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Applications Of Elzaki Transform For Solving Linear Volterra Integral Equation Of The First Kind

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DEPARTMENT OF MATHEMATICS

A PROJECT WORK ON

APPLICATIONS OF ELZAKI TRANSFORM FOR SOLVING LINEAR VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND

By

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JUNE, 2023

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APPLICATIONS OF ELZAKI TRANSFORM FOR SOLVING LINEAR VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND

A PROJECT WORK SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

By

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Approval of the project for oral defense

I hereby certify that I have supervised, read and evaluated this project entitled "**applications of Elzaki transform method for solving linear Volterra integral equation of the first kind**" by Yirga Mulugeta prepared under my guidance. I recommend that the project is submitted for oral defense.

Dr. Molalign Adam	____________________	________________
Advisor's name	Signature	Date
Dr. Getachew Mehabie	______________________	_________________
Department head's name	Signature	Date

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS

Approval of the Project for defense result

We hereby certify that we have examined this project entitled "**applications of Elzaki transform method for solving linear Volterra integral equation of the first kind**" by Yirga Mulugeta. We recommend that this project is approved for the degree of Master of Science in mathematics.

Board of Examiner's name

____________________________	_____________	_______________
External examiner's name	Signature	Date
____________________________	_____________	_______________
Internal examiner's name	Signature	Date
____________________________	_____________	_______________
Chair person's name	Signature	Date

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ABSTRACT

In this project report, we consider Volterra integral equations. We show the application of Elzaki transform method to solve Volterra integral equations, in particular linear Volterra integral equation of the first kind. We also examine properties of Elzaki transform and scope of the application of the proposed method. To show the applicability and efficiency of Elzaki transform we apply this method to some illustrative examples.

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CHAPTER ONE

INTRODUCTION AND PRELIMINARY CONCEPTS

1.1 Introduction

An integral equation is defined as an equation in which the unknown function to be determined appears under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems [1]. Many initial and boundary value problems associated with ordinary differential equation and partial differential equation can be transformed into problems of solving some approximate integral equations [2]. The most frequently used integral equations fall under two major classes, namely Volterra and Fredholm integral equations. Of course, we have to classify them as homogeneous or nonhomogeneous; and also linear or nonlinear [3].

The name integral equation was given by du Bois-Reymond in (1888). However, the Volterra integral equations can be derived from initial value problems. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name Volterra integral equation was first coined by Lalesco in 1908. Fredholm integral equations can be derived from boundary value problems. Erik Ivar Fredholm(1866-1927) is best remembered for his work on integral equations and spectral theory [2, 4, 5].

The origin of the integral transforms including the Laplace and Fourier transforms can be traced back to celebrated the works of Laplace (1749-1827) on probability theory in the 1780s [11]. The method of integral transforms is one of the most easy and effective methods for solving problems arising in applied mathematics, mathematical physics, and engineering science which are defined by differential equations, difference equations and integral equations. The main idea in the application of the integral transform methods is to transform the associated higher order differential equation to either a differential equation of lower order or an algebraic equation [13].

Literature showed that various problems of integral equations can be solved by different integral transform methods such as Laplace transform method, Elzaki transform method, Aboodh transform method, Mohand transform method, Kamal transform method, Mahgoub transform method, Sawi transform method, Shehu transform method, Tarig transform method, etc.

Aggarwal & Sharma [6] applied Laplace transform method for solving linear Volterra integral equation of the first kind. Aggarwal et.al [7, 8] used Kamal and Mahgoub transform method for solving linear Volterra integral equation of the first kind. Kumar et.al [9] applied Mohand transform method for solving linear Volterra integral equation of the first kind. The main purpose of this project is to determine the exact solution of a linear Volterra integral equation of the first kind using Elzaki transform without large computational work.

Elzaki transform is derived from the classical Fourier integral transform. It was introduced by Tarig Elzaki to facilitate the process of solving ordinary and partial differential equations in the time domain [10].

This project consists of two chapters. In the first chapter, we will present basic concepts on integral equations, particularly linear Volterra integral equations of the first kind, definitions and properties of Elzaki transform method and definitions and Elzaki transforms of Bessel's function of order zero, one, and two. The second chapter deals with applications of Elzaki transform method for solving linear Volterra integral equations of the first kind.

1.2 Integral Equation

Integral equation is an equation in which an unknown function appears under the integral sign.

The general form of an integral equation is given by:

$$\alpha(x)u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)h\big(u(t)\big)dt, \qquad x,t \in \mathbb{R}, \tag{1.1}$$

where, $a(x)$ and $b(x)$ are the limits of integration, λ is a nonzero constant parameter, $k(x, t)$ is a known function of two variables x and t called the kernel of the integral equation, $u(x)$ the unknown function to be determined, $f(x)$ and $\alpha(x)$ are known functions.

An integral equation can be classified as a linear or nonlinear integral equation, homogenous or non-homogenous, Fredholm or Volterra, etc.

1.2.1 Fredholm integral equation

An integral equation is said to be Fredholm integral equation if both the limits of integrations are constant.

The general form of Fredholm integral equation is given by:

$$\alpha(x)u(x) = f(x) + \lambda \int_{a}^{b} k(x,t)h\big(u(t)\big)dt\,, \qquad x,t \in \mathbb{R}, \tag{1.2}$$

Remark 1.2.1

1) If h is linear in u, then (1.2) is called linear Fredholm integral equation, otherwise, it is nonlinear Fredholm integral equation.

2) If $f(x) = 0$, then (1.2) is homogeneous Fredholm integral equation, otherwise, it is inhomogeneous.

3) A Fredholm integral equation can be classified as first kind, second kind, and third kind depending on $\alpha(x)$ is zero, nonzero constant, and non-constant respectively.

For example, the equation

$$u(x) = x + \int_{-1}^{1} (x-t)\, u(t)dt,$$

is non-homogenous linear Fredholm integral equation of the second kind.

1.2.2 Volterra integral equation

An integral equation is said to be Volterra integral equation if at least one of the limits of integrations is non-constant.

