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# A Study on Well-Posedness and Persistence of Spatial Analyticity to the Solution of Higher Order KdV-BBM Type Equations

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**COLLEGE OF SCIENCE**
**DEPARTMENT OF MATHEMATICS**

# A STUDY ON WELL-POSEDNESS AND PERSISTENCE OF SPATIAL ANALYTICITY TO THE SOLUTION OF HIGHER ORDER KDV-BBM TYPE EQUATIONS

BY
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# A Study on Well-Posedness and Persistence of Spatial Analyticity to the Solution of Higher Order KdV-BBM Type Equations

**A Dissertation Submitted to College of Science, Department of Mathematics, Bahir Dar University, in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics**

By

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## Declaration of the Authorship

I declare that the dissertation entitled “A Study on Well-Posedness and Persistence of Spatial Analyticity to the Solution of Higher Order KdV-BBM Type Equations” has been carried out originality by me, under the guidance and supervision of Dr.Achenef Tesfahun, Department of Mathematics, Nazarbayev University, Kazakhstan and Dr.Birilew Belayneh, Department of Mathematics, Bahir Dar University, Ethiopia, for the award of the degree of Doctor of philosophy in mathematics and that this work has not be submitted either in whole or in part for any degree, diploma, or fellowship at any university.

I hereby confirm the originality of the work and that there is no plagiarism in any part of the dissertation.

| Emawayish Tegegn | ________ | ______ |
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**Bahir Dar University,**
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## Approval of Dissertation for Oral Defense

We hereby certify that we have supervised, read, and evaluated this dissertation entitled "A Study on Well-Posedness and Persistence of Spatial Analyticity to the Solution of Higher Order KdV-BBM Type Equations" by Emawayish Tegegn under our guidance. We recommend the dissertation to be submitted for oral defense (mock-viva and viva voce).

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# Contents

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# List of Abbreviations

| | |
|---|---|
| BBM | Benjamin-Bona-Mahony. |
| GBBM | Generalized Benjamin- Bona- Mahony. |
| GKdV-BBM | Generalized Korteweg de- Vries Benjamin- Bona- Mahony. |
| ICs | Initial Conditions. |
| IVPs | Intial Value Problems. |
| KdV | Korteweg de- Vries. |
| KP | Kadomtsev-Petviashvili. |
| MRLWEs | Modified Regularized Long-Wave Equations. |
| NLEEs | Nonlinear Evolution Equations. |
| NLPDEs | Nonlinear Partial Differential Equations. |
| ODEs | Ordinary Differential Equations. |
| PDEs | Partial Differential Equations. |
| RHS | Right Hand Side. |
| RLWEs | Regularized Long-Wave Equations. |

# List of Symbols

| | |
|---|---|
| $G^{\sigma_1,\sigma_2,s_1,s_2}$ : | Anisotropic Gevrey Space. |
| $G^{\sigma,s}$ : | Analytic Function Space. |
| $C$ : | Generic Positive Constant. |
| $\mathcal{S}'(\mathbb{R}^d)$ : | Class of Tempered Distribution. |
| $\mathbb{R}^n$ : | n-dimentional Euclidean Space. |
| $A \hookrightarrow B$ : | A Embedded in B. |
| $\mathcal{F}^{-1}(f), \check{f}$ : | Inverse Fourier Transform. |
| $\mathcal{F}(f), \hat{f}$ : | Fourier Transform. |
| $C\big([0,T];G^{\sigma,s}\big)$ : | Space of Continuous Function. |
| $L^p$ : | $p^{th}$-Power Lebesgue-Integrable Function. |
| $H^s$ : | Sobolev Space. |
| $C^\infty$ : | Space of Smooth Function. |
| $L^2$ : | Square Integrable Lebesgue Space. |

# Publications

1. B. Belayneh, **E. Tegegn** and A. Tesfahun, Lower bound on the radius of analyticity of solution for fifth order KdV-BBM Equation, Nonlinear Differ. Equ. Appl. NoDEA, 29 (6) (2022). (**DOI:10.1007/s00030-021-00738-z**).
2. **E. Tegegn**, A. Tesfahun, and B. Belayneh, Lower bounds on the radius of spatial analyticity of solution for KdV-BBM type equations, Nonlinear Differ. Equ. Appl. NoDEA, 30 (4) (2023). (**DOI:10.1007/s00030-023-00855-x**).

# Conferences

- ✎ Lower bound on the radius of analyticity of solution for fifth order KdV-BBM equation, presented on "The $5^{th}$ National conference" organized by Haramaya University, May 27-28, 2022, Haramaya, Ethiopia.
- ✎ Lower bound on the radius of analyticity of solution for fifth order KdV-BBM equation, presented on "The $11^{th}$ annual science conference", organized by Bahir Dar University, March 24-25, 2023, Bahir Dar, Ethiopia.
- ✎ Lower bounds on the radius of spatial analyticity of solution for KdV-BBM type equations, presented on "The $6^{th}$ National conference" which was organized by ASTU in collaboration with EMPA, in the memory of the Late Dr. Shiferaw Feyissa, from May 12-13, 2023, Adama, Ethiopia.

# Abstract

In this dissertation, we investigate the well-posedness and persistence of spatial analyticity of the solution for nonlinear evolution dispersive higher order KdV-BBM-type equations which governs waves on shallow water surfaces.

We considered the initial value problem (IVP) associated with a fifth order KdV-BBM type model that describes the propagation of the unidirectional water wave. We show that the uniform radius of spatial analyticity $\sigma(t)$ of solution at time t cannot decay faster than $1/t$ for large $t > 0$, given initial data that is analytic with fixed radius $\sigma_0$. This significantly improve the previous result an exponential decay rate of $\sigma(t)$ for large $t$ obtained in [28].

We also considered the initial value problem (IVP) associated with generalized KdV-BBM equation and coupled system of generalized BBM equations, subject to initial data which is analytic in modified Gevrey space with a fixed radius $\sigma_0$. It is shown that the uniform radius of spatial analyticity of solutions for both problems can not decay faster than $ct^{-2/3}$ as $t \to \infty$.

We proved the global well-posedness result of Kadomtsev, Petviashvili - Benjamin, Bona, Mahony (KP-BBM II) equation in an anisotropic Gevrey space, which complements earlier results on the well-posedness of this equation in anisotropic Sobolev spaces. In addition, we analyzed the evolution of the radius of spatial analyticity of the solution and we obtained asymptotic lower bound for the radius of spatial analyticity of the solution for the KP-BBM II equation. We used the conservation law, contraction mapping principle and different multilinear estimates to obtain the results.

# Chapter 1

# Introduction

## 1.1 Background of the Study

Nonlinear evolution equations (NLEEs) are partial differential equations which contain time derivative has become significant tool for investigating the natural phenomena of science and engineering. NLEEs appear in an extensive diversity of applications in solitary wave theory, water waves, propagation of shallow water waves, theoretical physics, nuclear physics, plasma physics, chemical physics, hydrodynamics, fluid dynamics, theory of turbulence, meteorology, optical fibers, quantum mechanics, coastal engineering, ocean engineering, biomathematics and such many other applications [8, 50, 72, 73, 77, 91].

Waves are among the most extensive phenomena to be studied and described by partial differential equations (PDEs). Various fields in science such as optics, fluid dynamics, especially hydrodynamics and particle physics are the applications of wave theory and have been actively studied [46]. Waves can occur when a system of medium is disturbed from its equilibrium state and the disturbance travel or propagate from one region to another in the system. A wave with speed which is not affected by its amplitude is usually categorized as linear wave, while the one with the speed affected by its amplitude is categorized as nonlinear wave. Some common nonlinear wave equations are the Korteweg-de Vries (KdV), Boussinesq, Kadomtsev-Petviashvili (KP) and Benjamin-Bona-Mahony (BBM) equations.

Another interesting wave phenomena is the so-called soliton or sometimes known as solitary wave. Soliton is localized nonlinear wave which maintains its shape unchanged along the propagation with constant speed [1]. Soliton is one of the natural phenomena which appear everywhere in daily life. The dynamics of soliton have changed the daily lifestyle of all people across the world. For example, all internet activities, phone conversations are due to soliton transmission, over transcontinental and transoceanic distances, through optical fiber cables [60]. KdV and BBM equations generate soliton solutions, so does modified regularized long wave (MRLW) equation which represents the dispersed wave phenomenon such as shallow water wave and phonon packet on nonlinear crystal [73].

Propagation of waves on the surface of an ideal fluid under gravitational force is governed by the Euler equations [31]. But Euler equations frequently look to be more complex than what is required for the undertaking of the modeling issue in practice. Consequently, in literature several approximate models are derived, among them, Boussinesq equations and its regularized version are the most well known. In [18], the first and the second-order Boussinesq systems were derived from the original Euler equations using the first and the second-order approximations respectively. Both systems describe the two-way propagation of waves. In 1870 s, Boussinesq derived some model evolution equations that are applicable in principle to describe motions. From the Boussinesq systems, several one way models for long waves were derived. The most famous one way models are Korteweg-de Vries (KdV) equation, Benjamin-Bona-Mahony (BBM) equations, Korteweg-de Vries- Benjamin-Bona-Mahony (KdV-BBM) type model equations and etc.

A dispersive partial differential equation is a type of mathematical equation that describes how waves of different wavelengths travel at different speeds. Dispersive partial differential equations are often used to model phenomena such as water waves, light waves, sound waves, and quantum mechanics. Some examples of dispersive PDEs are the Schrödinger equation, which describes the wave function of a quantum system, the Korteweg–de Vries equation, which describes the propagation of long waves in shallow water, the sine–Gordon equation, which describes the motion of a pendulum chain with nonlinear coupling. Dispersive PDEs also model physical systems in which waves of different frequencies propagate through a medium at different velocities. For instance, as theoretical models in nonlinear optics [1], quantum many-body systems [38, 39], and water waves [87, 88].

The most illustrative consequence of dispersive effects is a rainbow, which occurs when light passes through rain droplets and is split into different colors.

**Definition 1.1.1.** We say that an evolution equation defined on $\mathbb{R}^{n+1}$ is dispersive if its dispersive relation $\dfrac{\omega(k)}{k} = g(k)$, is a real valued function, such that $g(k) \to \pm\infty$ as $k \to \pm\infty$, where plane-wave solutions are of the form:

$$u(x,t) = ae^{i(kx-\omega t)}, x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{1.1.1}$$

with $k$ is the wave number, $\omega$ is the angular frequency and $a$ is the amplitude.

As an example, consider the linearized Korteweg-de Vries (Airy) equation appears as a model for small-amplitude water waves with long wavelength

$$u_t + u_{xxx} = 0. \tag{1.1.2}$$

If nonlinear effects $uu_x$ add on the Airy equation, it gives nonlinear dispersive PDE, which is KdV equation.

Substituting the plane-wave solutions (1.1.1) into the (1.1.2), we find that,

$$\omega(k) = -k^3,$$

and therefore,

$$\frac{\omega(k)}{k} = -k^2,$$

which is called the dispersive relation and shows that the frequency is a real valued function of the wave number.

A simple wave of the form (1.1.1) that satisfies the equation (1.1.2) if and only if $\omega = -k^3$. If we denote the phase velocity by $v = \frac{\omega(k)}{k}$, the solution can be written $u(x,t) = ae^{ik(x-v(k)t)}$, and notice that the wave travels with velocity $k$. Thus, the wave propagates in such a way that waves with large wave numbers travel faster than smaller ones. For the heat equation $u_t - u_{xx} = 0$, we obtain that $\omega$ is complex valued and the wave solution decays exponential in time. On the other hand the transport equation $u_t - u_x = 0$, and the one dimensional wave equation $u_{tt} - u_{xx} = 0$, are traveling waves with constant velocity.

For developing algorithms or modeling engineering systems, analytical solutions often offer important advantages. Analytical solutions are presented as mathematical expressions, they offer a clear view into how variables and interactions between variables affect the result. Information about the domain of analyticity of a solution to a partial differential equation can be used to gain understanding of the structure of the equation, and to obtain insight into underlying physical processes [37]. The study of real-analytic solutions to nonlinear PDE has developed over the last three decades.

Starting with the works of Kato and Masuda [58] for dispersive wave-type equations, and Foias and Temam [42] for the Navier-Stokes equations, analytic function spaces have become popular tools to the study of a variety of questions connected with nonlinear evolution PDEs. In particular, the use of Gevrey-type spaces has given rise to a number of important results in the study of long time dynamics dissipative equation, such as estimating the asymptotic degrees of freedom, approximating the global inertial manifolds and a rigorous estimate of the Reynold's [70].

The analytic Gevrey space play an important role in various branches of partial and ordinary differential equations as intermediate spaces between the spaces of smooth functions $C^\infty$ and the analytic functions. In particular, whenever the properties of a certain operator differ in the $C^\infty$ and in the analytic framework, it is natural to test its behavior on the classes of the Gevrey function. A function in $G^{\sigma,s}(\mathbb{R})$ has radius of analyticity at least $\sigma$ at every point on the real line. This fact leads us to consider the following question. Given $u_0 \in G^{\sigma_0,s}(\mathbb{R})$ for some initial radius $\sigma_0 > 0$, how does the radius of analyticity of the solution $u$ evolve in time?. The reason for considering initial data in $G^{\sigma_0,s}(\mathbb{R})$ space is due to the analyticity properties of Gevrey functions. The main advantage of using Gevrey space is that they allow us to compare and contrast different types of differential operators or mappings that behave differently in smooth and analytic spaces. For instance, we can use Gevrey classes to show that some differential operators are stable under perturbations in one category but not in another. We can also use Gevrey classes to find sharp estimates for the regularity or solvability of certain differential equations.

Local and global well-posedness results of PDEs are important topics in the theory of PDEs, as they provide information about the existence, uniqueness, and regularity of solutions to various types of PDEs, as well as their stability and continuity with respect to the initial and boundary data. There are many methods and techniques to establish local and global well-posedness results for different types of PDEs, such as energy methods, contraction principles, Strichartz estimates, dispersive estimates, Nash-Moser iteration, and so on. These methods often rely on exploiting some special structures or properties of the PDEs, such as symmetries, conservation laws, scaling invariance, or integrability.

A global well-posedness result for a PDE is a mathematical statement that asserts the existence, uniqueness, and regularity of a solution to the PDE for all times, given some initial and boundary conditions. A global well-posedness result usually requires some assumptions on the coefficients, the forcing term, and the data of the PDE, as well as some compatibility conditions between them. It also implies the stability and continuity of the solution with respect to the data. Global well-posedness results are important for understanding the qualitative and quantitative behavior of solutions to PDEs, as well as their physical and geometric interpretations.

Local well-posedness result is a weaker notion than well-posedness, which only requires the existence, uniqueness, and stability of the solution in a small neighborhood of the initial data. This means that the solution may blow up or become ill-defined after some finite time, or that the solution may depend discontinuously on the data outside of a small neighborhood.

Usually iterative methods are used to construct local-in-time solutions to nonlinear PDEs, given suitable regularity assumptions on the initial data. These methods were perturbation in nature (Duhamel's formula used to approximate the nonlinear evolution by the linear) and thus do not work directly for large data and long times. However, for the cases of large data, one can use non-perturbative tools to gain enough control on the equation to prevent the solution from blowing up. The most important tool used for doing this is the conservation law.

A conservation law is a principle that states a certain physical quantity does not change in the course of time within an isolated system. A conservation law of partial differential equation is a divergence expression which vanishes on solutions of the PDE system. Furthermore, conservation laws have applications in the study of PDEs such as in showing existence and uniqueness of solutions. The partial differential equations, which arise in sciences, dynamics, fluid mechanics, electromagnetism, economics and so forth, express conservation of mass, momentum, energy, electric charge, or value of firm. All the conservation laws of partial differential equations may not have physical interpretation, but are essential in studying the integrability of the PDE. The high number of conservation laws for a partial differential equation guarantees that the equation is strongly integrable and can be linearized or explicitly solved [2]. Moreover, the conservation laws are used for analysis, particularly, development of numerical schemes and study the properties of partial differential equations.

Let us consider PDEs of the form:

$$\partial_t M(x,t,u_x,u_{xx},\dots) + \partial_x N(x,t,u_x,u_{xx},\dots) = 0, \tag{1.1.3}$$

where $M(x,t,u_x,u_{xx},\dots)$ is a density, $N(x,t,u_x,u_{xx},\dots)$ is the associated flux and $u$ is the solution of the PDE. Integrating equation (1.1.3) with respect to $x$, then if $N$ decays sufficiently at the ends, we have

$$\frac{d}{dt}\int_{\mathbb{R}} M\,dx = 0, \tag{1.1.4}$$

which implies that $\int_{\mathbb{R}} M\,dx$ is a constant of motion. Conservation laws can be associated with this constants of the motion. One of the main ways of proving global existence of solutions of dispersive PDEs is to use conserved quantities.

### 1.1.1 KdV and BBM Equations

The Korteweg-de Vries (KdV) and Benjamin-Bona-Mahony (BBM) equations are two typical examples associated with the effects of dispersion, nonlinearity and describe the propagation of water waves with small amplitude or soliton in other liquid medium. The KdV equation is a nonlinear partial differential equation of third order:

$$u_t + u_x + 6uu_x + u_{xxx} = 0, \tag{1.1.5}$$

where $u(x,t)$ denotes the elongation of the wave at $(x,t)$.

The KdV equation was used as a representation of the evolution of long waves with a moderate amplitude that propagate in a single direction in shallow water with uniform depth. Although equation (1.1.5) now bears the name KdV, it was apparently first obtained by Boussinesq [26]. The study of compressible fluids in fluid mechanics, the explanation of the characteristics of electron plasmas, the study of oceanic water waves, and the investigation of mass transport issues related to chemical compounds are all areas in which the KdV equation is crucial [89]. The KdV equation (1.1.5) is widely recognized as a paradigm for the description of weakly nonlinear long waves in many branches of physics and engineering.

The KdV equation owes its name to the famous paper of Korteweg and de Vries [63] published in 1895, in which they showed that small-amplitude long waves on the free surface of water could be described by the equation

$$u_t + cu_x + \frac{3c}{3h}uu_x + \frac{ch^2}{6}\upsilon u_{xxx} = 0, \tag{1.1.6}$$

where $u(x,t)$ is the elevation of the free surface relative to the undisturbed depth $h, c = gh/2$ is the linear long wave phase speed, and $\upsilon = 1 - \frac{3\rho}{gh^2}$, is the bond number measuring the effects of surface tension and $\rho$ is the water density.

The Benjamin-Bona-Mahony (BBM) equation also known as the Regularized Long-Wave equation (RLWE) is the partial differential equation given by

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{1.1.7}$$

This equation was studied by Benjamin, Bona, and Mahony in [10] as a modified KdV equation for modeling long surface gravity waves of small amplitude propagating unidirectionally in $(1+1)$-dimensions. The authors examined the stability and uniqueness of the solutions to the BBM equation. This contrasts with the KdV equation, which is unstable in its high wave number components.

The BBM equation is well known in physical applications. It describes the model for propagation of long waves which incorporates nonlinear and dissipative effects. It is used in the analysis of the surface waves of long wavelength in liquid, hydro magnetic waves, cold plasma, acoustic gravity waves in compressible fluids, and acoustic waves in harmonic crystals [9]. In certain theoretical investigations, the BBM equation is superior as a model for long waves, from the standpoint of existence and stability, the equation offers considerable technical advantages over the KdV equation. In addition to shallow water waves, the equation is applicable to the study of drift waves in Plasma or the Rossby waves in rotating fluids. Under certain conditions, it also provides a model of one-dimensional transmitted waves.

The main mathematical difference between KdV and BBM models is best understood by comparing the dispersion relation for the equivalent linearized equations. It is obvious that these linkages only result in similar reactions for waves with low wave numbers and result in radically different reactions for waves with high wave numbers. This is one of the reasons that the existence and regularity theory for the KdV equation is more complex than the theory of the BBM equation. The BBM equation, replaces the third-order derivative in (1.1.5) by a mixed derivative, $-u_{xxt}$, which, inturn, results in a bounded dispersion relation. This boundedness was utilized to prove existence, uniqueness, and regularity results for solutions of the BBM equation (1.1.7). Further, while the KdV equation has an infinite number of integrals of motion, the BBM equation has only three [69].

Bona and Smith [24] established another regularized long wave equation by the addition of linear dispersion term $-u_{xxt}$ with KdV equation, namely, KdV-BBM equation

$$u_t + u_x + \frac{3}{2}uu_x + \nu u_{xxx} - (\frac{1}{6} - \nu)u_{xxt} = 0. \tag{1.1.8}$$

### 1.1.2 KdV-BBM Equation and Coupled System of generalized BBM Equations

The starting point of the derivation of higher-order Korteweg-de Vries- Benjamin-Bona-Mahony (KdV-BBM) type equations are the papers [14] and [18], where several-parameter variant of the classical Boussinesq coupled system of equations was derived.

Formal derivations of first and second order Boussinesq systems depend on the small parameters say $\alpha_1$ and $\beta_1$. In dimensionless scaled variables, the family of first-order Boussinesq system has the form

$$\begin{cases} \eta_t + w_x + \alpha_1(w\eta)_x + \beta_1\left(aw_{xxx} - b\eta_{xxt}\right) = 0, \\ w_t + \eta_x + \alpha_1 ww_x + \beta_1\left(c\eta_{xxx} - dw_{xxt}\right) = 0, \end{cases} \tag{1.1.9}$$

where the constants $a, b, c$ and $d$ satisfy the following relations

$$\begin{cases} a = \frac{1}{2}\left(\theta_1^2 - \frac{1}{3}\right)\lambda_1, & b = \frac{1}{2}\left(\theta_1^2 - \frac{1}{3}\right)(1-\lambda_1), \\ c = \frac{1}{2}\left(1-\theta_1^2\right)\mu_1, & d = \frac{1}{2}\left(1-\theta_1^2\right)(1-\mu_1), \end{cases}$$

so that $a + b + c + d = \frac{1}{3}$ .

The second-order Boussinesq system is given by

$$\begin{cases} \eta_t + w_x + \beta_1\left(aw_{xxx} - b\eta_{xxt}\right) + \beta_1^2\left(a_1 w_{xxxxx} + b_1\eta_{xxxxt}\right) \\ \quad = -\alpha_1(\eta w)_x + \alpha_1\beta_1\left(b(\eta w)_{xxx} - \left(a + b - \frac{1}{3}\right)\left(\eta w_{xx}\right)_x\right) \\ w_t + \eta_x + \beta_1\left(c\eta_{xxx} - dw_{xxt}\right) + \beta_1^2\left(c_1\eta_{xxxxx} + d_1 w_{xxxxt}\right) \\ \quad = -\alpha_1 ww_x + \alpha_1\beta_1\left((c+d)ww_{xxx} - c\left(ww_x\right)_{xx} - \left(\eta\eta_{xx}\right)_x + (c+d-1)w_x w_{xx}\right) \end{cases} \tag{1.1.10}$$

where the additional constants $a_1, b_1, c_1$ and $d_1$ satisfy

$$\begin{cases} a_1 = -\frac{1}{4}\left(\theta_1^2 - \frac{1}{3}\right)^2(1-\lambda_1) + \frac{5}{24}\left(\theta_1^2 - \frac{1}{5}\right)^2\lambda_{11} \\ b_1 = -\frac{5}{24}\left(\theta_1^2 - \frac{1}{5}\right)^2(1-\lambda_{11}) \\ c_1 = \frac{5}{24}\left(1-\theta_1^2\right)\left(\theta_1^2 - \frac{1}{5}\right)(1-\mu_{11}) \\ d_1 = -\frac{1}{4}\left(1-\theta_1^2\right)^2\mu_1 - \frac{5}{24}\left(1-\theta_1^2\right)\left(\theta_1^2 - \frac{1}{5}\right)\mu_{11}, \end{cases} \tag{1.1.11}$$

with $\theta_1 \in [0,1]$, $\lambda_1$, $\mu_1$, $\lambda_{11}$ and $\mu_{11}$ are modeling parameters and can take any real number.

From the second order Boussinesq system (1.1.10), Bona et al. in [14] derived the unidirectional model, namely, fifth order KdV-BBM equation

$$\begin{aligned} &u_t + u_x - \gamma_1 u_{xxt} + \gamma_2 u_{xxx} + \delta_1 u_{xxxxt} + \delta_2 u_{xxxxx} + \tfrac{3}{2}uu_x + \gamma(u^2)_{xxx} \\ &-\tfrac{7}{48}(u_x^2)_x - \tfrac{1}{8}(u^3)_x = 0, \end{aligned} \tag{1.1.12}$$

where $u : \mathbb{R}^2 \to \mathbb{R}$. The parameters $\gamma_1$, $\gamma_2$, $\delta_1$, $\delta_2$ and $\gamma$ are constants that satisfy the following conditions.

$$\begin{cases} \gamma_1 = \frac{1}{2}(b + d - \rho), \\ \gamma_2 = \frac{1}{2}(a + c + \rho), \\ \delta_1 = \frac{1}{4}\left[2\left(b_1 + d_1\right) - (b - d + \rho)\left(\frac{1}{6} - a - d\right) - d(c - a + \rho)\right], \\ \delta_2 = \frac{1}{4}\left[2\left(a_1 + c_1\right) - (c - a + \rho)\left(\frac{1}{6} - a\right) + \frac{1}{3}\rho\right], \\ \gamma = \frac{1}{24}[5 - 9(b + d) + 9\rho]. \end{cases} \tag{1.1.13}$$

Such equation describes the unidirectional propagation of water waves.

From the first order Boussinesq system (1.1.9) Bona et al. in [14] derived, the KdV-BBM equation. The generalized version of KdV-BBM equation is given by

$$u_t + u_x + \frac{3}{2}u^p u_x + \nu u_{xxx} - (\frac{1}{6} - \nu)u_{xxt} = 0, \tag{1.1.14}$$

where $\nu = \frac{1}{4}\left[\theta_1^2(\lambda_1 - \mu_1) - \frac{1}{3}\lambda_1 + \mu_1\right]$. $\theta_1, \lambda_1$ and $\mu_1$ formally take any real value and $p$ is positive integer, (see, [14]).

The KdV-BBM equation analyses the evolution of long waves with modest amplitudes propagating in plasma physics and the motion of waves in fluids and other weakly dispersive mediums. Moreover, rogue waves and lumps occur in several scientific areas, such as fluid dynamics, optical fibers, dusty plasma, oceanography, water engineering, and other nonlinear sciences [64].

To model two-way propagation of waves in physical systems where nonlinear and dispersive effects are equally important, systems of nonlinear dispersive equations have been used. There are different nonlinear dispersive system of equations that describe different phenomena, and their fundamental properties of solutions are studied in different functional spaces. For instance, the model derived by Gear and Grimshaw [43] describes the strong interaction of weakly nonlinear long waves, the Majda-Biello system [66] arises as a model for the interaction of barotropic and baroclinic equatorial Rossby waves, the Maxwell-Dirac equation describes the interaction of an electron with its own electromagnetic field that play a major role in quantum electrodynamics [11].

In particular, the coupled system of generalized BBM equations which arises in water wave theory, climate modeling and other situations where wave propagation is important, is given by .

$$\begin{cases} u_t + u_x - u_{xxt} + \Big(P(u,v)\Big)_x = 0, \\ v_t + v_x - v_{xxt} + \Big(Q(u,v)\Big)_x = 0, \end{cases} \tag{1.1.15}$$

where $u$ and $v$ are real-valued functions of $x \in \mathbb{R}$ and $t \geq 0$. Here, $P$ and $Q$ are arbitrary homogeneous quadratic polynomials in the variables $u$ and $v$ given by

$$P(u,v) = \alpha u^2 + \beta uv + \gamma v^2,$$

$$Q(u,v) = \theta u^2 + \lambda uv + \psi v^2,$$

with real valued coefficients $\alpha, \beta, \gamma, \theta, \lambda$ and $\psi$.

### 1.1.3 KP-BBM Equation

Consider the mathematical model equations, namely Kadomtsev-Petviashvili (KP) equation

$$v_t + v_x + v_{xxx} + vv_x + \alpha\partial_x^{-1}v_{yy} = 0, \tag{1.1.16}$$

and the regularized version, the Kadomtsev-Petviashvili, Benjamin-Bona-Mahony (KP-BBM) equation

$$u_t - u_{txx} + u_x + uu_x + \alpha\partial_x^{-1}u_{yy} = 0, \tag{1.1.17}$$

where $\alpha = \pm 1$. These equations occur naturally in many physical contexts as universal models for unidirectional propagation of weakly nonlinear dispersive long waves with weak transverse effects. If $\alpha = 1$ in (1.1.16) and(1.1.17), the equations are known as the KP II and KP-BBM II equation respectively, while $\alpha = -1$, they are the KP I and KP-BBM I equation.

The Kadomtsev–Petviashvilli (KP) equation, arises in various contexts where nonlinear dispersive waves propagate principally along the x-axis, but with weak dispersive effects being felt in the direction parallel to the y-axis perpendicular to the main direction of propagation.

Note that, in case the wave motion does not vary at all with $y$, (1.1.16) and (1.1.17) reduce to the Korteweg–de Vries equation

$$u_t + u_x + \frac{3}{2}uu_x + \frac{1}{6}u_{xxx} = 0,$$

and the regularized long-wave equation or BBM-equation

$$u_t + u_x + \frac{3}{2}uu_x - \frac{1}{6}u_{xxt} = 0,$$

respectively, which govern the unidirectional propagation of small-amplitude long water waves in a channel where variation across the channel can be safely ignored [12, 35]. Hence, KP equation is the two-dimensional extensions of the KdV equation and KP-BBM equation is the two-dimensional extensions of the BBM equation.

The KP-BBM equation can be used to describe the dynamics of surface gravity waves in shallow water, such as tsunamis, tidal waves, and storm surges. It can also be applied to model the propagation of ion-acoustic waves in plasma physics, where the ions behave as a fluid and the electrons are assumed to be isothermal. It also used for studying the optical solitons in nonlinear media, where the light pulses can maintain their shape and speed due to the interplay of nonlinearity and dispersion.

In this dissertation, we study the well-posedness result and persistence of spatial analyticity of the solution to the initial value problems associated with nonlinear dispersive evolution equations we discussed above. The notion of well-posedness which is featured here was put forward by the well-known French mathematician Hadamard a century ago [48]. In his study, a problem is well-posed subject to given auxiliary data when there corresponds a unique solution which depends continuously on variations in the specified supplementary data. Hadamard in [48] pointed out that if the problem is lacking well-posedness properties, it will probably be useless in practical applications. Auxiliary data brought from real-world situations typically features at least a small amount of error. The well-posedness of PDE problems is an important concept in mathematics and physics. Well-posed problems are important in real practical applications because they ensure the stability and regularity of the solutions. If a problem is not well-posed, it may have no solution, infinitely many solutions, or solutions that are sensitive to small changes in the data. Such problems are ill-posed and they may have no real practical applications.

The local well-posedness result can be extended to global well-posedness result using almost conservation law for the problems we considered. We also study the persistence of spatial analyticity to the solution of these higher order dispersive partial differential equations in the class of analytic functions by providing explicit formulas to lower bounds for the radius of analyticity of the solution.

The persistence of spatial analyticity of the solution means that the solution remains analytic in space for some time, even if the initial data is only analytic in a strip around the real axis. The persistence of spatial analyticity for the solutions of PDE problems depends on several factors, such as the type of the PDE (elliptic, parabolic, hyperbolic, etc.), the coefficients of the PDE, the initial data, the boundary conditions, the dimension of space, conservation law etc.

