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Application of Elzaki Transform - Homotopy Perturbation Method for the Analytical Solution of Nonlinear þÿ F r a c t i o n a l H e a t L i k e E q u a t i o

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DEPARTMENT OF MATHEMATICS

A PROJECT REPORT
ON

APPLICATION OF ELZAKI TRANSFORM - HOMOTOPY PERTURBATION METHOD FOR THE ANALYTICAL SOLUTION OF NONLINEAR FRACTIONAL HEAT – LIKE EQUATIONS

BY

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Application of Elzaki Transform - Homotopy Perturbation Method for the Analytical Solution of Nonlinear Fractional Heat – Like Equations

A project submitted to the Department of Mathematics, in partial fulfilment of the requirements for the Degree of Masters of Science in Mathematics

By

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Declaration

I hereby declare that, this project is done by me under the supervision of Dr. Birlew Belayneh, Department of Mathematics, Bahir Dar University, in partial fulfilment of the requirements for the Degree of Master of Science in Mathematics. I am declaring that this project is my original work. I also declare that neither of this project nor any of its parts has been submitted to elsewhere for the award of any other degrees or certificates.

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Approval of the project for oral defence

I hereby certify that I have supervised, read and evaluated this project entitled **"Application of Elzaki Transform-Homotopy Perturbation Method for the Analytical Solution of Nonlinear Fractional Heat – Like Equations"** prepared by Fentahun Getachew under my guidance. I recommend that the project is submitted for oral defence.

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Approval of the project for defence result

We hereby certify that we have examined this project entitled **"Application of Elzaki Transform-Homotopy Perturbation Method for the Analytical Solution of Nonlinear Fractional Heat – Like Equations"** prepared and presented by Fentahun Getachew. We recommend that this project is approved for the Degree of Master of Science in Mathematics.

Board of Examiners

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Internal examiner's name	Signature	Date
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ABSTRACT

In this project, we apply the combination of Elzaki transform and homotopy perturbation method (ETHPM) to solve nonlinear fractional heat -like equations and system of equations. The linear term in the equation can be solved by using Elazaki Transform Method (ETM) and the nonlinear term can be handled by using Homotopy Perturbation Method (HPM). This method is very powerful and efficient technique for solving different kinds of linear and nonlinear fractional differential equations. The benefit of the combined ELzaki transform and homotopy perturbation method is more efficient and easier to handle nonlinear partial differential equations.

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CHAPTER ONE

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unify and generalize the notions of integer- order differentiation and n- fold integration.

Fractional differential equations are equations that relate some function with its fractional derivatives. Fractional differential equations are generalization of differential equations of integral order. They involve fractional derivatives of the form $\frac{d^{\alpha}}{dx^{\alpha}}$ which are defined for $\alpha > 0$, where α is not necessary an integer. Fractional differential equation of order α in variable t is given by

$$D_t^{\alpha} u(x,t) = f(x,t,u(x,t),u_x(x,t), \ . \ . \ . \ ), \quad n-1 < \alpha < n,$$

where n is a positive integer.

Nowadays, there is an increasing attention paid to fractional differential equation (Podlubuny, Fractional Differential Equations, 1999) and their application in different areas. Fractional differential equations(FDEs) with specified value of the unknown function play an important role in technology, science, economics and engineering application including population model, control engineering electrical network analysis, gravity, medicine, etc (Hesameddini, et.al, 2012).

Heat-like model can describe many physical problems in different fields of science and engineering. These physical problems describe some nonlinear phenomena, such as diffusion of alleles in population genetics. The fractional heat-like equation has been applied in modeling to describe practical sub-diffusive problems in fluid flow process and finance (Yousif & Hamed, 2014).

In recent years, many researchers mainly had paid attention to studying the solution of nonlinear fractional partial differential equations by using various methods. For instance; the Varational Iteration Method (Wu, 2011), Adomain Decomposition Method (Duan, et.al, 2012), projected differential transform method (Elzaki & Hilal, 2012) and Differential Transform Method (Chang & Chang, 2008).

Elzaki transform was introduced by Elzaki (2011) as a modification of the classical Sumudu Transform. Elzaki derived this transform for solving differential and integral equations.

The homotopy perturbation method, first proposed by He (1999), is relatively new approach to provide an analytical approximation to both linear and nonlinear differential equations without linearization or discretization.

The combination of Elzaki transform with the homotopy perturbation method is applicable to construct an analytical solution for nonlinear fractional equations. The advantage of this method is its capability of obtaining exact solutions for nonlinear partial differential equations (Abdelilah, 2016).