The general form of Volterra integral equation is given by

$$\alpha(x)u(x) = f(x) + \lambda \int_{a}^{b(x)} k(x,t)h\big(u(t)\big)dt\,, \qquad x \in \mathbb{R}, t \geq 0 \tag{1.3}$$

where b(x) is non constant function of x.

Remark 1.2.2

1) If h is linear in u, then (1.3) is called linear Volterra integral equation, otherwise, it is nonlinear Volterra integral equation.

2) If $f(x) = 0$, then (1.3) is homogeneous Volterra integral equation, otherwise, it is inhomogeneous.

3) A Volterra integral equation can be classified as the first kind, second kind, and third kind depending on $\alpha(x)$ is zero, nonzero constant, and non-constant respectively.
4) An integral equation that contains both Fredholm and Volterra integral is called Volterra-Fredholm integral equation.

For example, the equation

$$xe^{-x} = \int_0^x e^{x-t}\, u(t)dt,$$

is inhomogeneous linear Volterra integral equation of the first kind, whereas the equation

$$u(x) = \int_0^x (x-t)\, u(t)dt + \int_0^1 \sin\big(u(t)\big)\, dt,$$

is inhomogeneous nonlinear Volterra-Fredholm integral equation of the second kind.

1.3 Elzaki transform

Integral transforms are defined by integrals, which play an important role in solving linear differential equations, integral equations, integro-differential equations, etc. An integral transform maps an equation from its original domain into another domain. Manipulating and solving the equation in the target domain can be much easier than manipulating and solving in the original domain. The solution is then mapped back to the original domain with the inverse of the integral transform.

An integral transform is any transform T of the following form:

$$T\{f(t)\}(v) = \int_a^b f(t)k(t,v)dt \qquad v,t \in \mathbb{R} \tag{1.4}$$

where $k(t,v)$ is called its kernel. The function f and Tf are the input and output of its transform respectively. An integral transform is a particular kind of mathematical operator.

There are numerous useful integral transforms. Each is specified by a choice of the kernel function of the integral operator and the limits of integration. For example, (1.4) defines

1. Laplace transform if $(k(t,v) = e^{-vt},\ \ a = 0,\ \ b = +\infty)$,

2. Fourier transform if $\left(k(t,v) = (2\pi)^{\frac{-1}{2}} e^{-ivt}, \quad a = -\infty, \quad b = +\infty, \qquad i = \sqrt{-1}\right)$,

3. Sumudu transform if $\left(k(t,v) = \frac{1}{v} e^{\frac{-t}{v}}, \; a = 0, \quad b = +\infty\right)$,

4. Mohand transform if $(k(t,v) = v^2 e^{-vt}, \; a = 0, \; b = +\infty)$,

5. Aboodh transform if $\left(k(t,v) = \frac{1}{v} e^{-vt}, \; a = 0, \quad b = +\infty\right)$,

6. Mahgoub transform if $(k(t,v) = ve^{-vt}, \quad a = 0, \quad b = +\infty)$,

7. Kamal transform if $\left(k(t,v) = e^{\frac{-t}{v}}, \quad a = 0, \quad b = +\infty\right)$,

8. Elzaki transform if $\left(k(t,v) = ve^{\frac{-t}{v}}, \quad a = 0, \quad b = +\infty)\right)$,

Some kernels of the integral operator have an associated inverse kernel,$k^{-1}(t,v)$ which yields an inverse integral transform:

$$f(t) = \int_{v1}^{v2} (Tf)(v) k^{-1}(t,v) dv \tag{1.5}$$

This transform is used to map the solution back to the original domain of the problem of the differential equation.

Definition 1.3.1: A function f is said to be piecewise continuous in any interval, if it is continuous in each subinterval of the given interval. Then this gives a finite jumps as the only possible discontinuities.

Definition 1.3.2: A function f is said to be exponential order "a > 0", if there exists a finite positive constant M, that satisfies the growth restriction

$$|f(t)| \leq Me^{at}, \text{for all } t \geq 0$$

Definition 1.3.3: Let f be of exponential order and piece-wise continuous on $t \geq 0$. Then the Elzaki transform of f exists and is defined as

$$E\{f(t)\}(v) = T(v) = v \int_0^\infty f(t) e^{\frac{-t}{v}} dt, \tag{1.6}$$

If $E\{f(t)\} = T(v)$, then $f(t)$ is called the inverse Elzaki transform of $T(v)$ and mathematically it can be expressed as

$$f(t) = E^{-1}\big(T(v)\big)$$

where E^{-1} is the inverse Elzaki transform operator.

1.3.1 Elzaki transform of some elementary functions

As stated shown below, we have the following Elzaki transform of some elementary functions, which can be derived using the definition and properties of integrals.

$f(t)$	$E\{f(t)\} = F(v)$
c (constant)	cv^2
$t^n, \quad n \in \mathbb{Z}_0^+$	$n!\, v^{n+2}$
e^{at}	$\frac{v^2}{1 - av}$
$\sin(at)$	$\frac{av^3}{1 + a^2v^2}$
$\cos(at)$	$\frac{v^2}{1 + a^2v^2}$
$\sinh(at)$	$\frac{av^3}{1 - a^2v^2}$
$\cosh(at)$	$\frac{v^2}{1 - a^2v^2}$

By using definitions of Elzaki transform and properties of integral, we can proof the above as follows

1. $E\{c\} = v\int_0^\infty c\, e^{\frac{-t}{v}} dt = cv\int_0^\infty e^{\frac{-t}{v}}\, dt = cv\left(-ve^{\frac{-t}{v}}\Big|_0^\infty\right) = cv^2$

2. Using integration by parts

For $n = 1$,

$$E\{t\} = v\int_0^\infty te^{-\frac{t}{v}}dt = v\left[-vte^{\frac{-t}{v}}\right]_0^\infty + v\int_0^\infty e^{\frac{-t}{v}}dt = v^2\int_0^\infty e^{\frac{-t}{v}}dt = -v^3\left[e^{\frac{-t}{v}}\right]_0^\infty = v^3$$