The radius of analyticity of the solution is a measure of how smoothness of the solution is in the complex domain. It is defined as the largest distance from the real axis to the nearest complex singularity such that the solution can be extended as a holomorphic function. Studying the radius of analyticity of the solution can help us to understand the structure and properties of the equation, as well as the underlying physical processes.

There are many studies that investigate the radius of analyticity of the solution for different types of equations, such as the Navier-Stokes system [55], Schrödinger equation [84] and the semi-linear parabolic system [32] etc. These studies use various methods and techniques to obtain lower or upper bounds for the radius of analyticity, and to analyze how it changes over time. Some of the results showed that the radius of analyticity can decay exponentially or algebraically as time increases, depending on the equation and the initial data.

The study of well-posedness and persistence of spatial analyticity of the solutions of higher order nonlinear dispersive PDEs is challenging because of the presence of nonlinear and dispersive terms, which can cause the solution to blow up or lose regularity in finite time. Therefore, in this study we used various techniques and tools to analyze the problems, such as Fourier analysis, multilinear estimates, contraction mapping principle, and approximate conservation laws.

## 1.2 Objectives of the Study

In this study, the initial value problems associated with higher order KdV-BBM type equations, coupled system of generalized BBM equations and KP-BBM equation are taken into consideration. We investigate the well-posedness and persistence of spatial analyticity of the solution for these higher order nonlinear evolution dispersive PDEs. Lower bound for the radius of spatial analyticity of solutions also established for the problems under consideration.

### 1.2.1 General Objective

The main objective of this study is to investigate the well-posedness properties and persistence of spatial analyticity of the solution of nonlinear evolution dispersive higher order KdV-BBM- type equations in Gevrey space $G^{\sigma,s}(\mathbb{R})$ and modified Gevrey space $H^{\sigma,s}(\mathbb{R})$.

### 1.2.2 Specific Objectives

The specific objectives of this study are:

- ❖ to obtain local and global well-posedness results of higher order KdV- BBM type equations and coupled system of generalized BBM equations in Gevrey space and modified Gevrey space ,

- to obtain local and global well-posedness results of KP-BBM equation in modified anisotropic Gevrey space $H^{\sigma_1,\sigma_2,s_1,s_2}(\mathbb{R}^2)$,
- to improve the exponential lower bound of the radius of spatial analyticity to algebraic one for the solution of fifth order KdV-BBM equation,
- to analyze the evolution of radius of spatial analyticity to the solutions of nonlinear dispersive coupled systems of generalized BBM equations in analytic modified Gevrey space,
- to construct asymptotic lower bound for the radius of spatial analyticity to the solution of generalized KdV-BBM equation in modified Gevrey space $H^{\sigma,s}(\mathbb{R})$.

## 1.3 Significance of the Study

The concept of well-posedness and persistence of spatial analyticity of the solution is a significant topic in the field of nonlinear partial differential equations. The problems we considered in this dissertation are mathematical models of waves on shallow water surfaces. They have many applications in physics, engineering, biology, and other sciences. The well-posedness of these problems has important property to ensure the stability and predictability of the problems. The results of this study may have the following importance:

- To provide good understanding and insight for models governing physical process having a wave structure.
- To study the well-posedness of other classes of differential equations in different classes of functions.
- To identify well-posed problems which arise in practical applications.
- To provide background information for postgraduate students and other researchers who work on related area.

# Chapter 2

# Preliminaries

It is important to identify the functional spaces in which solutions to PDEs belong and to define convergence in those spaces in order to establish Theorems of existence, uniqueness, and continuous dependence. In this chapter, notations, Definitions, function spaces, and known results are presented that will be used in the subsequent sections and chapters.

Throughout this study, we use a positive constant C to denote a constant that may vary from one line to the next. If A and B are two non-negative quantities, the notation $A \lesssim B$ stands for $A \leq CB$, $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$. We also use $A \ll B$ to mean $A \leq cB$ for some small constant $c > 0$.

## 2.1 Space of Functions

In this section, we introduce some well-known function spaces and the space-time Fourier transform.

For $1 \leq p < \infty$, $L^p := L^p(\mathbb{R})$ denotes the set of $p^{th}$-power Lebesgue-integrable functions. The norm of a function $f \in L^p(\mathbb{R})$ is given by

$$\|f\|_{L^p(\mathbb{R})} = \left( \int_{\mathbb{R}} |f|^p dx \right)^{\frac{1}{p}}.$$

For the case $p = \infty$, $L^\infty(\mathbb{R})$ define with the norm

$$\|f\|_{L^\infty(\mathbb{R})} = ess \sup |f|,$$

where $ess \sup |f|$ is the essential supremum of $|f|$ which is defined as the minimal $c \in \mathbb{R}$ such that $|f(x)| \leq c$, almost everywhere, that is,

$$ess \sup |f| = \inf\{c \in \mathbb{R} : |f(x)| \leq c \quad a.e\}.$$

We also write $\|f\|_{L^p(\mathbb{R}^d)}$ if the dimension $d$ is clear from the context. The $L^p$–space is defined as

$$L^p(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbf{C} : \text{f is measurable and}, \|f\|_{L^p(\mathbb{R}^d)} < \infty\}.$$

**Theorem 2.1.1** (Hölder inequality [3]). *Suppose that* $1 \leqslant p \leqslant \infty$ *and* $\frac{1}{\mathrm{p}} + \frac{1}{q} = 1$. *If* f *and* g *are measurable functions on* $\mathbb{R}^{\mathrm{d}}$, *then*

$$\|fg\|_{\mathrm{L}^1} \leqslant \|f\|_{L^p} \|g\|_{L^q},$$

*that is, if* $f \in L^p$ *and* $g \in L^q$, *then* $fg \in L^1$.
*More generally, if* $1 \leqslant p, q \leqslant \infty$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *then*

$$\|fg\|_{L^r} \leqslant \|f\|_{L^p}\|g\|_{L^q}.$$

If $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, and $1 \leq q,\ p \leq \infty$, then the mixed space-time Lebesgue space $L_t^q L_x^p := L_t^q L_x^p(\mathbb{R}^d \times \mathbb{R})$ is defined via the norm

$$\|f\|_{L_t^q L_x^p(\mathbb{R}^d\times\mathbb{R})} = \left\| \|f(\cdot,t)\|_{L_x^p(\mathbb{R}^d)} \right\|_{L_t^q(\mathbb{R})} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |f(x,t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

with the usual modification when $p$ or $q$ is $\infty$, that is,

$$\|f\|_{L_t^\infty L_x^p(\mathbb{R}^d\times\mathbb{R})} = \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{L_x^p(\mathbb{R}^d)},$$

and

$$\|f\|_{L_t^q L_x^\infty(\mathbb{R}^d\times\mathbb{R})} = \left\| \sup_{x\in\mathbb{R}^d} |f(x,t)| \right\|_{L_t^q(\mathbb{R})}.$$

An analogous definition is used for the other mixed norms $L_x^p L_t^q$, with the order of integration in time and space interchangeably. If $p = q$, then we write $L_x^p L_t^q = L_{x,t}^p$.

The space of square-integrable, measurable functions defined on a measurable subset $\Omega$ of Euclidean space is denoted by $L^2(\Omega)$. In fact, throughout, $\Omega$ will always be $\mathbb{R}$ , $\mathbb{R}^2$ or $\mathbb{R}^3$ and we will usually not bother to display the set, but just write $L^2$ for $L^2(\mathbb{R}^d)$, $d = 1, 2, 3$.

For any Banach space $X$ and $T > 0$, $C(0,T;X)$ is the class of continuous maps from $[0,T]$ into $X$ with its usual norm

$$\|u\|_{C(0,T;X)} = \sup_{t\in[0,T]} \|u\|_X.$$

If $X$ and $Y$ are Banach spaces, then their Cartesian product $X \times Y$ is also a Banach space with product norm defined by

$$\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y,$$

and hence

$$\|(u,v)\|_{C(0,T;X\times Y)} = \sup_{t\in[0,T]} \|u\|_X + \sup_{t\in[0,T]} \|v\|_Y. \tag{2.1.1}$$

**Definition 2.1.2.** ([68]) Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces.
A map $T : X \to Y$ is called a contraction mapping, if there exists a constant $\theta \in [0, 1)$ such that

$$\|T(x) - T(y)\|_Y \leq \theta \|x - y\|_X,$$

for all $x, y \in X$.

**Theorem 2.1.3.** *(Banach's fixed point theorem [68]). Let $(X, \|.\|_X)$ be a Banach space. If $T : X \to X$ is a contraction map, then the fixed point equation $T(x) = x$ has a unique solution.*

Banach's fixed point theorem (also called the contraction principle) is one of the most important tool used to prove the well-posedness result of nonlinear evolution equations. It is also called an existence and uniqueness theorem for fixed points of certain mappings.

**Definition 2.1.4.** [51] We define a complex-valued smooth, rapidly decreasing functions

$$S(\mathbb{R}^d) = \{u \in C^\infty(\mathbb{R}^d) : \|u\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta u(x)| < \infty, \}$$

for all multi-indices's $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_d), \beta = (\beta_1, \beta_2, \beta_3, ..., \beta_d) \in \mathbb{N}_0^d$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The space of all Schwartz functions on $\mathbb{R}^d$ is denoted by $S(\mathbb{R}^d)$.

**Definition 2.1.5.** [51] The dual space of the Schwartz space $S(\mathbb{R}^d)$ is called the space of tempered distributions, denoted by $S'(\mathbb{R}^d)$.
We write

$$u(\phi) = \langle u, \phi \rangle,$$

for $u \in S'(\mathbb{R}^d)$, $\phi \in S(\mathbb{R}^d)$.

**Definition 2.1.6.** [83] The Fourier transform of $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ is denoted by $\mathcal{F}f$ or $\hat{f}$ and defined by

$$\mathcal{F}f = \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx,$$

where $x \cdot \xi = \sum_{i=1}^d x_i \xi_i$ for $x = (x_1, x_2, \ldots, x_d), \xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d$.
Then, the mapping $\hat{\cdot} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is an isomorphism with inverse transform, given by

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

The spatial Fourier transform $f(x) \mapsto \hat{f}(\xi)$ brings into view the oscillation of a function in space. In the analysis of dispersive PDE, it is also important to analyze the oscillation in time, which leads to the introduction of the space-time Fourier transform. If $f : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{C}$ is a complex scalar field, we can define its space-time Fourier transform $\hat{f} : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{C}$, formally as

$$\hat{f}(\xi, \tau) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} f(x,t) e^{-i(x\cdot\xi + t\tau)} dt dx.$$

**Definition 2.1.7.** The convolution of the functions $f, g \in L^1(\mathbb{R}^d)$ is denoted by $f * g$ and defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y)\, dy.$$

Note that the following important properties of the Fourier transform

(i) Parseval's relation

$$\int_{\mathbb{R}^d} f(x)\overline{h(x)} dx = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{h}(\xi)} d\xi.$$

(ii) Convolution

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

**Proposition 2.1.8.** *[51] $\mathcal{F}(L^2(\mathbb{R}^d)) = L^2(\mathbb{R}^d)$. Moreover, $\mathcal{F}|_{L^2(\mathbb{R}^d)} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$is a unitary operator and in particular*

$$\|\mathcal{F}u\|_{L^2(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)}, u \in L^2(\mathbb{R}^d).$$

Natural spaces to measure the regularity of the initial data in Cauchy problems are the classical Sobolev spaces $H^s(\mathbb{R}^d), s \in \mathbb{R}$. Sobolev spaces are named after the Russian mathematician Sergei Sobolev. Their importance comes from the fact that weak solutions of some important partial differential equations exist in appropriate Sobolev spaces, even when there are no strong solutions in spaces of continuous functions with the derivatives understood in the classical sense.

**Definition 2.1.9.** [62] Assume that $\Omega$ is an open subset of $\mathbb{R}^d$ , $u \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}_0$ be a multi-index. Then $v \in L^1_{loc}(\Omega)$ is the $\alpha^{th}$ weak partial derivative of $u$, written $D^\alpha u = v$, if

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v \varphi dx$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

**Definition 2.1.10.** [62] Assume that $\Omega$ is an open subset of $\mathbb{R}^d$. The Sobolev space $W^{k,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that for every multi-index $\alpha$ with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^p(\Omega)$. Thus

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \quad |\alpha| \leq k\}.$$

If $u \in W^{k,p}(\Omega)$, we define its norm

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and for $p = \infty$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} = \sum_{|\alpha| \leq k} ess \sup_{\Omega} |D^\alpha u|.$$

The Sobolev space $H^k(\Omega)$ is the space of functions $u$ in $L^2(\Omega)$ such that the $D^\alpha u$ for any $\alpha = 0, 1, ..., k$ are also in $L^2(\Omega)$, where the derivatives are interpreted in the sense of distributions. Thus,

$$H^k(\Omega) = \{u \in L^2 : D^\alpha u \in L^2, \text{where } \alpha = 0, 1, 2, ..., k\}.$$

The notation $W^{k,2}$ is also frequently used instead of $H^k$. One then regards $H^k$ as a member of a more general family of Sobolev spaces $W^{k,p}$.

We write $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ and define the Bessel potential operator

$$J^s : S^{'}(\mathbb{R}^d) \to S^{'}(\mathbb{R}^d), \quad \langle \mathcal{F} J^s u, \phi \rangle = \langle \mathcal{F} u, \langle \xi \rangle^s \phi \rangle, \phi \in C_0^\infty(\mathbb{R}^d).$$

**Definition 2.1.11.** [51] Let $s \in \mathbb{R}$. We define the $L^2$- based inhomogeneous Sobolev spaces $H^s(\mathbb{R})$ as

$$H^s(\mathbb{R}) = \left\{ u \in \mathcal{S}'(\mathbb{R}) : J^s u \in L^2(\mathbb{R}) \right\},$$

and we have

$$\|u\|^2_{H^s(\mathbb{R})} = \|J^s u\|^2_{L^2(\mathbb{R})} = \|\langle \xi \rangle^s \widehat{u}(\xi)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi.$$

**Definition 2.1.12.** ([40]) Let $\bar{s} = s_1, s_2, \ldots, s_d \in \mathbb{R}^d$. The anisotropic Sobolev spaces $H^{\bar{s}}(\mathbb{R}^d)$ defined as

$$H^{\bar{s}}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \prod_{i=1}^{d} \left(1 + |\xi_i|^2\right)^{\frac{s_i}{2}} \hat{u} \in L^2(\mathbb{R}^d) \right\},$$

where $\xi_i$ denotes the $i^{th}$ component of the Fourier variable $\xi_i \in \mathbb{R}^d$.

For $\bar{s} = s_1, s_1 \in \mathbb{R}^2$, we define anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ via norm

$$\begin{aligned} \|u\|^2_{H^{s_1,s_2}(\mathbb{R}^2)} &= \|\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\hat{u}(\xi_1,\xi_2,t)\|^2_{L^2(\mathbb{R}^2)} \\ &= \int_{\mathbb{R}^2}\langle\xi_1\rangle^{2s_1}\langle\xi_2\rangle^{2s_2}|\hat{u}(\xi_1,\xi_2,t)|^2 d\xi_1 d\xi_2, \end{aligned} \tag{2.1.2}$$

where $\hat{u}$ denotes the spatial Fourier transform

$$\hat{u}(\xi_1,\xi_2,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i(x\xi_1+y\xi_2)}u(x,y,t)dxdy.$$

**Theorem 2.1.13.** *(Sobolev Embedding Theorem [61]). If $s > k+\frac{1}{2}$, then $H^s(\mathbb{R}^d)$ is continuously embedding in $C^k(\mathbb{R}^d)$, the space of functions with $k$ times continuous differentiable vanishing at infinity. In other words, if $f \in H^s(\mathbb{R}^d)$, $s > k+\frac{1}{2}$, then $f \in C^k(\mathbb{R}^d)$ and*

$$\|f\|_{C^k(\mathbb{R}^d)} \le C\|f\|_{H^s(\mathbb{R}^d)}.$$

**Proposition 2.1.14** (Sobolev Lemma)**.** *([74]). For $s > \frac{1}{2}$, we have*

$$\|u\|_{L^\infty} \le C\|u\|_{H^s}, \quad u \in H^s,$$

*for some positive constant $C$ depending only on $s$.*

**Proposition 2.1.15** ([61])**.** *Let $s, s_1, s_2, s' \in \mathbb{R}$.*

1. *If $0 \le s < s'$, then $H^{s'}(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d)$.*

2. *$H^s(\mathbb{R}^d)$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_s$ defined as follow. If $f, g \in H^s(\mathbb{R}^d)$, then*

$$\langle f,g\rangle_s = \int_{\mathbb{R}^d}(1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi)\overline{(1+|\xi|^2)^{\frac{s}{2}}\hat{g}(\xi)}d\xi.$$

3. *The Schwartz space, $S(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$, $\forall s \in \mathbb{R}$.*

4. *If $s_1 \le s \le s_2$, with $s = \theta s_1 + (1-\theta)s_2$ for $0 \le \theta \le 1$, then*

$$\|f\|_{H^s} \le \|f\|^\theta_{H^{s_1}}\|f\|^{1-\theta}_{H^{s_2}}, \quad f \in H^s(\mathbb{R}^d),$$

which is known as interpolation inequality.

**Proposition 2.1.16** ([45])**.** *Let $1 < p < \infty$.*

*i. If* $0 < s < \frac{d}{p}$, *then the Sobolev space* $W^{s,p}\left(\mathbb{R}^d\right)$ *embeds continuously in* $L^q\left(\mathbb{R}^d\right)$ *for*

$$\frac{1}{p} - \frac{1}{q} = \frac{s}{d}. \tag{2.1.3}$$

*ii. If* $0 < s < \frac{d}{p}$, *then the Sobolev space* $W^{s,p}\left(\mathbb{R}^d\right)$ *embeds continuously in* $L^q\left(\mathbb{R}^d\right)$ *for any* $\frac{d}{s} < q < \infty$.

*iii. If* $\frac{d}{p} < s < \infty$, *then every element of* $W^{s,p}\left(\mathbb{R}^d\right)$ *can be modified on a set of measure zero so that the resulting function is bounded and uniformly continuous.*

**Theorem 2.1.17** (H-algebra)**.** *The Sobolev space,* $H^s(\mathbb{R})$ *is an algebra for* $s > \frac{1}{2}$. *i.e., if* $u, v \in H^s(\mathbb{R})$ *then* $uv \in H^s(\mathbb{R})$ *and there exists* $\mathrm{C} = \mathrm{C(s)} > 0$, *such that*

$$\|uv\|_{H^s(\mathbb{R})} \leq C\|u\|_{\mathrm{H^s(\mathbb{R})}}\|v\|_{\mathrm{H^s(\mathbb{R})}}.$$

In mathematics, an analytic function is a function that is locally given by a convergent power series. There exist both real analytic functions and complex analytic functions. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not generally hold for real analytic functions. A function is analytic if, and only if, its Taylor series about $x_0$ converges to the function in some neighborhood for every $x_0$ in its domain.

**Definition 2.1.18.** Let $f$ be a real-valued function defined on an open set $\Omega \subset \mathbb{R}^n$. We call $f$ is real analytic at $x_0$ if there is a neighborhood of $x_0$ within which $f$ can be represented as a Taylor series

$$f(x) = \sum_{i=0}^{\infty} C_i(x - x_0)^n, \tag{2.1.4}$$

where the coefficients $C_i$, $i = 1, 2, 3, \cdots$ are real numbers and the series is convergent to $f(x)$ for $x$ in a neighborhood of $x_0$.

A function $f(x)$ is said to be analytic at a point $x_0$ if $x_0$ is an interior point of some region where $f(x)$ is analytic. Hence the concept of analytic function at a point implies that the function is analytic in some neighborhood at this point. Thus, we say $f$ is real analytic in $\Omega$ if it is analytic at every point in $\Omega$. The symbol $C^\omega(\Omega)$ is used to denote the class of functions which are analytic in $\Omega$, whereas $C^\infty(\Omega)$ denotes functions which have derivatives of all orders. Obviously $C^\omega(\Omega) \subset C^\infty(\Omega)$.

Like holomorphic functions of a single complex variable, analytic functions have a unique continuation property. Real analytic functions can also be characterized as restrictions of complex analytic functions. Thus, every real analytic function can be extended into a subset of the complex plane, and since power series can be differentiated term by term, the extended function is differentiable.

**Definition 2.1.19.** Let $\sigma \geq 0$, $s \in \mathbb{R}$. Then Gevrey space $G^{\sigma,s}(\mathbb{R})$ is defined as the subspace of $L^2(\mathbb{R})$:

$$G^{\sigma,s}(\mathbb{R}) = \Big\{ u \in \mathcal{S}'(\mathbb{R}) : \langle D \rangle^s e^{\sigma|D|} f \in L^2(\mathbb{R}) \Big\},$$

with the norm

$$\|f\|^2_{G^{\sigma,s}(\mathbb{R})} = \|\langle \xi \rangle^s e^{\sigma|\xi|} \widehat{f}(\xi)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \langle \xi \rangle^{2s} e^{2\sigma|\xi|} |\widehat{f}(\xi)|^2 d\xi. \tag{2.1.5}$$

where $D = -i\partial_x$ with Fourier symbol $\xi$ and $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

For $\sigma = 0$, the Gevrey-space coincides with the Sobolev space $H^s(\mathbb{R})$. Observe that the Gevrey spaces satisfy the following embedding property:

$$G^{\sigma,s}(\mathbb{R}) \subset G^{\sigma',s'}(\mathbb{R}), \quad \text{for all} \quad \sigma > \sigma' \geq 0, \quad s, s' \in \mathbb{R}. \tag{2.1.6}$$

This implies that, there exist a constant $C$ such that

$$\|u\|_{G^{\sigma',s'}(\mathbb{R})} \leq C\|u\|_{G^{\sigma,s}(\mathbb{R})}, \quad \forall u \in G^{\sigma,s}(\mathbb{R}).$$

In particular, for $\sigma' = 0$, we have the embedding $G^{\sigma,s}(\mathbb{R}) \subset H^{s'}(\mathbb{R})$, for all $\sigma > 0$ and $s, s' \in \mathbb{R}$, that is

$$\|u\|_{H^{s'}(\mathbb{R})} \leq C\|u\|_{G^{\sigma,s}(\mathbb{R})}, \quad \forall \sigma > 0, \quad s, s' \in \mathbb{R}.$$

For $s_1, s_2 \in \mathbb{R}$, let $\bar{s} = s_1, s_2$ and $\sigma_1, \sigma_2 \geq 0$. We define anisotropic Gevrey space, $G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)$ via norm

$$\begin{aligned} \|u\|^2_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)} &= \|e^{\sigma_1|\xi_1|} e^{\sigma_2|\xi_2|} \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \hat{u}(\xi_1, \xi_2)\|^2_{L_2(\mathbb{R}^2)} \\ &= \int_{\mathbb{R}^2} e^{2\sigma_1|\xi_1|} e^{2\sigma_2|\xi_2|} \langle \xi_1 \rangle^{2s_1} \langle \xi_2 \rangle^{2s_2} |\hat{u}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2, \end{aligned} \tag{2.1.7}$$

where $\hat{u}$ denotes the spatial Fourier transform.

One of the key properties of the analytic Gevrey space is that every function in $G^{\sigma,s}(\mathbb{R})$ with $\sigma > 0$, has an analytic extension to the complex strip $S_\sigma = \{z = x + iy : x, y \in \mathbb{R}, |y| < \sigma\}$. This property is contained in the following theorem.

**Theorem 2.1.20** (Paley-Wiener Theorem,[59]). *Let $\sigma > 0$, $s \in \mathbb{R}$. Then the following are equivalent:*

*(1) $f \in G^{\sigma,s}(\mathbb{R})$.*

*(2) $f$ is the restriction to the real line of a function $F$ which is holomorphic in the strip*

$$S_\sigma = \{z = x + iy : x, y \in \mathbb{R}, |y| < \sigma\},$$

*and satisfying*

$$\sup_{|y|<\sigma} \|F(x+iy)\|_{H^s(\mathbb{R})} < \infty. \tag{2.1.8}$$

***Proof :*** Let us start the proof for $s = 0$

(i). If $f \in G^{\sigma,s}(\mathbb{R})$ then $f$ is the restriction to the real line of a function $F$ which is holomorphic in the strip $S_\sigma$, to show this,

Write

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(\xi) e^{i\xi z} d\xi. \tag{2.1.9}$$

Then by the inversion formula $F|_{\mathbb{R}} = f$ the function $F$ is well defined and holomorphic in $\{z : |y| < \sigma\}$. By Plancherel's theorem, we have

$$\int_{\mathbb{R}} |F(x+iy)|^2 \, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 e^{2\xi y} d\xi \leq \left\| \hat{f} e^{\sigma|\xi|} \right\|_{L^2(\mathbb{R})}^2 < \infty. \tag{2.1.10}$$

The converse of (i) also true, that means, if $f$ is the restriction to the real line of a function $F$ which is holomorphic in the strip $S_\sigma$ then $f \in G^{\sigma,s}(\mathbb{R})$, to show this

Write

$$f_y(x) = F(x+iy).$$

Then, we want to show that

$$\hat{f}_y(\xi) = \hat{f}(\xi) e^{-\xi y}. \tag{2.1.11}$$

By Plancherel's theorem and (2.1.8), we conclude that the integral $\int_{\mathbb{R}} |f(\xi)|^2 e^{2\xi y} dx$ is uniformly bounded in $|y| < \sigma$ , which clearly implies (2).

For $\lambda > 0$ and $z$ in the strip $\{z : |y| < \sigma\}$, put

$$G_\lambda(z) = \mathbf{K}_\lambda * F = \int_{-\infty}^{\infty} F(z-u) \mathbf{K}_\lambda(u) du. \tag{2.1.12}$$

where K denotes Fejer's kernel.

Clearly, $G_\lambda$ is holomorphic in the strip $\{z : |y| < \sigma\}$ and note that

$$g_{\lambda,y}(x) = G_\lambda(x + iy) = \mathbf{K}_\lambda * f_y.$$

Hence

$$\widehat{g_{\lambda,y}}(\xi) = \widehat{\mathbf{K}_\lambda}\hat{f}_y(\xi).$$

Now, since $\widehat{g_{\lambda,y}}(\xi)$ has a compact support (contained in $[-\lambda, \lambda]$), we have

$$\widehat{g_{\lambda,y}}(\xi) = \widehat{g_{\lambda,0}}(\xi)e^{-\xi y},$$

and consequently if $|\xi| < \lambda$, (2.1.11) holds. Since $\lambda > 0$ is arbitrary, (2.1.11) holds for all $\xi$ and the proof is complete. □

The modified Gevrey space, $H^{\sigma,s}(\mathbb{R})$ is obtained from the $G^{\sigma,s}(\mathbb{R})$ by replacing the exponential weight $e^{\sigma|\xi|}$ with the hyperbolic weight $\cosh(\sigma|\xi|)$, equipped with the norm

$$\|f\|^2_{H^{\sigma,s}(\mathbb{R})} = \|\cosh(\sigma|\xi|)\langle\xi\rangle^s \widehat{f}(\xi)\|^2_{L^2(\mathbb{R})}, \quad \sigma \geq 0, \tag{2.1.13}$$

where

$$\cosh(\sigma|\xi|) = \frac{e^{\sigma|\xi|} + e^{-\sigma|\xi|}}{2}.$$

Observe that, for large values of $\xi$ we have, $e^{-|\xi|} \approx 0$. From this fact and the definition of $\cosh(\xi)$, we have

$$\frac{1}{2}e^{\sigma|\xi|} \leq \cosh(\sigma|\xi|) \leq e^{\sigma|\xi|}. \tag{2.1.14}$$

Thus, the associated $H^{\sigma,s}(\mathbb{R})$ and $G^{\sigma,s}(\mathbb{R})$–norms are equivalent. That is

$$\|f\|_{H^{\sigma,s}(\mathbb{R})} \sim \|f\|_{G^{\sigma,s}(\mathbb{R})}. \tag{2.1.15}$$

Paley-Wiener Theorem still holds for functions in $H^{\sigma,s}(\mathbb{R})$. Note also that, $G^{0,s}(\mathbb{R}) = H^{0,s}(\mathbb{R}) = H^s(\mathbb{R})$.

The reason for considering the modified Gevrey space, $H^{\sigma,s}(\mathbb{R})$ is due to the decay rate of exponential weight in $G^{\sigma,s}(\mathbb{R})$-norm. In fact, in Gevrey space, $G^{\sigma,s}(\mathbb{R})$ the desired decay rate-in-time of the radius of analyticity $\sigma$ is obtained from the algebraic estimate

$$e^{\sigma|\xi|} - 1 \leq (\sigma|\xi|)^\rho e^{\sigma|\xi|}, \quad \rho \in [0, 1],$$

which could provide a decay rate of order $t^{-1/\rho}$ for $\rho \in (0, 1]$. In the new space $H^{\sigma,s}(\mathbb{R})$, the desired decay rate-in-time of the radius of analyticity $\sigma$ is obtained from the estimate

$$\cosh(\sigma|\xi|) - 1 \leq (\sigma|\xi|)^{2\rho} \cosh(\sigma|\xi|), \quad \rho \in [0,1], \tag{2.1.16}$$

which could provide a decay rate of order $t^{-1/2\rho}$ for $\rho \in (0,1]$.

The estimate (2.1.16) follows from

$$\cosh x - 1 \leq \cosh x \quad \textit{and} \quad \cosh x - 1 \leq x^2 \cosh x, \quad x \in \mathbb{R}.$$

From the embedding property (2.1.6) and (2.1.15), we have

$$\|f\|_{H^s(\mathbb{R})} \leq C\|f\|_{H^{\sigma,s}(\mathbb{R})}, \quad \sigma > 0. \tag{2.1.17}$$

## 2.2 Cauchy Problems and Well-posedness

A PDE with its domain and all required boundary and/or initial conditions is called a PDE problem. The condition at initial time $t = 0$ for a time-dependent problem are also known as initial conditions. Some boundary conditions are unsuitable for certain types of PDE in that they can lead to unphysical behaviour. For example, Cauchy type conditions are unsuitable for the Laplace equation and Dirichlet conditions are unsuitable for the wave equation. This leads to the notion of a well-posed problem. Problems which arise in practical applications are usually well-posed boundary value problems (for PDEs in space only) or well-posed Cauchy problems (for PDEs in time and space).

**Definition 2.2.1.** [48] A problem for PDEs is well-posed if and only if

($i$) a solution exists (existence),

($ii$) for given data there is only one solution (uniqueness), and

($iii$) a small change in the data (boundary data, initial data, source terms) produces only a small change in the solution(continuous dependence on the data).

Hadamard, 1902 pointed out that, if the problem is lacking conditions in the Definition 2.2.1, it will probably be useless in practical applications.

**Definition 2.2.2** (Local well-posedness (LWP))**.** The Cauchy problem with initial data $u_0$ is said to be locally well-posed in $H^s(\mathbb{R}^d)$ if the following conditions are satisfied:

(*i*) For every $u_0 \in H^s(\mathbb{R})$, there exists a time $T > 0$, and a solution $u$ in some Banach subspace $X$ of $C([0,T);H^s(\mathbb{R}))$, i.e.,

$$u \in X \subset C([0,T);H^s(\mathbb{R})).$$

(*ii*) The solution is unique in $X$.

(*iii*) The map from data to solution, $u_0 \mapsto u$ defined from $H^s(\mathbb{R})$ to $X$ is continuous.

A PDE is said to be locally well-posed if there exists a time interval and a unique solution that depends continuously on the initial data which belongs to given class of functions. This means that small perturbations in the initial data will result in small changes in the solution.