The purpose of this project is to show application of Elzaki transform - Homotopy perturbation method to obtain analytic solution of nonlinear fractional heat – like equations and the corresponding coupled systems.

This project consists of two chapters. The first chapter focuses on facts about fractional differential equations, special functions, Elzaki transform and Homotopy perturbation methods. In the second chapter, we will discuss the application of Elzaki transform and homotopy perturbation method to construct an analytical solution for nonlinear fractional heat-like equation and systems.

1.2. Special Functions

In this section some basic theory of the special functions are considered. Here recall that definition and some properties of the Gamma functions and Mittage-Leffler functions.

1.2.1 Gamma Function

One of the basic functions of the fractional calculus is Euler's Gamma function (Podlubuny, Fractional Differential Equations, 1999), which generalizes the factorial n! and allows n to take also non- integer and even complex values. We will recall in this subsection some results on the gamma function which are important for the subsequent sections and chapter.

Definition 1.1: The Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \qquad (1.1)$$

which converges in the right half of the complex plane Re(z) > 0.

One of the basic properties of the Gamma function is that it satisfies the following functional equation

$$\Gamma(z+1) = z\Gamma(z), \qquad (1.2)$$

which can be easily proved by integrating by parts:

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^{z}dt = [-e^{-t}t^{z}]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}dt = z\Gamma(z).$$

Obviously, Γ(l) =1, and using (1.2) we obtain for z = 2, 3, 4, . . .

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2.\Gamma(2) = 2.1 = 2!$$

$$\Gamma(4) = \Gamma(3+1) = 3.\Gamma(3) = 3.2! = 3!$$

Continuing the same way leads to

$$\Gamma(n+1) = n.\Gamma(n) = n.(n-1)! = n!. \quad (1.3)$$

Another important property of the Gamma function is that it has simple poles at the point's z = -n, (n = 0, 1, 2 . . .). To demonstrate this, let us rewrite 1.1 in the form:

$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \; . \quad (1.4)$$

The first integral in (1.4) can be evaluated by using the series expansion for the exponential function.

$$\int_0^1 e^{-t} t^{z-1} dt = \int_0^1 \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} t^{z-1} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 t^{k+z-1} dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)} \, . \quad (1.5)$$

The second integral in (1.4) defines an entire function of the complex variable z. Indeed, let us rewrite as

$$\int_1^\infty e^{-t} t^{z-1} dt = \int_1^\infty e^{-t+(z-1)\ln t} dt \, . \quad (1.6)$$

The function $e^{-t+(z-1)\ln t}$ is a continuous function of z and t for arbitrary z & t ≥ 1.

Moreover, if $t \geq 1$ and therefore $\ln t \geq 0$, then it is an entire function of z.

Bringing together (1.5) and (1.6), we get

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)} + \int_1^\infty e^{-t} t^{z-1} dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)} + \text{ entire function,}$$

and, indeed, Γ(z) has only simple poles at the points z = - k, k = 0, 1, 2, . . .

1.2.2 Mittage-Leffler Function

Mittag-Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations. In 1903, the Swedish mathematician Gosta Mittag-Leffler (Mittag-Leffler, 1903) introduced the function $E_\alpha(z)$, defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \, , \quad \alpha \geq 0, \quad (1.7)$$

where z is a complex variable and Γ(z) is a Gamma function. The Mittag-Leffler function is a direct generalisation of the exponential function to which it reduces for α = 1. For 0 < α <1 it interpolates between the pure exponential and a hyper geometric function $(1-z)^{-1}$.

The generalisation of $E_\alpha(z)$ was studied by Wiman in 1905 and he defined the function as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in \mathbb{C}; \; Re(\alpha) > 0, \qquad Re(\beta) > 0), \tag{1.8}$$

which is known as Wiman's function or generalised Mittag-Leffler function.

For particular values of parameters the Mittag-Leffler function coincides with some elementary functions and simple special functions. For instance,

1. $E_0(z) = \frac{1}{z}, \quad |z| < 1$
2. $E_{1,1}(z) = E_1(z) = e^z$
3. $E_{1,2}(z) = \frac{e^z - 1}{z}$
4. $E_{2,1}(z^2) = E_2(z^2) = coshz$
5. $E_{2,1}(-z^2) = E_2(-z^2) = cosz$
6. $E_{\alpha,1}(z) = E_\alpha(z), \forall \alpha > 0$

1.3 Nonlinear Fractional Differential Equation

Nonlinear differential equations with integer or fractional order have played a very important role in various fields of science and engineering, such as mechanics, electricity, chemistry, biology, control theory, signal processing and image processing (Podlubuny, 1999). In all these fields, it is important to obtain exact or approximate solutions of nonlinear fractional differential equations. But in general, there exists no method that gives an exact solution and most of the obtained solutions are only approximations.

