) = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) = max\{\nu_A(x), \nu_A(y)\}$.

Case(ii). If $x \in S, y \notin S$, then we have $\mu_A(x) = t, \mu_A(y) = 0$ and $\nu_A(x) = s, \nu_A(y) = 1$.
Thus, $\mu_A(x \star y) \geq 0 = min\{t, 0\} = min\{\mu_A(x), \mu_A(y)\}$ which implies that $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) \leq 1 = max\{s, 1\} = max\{\nu_A(x), \nu_A(y)\}$ implies $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\}$.

Case(iii). If $x \notin S, y \in S$, then interchanging the roles of x and y in Case (ii), yields similar results $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\}$.

Case(iv). If $x, y \notin S$ then $\mu_A(x) = 0 = \mu_A(y)$ and $\nu_A(x) = 1 = \nu_A(y)$. Then
$\mu_A(x \star y) \geq 0 = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) \leq 1 = max\{\nu_A(x), \nu_A(y)\}$ So, in all cases we get $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\}$,for all $x, y \in X$.

Thus, A is an intuitionistic fuzzy PMS-subalgebra of X. □

Theorem 2.2.8. *Let $\{S_i\}$ be any family of a PMS-subalgebra of a PMS-algebra X such that $S_0 \subset S_1 \subset S_2 \subset ... \subset S_n = X$, then there exists an intuitionistic fuzzy PMS-subalgebra $A = (\mu_A, \nu_A)$ of X whose level PMS-subalgebras are exactly the PMS-subalgebras $\{S_i\}$.*

Proof. Suppose $t_0 > t_1 > t_2 > ... > t_n$ and $s_0 < s_1 < s_2... < s_n$ where each $t_i,\ s_i \in [0, 1]$ with $0 \leq t_i + s_i \leq 1$. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set defined by:

$$\mu_A(x) = \begin{cases} t_0, & \text{if } x \in S_0 \\ t_i, & \text{if } x \in S_i - S_{i-1}, 0 < i \leq n \end{cases} \quad \text{and } \nu_A(x) = \begin{cases} s_0, & \text{if } x \in S_0 \\ s_i, & \text{if } x \in S_i - S_{i-1}, 0 < i \leq n \end{cases}.$$

Now, We claim that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X and $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 \leq i \leq n$.

Let $x, y \in X$ Then, we consider the following two cases.

Case (i): Let $x, y \in S_i - S_{i-1}$. Therefore by the definition of $A = (\mu_A, \nu_A)$, we have $\mu_A(x) = t_i = \mu_A(y)$ and $\nu_A(x) = s_i = \nu_A(y)$. Since S_i is a PMS-subalgebra of X, it follows that $x \star y \in S_i$, and so either $x \star y \in S_i - S_{i-1}$ or $x \star y \in S_{i-1}$ or $x \star y \in S_{i-1} - S_{i-2}$.

$\Rightarrow \mu_A(x) = t_i$ or $\mu_A(x) = t_{i-1} > t_i$ and $\nu_A(x) = s_i$ or $\nu_A(x) = s_{i-1} < s_i$.

In any case we conclude that $\mu_A(x \star y) \geq t_i = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) \leq s_i = max\{\nu_A(x), \nu_A(y)\}$.

Case (ii): For $i > j, t_j > t_i, s_j < s_i$ and $S_j \subset S_i$.

Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$ Then, $\mu_A(x) = t_i, \mu_A(y) = t_j > t_i, \nu_A(x) = s_i$ and $\nu_A(y) = s_j < s_i$.

Then $x \star y \in S_i$, since S_i is a PMS-subalgebra of X and $S_j \subset S_i$.

Hence, $\mu_A(x \star y) \geq t_i = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) \leq s_i = max\{\nu_A(x), \nu_A(y)\}$ by case (i).

Thus, $A = (\mu_A, \nu_A))$ is an intuitionistic fuzzy PMS-subalgebra of X.

Also, from the definition of $A = (\mu_A, \nu_A)$, it follows that $Im(\mu_A) = \{t_0, t_1, ..., t_n\}$ and $Im(\nu_A) = \{s_0, s_1, ..., s_n\}$. So, $U(\mu_A, t_i)$ and $L(\nu_A, s_i)$ are the level subalgebras of A for $0 \leq i \leq n$, and form the chains, $U(\mu_A, t_0) \subset U(\mu_A, t_1) \subset ... \subset U(\mu_A, t_n) = X$ and $L(\nu_A, s_0) \subset L(\nu_A, s_1) \subset ... \subset L(\nu_A, s_n) = X$, for $0 \leq i \leq n$.

Now, $U(\mu_A, t_0) = \{x \in X \mid \mu_A(x) \geq t_0\} = S_0 = \{x \in X \mid \nu_A(x) \leq s_0\} = L(\nu_A, s_0)$.

Finally, we prove that $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$.