*Remark* 2.2.3. In Definition 2.2.2,

(*i*) If the data to the solution map, $u_0 \to u$ is uniformly continuous, we call the problem is semilinear. If it is only continuous, the problem is called a quasilinear problem.

(*ii*) If $T$ is arbitrarily large, we say that the problem is globally well-posed.

(*iii*) If $X = C([0,T);H^s(\mathbb{R}))$, the problem is said to be unconditionally well-posed.

# Chapter 3

# Literature Reviews

In this chapter, we investigate previous studies conducted on different nonlinear partial differential equations that describe the propagation of long waves in shallow water. The aim of this review is to answer the following research question: Under what conditions does the solution of the given problems remain spatially analytic for all time?

Here, we analyze the existing literature on the initial value problems for fifth order KdV-BBM equation, generalized KdV-BBM equation, coupled system of generalized BBM equations and KP-BBM equation. We identify previous well-posedness results and persistence of spatial analyticity to the solution of these problems, as well as the gaps and limitations of the study.

## 3.1 Fifth Order KdV-BBM Equation

Local and global well-posedness of the initial value problem (IVP) associated to a fifth order KdV-BBM type model was proved in the space of the analytic functions, so called Gevrey space. They also analyze the evolution of radius of analyticity in such class by providing explicit exponential upper and lower bound formulas for the radius of analyticity of the solution.

Consider the IVP for fifth order KdV-BBM equation [14]

$$\begin{cases} u_t + u_x - \gamma_1 u_{xxt} + \gamma_2 u_{xxx} + \delta_1 u_{xxxxt} + \delta_2 u_{xxxxx} + \frac{3}{2} u u_x + \gamma (u^2)_{xxx} \\ -\frac{7}{48}(u_x^2)_x - \frac{1}{8}(u^3)_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{3.1.1}$$

where $u$ is real valued functions of $x$ and $t$ and $u_0(x)$ is the initial data at $t = 0$.

In the case, $\gamma = \frac{7}{48}$, the energy of the solution given by

$$\mathcal{E}[u(t)] = \frac{1}{2} \int_{\mathbb{R}} \left( u^2 + \gamma_1 u_x^2 + \delta_1 u_{xx}^2 \right) dx, \tag{3.1.2}$$

is conserved [14], that is,

$$\mathcal{E}[u(t)] = \mathcal{E}[u(0)] \quad \forall t \in \mathbb{R}.$$

For the local well-posedness theory of fifth order KdV-BBM equation, it is important that the coefficients $\gamma_1$ and $\delta_1$ appearing, respectively, in front of $u_{xxt}$ and $u_{xxxxt}$ terms be nonnegative [29]. The problem (3.1.1) is ill-posed if this is not the case. The special cases where $\delta_1 = 0$ and $\gamma_1 > 0$ is also locally well-posed [14]. The local theory does not depend upon special choices of the parameters in the problem other than the positivity of $\gamma_1$ and $\delta_1$.

In general, the fifth order KdV-BBM equation does not have an obvious Hamiltonian structure. However, by suitably choosing the parameters, it can be put into Hamiltonian form. The Hamiltonian structure allows one to infer bounds on solutions that lead to global well-posedness. As seen in [17], lack of Hamiltonian structure often seems to go along with lack of global well-posedness for arbitrarily sized data. To obtain a global well-posedness result for initial data with lower-order Sobolev regularity, the authors in [6, 76] used a high-low frequency splitting technique. This technique was applied in [13, 15] in the context of BBM - type equations, to obtain sharper well-posedness results.

For $\gamma_1, \delta_1 > 0$, the local well-posedness of (3.1.1) in Sobolev space $H^s(\mathbb{R})$ for $s \geq 0$ was studied by the authors in [14]. For $\gamma = \frac{7}{48}$, the conservation energy (3.1.2) was used to prove the global well-posedness for data in $H^s(\mathbb{R})$ for $s \geq 2$, in the case $\gamma \neq \frac{7}{48}$, the corresponding energy has no positive sign, and therefore not useful to prove global well-posedness of (3.1.1). While, for data with regularity $\frac{3}{2} \leq s < 2$, splitting to high-low frequency technique was used in [14] to get the global well-posedness result. This global well-posedness result was further improved in [27] for initial data with Sobolev regularity $s \geq 1$. Furthermore, the authors in [27] showed that the well-posedness result is sharp by proving that the mapping data-solution fails to be continuous at the origin for $s < 1$. For similar results in the periodic case, we refer to [30].

Most recently, an exponential lower bounds on the width of the strip was presented in [28] via a Gevrey-class technique. The authors studied the property of spatial analyticity of the solution $u(x,t)$ to (3.1.1) given that the initial data $u_0(t)$ is real-analytic with uniform radius of analyticity $\sigma_0$, so there is a holomorphic extension to a complex strip

$$S_{\sigma_0} = \{x + iy : x, y \in \mathbb{R} \quad |y| < \sigma_0\}.$$

The authors proved that, for short times, the radius of analyticity $\sigma(t)$ of the solution remains at least as large as the initial radius, i.e, one can take $\sigma(t) = \sigma_0$. On the other hand, for large times, they proved that $\sigma(t)$ decays exponentially in t and they got, exponential lower bounds given by

$$\sigma_0 \exp\left\{-\left(\|u_0\|_{G^{\sigma,2}} + 2\|u_0\|^2_{G^{\sigma,2})}\right)t - \frac{3}{2}t^{3/2}\left(\|u_0\|^{3/2}_{H^2} + \|u_0\|^2_{H^2}\right)\right\}$$
$$\exp\left\{-t^2\left(\|u_0\|^{3/2}_{H^2} + \|u_0\|^2_{H^2}\right)^2\right\},$$

and exponential upper bound given by

$$C\sigma_0 \exp\left\{-\|u_0\|^2_{H^2} t\right\}.$$

Moreover

$$\|u(t)\|_{G^{\sigma,2}} \leq \|u_0\|_{G^{\sigma,2}} + Ct^{1/2}\left(\|u_0\|^{3/2}_{H^2} + \|u_0\|^2_{H^2}\right).$$

As stated previously, (3.1.1) is well-posed in Sobolev spaces as well as in the spaces of analytical functions, the so-called Gevrey class of functions and explicit exponential lower bound for the radius of analyticity to the solution has also been obtained. However, the exponential lower bound for the radius of analyticity $\sigma(t)$, decays more quickly as $t$ approaches infinity, so we are interested to improve the result from exponential lower bound to algebraic lower bound for $\sigma(t)$.

Particularly, our main interest is to find solutions $u(x,t)$ of the IVP (3.1.1) with real-analytic initial data $u_0$ which admit extension as an analytic function to a complex strip $S_{\sigma_0} = \{x + iy : x, y \in \mathbb{R} \quad |y| < \sigma_0\}$, for some $\sigma_0 > 0$ at least for a short time. After getting this result, a natural question one may ask is whether this property holds globally in time, but with a possibly smaller radius of analyticity $\sigma(t) > 0$. In other words, is the solution $u(x,t)$ of the IVP (3.1.1) with real-analytic initial data $u_0$ analytic in $S_{\sigma(t)}$ for all $t$?. What is the lower bound of $\sigma(t)$ ?. These questions will also be addressed in chapter four of this dissertation.

The present work, presuming that a specific Sobolev norm of the solution remains finite, focuses on examining the asymptotic of the breadth $\sigma$ of the strip of analyticity for large $t$. Fixed point principle and different multilinear estimates used to prove the local well-posedness of the solution in analytic Gevrey space. Several scholars applied the method approximate conservation law to prove the global well-posedness results for various problems [4, 5, 56, 79, 81, 85, 86].

The approximate conservation law enable us to repeat the local result on successive short time intervals to reach any target time $T > 0$, by adjusting the strip width parameter $\sigma$ according to the size of $T$, and to analyze the evolution of radius of analyticity in Gevrey space by providing explicit formulas for lower bound.

## 3.2 Generalized KdV-BBM Equation

Another interesting initial value problem is generalized KdV-BBM equation

$$u_t + u_x + \frac{3}{2}u^p u_x + \nu u_{xxx} - (\frac{1}{6} - \nu)u_{xxt} = 0, \tag{3.2.1}$$

subject to

$$u(x, 0) = u_0(x), \tag{3.2.2}$$

where $u : \mathbb{R}^2 \to \mathbb{R}$, the parameter $\nu$ is constant that satisfy certain constraints [14] and $p$ is positive integer. Equation (3.2.1) is generalized BBM equation when $\nu = 0$ and generalized KdV equation when $\nu = \frac{1}{6}$.

The energy of the solution of (3.2.1) is conserved for $\nu < \frac{1}{6}$, that is

$$\mathcal{A}[u(t)] = \frac{1}{2}\int_{\mathbb{R}} [u^2 + (\frac{1}{6} - \nu)u_x^2]\, dx = \mathcal{A}[u(0)], \quad \forall t \in \mathbb{R}. \tag{3.2.3}$$

The KdV-BBM equation can be derived by expanding the Dirichlet–Neumann operator [65].

Mancas and Adams in [67] studied the local and global well-posedness of the solution for KdV–BBM type equation. The locally well-posedness was proved in Sobolev space $H^s(\mathbb{R})$ for $s \geq 1$ and $p = 1$. This local well-posedness is established using a contraction mapping type argument combined with multilinear estimates. For $\nu < \frac{1}{6}$, the conserved energy (3.2.3) was used to prove the global well-posedness of (3.2.1)-(3.2.2) with initial data in $H^s(\mathbb{R})$ for $s \geq 1$. The global well-posedness of (3.2.1)-(3.2.2) is established in Sobolev spaces $H^s(\mathbb{R})$, which relies on the local results with energy type estimates.

The global analytic theory of nonlinear evolution PDE started with the work of Kato and Masuda [58] and has recently received a lot of attention for the Korteweg-de Vries (KdV) equation [20, 22, 56, 79, 85]. See also a recent related result for the quartic generalized KdV equation [82], periodic Benjamin-Bona-Mahony (BBM) equation [52]. For earlier studies concerning properties of spatial analyticity of solutions for a large class of nonlinear partial differential equations [21, 22, 41, 44, 53, 54, 59, 71, 81]. But as far as we know, there is currently no research being done on the persistence of spatial analyticity of the solution for gKdV-BBM. Motivated by the work of Kato and Masuda, we are concerned with the persistence of spatial analyticity for the solutions of (3.2.1)-(3.2.2), given initial data in analytic modified Gevrey space $H^{\sigma,s}(\mathbb{R})$ introduced by Foias and Temam [42]. We focus on the situation where we consider a real-analytic initial data with uniform radius of analyticity $\sigma_0 > 0$, so there is a holomorphic extension to a complex strip

$$S_{\sigma_0} = \{x + iy : x, y \in \mathbb{R} \quad |y| < \sigma_0\}.$$

We need to show whether analyticity of the initial data can continue or not to a solution at all later time $t$ in complex strip $S_{\sigma(t)}$. We examine the well-posedness result for (3.2.1)-(3.2.2), given data in $H^{\sigma,s}$ and analyze how $\sigma = \sigma(t)$ evolves in time. We also find the explicit algebraic lower bound for the radius of analyticity $\sigma(t)$ by applying approximate conservation law introduced in [80]. The reason for considering initial data in the space $H^{\sigma_0,s}(\mathbb{R})$ is due to the analyticity properties of modified Gevrey functions. A function in $H^{\sigma,s}(\mathbb{R})$ is a restriction to the real axis of a function analytic on a symmetric strip of width $2\sigma$.

## 3.3 Coupled System of Generalized BBM Equations

Consider the initial value problem for coupled system of generalized BBM equations

$$\begin{cases} u_t + u_x - u_{xxt} + \Big(P(u,v)\Big)_x = 0, \\ v_t + v_x - v_{xxt} + \Big(Q(u,v)\Big)_x = 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), \end{cases} \tag{3.3.1}$$

where $u$ and $v$ are real-valued functions of $x \in \mathbb{R}$ and $t \geq 0$ and $u_0$, $v_0$ are the initial data. $P$ and $Q$ are arbitrary homogeneous quadratic polynomials in the variables $u$ and $v$ given by

$$P(u,v) = \alpha u^2 + \beta uv + \gamma v^2,$$

$$Q(u,v) = \theta u^2 + \lambda uv + \psi v^2,$$

with real valued coefficients $\alpha, \beta, \gamma, \theta, \lambda$ and $\psi$. This type of system arises in water wave theory, climate modeling and other situations where wave propagation is important.

The energy obtained from (3.3.1) is given by

$$\mathcal{E}(u,v) := \int_{\mathbb{R}} (au^2 + buv + cv^2 + du_x^2 + eu_xv_x + fv_x^2)\, dx, \tag{3.3.2}$$

where $a, b, c, d, e$ and $f$ are real numbers that will be determined later.

Differentiating the energy of the solution $\mathcal{E}(u,v)$ in (3.3.2) with respect to the time $t$ and applying integration by parts with the assumption that $(u,v)$ is the solution of the system, $u$, $v$ and their derivatives do not make any contribution as $|x| \to \infty$, leads to

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(u,v) &= \int_{\mathbb{R}} (2auu_t + bu_tv + buv_t + 2cvv_t + 2du_xu_{xt} + eu_{xt}v_x + eu_xv_{xt} + 2fv_xv_{xt})\,dx \\
&= \int_{\mathbb{R}} (2auu_t + bu_tv + buv_t + 2cvv_t - 2duu_{xxt} - eu_{xxt}v - euv_{xxt} - 2fvv_{xxt})\,dx \\
&=2\int_{\mathbb{R}} u\,(au_t - du_{xxt})\,dx + \int_{\mathbb{R}} v\,(bu_t - eu_{xxt})\,dx \\
&\quad + \int_{\mathbb{R}} u\,(bv_t - euv_{xxt})\,dx + 2\int_{\mathbb{R}} v\,(cv_t - fv_{xxt})\,dx \\
&=2\int_{\mathbb{R}} (a-d)uu_tdx - 2d\int_{\mathbb{R}} uP_xdx + \int_{\mathbb{R}} (b-e)vu_tdx - e\int_{\mathbb{R}} vP_xdx \\
&\quad + \int_{\mathbb{R}} (b-e)uv_tdx - e\int_{\mathbb{R}} uQ_xdx + 2\int_{\mathbb{R}} (c-f)vv_tdx - 2f\int_{\mathbb{R}} vQ_xdx.
\end{aligned}$$

For simplicity, assume $a = d$, $b = e$ and $c = f$. Expanding the spacial derivative of quadratic polynomials $P$ and $Q$ the terms in the sum above may be rewritten in the form

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(u,v) = &- 2a\int_{\mathbb{R}} \left(2\alpha u^2u_x + \beta uvu_x + \beta u^2v_x + 2\gamma uvv_x\right)dx \\
&- b\int_{\mathbb{R}} \left(2\alpha uvu_x + \beta uvv_x + \beta v^2u_x + 2\gamma v^2v_x\right)dx \\
&- b\int_{\mathbb{R}} \left(2\theta u^2u_x + \lambda u^2v_x + \lambda uvu_x + 2\psi uvv_x\right)dx \\
&- 2c\int_{\mathbb{R}} \left(2\theta uvu_x + \lambda v^2u_x + \lambda uvv_x + 2\psi v^2v_x\right)dx \\
= &\int_{\mathbb{R}} (-4\alpha a - 2\theta b)u^2u_xdx + \int_{\mathbb{R}} (2\gamma b - 4\psi c)v^2v_xdx \\
&- \int_{\mathbb{R}} \left[(2\beta a + \lambda b + 2\alpha b + 4\theta c)uvu_x + (2\beta a + \lambda b)u^2v_x\right]dx \\
&- \int_{\mathbb{R}} \left[(\beta b + 2\lambda c + 4\gamma a + 2\psi b)uvv_x + (\beta b + 2\lambda c)v^2u_x\right]dx.
\end{aligned}$$

In the fifth line, the two integrals tends to zero at infinity ($|x| \to \infty$), for smooth solutions without further assumptions. The last two integrals in the sum would vanish, if and only if the following hold.

$$2\beta a + \lambda b = 2\alpha b + 4\theta c,$$
$$\beta b + 2\lambda c = 4\gamma a + 2\psi b.$$

Rearrange these equations

$$\begin{cases} 2\beta a + (\lambda - 2\alpha)b - 4\theta c = 0, \\ 4\gamma a + (2\psi - \beta)b - 2\lambda c = 0. \end{cases} \tag{3.3.3}$$

Thus, $a, b$ and $c$ must solve the system (3.3.3). For the solution {(a,b,c)} of (3.3.3), the time derivative of the energy $\mathcal{E}(u, v)$ is zero $\left(\frac{d}{dt}\mathcal{E}(u, v) = 0\right)$, which implies the energy of the solutions is conserved. This system always has a non-trivial solution.

Set, the coefficient matrix of the system (3.3.3) by

$$S = \begin{pmatrix} 2\beta & \lambda - 2\alpha & -4\theta \\ 4\gamma & 2\psi - \beta & -2\lambda \end{pmatrix}.$$

Then, for $s \geq 0$, the BBM system (3.3.1) with initial data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ has global solutions in the following two cases (see, [49])
The first case, if rank S = 2 and

$$\begin{aligned} 4ac =& \beta\lambda(\lambda - 2\alpha)(\beta - 2\psi) + 2\gamma\lambda(\lambda - 2\alpha)^2 + 2\beta\theta(\beta - 2\psi)^2 \\ &+ 4\gamma\theta(\lambda - 2\alpha)(\beta - 2\psi) \\ =& (\beta\lambda + 4\gamma\theta)(\lambda - 2\alpha)(\beta - 2\psi) + 2\gamma\lambda(\lambda - 2\alpha)^2 + 2\beta\theta(\beta - 2\psi)^2 \\ &> (\beta\lambda - 4\gamma\theta)^2 = b^2, \end{aligned}$$

which implies $4ac - b^2 > 0$.
The second case, if rank S = 1 and either

$$(\lambda - 2\alpha)^2 + 8\beta\theta \geq 0,$$

or

$$(2\psi - \beta)^2 + 8\gamma\lambda \geq 0.$$

If rank $S = 2$, then $a, b, c$ are given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda(\lambda - 2\alpha) + 2\theta(\beta - 2\psi) \\ -2\beta\lambda + 8\gamma\theta \\ \beta(\beta - 2\psi) + 2\gamma(\lambda - 2\alpha) \end{pmatrix}.$$

The global well- posedness result of (3.3.1) depends on the solution {(a,b,c)} of the system (3.3.3) [49].

A few particular examples of the choices of coefficients are given below.
If $\beta = \gamma = \theta = \lambda = 0$ and $\alpha, \psi \neq 0$, then both quantities $(\lambda - 2\alpha)^2 + 8\beta\theta$ and $(2\psi - \beta)^2 + 8\gamma\lambda$ are strictly positive, and the IVP (3.3.1) is decoupled to BBM equations.

$$
\begin{aligned}
&u_t + u_x - u_{xxt} + \alpha(u^2)_x = 0, \\
&v_t + v_x - v_{xxt} + \psi(v^2)_x = 0.
\end{aligned}
.
$$

By choosing $a = 1$, $b = 0$ and $c = 1$, one obtains a time invariant energy $\mathcal{E}(x,t)$ under the flow generated by (3.3.1). In this case, the local and global well-posedness results hold true.
If $\alpha = \frac{1}{2}, \beta = \gamma = \theta = \psi = 0$, and $\lambda = 1$, then both quantities $(\lambda - 2\alpha)^2 + 8\beta\theta$ and $(2\psi - \beta)^2 + 8\gamma\lambda$ are equal to zero. The IVP (3.3.1) reduced to

$$
\begin{cases}
u_t + u_x - u_{xxt} + uu_x = 0, \\
v_t + v_x - v_{xxt} + (uv)_x = 0, \\
u(x,0) = u_0(x), v(x,0) = v_0(x),
\end{cases}
$$

for which the local and global well-posedness results hold true.
If $\alpha = \frac{1}{2}, \beta = \theta = \psi = 0, \gamma = -\frac{1}{2}$ and $\lambda = 1$, then $(\lambda - 2\alpha)^2 + 8\beta\theta$ and $(2\psi - \beta)^2 + 8\gamma\lambda$ are negative. The IVP (3.3.1) becomes

$$
\begin{cases}
u_t + u_x - u_{xxt} + uu_x - vv_x = 0, \\
v_t + v_x - v_{xxt} + (uv)_x = 0, \\
u(x,0) = u_0(x), v(x,0) = v_0(x),
\end{cases}
$$

which is not globally well-posed.

Bona et al.[34] improved the global existence results obtained by Ash et al.[7] for system of KdV equations with quadratic nonlinearities. They established conditions on the coefficients of the quadratic nonlinear terms so that the problem is globally well-posed in the $L^2$-based Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s > -3/4$. In [34], the authors also improved the global existence results of the Gear–Grimshaw system [43] and the Majda–Biello system [66]. The global well-posedness of the BBM equation in Sobolev space $H^s(\mathbb{R})$ for $s \geq 0$ was studied in [13]. In [16], the authors studied the well-posedness results for generalized BBM-type equations in $L^p$ spaces.

The local well-posedness of (3.3.1) does not depend on the coefficients $\alpha$, $\beta$, $\gamma$, $\theta$, $\lambda$ and $\psi$, but, global in time well-posedness depends on the choices of $\alpha, \beta, \gamma, \theta, \lambda$

and $\psi$. The local well-posedness of (3.3.1) was proved in [49] in the Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 0$. The authors established conditions on the coefficients $\alpha$, $\beta$, $\gamma$, $\theta$, $\lambda$ and $\psi$ so that (3.3.1) is globally well-posed in the $L^2$-based Sobolev space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 0$. The time derivative of $\mathcal{E}(u, v)$ is zero, for nontrivial solutions $\{(a, b, c)\}$ of the system (3.3.3) for $4ac - b^2 > 0$, with assumption $a = d, b = e$ and $c = f$, ( see, [49]). The invariants of $\mathcal{E}(u, v)$ was used in [33, 49] to prove global well-posedness of (3.3.1) in the space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 0$, with different choice of the coefficients $\alpha, \beta, \gamma, \theta, \lambda$ and $\psi$.

The persistence of spatial analyticity for the solution of nonlinear evolution PDEs introduced in [58] applied to numerous single nonlinear dispersive PDEs, including the Kadomtsev-Petviashvili equation, KdV type equations, BBM equation, Schrödinger equation, and Klein-Gordon equation, among others. Moreover, there have been research on the spatial analyticity of the solutions of nonlinear dispersive systems, such as the Dirac-Klein-Gordon system in $1d$ and $2d$ [78, 81]. However, to the best of our knowledge, the persistence of spatial analyticity of the solution for nonlinear dispersive systems is not being looked into anymore. Thus, our focus is on the persistence of spatial analyticity to the solutions of coupled system of generalized BBM equations, given initial data in modified Gevrey space $H^{\sigma,s}(\mathbb{R}) \times H^{\sigma,s}(\mathbb{R})$.

## 3.4 KP-BBM Equation

Now, consider the IVP associated with KP-BBM II equation, which was derived by Wazwaz in [90]

$$\begin{cases} u_t - u_{txx} + u_x + uu_x + \partial_x^{-1} u_{yy} = 0, \\ u(x, y, 0) = u_0(x, y) \in H^{\sigma_1,\sigma_2}(\mathbb{R}^2), \end{cases} \tag{3.4.1}$$

where $u = u(x, y, t)$ and $(x, y, t) \in \mathbb{R}^3$.

The energy obtained from (3.4.1) is conserved, that is

$$\mathcal{A}[u(x, y, t)] = \int_{\mathbb{R}^2} \Big(u^2(x, y, t) + u_x^2(x, y, t)\Big) dxdy = \mathcal{A}[u(x, y, 0)], \tag{3.4.2}$$

for all $t \in \mathbb{R}$.

The well-posedness of (3.4.1) were studied in [23]. It has been proven that (3.4.1) can be solved by iteration, yielding to local and global well-posedness results. Saut and Tzvetkov in [75] proved the global well-posedness results in Sobolev space $H^s(\mathbb{R}^2)$ by using the conservation law in (3.4.2).

The well-posedness of (3.4.1) in Sobolev space $H^s(\mathbb{R}^2)$ is well developed, but to the best of our knowledge the radius of analyticity of the solution is yet not studied. Thus, the main concern is to study the persistent of spacial analyticity of the solution $u(x, y, t)$ to (3.4.1), given a real analytic initial data $u_0(x, y)$ in anisotropic modified Gevrey spaces $H^{\sigma_1,\sigma_2}(\mathbb{R}^2)$ with uniform radius of analyticity $\sigma_0$, so that there is a holomorphic extension to a complex strip

$$S_{\sigma_0} = \{x + iy \in \mathbb{C} : |y| < \sigma_0\}.$$

The approach we used was first introduced by Selberg and Tesfahun [81] in the context of the Dirac-Klein-Gordon equations, which is based on an approximate conservation laws and Bourgan's Fourier restriction technique.

In applications of PDEs to physical problems, the dependent variable $u$ is usually real-valued. However, for several reasons, complex-valued solutions have attracted interest lately. It should be noted that there are situations where analytic solutions emanate from non-analytic initial data [36, 57].

In [57], it is proved that for the KdV equation, a certain class of initial data with a single point singularity yields analytic solutions. However, these results do not produce explicit estimates on a radius $\sigma$ of spatial analyticity of solutions. On the other hand, if the initial datum $u_0$ is analytic in a symmetric strip around the real axis, it has recently been established that the solution will remain analytic in the same strip at least for a small time interval [47].

Jean Bourgain has been the first to observe the local smoothing effect related to the bilinear estimate and establish the well-posedness result for low regularity. In [25], he showed global well-posedness for initial data in $H^s$ for $s \geq 0$. More precisely, this has been the first well-posedness result in $H^s$ with $s < 3/2$ for periodic KdV equation. This result improved in [61] and obtained local well-posedness results in $H^s$ for $s > -1/2$. Well-posedness for the non-periodic gKdV equation in spaces of analytic functions, $G^{\sigma,s}(\mathbb{R})$ has been proved by Gruji´c and Kalisch [47]. They showed that for given real initial data that are analytic in a symmetric strip $S_{\sigma(t)} = \{z = x + iy : |y| < \sigma\}$ in the complex plane of width $2\sigma$, there exists a time $T$ such that the corresponding KdV solution is analytic in the same strip in the time interval [0,T].

Generally, in this dissertation, we deal with the nonlinear evolution dispersive PDEs discussed above, with real analytic initial data at $t = 0$, if this data has a uniform radius of analyticity $\sigma_0$, in the sense that there exists a holomorphic extension to the complex strip of width $\sigma_0$, then we ask whether the solution at some later time $t > 0$ also has a uniform radius of analyticity $\sigma = \sigma(t) > 0$, in which case we have an explicit lower bound for the radius $\sigma(t)$.

In particular, our interest is focused on the solutions of higher order KdV-BBM type equations, coupled system of generalized BBM equations and KP-BBM equations which admit an extension as an analytic function to a complex strip $S_{\sigma(t)} = \{x + iy : x, y \in \mathbb{R}, |y| < \sigma\}$ at least for small values of $\sigma$. It also focused on studying the asymptotic property of the width $\sigma$ of the strip of analyticity for large $t$, assuming that a certain Sobolev norm of the solution remains finite.

The main tools in the proof of our results are contraction mapping principle, multilinear estimates and approximate conservation law in Gevrey spaces and modified Gevrey spaces.

# Chapter 4

# Fifth Order KdV-BBM Equation

In this chapter, we study the well-posedness of the initial value problem (IVP) associated with the fifth order KdV-BBM equation. We prove the local well-posedness in the space of the analytic functions. We also analyze the evolution of radius of analyticity of the solution in analytic Gevrey space.

## 4.1 Problem Statement

Consider initial value problem associated with the fifth order KdV-BBM equation, given by

$$\begin{cases} u_t + u_x - \gamma_1 u_{xxt} + \gamma_2 u_{xxx} + \delta_1 u_{xxxxt} + \delta_2 u_{xxxxx} + \frac{3}{2} u u_x + \gamma (u^2)_{xxx} \\ -\frac{7}{48}(u_x^2)_x - \frac{1}{8}(u^3)_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{4.1.1}$$

where the unknown function $u : \mathbb{R}^{1+1} \to \mathbb{R}$. The parameters appearing in (4.1.1) satisfy

$$\gamma_1 + \gamma_2 = \frac{1}{6},$$

$$\gamma = \frac{1}{24}(5 - 18\gamma_1),$$

$$\delta_2 - \delta_1 = \frac{19}{360} - \frac{1}{6}\gamma_1,$$

$$\gamma_1 + \delta_1 + \gamma_2 + \delta_2 = \frac{1}{3},$$

with $\gamma_1, \delta_1 > 0$.

The propagation of unidirectional water waves is described by a higher order model, equation (4.1.1). It was recently introduced by Bona et al. [14] by using the second order approximation in the two-way model, the so -called abcd - system derived in [18, 19].

For $\gamma = \frac{7}{48}$, a solution to (4.1.1) satisfies the the following conservation of energy [14]

$$\mathcal{E}[u(t)] = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + \gamma_1 u_x^2 + \delta_1 u_{xx}^2\right) dx = \mathcal{E}[u(0)], \forall t \in \mathbb{R}. \tag{4.1.2}$$

The main result of this work gives an algebraic lower bound on the radius of analyticity $\sigma(t)$ of the solution as the time $t$ tends to infinity. More precisely, we have the following global well-posedness result.

**Theorem 4.1.1.** *Assume that $\gamma_1, \delta_1 > 0$ and $\gamma = \frac{7}{48}$. Suppose that $u$ is the global solution of (4.1.1) with $u_0 \in G^{\sigma_0,2}(\mathbb{R})$ for $\sigma_0 > 0$ . Then*

$$u(t) \in G^{\sigma(t),2}(\mathbb{R}), \quad \forall t > 0,$$

*with the radius of analyticity $\sigma(t)$ satisfying the asymptotic lower bound*

$$\sigma(t) \geq \frac{c}{t} \quad \text{as} \quad t \to +\infty,$$

*where $c > 0$ is a constant depending on $\|u_0\|_{G^{\sigma_0,2}(\mathbb{R})}$.*

Thus, the solution $u(t)$ is analytic in the strip $S_{\sigma(t)}$ at any time $t$.

The first step in the proof of Theorem 4.1.1 is to show that in a short time interval $0 \leq t \leq T$, where $T > 0$ depends on the norm of the initial data, the radius of analyticity remains strictly positive. This is proved by a contraction argument and a multilinear estimate which will be given in the next section.

The next step is to improve the control on the growth of the solution in the time interval $[0, T]$. The approximate conservation law will allow us to iterate the local result and prove Theorem 4.1.1.

## 4.2 Multilinear Estimates

Various multilinear estimates are now established that will be useful in the proof of the local well-posedness results. We start by writing (4.1.1) in an equivalent integral equation. Taking the Fourier transform of the first equation in (4.1.1) with respect to the spatial variable and rearranging terms gives

$$(1+\gamma_1\xi^2+\delta_1\xi^4)i\widehat{u_t} = \xi(1-\gamma_2\xi^2+\delta_2\xi^4)\widehat{u}+\frac{1}{4}(3\xi-4\gamma\xi^3)\widehat{u^2}-\frac{1}{8}\xi\widehat{u^3}-\frac{7}{48}\xi\widehat{u_x^2}. \tag{4.2.1}$$

The fourth-order polynomial

$$\varphi(\xi) = 1 + \gamma_1\xi^2 + \delta_1\xi^4,$$

is strictly positive because $\gamma_1$, and $\delta_1$ are taken to be positive.