There exists a vast literature on different definitions of fractional derivatives and integrals. The most popular ones are the Riemann–Liouville and the Caputo derivatives. But in this project we use Caputo derivative.

Definition 1.2: Let $\Omega = \mathbb{R} \times \mathbb{R}^+$. Then

a. A set of functions $C_\mu(\Omega)$ is defined as

$$C_\mu(\Omega) = \{u: \Omega \to \mathbb{R} | u(x,t) = t^p h(x,t) \;, \qquad p > \mu \;\},$$

for real function $h(x,t)$.

b. A set of functions $C_\mu^\alpha(\Omega)$ is defined as

$$C_\mu^\alpha(\Omega) = \left\{u: \Omega \to \mathbb{R} \middle| cD_t^\alpha u(x,t) \in C_\mu(\Omega), m-1 < \alpha < m, m \in \mathbb{N} \right\}.$$

Definition 1.3: The Riemann-Liouville fractional integral of order α, $\alpha \geq 0$ of a function $u(x,t) \in C_\mu(\Omega)$, $\mu \geq -1$, $t > 0$ is denoted by $aI_t^\alpha u(x,t)$ and defined as

$$aI_t^{\alpha}u(x,t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}u(x,s)ds\ , & \alpha>0,\\ u(x,t), & \alpha=0\ ,\end{cases}$$

where $\Gamma(.)$ is Gamma function.

Using Definition 1.3, one can derive the following properties of fractional integral operator aI_t^{α}:

1. $I_t^{\alpha}I_t^{\beta}u(x,t)=I_t^{\alpha+\beta}u(x,t)\beta.$
2. $I_t^{\alpha}I_t^{\beta}u(x,t)=I_t^{\beta}I_t^{\alpha}u(x,t).$
3. $I_t^{\alpha}x^nt^m=\frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)}x^nt^{\alpha+m}.$

Definition 1.4: The Riemann-Liouville fractional derivative of order α, $\alpha>0, n\in\mathbb{N}$ of a function $u(x,t)\in C_{\mu}^{n}(\Omega)$, $\mu\geq-1$, is denoted by ${}_a^RD_t^{\alpha}u(x,t)$ and defined as

$${}_a^RD_t^{\alpha}u(x,t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}D_t^n\int_a^t(t-s)^{n-\alpha-1}u(x,s)ds, & n-1<\alpha<n,\\ D_t^nu(x,t), & \alpha=n\ ,\end{cases}$$

where D_t^n is n^{th} order partial derivative of u with respect to t.

The Riemann-Liouville fractional derivative operator ${}_a^RD_t^{\alpha}$ has the following properties.

1. ${}_a^RD_t^{\alpha}x^nt^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^nt^{\beta-\alpha}$
2. ${}_a^RD_t^{\alpha}[f(x,t)g(x,t)]=g(x,t){}_a^RD_t^{\alpha}f(x,t)+f(x,t){}_a^RD_t^{\alpha}g(x,t)$
3. ${}_a^RD_t^{\alpha}{}_a^RD_t^{\beta}u(x,t)={}_a^RD_t^{\alpha+\beta}u(x,t)$

Definition 1.5: The Caputo fractional derivative of a function $u(x,t)\in C_{\mu}^{m}(\Omega)$, $\mu\geq-1$, $m\in\mathbb{N}$,w.r.t ''t'', denoted by $cD_t^{\alpha}u(x,t)$ and defined as

$$cD_t^{\alpha}u(x,t)=\begin{cases}D_t^mu(x,t), & \alpha=m,\\ \frac{1}{\Gamma(m-\alpha)}\int_0^t(t-s)^{m-\alpha-1}D_s^mu(x,s)ds\,, & m-1<\alpha<m\,,\end{cases}$$

where D_t^m is m^{th} order partial derivative of u with respect to t.

The Caputo fractional differential operator cD_t^{α} has the following properties.

1. $cD_t^{\alpha}\,k=0$, where k is constant.
2. $cD_t^{\alpha}\,x^nt^{\beta}=\dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^nt^{\beta-\alpha}.$
3. $cD_t^{\alpha}cD_t^{\beta}u(x,t)=cD_t^{\alpha+\beta}u(x,t).$
4. $cD_t^{\alpha}aI_t^{\alpha}u(x,t)=u(x,t).$
5. $aI_t^{\alpha}cD_t^{\alpha}u(x,t)=u(x,t)-\sum_{k=0}^{m-1}\dfrac{t^k}{k!}D_t^ku(x,0).$

This project focuses on nonlinear fractional heat-like equation. The fractional heat-like equations take the form:

$$cD_t^{\alpha}u(x,t) = k(x)cD_x^{\beta}u(x,t) + f(x,t,u,u_x),$$

with initial condition

$$u(x,0) = g(x)$$

where $0 < \alpha \leq 1$ is a parameter describing the order of the Caputo-fractional time derivative and $1 < \beta \leq 2$ is a parameter describing the order of the Caputo-fractional space derivative.