For $n = 2$, $E\{t^2\} = v\int_0^\infty t^2 e^{-\frac{t}{v}} dt = v\left[-t^2 v e^{\frac{-t}{v}}\right]_0^\infty + 2v^2\int_0^\infty t e^{\frac{-t}{v}} dt = 2vE\{t\} = 2\,v^4$

For $n = 3$, $E\{t^3\} = v\int_0^\infty t^3 e^{-\frac{t}{v}} dt = v\left[-t^3 \nu e^{\frac{-t}{v}}\right]_0^\infty + 3v^2\int_0^\infty t^2 e^{\frac{-t}{v}} dt = 3vE\{t^2\} = 6\,v^5$

By continuing in the same pattern, we get the result $E\{t^n\} = n!\,v^{n+2}$

3. $E\{e^{at}\} = v\int_0^\infty e^{at} e^{-t/v}\, dt = v\int_0^\infty e^{-t\left(\frac{1}{v}-a\right)}\, dt = \frac{-v}{\frac{1}{v}-a}\left[e^{-t\left(\frac{1}{v}-a\right)}\right]_0^\infty = \frac{v^2}{1-av}$

4. $E\{\sin(at)\} = E\left\{\frac{e^{iat}-e^{-iat}}{2i}\right\} = v\int_0^\infty \left[\frac{e^{iat}-e^{-iat}}{2i}\right] e^{\frac{-t}{v}} dt$

$$= \frac{1}{2i}\left[v\int_0^\infty e^{iat} e^{\frac{-t}{v}} dt - v\int_0^\infty e^{-iat} e^{\frac{-t}{v}} dt\right] = \frac{1}{2i}\left(E\{e^{iat}\} - E\{e^{-iat}\}\right)$$

$$= \frac{1}{2i}\left[\frac{v^2}{1-iav} - \frac{v^2}{1+iav}\right] = \frac{1}{2i}\left[\frac{v^2+iav^3-v^2+iav^3}{1+iav-iav+a^2v^2}\right]$$

$$= \frac{1}{2i}\left[\frac{2iav^3}{1+a^2v^2}\right] = \left[\frac{av^3}{1+a^2v^2}\right]$$

5. $E\{\cos(at)\} = E\left\{\frac{e^{iat}+e^{-iat}}{2}\right\} = v\int_0^\infty \left[\frac{e^{iat}+e^{-iat}}{2}\right] e^{\frac{-t}{v}} dt$

$$= \frac{1}{2}[v\int_0^\infty e^{iat} e^{\frac{-t}{v}} dt + v\int_0^\infty e^{-iat} e^{\frac{-t}{v}} dt] = \frac{1}{2}\left(E\{e^{iat}\} - E\{e^{-iat}\}\right)$$

$$= \frac{1}{2}\left[\frac{v^2}{1-iav} + \frac{v^2}{1+iav}\right] = \frac{1}{2}\left[\frac{v^2+iav^3+v^2-iav^3}{1+iav-iav+a^2v^2}\right]$$

$$= \frac{1}{2}\left[\frac{2v^2}{1+a^2v^2}\right] = \left[\frac{v^2}{1+a^2v^2}\right]$$

$$6.\ E\{\sinh(at)\} = v\int_0^{\infty}\left[\frac{e^{at}-e^{-at}}{2}\right]e^{\frac{-t}{v}}dt = \frac{1}{2}[v\int_0^{\infty} e^{at}e^{\frac{-t}{v}}dt - v\int_0^{\infty} e^{-at}e^{\frac{-t}{v}}dt]$$

$$= \frac{1}{2}(E\{e^{at}\} - E\{e^{-at}\}) = \frac{1}{2}\left[\frac{v^2}{1-av} - \frac{v^2}{1+av}\right]$$

$$= \frac{1}{2}\left[\frac{v^2+av^3-v^2+av^3}{1+av-av-a^2v^2}\right] = \frac{1}{2}\left[\frac{2av^3}{1-a^2v^2}\right]$$

$$= \left[\frac{av^3}{1-a^2v^2}\right]$$

$$7.\ E\{\cosh(at)\} = v\int_0^{\infty} \cosh(at)\, e^{\frac{-t}{v}}dt = v\int_0^{\infty}\left[\frac{e^{at}+e^{-at}}{2}\right]e^{\frac{-t}{v}}dt$$

$$= \frac{1}{2}\left[v\int_0^{\infty} e^{at}e^{\frac{-t}{v}}dt + v\int_0^{\infty} e^{-at}e^{\frac{-t}{v}}dt\right] = \frac{1}{2}(E\{e^{at}\} + E\{e^{-at}\})$$

$$= \frac{1}{2}\left[\frac{v^2}{1-av} + \frac{v^2}{1+av}\right] = \frac{1}{2}\left[\frac{v^2+av^3+v^2-av^3}{1+av-av-a^2v^2}\right]$$

$$= \frac{1}{2}\left[\frac{2v^2}{1-a^2v^2}\right] = \left[\frac{v^2}{1-a^2v^2}\right]$$

Inverse Elzaki transform of some elementary functions

$F(v)$	$f(t) = E^{-1}\{F(v)\}$
v^2	1
v^3	t
v^4	$\frac{t^2}{2!}$
$v^{n+2}, n \in N$	$\frac{t^n}{n!}$
$v^{n+2}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
$\frac{v^2}{1-av}$	e^{at}

$\frac{v^3}{1+a^2v^2}$	$\frac{\sin(at)}{a}$
$\frac{v^2}{1+a^2v^2}$	$\cos(at)$
$\frac{v^3}{1-a^2v^2}$	$\frac{\sinh(at)}{a}$
$\frac{v^2}{1-a^2v^2}$	$\cosh(at)$

Example: Consider the function

$$f(t) = \sin(t) + \cosh(3t) - 3e^{-2t} + t^4 + 6$$

By definition of Elzaki transform

$E\{f(t)\} = E\{\sin(t) + \cosh(3t) - 3e^{-2t} + t^4 + 6\}$

$$= v\int_0^\infty (\sin(t) + \cosh(3t) - 3e^{-2t} + t^4 + 6)\, e^{\frac{-t}{v}} dt$$

$$= v\int_0^\infty \sin(t)\, e^{\frac{-t}{v}} dt + v\int_0^\infty \cosh(3t)\, e^{\frac{-t}{v}} dt - 3v\int_0^\infty e^{-2t}\, e^{\frac{-t}{v}} dt + v\int_0^\infty t^4\, e^{\frac{-t}{v}} dt + v\int_0^\infty 6\, e^{\frac{-t}{v}} dt$$