Now, we define the Fourier multiplier operators $\phi(D_x)$, $\psi(D_x)$ and $\tau(D_x)$ as follow

$$\begin{aligned} \phi(D_x)f(\xi) &= \mathcal{F}^{-1}\Big(\phi(\xi)\widehat{f}(\xi)\Big), \\ \psi(D_x)f(\xi) &= \mathcal{F}^{-1}\Big(\psi(\xi)\widehat{f}(\xi)\Big), \\ \tau(D_x)f(\xi) &= \mathcal{F}^{-1}\Big(\tau(\xi)\widehat{f}(\xi)\Big), \end{aligned} \tag{4.2.2}$$

where the Fourier symbols are given by

$$\phi(\xi) = \frac{\xi(1-\gamma_2\xi^2+\delta_2\xi^4)}{\varphi(\xi)}, \quad \psi(\xi) = \frac{\xi}{\varphi(\xi)}, \quad \tau(\xi) = \frac{3\xi - 4\gamma\xi^3}{4\varphi(\xi)}.$$

With this notation, (4.1.1) can be rewritten as

$$\begin{cases} iu_t = \phi(D_x)u + \tau(D_x)u^2 - \frac{1}{8}\psi(D_x)u^3 - \frac{7}{48}\psi(D_x)u_x^2, \\ u(x,0) = u_0(x). \end{cases} \tag{4.2.3}$$

Consider first the following linear IVP associated to (4.2.3)

$$\begin{cases} iu_t = \phi(D_x)u, \\ u(x,0) = u_0(x), \end{cases} \tag{4.2.4}$$

whose solution is given by

$$u(t) = S(t)u_0,$$

where

$$\widehat{S(t)u_0} = e^{-i\phi(\xi)t}\widehat{u_0},$$

is defined via its Fourier transform. S(t) is a unitary operator on $H^s(\mathbb{R})$ and $G^{\sigma,s}(\mathbb{R})$ for any $s \in \mathbb{R}$, since the modulus of $e^{-i\phi(\xi)t}$ equal to one, so that

$$\|S(t)u_0\|_{H^s(\mathbb{R})} = \|u_0\|_{H^s(\mathbb{R})}, \qquad \|S(t)u_0\|_{G^{\sigma,s}(\mathbb{R})} = \|u_0\|_{G^{\sigma,s}(\mathbb{R})}, \forall t > 0. \tag{4.2.5}$$

Duhamel's formula allows us to rewrite (4.2.3) in an equivalent integral equation of the form,

$$u(x,t) = S(t)u_0 - i\int_0^t S(t-t^{'})\big(\tau(D_x)u^2 - \frac{1}{8}\psi(D_x)u^3 - \frac{7}{48}\psi(D_x)u_x^2\big)(x,t^{'})\,dt^{'}. \tag{4.2.6}$$

The following Lemmas gives multilinear estimate for nonlinear terms in the above integral

**Lemma 4.2.1.** *For $s \geq 0$, there is a constant $C = C_s$ for which*

$$\|\omega(D_x)(uv)\|_{G^{\sigma,s}} \leq C\|u\|_{G^{\sigma,s}}\|v\|_{G^{\sigma,s}}, \tag{4.2.7}$$

*where $\omega(D_x)$ is the Fourier multiplier operator with symbol*

$$\omega(\xi) = \frac{|\xi|}{1+\xi^2}.$$

***Proof:*** Using the definition of the $G^{\sigma,s}$-norm and convolution of functions, one can obtain

$$\begin{aligned} \|\omega(D_x)(uv)\|^2_{G^{\sigma,s}} &= \|\langle\xi\rangle^s e^{\sigma\langle\xi\rangle}\omega\widehat{(D_x)(uv)}\|^2_{L^2} \\ &= \|\langle\xi\rangle^s e^{\sigma\langle\xi\rangle}\omega(\xi)\widehat{u} * \widehat{v}(\xi)\|^2_{L^2} \\ &= \int_{\mathbb{R}} \langle\xi\rangle^{2s} e^{2\sigma\langle\xi\rangle}\frac{\xi^2}{(1+\xi^2)^2}\Big(\int_{\mathbb{R}} \widehat{u}(\xi-\xi_1)\widehat{v}(\xi_1)d\xi_1\Big)^2 d\xi. \end{aligned} \tag{4.2.8}$$

Now, for $s \geq 0$, we have

$$\langle\xi\rangle^s \leq \langle\xi-\xi_1\rangle^s\langle\xi_1\rangle^s,$$

and

$$e^{\sigma\langle\xi\rangle} \leq e^{\sigma\langle\xi-\xi_1\rangle}e^{\sigma\langle\xi_1\rangle}.$$

Using these facts from (4.2.8), we get

$$\|\omega(D_x)(uv)\|^2_{G^{\sigma,s}} \leq \int_{\mathbb{R}} \frac{\xi^2}{(1+\xi^2)^2}\Big(\int_{\mathbb{R}} \langle\xi-\xi_1\rangle^{2s}e^{2\sigma\langle\xi-\xi_1\rangle}\widehat{u}(\xi-\xi_1)\langle\xi_1\rangle^{2s}e^{2\sigma\langle\xi_1\rangle}\widehat{v}(\xi_1)d\xi_1\Big)^2 d\xi. \tag{4.2.9}$$

Then, using $\frac{\xi^2}{(1+\xi^2)^2} \leq \frac{1}{1+\xi^2}$, the Cauchy-Schwartz inequality and the definition of the $G^{\sigma,s}$ norm, we obtain from (4.2.9) that

$$\begin{aligned} \|\omega(D_x)(uv)\|^2_{G^{\sigma,s}} &\leq \int_{\mathbb{R}} \frac{1}{(1+\xi^2)}d\xi\|u\|^2_{G^{\sigma,s}}\|v\|^2_{G^{\sigma,s}} \\ &\leq C\|u\|^2_{G^{\sigma,s}}\|v\|^2_{G^{\sigma,s}}, \end{aligned} \tag{4.2.10}$$

and this completes the proof of Lemma 4.2.1. □

**Lemma 4.2.2.** *For any $s \geq 0$ and $\sigma > 0$, there is a constant $C = C_s$ such that the inequality*

$$\|\tau(D_x)u^2\|_{G^{\sigma,s}} \leq C\|u\|^2_{G^{\sigma,s}} \tag{4.2.11}$$

*holds, where the operator $\tau(D_x)$ is defined as in (4.2.2).*

***Proof:*** Since $\delta_1 > 0$, one can easily verify that $|\tau(\xi)| \leq \omega(\xi)$ for some constant $C > 0$. Using this fact, definition of the $G^{\sigma,s}$ norm and the estimate (4.2.7), one can obtain

$$\begin{aligned}
\|\tau(D_x)u^2\|_{G^{\sigma,s}} &= \|\langle\xi\rangle^s e^{\sigma\langle\xi\rangle}\widehat{\tau(D_x)u^2}\|_{L^2} \\
&\leq \|\langle\xi\rangle^s e^{\sigma\langle\xi\rangle}\tau(\xi)\widehat{u} * \widehat{u}\|_{L^2} \\
&\leq \|\langle\xi\rangle^s e^{\sigma\langle\xi\rangle}\omega(\xi)\widehat{u} * \widehat{u}(\xi)\|_{L^2} \\
&\leq \|u\|^2_{G^{\sigma,s}}.
\end{aligned}$$

**Lemma 4.2.3.** *For $s \geq \frac{1}{6}$, there is a constant $C = C(s)$ such that*

$$\|\psi(D_x)u^3\|_{G^{\sigma,s}} \leq C\|u\|^3_{G^{\sigma,s}}. \tag{4.2.12}$$

***Proof:*** Consider first the case $\frac{1}{6} \leq s < \frac{5}{2}$, In this case, it appears that

$$|(1+|\xi|)^s\psi(\xi)| = \left|\frac{(1+|\xi|)^s\xi}{(1+\gamma_1\xi^3+\delta_1\xi^4)}\right| \leq C\frac{1}{(1+|\xi|)^{3-s}}.$$

From the definition of Gevrey space norm and the last inequality, we have

$$\begin{aligned}
\|\psi(D_x)u^3\|_{G^{\sigma,s}} &= \left\|(1+|\xi|)^s\psi(\xi)e^{\sigma\langle\xi\rangle}\widehat{u^3}(\xi)\right\|_{L^2} \\
&\leq C\left\|\frac{1}{(1+|\xi|)^{3-s}}e^{\sigma\langle\xi\rangle}\widehat{u^3}(\xi)\right\|_{L^2} \\
&\leq C\left\|\frac{1}{(1+|\xi|)^{3-s}}\right\|_{L^2}\left\|e^{\sigma\langle\xi\rangle}\widehat{u^3}(\xi)\right\|_{L^\infty}.
\end{aligned} \tag{4.2.13}$$

Set

$$\widehat{f}(\xi) = e^{\sigma\langle\xi\rangle}\widehat{u}(\xi).$$

Using $e^{\sigma\langle\xi\rangle} \leq e^{\sigma\langle\xi-\xi_1-\xi_2\rangle}e^{\sigma\langle\xi_1\rangle}e^{\sigma\langle\xi_2\rangle}$, we get

$$e^{\sigma\langle\xi\rangle}\widehat{u^3}(\xi) \leq \int_{\mathbb{R}^2} e^{\sigma\langle\xi-\xi_1-\xi_2\rangle}\widehat{u}(\xi-\xi_1-\xi_2)e^{\sigma\langle\xi_1\rangle}\widehat{u_1}e^{\sigma\langle\xi_2\rangle}\widehat{u_2}d\xi_1 d\xi_2 = \widehat{f^3}(\xi). \tag{4.2.14}$$

By (4.2.13) and the fact that $\left\|\frac{1}{(1+|\xi|)^{3-s}}\right\|_{L^2}$ is bounded for $s < \frac{5}{2}$, we obtain from (4.2.14) that

$$\|\psi(D_x)u^3\|_{G^{\sigma,s}} \leq \|\widehat{f^3}(\xi)\|_{L^\infty} \leq \|f\|^3_{L^3}. \tag{4.2.15}$$

From one dimensional Sobolev embedding, we have

$$\|f\|_{L^3} \leq C\|f\|_{H^{\frac{1}{6}}} = C\|u\|_{G^{\sigma,s}}. \tag{4.2.16}$$

Therefore, for $\frac{1}{6} \leq s < \frac{5}{2}$, from (4.2.15) and (4.2.16), we obtain

$$\|\psi(D_x)u^3\|_{G^{\sigma,s}} \leq \|u\|^3_{G^{\sigma,s}}. \tag{4.2.17}$$

For the case $s \geq \frac{5}{2}$, we observe that $G^{\sigma,s}$ is a Banach algebra. Also, note that $|\psi(\xi)| \leq C\frac{|\xi|}{1+\xi^2}$. So, using the same procedure as in Lemma 4.2.2, we obtain

$$\|\psi(D_x)uu^2\|_{G^{\sigma,s}} \leq C\|u\|_{G^{\sigma,s}}\|u^2\|_{G^{\sigma,s}} \leq C\|u\|^3_{G^{\sigma,s}}.$$

**Lemma 4.2.4.** *For $s \geq 1$, the following inequality holds*

$$\|\psi(D_x)u_x^2\|_{G^{\sigma,s}} \leq C\|u\|^2_{G^{\sigma,s}}. \tag{4.2.18}$$

***Proof:*** Observe that

$$\psi(\xi) \leq C\omega(\xi)\frac{1}{1+|\xi|}.$$

Since $s-1 \geq 0$, the inequality (4.2.7) allows the conclusion

$$\|\psi(D_x)u_x^2\|_{G^{\sigma,s}} \leq C\|\omega(D_x)u_x^2\|_{G^{\sigma,s-1}} \leq C\|u_x\|_{G^{\sigma,s-1}}\|u_x\|_{G^{\sigma,s-1}} \leq \|u\|^2_{G^{\sigma,s}}.$$

## 4.3 Local Well-posedness Result in Gevrey Space

In what follows, we use the multilinear estimates in section 4.2 to prove local well-posedness result in the $G^{\sigma_0,s}$ space for $s \geq 1$.

**Theorem 4.3.1.** *Let $s \geq 1$, $\sigma_0 > 0$ and $u_0 \in G^{\sigma_0,s}(\mathbb{R})$ be given. Then there exist a time $T = T(\|u_0\|)_{G^{\sigma_0,s}} > 0$ and a unique solution*

$$u \in C([0,T]; G^{\sigma_0,s}),$$

*satisfying (4.1.1), and we have*

$$T \sim (1+\|u_0\|)^{-2}_{G^{\sigma_0,s}}. \tag{4.3.1}$$

*Moreover,*

$$\|u\|_{L^\infty_T G^{\sigma_0,s}} \lesssim \|u_0\|_{G^{\sigma_0,s}}. \tag{4.3.2}$$

*Here we use the notation*

$$L^\infty_T G^{\sigma_0,s} = L^\infty_t G^{\sigma_0,s}\big([0,T]\times\mathbb{R}\big).$$

***Proof:*** Taking into account the Duhamel's formula (4.2.6), we define a mapping

$$\Psi u(x,t) = S(t)u_0 - i\int_0^t S(t-t^{'})\Big(\tau(D_x)u^2 - \frac{1}{8}\psi(D_x)u^3 - \frac{7}{48}\psi(D_x)u_x^2\Big)(x,t^{'})\,dt'. \tag{4.3.3}$$

Now, we choose the Banach space

$$X = C([0,T];G^{\sigma,s}).$$

equipped with norm

$$\|u\|_X = \sup_{0\le t\le T}\|u(t)\|_{G^{\sigma,s}}$$

where T is to be determined later.

The strategy is to prove that $\Psi$ is a contraction map in the space X for T sufficiently small. To this end, consider the ball $B_r$ of radius $r$ in X:

$$B_r = \{u\in X : \|u\|_X \le r\}.$$

Then, $B_r$ is a closed subset of the Banach space $X$ and hence it is a Banach space. The goal is to prove the following

(1) $\Psi$ maps $B_r$ into $B_r$ i.e,

$$\|\Psi(u)\|_X \le r, \quad u\in X,$$

(2) $\Psi$ is a contraction map i.e,

$$\|\Psi(u)-\Psi(\mu)\|_X \le \theta\|u-\mu\|_X,$$

for all $u$,$\mu \in B_r$ and some $\theta\in[0,1)$.

From (4.2.5), we know that S(t) is a unitary group in $G^{\sigma,s}(\mathbb{R})$. Using this fact, we obtain

$$\|\Psi u\|_{G^{\sigma,s}} = \|u_0\|_{G^{\sigma,s}} + CT\Big\|\tau(D_x)u^2 - \frac{1}{8}\psi(D_x)u^3 - \frac{7}{48}\psi(D_x)u_x^2\Big\|_X. \tag{4.3.4}$$

In view of the inequalities (4.2.11), (4.2.12) and (4.2.18), we obtain from (4.3.4) that

$$\|\Psi u\|_{G^{\sigma,s}} \le \|u_0\|_{G^{\sigma,s}} + CT\Big[\|u\|_X^2 + \|u\|_X^3 + \|u\|_X^2\Big]. \tag{4.3.5}$$

Now, consider $u\in B_r$, then (4.3.5) yields

$$\|\Psi u\|_{G^{\sigma,s}} \le \|u_0\|_{G^{\sigma,s}} + CT[2r + r^2]r.$$

If we choose

$$r = 2\|u_0\|_{G^{\sigma,s}}, \quad T = \frac{1}{2Cr(2+r)},$$

then $\|\Psi u\|_{G^{\sigma,s}} \leq r$, showing that $\Psi$ maps the closed ball $B_r$ onto itself.
With the same choice of $r$ and $T$ and the same estimates, one can easily show that $\Psi$ is a contraction map on $B_r$
Let $u$, $\mu \in X = C([0,T] : G^{\sigma,s})$. Then

$$\begin{aligned}
\|\Psi u - \Psi \mu\|_{G^{\sigma,s}} &\leq CT\Big[\|u-\mu\|_X\|u+\mu\| + \|u^2 + u\mu + \mu^2\|_X\Big] \\
&\leq CT\|u-\mu\|_X(2r + r^2) \\
&\leq \frac{1}{2}\|u-\mu\|_X.
\end{aligned}$$

Thus, $\Psi$ is a contraction map on $B_r$ with a contraction constant $\frac{1}{2}$. So by contraction mapping principle (4.1.1) has a unique solution. □

*Remark* 4.3.2. The following properties follow immediately from the proof of the Theorem 4.1.1:

(1) The maximal existence time $T^*$ of the solution satisfies

$$T^* \geq T = \frac{1}{8C_s\|u_0\|_{G^{\sigma,s}}(1 + \|u_0\|_{G^{\sigma,s}})}, \tag{4.3.6}$$

where the constant $C_s$ depends only on $s$.

(2) The solution can not grow too much on the interval $[0, T]$ since

$$\|u(\cdot, t)\|_{G^{\sigma,s}} \leq r = 2\|u_0\|_{G^{\sigma,s}}, \tag{4.3.7}$$

for $t$ in this interval, where $T$ is as above in (4.3.1).

## 4.4 Evolution of Radius of Analyticity

Evolution of radius of analyticity deals with how the size of the region where a function is analytic changes over time. The radius of analyticity of the solution of PDEs is a measure of how smooth the solution is in the complex plane. It is defined as the largest radius of a disk centered at a point where the solution is analytic. The radius of analyticity can depend on both the initial data and the time evolution of the solution.The radius of analyticity can be used to study the regularity and stability of the solutions of PDEs, that arise from physical phenomena, such as fluid dynamics, wave propagation, heat conduction, etc.

One of the interesting problem is to investigate how the radius of analyticity evolves in time for different initial data. If the radius of analyticity shrinks to zero in finite time, then the solution becomes singular and loses its physical meaning. On the other hand, if the radius of analyticity remains positive for all time, then the solution stays smooth and well-defined.

Almost conservation law will allow us to repeat the local result on successive short-time intervals to reach any target time $T^* > 0$, by adjusting the strip width parameter $\sigma$ according to the size of $T^*$.

**Lemma 4.4.1** ([79]). *Let $0 \le \rho \le 1$, $\sigma > 0$ and $\alpha$ , $\beta \in \mathbb{R}$, then*

$$e^{\sigma|\alpha|}e^{\sigma|\beta|} - e^{\sigma|\alpha+\beta|} \le \Big(2\sigma \min\{|\alpha|, |\beta|\}\Big)^{\rho} e^{\sigma|\alpha|}e^{\sigma|\beta|}.$$

Let as fix $\gamma_1, \delta_1 > 0$ and $\gamma = \frac{7}{48}$ in (4.1.1) and set

$$v(x,t) := \Lambda_\sigma u(x,t),$$

where

$$\Lambda_\sigma := e^{\sigma|D_x|}.$$

Then $u(x,t) = \Lambda_{-\sigma} v(x,t)$. Note also that $v_0 := v(x,0) = \Lambda_\sigma u_0$.

Applying the operator $\Lambda_\sigma$ to the first equation of (4.1.1) gives

$$\begin{aligned} &v_t + v_x - \gamma_1 v_{xxt} + \gamma_2 v_{xxx} + \delta_1 v_{xxxxt} + \delta_2 v_{xxxxx} + \tfrac{3}{2} v v_x + \gamma (v^2)_{xxx} \\ &-\tfrac{7}{48}(v_x^2)_x - \tfrac{1}{8}(v^3)_x = N(v), \end{aligned} \tag{4.4.1}$$

where

$$N(v) = \Big(\frac{3}{4} + \gamma\partial_x^2\Big)\partial_x N_1(v) - \gamma\partial_x N_2(v) - \frac{1}{8}\partial_x N_3(v) \tag{4.4.2}$$

with

$$\begin{aligned} N_1(v) &= v^2 - \Lambda_\sigma\big[(\Lambda_{-\sigma} v)^2\big], \\ N_2(v) &= v_x^2 - \Lambda_\sigma\big[(\Lambda_{-\sigma} v_x)^2\big], \\ N_3(v) &= v^3 - \Lambda_\sigma\big[(\Lambda_{-\sigma} v)^3\big]. \end{aligned} \tag{4.4.3}$$

Define the modified energy

$$\mathcal{E}_\sigma[v(t)] = \frac{1}{2}\int_{\mathbb{R}} \Big(v^2 + \gamma_1(v_x)^2 + \delta_1(v_{xx})^2\Big)dx.$$

Observe that for $\sigma = 0$, we have $v = u$, and therefore the energy is conserved, i.e.,

$$\mathcal{E}_0[v(t)] = \mathcal{E}_0[v(0)], \quad \text{for all t.}$$

However, this fails to hold for $\sigma > 0$. In what follows we will nevertheless prove almost conservation law

$$\sup_{0\le t\le T} \mathcal{E}_\sigma[v(t)] \le \mathcal{E}_\sigma[v(0)] + \sigma C\Big(1 + \mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big)\mathcal{E}_{\sigma_0}^{\frac{3}{2}}[v(0)],$$

we have for $T$ as in Theorem 4.3.1.
Thus, in the limit as $\sigma \to 0$, we recover the conservation

$$\mathcal{E}_0[v(t)] = \mathcal{E}_0[v(0)].$$

Indeed, using integration by parts and (4.4.1)-(4.4.3),

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_\sigma[v(t)] &= \int_{\mathbb{R}} \Big(v\partial_t v + \gamma_1 \partial_x v \partial_t \partial_x v + \delta_1 \partial_x^2 v \partial_t \partial_x^2 v\Big)\, dx \\
&= \int_{\mathbb{R}} v\Big((\partial_t v - \gamma_1 \partial_t \partial_x^2 v + \delta_1 \partial_t \partial_x^4 v\Big)\, dx \\
&= -\int_{\mathbb{R}} v\Big(\partial_x v + \gamma_2 \partial_x^3 v + \delta_2 \partial_x^5 v + \frac{3}{4}\partial_x(v^2) + \gamma \partial_x^3(v^2) \\
&\quad - \gamma \partial_x (u_x)^2 - \frac{1}{8}\partial_x(v^3)\Big)\, dx + \int_{\mathbb{R}} vN(v)\, dx.
\end{aligned} \tag{4.4.4}$$

The third integral of (4.4.4) is zero due to the following identities

$$v\partial_x v = \frac{1}{2}(v)_x^2, \quad v\partial_x^3 v = (vv_{xx})_x - \frac{1}{2}(v_x^2)_x,$$

$$v\partial_x^5 v = (v\partial_x^4 v)_x - (\partial_x v \partial_x^3 v)_x + \frac{1}{2}(v_{xx}^2)_x,$$

$$v\partial_x(v^2) = \frac{2}{3}(v^3)_x, \quad v\partial_x(v^3) = \frac{3}{4}(v^4)_x,$$

and

$$v\partial_x^3(v^2) = 2(v^2 v_{xx})_x + v(v_x^2)_x.$$

Therefore,

$$\frac{d}{dt}\mathcal{E}_\sigma[v(t)] = \int_{\mathbb{R}} vN(v)\, dx. \tag{4.4.5}$$

Consequently, integrating (4.4.5) in time interval $[0, t]$ yields

$$\mathcal{E}_\sigma[v(t)] = \mathcal{E}_\sigma[v(0)] + \int_0^t \int_{\mathbb{R}} v(x, s) N(v(x, s))\, dxds. \tag{4.4.6}$$

The following Lemma gives the estimate of the inner integral in (4.4.6).

**Lemma 4.4.2.** *Let $N(v)$ is as in (4.4.2)-(4.4.3). Then we have*

$$\left| \int_{\mathbb{R}} vN(v)dx \right| \leq C\sigma \Big[ 1 + \|v\|_{H_x^2} \Big] \|v\|_{H_x^2}^3, \tag{4.4.7}$$

*for all $v \in H_x^2$.*

***Proof:*** Using (4.4.1)-(4.4.3), Plancherel Theorem and Cauchy-Schwartz inequality, we get

$$\begin{aligned}
\int_{\mathbb{R}} vN(v)\,dx &= \int_{\mathbb{R}} v\left(\frac{3}{4} + \gamma\partial_x^2\right)\partial_x N_1(v)\,dx - \gamma \int_{\mathbb{R}} v\partial_x N_2(v)\,dx - \frac{1}{8}\int_{\mathbb{R}} v\partial_x N_3(v)\,dx \\
&= \int_{\mathbb{R}} \left(\frac{3}{4} + \gamma\partial_x^2\right) v.\partial_x N_1(v)\,dx + \gamma \int_{\mathbb{R}} \partial_x v.N_2(v)\,dx + \frac{1}{8}\int_{\mathbb{R}} \partial_x v.N_3(v)\,dx \\
&\leq \left\| \left(\frac{3}{4} + \gamma\partial_x^2\right) v \right\|_{L_x^2} \|\partial_x N_1(v)\|_{L_x^2} + \gamma \|\partial_x v\|_{L_x^2} \|N_2(v)\|_{L_x^2} + \frac{1}{8}\|\partial_x v\|_{L_x^2} \|N_3(v)\|_{L_x^2} \\
&\leq \|v\|_{H_x^2} \|\partial_x N_1(v)\|_{L_x^2} + \gamma \|v\|_{H_x^1} \|N_2(v)\|_{L_x^2} + \frac{1}{8}\|v\|_{H_x^1} \|N_3(v)\|_{L_x^2} \\
&\leq \|v\|_{H_x^2} \|\partial_x N_1(v)\|_{L_x^2} + \gamma \|v\|_{H_x^2} \|N_2(v)\|_{L_x^2} + \frac{1}{8}\|v\|_{H_x^2} \|N_3(v)\|_{L_x^2}.
\end{aligned}$$

The estimate (4.4.7) follows from the following estimates

$$\|\partial_x N_1(v)\|_{L_x^2} \leq C\sigma \|v\|_{H_x^2}^2. \tag{4.4.8}$$

$$\|N_2(v)\|_{L_x^2} \leq C\sigma \|v\|_{H_x^2}^2. \tag{4.4.9}$$

$$\|N_3(v)\|_{L_x^2} \leq C\sigma \|v\|_{H_x^2}^3. \tag{4.4.10}$$

***Proof of* (4.4.8)*:*** By taking the Fourier transform of $\partial_x N_1(v)$, where $N_1(v)$ is as defined in (4.4.3), we obtain

$$\begin{aligned}
\widehat{\partial_x N_1(v)}(\xi) &= i \int_{\xi=\xi_1+\xi_2} \xi \Big( e^{\sigma(|\xi_1|+|\xi_2|)} - e^{\sigma|\xi|} \Big) \widehat{u}(\xi_1)\widehat{u}(\xi_2)\, d\xi_1 d\xi_2 \\
&= i \int_{\xi=\xi_1+\xi_2} \xi \mathcal{P}_\sigma(\xi_1,\xi_2) \widehat{v}(\xi_1)\widehat{v}(\xi_2)\, d\xi_1 d\xi_2,
\end{aligned} \tag{4.4.11}$$

where

$$\mathcal{P}_\sigma(\xi_1,\xi_2) = 1 - exp\Big[ -\sigma\Big( |\xi_1| + |\xi_2| - |\xi_1+\xi_2| \Big) \Big].$$

Since $1 - e^{-r} \leq r$, for all $r \geq 0$, we have

$$|\mathcal{P}_\sigma(\xi_1, \xi_2)| \leq \sigma[(|\xi_1| + |\xi_2|) - |\xi_1 + \xi_2|] \\ = \sigma \frac{(|\xi_1| + |\xi_2|)^2 - |\xi_1 + \xi_2|^2}{|\xi_1| + |\xi_2| + |\xi_1 + \xi_2|} \\ \leq 2\sigma \min \{|\xi_1|, |\xi_2|\}. \tag{4.4.12}$$

By symmetry, we may assume $|\xi_1| \leq |\xi_2|$. This implies that

$$|\xi| \leq 2|\xi_2|.$$

Let

$$V = \mathcal{F}_x^{-1}(|\widehat{v}|).$$

Now, using (4.4.12),from (4.4.11), we get

$$|\widehat{\partial_x N_1(v)}(\xi)| = 4\sigma \int_{\xi=\xi_1+\xi_2} |\xi_1||\widehat{v}|(\xi_1).|\xi_2||\widehat{v}|(\xi_2)\, d\xi_1 d\xi_2 \\ = 4\sigma \int_{\xi=\xi_1+\xi_2} |\xi_1|\widehat{V}(\xi_1).|\xi_2|\widehat{V}(\xi_2)\, d\xi_1 d\xi_2 \\ = 4\sigma \mathcal{F}_x\{|D_x|V.|D_x|V\}(\xi).$$

Using Plancherel Theorem, Hölder inequality and Sobolev inequality, we get

$$\|\widehat{\partial_x N_1(v)}(\xi)\|_{L^2_x} = \|\partial_x N_1(v)(\xi)\|_{L^2_x} \leq 4\sigma \||D_x|V.|D_x|V\|_{L^2_x} \\ \leq 4\sigma \||D_x|V\|_{L^2_x} \||D_x|V\|_{L^\infty_x} \\ \lesssim \sigma \|V\|^2_{H^2_x} \sim \sigma \|v\|^2_{H^2_x},$$

as desired.

***Proof of*** (4.4.9)***:*** By taking the Fourier transform of $N_2(v)$, we have

$$\widehat{N_2(v)}(\xi) = \int_{\xi=\xi_1+\xi_2} \left(e^{\sigma(|\xi_1|+|\xi_2|)} - e^{\sigma|\xi|}\right) \widehat{u_x}(\xi_1)\widehat{u_x}(\xi_2)\, d\xi_1 d\xi_2 \\ = \int_{\xi=\xi_1+\xi_2} \xi_1\xi_2 \mathcal{P}_\sigma(\xi_1, \xi_2)\widehat{v}(\xi_1)\widehat{v}(\xi_2)\, d\xi_1 d\xi_2. \tag{4.4.13}$$

By symmetry $|\xi_1| \leq |\xi_2|$, we have

$$|\xi_1\xi_2\mathcal{P}_\sigma(\xi_1, \xi_2)| \leq 2\sigma|\xi_1|^2|\xi_2|.$$

Thus, from (4.4.13), we get

$$\begin{aligned}
\big|\widehat{N_2(v)}(\xi)\big| &\le 2\sigma \int_{\xi=\xi_1+\xi_2} |\xi_1|^2 |\widehat{v}|(\xi_1)|\xi_2||\widehat{v}|(\xi_2)\, d\xi_1 d\xi_2 \\
&= 2\sigma \int_{\xi=\xi_1+\xi_2} |\xi_1|^2 \widehat{V}(\xi_1).|\xi_2|\widehat{V}(\xi_2)\, d\xi_1 d\xi_2 \\
&= 2\sigma \mathcal{F}_x\{|D_x|^2 V.|D_x|V\}(\xi).
\end{aligned}$$

By applying Plancherel Theorem, Hölder inequality and Sobolev inequality we get

$$\begin{aligned}
\|\widehat{N_2(v)}(\xi)\|_{L^2_x} = \|N_2(v)(\xi)\|_{L^2_x} &\le 2\sigma \||D_x|V.|D_x|V\|_{L^2_x} \\
&\le 2\sigma \||D_x|^2 V\|_{L^2_x} \||D_x|V\|_{L^\infty_x} \\
&\lesssim \sigma \|V\|^2_{H^2_x} \sim \sigma \|v\|^2_{H^2_x},
\end{aligned}$$

which shows (4.4.9).