1.4 Elzaki Transform Method (ETM)

In this section, we will present definition and properties of Elzaki transform of a function of two variables with respect to one of the independent variable. Conditions for the existence of such transform will be considered.

Like other integral transforms, the Elzaki transform of the function exists if it is piecewise continuous and of exponential order. These conditions are only sufficient conditions for existence of Elzaki transform of a function.

Definition 1.6: A function is called piecewise continuous on a given interval if the interval can be subdivided into a finite number of subintervals on which a function is continuous on each open subinterval and has a finite limit at end point of each subinterval.

Definition 1.7: A function f is said to be of exponential order α if there exist positive constants M and α such that

$$|f(t)| < Me^{\alpha t}, \;\; t \geq t_o, \tag{1.9}$$

for some $t_o \geq 0$.

Example 1.1: Function of exponential order.

1. e^{-x} is of exponential order of -1.
2. t^n is of exponential order α for any $\alpha \geq 0$ for any $n \in \mathbb{N}$.
3. Any bounded function like $\sin x, \cos x$ are of exponential order 0.
4. e^{t^2} is not of exponential order α for any $\alpha \in \mathbb{R}$.

Definition 1.8: For a piecewise continuous function f defined on $\mathbb{R}^2$ if there exist constants $0 < M < \infty$ and $0 < k_i < \infty, i = 1, 2$, such that:

$$|f(x,t)| < Me^{\frac{|t|}{k_i}}, \qquad t \in (-1)^i \times [0,\infty),$$

then, Elzaki transform of the function f with respect to variable t for fixed x is define as

$$E_t[f(x,t)] = T(x,v) = v\int_0^\infty f(x,t)e^{\frac{-t}{v}}dt = v^2\int_0^\infty f(x,tv)e^{-t}dt, \quad (1.10)$$

where $t \geq 0, k_1 \leq v \leq k_2$. In (1.10) the variable v is used to factor the variable t in the argument of function f.

If $E_t[f(x,t)] = T(x,v)$, then the inverse of Elzaki transform of $T(x,v)$ is defined as:

$$f(x,t) = E_t^{-1}[T(x,v)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} T\left(x,\frac{1}{v}\right)e^{tv}v\,dv$$

$$= \sum Res\left[T\left(x,\frac{1}{v}\right)e^{tv}v\right],$$

where Res (.) is residues of a function .

Similarly, one can also define the Elzaki transform of the function f with respect to the variable x for fixed t.

Example 1.2: Elzaki transform of elementary functions.

1. If f (x, t) = 1, then by (1.10) , we have

$$E_t(1) = v\int_0^\infty e^{\frac{-t}{v}}dt = v\lim_{r\to 0^+}\left[-ve^{\frac{-t}{v}}\right]\Big|_r^\infty = v^2, \qquad v > 0.$$

2. If f (x, t) = e^{ax+bt}, then

$$E_t(e^{ax+bt}) = v\int_0^\infty e^{ax+bt}e^{\frac{-t}{v}}dt = ve^{ax}\int_0^\infty e^{\left(\frac{bv-1}{v}\right)t}dt$$

$$= ve^{ax}\lim_{r\to 0^+}\frac{v}{bv-1}e^{\left(\frac{bv-1}{v}\right)t}\Big|_r^\infty = \frac{v^2e^{ax}}{1-bv}, \qquad v < \frac{1}{b}, b \neq 0.$$

3. If f (x, t) = $\sin(ax+bt))$, then

$$E_t[\sin(ax+bt)] = E_t\left[\frac{e^{i(ax+bt)} - e^{i(ax+bt)}}{2i}\right]$$

$$= \frac{1}{2i}\left[E_t\left(e^{i(ax+bt)}\right) - E_t\left(e^{-i(ax+bt)}\right)\right]$$

$$= \frac{1}{2i}e^{iax}\left(\frac{v^2}{1-ibv}\right) - \frac{1}{2i}e^{-iax}\left(\frac{v^2}{1+ibv}\right)$$

$$= \frac{v^2}{2i}\left(\frac{e^{iax}(1+ibv) - e^{-iax}(1-ibv)}{1+b^2v^2}\right)$$