Now let $x \in S_i$ then $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$. This implies $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$. Hence, $S_i \subseteq U(\mu_A, t_i)$ and $S_i \subseteq L(\nu_A, s_i)$. If $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$ then $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$ which implies that $x \notin S_j$ for $j > i$. For otherwise, if $x \in S_j$, then $\mu_A(x) \geq t_j$ and $\nu_A(x) \leq s_j$, which implies $t_i > \mu_A(x) \geq t_j$ and $s_i < \nu_A(x) \leq s_j$. This contradicts the assumption that $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$.

Hence, $\mu_A(x) \in \{t_0, t_1, ..., t_n\}$ and $\nu_A(x) \in \{s_0, s_1, ..., s_n\}$. So, $x \in S_k$ for some $k \leq i$. As $S_k \subseteq S_i$, it follows that $x \in S_i$. Hence, $U(\mu_A, t_i) \subseteq S_i$ and $L(\nu_A, s_i) \subseteq S_i$.

Therefore, $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$. □

Theorem 2.2.9. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X,then* *(i). The upper level PMS-subalgebras $U(\mu_A, t_1)$ and $U(\mu_A, t_2)$,(with $t_1 < t_2$) of an intuitionistic fuzzy PMS-subalgebra A are equal if and only if there is no $x \in X$ such that $t_1 \leq \mu_A(x) < t_2$.*

(ii). The lower level PMS-sub algebras $L(\nu_A, s_1)$ and $L(\nu_A, s_2)$,(with $s_1 > s_2$) of an intuitionistic fuzzy PMS-subalgebra A are equal if and only if there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X. Since the proofs for both (i) and (ii) are similar, here we prove for only (ii).

Suppose that $L(\nu_A, s_1) = L(\nu_A, s_2)$, for $s_1 > s_2$. Then we claim that there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$. Assume that there exists $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.

$\Rightarrow x \in L(\nu_A, s_1)$ but $x \notin L(\nu_A, s_2)$.

$\Rightarrow L(\nu_A, s_2)$ is a proper subset of $L(\nu_A, s_1)$.

This contradicts to the assumption that $L(\nu_A, s_1) = L(\nu_A, s_2)$.

Hence, there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.

Conversely, suppose that there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$. Then we prove that $L(\nu_A, s_1) = L(\nu_A, s_2)$.Since $s_1 > s_2$, we get $L(\nu_A, s_2) \subseteq L(\nu_A, s_1)$ .... (1)

Now, $x \in L(\nu_A, s_1) \Rightarrow \nu_A(x) \leq s_1$.

$\Rightarrow \nu_A(x) \leq s_2$, (Since $\nu_A(x)$ does not lie between s_1 and s_2).

$\Rightarrow x \in L(\nu_A, s_2)$.

Hence, $L(\nu_A, s_1) \subseteq L(\nu_A, s_2)$ .... (2)

From (1) and (2) we get $L(\nu_A, s_1) = L(\nu_A, s_2)$. □

Remark 2.2.10. *As the consequence of Theorem 2.2.9, the level subalgebras of an intuitionistic fuzzy PMS-algebra $A = (\mu_A, \nu_A)$ of a finite PMS-algebra X form a chain, $U(\mu_A, t_0) \subset U(\mu_A, t_1) \subset ... \subset U(\mu_A, t_n) = X$ and $L(\nu_A, s_0) \subset L(\nu_A, s_1) \subset ... \subset L(\nu_A, s_n) = X$, where $t_0 > t_1 > ... > t_n$ and $s_0 < s_1 < ... < s_n$.*

Corollary 2.2.11. *Let X be a finite PMS-algebra and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X.*

(i). If $Im(\mu_A) = \{t_1, ..., t_n\}$, then the family of PMS-subalgebras$\{U(\mu_A, t_i) \mid 1 \leq i \leq n\}$, constitutes all the upper level PMS-subalgebras of A in X.

(ii). If $Im(\nu_A) = \{s_1, ..., s_n\}$, then the family of PMS-subalgebras $\{L(\nu_A, s_i) \mid 1 \leq i \leq n\}$, constitutes all the lower level PMS-subalgebras of A in X.

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X such that $Im(\mu_A) = \{t_1, \ ..., \ t_n\}$ with $t_1 < t_2 < \ ... \ < t_n$ and $Im(\nu_A) = \{s_1, \ ..., \ s_n\}$ with $s_1 > s_2 > \ ... \ > s_n$.

(i). Let $t \in [0, 1]$ and $t \notin Im(\mu_A)$. Now, we can consider the following cases.

Case (1). If $t \leq t_1$, then $U(\mu_A, t_1) = X = U(\mu_A, t)$.

Case (2). If $t > t_n$, then $U(\mu_A, t) = \{x \in X \mid \mu_A(x) \geq t\} = \{x \in X \mid \mu_A(x) > t_n\} = \emptyset$

Case (3). If $t_{i-1} < t < t_i$, then $U(\mu_A, t) = U(\mu_A, t_i)$ by Theorem 2.2.9 (i), since there is no $x \in X$ such that $t \leq \mu_A(x) < t_i$.