***Proof of*** (4.4.10)***:*** Taking the Fourier transform of $N_3(v)$, we get

$$\begin{aligned}
\widehat{N_3(v)}(\xi) &= \int_{\xi=\xi_1+\xi_2+\xi_3} \left( e^{\sum_j^3 \sigma(|\xi_j|} - e^{\sigma|\xi|} \right) \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\, d\xi_1 d\xi_2 d\xi_3 \\
&= \int_{\xi=\xi_1+\xi_2+\xi_3} \mathcal{K}_\sigma(\xi_1,\xi_2,\xi_3)\widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3)\, d\xi_1 d\xi_2 d\xi_3,
\end{aligned} \tag{4.4.14}$$

where

$$\mathcal{K}_\sigma(\xi_1,\xi_2,\xi_3) = 1 - exp\Big[ -\sigma\Big( \sum_{j=1}^3 |\xi_j| - |\sum_{j=1}^3 \xi_j| \Big) \Big].$$

Using similar argument as (4.4.12), we estimate

$$\begin{aligned}
\mathcal{K}_\sigma(\xi_1,\xi_2,\xi_3) &\le \sigma \Big[ \sum_j^3 |\xi_j| - |\sum_{j=1}^3 \xi_j| \Big] \\
&= \sigma \frac{(\sum_j^3 |\xi_j|)^2 - |\sum_{j=1}^3 \xi_j|^2}{\sum_j^3 |\xi_j| + |\sum_{j=1}^3 \xi_j|} \\
&\le 12\sigma \,\mathrm{med}\,\big\{|\xi_1|, |\xi_2|, |\xi_3|\big\}.
\end{aligned} \tag{4.4.15}$$

where $\mathrm{med}\,\big\{|\xi_1|, |\xi_2|, |\xi_3|\big\} = |\xi_2|$

By symmetry, assume $|\xi_1| \le |\xi_2| \le |\xi_3|$. Using (4.4.15), from (4.4.14), we set

$$
\begin{aligned}
|\widehat{N_3(v)}(\xi)| &= 12\sigma \int_{\xi=\xi_1+\xi_2+\xi_3} |\widehat{v}|(\xi_1).|\xi_2||\widehat{v}|(\xi_2)|\widehat{v}|(\xi_3)\, d\xi_1 d\xi_2 d\xi_3 \\
&= 12\sigma \int_{\xi=\xi_1+\xi_2+\xi_2} \widehat{V}(\xi_1).|\xi_2|\widehat{V}(\xi_2).\widehat{V}(\xi_3)\, d\xi_1 d\xi_2 d\xi_3 \\
&= 12\sigma \mathcal{F}_x\{V.|D_x|V.V\}(\xi).
\end{aligned}
$$

Applying, Plancherel Theorem, Hölder inequality and Sobolev inequality gives

$$
\begin{aligned}
\|\widehat{N_3(v)}(\xi)\|_{L^2_x} = \|N_3(v)(\xi)\|_{L^2_x} &\leq 12\sigma\|V.|D_x|V.V\|_{L^2_x} \\
&\leq 12\sigma\|V\|_{L^\infty_x}\||D_x|V\|_{L^2_x}\|V\|_{L^\infty_x} \\
&\lesssim \sigma\|V\|^3_{H^2_x} \sim \sigma\|v\|^3_{H^2_x},
\end{aligned}
$$

which completes the proof of (4.4.10) □

Therefore, in view of (4.4.6) and (4.4.7), we have the apriori energy estimate

$$
\mathcal{E}_\sigma[v(t)] \leq \mathcal{E}_\sigma[v(0)] + \sigma T C\Big[1 + \|v\|_{L^\infty_T H^2_x}\Big]\|v\|^3_{L^\infty_T H^2_x}, \tag{4.4.16}
$$

where

$$
v \in L^\infty_T H^2_x := L^\infty_t H^2_x([0,T];\mathbb{R}).
$$

Combing the estimate in (4.4.16) with the local existence result in Theorem 4.3.1 gives an almost conservation law to the modified energy.

**Lemma 4.4.3.** *[Almost conservation law] Let $u_0 \in G^{\sigma,2}$. Suppose that $u \in C\big([0,T];G^{\sigma,2}\big)$ is the local-in-time solution to the Cauchy problem (4.1.1). Then*

$$
\sup_{0\leq t\leq T} \mathcal{E}_\sigma[v(t)] \leq \mathcal{E}_\sigma[v(0)] + C\sigma\Big(1 + \mathcal{E}_\sigma^{\frac{1}{2}}[v(0)]\Big)\mathcal{E}_\sigma^{\frac{3}{2}}[v(0)]. \tag{4.4.17}
$$

***Proof:*** By Theorem 4.3.1, we have the bound

$$
\|v\|_{L^\infty_T H^2_x} = \|u\|_{L^\infty_T G^{\sigma,2}} \leq C\|u_0\|_{G^{\sigma,2}} = C\|v_0\|_{H^2_x}. \tag{4.4.18}
$$

where T is as in (4.3.1).
On the other hand, for fixed constants $\gamma_1, \delta_1 > 0$, we have

$$
\mathcal{E}_\sigma[v(t)] = \frac{1}{2}\int_{\mathbb{R}} \Big(v_0^2 + \gamma_1(v_0')^2 + \delta_1(v_0'')^2\Big)dx \sim \|v_0\|^2_{H^2_x}. \tag{4.4.19}
$$

which implies that $\|v_0\|_{H_x^2} = \mathcal{E}_\sigma^{\frac{1}{2}}[v(t)]$.

Then, combining (4.4.18) and (4.4.19) with (4.4.16) yields the desired estimate (4.4.17). □

**Proof of Theorem 4.1.1:** In this subsection, we study the evolution of the radius of analyticity $\sigma(t)$ as the time $t$ grows. We have established the existence of local solutions, and apply the local result repeatedly using almost conservation law to cover time intervals of arbitrary length.

To prove the Theorem, first we consider the case $s = 2$, then the general case , $s \in \mathbb{R}$ will essentially reduce to the case $s = 2$.

**The Case $s = 2$:** Suppose that $u_0 = u(x,0) \in G^{\sigma_0,2}(\mathbb{R})$ for some $\sigma_0 > 0$. Then there exists a unique solution

$$u(x,t) \in \Big([0,T]; G^{\sigma_0,2}(\mathbb{R})\Big),$$

of (4.1.1) constructed in Theorem 4.3.1 with existence time $T$ as given in (4.3.1).

Note that

$$v_0 = e^{\sigma_0|D_x|}u_0 \in H_x^2.$$

and we have

$$\mathcal{E}_\sigma[v_0] \sim \|v_0\|_{H^2}^2 < \infty.$$

Now, following the argument in [84], we can construct a solution on $[0,T^*]$ for arbitrarily large time $T^*$, by applying the almost conservation law in Lemma 4.4.3, so as to repeat the local result on successive short time intervals of size $T$ to reach $T^*$ by adjusting the strip width parameter $\sigma$ according to the size of $T^*$. Doing so, we establish the bound

$$\sup_{t\in[0,T^*]} \mathcal{E}_\sigma[v(t)] \leq 2\mathcal{E}_{\sigma_0}[v(0)], \tag{4.4.20}$$

for $\sigma$ satisfying

$$\sigma(t) \geqslant \frac{c}{T^*}, \tag{4.4.21}$$

where $c > 0$ is a constant depending on $\|u_0\|_{G^{\sigma_0,2}}$ and $\sigma_0$. By Theorem 4.3.1, there is a solution $u$ to (4.1.1) satisfying

$$u(x,t) \in G^{\sigma_0,2}, \quad \forall t \in [0,T],$$

where

$$T \sim (1 + \|u_0\|)^{-2}_{G^{\sigma_0,s}}.$$

Thus, $\mathcal{E}_\sigma[v(t)] < \infty$ for $t \in [0, T^*]$, which inturn implies

$$u(x,t) \in G^{\sigma(t),2}, \quad \text{for all} \quad t \in [0, T^*].$$

It remains to prove (4.4.20). Choose $n \in \mathbb{N}$ so that $T^* \in [nT, (n+1)T]$. Using induction we can show for any $k \in \{1, 2, 3, ..., n+1\}$ that

$$\sup_{t\in[0,kT]} \mathcal{E}_\sigma[v(t)] \leq \mathcal{E}_\sigma[v(0)] + kC\sigma\mathcal{E}_{\sigma_0}^{\frac{3}{2}}[v(0)]\Big(1 + \mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big), \tag{4.4.22}$$

$$\sup_{t\in[0,kT]} \mathcal{E}_\sigma[v(t)] \leq 2\mathcal{E}_{\sigma_0}[v(0)], \tag{4.4.23}$$

provided that $\sigma$ satisfies

$$\frac{2T^*}{T} C\sigma\mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big(1 + \mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big) \leq 1, \tag{4.4.24}$$

Indeed, for $k = 1$, from Lemma 4.4.3, we have

$$\begin{aligned}\sup_{t\in[0,T]} \mathcal{E}_\sigma[v(t)] &\leq \mathcal{E}_\sigma[v(0)] + C\sigma\mathcal{E}_{\sigma}^{\frac{3}{2}}[v(0)]\Big(1 + \mathcal{E}_{\sigma}^{\frac{1}{2}}[v(0)]\Big) \\ &\leq \mathcal{E}_{\sigma_0}[v(0)] + C\sigma\mathcal{E}_{\sigma_0}^{\frac{3}{2}}[v(0)]\Big(1 + \mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big).\end{aligned}$$

For $\sigma < \sigma_0$,

$$\mathcal{E}_\sigma[v(0)] \leq \mathcal{E}_{\sigma_0}[v(0)].$$

This inturn implies (4.4.23) holds provided that

$$C\sigma\mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big(1 + \mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big) \leq 1.$$

Now, assume that (4.4.22) and (4.4.23) hold for some $k \in \{1, 2, 3, ..., n\}$ and $\sigma$ satisfies (4.4.24).

Then, we need to show that (4.4.22) and (4.4.23) hold for $k = n + 1$.

Applying Lemma 4.4.3, (4.4.23) and (4.4.22) for $k = n+1$, respectively, we obtain

$$\begin{aligned}\sup_{t\in[kT,(k+1)T]} \mathcal{E}_\sigma[v(t)] &\leq \mathcal{E}_\sigma[v(kT)] + C\sigma\mathcal{E}_{\sigma}^{\frac{3}{2}}[v(kT)]\Big(1 + \mathcal{E}_{\sigma}^{\frac{1}{2}}[v(kT)]\Big) \\ &\leq \mathcal{E}_\sigma[v(kT)] + C\sigma\mathcal{E}_{\sigma_0}^{\frac{3}{2}}[v(0)]\Big(1 + \mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big) \\ &\leq \mathcal{E}_\sigma[v(0)] + C\sigma(k+1)\mathcal{E}_{\sigma_0}^{\frac{3}{2}}[v(0)]\Big(1 + \mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big).\end{aligned} \tag{4.4.25}$$

Combining (4.4.25) with the induction hypothesis (4.4.22), we obtain

$$\sup_{t\in[0,(k+1)T]} \mathcal{E}_\sigma[v(t)] \leq \mathcal{E}_\sigma[v(0)] + C\sigma(k+1)\mathcal{E}_{\sigma_0}^{\frac{3}{2}}[v(0)]\Big(1 + \mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big),$$

which proves (4.4.22) for $k = n+1$. This also implies (4.4.23) holds for $k = n+1$ provided that

$$C\sigma(k+1)\mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big(1+\mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big) \le 1.$$

However, the later follows from (4.4.22), since

$$k+1 \le n+1 \le \frac{T^*}{T} + 1 \le \frac{2T^*}{T}.$$

Finally, the condition (4.4.24) is satisfied for $\sigma$ such that

$$\sigma = \frac{c_1}{T^*},$$

where

$$c_1 = \frac{T}{C\mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)]\Big(1+(\mathcal{E}_{\sigma_0}^{\frac{1}{2}}[v(0)])\Big)},$$

which gives (4.4.21) if we choose $c \le c_1$.

**The general case $s \in \mathbb{R}$:** For any $s \in \mathbb{R}$, we use the embedding (2.1.6) to get

$$u_0 \in G^{\sigma_0,s} \subset G^{\frac{\sigma_0}{2},2}.$$

From the local result, there is a time $T = T(\mathcal{E}_{\frac{\sigma_0}{2}}[v(0)])$ such that

$$v(t) \in G^{\frac{\sigma_0}{2},2}, \quad 0 \le t \le T.$$

Fix an arbitrarily large $T^*$. From the case $s = 2$, we have

$$v(t) \in G^{2a_0T^*,2}, \quad 0 \le t \le T^*,$$

where $a_0 > 0$ depends on $\mathcal{E}_{\frac{\sigma_0}{2}}[v(0)])$ and $\sigma_0$.
Applying again the embedding (2.1.6) we conclude that

$$u(x,t) \in G^{a_0T^{-1},s}, \quad 0 \le t \le T^*.$$

This completes the proof of Theorem 4.1.1. □

# Chapter 5

# Generalized KdV-BBM Equation and Coupled System of Generalized BBM Equations

In this chapter, we study the evolution of the radius of spatial analyticity to the solutions of generalized KdV-BBM equation and coupled system of generalized BBM equations, subject to initial data in modified Gevrey space $H^{\sigma_0,s}$ with a fixed radius $\sigma_0$. It is shown that the uniform radius of spatial analyticity of solutions for both problems can not decay faster than $ct^{-2/3}$ as $t \to \infty$. We used contraction mapping principle, multilinear estimates and higher order approximate conservation law in modified Gevrey space to establish the results.

## 5.1 Problem Statement

Consider the Cauchy problem for generalized KdV-BBM equation

$$\begin{cases} u_t + u_x + \frac{3}{2}u^p u_x + \nu u_{xxx} - (\frac{1}{6} - \nu)u_{xxt} = 0, & p \geq 1 \\ u(x,0) = u_0(x), \end{cases} \tag{5.1.1}$$

and for coupled system of BBM equations

$$\begin{cases} u_t + u_x - u_{xxt} + \frac{1}{2}(v^2)_x = 0, \\ v_t + v_x - v_{xxt} + (uv)_x = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{5.1.2}$$

where $u$ and $v$ are real-valued functions of $x \in \mathbb{R}$ and $t \geq 0$.

The energy obtained from (5.1.1) is conserved for $\nu < \frac{1}{6}$, that is

$$\mathcal{A}[u(t)] = \frac{1}{2}\int_{\mathbb{R}} \left[u^2 + (\frac{1}{6} - \nu)u_x^2\right] dx = \mathcal{A}[u(0)], \quad \forall t \in \mathbb{R}. \tag{5.1.3}$$

The Cauchy problem (5.1.2) is the particular case of the coupled system of generalized BBM equations

$$\begin{cases} u_t + u_x - u_{xxt} + (\alpha u^2 + \beta uv + \gamma v^2)_x = 0 \\ v_t + v_x - v_{xxt} + (\theta u^2 + \lambda uv + \psi v^2)_x = 0 \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{5.1.4}$$

where $\alpha = \beta = \theta = \psi = 0, \gamma = \frac{1}{2}, \lambda = 1$.

The energy obtained from (5.1.2) also conserved, which is given by

$$\mathcal{E}(u,v) := \int_{\mathbb{R}} (u^2 + v^2 + u_x^2 + v_x^2)\, dx = \mathcal{E}(u(0), v(0)). \tag{5.1.5}$$

This chapter focuses on analyticity properties of solutions of (5.1.1) and (5.1.2). For a given real analytic initial data that has analytic extension to complex strip $S_{\sigma_0} = \{x + iy, x, y \in \mathbb{R}, |y| < \sigma_0\}$, we need to check whether the solution is analytic or not in complex strip $S_{\sigma(t)}$ for $t \to \infty$. We also analyze the evolution of the radius of spacial analyticity $\sigma(t)$ at each time $t$. The space of functions we used to study the spatial analyticity of the solution is modified Gevrey space, $H^{\sigma,s}(\mathbb{R})$ which was introduced in [37].

A function in the Gevrey class $G^{\sigma,s}(\mathbb{R})$ is a restriction to the real axis of a function which is analytic in a symmetric strip of width $2\sigma$. This property is described in the Paley-Wiener Theorem (see,Theorem 2.1.20).

The modified Gevrey space, $H^{\sigma,s}(\mathbb{R})$ is obtained from the Gevrey space, $G^{\sigma,s}(\mathbb{R})$ by replacing the exponential weight $e^{\sigma|\xi|}$ with the hyperbolic weight $\cosh(\sigma|\xi|)$, equipped with the norm

$$\|f\|^2_{H^{\sigma,s}(\mathbb{R})} = \|\cosh(\sigma|\xi|)\langle\xi\rangle^s \widehat{f}(\xi)\|^2_{L^2(\mathbb{R})}, \quad \sigma \geq 0. \tag{5.1.6}$$

Observe that, for large values of $\xi$ we have, $e^{-|\xi|} \approx 0$. From this fact and the definition of $\cosh(\xi)$, we have

$$\frac{1}{2} e^{\sigma|\xi|} \leq \cosh(\sigma|\xi|) \leq e^{\sigma|\xi|}. \tag{5.1.7}$$

Thus, the associated $H^{\sigma,s}(\mathbb{R})$ and $G^{\sigma,s}(\mathbb{R})$–norms are equivalent.That is

$$\|f\|_{H^{\sigma,s}(\mathbb{R})} \sim \|f\|_{G^{\sigma,s}(\mathbb{R})}. \tag{5.1.8}$$

Paley-Wiener Theorem still holds for functions in $H^{\sigma,s}(\mathbb{R})$. Note also that, $G^{0,s}(\mathbb{R}) = H^{0,s}(\mathbb{R}) = H^s(\mathbb{R})$. The reason for considering the $H^{\sigma,s}(\mathbb{R})$ is due to its advantage, since $\cosh(\sigma|\xi|)$ satisfies the estimate

$$\cosh(\sigma|\xi|) - 1 \leq (\sigma|\xi|)^{2\rho} \cosh(\sigma|\xi|), \quad \rho \in [0,1]. \tag{5.1.9}$$

The estimate (5.1.9) follows from

$$\cosh x - 1 \leq \cosh x \quad and \quad \cosh x - 1 \leq x^2 \cosh x, \quad x \in \mathbb{R}.$$

From the embedding property (2.1.6) and (5.1.8), we have

$$\|f\|_{H^s(\mathbb{R})} \leq C\|f\|_{H^{\sigma,s}(\mathbb{R})}, \quad \sigma > 0. \tag{5.1.10}$$

By virtue of (5.1.10) and the existing well-posedness theory in $H^s(\mathbb{R})$, (5.1.1) and (5.1.2) have a unique and global in time solution, with initial data $u_0 \in H^{\sigma_0,s}(\mathbb{R})$ and $(u_0, v_0) \in H^{\sigma_0,s}(\mathbb{R}) \times H^{\sigma_0,s}(\mathbb{R})$ respectively for all $\sigma_0 \geq 0$ and $s \in \mathbb{R}$. Our main results of this chapter are given in the following Theorems.

**Theorem 5.1.1.** *Let $\nu < \frac{1}{6}$, $\sigma_0 > 0$ and $u_0 \in H^{\sigma_0,1}(\mathbb{R})$. Then, the solution $u$ of (5.1.1) with initial data $u_0$ remains analytic in complex strip $S_{\sigma(t)}$ for all $t > 0$ $\big(u(t) \in H^{\sigma(t),1}(\mathbb{R}), \forall t > 0\big)$ with the radius of analyticity $\sigma(t)$ satisfying the lower bound*

$$\sigma(t) \geq ct^{-2/3} \quad as \quad t \to +\infty,$$

*where $c > 0$ is a constant depending on $\|u_0\|_{H^{\sigma_0,1}(\mathbb{R})}$ and $\sigma_0$.*

**Theorem 5.1.2.** *Let $\sigma_0 > 0$ and $(u_0, v_0) \in H^{\sigma_0,1}(\mathbb{R}) \times H^{\sigma_0,1}(\mathbb{R})$. Then, the solution $(u, v)$ of (5.1.2) with initial value $(u_0, u_0)$ is analytic in the space $H^{\sigma(t),1}(\mathbb{R}) \times H^{\sigma(t),1}(\mathbb{R})$ for all time $t \to \infty$, where $\sigma(t)$ satisfying the lower bound*

$$\sigma(t) \geq ct^{-2/3}, \quad \forall t \geq 0,$$

*where $c > 0$ is a constant depending on $\|(u_0, v_0)\|_{H^{\sigma_0,1}(\mathbb{R}) \times H^{\sigma_0,1}(\mathbb{R})}$ and $\sigma_0$.*

*Remark* 5.1.3. If (3.3.3) has a non zero solution { (a, b, c) } such that $4ac - b^2 > 0$, then (5.1.4) is globally well-posed in the space $H^{\sigma(t),1}(\mathbb{R}) \times H^{\sigma(t),1}(\mathbb{R})$ for all $t \to \infty$. The radius of analyticity of the solution has the same lower bounds as in Theorem 5.1.2.

The first step in the proof of Theorem 5.1.1 and Theorem 5.1.2 is to show that in a short time interval $0 \leq t \leq T$, where $T > 0$ depends on the norm of the initial data, the radius of analyticity remains strictly positive. This is proved by a contraction mapping argument and a multilinear estimate which will be given in the next section.

The next step is to prove the approximate conservation law in the time interval $[0, T]$, measured in the data norm $H^{\sigma,s}(\mathbb{R})$. This approximate conservation law will allow us to iterate the local results and obtain the results in Theorem 5.1.1 and 5.1.2.

## 5.2 Local Well-posedness Results

Before stating and proving local well-posedness of (5.1.1) and (5.1.2), let us recall bilinear estimate stated and proved in Lemma 4.2.1 of chapter 4. This estimate will be useful in the proof of the local well-posedness results.

**Lemma 5.2.1.** *For $s \geq 0$ and $\sigma > 0$, there is a constant $C = C(s)$ such that*

$$\|\varphi(D_x)(uv)\|_{G^{\sigma,s}(\mathbb{R})} \leq C\|u\|_{G^{\sigma,s}(\mathbb{R})}\|v\|_{G^{\sigma,s}(\mathbb{R})}, \tag{5.2.1}$$

*and*

$$\|\varphi(D_x)u\|_{G^{\sigma,s}(\mathbb{R})} \leq C\|u\|_{G^{\sigma,s}(\mathbb{R})}, \tag{5.2.2}$$

*where $\varphi(D_x)$ is the Fourier multiplier operator with symbol*

$$\varphi(\xi) = \frac{|\xi|}{1+\xi^2}.$$

By (2.1.5) and the multiplier estimate $\frac{|\xi|}{1+\xi^2} \leq 1$, the estimate in (5.2.2) holds.

*Remark* 5.2.2. By virtue of (5.1.8), the estimates in Lemma 5.2.1 hold in $H^{\sigma,s}$.

#### Local Well-posedness of Cauchy Problem for Generalized KdV-BBM Equation

In this subsection, we will study the local well-posedness of the Cauchy problem (5.1.1) in the modified Gevrey space $H^{\sigma,s}(\mathbb{R})$. The theory begins by converting the original initial-value problem into an associated integral equation. By taking the Fourier transform of (5.1.1) with respect to the spatial variable, we get

$$\widehat{u}_t + i\xi\widehat{u} + \frac{3i\xi}{2(p+1)}\widehat{u^{p+1}} - i\xi^3\nu\widehat{u} + (\frac{1}{6}-\nu)\xi^2\widehat{u}_t = 0.$$

Now, for $\nu < \frac{1}{6}$, we have $\left(1 + (\frac{1}{6}-\nu)\xi^2\right) > 0$. Then, rearranging the terms gives

$$\left(1 + (\frac{1}{6}-\nu)\xi^2\right)i\widehat{u}_t - \xi(1-\nu\xi^2)\widehat{u} = \frac{3\xi}{2(p+1)}\widehat{u^{p+1}}.$$

Dividing by $\left(1 + (\frac{1}{6}-\nu)\xi^2\right)$, we get

$$i\widehat{u}_t - \phi(\xi)\widehat{u} = \psi(\xi)\widehat{u^{p+1}}, \tag{5.2.3}$$

where

$$\phi(\xi) = \frac{\xi(1-\nu\xi^2)}{1+\left(\frac{1}{6}-\nu\right)\xi^2}, \quad \psi(\xi) = \frac{3\xi}{2(p+1)\left[1+(\frac{1}{6}-\nu)\xi^2\right]}.$$

Next, we define the Fourier multiplier operators $\phi(D_x)$ and $\psi(D_x)$ as follows

$$\phi(D_x)f(\xi) = \mathcal{F}^{-1}\Big(\phi(\xi)\widehat{f}(\xi)\Big), \quad \psi(D_x)f(\xi) = \mathcal{F}^{-1}\Big(\psi(\xi)\widehat{f}(\xi)\Big). \tag{5.2.4}$$

By the inverse Fourier transform of (5.2.3), equation (5.1.1) can be rewritten in an operator form as

$$\begin{cases} iu_t - \phi(D_x)u = \psi(D_x)u^{p+1}, \\ u(x,0) = u_0(x). \end{cases} \tag{5.2.5}$$

First, consider the following linear initial value problem associated with (5.2.5)

$$\begin{cases} iu_t - \phi(D_x)u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{5.2.6}$$

whose solution is given by

$$u(t) = S(t)u_0,$$

where $S(t) = e^{-it\phi(\partial_x)}$ and

$$\widehat{S(t)u_0} = e^{-i\phi(\xi)t}\widehat{u}_0 = \int_{\mathbb{R}} e^{-i\left(x\xi + t\phi(\xi)\right)} u_0(x)\,dx.$$

The operator $S(t)$ is unitary in $H^{\sigma,s}(\mathbb{R})$ for any $s \in \mathbb{R}$ and hence

$$\|S(t)u_0\|_{H^{\sigma,s}(\mathbb{R})} = \|u_0\|_{H^{\sigma,s}(\mathbb{R})}, \quad \forall t > 0. \tag{5.2.7}$$

By Duhamel's principle, the integral form of (5.2.5) is given by

$$u(x,t) = S(t)u_0 - i\int_0^t S(t-s')\psi(D_x)u^{p+1}(x,s')\,ds'. \tag{5.2.8}$$

**Theorem 5.2.3.** *Suppose that $\nu < \frac{1}{6}$. Let $\sigma_0 > 0$ and $u_0 \in H^{\sigma_0,1}(\mathbb{R})$ be given. Then, there exist a time $T = T\big(\|u_0\|_{H^{\sigma_0,1}(\mathbb{R})}\big) > 0$, and a unique solution $u(t) \in C([0,T]; H^{\sigma(t),1}(\mathbb{R}))$ of (5.1.1), where*

$$T = \frac{1}{2^{p+1}C(\|u_0\|_{H^{\sigma_0,1}(\mathbb{R})})^p} \tag{5.2.9}$$

*where $C$ is a constant depends on $s$.*
*Moreover, the map from $u_0 \in H^{\sigma_0,1}(\mathbb{R}) \to u \in C([0,T]; H^{\sigma(t),1}(\mathbb{R}))$ is continuous. That is*

$$\|u(\cdot,t)\|_X = \sup_{0\le t\le T} \|u(\cdot,t)\|_{H^{\sigma(t),1}(\mathbb{R})} \le C\|u_0\|_{H^{\sigma_0,1}(\mathbb{R})}, \tag{5.2.10}$$

*where*

$$X := C\Big([0,T]; H^{\sigma(t),1}(\mathbb{R})\Big),$$

*is Banach space.*

Since $S(t)$ is a unitary group in $H^{\sigma,s}(\mathbb{R})$, applying $\cosh(\sigma|D_x|)$ to the integral (5.2.8) and taking the $H^1$-norm on both sides yields the energy inequality

$$\sup_{0\leq t\leq T}\|u\|_{H^{\sigma,1}} \leq \|u_0\|_{H^{\sigma,1}} + \int_0^t \|\psi(D_x)u^{p+1}\|_{H^{\sigma,1}}\,ds', \tag{5.2.11}$$

Now, we use the following Lemma to estimate the nonlinear term in (5.2.11).

**Lemma 5.2.4.** *For $\sigma > 0$, we have nonlinear estimate*

$$\|\psi(D_x)u^{p+1}\|_{H^{\sigma,0}} \lesssim \|u\|^{p+1}_{H^{\sigma,1}}. \tag{5.2.12}$$

***Proof:*** Note that, $\psi(\xi) \leq C\omega(\xi)$ for some constant $C > 0$. From (5.2.2) and (5.1.8), we have

$$\|\psi(D_x)u^{p+1}\|_{H^{\sigma,0}} \lesssim \|u^{p+1}\|_{H^{\sigma,0}}, \tag{5.2.13}$$

which is also equivalent to

$$\|\psi(D_x)u^{p+1}\|_{G^{\sigma,0}} \lesssim \|u^{p+1}\|_{G^{\sigma,0}}. \tag{5.2.14}$$

Setting $U := e^{\sigma|D_x|}u$ in (5.2.14) and applying Plancherel Theorem, leads to

$$\begin{aligned}
\|(e^{-\sigma|D_x|}U)^{p+1}\|_{G^{\sigma,0}} &= \left\|e^{\sigma|D_x|}(e^{-\sigma|D_x|}U)^{p+1}\right\|_{L^2_x} \\
&= \left\|\mathcal{F}_x\left(e^{\sigma|D_x|}(e^{-\sigma|D_x|}U)^{p+1}\right)(\xi)\right\|_{L^2_\xi} \\
&= \left\|\int_{\xi=\sum_{i=1}^{p+1}\xi_i} e^{\sigma\left(|\xi|-\sum_{i=1}^{p+1}|\xi_i|\right)}\prod_{i=1}^{p+1}\widehat{U(\xi_i)}d\xi_1\dots d\xi_{p+1}\right\|_{L^2_\xi} \\
&\leq \left\|\int_{\xi=\sum_{i=1}^{p+1}\xi_i}\prod_{i=1}^{p+1}|\widehat{U(\xi_i)}|d\xi_1\dots d\xi_{p+1}\right\|_{L^2_\xi} \\
&= \|W^{p+1}\|_{L^2_x},
\end{aligned} \tag{5.2.15}$$

where $W = \mathcal{F}^{-1}(|\widehat{U}|)$ and $|\xi| \leq \sum_{i=1}^{p+1}|\xi_i|$, which follows from the triangle inequality.

Now, by Sobolev embedding, we obtain

$$\|W^{p+1}\|_{L^2_x} = \|W\|^{p+1}_{L^{2(p+1)}_x} \leq \|W\|^{p+1}_{H^1} = \|u\|^{p+1}_{H^{\sigma,1}}.$$

Thus, from (5.2.13) and (5.2.15) the estimate (5.2.12) holds.

***Proof of Theorem 5.2.3.*** Taking into account the Duhamel's formula (5.2.8), we define a mapping

$$\Gamma u(x,t) = S(t)u_0 - i\int_0^t S(t-s')\psi(D_x)u^{p+1}(x,s')\,ds'. \tag{5.2.16}$$

Consider the closed ball $\mathcal{B}_r$ in X:

$$\mathcal{B}_r = \left\{ u \in X : \sup_{0\le t\le T} \|u\|_{H^{\sigma(t),1}} \le 2\|u_0\|_{H^{\sigma_0,1}} \right\},$$

where

$$X := C\Big(0,T; H^{\sigma(t),1}(\mathbb{R})\Big).$$

Using the energy estimate (5.2.11) and Lemma 5.2.4, we obtain

$$\begin{aligned} \sup_{0\le t\le T} \|\Gamma u\|_{H^{\sigma(t),1}(\mathbb{R})} &\le \|u_0\|_{H^{\sigma_0,1}(\mathbb{R})} + \int_0^t \|\psi(D_x)u^{p+1}\|_{H^{\sigma(t),1}} \\ &\le \|u_0\|_{H^{\sigma_0,1}(\mathbb{R})} + CT\sup_{0\le t\le T}\|u\|^{p+1}_{H^{\sigma(t),1}(\mathbb{R})}. \end{aligned} \tag{5.2.17}$$

For $u \in \mathcal{B}_r$, (5.2.17) becomes

$$\sup_{0\le t\le T} \|\Gamma u\|_{H^{\sigma(t),1}(\mathbb{R})} \le \|u_0\|_{H^{\sigma_0,1}(\mathbb{R})} + CT2^{p+1}\|u_0\|^{p+1}_{H^{\sigma_0,1}(\mathbb{R})}.$$

Now, choose T so small, such that

$$T = \frac{1}{2C(2\|u_0\|_{H^{\sigma_0,1}(\mathbb{R})})^p}.$$

Then, $\|\Gamma u\|_X \le 2\|u_0\|_{H^{\sigma_0,1}}$ showing that $\Gamma$ maps $\mathcal{B}_r$ into itself.