By Elzaki transform of hyperbolic, trigonometric, exponential and polynomial functions, we know that

$$v\int_0^\infty \sin(t)\, e^{\frac{-t}{v}} dt = E\{\sin(t)\} = \frac{v^3}{1+v^2}, \quad v\int_0^\infty \cosh(3t)\, e^{\frac{-t}{v}} dt = E\{\cosh(3t)\} = \frac{v^2}{1-9v^2}$$

$$3v\int_0^\infty e^{-2t}\, e^{\frac{-t}{v}} dt = 3E\{e^{-2t}\} = \frac{3v^2}{1+2v}, \quad v\int_0^\infty t^4\, e^{\frac{-t}{v}} dt = E\{t^4\} = 24v^6$$

$$v\int_0^\infty 6\, e^{\frac{-t}{v}} dt = E\{6\} = 6v^2$$

Therefore, by adding and re arranging the above terms we obtain

$$E\{f(t)\} = \frac{v^3}{1+v^2} + \frac{v^2}{1-9v^2} - \frac{3v^2}{1+2v} + 24v^6 + 6v^2$$

1.3.2 Properties of Elzaki transform

The following properties of Elzaki transform are derived from the definition and properties of integrals.

Let $E\{f(t)\} = F(v)$ and $E\{g(t)\} = G(v)$, then for arbitrary constant a and b, we have the following properties.

Property 1(Linearity property):

$$E\{af(t) + bg(t)\} = aF(v) + bG(v)$$

Proof: By the definition of Elzaki transform, we obtain

$$E\{af(t) + bg(t)\} = v\int_0^\infty [af(t) + bg(t)]e^{-\frac{t}{v}}\,dt = v\int_0^\infty af(t)e^{-\frac{t}{v}}dt + v\int_0^\infty bg(t)e^{-\frac{t}{v}}dt$$

$$= av\int_0^\infty f(t)e^{-\frac{t}{v}}dt + bv\int_0^\infty g(t)e^{-\frac{t}{v}}dt$$

$$= aF(v) + bG(v)$$

Property 2 (Change of scale property):

$$E\{f(at)\} = \frac{1}{a^2}F(av)$$

Proof: By the definition of Elzaki transform, we have

$$E\{f(at)\} = v\int_0^\infty f(at)e^{-\frac{t}{v}}dt,$$

Put $p = at \Rightarrow dt = \frac{dp}{a}$

$$E\{f(at)\} = \frac{v}{a}\int_0^\infty f(p)e^{\frac{-p}{av}}\,dp = \frac{1}{a^2}av\int_0^\infty f(p)\,e^{\frac{-p}{av}}dp = \frac{1}{a^2}F(av)$$

Property 3 (Shifting property):

$$E\{e^{at}f(t)\} = (1 - av)F\left(\frac{v}{1 - av}\right)$$

Proof: By the definition of Elzaki transform, we have

$$E\{e^{at}f(t)\} = v\int_0^\infty e^{at}f(t)e^{-\frac{t}{v}}dt = v\int_0^\infty f(t)\,e^{-\left(\frac{1}{v}-a\right)t}dt = v\int_0^\infty f(t)\,e^{\frac{\frac{-t}{v}}{1-av}}dt$$

$$= (1-av)\frac{v}{1-av}\int_0^\infty f(t)\,e^{\frac{\frac{-t}{v}}{1-av}}dt = (1-av)F\left(\frac{v}{1-av}\right)$$

Property 4 (Elzaki transform for derivatives):

$$E\{f^{(n)}(t)\} = \frac{F(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k}\,f^{(k)}(0)$$

Proof: Proof by mathematical induction

In case of $n = 1$,

$$E\{f'(t)\} = v\int_0^\infty f'(t)\,e^{-\frac{t}{v}}dt = v\left(f(t)e^{-\frac{t}{v}}\Big|_0^\infty + \frac{1}{v}\int_0^\infty f(t)\,e^{-\frac{t}{v}}dt\right) = \frac{1}{v}F(v) - vf(0).$$

Now suppose it holds for n=m, i.e.,

$$E\{f^{(m)}(t)\} = \frac{1}{v^m}F(v) - \sum_{k=0}^{m-1} v^{2-m+k}\,f^{(k)}(0)$$

and suppose n=m+1:

$$E\{f^{(m+1)}(t)\} = E\left\{\left(f^{(m)}(t)\right)'\right\} = \frac{1}{v}E\{f^{(m)}(t)\} - vf^{(m)}(0)$$

$$= \frac{1}{v}\left(\frac{1}{v^m}F(v) - \sum_{k=0}^{m-1} v^{2-m+k}\,f^{(k)}(0)\right) - vf^{(m)}(0)$$

$$= \frac{1}{v^{m+1}}F(v) - \frac{1}{v}\sum_{k=0}^{m-1} v^{2-m+k}\,f^{(k)}(0) - vf^{(m)}(0)$$

$$= \frac{1}{v^{m+1}}F(v) - \sum_{k=0}^{m-1} v^{1-m+k}\,f^{(k)}(0) - vf^{(m)}(0)$$

$$= \frac{1}{v^{m+1}}F(v) - \sum_{k=0}^{m} v^{1-m+k}\,f^{(k)}(0)$$

Hence, this property is valid at an arbitrary natural number n.