$$= \frac{v^2}{1+b^2v^2}\left(\frac{e^{iax} + ibve^{iax} - e^{-iax} + ibve^{-iax}}{2i}\right)$$

$$= \frac{v^2}{1+b^2v^2}\left(\frac{e^{iax} - e^{-iax}}{2i} + vb\frac{e^{iax} + e^{-iax}}{2}\right)$$

$$= \frac{v^2 \sin ax + bv^3 \cos ax}{1+b^2v^2}, \qquad v > 0\,.$$

Similarly,

$$E_t[\cos(ax+bt)] = \frac{v^2(\cos ax - bv \sin ax)}{1+b^2v^2}, \qquad v > 0.$$

4. If f (x, t) = $x^m t^n$, then

$$E_t[x^m t^n] = v\int_0^\infty x^m t^n e^{\frac{-t}{v}}\, dt = vx^m \int_0^\infty t^n e^{\frac{-t}{v}}\, dt$$

$$= vx^m e^{\frac{-t}{v}}\left(vt^n - \sum_{i=0}^{n} \frac{n!}{(n-i)!} v^{i+1} t^{n-i}\right)\Bigg|_{t=0}^{t=\infty}$$

$$= n!\, x^m v^{n+2} = \Gamma(n+1) x^m v^{n+2}, \qquad v > 0\ .$$

Generally, (Salman, 2022)for $\alpha \in \mathbb{R}_0^+$

$$E_t[t^\alpha] = v^{\alpha+2}\Gamma(\alpha+1).$$

Definition 1.9: For a piecewise continuous function f defined on $\mathbb{R}^2$ if there exist constants $0 < M < \infty$ and $0 < k_i < \infty, i = 1, 2$ such that:

$$|f(x,t)| < Me^{\frac{|t|}{k_i}}, \qquad t \in (-1)^i \times [0, \infty),$$

then, Laplce transform of the function f with respect to variable t for fixed x is define as

$$L_t[f(x,t)] = F(x,s) = \int_0^\infty f(x,t) e^{-st} dt. \tag{1.11}$$

Theorem 1.1: Let $f(x,t)$ be a piecewise continuous and of exponential order with Laplace transform $F(x,s)$. Then the Elzaki transform of $f(x,t)$ is given by:

$$T(x,v) = vF\left(x, \frac{1}{v}\right). \tag{1.12}$$

Proof: By (1.10), we have

$$E_t[f(x,t)] = T(x,v) = v\int_0^\infty e^{\frac{-t}{v}} f(x,t)dt = v^2\int_0^\infty e^{-t} f(x,tv)dt.$$

Let $s = vt$ and $ds = vdt$, we rewrite the above equation as:

$$T(x,v) = v^2\int_0^\infty f(x,tv)e^{-t}dt = v^2\int_0^\infty e^{-\frac{s}{v}} f(x,s)\frac{1}{v}ds$$

$$= v\int_0^\infty e^{-\frac{s}{v}} f(x,s)\,ds = vF\left(x, \frac{1}{v}\right).$$

Also we have that $T(1) = F(1) = 1$ so that both the ELzaki and Laplace transforms must coincide at $v = s = 1$.

Definition 1.10: The Laplace transform of the Caputo fractional derivative of the function $f(x,t)$ with respect to variable t is defined as

$$L_t[cD_t^\alpha f(x,t)] = s^\alpha F(x,s) - \sum_{k=0}^{m-1} s^{\alpha-(k+1)} D_t^k f(x,0), \qquad m-1 < \alpha < m. \tag{1.13}$$

Some important properties of Elzaki transform which will be used in the subsequent chapter are derived from the definition.

Property 1: Linearity property

Let $E_t[f(x,t)] = T(x,v)\ and\ E_t[g(x,t)] = L(x,v)$. For arbitrary constants a and b, we have

$$E_t[af(x,t) + bg(x,t)] = aT(x,v) + bL(x,v).$$

Proof: By definition of Elzaki transform and properties of integration, we get

$$E_t[af(x,t) + bg(x,t)] = v\int_0^\infty [af(x,t) + bg(x,t)]e^{\frac{-t}{v}}dt$$

$$= av\int_0^\infty f(x,t)e^{\frac{-t}{v}}dt + bv\int_0^\infty g(x,t)e^{\frac{-t}{v}}dt$$