Thus, for any $t \in [0, 1]$ the level PMS-subalgebra is one of $\{U(\mu_A, t_i) \mid i = 1, 2, ..., n\}$.

(ii). proof of (ii) is similar to (i). □

Corollary 2.2.12. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS–subalgebra of X with finite images.*

(i). If $U(\mu_A, t_i) = U(\mu_A, t_j)$ for any $t_i, t_j \in Im(\mu_A)$, then $t_i = t_j$.

(ii). If $L(\nu_A, s_i) = L(\nu_A, s_j)$ for any $s_i, s_j \in Im(\nu_A)$, then $s_i = s_j$.

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X with finite images. Here we only prove (ii).the prove of (i) can be done similarly.

Assume $L(\nu_A, s_i) = L(\nu_A, s_j)$ for $s_i, s_j \in Im(\nu_A)$.

So to show that $s_i = s_j$ assume on contrary,that is, $s_i \neq s_j$. Without loss of generality assume $s_i > s_j$. Let $x \in L(\nu_A, s_j)$, then $\nu_A(x) \leq s_j < s_i$.

$\Rightarrow \nu_A(x) < s_i$.

$\Rightarrow x \in L(\nu_A, s_i)$.

Let $x \in X$ such that $s_i > \nu_A(x) > s_j$. Then $x \in L(\nu_A, s_i)$ but $x \notin L(\nu_A, s_j)$

$\Rightarrow L(\nu_A, s_j) \subset L(\nu_A, s_i)$.

$\Rightarrow L(\nu_A, s_i) \neq L(\nu_A, s_j)$. which contradicts the hypothesis that $L(\nu_A, s_i) = L(\nu_A, s_j)$.

Therefore, $s_i = s_j$. □

2.3 Homomorphism on IF PMS-subalgebras of a PMS-algebra

Theorem 2.3.1. *Let $f : X \to Y$ be an epimorphism of PMS-algebras. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X with sup-inf property, then f(A) is an intuitionistic fuzzy PMS-subalgebra of Y.*

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of and let $a, b \in Y$ with $x_0 \in f^{-1}(a)$ and $y_0 \in f^{-1}(b)$ such that $\mu_A(x_0) = \sup\limits_{x \in f^{-1}(a)} \mu_A(x), \mu_A(y_0) = \sup\limits_{x \in f^{-1}(b)} \mu_A(x)$, and $\nu_A(x_0) = \inf\limits_{x \in f^{-1}(a)} \nu_A(x)$, $\nu_A(y_0) = \inf\limits_{x \in f^{-1}(b)} \nu_A(x)$, then by Definition 1.2.19 and 1.2.20 we have;

$$\begin{aligned}
\mu_{f(A)}(a \star b) &= \sup_{x \in f^{-1}(a \star b)} \mu_A(x) \\
&= \mu_A(x_0 \star y_0) \\
&\geq min\{\mu_A(x_0), \mu_A(y_0)\} \\
&= min\{\sup_{x \in f^{-1}(a)} \mu_A(x), \sup_{x \in f^{-1}(b)} \mu_A(x)\} \\
&= min\{\mu_{f(A)}(a), \mu_{f(A)}(b)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_{f(A)}(a \star b) &= \inf_{x \in f^{-1}(a \star b)} \nu_A(x) \\
&= \nu_A(x_0 \star y_0) \\
&\leq max\{\nu_A(x_0, \nu_A(y_0)\} \\
&= max\{\inf_{x \in f^{-1}(a)} \nu_A(x), \inf_{x \in f^{-1}(b)} \nu_A(x)\} \\
&= max\{\nu_{f(A)}(a), \nu_{f(A)}(b)\}.
\end{aligned}$$

Hence, f(A) is an intuitionistic fuzzy PMS-subalgebra of Y . □

Theorem 2.3.2. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. If $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of Y , then $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Assume that $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of Y and let

$x, y \in X$. Then,

$$\begin{aligned}
\mu_{f^{-1}(B)}(x \star y) &= \mu_B(f(x \star y)) \\
&= \mu_B(f(x) \star f(y)) \\
&\geq min\{\mu_B(f(x)), \mu_B(f(y))\} \\
&= min\{\mu_{f^{-1}(B)}(x), \mu_{f^{-1}(B)}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_{f^{-1}(B)}(x \star y) &= \nu_B(f(x \star y)) \\
&= \nu_B(f(x) \star f(y)) \\
&\leq max\{\nu_B(f(x)), \nu_B(f(y))\} \\
&= max\{\nu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(y)\}.
\end{aligned}$$

Therefore, $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of X. □

Theorem 2.3.3. *Let $f : X \to Y$ be an epimorphism of PMS-algebras and $B = (\mu_B, \nu_B)$ is a fuzzy set in Y . If $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of X, then $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of Y*