Property 5 (Elzaki transform for integrals):

$$E\left\{\int_0^t f(x)dx\right\} = vF(v)$$

Proof: By definition of Elzaki transform, Elzaki transform for derivatives and fundamental theorem of calculus, we have

Let $h(t) = \int_0^t f(x)dx \Rightarrow h'(t) = f(t)$ and $h(0) = 0$, then

$$E\{h'(t)\} = \frac{1}{v}E\{h(t)\} - vh(0) = \frac{1}{v}E\{h(t)\}$$

$$\Rightarrow E\left\{\int_0^t f(x)dx\right\} = E\{h(t)\} = vE\{h'(t)\} = vE\{f(t)\} = vF(v$$

Property 6 (Elzaki transforms of multiplication by t^n):

Let, $$E\{f(t)\} = v\int_0^\infty f(t)e^{-\frac{t}{v}}dt = F(v) \qquad (i)$$

For $n = 1$, $E\{tf(t)\} = v^2\left(\frac{d}{dv} - \frac{1}{v}\right)F(v)$

Proof:

By differentiating (i) with respect to v, we have

$$\frac{d}{dv}F(v) = \frac{d}{dv}\int_0^\infty ve^{-\frac{t}{v}}f(t)dt = \int_0^\infty f(t)\left(\frac{d}{dv}ve^{-\frac{t}{v}}\right)dt$$

$$= \frac{1}{v}\int_0^\infty tf(t)e^{-\frac{t}{v}}dt + \int_0^\infty f(t)e^{-\frac{t}{v}}dt = \frac{1}{v^2}v\int_0^\infty tf(t)e^{-\frac{t}{v}}dt + \frac{1}{v}v\int_0^\infty f(t)e^{-\frac{t}{v}}dt$$

$$= \frac{1}{v^2}E\{tf(t)\} + \frac{1}{v}F(v)$$

$$\Rightarrow E\{tf(t)\} = v^2\left(\frac{d}{dv} - \frac{1}{v}\right)F(v)$$

For $n = 2$, $\quad E\{t^2 f(t)\} = v^4 \frac{d^2}{dv^2} F(v)$

Proof:

By differentiating (i) twice with respect to v, we have

$$\frac{d^2}{dv^2} F(v) = \int_0^\infty f(t)\left(\frac{d^2}{dv^2} v e^{-\frac{t}{v}}\right) dt = \frac{1}{v^3}\int_0^\infty t^2 f(t) e^{-\frac{t}{v}} dt = \frac{1}{v^4} v \int_0^\infty t^2 f(t) e^{-\frac{t}{v}} dt = \frac{1}{v^4} E\{t^2 f(t)\}$$

$$\Rightarrow E\{t^2 f(t)\} = v^4 \frac{d^2}{dv^2} F(v)$$

By continuing in the same manner, we can determine $E\{t^n f(t)\}$, $\quad n \in \mathbb{N}$.

Property 7(Elzaki transform for convolution):

$$E\{(f * g)(t)\} = \frac{1}{v} E\{f(t)\} E\{g(t)\} = \frac{1}{v} F(v) G(v)$$

where the convolution $(f * g)(t)$ is defined as

$$(f * g)(t) = \int_0^t f(u) g(t-u) du = \int_0^t f(t-u) g(u) du$$

Proof:

Let, $\quad F(v) = E\{f(t)\} = v\int_0^\infty f(t)\, e^{-\frac{t}{v}} dt$, and $G(v) = E\{g(x)\} = v\int_0^\infty g(x)\, e^{-\frac{x}{v}} dx$,

$$F(v)G(v) = \left(v\int_0^\infty f(t)\, e^{-\frac{t}{v}} dt\right)\left(v\int_0^\infty g(x)\, e^{-\frac{x}{v}} dx\right) = v^2 \int_0^\infty f(t) \int_0^\infty g(x)\, e^{-\frac{(x+t)}{v}} dx dt$$

Let us put $\tau = x + t, x = \tau - t$, we get

$$F(v)G(v) = v^2 \int_0^\infty f(t) \int_t^\infty g(\tau - t)\, e^{-\frac{\tau}{v}} d\tau dt = v^2 \int_0^\infty e^{-\frac{\tau}{v}} d\tau \int_0^t f(t) g(\tau - t)\, dt$$

$$= v^2 \int_0^\infty e^{-\frac{\tau}{v}} \int_0^t f(t) g(\tau - t)\, dt d\tau = v^2 \int_0^\infty e^{-\frac{\tau}{v}} (f * g) d\tau = vE\{(f * g)(t)\}$$

$$\Rightarrow E\{(f * g)(t)\} = \frac{1}{v} F(v) G(v)$$

Example: Consider the following function

$$\mathrm{g}\big(\mathrm{t}, \mathrm{f}(\mathrm{t}), \mathrm{f}'(\mathrm{t})\big) = \mathrm{f}(\mathrm{t}) - 3\mathrm{f}'^{(\mathrm{t})} + \mathrm{e}^{\mathrm{t}}\mathrm{t}^2 + \cos 2\mathrm{t} + \int_0^{\mathrm{t}} (\mathrm{x} - \mathrm{s})\mathrm{e}^{\mathrm{s}}\, \mathrm{ds} - 4\mathrm{t}$$

Let, $\mathrm{E}\{\mathrm{f}(\mathrm{t})\} = \mathrm{F}(\mathrm{v})$

By definition of Elzaki transform

$$\mathrm{E}\big\{\mathrm{g}\big(\mathrm{t}, \mathrm{f}(\mathrm{t}), \mathrm{f}'(\mathrm{t})\big)\big\} = \mathrm{E}\left\{\mathrm{f}(\mathrm{t}) - 3\mathrm{f}'(\mathrm{t}) + \mathrm{e}^{\mathrm{t}}\mathrm{t}^2 + \cos 2\mathrm{t} + \int_0^{\mathrm{t}} (\mathrm{x} - \mathrm{s})\mathrm{e}^{\mathrm{s}}\, \mathrm{ds} - 4\mathrm{t}\right\}$$

Applying linearity property of Elzaki transform gives

$$\mathrm{E}\big\{\mathrm{g}\big(\mathrm{t}, \mathrm{f}(\mathrm{t}), \mathrm{f}'(\mathrm{t})\big)\big\} = \mathrm{E}\{\mathrm{f}(\mathrm{t})\} - 3\mathrm{E}\{\mathrm{f}'(\mathrm{t})\} + \mathrm{E}\{\mathrm{e}^{\mathrm{t}}\mathrm{t}^2\} + \mathrm{E}\{\cos 2\mathrm{t}\} + \mathrm{E}\left\{\int_0^{\mathrm{t}} (\mathrm{x} - \mathrm{s})\mathrm{e}^{\mathrm{s}}\, \mathrm{ds}\right\} - \mathrm{E}\{4\mathrm{t}\}$$