$$= aT(x,v) + bL(x,v).$$

Example 1.3: Consider the function

$$f(x,t) = t + e^{2x+t} + \sin(x+2t)$$

By property 1, we have

$$E_t f(x,t) = T(x,v) = E_t[t + e^{2x+t} + \sin(x+2t)]$$

$$= E_t[t] + E_t[e^{2x+t}] + E_t[\sin(x+2t)]$$

$$= v^3 + \frac{v^2 e^{2x}}{1-v} + \frac{v^2 \sin x + 2v^3 \cos x}{1+4v^2}.$$

Property 2: Elzaki transform of integer order derivatives

Let $E_t[f(x,t)] = T(x,v)$, then we get

$$E_t[D_t^n f(x,t)] = \frac{T(x,v)}{v^n} - \sum_{i=0}^{n-1} \frac{1}{v^{n-(i+2)}} cD_t^i f(x,0),$$

and

$$E_t[D_x^n f(x,t)] = D_x^n T(x,v).$$

Proof: To obtain Elzaki transform of partial derivative with respect to t, we use integration by part and mathematical induction as follows.

$$E_t[D_t f(x,t)] = v\int_0^\infty D_t f(x,t) e^{\frac{-t}{v}} dt = v \lim_{p\to\infty, q\to 0^+} \int_q^p D_t f(x,t) e^{\frac{-t}{v}} dt$$

$$= \lim_{p\to\infty, q\to 0^+} \left\{ \left[v e^{\frac{-t}{v}} f(x,t) \right] \Big|_q^p + \int_q^p e^{\frac{-t}{v}} f(x,t) dt \right\}$$

$$= \frac{1}{v} \lim_{p\to\infty, q\to 0^+} v \int_q^p e^{\frac{-t}{v}} f(x,t) dt - vf(x,0)$$

$$= \frac{T(x,v)}{v} - vf(x,0) .$$

To find $E_t(D_t^2 f(x,t))$, let $D_t f(x,t) = g(x,t)$, then we have

$$E_t[D_t^2 f(x,t)] = E_t[D_t g(x,t)] = \frac{E_t[g(x,t)]}{v} - vg(x,0)$$

$$= \frac{T(x,v)}{v^2} - f(x,0) - vD_t f(x,0).$$

Using mathematical induction, we can extended this property to the n[th] order partial derivative to get

$$E_t[D_t^n f(x,t)] = \frac{E_t[f(x,t)]}{v^n} - \frac{f(x,0)}{v^{n-2}} - \frac{D_t f(x,0)}{v^{n-3}} - \ \dots - D_t^{n-2} f(x,0) - v D_t^{n-1} f(x,0)$$

$$= \frac{T(x,v)}{v^n} - \sum_{i=0}^{n-1} \frac{1}{v^{n-(i+2)}} D_t^i f(x,0).$$

The Elzaki transform of integer order derivative with respect to x is:

$$E_t[D_x^n f(x,t)] = v\int_0^\infty D_x^n f(x,t) e^{\frac{-t}{v}} dt = D_x^n \left(v\int_0^\infty f(x,t) e^{\frac{-t}{v}} dt \right) = D_x^n T(x,v).$$

Property 3: Elzaki transform of fractional order derivative

If $T(x,v)$ is Elzaki transform of $f(x,t)$, then Elzaki transform of the Caputo fractional derivatives defined as follows:

$$E_t[cD_t^\alpha f(x,t)] = v^{-\alpha}\left[T(x,v) - \sum_{k=1}^{n} v^{2+k} D_t^k f(x,0)\right], \qquad -1 < n-1 < \alpha < n\,. \quad (1.14)$$

Proof: By virtue of Theorem 1.1 and definition 1.10 we have

$$T(x,v) = vF\left(x,\frac{1}{v}\right),$$

and

$$L[cD_t^\alpha f(x,t)] = S^\alpha F(x,s) - \sum_{k=0}^{n-1} S^{\alpha-k-1}[D_t^k f(x,0)].$$

Then,

$$E_t[cD_t^\alpha f(x,t)] = vF^\alpha\left(x,\frac{1}{v}\right) = v\left[\frac{1}{v^\alpha}F\left(x,\frac{1}{v}\right) - \sum_{k=0}^{n-1}\left(\frac{1}{v}\right)^{\alpha-k-1} D_t^k f(x,0)\right]$$

$$= \frac{1}{v^\alpha} vF\left(x,\frac{1}{v}\right) - v\sum_{k=0}^{n-1}(v)^{1-\alpha+k} D_t^k f(x,0)$$

$$= v^{-\alpha} T(x,v) - \sum_{k=0}^{n-1} v^{2-\alpha+k} D_t^k f(x,0).$$

Property 4: Elzaki transform of integrals:

Let $E_t[f(x,t)] = T(x,v)$, then the Elzaki transform of $g(x,t) = \int_0^t f(x,s)ds$ is given by

$$E_t[g(x,t)] = vT(x,v).$$

Proof: By fundamental Theorem of Calculus I, we have

$$D_t g(x,t) = f(x,t)$$

and g (x, 0) = 0.