Proof. Assume that f is an epimorphism of PMS-algebras and $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of X. Let $y_1, y_2 \in Y$. Since f is an epimorphism of PMS-algebras, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now,

$$\begin{aligned}
\mu_B(y_1 \star y_2) &= \mu_B(f(x_1) \star f(x_2)) \\
&= \mu_B(f(x_1 \star x_2)) \\
&= \mu_{f^{-1}(B)}(x_1 \star x_2) \\
&\geq min\{\mu_{f^{-1}(B)}(x_1), \mu_{f^{-1}(B)}(x_2)\} \\
&= min\{\mu_B f(x_1), \mu_B f(x_2)\} \\
&= min\{\mu_B(y_1), \mu_B(y_2)\}
\end{aligned}$$

and

$$\begin{aligned}\nu_B(y_1 \star y_2) &= \nu_B(f(x_1) \star f(x_2)) \\ &= \nu_B(f(x_1 \star x_2)) \\ &= \nu_{f^{-1}(B)}(x_1 \star x_2) \\ &\leq max\{\nu_{f^{-1}(B)}(x_1), \nu_{f^{-1}(B)}(x_2)\} \\ &= max\{\nu_B f(x_1), \nu_B f(x_2)\} \\ &= max\{\nu_B(y_1), \nu_B(y_2)\}.\end{aligned}$$

Hence, $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-Subalgebra of Y . □

Definition 2.3.4. *Let $f : X \to Y$ be a homomorphism of PMS-algebras for any intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in Y. We define an intuitionistic fuzzy set $A^f = (\mu_A^f, \nu_A^f)$ in X by $\mu_A^f(x) = \mu_A(f(x))$ and $\nu_A^f(x) = \nu_A(f(x))$, $\forall x \in X$.*

Theorem 2.3.5. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. If the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of Y, then the intuitionistic fuzzy set $A^f = (\mu_A^f, \nu_A^f)$ in X is an intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Let f be a homomorphism of PMS-algebras and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of Y. Let $x, y \in X$. Then

$$\begin{aligned}\mu_A^f(x \star y) &= \mu_A(f(x \star y)) \\ &= \mu_A(f(x) \star f(y)) \\ &\geq min\{\mu_A(f(x)), \mu_A(f(y))\} \\ &= min\{\mu_A^f(x), \mu_A^f(y)\}\end{aligned}$$

and

$$\begin{aligned}\nu_A^f(x \star y) &= \nu_A(f(x \star y)) \\ &= \nu_A(f(x) \star f(y)) \\ &\leq max\{\nu_A(f(x)), \nu_A(f(y))\} \\ &= max\{\nu_A^f(x), \nu_A^f(y)\}.\end{aligned}$$

Hence, $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of X. □

Theorem 2.3.6. *Let $f : X \to Y$ be an epimorphism of PMS-algebra. If $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of X, then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of Y .*

Proof. Let $A^f = (\mu_A^f, \nu_A^f)$ be an intuitionistic fuzzy PMS-subalgebra in X and let $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Now we have,

$$
\begin{aligned}
\mu_A(x \star y) &= \mu_A(f(a) \star f(b)) \\
&= \mu_A(f(a \star b)) \\
&= \mu_A^f(a \star b) \\
&\geq min\{\mu_A^f(a), \mu_A^f(b)\} \\
&= min\{\mu_A(f(a)), \mu_A(f(b))\} \\
&= min\{\mu_A(x), \mu_A(y)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_A(x \star y) &= \nu_A(f(a) \star f(b)) \\
&= \nu_A(f(a \star b)) \\
&= \nu_A^f(a \star b) \\
&\leq max\{\nu_A^f(a), \nu_A^f(b)\} \\
&= max\{\nu_A(f(a)), \nu_A(f(b))\} \\
&= max\{\nu_A(x), \nu_A(y)\}.
\end{aligned}
$$

Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of Y . □

Theorem 2.3.7. *Let $f : X \to Y$ be an epimorphism of PMS-algebra. Then $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of X if and only if $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of Y .*

2.4 Cartesian Product of IF PMS-subalgebras

Lemma 2.4.1. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-subalgebras of X and Y respectively. Then $\mu_{A\times B}(0, 0) \geq \mu_{A\times B}(x, y)$ and $\nu_{A\times B}(0, 0) \leq \nu_{A\times B}(x, y), \forall (x, y) \in X \times Y$*

Proof. Let $(x, y) \in X \times Y$. Then,

$$\begin{aligned}\mu_{A\times B}(0,0) &= min\{\mu_A(0), \mu_B(0)\}\\ &\geq min\{\mu_A(x), \mu_B(y)\}\\ &= \mu_{A\times B}(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\times B}(0,0) &= max\{\nu_A(0), \nu_B(0)\}\\ &\leq max\{\nu_A(x), \nu_B(y)\}\\ &= \nu_{A\times B}(x,y).\end{aligned}$$