Next, with the same choice of $T$, we can show that $\Gamma$ is a contraction map on $\mathcal{B}_r$. For $u$, $v \in \mathcal{B}_r$, by (5.2.11) and Lemma 5.2.4, we have

$$\begin{aligned} \sup_{0\le t\le T} \Big\|\Gamma u - \Gamma v\Big\|_{H^{\sigma(t),1}} &\le \left\| \int_0^t S(t-s)\psi(\partial_x)\big(u^{p+1} - v^{p+1}\big)(x,s)\,ds \right\|_X \\ &\le CT\sup_{0\le t\le T}\|u-v\|_{H^{\sigma,1}(\mathbb{R})}\|u^p + u^{p-1}v + \ldots + v^p\|_{H^{\sigma,1}(\mathbb{R})} \\ &\le CT(2\|u_0\|_{H^{\sigma_0,1}})^p \sup_{0\le t\le T}\|u-v\|_{H^{\sigma,1}(\mathbb{R})} \\ &\le \frac{1}{2}\sup_{0\le t\le T}\|u-v\|_{H^{\sigma,1}(\mathbb{R})}. \end{aligned} \tag{5.2.18}$$

Hence, $\Gamma$ is a contraction map on $\mathcal{B}_r$.

Therefore by contraction mapping principle, (5.1.1) has unique solution in $X$. Continuous dependence on the initial data can be shown in a similar way using the difference estimate. This concludes the proof of Theorem 5.2.3.

**Local Well-posedness of Cauchy Problem for Coupled System of BBM Equations**

Here, we will prove the local well-posedness of (5.1.2) by converting the system to an equivalent system of integral equations.

Let us denote vector of dependent variables, vector of initial data and vector of nonlinearities of (5.1.2) respectively as follows:

$$W = \begin{pmatrix} u \\ v \end{pmatrix}, \quad W_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad M(W) = \begin{pmatrix} \frac{1}{2}v^2 \\ uv \end{pmatrix}.$$

Then, the system (5.1.2) can be rewritten as

$$\begin{cases} W_t + W_x - W_{xxt} + \partial_x M(W) = 0, \\ W(x,0) = W_0(x). \end{cases} \tag{5.2.19}$$

By rearranging terms in the first equation of (5.2.19) and taking its Fourier transform, we get

$$(1-\partial_x^2)W_t + \partial_x W = -\partial_x M(W)$$

$$iW_t - \frac{D_x}{1+D_x^2}W = \frac{D_x}{1+D_x^2}M(W)$$

$$i\widehat{W_t} - \varphi(\xi)\widehat{W} = \varphi(\xi)\widehat{M(W)}, \tag{5.2.20}$$

where $D_x = -i\partial_x$ with Fourier symbol $\xi$ and $\varphi(\xi)$ is given by

$$\varphi(\xi) = \frac{\xi}{1+\xi^2}.$$

Now, define the Fourier multiplier operator $\varphi(D_x)$ as

$$\varphi(D_x)f = \mathcal{F}^{-1}\Big(\varphi(\xi)\widehat{f}(\xi)\Big),$$

By taking the inverse Fourier transform of (5.2.20), and using the resulting equation, (5.2.19) can be rewritten as

$$\begin{cases} iW_t - \varphi(D_x)W = \varphi(D_x)M(W), \\ W(x,0) = W_0(x). \end{cases} \tag{5.2.21}$$

First, let us solve the linear part of (5.2.21),i.e

$$\begin{cases} iW_t - \varphi(D_x)W = 0, \\ W(x,0) = W_0(x). \end{cases} \tag{5.2.22}$$

Set $S(t) = e^{-it\varphi(D_x)}$ be the unitary group in $G^{\sigma,s}(\mathbb{R})$. Then $S(t)W_0(x)$ solves (5.2.22). Using Duhamel's formula, (5.2.21) can be rewritten as the integral equation

$$W(x,t) = S(t)W_0(x) - i\int_0^t S(t-\tau)\varphi(D_x)M(W(x,\tau))\,d\tau. \tag{5.2.23}$$

Define the operator $\mathcal{K}$ by

$$\mathcal{K}(W(x,t)) = S(t)W_0(x) - i\int_0^t S(t-\tau)\varphi(D_x)M(W(x,\tau))\,d\tau. \tag{5.2.24}$$

We have the following local well posedness result.

**Theorem 5.2.5.** *Let $\sigma_0 > 0$, $s \geq 0$ and the initial data $W_0 \in H^{\sigma_0,s}(\mathbb{R}) \times H^{\sigma_0,s}(\mathbb{R})$. Then, there exist a time $T = T\big(\|W_0\|_{H^{\sigma_0,s}(\mathbb{R})\times H^{\sigma_0,s}(\mathbb{R})}\big) > 0$ and a unique solution*

$$W \in X = C\Big([0,T]; H^{\sigma(t),s}(\mathbb{R}) \times H^{\sigma(t),s}(\mathbb{R})\Big)$$

*of the Cauchy problem (5.2.21), and we have*

$$T = \frac{1}{4C\|W_0\|_{H^{\sigma_0,s}(\mathbb{R})\times H^{\sigma_0,s}(\mathbb{R})}}, \tag{5.2.25}$$

*where $C > 0$ is a constant depends on $s$.*

*Moreover, the map from $W_0 \in H^{\sigma_0,s}(\mathbb{R})\times H^{\sigma_0,s}(\mathbb{R}) \to W \in C\Big([0,T]; H^{\sigma(t),s}(\mathbb{R})\times H^{\sigma(t),s}(\mathbb{R})\Big)$ is continuous.i.e.,*

$$\|W(.,t)\|_X = \sup_{0\leq t\leq T}\|W(.,t)\|_{H^{\sigma,s}(\mathbb{R})\times H^{\sigma,s}(\mathbb{R})} \leq C\|W_0\|_{H^{\sigma_0,s}(\mathbb{R})\times H^{\sigma_0,s}(\mathbb{R})}. \tag{5.2.26}$$

This implies for short time interval $0 \leq t \leq T$, the solution remains analytic in the initial strip $S_{\sigma_0}$ with radius of analyticity $\sigma(t) = \sigma_0$.

***Proof:*** Using similar procedure as in the proof of Theorem 5.2.3, applying the contraction mapping argument to (5.2.24) and using bilinear estimates (5.2.1) and (5.2.2), we can prove the local well-posedness in analytic space $H^{\sigma,s}(\mathbb{R}) \times H^{\sigma,s}(\mathbb{R})$.

Denote

$$X := C\Big(0,T; H^{\sigma,s}(\mathbb{R}) \times H^{\sigma,s}(\mathbb{R})\Big),$$

equipped with norm

$$\|W\|_X = \sup_{t\in[0,T]} \|W\|_{H^{\sigma,s}(\mathbb{R})\times H^{\sigma,s}(\mathbb{R})} = \sup_{t\in[0,T]} \|u\|_{H^{\sigma,s}(\mathbb{R})} + \sup_{t\in[0,T]} \|v\|_{H^{\sigma,s}(\mathbb{R})}.$$

Note that, $(X, \|.\|_X)$ is a Banach space. To establish local existence via the contraction mapping principle the domain for the operator $\mathcal{K}$ will be restricted to the closed subset $B_r(0)$ of $X$, given by

$$B_r = \{W \in X : \|W\|_X \le r\}.$$

Our claim is to show the nonlinear operator $\mathcal{K}$ defined in (5.2.24) is a contraction or not in the closed ball $B_r$ for $T$ sufficiently small, which means $\mathcal{K}$ maps $B_r$ into $B_r$. i.e,

$$\|\mathcal{K}W\|_X \le r,$$

and

$$\|\mathcal{K}(W) - \mathcal{K}(\bar{W})\|_X \le \vartheta\|W - \bar{W}\|_X,$$

where $W, \bar{W} \in X$ and $\vartheta \in [0, 1)$.

Next, we go back to (5.2.24) and its norm

$$\|\mathcal{K}W(x,t)\|_X = \|S(t)W_0\|_{H^{\sigma,s}(\mathbb{R})\times H^{\sigma,s}(\mathbb{R})} + \Big\|\int_0^t S(t-\tau)\varphi(D_x)M(W(x,\tau))\,d\tau\Big\|_X. \tag{5.2.27}$$

Since S(t) is a unitary operator on $H^{\sigma,s}(\mathbb{R}) \times H^{\sigma,s}(\mathbb{R})$ for any $s \in \mathbb{R}$, we have

$$\|S(t)W_0\|_{H^{\sigma,s}(\mathbb{R})\times H^{\sigma,s}(\mathbb{R})} = \|W_0\|_{H^{\sigma,s}(\mathbb{R})\times H^{\sigma,s}(\mathbb{R})}, \quad \forall t > 0. \tag{5.2.28}$$

By virtue of Lemma 5.2.1, we have

$$\begin{aligned}
\|\mathcal{K}W(x,t)\|_X &\le \|W_0\|_{H^{\sigma,s}(\mathbb{R})\times H^{\sigma,s}(\mathbb{R})} + \int_0^t \Big\|\varphi(D_x)M(W(x,\tau))\Big\|_X d\tau \\
&\le \|W_0\|_{H^{\sigma,s}\times H^{\sigma,s}} + T\Big\|\varphi(D_x)M(W(x,\tau))\Big\|_X \\
&\le \|W_0\|_{H^{\sigma,s}\times H^{\sigma,s}} + T\Big\|\varphi(D_x)\Big(\frac{1}{2}v^2, uv\Big)\Big\|_X \\
&\le \|W_0\|_{H^{\sigma,s}\times H^{\sigma,s}} + T\sup_{0\le t\le T}\Big\|\varphi(D_x)\Big(\frac{1}{2}v^2, uv\Big)\Big\|_{H^{\sigma,s}\times H^{\sigma,s}} \\
&\le \|W_0\|_{H^{\sigma,s}\times H^{\sigma,s}} + T\Big[\sup_{0\le t\le T}\Big\|\varphi(D_x)(\frac{1}{2}v^2)\Big\|_{H^{\sigma,s}} + \sup_{0\le t\le T}\Big\|\varphi(D_x)(uv)\Big\|_{H^{\sigma,s}}\Big] \\
&\le \|W_0\|_{H^{\sigma,s}\times H^{\sigma,s}} + TC\sup_{0\le t\le T}\Big[\|v\|^2_{H^{\sigma,s}} + \|u\|_{H^{\sigma,s}(\mathbb{R})}\|v\|_{H^{\sigma,s}}\Big] \\
&\le \|W_0\|_{H^{\sigma,s}\times H^{\sigma,s}} + TCr^2.
\end{aligned} \tag{5.2.29}$$

If we choose

$$r = 2\|W_0\|_{H^{\sigma,s}\times H^{\sigma,s}}, \quad T = \frac{1}{2Cr} = \frac{1}{4C\|W_0\|_{H^{\sigma,s}\times H^{\sigma,s}}}.$$

Then, $\|\mathcal{K}W(x,t)\|_X \le r$ showing that $\mathcal{K}$ maps the closed ball $B_r(0)$ in $C\big([0,T]; H^{\sigma,s}\times H^{\sigma,s}\big)$ onto itself.

To show the map $\mathcal{K} : X \longrightarrow X$ be contraction mapping, let $W, \bar{W} \in X = C\big(0,T; H^{\sigma,s}\times H^{\sigma,s}\big)$, With the same choice of $T$ as above, by applying (5.2.2), we obtain

$$\begin{aligned}
\Big\|\mathcal{K}W - \mathcal{K}\bar{W}\Big\|_X &= \Big\| \int_0^t S(t-\tau)\big[\varphi(D_x)M(W(x,\tau)) - \varphi(D_x)M(\bar{W}(x,\tau))\big]\, d\tau\Big\|_X \\
&\le \int_0^t \Big\|\varphi(D_x)M(W(x,\tau)) - \varphi(D_x)M(\bar{W}(x,\tau))\Big\|_X d\tau \\
&\le \int_0^t \Big\|\varphi(D_x)\Big[\Big(\frac{1}{2}v^2, uv\Big) - \Big(\frac{1}{2}\bar{v}^2, \bar{u}\bar{v}\Big)\Big]\Big\|_X d\tau \\
&\le T\Big[\Big\|\varphi(D_x)\Big(\frac{1}{2}(v-\bar{v})(v+\bar{v})\Big)\Big\|_{H^{\sigma,s}} \\
&+ \Big\|\varphi(D_x)\Big(v(u-\bar{u}) + \bar{u}(v-\bar{v})\Big)\Big\|_{H^{\sigma,s}}\Big] \\
&\le CT\Big[\frac{1}{2}\|v-\bar{v}\|_{H^{\sigma,s}}\|v+\bar{v}\|_{H^{\sigma,s}} \\
&+ \|u-\bar{u}\|_{H^{\sigma,s}}\|v\|_{H^{\sigma,s}} + \|v-\bar{v}\|_{H^{\sigma,s}}\|\bar{u}\|\|_{H^{\sigma,s}}\Big)\Big] \\
&\le CTr\Big[\|u-\bar{u}\|_{H^{\sigma,s}} + \|v-\bar{v}\|_{H^{\sigma,s}}\Big] \\
&\le CTr\|W-\bar{W}\|_{H^{\sigma,s}\times^{\sigma,s}}.
\end{aligned} \tag{5.2.30}$$

Hence

$$\|\mathcal{K}W - \mathcal{K}\bar{W}\|_X \le \frac{1}{2}\|W-\bar{W}\|_{H^{\sigma,s}\times H^{\sigma,s}}.$$

This shows that $\mathcal{K}$ is a contraction on $B_r$ with a contraction constant $\frac{1}{2}$. So by contraction mapping principle, initial value problem (5.2.19) has unique solution.

To see the continuous dependence of the solution on the initial data, let $W$ and $\bar{W}$ be the solutions with initial data $W_0$ and $\bar{W}_0$ respectively, then

$$\begin{aligned}
\|W-\bar{W}\|_X &\le \|W_0-\bar{W}_0\|_{H^{\sigma,s}\times H^{\sigma,s}} + \tfrac{1}{2}\|W-\bar{W}\|_{H^{\sigma,s}\times H^{\sigma,s}}, \\
&\le 2\|W_0-\bar{W}_0\|_{H^{\sigma,s}\times H^{\sigma,s}}.
\end{aligned}$$

## 5.3 Almost Conservation Law

Almost conservation law enables us to prove the existence of global solution by repeating the local result obtained in section 5.2

**Almost Conservation Law to (5.1.1):** Let $u$ be the solution of (5.1.1). Define

$$w(x,t) = \cosh(\sigma|D_x|)u.$$

Applying the operator $\cosh(\sigma|D_x|)$ to the first equation in (5.1.1), we obtain

$$w_t + w_x + \frac{3}{2(p+1)}(w^{p+1})_x + \nu w_{xxx} - (\frac{1}{6} - \nu)w_{xxt} = f(w), \tag{5.3.1}$$

where

$$f(w) = \frac{3}{2(p+1)}\partial_x\Big[w^{p+1} - \cosh(\sigma|D_x|)\Big(\text{sech}(\sigma|D_x|)w\Big)^{p+1}\Big]. \tag{5.3.2}$$

The modified energy, given by

$$\mathcal{A}_\sigma[w(t)] = \frac{1}{2}\int_{\mathbb{R}} \big[w^2 + (\frac{1}{6} - \nu)w_x^2\big]\,dx. \tag{5.3.3}$$

For $\sigma = 0$, we have $w = u$, and therefore the energy is conserved. i.e.

$$\mathcal{A}_0[w(t)] = \mathcal{A}_0[w(0)].$$

However, this fails to hold for $\sigma > 0$.

Differentiating $\mathcal{A}_\sigma[w(t)]$ with respect to $t$, then applying integration by parts and using (5.3.1) and (5.3.2), gives

$$\begin{aligned}
\frac{d}{dt}\mathcal{A}_\sigma[w(t)] &= \int_{\mathbb{R}} \big(ww_t + (\frac{1}{6} - \nu)w_x w_{xt}\big)\,dx \\
&= \int_{\mathbb{R}} \big(ww_t - (\frac{1}{6} - \nu)ww_{xxt}\big)\,dx \\
&= -\int_{\mathbb{R}} \big(ww_x + \frac{3}{2(p+1)}w(w^{p+1})_x + \nu ww_{xxx} - wf(w)\big)\,dx \\
&= -\frac{1}{2}\int_{\mathbb{R}} \big[(w^2)_x - \frac{3}{(p+1)(p+2)}(w^{p+2})_x - \nu(w_x^2)_x\big]\,dx + \int_{\mathbb{R}} wf(w)\,dx.
\end{aligned}$$

By assumption, $w$ and its all spatial derivatives decay to zero as $|x| \to \infty$. Consequently, the first integral in the last line is vanished. Therefore

$$\frac{d}{dt}\mathcal{A}_\sigma[w(t)] = \int_{\mathbb{R}} wf(w)\,dx.$$

Integrating over $[0, t]$, gives

$$\mathcal{A}_\sigma[w(t)] = \mathcal{A}_\sigma[w(0)] + \int_0^t \int_{\mathbb{R}} w(x,s) f(w(x,s))\, dxds. \tag{5.3.4}$$

The following Lemmas are important to prove an almost conservation law to (5.1.1) and (5.1.2) .

**Lemma 5.3.1.** *Let* $\xi = \sum_{i=1}^{p} \xi_i \quad for \quad \xi_i \in \mathbb{R}$ *and* $p$ *is positive integer. Then*

$$\left| 1 - \cosh|\xi| \prod_{i=1}^{p} \operatorname{sech}|\xi_i| \right| \le 2^p \sum_{i \neq j = 1}^{p} |\xi_i||\xi_j|. \tag{5.3.5}$$

The proof is found in [37].

**Lemma 5.3.2.** *Let* $f(w)$ *be defined as in* (5.3.2)*, then we have*

$$\left| \int_{\mathbb{R}} w f(w) dx \right| \le C\sigma^{\frac{3}{2}} \|w\|_{H^1(\mathbb{R})}^{p+2}, \tag{5.3.6}$$

*for all* $w \in H^1(\mathbb{R})$.

***Proof:*** Using Hölder inequality and Sobolev embedding, we get

$$\begin{aligned} \left| \int_{\mathbb{R}} w f(w)\, dx \right| &= \left| \int_{\mathbb{R}} \langle D_x \rangle w \langle D_x \rangle^{-1} f(w)\, dx \right| \\ &\le \|\langle D_x \rangle w\|_{L^2(\mathbb{R})} \|\langle D_x \rangle^{-1} f(w)\|_{L^2(\mathbb{R})} \\ &\le \|w\|_{H^1(\mathbb{R})} \|\langle D_x \rangle^{-1} f(w)\|_{L^2(\mathbb{R})}. \end{aligned} \tag{5.3.7}$$

By the Fourier transform of $\langle D_x \rangle^{-1} f(w)$ where $f(w)$, we obtain

$$\begin{aligned} \left| \mathcal{F}_x\Big(\langle D_x \rangle^{-1} f(w)\Big)(\xi) \right| &= \left| \langle \xi \rangle^{-1} \frac{3i\xi}{2(p+1)} \int_\xi \Big[ 1 - \cosh(\sigma|\xi|) \prod_{j=1}^{p+1} \operatorname{sech}(\sigma|\xi_j|) \Big] \prod_{j=1}^{p+1} \widehat{w}(\xi_j) d\xi \right| \\ &\le \frac{3}{2(p+1)} \int_\xi \left| 1 - \cosh(\sigma|\xi|) \prod_{j=1}^{p+1} \operatorname{sech}(\sigma|\xi_j|) \right| \prod_{j=1}^{p+1} |\widehat{w}(\xi_j)|\, d\xi \\ &\le \frac{3}{2(p+1)} \int_\xi |K| \prod_{j=1}^{p+1} |\widehat{w}(\xi_j)|\, d\xi, \end{aligned} \tag{5.3.8}$$

where

$$\xi = \sum_{j=1}^{p+1} \xi_j, \qquad d\xi = \prod_{j=1}^{p+1} d\xi_j,$$

and

$$K = 1 - \cosh(\sigma|\xi|)\prod_{j=1}^{p+1}\operatorname{sech}(\sigma|\xi_j|). \tag{5.3.9}$$

Note that $|K| \leq 1$.

By symmetry, we may assume $|\xi_1| \leq |\xi_2| \leq |\xi_3| \leq \cdots \leq |\xi_p| \leq |\xi_{p+1}|$. Then by Lemma 5.3.1, we have

$$\begin{aligned} |K| &= \left|1 - \cosh(\sigma|\xi|)\prod_{j=1}^{P+1}\operatorname{sech}(\sigma|\xi_j|\right| \\ &\leq \sigma^2 2^{P+1}\sum_{i\neq j=1}^{p+1}\left|\xi_i\right|\left|\xi_j\right| \\ &\leq C(p)\sigma^2|\xi_p||\xi_{p+1}|, \end{aligned}$$

where

$$C(p) = 2^{p+1}(p^2 - p).$$

For $0 \leq \beta \leq 1$

$$|K| \leq C(p)\sigma^2|\xi_p||\xi_{p+1}| \leq C(p)\sigma^{2\beta}|\xi_p|^\beta|\xi_{p+1}|^\beta.$$

Choosing $\beta = \frac{3}{4}$, we get

$$|K| \leq C(p)\sigma^{\frac{3}{2}}|\xi_p|^{\frac{3}{4}}|\xi_{p+1}|^{\frac{3}{4}}. \tag{5.3.10}$$

Now, let

$$W = \mathcal{F}_x^{-1}(|\widehat{w}|).$$

From (5.3.8) and (5.3.10), we obtain

$$\begin{aligned} \left|\mathcal{F}_x\Big(\langle D_x\rangle^{-1}F(w)\Big)(\xi)\right| &\leq C(p)\sigma^{\frac{3}{2}}\int_\xi\Big(\prod_{j=1}^{p-1}\left|\widehat{w}(\xi_j)\right|\Big)\left|\xi_p\right|^{\frac{3}{4}}\left|\widehat{w}(\xi_p)\right|\left|\xi_{p+1}\right|^{\frac{3}{4}}\left|\widehat{w}(\xi_{p+1})\right|d\xi \\ &\leq C(p)\sigma^{\frac{3}{2}}\int_\xi\Big(\prod_{j=1}^{p-1}\widehat{W}(\xi_j)\Big)\left|\xi_p\right|^{\frac{3}{4}}\widehat{W}(\xi_p)\left|\xi_{p+1}\right|^{\frac{3}{4}}\widehat{W}(\xi_{p+1})\,d\xi \\ &\leq C(p)\sigma^{\frac{3}{2}}\mathcal{F}_x\big[W^{p-1}\cdot|D_x|^{\frac{3}{4}}W\cdot|D_x|^{\frac{3}{4}}W\big](\xi). \end{aligned}$$

Using Plancherel Theorem , Hölder inequality and one dimensional Sobolev embedding, we get

$$\begin{aligned}\Big\|\mathcal{F}_x\Big(\langle D_x\rangle^{-1}F(w)\Big)(\xi)\Big\|_{L^2_x(\mathbb{R})} &\leq C(p)\sigma^{\frac{3}{2}}\|W^{p-1}(|D_x|^{\frac{3}{4}}W)^2\|_{L^2_x(\mathbb{R})}\\ &\leq C(p)\sigma^{\frac{3}{2}}\|W^{p-1}\|_{L^\infty_x(\mathbb{R})}\|(|D_x|^{\frac{3}{4}}W)^2\|_{L^2_x(\mathbb{R})}\\ &\leq C(p)\sigma^{\frac{3}{2}}\|W\|^{p-1}_{L^\infty(\mathbb{R})}\||D_x|^{\frac{3}{4}}W\|^2_{L^4(\mathbb{R})}\\ &\leq C(p)\sigma^{\frac{3}{2}}\|W\|^{p-1}_{H^1(\mathbb{R})}\||D_x|^{\frac{3}{4}}W\|^2_{H^{\frac{1}{4}}(\mathbb{R})}\\ &\leq C(p)\sigma^{\frac{3}{2}}\|W\|^{p-1}_{H^1(\mathbb{R})}\|W\|^2_{H^1(\mathbb{R})}\\ &\sim C(p)\sigma^{\frac{3}{2}}\|w\|^{p+1}_{H^1(\mathbb{R})}\end{aligned} \tag{5.3.11}$$

Then, the desired result (5.3.6) followed from (5.3.7) and (5.3.11). □

In view of (5.3.4) and (5.3.6), we have energy estimate

$$\mathcal{A}_\sigma[w(t)] \leq \mathcal{A}_\sigma[w(0)] + C\sigma^{\frac{3}{2}}T\|w(t)\|^{p+2}_{H^1(\mathbb{R})}. \tag{5.3.12}$$

**Lemma 5.3.3.** *[Almost conservation law]. Let $w_0 \in H^1(\mathbb{R})$. Suppose that $w \in C\big([0,T];H^1(\mathbb{R})\big)$ is the local-in-time solution to the Cauchy problem (5.1.1). Then*

$$\sup_{0\leq t\leq T} \mathcal{A}_\sigma[w(t)] \leq \mathcal{A}_\sigma[w(0)] + C\sigma^{\frac{3}{2}}\|w\|^{p+2}_{L^\infty_T H^1(\mathbb{R})}, \tag{5.3.13}$$

$$\sup_{0\leq t\leq T} \mathcal{A}_\sigma[w(t)] \leq \mathcal{A}_\sigma[w(0)] + C\sigma^{\frac{3}{2}}\Big(\mathcal{A}_\sigma[w_0]\Big)^{\frac{p+2}{2}}, \tag{5.3.14}$$

*where $C$ is a positive constant depends on $\|w_0\|_{H^1(\mathbb{R})}$ , $s$ and $p$.*

***Proof:*** By combining (5.3.12) with the local existence theory, we obtain (5.3.13). By (5.2.10), we have the bound

$$\|w\|_{L^\infty_T H^1(\mathbb{R}} = \|u\|_{L^\infty_T H^{\sigma,1}(\mathbb{R})\big)} \leq C\|u_0\|_{H^{\sigma_0,1}(\mathbb{R})} = C\|w_0\|_{H^1(\mathbb{R})}, \tag{5.3.15}$$

where $T$ is as in (5.2.9) and

$$L^\infty_T H^1(\mathbb{R}) = L^\infty_t H^1\Big([0,T]\times\mathbb{R}\Big).$$

For $\nu < \frac{1}{6}$, we have

$$\mathcal{A}_\sigma[w_0] = \frac{1}{2}\int_{\mathbb{R}} \big[w_0^2 + (\frac{1}{6}-\nu)(w_0')^2\big]\,dx \sim \|w_0\|^2_{H^1(\mathbb{R})}. \tag{5.3.16}$$

Hence

$$\|w_0\|_{H^1(\mathbb{R})} \sim (\mathcal{A}_\sigma[w_0])^{\frac{1}{2}}.$$

Then, using (5.3.15) and (5.3.16), we obtain the desired estimate (5.3.14). □

**Almost Conservation Law to (5.1.2):** For $\alpha = \beta = \theta = \psi = 0, \gamma = \frac{1}{2}, \lambda = 1$, the solution of (3.3.3) is $a = 1, b = 0$ and $c = 1$, that satisfy $4ac - b^2 > 0$. In this case, the energy of (5.1.2)

$$\mathcal{E}\Big(u(t), v(t)\Big) := \int_{\mathbb{R}} \Big(u^2 + v^2 + u_x^2 + v_x^2\Big)\, dx \tag{5.3.17}$$

is conserved. i.e.,

$$\mathcal{E}\Big(u(t), v(t)\Big) = \mathcal{E}\Big(u(0), v(0)\Big), \quad \forall t \geq 0.$$

Let $(u, v)$ be the solution of (5.1.2). Then define

$$u_\sigma(x, t) = \cosh(\sigma|D_x|)u(x, t),$$

$$v_\sigma(x, t) = \cosh(\sigma|D_x|)v(x, t), \quad \sigma \geq 0,$$

The modified conserved energy is given by

$$\mathcal{E}_\sigma\Big(u_\sigma(t), v_\sigma(t)\Big) = \int_{\mathbb{R}} \Big(u_\sigma^2 + v_\sigma^2 + (\partial_x u_\sigma)^2 + (\partial_x v_\sigma)^2\Big)\, dx. \tag{5.3.18}$$

For $\sigma = 0$, $u_\sigma = u$ and $v_\sigma = v$, then

$$\mathcal{E}_0\big(u_0(t), v_0(t)\big) = \mathcal{E}\big(u(t), v(t)\big) = \mathcal{E}\big(u(0), v(0)\big). \tag{5.3.19}$$

However, for $\sigma > 0$, the energy is not conserved.