By Elzaki transform for derivatives, we have

$$\mathrm{E}\{\mathrm{f}'(\mathrm{t})\} = \frac{1}{\mathrm{v}}\mathrm{F}(\mathrm{v}) - \mathrm{v}\mathrm{f}(0)$$

By change of scale property for Elzaki transform, we have

$$\mathrm{E}\{\cos 2\mathrm{t}\} = \frac{\mathrm{v}^2}{1 + 4\mathrm{v}^2}$$

By shifting property for Elzaki transform, we have

$$\mathrm{E}\{\mathrm{e}^{\mathrm{t}}\mathrm{t}^2\} = \frac{2\mathrm{v}^4}{(1 - \mathrm{v})^3}$$

By convolution theorem for Elzaki transform, we have

$$\mathrm{E}\left\{\int_0^{\mathrm{t}} (\mathrm{t} - \mathrm{s})\mathrm{e}^{\mathrm{s}}\, \mathrm{ds}\right\} = \frac{1}{v}\mathrm{E}\{\mathrm{t}\}\mathrm{E}\{\mathrm{e}^{\mathrm{t}}\} = \frac{\mathrm{v}^4}{1 - v}$$

By property 6, we obtain

$$\mathrm{E}\{4\mathrm{t}\} = \mathrm{v}^2\left(\frac{d}{dv}\mathrm{v}^2 - \frac{1}{v}\mathrm{v}^2\right) = \mathrm{v}^3$$

Therefore, by adding and rearranging all the above terms, we have

$$\mathrm{E}\big\{\mathrm{g}\big(\mathrm{t}, \mathrm{f}(\mathrm{t}), \mathrm{f}'(\mathrm{t})\big)\big\} = \mathrm{F}(\mathrm{v}) - \frac{3}{\mathrm{v}}\mathrm{F}(\mathrm{v}) + 3\mathrm{v}\mathrm{f}(0) + \frac{2\mathrm{v}^4}{(1 - \mathrm{v})^3} + \frac{\mathrm{v}^2}{1 + 4\mathrm{v}^2} + \frac{\mathrm{v}^4}{1 - v} - \mathrm{v}^3$$

1.4 Elzaki transform of Bessel's functions

Bessel's function of order n, where $n \in \mathbb{N}$ is given by

$$J_n(t) = \frac{t^n}{2^n n!}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \frac{t^6}{2.4.6.(2n+2)(2n+4)(2n+6)} + \cdots\right] \quad (1.7)$$

In particular, when $n = 0$, we have Bessel's function of zero order and it is denoted by $J_0(t)$ and it is given by the infinite power series

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots$$

For $n = 1$, we have Bessel's function of order one and it is denoted by $J_1(t)$ and it is given by

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^2.4} + \frac{t^5}{2^2.4^2.6} - \frac{t^7}{2^2.4^2.6^2.8} + \cdots$$

$$= \frac{t}{2} - \frac{t^3}{2^3.2!} + \frac{t^5}{2^5.2!.3!} - \frac{t^7}{2^7.3!.4!} + \cdots$$

For $n = 2$, we have Bessel's function of order two and it is denoted by $J_2(t)$ and it is given by

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{2^2.4.6} + \frac{t^6}{2^2.4^2.6.8} - \frac{t^8}{2^2.4^2.6^2.8.10} + \cdots$$

In the same manner, we can determine $J_n(t), n > 2$.

Remark 1.4.1: We have the following relation of Bessel's functions

1. Relation between $J_0(t)$ and $J_1(t)$ [14, 15].

$$J_1(t) = -\frac{d}{dt}J_0(t)$$

2. Relation between $J_0(t)$ and $J_2(t)$ [15].

$$J_2(t) = J_0(t) + 2\frac{d^2}{dt^2}J_0(t)$$

Elzaki transform of Bessel's functions of zero order, order one and order two

1. Elzaki transform of $J_0(t)$

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots$$

$$\Rightarrow E\{J_0(t)\} = E\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots\right\}$$

$$= E\{1\} - \frac{1}{2^2}E\{t^2\} + \frac{1}{2^2 4^2}E\{t^4\} - \frac{1}{2^2 4^2 6^2}E\{t^6\} + \cdots$$

$$= v^2 - \frac{1}{2^2}(2!\,v^4) + \frac{1}{2^2 4^2}(4!\,v^6) - \frac{1}{2^2 4^2 6^2}(6!\,v^8) + \cdots$$

$$= v^2\left(1 - \frac{1}{2}v^2 + \frac{1.3}{2.4}(v^2)^2 - \frac{1.3.5}{2.4.6}(v^2)^3 + \cdots\right)$$

$$= v^2(1+v^2)^{\frac{-1}{2}} = \frac{v^2}{\sqrt{1+v^2}}$$

2. Elzaki transform of $J_1(t)$

$$J_1(t) = -\frac{d}{dt}J_0(t)$$

$$\Rightarrow E\{J_1(t)\} = -E\left\{\frac{d}{dt}J_0(t)\right\}$$

Now applying the property, Elzaki transform of derivative of the function on equation, we have

$$E\{J_1(t)\} = -\left[\frac{1}{v}E\{J_0(t)\} - vJ_0(0)\right] = -\frac{1}{v}E\{J_0(t)\} + v = v - \frac{v}{\sqrt{1+v^2}}$$

3. Elzaki transform of $J_2(t)$

$$J_2(t) = J_0(t) + 2J''_0(t)$$

$$\Rightarrow E\{J_2(t)\} = E\{J_0(t) + 2J''_0(t)\}$$

$$= E\{J_0(t)\} + 2E\{J''_0(t)\}$$

Now applying the property, Elzaki transform of derivative of the function on equation, we have

$$E\{J_2(t)\} = E\{J_0(t)\} + 2\left[\frac{1}{v^2}E\{J_0(t)\} - J_0(0) - vJ'_0(0)\right]$$

$$= \frac{v^2}{\sqrt{1+v^2}} + 2\left[\frac{1}{v^2}\frac{v^2}{\sqrt{1+v^2}} - 1 - vJ_1(0)\right]$$

$$= \frac{v^2}{\sqrt{1+v^2}} + \frac{2}{\sqrt{1+v^2}} - 2$$

$$= \frac{v^2 + 2 - 2\sqrt{1+v^2}}{\sqrt{1+v^2}}$$

CHAPTER TWO

APPLICATIONS OF ELZAKI TRANSFORM METHOD

In this chapter, we use Elzaki transform method to solve linear Volterra integral equation of the first kind involving difference kernels.