□

Theorem 2.4.2. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-subalgebras of X and Y respectively. Then $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then,

$$\begin{aligned}\mu_{A\times B}(x_1,y_1)\star(x_2,y_2)) &= \mu_{A\times B}(x_1\star x_2, y_1\star y_2)\\ &= min\{\mu_A(x_1\star x_2), \mu_B(y_1\star y_2)\}\\ &\geq min\{min\{\mu_A(x_1), \mu_A(x_2)\}, min\{\mu_B(y_1), \mu_B(y_2)\}\}\\ &= min\{min\{\mu_A(x_1), \mu_B(y_1)\}, min\{\mu_A(x_2), \mu_B(y_2)\}\}\\ &= min\{\mu_{A\times B}(x_1,y_1), \mu_{A\times B}(x_2,y_2)\}\end{aligned}$$

and

$$\begin{aligned}\nu_{A\times B}(x_1,y_1)\star(x_2,y_2)) &= \nu_{A\times B}(x_1\star x_2, y_1\star y_2)\\ &= max\{\nu_A(x_1\star x_2), \nu_B(y_1\star y_2)\}\\ &\geq max\{max\{\nu_A(x_1), \nu_A(x_2)\}, max\{\nu_B(y_1), \nu_B(y_2)\}\}\\ &= max\{max\{\nu_A(x_1), \nu_B(y_1)\}, max\{\nu_A(x_2), \nu_B(y_2)\}\}\\ &= max\{\mu_{A\times B}(x_1,y_1), \nu_{A\times B}(x_2,y_2)\}\end{aligned}$$

Hence, $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. □

Theorem 2.4.3. *Let A and B be intuitionistic fuzzy subsets of the PMS-algebras X and Y respectively. Suppose that 0 and $0'$ are the constant elements of X and Y respectively. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebras of $X \times Y$, then at least one of the following two statements holds.*

(i). $\mu_A(x) \leq \mu_B(0')$ and $\nu_A(x) \geq \nu_B(0')$,for all $x \in X$,

(ii). $\mu_B(y) \leq \mu_A(0)$ and $\nu_B(y) \geq \nu_A(0)$, for all $y \in Y$.

Proof. Let $A \times B$ be an intuitionistic fuzzy PMS-subalgebra of $X \times Y$.
Suppose that none of the statements (i) and (ii) holds. Then we can find $x \in X$ and $y \in Y$ such that $\mu_A(x) > \mu_B(0'), \nu_A(x) < \nu_B(0')$ and $\mu_B(y) > \mu_A(0), \nu_B(y) < \nu_A(0)$. Then we have;

$$\begin{aligned} \mu_{A\times B}(x,y) &= min\{\mu_A(x), \mu_B(y)\} \\ &> min\{\mu_B(0'), \mu_A(0)\} \\ &= \mu_{A\times B}(0,0') \end{aligned}$$

and

$$\begin{aligned} \nu_{A\times B}(x,y) &= max\{\nu_A(x), \nu_B(y)\} \\ &< max\{\nu_B(0'), \nu_A(0)\} \\ &= \nu_{A\times B}(0,0') \end{aligned}$$

which leads to $\mu_{A\times B}(x,y) > \mu_{A\times B}(0,0')$ and $\nu_{A\times B}(x,y) < \nu_{A\times B}(0,0')$. This contradicts Lemma 2.4.1. Hence, either (i) or (ii) holds □

Theorem 2.4.4. *Let A and B be intuitionistic fuzzy subsets of PMS-algebras X and Y respectively such that $\mu_A(x) \leq \mu_B(0')$ and $\nu_A(x) \geq \nu_B(0')$ for all $x \in X$, where $0'$ is a constant in Y. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then A is an intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Let $x, y \in X$. Then $(x, 0'), (y, 0') \in X \times Y$. Since $\mu_A(x) \leq \mu_B(0')$ and $\nu_A(x) \geq$

$\nu_B(0')$ for all $x \in X$, then for all $x, y \in X$ we get,

$$\begin{aligned}
\mu_A(x \star y) &= min\{\mu_A(x \star y), \mu_B(0' \star 0')\} \\
&= \mu_{A\times B}(x \star y, 0' \star 0') \\
&= \mu_{A\times B}((x, 0') \star (y, 0')) \\
&\geq min\{\mu_{A\times B}(x, 0'), \mu_{A\times B}(y, 0')\} \\
&= min\{min\{\mu_A(x), \mu_B(0')\}, min\{\mu_A(y), \mu_B(0')\}\} \\
&= min\{\mu_A(x), \mu_A(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x \star y) &= max\{\nu_A(x \star y), \nu_B(0' \star 0')\} \\
&= \nu_{A\times B}(x \star y, 0' \star 0')) \\
&= \nu_{A\times B}((x, 0') \star (y, 0') \\
&\leq max\{\nu_{A\times B}(x, 0'), \nu_{A\times B}(y, 0')\} \\
&= max\{max\{\nu_A(x), \nu_B(0')\}, max\{\nu_A(y), \nu_B(0')\}\} \\
&= max\{\nu_A(x), \nu_A(y)\}
\end{aligned}$$

Hence, $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \star y) \leq max\{\nu_A(x), \nu_A(y)\}$.
Therefore, A is an intuitionistic fuzzy PMS-subalgebra of X. □