**Theorem 5.3.4** (Almost conservation law)**.** *Let $\sigma_0 > 0$, $\big(u(0), v(0)\big) \in H^{\sigma_0,1}(\mathbb{R}) \times H^{\sigma_0,1}(\mathbb{R})$ and suppose that $(u, v) \in H^{\sigma(t),1}(\mathbb{R}) \times H^{\sigma(t),1}(\mathbb{R})$ is the local solution of the Cauchy problem (5.1.2) that is constructed in Theorem 5.2.5 on the time interval $[0, T]$. Then we have the estimate*

$$\sup_{t\in[0,T]} \mathcal{E}_\sigma\big(u_\sigma(t), v_\sigma(t)\big) \leq \mathcal{E}_\sigma\big(u_\sigma(0), v_\sigma(0)\big) + \sigma^{\frac{3}{2}} C_* \|u_\sigma(t)\|_{L_T^\infty H^1(\mathbb{R})} \|v_\sigma(t)\|^2_{L_T^\infty H^1(\mathbb{R})}. \tag{5.3.20}$$

*Moreover, by (5.2.26), we get*

$$\sup_{t\in[0,T]} \mathcal{E}_\sigma(u_\sigma(t), v_\sigma(t)) \leq \mathcal{E}_\sigma\big(u_\sigma(0), v_\sigma(0)\big) + \sigma^{\frac{3}{2}} C_* \|u_{\sigma_0}(0)\|_{H^1(\mathbb{R})} \|v_{\sigma_0}(0)\|^2_{H^1(\mathbb{R})}. \tag{5.3.21}$$

***Proof:*** Applying the operator $\cosh(\sigma|D_x|)$ to the first equation of (5.1.2) with

$$u_\sigma = \cosh(\sigma|D_x|)u, \quad \widehat{u}(\xi) = \operatorname{sech}(\sigma|\xi|)\widehat{u_\sigma}(\xi).$$

and

$$v_\sigma = \cosh(\sigma|D_x|)v, \quad \widehat{v}(\xi) = \operatorname{sech}(\sigma|\xi|)\widehat{u_\sigma}(\xi).$$

we have

$$\partial_t u_\sigma + \partial_x u_\sigma - \partial_t \partial_x^2 u_\sigma + v_\sigma \partial_x v_\sigma = F_1(v_\sigma), \tag{5.3.22}$$

where

$$F_1(v_\sigma) = \frac{1}{2}\partial_x\Big[v_\sigma^2 - \cosh(\sigma|D_x|)\Big(\operatorname{sech}(\sigma|D_x|)v_\sigma\Big)^2\Big]. \tag{5.3.23}$$

Similarly, applying the operator $\cosh(\sigma|D_x|)$ to the second equation of (5.1.2), yields

$$\partial_t v_\sigma + \partial_x v_\sigma - \partial_t \partial_x^2 v_\sigma + \partial_x(u_\sigma v_\sigma) = F_2(u_\sigma, v_\sigma), \tag{5.3.24}$$

where

$$F_2(u_\sigma, v_\sigma) = \partial_x\Big[u_\sigma v_\sigma - \cosh(\sigma|D_x|)\Big(\operatorname{sech}(\sigma|D_x|)u_\sigma \operatorname{sech}(\sigma|D_x|)v_\sigma\Big)\Big]. \tag{5.3.25}$$

Now, differentiating $\mathcal{E}_\sigma(u_\sigma, v_\sigma)$ with respect to $t$, then applying integration by parts and using (5.3.22) and (5.3.24), gives

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_\sigma\big(u_\sigma(t), v_\sigma(t)\big) &= \int_{\mathbb{R}} \Big(2u_\sigma\partial_t u_\sigma + 2v_\sigma\partial_t v_\sigma + 2\partial_x u_\sigma \partial_t\partial_x u_\sigma + 2\partial_x v_\sigma \partial_t \partial_x v_\sigma\Big)\,dx \\
&= \int_{\mathbb{R}} \Big(2u_\sigma\partial_t u_\sigma + 2v_\sigma\partial_t v_\sigma - 2u_\sigma\partial_t\partial_x^2 u_\sigma - 2v_\sigma\partial_t\partial_x^2 v_\sigma\Big)\,dx \\
&= -2\int_{\mathbb{R}} \Big[u_\sigma\Big(\partial_x u_\sigma + v_\sigma\partial_x v_\sigma - F_1(v_\sigma)\Big) \\
&\quad + v_\sigma\Big(\partial_x v_\sigma + \partial_x(u_\sigma v_\sigma) - F_2(u_\sigma, v_\sigma)\Big)\Big]\,dx \\
&= -\int_{\mathbb{R}} \Big(\partial_x(u_\sigma^2) + \partial_x(v_\sigma^2) + 2\partial_x(u_\sigma)v_\sigma^2 + 2u_\sigma\partial_x(v_\sigma^2)\Big)\,dx \\
&\quad + \int_{\mathbb{R}} \Big(2u_\sigma F_1(v_\sigma) + 2v_\sigma F_2(u_\sigma, v_\sigma)\Big)\,dx \\
&= 2\int_{\mathbb{R}} u_\sigma F_1(v_\sigma)\,dx + 2\int_{\mathbb{R}} v_\sigma F_2(u_\sigma, v_\sigma)]\,dx.
\end{aligned} \tag{5.3.26}$$

Due to integration by parts of smooth functions, the integral in the fifth line is vanished.

Now, integrating (5.3.26) over $[0,t]$, we get

$$\mathcal{E}_\sigma\big(u_\sigma, v_\sigma\big) = \mathcal{E}_\sigma\big(u_\sigma(0), v_\sigma(0)\big) + 2\int_0^t\int_{\mathbb{R}} u_\sigma F_1(v_\sigma)\,dx\,dt + 2\int_0^t\int_{\mathbb{R}} v_\sigma F_2(u_\sigma, v_\sigma)\big]\,dx\,dt. \tag{5.3.27}$$

Let

$$I_1 = \int_{\mathbb{R}} u_\sigma F_1(v_\sigma)\,dx, \quad I_2 = \int_{\mathbb{R}} v_\sigma F_2(u_\sigma, v_\sigma)\big]\,dx.$$

By Hölder inequality, we get

$$\begin{aligned} |I_1| &= \Big|\int_{\mathbb{R}} \langle D_x\rangle u_\sigma \langle D_x\rangle^{-1} F_1(v_\sigma)\,dx\Big| \\ &\le C\|\langle D_x\rangle u_\sigma\|_{L^2(\mathbb{R})}\|\langle D_x\rangle^{-1}F_1(v_\sigma)\|_{L^2(\mathbb{R})} \\ &\le C\|u_\sigma\|_{H^1(\mathbb{R})}\|\langle D_x\rangle^{-1}F_1(v_\sigma)\|_{L^2(\mathbb{R})}. \end{aligned} \tag{5.3.28}$$

By taking the Fourier Transform of $\langle D_x\rangle^{-1}F_1(v_\sigma)$, where $F_1(v_\sigma)$ is as stated in (5.3.23), we obtain

$$\begin{aligned} \Big|\mathcal{F}_x\Big(\langle D_x\rangle^{-1}F_1(v_\sigma)\Big)(\xi)\Big| &= \Big|\langle\xi\rangle^{-1}\frac{i\xi}{2}\int_\xi\Big(1 - \frac{\cosh(\sigma|\xi|)}{\cosh(\sigma|\xi_1|)\cosh(\sigma|\xi_2|)}\Big)\widehat{v}_\sigma(\xi_1)\widehat{v}_\sigma(\xi_2)\,d\xi\Big| \\ &\le \frac{1}{2}\int_\xi\Big|1 - \frac{\cosh(\sigma|\xi|)}{\cosh(\sigma|\xi_1|)\cosh(\sigma|\xi_2|)}\Big|\Big|\widehat{v}_\sigma(\xi_1)\Big|\Big|\widehat{v}_\sigma(\xi_2)\Big|\,d\xi, \end{aligned} \tag{5.3.29}$$

where

$$\xi = \sum_{j=1}^{2}\xi_j, \qquad d\xi = \prod_{j=1}^{2} d\xi_j$$

$$M := 1 - \frac{\cosh(\sigma|\xi|)}{\cosh(\sigma|\xi_1|)\cosh(\sigma|\xi_2|)}. \tag{5.3.30}$$

Note that, $|M| \le 1$.

Next, we estimate $|M|$ by applying Lemma 5.3.1, for p = 2.

$$\begin{aligned} |M| &= \Big|1 - \cosh(\sigma|\xi|)\prod_{j=1}^{2}\operatorname{sech}(\sigma|\xi_j|)\Big| \\ &\le 4\sigma^2\sum_{i\ne j=1}^{2}|\xi_i||\xi_j| \\ &\le 8\sigma^2|\xi_1||\xi_2| \end{aligned}$$

For $0 \le \rho \le 1$

$$|M| \le 8\sigma^2|\xi_1||\xi_2| \le 8\sigma^{2\rho}|\xi_1|^\rho|\xi_2|^\rho.$$

By choosing $\rho = \frac{3}{4}$, we get

$$|M| \leq 8\sigma^{\frac{3}{2}}|\xi_1|^{\frac{3}{4}}|\xi_2|^{\frac{3}{4}}. \tag{5.3.31}$$

From (5.3.29) and (5.3.31), we have

$$\begin{aligned}\left|\mathcal{F}_x\Big(\langle D_x\rangle^{-1}F_1(v_\sigma)\Big)(\xi)\right| &\leq \frac{1}{2}\int_\xi |M|\big|\widehat{v}_\sigma(\xi_1)\big|\big|\widehat{v}_\sigma(\xi_2)\big|\,d\xi \\ &\leq 4\sigma^{\frac{3}{2}}\int_\xi \big|\xi_1\big|^{\frac{3}{4}}\big|\widehat{v}_\sigma(\xi_1)\big|\big|\xi_2\big|^{\frac{3}{4}}\big|\widehat{v}_\sigma(\xi_2)\big|\,d\xi.\end{aligned} \tag{5.3.32}$$

Set

$$V := \mathcal{F}_x^{-1}|\widehat{v}_\sigma(\xi)|, \quad \widehat{V} = |\widehat{v}_\sigma(\xi)|.$$

Then

$$\begin{aligned}\left|\mathcal{F}_x\Big(\langle D_x\rangle^{-1}F_1(v_\sigma)\Big)(\xi)\right| &\leq 4\sigma^{\frac{3}{2}}\int_\xi |\xi_1|^{\frac{3}{4}}\widehat{V}(\xi_1).|\xi_2|^{\frac{3}{4}}\widehat{V}(\xi_2)\,d\xi \\ &\leq 4\sigma^{\frac{3}{2}}\mathcal{F}_x\big(|D_x|^{\frac{3}{4}}V.|D_x|^{\frac{3}{4}}V\big)(\xi).\end{aligned} \tag{5.3.33}$$

Using Plancherel Theorem and one dimensional Sobolev embedding

$$H^s \subset L^P, \qquad s = \frac{1}{2} - \frac{1}{p} \qquad (2 \leq p < \infty)$$

we obtain

$$\begin{aligned}\left\|\mathcal{F}_x\left(\langle D_x\rangle^{-1}F_1(v_\sigma)\right)(\xi)\right\|_{L^2(\mathbb{R})} &\leq 4\sigma^{\frac{3}{2}}\||D_x|^{\frac{3}{4}}V.|D_x|^{\frac{3}{4}}V\big)\|_{L^2(\mathbb{R})} \\ &\lesssim \sigma^{\frac{3}{2}}\||D_x|^{\frac{3}{4}}V\|^2_{L^4(\mathbb{R})} \\ &\lesssim \sigma^{\frac{3}{2}}\||D_x|^{\frac{3}{4}}V\|^2_{H^{\frac{1}{4}}(\mathbb{R})} \\ &\lesssim \sigma^{\frac{3}{2}}\|V\|^2_{H^1(\mathbb{R})} \\ &\sim \sigma^{\frac{3}{2}}\|v_\sigma\|^2_{H^1(\mathbb{R})}.\end{aligned} \tag{5.3.34}$$

From (5.3.28) and (5.3.34), we get

$$|I_1| = \left|\int_{\mathbb{R}} u_\sigma F_1(v_\sigma)\,dx\right| \lesssim \sigma^{\frac{3}{2}}\|u_\sigma\|_{H^1(\mathbb{R})}\|v_\sigma\|^2_{H^1(\mathbb{R})}. \tag{5.3.35}$$

Similarly, by Hölder inequality, we have

$$\begin{aligned}|I_2| = \left|\int_{\mathbb{R}} v_\sigma F_2(u_\sigma, v_\sigma)\,dx\right| &= \left|\int_{\mathbb{R}}\langle D_x\rangle v_\sigma\langle D_x\rangle^{-1}F_2(u_\sigma, v_\sigma)\,dx\right| \\ &\leq C\|\langle D_x\rangle v_\sigma\|_{L^2(\mathbb{R})}\|\langle D_x\rangle^{-1}F_2(u_\sigma, v_\sigma)\|_{L^2(\mathbb{R})} \\ &\leq C\|v_\sigma\|_{H^1(\mathbb{R})}\|\langle D_x\rangle^{-1}F_2(u_\sigma, v_\sigma)\|_{L^2(\mathbb{R})}.\end{aligned} \tag{5.3.36}$$

Taking the Fourier Transform of $\langle D_x\rangle^{-1}F_2(u_\sigma, v_\sigma)$, where $F_2$ as stated in (5.3.25)

$$\begin{aligned}\Big|\mathcal{F}_x\Big(\langle D_x\rangle^{-1}F_2(u_\sigma,v_\sigma)(\xi)\Big)\Big| &= \Big|\langle\xi\rangle^{-1}i\xi\int_\xi\Big(1-\frac{\cosh(\sigma|\xi|)}{\cosh(\sigma|\xi_1|)\cosh(\sigma|\xi_2|)}\Big)\widehat{u}_\sigma(\xi_1)\widehat{v}_\sigma(\xi_2)\,d\xi\Big|\\ &\le \int_\xi\Big|1-\frac{\cosh(\sigma|\xi|)}{\cosh(\sigma|\xi_1|)\cosh(\sigma|\xi_2|)}\Big|\Big|\widehat{u}_\sigma(\xi_1)\Big|\Big|\widehat{v}_\sigma(\xi_2)\Big|\,d\xi\\ &\le \int_\xi\Big|M(\xi_1,\xi_2)\Big|\Big|\widehat{u}_\sigma(\xi_1)\Big|\Big|\widehat{v}_\sigma(\xi_2)\Big|\,d\xi\\ &\le 8\sigma^{\frac{3}{2}}\int_\xi|\xi_1|^{\frac{3}{4}}\big|\widehat{u}_\sigma(\xi_1)\big||\xi_2|^{\frac{3}{4}}\big|\widehat{v}_\sigma(\xi_2)\big|\,d\xi,\end{aligned} \tag{5.3.37}$$

where $M$ is as stated in (5.3.30) and the same choice of $\rho = \frac{3}{4}$.

$$|M| \le 8\sigma^{2\rho}|\xi_1|^\rho|\xi_2|^\rho \le 8\sigma^{\frac{3}{2}}|\xi_1|^{\frac{3}{4}}|\xi_2|^{\frac{3}{4}}.$$

Let

$$U = \mathcal{F}_x^{-1}|\widehat{u}_\sigma(\xi)|, \quad V = \mathcal{F}_x^{-1}|\widehat{v}_\sigma(\xi)|.$$

Then

$$\begin{aligned}\Big|\mathcal{F}_x\Big(\langle D_x\rangle^{-1}F_2(u_\sigma,v_\sigma)(\xi)\Big)\Big| &\le 8\sigma^{\frac{3}{2}}\int_\xi|\xi_1|^{\frac{3}{4}}\widehat{U}(\xi_1).|\xi_2|^{\frac{3}{4}}\widehat{V}(\xi_2)\,d\xi\\ &\le 8\sigma^{\frac{3}{2}}\mathcal{F}_x\big(|D_x|^{\frac{3}{4}}U.|D_x|^{\frac{3}{4}}V\big)(\xi).\end{aligned} \tag{5.3.38}$$

By using Plancherel Theorem, Hölder inequality and one dimensional Sobolev embedding, we obtain

$$\begin{aligned}\Big\|\mathcal{F}_x\Big(\langle D_x\rangle^{-1}F_2(u_\sigma,v_\sigma)(\xi)\Big)\Big\|_{L^2(\mathbb{R})} &\le 8\sigma^{\frac{3}{2}}\||D_x|^{\frac{3}{4}}U.|D_x|^{\frac{3}{4}}V)\|_{L^2(\mathbb{R})}\\ &\lesssim \sigma^{\frac{3}{2}}\||D_x|^{\frac{3}{4}}U\|_{L^4(\mathbb{R})}\||D_x|^{\frac{3}{4}}V\|_{L^4(\mathbb{R})}\\ &\lesssim \sigma^{\frac{3}{2}}\||D_x|^{\frac{3}{4}}U\|_{H^{\frac{1}{4}}(\mathbb{R})}\||D_x|^{\frac{3}{4}}V\|_{H^{\frac{1}{4}}(\mathbb{R})}\\ &\lesssim \sigma^{\frac{3}{2}}\|U\|_{H^1(\mathbb{R})}\|V\|_{H^1(\mathbb{R})}\\ &\sim \sigma^{\frac{3}{2}}\|u_\sigma\|_{H^1(\mathbb{R})}\|v_\sigma\|_{H^1(\mathbb{R})}.\end{aligned} \tag{5.3.39}$$

From (5.3.36) and (5.3.39), we obtain

$$\begin{aligned}
|I_2| &= \left| \int_{\mathbb{R}} v_\sigma F_2(u_\sigma, v_\sigma)\, dx \right| \\
&= \left| \int_{\mathbb{R}} \langle D_x \rangle v_\sigma \langle D_x \rangle^{-1} F_2(u_\sigma, v_\sigma)\, dx \right| \\
&\lesssim \sigma^{\frac{3}{2}} \|u_\sigma\|_{H^1(\mathbb{R})} \| |v_\sigma\|^2_{H^1(\mathbb{R})}.
\end{aligned} \tag{5.3.40}$$

Therefore, the inequality in (5.3.20) follows from local well-posedness result, (5.3.27), (5.3.35) and (5.3.40). □

## 5.4 Evolution of Radius of Analyticity

This section is devoted to prove the maim Theorems stated in chapter 5 of section 5.1

***Proof of Theorem 5.1.1:*** Let $u_0 = u(0) \in H^{\sigma_0,1}(\mathbb{R})$ for some $\sigma_0 > 0$ and note that

$$w_0 = \cosh(\sigma_0 |D_x|) u_0 \in H^1(\mathbb{R}).$$

By Theorem 5.2.3, there is a solution $u$ to (5.1.1) satisfying

$$u(t) \in H^{\sigma_0,1}(\mathbb{R}), \quad \forall t \in [0, T],$$

where $T$ is as in (5.2.9).

For arbitrarily large $T_l$ , we want to show that the solution $u$ to (5.1.1) satisfies

$$u(t) \in H^{\sigma(t),1}(\mathbb{R}), \quad \forall t \in [0, T_l], \tag{5.4.1}$$

for

$$\sigma(t) \geqslant \frac{c}{T_l^{2/3}}, \tag{5.4.2}$$

where $c > 0$ is a constant depending on $\|u_0\|_{H^{\sigma_0,1}}$ and $\sigma_0$.

From (5.3.15) and (5.3.16), we have

$$\mathcal{A}_\sigma[w(t)] \le C\mathcal{A}_{\sigma_0}[w_0] < \infty,$$

for all $t \in [0, T_l]$.

Now, fix $T_l$ arbitrarily large. It suffices to show that

$$\sup_{t\in[0,T_l]} \mathcal{A}_\sigma[w(t)] \le 2\mathcal{A}_{\sigma_0}[w_0], \quad \forall t \in [0, T_l], \tag{5.4.3}$$

for $\sigma$ satisfying (5.4.2), which inturn implies $u(t) \in H^{\sigma(t),1}$ for all $t \in [0, T_l]$ as desired.

To prove (5.4.3), we apply almost conservation law and local well-posedness result repeatedly on successive short time intervals to reach $T_l$.

Now, choose $n \in \mathbb{N}$ so that $T_l \in [nT, (n+1)T]$. Using induction, we can show for any $k = n+1, n \in \mathbb{N}$ that

$$\sup_{t\in[0,kT]} \mathcal{A}_\sigma[w(t)] \le \mathcal{A}_\sigma[w(0)] + kC\sigma^{\frac{3}{2}}\big(\mathcal{A}_\sigma[w(0)]\big)^{\frac{p+2}{2}}, \tag{5.4.4}$$

which implies

$$\sup_{t\in[0,kT]} \mathcal{A}_\sigma[w(t)] \le 2\mathcal{A}_{\sigma_0}[w(0)], \tag{5.4.5}$$

provided that $\sigma$ satisfies

$$\frac{2T_l}{T} C\sigma^{\frac{3}{2}}\big(\mathcal{A}_{\sigma_0}[w(0)]\big)^{\frac{p}{2}} \le 1. \tag{5.4.6}$$

For $k = 1$, by virtue of Lemma 5.3.3 and the fact $\mathcal{A}_\sigma[w(0)] \le \mathcal{A}_{\sigma_0}[w(0)]$ for $\sigma < \sigma_0$, we have

$$\begin{aligned}\sup_{t\in[0,T]} \mathcal{A}_\sigma[w(t)] &\le \mathcal{A}_\sigma[w(0)]) + C\sigma^{\frac{3}{2}}\Big(\mathcal{A}_\sigma[w(0)]\Big)^{\frac{p+2}{2}} \\ &\le \mathcal{A}_{\sigma_0}[w(0)] + C\sigma^{\frac{3}{2}}\Big(\mathcal{A}_{\sigma_0}[w(0)]\Big)^{\frac{p+2}{2}}.\end{aligned}$$

This implies (5.4.5) holds by (5.4.6) provided that $\sigma$ satisfies

$$C\sigma^{\frac{3}{2}}\Big(\mathcal{A}_{\sigma_0}[w(0)]\Big)^{\frac{p}{2}} \le 1.$$

Now, assume that (5.4.4) implies (5.4.5) for $k = n$ and $\sigma$ satisfies (5.4.6). Then our claim is to show that (5.4.4) and (5.4.5) hold for $k = n+1$.
Applying Lemma 5.3.3, (5.4.5) and (5.4.4) within time interval $nT \le t \le (n+1)T$, we obtain

$$\begin{aligned}\sup_{t\in[nT,(n+1)T]} \mathcal{A}_\sigma[w(t)] &\le \mathcal{A}_\sigma[w(nT)] + C\sigma^{\frac{3}{2}}\Big(\mathcal{A}_\sigma[w(nT)]\Big)^{\frac{p+2}{2}} \\ &\le \mathcal{A}_\sigma[w(nT)] + C\sigma^{\frac{3}{2}}\Big(2\mathcal{A}_{\sigma_0}[w(0)]\Big)^{\frac{p+2}{2}} \\ &\le \mathcal{A}_\sigma[w(0)] + C\sigma^{\frac{3}{2}}(n+1)\Big(\mathcal{A}_{\sigma_0}[w(0)]\Big)^{\frac{p+2}{2}}.\end{aligned} \tag{5.4.7}$$

Combining (5.4.7) with the induction hypothesis of (5.4.4) for $k = n$, we obtain

$$\sup_{t\in[0,(n+1)T]} \mathcal{A}_\sigma[w(t)] \leq \mathcal{A}_\sigma[w(0)] + C\sigma^{\frac{3}{2}}(n+1)\Big(\mathcal{A}_{\sigma_0}[w(0)]\Big)^{\frac{p+2}{2}}.$$

This proves (5.4.4) for $k = n+1$. Consequently, (5.4.5) holds for $k = n+1$ provided that

$$C\sigma^{\frac{3}{2}}(n+1)\big(\mathcal{A}_{\sigma_0}[w(0)]\big)^{\frac{p}{2}} \leq 1,$$

which follows from (5.4.6).

Since

$$n+1 \leq \frac{T_l}{T} + 1 = \frac{2T_l}{T},$$

we have

$$\frac{2T_l}{T} C\sigma^{\frac{3}{2}}\big(\mathcal{A}_{\sigma_0}[w(0)]\big)^{\frac{p}{2}} = 1.$$

Thus

$$\sigma(t) = \left(\frac{c_1}{T_l}\right)^{\frac{2}{3}},$$

where

$$c_1 = \frac{T}{2C\big(\mathcal{A}_{\sigma_0}[w(0)]\big)^{\frac{p}{2}}},$$

Therefore,

$$\sigma(t) \geq \left(\frac{c}{T_l}\right)^{\frac{2}{3}}$$

for $c \leq c_1$ which gives (5.4.2), where T is as in (5.2.9). This completes the proof of Theorem 5.1.1. □

***Proof of Theorem 5.1.2:*** In the course of the proof, we need to apply repeatedly the local result obtained in Theorem 5.2.5 using the approximate conservation law obtained in Theorem 5.3.4 to cover time intervals of arbitrary length. To construct a solution on $[0, T^*]$ for arbitrarily large $T^*$, consider the following two possible cases.

The first case is $T^* = \infty$, which means that $(u, v) \in C\big([0,\infty); H^{\sigma_0,1}(\mathbb{R}) \times H^{\sigma_0,1}(\mathbb{R})\big)$. In this case the uniform radius of spatial analyticity of $(u, v)$ persistence for all time $t$. That is, $\sigma(t) = \sigma_0$ which proves Theorem 5.1.2.

The second case is $T^* < \infty$, which means $(u, v) \in C\big([0, T^*]; H^{\sigma,1}(\mathbb{R}) \times H^{\sigma,1}(\mathbb{R})\big)$ and

$$\frac{1}{4C_*\|W_0\|_{H^{\sigma,s}(\mathbb{R})\times H^{\sigma,s}(\mathbb{R})}} = T \leq T^* < \infty.$$

To prove Theorem 5.1.2 in the second case, we apply the approximate conservation law in Theorem 5.3.4, so as to repeat the local result on successive short time intervals to reach $T^*$ by adjusting the strip width parameter $\sigma$ according to the size of $T^*$.

First, we consider the case $s = 1$. By the embedding (2.1.6) the general case, $s \in \mathbb{R}$ will essentially reduce to $s = 1$.

**The Case: $\mathbf{s = 1}$**: Let $(u(0), v(0)) \in H^{\sigma_0,1}(\mathbb{R}) \times H^{\sigma_0,1}(\mathbb{R})$, for some $\sigma_0 > 0$. Then there is a unique solution

$$(u, v) \in H^{\sigma(t),1}(\mathbb{R}) \times H^{\sigma(t),1}(\mathbb{R}), \forall t \in [0, T],$$

of (5.1.2) constructed in Theorem 5.2.5 with existence time T as in (5.2.25).

Note that, since

$$(u_{\sigma_0}, v_{\sigma_0}) = e^{\sigma_0|D_x|}u_0 \times e^{\sigma_0|D_x|}v_0 \in H^1(\mathbb{R}) \times H^1(\mathbb{R}),$$

from (5.2.26) and (5.3.18) we have

$$\begin{aligned} \mathcal{E}_\sigma(u_\sigma, v_\sigma) &= \int_{\mathbb{R}} u_\sigma^2 + v_\sigma^2 + (\partial_x u_\sigma^2) + (\partial_x v_\sigma^2)\, dx \\ &\lesssim \|(u(0), v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})} < \infty. \end{aligned} \tag{5.4.8}$$

For arbitrarily large $T^*$, we want to show that the solution $(u, v)$ to (5.1.2) satisfies

$$(u, v) \in H^{\sigma(t),1}(\mathbb{R}) \times H^{\sigma(t),1}(\mathbb{R}), \quad \forall t \in [0, T^*]. \tag{5.4.9}$$

and

$$\sigma(t) \geqslant \frac{c}{T^*}, \tag{5.4.10}$$

where $c > 0$ is a constant depending on the norm of the initial data $(u_0, v_0)$ and $\sigma_0$.

From (5.4.8), we have

$$\|(u, v)\|^2_{H^{\sigma(t),1}(\mathbb{R})\times H^{\sigma(t),1}(\mathbb{R})} < \infty, \quad t \in [0, T^*].$$

Now, fix $T^*$ arbitrarily large. It suffices to show that

$$\sup_{t\in[0,T^*]} \|(u, v)\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \le 2\|(u(0), v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}, \tag{5.4.11}$$

which inturn implies $(u, v) \in H^{\sigma(t),1}(\mathbb{R}) \times H^{\sigma(t),1}(\mathbb{R})$, for $\sigma$ satisfying (5.4.10). To prove (5.4.11), we use induction as follows.

Choose $n \in \mathbb{N}$, so that $T^* \in [nT, (n+1)T]$. Using induction, we can show for any $k \in \{1, 2, 3, ..., n+1\}$ that

$$\sup_{t\in[0,kT]} \|(u,v)\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \le \|(u(0),v(0))\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} + k\sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})}, \tag{5.4.12}$$

$$\sup_{t\in[0,kT]} \|(u,v)\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \le 2\|(u(0),v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}, \tag{5.4.13}$$

provided that $\sigma$ satisfies

$$\frac{2T^*\sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})}}{T\|(u(0),v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}} \le 1. \tag{5.4.14}$$

For $k = 1$, from Theorem 5.3.4 we have

$$\sup_{t\in[0,T]} \|(u,v)\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \le \|(u(0),v(0))\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} + \sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})}, \tag{5.4.15}$$

Since $\sigma < \sigma_0$, we have

$$\|(u(0),v(0)))\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \le \|(u(0),v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}.$$

Then, it follows that

$$\frac{\sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})}}{\|(u(0),v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}} \le 1, \tag{5.4.16}$$

which holds by (5.4.14).
Now, combining (5.4.15) and (5.4.16) leads to

$$\begin{aligned}\sup_{t\in[0,kT]} \|(u,v)\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} &\le \|(u(0),v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})} \\ &\quad + \|(u(0),v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}, \\ &\le 2\|(u(0),v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})},\end{aligned} \tag{5.4.17}$$

which proves (5.4.13).

Next, assume that (5.4.12) and (5.4.13) hold for some $k \in \{1, 2, 3, ..., n\}$. Then, we need to show that (5.4.12) and (5.4.13) hold for $k = n + 1$ .

By applying Theorem 5.3.4, (5.4.13) and (5.4.12), we obtain

$$
\begin{aligned}
\sup_{t\in[kT,(k+1)T]} \|(u(t),v(t))\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} &\leq \|(u(kT),v(kT))\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \\
&+ \sigma C_* \|u(kT)\|_{H^{\sigma_0,1}} \|v(kT)\|^2_{H^{\sigma_0,1}} \\
&\leq \|(u(kT),v(kT)\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \\
&+ \sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})} \\
&\leq \|(u(0),v(0))\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \\
&+ (k+1)\sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})}
\end{aligned}
\tag{5.4.18}
$$

Combining (5.4.18) with the induction hypothesis (5.4.12) for $k = n$, we obtain

$$
\begin{aligned}
\sup_{t\in[0,(k+1)T]} \|(u(t),v(t))\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} &\leq \|(u(0),v(0))\|^2_{H^{\sigma,1}(\mathbb{R})\times H^{\sigma,1}(\mathbb{R})} \\
&+ (k+1)\sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})},
\end{aligned}
\tag{5.4.19}
$$

which proves (5.4.12) for $k = n+1$.

Since

$$k+1 \leq n+1 \leq \frac{T^*}{T} + 1 \leq \frac{2T^*}{T},$$

inequality (5.4.13) follows from (5.4.14) for $k = n+1$, provided that

$$\frac{(k+1)\sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})}}{\|(u_0,v_0)\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}} \leq 1.$$

Finally, the condition (5.4.14) is satisfied for $\sigma$ such that

$$\frac{2T^*\sigma C_* \|u(0)\|_{H^{\sigma_0,1}(\mathbb{R})} \|v(0)\|^2_{H^{\sigma_0,1}(\mathbb{R})}}{T\|(u(0),v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}} = 1.$$

Thus

$$\sigma(t) = \frac{c_1}{T^*},$$

where

$$c_1 = \frac{T\|(u(0),v(0))\|^2_{H^{\sigma_0,1}(\mathbb{R})\times H^{\sigma_0,1}(\mathbb{R})}}{\sigma C_* 2^{\frac{5}{2}} \|u_0\|_{H^{\sigma_0,1}(\mathbb{R})} \|v_0\|^2_{H^{\sigma_0,1}(\mathbb{R})}}.$$

Therefore, (5.4.10) holds for $c \le c_1$, and $T$ is as in (5.2.25).

**The general case $s \in \mathbb{R}$:** For any $s \in \mathbb{R}$, we use the embedding (2.1.6) to get

$$(u(0), v(0)) \in H^{\sigma_0,s}(\mathbb{R}) \times H^{\sigma_0,s}(\mathbb{R}) \subset H^{\frac{\sigma_0}{2},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{2},1}(\mathbb{R}).$$

From the local existence theory for every $(u(0), v(0)) \in H^{\frac{\sigma_0}{2},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{2},1}(\mathbb{R})$, there exist

$$T = T\Big(\|(u_0, v_0)\|_{G^{\frac{\sigma_0}{2},1}(\mathbb{R}) \times G^{\frac{\sigma_0}{2},1}(\mathbb{R})}\Big) > 0,$$

and a solution

$$(u(t), v(t)) \in C\big([0, T]; H^{\frac{\sigma_0}{2},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{2},1}(\mathbb{R})\big).$$

For arbitrarily large $T^*$, from the case $s = 1$, we have

$$(u(t), v(t)) \in H^{2b_*(T^*)^{-1},1}(\mathbb{R}) \times H^{2b_*(T^*)^{-1},1}(\mathbb{R}), \quad 0 \le t \le T^*,$$

where $b_* > 0$ depends on $\|(u(0), v(0))\|_{H^{\frac{\sigma_0}{2},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{2},1}(\mathbb{R})}$.