2.1 Description of the method

The Elzaki transform method, whose fundamental properties are presented in the first chapter plays a great role in solving differential equations, integral equations, integro-differential equations etc.

Consider the general linear Volterra type integral equation of the first kind involving difference kernel

$$f(x) = \int_0^x k(x-t)u(t)dt \tag{2.1}$$

To solve the linear Volterra integral equation of the first kind given by (2.1) using Elzaki transform method, we can follow the following steps.

Step 1: Applying Elzaki transform to both sides of equation (2.1), we have

$$E\{f(x)\} = E\left\{\int_0^x k(x-t)u(t)dt\right\} \tag{2.2}$$

Step 2: By using convolution theorem of Elzaki transform to equation (2.2), we have

$$E\{f(x)\} = \frac{1}{v}E\{k(x)\}E\{u(x)\}$$

$$E\{u(x)\} = \frac{vE\{f(x)\}}{E\{k(x)\}} \tag{2.3}$$

Step3: Operating Inverse Elzaki transform on both sides of (2.3), we have

$$u(x) = E^{-1}\left\{\frac{vE\{f(x)\}}{E\{k(x)\}}\right\}, \tag{2.4}$$

which is the required solution of (2.1).

2.2. Application of the method

In this section, we consider some examples whose kernels containing exponential function, hyperbolic function, trigonometric function, and polynomial function, and Bessel's function in order to demonstrate the effectiveness of Elzaki transform for solving linear Volterra integral equations of the first kind.

Example 2.1. Consider the linear Volterra integral equation of the first kind whose kernel contains exponential function

$$x = \int_0^x e^{x-t}\, u(t)dt \tag{2.5}$$

Applying the Elzaki transform to both sides of (2.5), we have

$$E\{x\} = E\left\{\int_0^x e^{x-t}\, u(t)dt\right\} \tag{2.6}$$

Using convolution theorem of Elzaki transform on (2.6), we have

$$v^3 = \frac{1}{v}E\{e^x\}E\{u(x)\} \Rightarrow\ v^3 = \frac{1}{v}\left(\frac{v^2}{1-v}\right)E\{u(x)\}$$

$$E\{u(x)\} = v^2 - v^3 \tag{2.7}$$

Operating inverse Elzaki transform on both sides of (2.7), we have

$$u(x) = 1 - x,$$

which is the required exact solution of (2.5).

Example 2.2: Consider the linear Volterra integral equation of the first kind whose kernel contains exponential function

$$\sin x = \int_0^x e^{3(x-t)}\, u(t)dt \tag{2.8}$$

Applying the Elzaki transform to both sides of (2.8), we have

$$E\{\sin x\} = E\left\{\int_0^x e^{3(x-t)}\, u(t)dt\right\} \tag{2.9}$$

Using convolution theorem of Elzaki transform on (2.9), we have

$$\frac{\mathrm{v}^3}{1+\mathrm{v}^2} = \frac{1}{\mathrm{v}}\mathrm{E}\{\mathrm{e}^{3\mathrm{x}}\}\mathrm{E}\{\mathrm{u}(\mathrm{x})\}$$

$$\Rightarrow \frac{\mathrm{v}^3}{1+\mathrm{v}^2} = \frac{1}{\mathrm{v}}\left(\frac{\mathrm{v}^2}{1-3\mathrm{v}}\right)\mathrm{E}\{\mathrm{u}(\mathrm{x})\}$$

$$\mathrm{E}\{\mathrm{u}(\mathrm{x})\} = \frac{\mathrm{v}^2(1-3\mathrm{v})}{1+\mathrm{v}^2} = \frac{\mathrm{v}^2}{1+\mathrm{v}^2} - \frac{3\mathrm{v}^3}{1+\mathrm{v}^2} \tag{2.10}$$

Operating inverse Elzaki transform on both sides of (2.10), we have

$$\mathrm{u}(\mathrm{x}) = \mathrm{E}^{-1}\left\{\frac{\mathrm{v}^2}{1+\mathrm{v}^2}\right\} - 3\mathrm{E}^{-1}\left\{\frac{\mathrm{v}^3}{1+\mathrm{v}^2}\right\}$$

$$\mathrm{u}(\mathrm{x}) = \cos\mathrm{x} - 3\sin\mathrm{x},$$

which is the required exact solution of (2.8).

Example 2.3: Consider the linear Volterra integral equation of the first kind whose kernel is a linear function

$$\mathrm{x}^2 = \frac{1}{3}\int_0^{\mathrm{x}} (\mathrm{x}-\mathrm{t})\mathrm{u}(\mathrm{t})d\mathrm{t} \tag{2.11}$$

Applying the Elzaki transform to both sides of (2.11), we have

$$\mathrm{E}\{\mathrm{x}^2\} = \frac{1}{3}\mathrm{E}\left\{\int_0^{\mathrm{x}} (\mathrm{x}-\mathrm{t})\mathrm{u}(\mathrm{t})d\mathrm{t}\right\} \tag{2.12}$$

Using convolution theorem of Elzaki transform on (2.12), we have

$$2!\,\mathrm{v}^4 = \frac{1}{3}\frac{1}{\mathrm{v}}\mathrm{E}\{\mathrm{x}\}\mathrm{E}\{\mathrm{u}(\mathrm{x})\} \Rightarrow\ 2\mathrm{v}^4 = \frac{1}{3}\frac{1}{\mathrm{v}}[\mathrm{v}^3]\mathrm{E}\{\mathrm{u}(\mathrm{x})\}$$

$$\mathrm{E}\{\mathrm{u}(\mathrm{x})\} = 6\mathrm{v}^2 \tag{2.13}$$

Operating inverse Elzaki transform on both sides of (2.13), we have

$$\mathrm{u}(\mathrm{x}) = 6\mathrm{E}^{-1}\{\mathrm{v}^2\} = 6,$$

which is the required exact solution of (14).