Theorem 2.4.5. *Let A and B be intuitionistic fuzzy subsets of PMS-algebras X and Y respectively such that $\mu_B(y) \leq \mu_A(0)$ and $\nu_B(y) \geq \nu_A(0)$ for all $y \in Y$, where 0 is a constant in X. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then B is an intuitionistic fuzzy PMS-subalgebraof Y.*

Proof. Let $x, y \in Y$. Then $(0, x), (0, y) \in X \times Y$. Since $\mu_B(y) \leq \mu_A(0)$ and $\nu_B(y) \geq \nu_A(0)$ for all $y \in Y$, then for all $x, y \in Y$ we get,

$$
\begin{aligned}
\mu_B(x \star y) &= min\{\mu_A(0 \star 0), \mu_B(x \star y)\} \\
&= \mu_{A\times B}(0 \star 0, x \star y) \\
&= \mu_{A\times B}((0,x) \star (0,y)) \\
&\geq min\{\mu_{A\times B}(0,x), \mu_{A\times B}(0,y)\} \\
&= min\{min\{\mu_A(0), \mu_B(x)\}, min\{\mu_A(0), \mu_B(y)\}\} \\
&= min\{\mu_B(x), \mu_B(y)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_B(x \star y) &= max\{\nu_A(0 \star 0), \nu_B(x \star y)\} \\
&= \nu_{A\times B}(0 \star 0, x \star y) \\
&= \nu_{A\times B}((0,x) \star (0,y)) \\
&\leq max\{\nu_{A\times B}(0,x), \nu_{A\times B}(0,y)\} \\
&= max\{max\{\nu_A(0), \nu_B(x)\}, max\{\nu_A(0), \nu_B(y)\}\} \\
&= max\{\nu_B(x), \nu_B(y)\}
\end{aligned}
$$

Hence, $\mu_B(x \star y) \geq min\{\mu_B(x), \mu_B(y)\}$ and $\nu_B(x \star y) \leq max\{\nu_B(x), \nu_B(y)\}$.
Therefore, B is an intuitionistic fuzzy PMS-subalgebra of Y . □

From Theorems 2.4.3 , 2.4.4 and 2.4.5, we have the following:

Corollary 2.4.6. *Let A and B be intuitionistic fuzzy subsets of PMS-algebras X and Y respectively. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then either A is an intuitionistic fuzzy PMS-subalgebra of X or B is an intuitionistic fuzzy PMS-subalgebra of Y .*

Proof. Since $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$,

$$\mu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \geq min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\} \quad (1)$$

$$\nu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \leq max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}B(x_2, y_2)\} \quad (2)$$

If we put $x_1 = 0 = x_2$ in (1), we get; $\mu_{A\times B}((0, y_1)\star(0, y_2)) \geq min\{\mu_{A\times B}(0, y_1), \mu_{A\times B}(0, y_2)\}$

$$
\begin{aligned}
&\Rightarrow \mu_{A\times B}(0 \star 0, y_1 \star y_2) \geq min\{\mu_{A\times B}(0, y_1), \mu_{A\times B}(0, y_2)\} \\
&\Rightarrow \mu_{A\times B}(0, y_1 \star y_2) \geq min\{\mu_{A\times B}(0, y_1), \mu_{A\times B}(0, y_2)\} \\
&\Rightarrow min\{\mu_A(0), \mu_B(y_1 \star y_2)\} \geq min\{min\{\mu_A(0), \mu_B(y_1)\}, min\{\mu_A(0), \mu_B(y_2)\}\}
\end{aligned}
$$

Hence, $\mu_B(y_1 \star y_2) \geq min\{\mu_B(y_1), \mu_B(y_2)\}$.

Also,if we put $x_1 = 0 = x_2$ in (2), we get; $\nu_{A\times B}((0, y_1)\star(0, y_2)) \leq max\{\nu_{A\times B}(0, y_1), \nu_{A\times B}(0, y_2)\}$

$$\Rightarrow \nu_{A\times B}(0 \star 0, y_1 \star y_2) \leq max\{\nu_{A\times B}(0, y_1), \nu_{A\times B}(0, y_2)\}$$

$$\Rightarrow \nu_{A\times B}(0, y_1 \star y_2) \leq max\{\nu_{A\times B}(0, y_1), \nu_{A\times B}(0, y_2)\}$$

$$\Rightarrow max\{\nu_A(0), \nu_B(y_1 \star y_2)\} \leq max\{max\{\nu_A(0), \nu_B(y_1)\}, max\{\nu_A(0), \nu_B(y_2)\}\}.$$

Hence, $\nu_B(y_1 \star y_2) \leq max\{\nu_B(y_1), \nu_B(y_2)\}$ and B is an intuitionistic fuzz PMS-subalgebra of Y .