Applying again the embedding property (2.1.6), we get

$$(u(0), v(0)) \in H^{\sigma_0,s}(\mathbb{R}) \times H^{\sigma_0,s}(\mathbb{R}) \subset H^{\frac{\sigma_0}{2},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{2},1}(\mathbb{R}) \subset H^{\frac{\sigma_0}{3},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{3},1}(\mathbb{R}),$$

for $0 \le \frac{\sigma_0}{3} < \frac{\sigma_0}{2}$.

Now, from Theorem 5.2.5, for every $(u(0), v(0)) \in G^{\frac{\sigma_0}{3},1}(\mathbb{R}) \times G^{\frac{\sigma_0}{3},1}(\mathbb{R})$, there exist

$$T = T\Big(\|(u(0), v(0))\|_{H^{\frac{\sigma_0}{3},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{3},1}(\mathbb{R})}\Big) > 0$$

and

$$(u(t), v(t)) \in H^{\frac{\sigma_0}{3},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{3},1}(\mathbb{R}), \quad t \in [0, T].$$

For arbitrarily large $T^*$, again from the case $s = 1$, we have

$$(u(t), v(t)) \in H^{3b_*(T^*)^{-1},1}(\mathbb{R}) \times H^{3b_*(T^*)^{-1},1}(\mathbb{R}), \quad t \in [0, T^*],$$

where $b_* > 0$ depends on $\|(u_0, v_0)\|_{H^{\frac{\sigma_0}{3},1}(\mathbb{R}) \times H^{\frac{\sigma_0}{3},1}(\mathbb{R})}$.

Applying the embedding property (2.1.6) again and again, we reduce the general case to the case $s = 1$, and conclude that

$$(u(t), v(t)) \in H^{b_*(T^*)^{-1},1}(\mathbb{R}) \times H^{b_*(T^*)^{-1},1}(\mathbb{R}), \quad t \in [0, T^*].$$

This completes the proof of Theorem 5.1.2. □

# Chapter 6

# KP-BBM Equation in Anisotropic Gevrey Spaces

## 6.1 Problem Statement

Consider a Cauchy problem for Kadomtsev, Petviashvili - Benjamin, Bona, Mahony (KP-BBM) equation with data in anisotropic Gevrey spaces

$$\begin{cases} u_t - u_{xxt} + u_x + uu_x + \partial_x^{-1} u_{yy} = 0, \\ u(x,y,0) = u_0(x,y) \in G^{\sigma_1,\sigma_2 \bar{s}}(\mathbb{R}^2), \end{cases} \tag{6.1.1}$$

where $u = u(x,y,t) \in \mathbb{R}^{2+1}$. KP-BBM equation describes the unidirectional propagation of dispersive long waves with weak transverse effects. It was derived by Wazwaz in [90].

Recall, conservation law obtained from (6.1.1) is given by

$$\mathcal{A}[u(x,y,t)] = \int_{\mathbb{R}^2} \Big( u^2(x,y,t) + u_x^2(x,y,t) \Big) dxdy = \mathcal{A}[u(x,y,0)], \tag{6.1.2}$$

for all $t \in \mathbb{R}$.

In the present chapter, we will study the property of spatial analyticity of the solution $u(x,y,t)$ to (6.1.1), given that the initial data $u_0(x,y)$ is real-analytic with uniform radius of analyticity $\sigma_0$, so there is a holomorphic extension to a complex strip

$$\mathrm{S}_{\sigma_0} = \{ x + iy \in \mathbb{C} : |y| < \sigma_0 \} .$$

Since the radius of analyticity can be related to the asymptotic decay of the Fourier transform, it is natural to use Fourier methods to study spatial analyticity of solution to problem of type (6.1.1).

For $s_1, s_2 \in \mathbb{R}$, let $\bar{s} = s_1, s_2$ and $\sigma_1, \sigma_2 \geq 0$. We define anisotropic Gevrey space, $G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)$ via norm

$$
\begin{aligned}
\|u\|^2_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)} &= \|e^{\sigma_1|\xi|}e^{\sigma_2|\eta|}\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{u}(\xi,\eta,t)\|^2_{L_2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}^2} e^{2\sigma_1|\xi|}e^{2\sigma_2|\eta|}\langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2}|\hat{u}(\xi,\eta,t)|^2 d\xi d\eta,
\end{aligned}
\tag{6.1.3}
$$

where $\hat{u}$ denotes the spatial Fourier transform, given by

$$
\hat{u}(\xi,\eta,t) = \int_{\mathbb{R}^2} u(x,y,t)e^{-i(x\xi+y\eta)}dxdy. \tag{6.1.4}
$$

For functions in anisotropic Gevrey $G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)$, if one of the variables is fixed, the resulting function in the other variable will have a holomorphic extension satisfying the stated bounds in Paley-Wiener Theorem (see, Theorem 2.1.20).

In addition to the holomorphic extension property, anisotropic Gevrey spaces satisfy the embeddings $G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2) \hookrightarrow G^{\sigma_1',\sigma_2',\bar{s}'}(\mathbb{R}^2)$ for $0 \le \sigma_i' < \sigma_i$, for $i = 1, 2$ and $s_1, s_2 \in \mathbb{R}$ which follow from the corresponding estimate

$$
\|f\|_{G^{\sigma_1',\sigma_2',\bar{s}'}(\mathbb{R}^2)} \lesssim \|f\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)}. \tag{6.1.5}
$$

In particular, for $\sigma_1' = \sigma_2' = 0$, $\|f\|_{H^{\bar{s}'}} \lesssim \|f\|_{G^{\sigma_1,\sigma_2,\bar{s}}}$, $H^{\bar{s}}$ is anisotropic Sobolev space.

Our main result yields an estimate on how the width of the strip of the radius of the spatial analyticity decay with time in the x - direction.

**Theorem 6.1.1** (Lower bound for radius of spatial analyticity). *Let $\sigma_{1_0} > 0$ and $\sigma_{2_0} \ge 0$, and assume $u_0 \in H^{\sigma_{1_0},\sigma_{2_0},\bar{s}}$. Then, the solution $u$ to (6.1.1) is globally well-posed in time, and for any $T^* > 0$, we have*

$$
u \in C\left([0,T^*]; H^{\sigma_1(t),0,\bar{s}}(\mathbb{R}^2)\right)
$$

*with lower bound for radius of spatial analyticity*

$$
\sigma_1(t) \ge ct^{-1},
$$

*where $c > 0$, is a constant depending on $\|u_0\|_{G^{\sigma_{1_0},\sigma_{2_0},\bar{s}}}$, $\sigma_{1_0}$ and $\sigma_{2_0}$.*

The method used here for proving lower bounds of the radius of analyticity was introduced in [81] in the study of the 1D Dirac-Klein-Gordon equations.

**Theorem 6.1.2** (Sobolev embedding theorem [45] ). *Let $1 < p < \infty$ and $0 < s < \frac{d}{p}$. Then, the Sobolev space $W^{s,p}\left(\mathbb{R}^d\right)$ embeds continuously in $L^q\left(\mathbb{R}^d\right)$ for $q$ such that*

$$
\frac{1}{p} - \frac{1}{q} = \frac{s}{d}, \tag{6.1.6}
$$

*or for any*

$$\frac{d}{s} < q < \infty.$$

Let $\sigma_i \geq 0$, for $i = 1, 2$ and define a Fourier multiplier operator

$$I^{\sigma_1,\sigma_2} u = \mathcal{F}^{-1}\big(m(\xi)m(\eta)\widehat{u}(\xi, \eta, t)\big), \tag{6.1.7}$$

where $\mathcal{F}^{-1}$ denotes the inverse of Fourier transform, and the symbol

$$m(\xi) = \cosh(\sigma_1|\xi|) = \frac{1}{2}\left(e^{\sigma_1|\xi|} + e^{-\sigma_1|\xi|}\right),$$

$$m(\eta) = \cosh(\sigma_2|\eta|) = \frac{1}{2}\left(e^{\sigma_2|\eta|} + e^{-\sigma_2|\eta|}\right), \quad \xi, \eta \in \mathbb{R}$$

The modified anisotropic Gevrey space, $H^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)$ is obtained from the anisotropic Gevrey space, $G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)$ by replacing the exponential weight $e^{\sigma_1|\xi|}e^{\sigma_2|\eta|}$ with the hyperbolic weight $\cosh(\sigma_1|\xi|)\cosh(\sigma_2|\eta|)$, equipped with the norm

$$\|f\|^2_{H^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R})} = \|\cosh(\sigma_1|\xi|)\cosh(\sigma_2|\eta|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\widehat{f}(\xi,\eta)\|^2_{L^2(\mathbb{R}^2)}, \tag{6.1.8}$$

for $\sigma_1, \sigma_2 \geq 0$.
Clearly, we have the bound

$$\frac{1}{2}e^{\sigma|\xi|} \leq m(\xi) \leq e^{\sigma|\xi|}, \quad \xi \in \mathbb{R}.$$

Then, it follows that

$$\frac{1}{2}\|u\|_{G^{\sigma_1,\sigma_2}(\mathbb{R}^2)} \leq \|I^{\sigma_1,\sigma_2}u\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{G^{\sigma_1,\sigma_2}(\mathbb{R}^2)},$$

which implies that $\|I^{\sigma_1,\sigma_2}u\|_{L^2(\mathbb{R}^2)}$ is an equivalent norm with $\|u\|_{G^{\sigma_1,\sigma_1}(\mathbb{R}^2)}$.

## 6.2 Local Well-posedness Result

Consider the integral form of (6.1.1), which is given by

$$u(t) = S(t)u_0 - \frac{1}{2}\int_0^t S(t-\tau)\varphi\left(D_x\right)\left(u^2(\tau)\right)d\tau, \tag{6.2.1}$$

where

$$\varphi(D_x) := \partial_x\left(1 - \partial_x^2\right)^{-1},$$

and $S(t)$ is a Fourier multiplier given by

$$S(t) = \mathcal{F}^{-1}_{x,y}\left(e^{\mathrm{i}t\left(1+\xi^2\right)^{-1}\left(\xi+\xi^{-1}\eta^2\right)}\right).$$

Since the modulus of $e^{\mathrm{i}t\left(1+\xi^2\right)^{-1}\left(\xi+\xi^{-1}\eta^2\right)}$ is 1, we have

$$\|S(t)u\|_{H^{\bar{s}}(\mathbb{R}^2)} = \|u\|_{H^{\bar{s}}(\mathbb{R}^2)}, \quad \forall t \in \mathbb{R}, \bar{s} = s_1, s_2 \in \mathbb{R}^2.$$

We need the following bilinear estimate to prove local well-posedness results.

**Lemma 6.2.1.** *For $\bar{s} = s_1, s_2$ such that $s_1, s_2, \geq 0$ and $\sigma_1, \sigma_2 \geq 0$, we have*

$$\|\varphi(D_x)u^2\|^2_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)} \leq C\|u\|^2_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)}. \tag{6.2.2}$$

***Proof:*** Using (2.1.7), we have

$$\begin{aligned}\|\varphi(D_x)u^2\|^2_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)} &= \|\langle\xi\rangle^{s_1}e^{\sigma_1|\xi|}\langle\eta\rangle^{s_2}e^{\sigma_2|\eta|}\widehat{\varphi(D_x)(u^2)}(\xi,\eta)\|^2_{L^2(\mathbb{R}^2)} \\ &= \|\langle\xi\rangle^{s_1}e^{\sigma_1|\xi|}\langle\eta\rangle^{s_2}e^{\sigma_2|\eta|}\varphi(\xi)(\widehat{u}*\widehat{u})(\xi,\eta)\|^2_{L^2(\mathbb{R}^2)} \\ &= \int_{\mathbb{R}^2}\Bigg[\langle\xi\rangle^{2s_1}e^{2\sigma_1|\xi|}\langle\eta\rangle^{2s_2}e^{2\sigma_2|\eta|}\frac{\xi^2}{(1+\xi^2)^2} \\ &\quad \left(\int_{\mathbb{R}^2}\widehat{u}(\xi-\xi_1,\eta-\eta_1)\widehat{u}(\xi_1,\eta_1)\,d\xi_1d\eta_1\right)^2\Bigg]d\xi d\eta.\end{aligned} \tag{6.2.3}$$

Now, for $s_1, s_2 \geq 0$, we have

$$\langle\xi\rangle^{s_1} \leq \langle\xi-\xi_1\rangle^{s_1}\langle\xi_1\rangle^{s_1}, \quad \langle\eta\rangle^{s_2} \leq \langle\eta-\eta_1\rangle^{s_2}\langle\eta_1\rangle^{s_2}.$$

and

$$e^{\sigma_1|\xi|} \leq e^{\sigma_1|\xi-\xi_1|}e^{\sigma_1|\xi_1|}, \quad e^{\sigma_2|\eta|} \leq e^{\sigma_2|\eta-\eta_1|}e^{\sigma_2|\eta_1|},$$

for $\xi_1, \eta_1 \in \mathbb{R}$.
Then, it follows that

$$\begin{aligned}\|\varphi(D_x)u^2\|^2_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)} \leq \int_{\mathbb{R}^2}\Bigg[\frac{\xi^2}{(1+\xi^2)}\int_{\mathbb{R}^2}&\Big(\langle\xi-\xi_1\rangle^{2s}e^{2\sigma|\xi-\xi_1|} \\ &\langle\eta-\eta_1\rangle^{2s}e^{2\sigma|\eta-\eta_1|}\widehat{u}^2(\xi-\xi_1,\eta-\eta_1)\Big) \\ &\Big(\langle\xi_1\rangle^{2s}\langle\eta_1\rangle^{2s}e^{2\sigma_1|\xi_1|}e^{2\sigma_2|\eta_1|}\widehat{u}^2(\xi_1,\eta_1)\;d\xi_1\,d\eta_1\Big)\Bigg]\,d\xi\,d\eta.\end{aligned} \tag{6.2.4}$$

Since $\xi^2/(1+\xi^2)^2 \le 1/(1+\xi^2)$, by the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\|\varphi(D_x)u^2\|^2_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)} &\le \int_{\mathbb{R}^2} \|u\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)}\|u\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)}\frac{1}{1+\xi^2}\,d\xi\,d\eta \\ &\le C\|u\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)}\|u\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)} \\ &\le C\|u\|^2_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)},\end{aligned} \tag{6.2.5}$$

which completes the proof of Lemma 6.2.1. □

**Theorem 6.2.2.** *Let $\sigma_i \ge 0$ and $\bar{s} = s_i \ge 0$ for $i = 1, 2$. Then for all initial data $u_0(x,y) \in G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)$, there exist $T = T\left(\|u_0\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)}\right) > 0$ and a unique solution $u$ of (6.1.1) in the time interval $[0,T]$ for $T > 0$ such that*

$$u \in C\left([0,T], G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)\right).$$

*Moreover the solution depends continuously on the data $u_0$. In particular, the time of existence can be chosen to satisfy*

$$T = \frac{c_0}{\left(1 + \|u_0\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)}\right)}, \tag{6.2.6}$$

*for some constants $c_0 > 0$. And the solution $u$ satisfies*

$$\sup_{t\in[0,T]} \|u(t)\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)} \le 2\|u_0\|_{G^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)}. \tag{6.2.7}$$

It is straightforward to prove Theorem 6.2.2. Applying the contraction mapping principle and multilinear estimate in Lemma 6.2.1 for the integral equation (6.2.1) and following similar argument as the proof of Theorem 4.3.1 and 5.2.3, leads to the desired result.

## 6.3 Almost Conservation Law

In this section, we will prove an almost conservation law of the KP-BBM equation (6.1.1) in modified an isotropic Gevrey space $H^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)$. This plays a key role in the proof of Theorem 6.1.1. The method we used here previously applied in [37] for the Beam equation.

To proof almost conservation law, let us first state the following lemma.

**Lemma 6.3.1** ([37])**.** *Let $\xi = \sum_{i=1}^{p} \xi_i$ for $\xi_i \in \mathbb{R}$ and $p$ is positive integer. Then*

$$\left|1 - \cosh|\xi| \prod_{i=1}^{p} \operatorname{sech}|\xi_i|\right| \le 2^p \sum_{i\ne j=1}^{p} |\xi_i||\xi_j|. \tag{6.3.1}$$

**Theorem 6.3.2** (Almost conservation law). *Let $\sigma_1 > 0$, $\sigma_2 \geq 0$, $u_0 \in H^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)$ and $I^{\sigma_1,\sigma_2}$ be the Fourier multiplier given by (6.1.7) and $u \in C\left([0,T]; H^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2)\right)$ be the local solution obtained in Theorem 6.2.2. Then the solution of KP-BBM equation satisfies*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{H^{\sigma_1,0,1,0}(\mathbb{R}^2)} \leq \|u_0\|^2_{H^{\sigma_1,0,1,0}(\mathbb{R}^2)} + C\sigma_1 \|u_0\|^3_{H^{\sigma_1,0,1,0}(\mathbb{R}^2)}. \tag{6.3.2}$$

***Proof:*** By the embedding property in (6.1.5) it suffices to consider the case $\sigma_2 = 0$, since

$$H^{\sigma_1,\sigma_2,\bar{s}}(\mathbb{R}^2) \hookrightarrow H^{\sigma_1,0,\bar{s}}(\mathbb{R}^2), \sigma_1, \sigma_2 > 0, \bar{s} \in \mathbb{R}^2.$$

Applying the operator $I^{\sigma_1,0}$ to (6.1.1) and set

$$\lambda = I^{\sigma_1,0}u,$$

then (6.1.1) becomes

$$\lambda_t - \lambda_{xxt} + \lambda_x + \partial_x^{-1}\lambda_{yy} + \lambda\lambda_x = F(\lambda), \tag{6.3.3}$$

where

$$F(\lambda) = \lambda\lambda_x - I^{\sigma_1,0}\big(I^{-(\sigma_1,0)}\big)^2\lambda\lambda_x. \tag{6.3.4}$$

Multiplying (6.3.3) by $\lambda$ and integrating with respect to the spatial variables, we obtain

$$\int_{\mathbb{R}^2} \Big(\lambda\lambda_t - \lambda\lambda_{xxt} + \lambda\lambda_x + \lambda\partial_x^{-1}\lambda_{yy} + \lambda^2\lambda_x\Big)dxdy = \int_{\mathbb{R}^2} \lambda F(\lambda)dxdy. \tag{6.3.5}$$

If we apply integration by parts, we may rewrite (6.3.5) as

$$\int_{\mathbb{R}^2} \Big(\lambda\lambda_t + \lambda_x\lambda_{xt} + \lambda\lambda_x - \lambda_y\partial_x^{-1}\lambda_y + \lambda^2\lambda_x\Big)dxdy = \int_{\mathbb{R}^2} \lambda F(\lambda)dxdy,$$

which implies that

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^2} (\lambda^2 + \lambda_x^2)dxdy + \frac{d}{dx}\int_{\mathbb{R}^2} (\frac{1}{2}\lambda^2 - \frac{1}{2}(\partial_x^{-1}\lambda_y)^2 + \frac{1}{3}\lambda^3)dxdy = \int_{\mathbb{R}^2} \lambda F(\lambda)dxdy.$$

For $\lambda$ and its spatial derivatives vanishing at infinity, we thus obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} (\lambda^2 + \lambda_x^2)(x,y,t)dxdy = 2\int_{\mathbb{R}^2} \lambda F(\lambda)(x,y,t)dxdy. \tag{6.3.6}$$

Integrating (6.3.6) with respect to time interval $[0,t]$, yields

$$\begin{aligned}\int_{\mathbb{R}^2} (\lambda^2 + \lambda_x^2)(x,y,t)dxdy = &\int_{\mathbb{R}^2} (\lambda^2 + \lambda_x^2)(x,y,0)dxdy \\ &+ 2\int_0^t \int_{\mathbb{R}^2} \lambda\,(x,y,\tau)\,F(\lambda)\,(x,y,\tau)\,dxdyd\tau.\end{aligned} \tag{6.3.7}$$

Using Hölder inequality and Sobolev embedding, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\lambda F(\lambda)dxdy\right| &= \left|\int_{\mathbb{R}^2}\langle D_x\rangle\lambda\langle D_x\rangle^{-1}F(\lambda)dxdy\right| \\ &\le \|\langle D_x\rangle\lambda\|_{L^2_{x,y}(\mathbb{R}^2)}\|\langle D_x\rangle^{-1}F(\lambda)\|_{L^2_{x,y}(\mathbb{R}^2)} \\ &\le \|\lambda\|_{H^{1,0}(\mathbb{R}^2)}\|\langle D_x\rangle^{-1}F(\lambda)\|_{L^2_{x,y}(\mathbb{R}^2)}.\end{aligned} \tag{6.3.8}$$

Taking the Fourier Transform of $\langle D_x\rangle^{-1}F(\lambda)$, we obtain

$$\begin{aligned}\left|\mathcal{F}\Big(\langle D_x\rangle^{-1}F(\lambda)\Big)(\xi,\eta)\right| &= \left|\langle\xi\rangle^{-1}\frac{i\xi}{2}\int_{\xi,\eta}\Big(1-\frac{\cosh(\sigma_1|\xi|)}{\cosh(\sigma_1|\xi_1|)\cosh(\sigma_1|\xi_2|)}\Big)\widehat{\lambda}(\xi_1,\eta_1)\widehat{\lambda}(\xi_2,\eta_2)\,d\xi\,d\eta\right| \\ &\le \frac{1}{2}\int_{\xi,\eta}\left|1-\frac{\cosh(\sigma_1|\xi|)}{\cosh(\sigma_1|\xi_1|)\cosh(\sigma_1|\xi_2|)}\right|\left|\widehat{\lambda}(\xi_1,\eta_2)\right|\left|\widehat{\lambda}(\xi_2,\eta_2)\right|\,d\xi\,d\eta,\end{aligned} \tag{6.3.9}$$

where

$$\xi=\sum_{j=1}^{2}\xi_j,\quad \eta=\sum_{j=1}^{2}\eta_j,\qquad d\xi d\eta=\prod_{j=1}^{2}d\xi_j d\eta_j.$$

Note that

$$\left|1-\frac{\cosh(\sigma_1|\xi|)}{\cosh(\sigma_1|\xi_1|)\cosh(\sigma_1|\xi_2|)}\right|\le 1.$$

By applying Lemma 6.3.1, for p = 2, we obtain

$$\begin{aligned}\left|1-\frac{\cosh(\sigma_1|\xi|)}{\cosh(\sigma|\xi_1|)\cosh(\sigma_1|\xi_2|)}\right| &= \left|1-\cosh(\sigma_1|\xi|)\prod_{j=1}^{2}\operatorname{sech}(\sigma_1|\xi_j|)\right| \\ &\le 4\sigma_1^2\sum_{i\neq j=1}^{2}|\xi_i||\xi_j|\le 8\sigma_1^2|\xi_1||\xi_2|.\end{aligned} \tag{6.3.10}$$

By choosing $\theta=\frac{1}{2}$ in $[0,1]$

$$8\sigma^2|\xi_1||\xi_2|\le 8\sigma^{2\theta}|\xi_1|^{\theta}|\xi_2|^{\theta}\le 8\sigma|\xi_1|^{\frac{1}{2}}|\xi_2|^{\frac{1}{2}}. \tag{6.3.11}$$

Plugging (6.3.11) in to (6.3.9), we obtain

$$\begin{aligned}\left|\mathcal{F}\Big(\langle D_x\rangle^{-1}F(\lambda)\Big)(\xi,\eta)\right| &\le \frac{1}{2}\int_{\xi,\eta}\left|1-\frac{\cosh(\sigma_1|\xi|)}{\cosh(\sigma_2|\xi_1|)\cosh(\sigma|\xi_2|)}\right|\left|\widehat{\lambda}(\xi_1,\eta_1)\widehat{\lambda}(\xi_2,\eta_2)\,d\xi d\eta\right| \\ &\le 4\sigma_1\int_{\xi,\eta}|\xi_1|^{\frac{1}{2}}|\widehat{\lambda}(\xi_1,\eta_1)||\xi_2|^{\frac{1}{2}}|\widehat{\lambda}(\xi_2,\eta_2)|\,d\xi d\eta,\end{aligned} \tag{6.3.12}$$

Set

$$\Lambda := \mathcal{F}^{-1}_{x,y}|\widehat{\lambda}(\xi,\eta)|, \quad \widehat{\Lambda} = |\widehat{\lambda}(\xi,\eta)|.$$

Then

$$\begin{aligned} \left|\mathcal{F}\Big(\langle D_x\rangle^{-1}F(\lambda)\Big)(\xi,\eta)\right| &\leq 4\sigma_1\int_{\xi,\eta}|\xi_1|^{\frac{1}{2}}\widehat{\Lambda}(\xi_1,\eta_1)|\xi_2|^{\frac{1}{2}}\widehat{\Lambda}(\xi_2,\eta_2)\,d\xi d\xi \\ &\leq 4\sigma_1\mathcal{F}_{x,y}\big(|D_x|^{\frac{1}{2}}\Lambda.|D_x|^{\frac{1}{2}}\Lambda\big)(\xi,\eta). \end{aligned} \tag{6.3.13}$$

Using Plancherel Theorem and Sobolev embedding Theorem,

$$H^s(\mathbb{R}^d)\subset L^P(\mathbb{R}^d), \quad \frac{s}{d}=\frac{1}{2}-\frac{1}{p}, \quad (1<p<\infty),$$

we obtain

$$\begin{aligned} \left\|\mathcal{F}\Big(\langle D_x\rangle^{-1}F(\lambda)\Big)(\xi,\eta)\right\|_{L^2_{x,y}(\mathbb{R}^2)} &\leq 4\sigma_1\left\|\mathcal{F}_{x,y}\big(|D_x|^{\frac{1}{2}}\Lambda.|D_x|^{\frac{1}{2}}\Lambda\big)(\xi,\eta)\right\|_{L^2_{x,y}(\mathbb{R}^2)} \\ &\lesssim \sigma_1\||D_x|^{\frac{1}{2}}\Lambda\|^2_{L^4_xL^2_y(\mathbb{R}^2)} \\ &\lesssim \sigma_1\||D_x|^{\frac{1}{2}}\Lambda\|^2_{H^{\frac{1}{2}}_xH^0_y(\mathbb{R}^2)} \\ &\lesssim \sigma_1\|\Lambda\|^2_{H^{1,0}(\mathbb{R}^2)} \\ &\sim \sigma_1\|\lambda\|^2_{H^{1,0}(\mathbb{R}^2)}. \end{aligned} \tag{6.3.14}$$

From (6.3.7),(6.3.8) and (6.3.14), we obtained the desired result (6.3.2)

$$\sup_{t\in[0,\delta]}\|u(t)\|^2_{H^{\sigma_1,0,1,0}} \leq \|u_0\|^2_{H^{\sigma_1,0,1,0}} + CT\sigma_1\,\|u_0\|^3_{H^{\sigma_1,0,1,0}}\,.$$

which complete the proof of Theorem 6.3.2 □

## 6.4 Lower Bound for the Radius of Analyticity

***Proof of Theorem 6.1.1:*** Suppose $u_0(x,y)\in G^{\sigma_{1_0},\sigma_{2_0},\bar{s}}(\mathbb{R}^2)$, for $\sigma_{1_0},\sigma_{2_0}>0$. Then there exist a unique solution of (6.1.1) constructed in Theorem 6.2.2 with existence time T as in (6.2.6).

Note that

$$\lambda_0 = \cosh(\sigma_{1_0}|D_x)|\cosh(\sigma_{1_0}|D_y)u_0.$$

and

$$u \in H^{\sigma_1(t),0,1,0}(\mathbb{R}^2), \quad \forall t \in [0,T].$$

From (6.2.7) and the modified energy, we have

$$E[\lambda(t)] = \int_{\mathbb{R}^2} \left(\lambda^2(x,y,t) + \lambda_x^2(x,y,t)\right) dxdy \leq 2\|u_0\|_{G^{\sigma_{1_0},\sigma_{2_0},\bar{s_0}}(\mathbb{R}^2)} < \infty. \quad (6.4.1)$$

Now, we can construct a solution on $[0,T^*]$ for arbitrarily large time $T^*$ by applying the almost conservation law so as to repeat the local result on successive short time intervals $[0,T],[T,2T],[2T,3T]$ etc of size T to reach large time $T^*$ by adjusting the strip width parameter $\sigma_1$ according to the size of $T^*$. Doing so, we establish the bound

$$\sup_{t\in[0,T^*]} \|u\|^2_{H^{\sigma_1,0,1,0}(\mathbb{R}^2)} \leq 2\|(u(0)\|^2_{H^{\sigma_{1_0},0,1,0}}, \quad (6.4.2)$$

for $\sigma_1$ satisfying

$$\sigma_1(t) \geq ct^{-1}, \quad \forall t \geq 0. \quad (6.4.3)$$

Thus, from (6.4.1), we have

$$\|u\|^2_{H^{\sigma_1(t),0,1,0}(\mathbb{R}^2)} < \infty, \quad t \in [0,T^*],$$

which inturn implies $u \in H^{\sigma_1(t),0,1,0}(\mathbb{R}^2)$ for all $t \in [0,T^*]$. □

# General Conclusion and Future Work

### General Conclusion

In this dissertation, we examined the existence of local and global well-posedness results of higher order KdV-BBM type equations in Gevrey spaces $G^{\sigma,s}(\mathbb{R})$ and modified Gevrey spaces $H^{\sigma,s}(\mathbb{R})$. We established the existence of local solutions using the contraction mapping principal and different multi-linear estimates. The local well-posedness result can be extended to global well-posedness result using almost conservation law for the problems we considered. We also study the persistence of spatial analyticity to the solution of these higher order dispersive partial differential equations in the class of analytic functions by providing explicit formulas to lower bounds for the radius of spatial analyticity of the solution. The persistence of spatial analyticity for the solutions of PDE problems depends on several factors, such as the type of the PDE (elliptic, parabolic, hyperbolic, etc.), the coefficients of the PDE, the initial data, the boundary conditions, the dimension of space, conservation law etc. We used various techniques and tools to prove persistence of spatial analyticity of the solution of the problems we considered, such as Fourier analysis, multilinear estimates, contraction mapping principle, and approximate conservation laws.

The lower bound of the radius of analyticity of the solution for higher order KdV-BBM type equations and coupled system of generalized BBM equations in modified Gevrey spaces $H^{\sigma,s}(\mathbb{R})$ also analyzed. The local and global well-posedness of KP-BBM equation in anisotropic Gevrey space were studied. For existence of global solution, we apply approximate conservation law to repeat the local result on successive short-time intervals to reach any large time $T^*$, by adjusting the strip width parameter $\sigma$ according to the size of $T^*$.

Studying the radius of analyticity of the solution can be used to prove the regularity and stability of the solutions of PDEs, since the radius of analyticity of the solution is a measure of how smooth the solution is in the complex plane. It is defined as the largest radius of a disk centered at a point where the solution is analytic.

Analytical solutions are presented as mathematical expressions, they offer a clear view into how variables and interactions between variables affect the result. Information about the domain of analyticity of a solution to a partial differential equation can be used to gain understanding of the structure of the equation, and to obtain insight into underlying physical processes. For developing algorithms or modeling engineering systems, analytical solutions often offer important advantages.

**Future Work**

In future work, we recommend to examine the well-posedness of problems for the KdV-BBM equation with initial data $u_0 \in G^{\sigma,s}(\mathbb{T})$ that are analytic on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. It is also interesting to study the global well-posedness of the coupled system of generalized BBM equations whose nonlinearities are cubic and nonhomogeneous polynomials.

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