Example 2.4: Consider a linear Volterra integral equation of the first kind whose kernel contains hyperbolic function

$$\sin x = \int_0^x \cosh(x - t)\, u(t) dt \tag{2.14}$$

Applying the Elzaki transform to both sides of (2.14), we have

$$E\{\sin x\} = E\left\{\int_0^x \cosh(x - t)\, u(t) dt\right\} \tag{2.15}$$

Using convolution theorem of Elzaki transform on (2.15), we have

$$\frac{v^3}{1 + v^2} = \frac{1}{v} E\{\cosh x\} E\{u(x)\}$$

$$\Rightarrow \frac{v^3}{1 + v^2} = \frac{1}{v}\left(\frac{v^2}{1 - v^2}\right) E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = \frac{v^2(1 - v^2)}{(1 + v^2)} = \frac{2v^2}{1 + v^2} - v^2 \tag{2.16}$$

Operating inverse Elzaki transform on both sides of (2.16), we have

$$u(x) = E^{-1}\left\{\frac{2v^2}{1 + v^2} - v^2\right\} = 2E^{-1}\left\{\frac{v^2}{1 + v^2}\right\} - E^{-1}\{v^2\}$$

$$u(x) = 2\cos x - 1,$$

which is the required exact solution of (2.14).

Example 2.5: Consider linear Volterra integral equation of the first kind whose kernel contains trigonometric function

$$x = \int_0^x \cos(x - t)\, u(t) dt \tag{2.17}$$

Applying the Elzaki transform to both sides of ((2.17)), we have

$$E\{x\} = E\left\{\int_0^x \cos(x - t)\, u(t) dt\right\} \tag{2.18}$$

Using convolution theorem of Elzaki transform on (2.18), we have

$$v^3 = \frac{1}{v} E\{\cos x\} E\{u(x)\} \Rightarrow \ v^3 = \frac{1}{v}\left(\frac{v^2}{1+v^2}\right) E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = v^2 + v^4 \tag{2.19}$$

Operating inverse Elzaki transform on both sides of (2.19), we have

$$u(x) = E^{-1}\{v^2 + v^4\} = E^{-1}\{v^2\} + E^{-1}\{v^4\}$$

$$u(x) = 1 + \frac{x^2}{2},$$

which is the required exact solution of (2.17).

Example 2.6: Consider a linear Volterra integral equation of the first kind whose kernel contains Bessel function of zero order

$$J_0(x) - \cos x = \int_0^x J_0(x-t)\, u(t) dt \tag{2.20}$$

Applying the Elzaki transform to both sides of (2.20), we have

$$E\{J_0(x) - \cos x\} = \ E\left\{\int_0^x J_0(x-t)\, u(t) dt\right\} \tag{2.21}$$

By applying linearity property of Elzaki transform to (2.21), we have

$$E\{J_0(x)\} - E\{\cos x\} = E\left\{\int_0^x J_0(x-t)\, u(t) dt\right\} \tag{2.22}$$

Using convolution theorem of Elzaki transform on (2.22), we have

$$\frac{v^2}{\sqrt{1+v^2}} - \frac{v^2}{1+v^2} = \frac{1}{v} E\{J_0(x)\} E\{u(x)\}$$

$$\frac{v^2}{\sqrt{1+v^2}} - \frac{v^2}{1+v^2} = \frac{1}{v} \frac{v^2}{\sqrt{1+v^2}} E\{u(x)\}$$

$$E\{u(x)\} = v - \frac{v}{\sqrt{1+v^2}} \tag{2.23}$$

Operating inverse Elzaki transform on both sides of (2.23), we have

$$u(x) = E^{-1}\left\{v - \frac{v}{\sqrt{1+v^2}}\right\} = J_1(x),$$

which is the required exact solution of equation (2.20).

Example 2.7: Consider linear Volterra integral equation of first kind whose kernel containing Bessel function of order one

$$J_0(x) - \cos x = \int_0^x J_1(x-t)\, u(t)dt \qquad (2.24)$$

Applying the Elzaki transform to both sides of (2.24), we have

$$E\{J_0(x) - \cos x\} = E\left\{\int_0^x J_1(x-t)\, u(t)dt\right\} \qquad (2.25)$$

By applying linearity property of Elzaki transform to (2.25), we have

$$E\{J_0(x)\} - E\{\cos x\} = E\left\{\int_0^x J_1(x-t)\, u(t)dt\right\} \qquad (2.26)$$

Using convolution theorem of Elzaki transform on (2.26), we have

$$\frac{v^2}{\sqrt{1+v^2}} - \frac{v^2}{1+v^2} = \frac{1}{v}E\{J_1(x)\}E\{u(x)\}$$

$$\frac{v^2}{\sqrt{1+v^2}} - \frac{v^2}{1+v^2} = \frac{1}{v}\left(v - \frac{v}{\sqrt{1+v^2}}\right)E\{u(x)\}$$

$$E\{u(x)\} = \frac{v^2}{\sqrt{1+v^2}} \qquad (2.27)$$

Operating inverse Elzaki transform on both sides of (2.27), we have

$$u(x) = E^{-1}\left\{\frac{v^2}{\sqrt{1+v^2}}\right\} = J_0(x)$$

which is the required exact solution of equation (2.24).

CONCLUSION

In this project, we have successfully discussed the application of Elzaki transform method for solving linear Volterra integral equations of the first kind involving difference kernel .The given applications showed that very less computational work and a very little time needed for finding the exact solution of linear Voltera integral equations of the first kind. The proposed method can be applied for other linear Volterra integral equations and their systems.

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