Similarly, we prove that A is an intuitionistic fuzzy PMS-subalgebra of X by putting $y_1 = 0 = y_2$ in (1) and (2). □

Theorem 2.4.7. *Let A and B be any intuitionistic fuzzy subsets of X and Y respectively. Then $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$ if and only if $\mu_{A\times B}$ and $\nu^-_{A\times B}$ are fuzzy PMS-subalgebra of $X \times Y$,where $\nu^-_{A\times B}$ is the complement of $\nu_{A\times B}$.*

Proof. Let $A \times B$ be an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. Then by Definition 2.1.1 $\mu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \geq min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}$ and $\nu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \leq max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\}, \forall (x_1, y_1), (x_2, y_2) \in X \times Y$. Hence, $\mu_{A\times B}$ is a fuzzy PMS-subalgebra of $X \times Y$ by Definition 1.2.14. Now for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

$$\begin{aligned}
\nu^-_{A\times B}((x_1, y_1) \star (x_2, y_2)) &= 1 - \nu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \\
&\geq 1 - max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\} \\
&= min\{1 - \nu_{A\times B}((x_1, y_1), 1 - \nu_{A\times B}((x_2, y_2)\} \\
&= min\{(\nu^-_{A\times B}(x_1, y_1), \nu^-_{A\times B}(x_2, y_2)\}
\end{aligned}$$

Hence, $\nu^-_{A\times B}((x_1, y_1) \star (x_2, y_2)) \geq min\{(\nu^-_{A\times B}(x_1, y_1), \nu^-_{A\times B}(x_2, y_2)\}$.

Thus, $\nu^-_{A\times B}$ is a fuzzy PMS subalgebra of $X \times Y$.

Conversely, assume $\mu_{A\times B}$ and $\nu^-_{A\times B}$ are fuzzy PMS-subalgebra of $X \times Y$.

Then we have that $\mu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \geq min\{(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}$ and $\nu^-_{A\times B}((x_1, y_1) \star (x_2, y_2)) \geq min\{(\nu^-_{A\times B}(x_1, y_1), \nu^-_{A\times B}(x_2, y_2)\}$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. So we need to show that $\nu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \leq max\{(\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\}$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Now,

$$
\begin{aligned}
1-\nu_{A\times B}((x_1,y_1)\star(x_2,y_2)) &= \nu^{-}_{A\times B}((x_1,y_1)\star(x_2,y_2))\\
&\geq min\{(\nu^{-}_{A\times B}(x_1,y_1),\nu^{-}_{A\times B}(x_2,y_2)\}\\
&= min\{1-\nu_{A\times B}((x_1,y_1),1-\nu_{A\times B}((x_2,y_2)\}\\
&= 1-max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}
\end{aligned}
$$

and so $\nu_{A\times B}((x_1,y_1)\star(x_2,y_2)) \leq max\{(\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}$.
Hence $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. $\square$

Theorem 2.4.8. *Let A and B be any intuitionistic fuzzy subsets of X and Y respectively, then $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$ if and only if $\Box(A\times B)$ and $\Diamond(A\times B)$ are intuitionistic fuzzy PMS-subalgebra of $X\times Y$.*

Proof. Suppose $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. Then $\mu_{A\times B}((x_1,y_1)\star(x_2,y_2)) \geq min\{(\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}$ and
$\nu_{A\times B}((x_1,y_1)\star(x_2,y_2)) \leq max\{(\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}$ for all $(x_1,y_1),(x_2,y_2)\in X\times Y$.
(i) To prove $\Box(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$,it suffices to show that for $(x_1,y_1),(x_2,y_2)\in X\times Y$, $\mu^{-}_{A\times B}((x_1,y_1)\star(x_2,y_2)) \leq min\{(\mu^{-}_{A\times B}(x_1,y_1),\mu^{-}_{A\times B}(x_2,y_2)\}$.
Now let $(x_1,y_1),(x_2,y_2)\in X\times Y$

$$
\begin{aligned}
\mu^{-}_{A\times B}((x_1,y_1)\star(x_2,y_2)) &= 1-\mu_{A\times B}((x_1,y_1)\star(x_2,y_2))\\
&\leq 1-min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\\
&= max\{1-\mu_{A\times B}((x_1,y_1),1-\mu_{A\times B}((x_2,y_2)\}\\
&= max\{(\mu^{-}_{A\times B}(x_1,y_1),\mu^{-}_{A\times B}(x_2,y_2)\}
\end{aligned}
$$

whence, $\mu^{-}_{A\times B}((x_1,y_1)\star(x_2,y_2)) \leq max\{(\mu^{-}_{A\times B}(x_1,y_1),\mu^{-}_{A\times B}(x_2,y_2)\}$ follows.
Hence, $\Box(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$.