Taking into account property 2, applying Elzaki transform with respect to t, gives

$$E_t[D_t g(x,t)] = \frac{E_t[g(x,t)]}{v} - g(x,0) = E_t[f(x,t)].$$

This implies that

$$E_t[g(x,t)] = vE_t[f(x,t)] = vT(x,v).$$

Property 5: Shifting property:

Let $E_t[f(x,t] = T(x,v)$, then

$$E_t[e^{at} f(x,t] = (1-av)T\left(x, \frac{v}{1-av}\right), \qquad a < \frac{1}{v}.$$

Proof: Since $E_t[f(x,t] = T(x,v)$, and

$$T\left(x, \frac{v}{1-av}\right) = \frac{v}{1-av}\int_0^\infty f(x,t)e^{\frac{\frac{-t}{v}}{1-av}}dt = \frac{v}{1-av}\int_0^\infty f(x,t)e^{\frac{-t}{v}(1-av)}dt$$

$$= \frac{1}{1-av} v\int_0^\infty f(x,t)e^{\frac{-t}{v}}e^{at}dt = \frac{1}{1-av}E_t[e^{at}f(x,t].$$

This implies that

$$E_t[e^{at} f(x,t] = (1-av)T\left(x, \frac{v}{1-av}\right).$$

1.5 Homotopy Perturbation Method (HPM)

Two continuous functions from one topological space to another topological space are called homotopic (Greek, homos = identical, same, similar and topos = place) if one can be ‘‘continuously deformed’’ into the other, and such a deformation being called a homtopy between the two function.

Definition 1.11: A homotopy between two continuous function f(x) and g(x) from a topological space X to topological space Y is formally defined to be a continuous function $H: X \times [0,1] \to Y$ such that, if $x \in X$, then $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Example 1.4: For continuous real valued functions $f, g: \mathbb{R} \to \mathbb{R}$ with $f(x) \neq g(x)$, define a function $H: \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$H(x,p) = (1-p)f(x) + \ pg(x), \qquad p \in [0,1].$$

As H is a composite of continuous functions, it is continuous and satisfies:

$$H(x, 0) = (1 - 0)f(x) + 0. g(x) = f(x),$$

$$H(x, 1) = (1 - 1)f(x) + 1g(x) = g(x).$$

Thus, H is a homotopy between $f(x)$ and $g(x)$.

The embedding parameter p ∈ [0,1] in a homotopy of functions or equation is called homotopy parameter. Perturbation method is a class of analytical methods used for determination approximate solutions of nonlinear equations. It leads to an expression for the desired solution in terms of a formal power series in small parameter (ϵ), that quantifies the deviation from the exactly solvable problem. Consider, the series

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

Here, x_0 be the known solution to the exactly solvable initial problem and x_1, x_2, ... are the higher order terms. For small ϵ these higher order terms are successively smaller. An approximate perturbation solution is obtained truncating the series, usually by keeping only the first two terms.

Definition 1.12: Homotopy perturbation Method (HPM) is the coupling of the perturbation method and the Homotopy methods, which has eliminated the limitation of traditional perturbation method.

Most perturbation methods assume a small parameter. Many methods such as variational method, variational iteration method and others are proposed to eliminate the short comings arising in the small parameter assumptions. Recently, the applications of homotopy perturbation method have appeared in the works of many authors which has become a powerful mathematical tool (Afrouzi, et.al., 2011).

To illustrate the basic concept of homotopy perturbation method, consider the following nonlinear functional equation.

$$A(u) - f(r) = 0, \qquad r \in \Omega\,, \tag{1.15}$$

with boundary conditions

$$B(u, u_r) = 0, \qquad r \in \partial\Omega, \tag{1.16}$$

where A is a general operator, B is a boundary operator, f(r) is a known analytic function, and $\partial\Omega$ is a boundary of domain Ω.

Generally speaking, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear operator. Equation (1.15) can be rewrite as follows:

$$L(u) + N(u) - f(r) = 0\ . \tag{1.17}$$

We construct a homotopy $v(r,p): \Omega \times [0,1] \rightarrow \mathbb{R}$ which satisfy

$$H(v,p) = (1-p)[L(v) - L(u_o] + p[L(v) + N(v) - f(r)] = 0.$$

Thus

$$L(v) - L(u_o) + pL(u_o) + p[N(v) - f(r)] = 0\,. \tag{1.18}$$

where $p \in [0,1]$ is an embedding parameter and u_o is an initial approximation for the equation (1.15), which satisfies the boundary conditions.