(ii) To prove $\Diamond(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$,it suffices to show that $\nu^{-}_{A\times B}((x_1,y_1)\star(x_2,y_2)) \geq min\{(\nu^{-}_{A\times B}(x_1,y_1),\nu^{-}_{A\times B}(x_2,y_2)\}$. Now let

$(x_1, y_1), (x_2, y_2) \in X \times Y$, then

$$\begin{aligned}\nu^{-}_{A\times B}((x_1, y_1) \star (x_2, y_2)) &= 1 - \nu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \\ &\geq 1 - max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\} \\ &= min\{1 - \nu_{A\times B}((x_1, y_1), 1 - \nu_{A\times B}((x_2, y_2)\} \\ &= min\{(\nu^{-}_{A\times B}(x_1, y_1), \nu^{-}_{A\times B}(x_2, y_2)\},\end{aligned}$$

whence, $\nu^{-}_{A\times B}((x_1, y_1) \star (x_2, y_2)) \geq min\{(\nu^{-}_{A\times B}(x_1, y_1), \nu^{-}_{A\times B}(x_2, y_2)\}$ follows.
Hence, $\Diamond(A \times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$.
The proof of the converse is trivial. □

Definition 2.4.9. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy subset of PMS-algebras X and Y reapectively. For $t, s \in [0, 1]$ satisfying the condition $t + s \leq 1$, the set $U(\mu_{A\times B}, t) = \{(x, y) \in X \times Y \mid \mu_{A\times B}(x, y) \geq t\}$ is called upper t-level set of $A \times B$ and the set $L(\nu_{A\times B}, s) = \{(x, y) \in X \times Y \mid \nu_{A\times B}(x, y) \leq s\}$ is called lower s-level set of $A \times B$.*

Theorem 2.4.10. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy subsets of X and Y reapectively. Then $A \times B$ is an intuitionistic fuzzy PMS-subalgebras of $X \times Y$ if and only if the nonempty upper t-level set $U(\mu_{A\times B}, t)$ and the nonempty lower s-level set $L(\nu_{A\times B}, s)$ are PMS-subalgebras of $X \times Y$ for any $t, s \in [0, 1]$ with $t + s \leq 1$.*

Proof. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sub-sets of X and Y respectively. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $(x_1, y_1), (x_2, y_2) \in U(\mu_{A\times B}, t)$ for $t \in [0, 1]$. Then $\mu_{A\times B}(x_1, y_1) \geq t$ and $\mu_{A\times B}(x_2, y_2) \geq t$. Since $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, we have; $\mu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \geq min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\} \geq min\{t, t\} = t$.
Therefore, $(x_1, y_1) \star (x_2, y_2) \in U(\mu_{A\times B}, t)$. Hence, $U(\mu_{A\times B}, t)$ is a PMS-subalgebra of $X \times Y$.
Also, Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $(x_1, y_1), (x_2, y_2) \in L(\nu_{A\times B}, s)$ for $s \in [0, 1]$. Then $\nu_{A\times B}(x_1, y_1) \leq s$ and $\nu_{A\times B}(x_2, y_2) \leq s$. Since $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, we have $\nu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \leq max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\} \leq max\{s, s\} = s$.
Therefore, $(x_1, y_1) \star (x_2, y_2) \in L(\nu_{A\times B}, s)$. Hence, $L(\nu_{A\times B}, s)$ is a PMS-subalgebra of $X \times Y$.

Conversely, Suppose $U(\mu_{A\times B}, t)$ and $L(\nu_{A\times B}, s)$ are PMS-subalgebra of $X \times Y$ for any $t, s \in [0,1]$ with $t + s \leq 1$. Assume that $A \times B$ is not an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. Then there exist $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $\mu_{A\times B}((x_1, y_1) \star (x_2, y_2)) < min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}$.

Then by taking $t_0 = \dfrac{1}{2}\{\mu_{A\times B}((x_1, y_1) \star (x_2, y_2)) + min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}\}$ we get:

$$\mu_{A\times B}((x_1, y_1) \star (x_2, y_2)) < t_0 < min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}.$$

Hence, $(x_1, y_1)\star(x_2, y_2) \notin U(\mu_{A\times B}, t_0)$ but $(x_1, y_1) \in U(\mu_{A\times B}, t_0)$ and $(x_2, y_2) \in U(\mu_{A\times B}, t_0)$.

This implies $U(\mu_{A\times B}, t_0)$ is not a PMS-subalgebra of $X \times Y$, which is a contradiction.

Therefore, $\mu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \geq min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)$.

Similarly, $\nu_{A\times B}((x_1, y_1) \star (x_2, y_2)) \leq max\{\mu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\}$.

Hence, $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. $\square$

Conclusion

In this project, we introduced the notion of intuitionistic fuzzy PMS-subalgebras of PMS-algebras and some results are obtained. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between an intuitionistic fuzzy sets in a PMS-algebra and their level sets is discussed and some interesting results are obtained. We discussed the concept of intuitionistic fuzzy PMS-subalgebra under homomorphism and Cartesian product in a PMS-algebra. We confirmed that the homomorphic image and the homomorphic inverse image of an intuitionistic fuzzy PMS-subalgebra in a PMS-algebra are intuitionistic fuzzy PMS-subalgebras. We also proved that the Cartesian product of the intuitionistic fuzzy PMS-subalgebras of a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra. Furthermore, we characterized the Cartesian products of intuitionistic fuzzy PMS-subalgebras in terms of their level sets.

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