From equation (1.18) we have:

$$H(v,0) = L(v) - L(u_o) = 0. \tag{1.19}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{1.20}$$

It is obvious that when $p = 0$, equation (1.18) becomes a linear equation (1.19); when $p = 1$ it becomes the original nonlinear equation (1.20). So the changing process of p from zero to unity is just that of $L(v) - L(u_o) = 0$ to $A(v) - f(r) = 0$. The imbedding parameter p monotonically increase from zero to unity as the trivial problem $L(v) - L(u_o) = 0$ is continuously deformed to the problem $A(v) - f(r) = 0$. This is basic idea of homotopy method which is to continuously deform a simple problem easy to solve into the difficult problem under the study. According to HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of equation (1.18) can be written as power series in p:

$$v = v_0 + v_1 p + v_1 p^2 + \ \dots = \sum_{i=0}^{\infty} v_i p^i \tag{1.21}$$

Considering p = 1, the approximate solution of (1.15) will be obtained as follows

$$u = \lim_{p \to 1} \sum_{i=0}^{\infty} v_i p^i = \ \sum_{i=0}^{\infty} v_i. \tag{1.22}$$

The series in equation (1.22) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$.

Let's rewrite the equation (1. 18) as follows

$$L(v) - L(u_o) = p[f(r) - L(u_o) - N(v)]\,. \tag{1.23}$$

Substituting equation (1.21) into equation (1.23) leads to:

$$L\left(\sum_{i=0}^{\infty} v_i p^i\right) - L(u_o) = p\left[f(r) - L(u_o) - N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]\ . \tag{1.24}$$

By linearity property of L, it follows that

$$\sum_{i=0}^{\infty} p^i L(v_i) - L(u_o) = p\left[f(r) - L(u_o) - N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]. \tag{1.25}$$

According to Maclaurin expansion $N\left(\sum_{i=0}^{\infty} v_i p^i\right)$ with respect to p, we have,

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!} D_p^n N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right). \tag{1.26}$$

Now, set

$$H_n(v_0,\ v_1, v_2, .\ .\ .) = \left[\frac{1}{n!} c D_p^n N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]_{p=0}, \qquad n = 0, 1, 2, \dots \tag{1.27}$$

where H_n is called He's polynomial. Then

$$N[v(x,t)] = \sum_{i=0}^{\infty} p^i H_i\ . \tag{1.28}$$

Substituting (1.28) into (1.25), we drive:

$$\sum_{i=0}^{\infty} p^i L(v_i) - L(u_o) = p\left[f(r) - L(u_o) - N\left(\sum_{i=0}^{\infty} p^i H_i\right)\right]. \quad (1.29)$$

By equating the term with the identical powers in p, we get

$$p^0:\ L(v_0) - L(u_0) = 0$$

$$p^1:\ L(v_1) = f(r) - L(u_0) - H_0$$

$$p^2:\ L(v_2) = -H_1 \quad .$$

In general,

$$p^{n+1}:\ L(v_{n+1}) = -H_n \ . \quad (1.30)$$

Solving for v_i, $i = 0, 1, 2, \ldots$, we obtain

$$v_0 = u_0 \ ,$$

$$v_1 = L^{-1}[f(r)] - u_0 - L^{-1}(H_0),$$

$$v_2 = -L^{-1}(H_1),$$

and continuing the same manner, we have

$$v_{n+1} = -L^{-1}(H_n) \quad . \quad (1.31)$$

In view of (1.31), the solution of (1.15) is given by (1.22).

CHAPTER TWO
APPLICATION OF ELZAKI TRANSFORM-HOMOTOPY PERTURBATION METHOD

2.1 Introduction

There are many integral transforms used in solving differential equations and integral equations. The fact that makes the integral transforms is effective to convert differential equations and integral equations into simpler or algebraic equations. So the integral transform is useful to allow one to convert a problem into a simpler one.

Elzaki transform is one type of integral transform and it is insufficient to handle the nonlinear equations due to nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities, one of these combinations of homotopy perturbation method and Elzaki transform. The linear term in the equation can be solved by using Elazaki transform method (ETM) and the nonlinear terms in the equations can be handled by using homotopy perturbation method (HPM).

We follow the following steps or procedures to apply Elzaki transform-homotopy perturbation method to solve nonlinear equations.

Step 1: Apply Elzaki transform to differential equation with respect to one of the independent variables to get a simpler equation.

Step 2: Solve the transformed differential equation with respect to transformed variable using differential property and given initial condition.

Step 3: Take the inverse of the Elzaki transform which gives term arising from the known function and the prescribed initial condition and inverse of the nonlinear part.

Step 4: Apply the Homotopy perturbation method to decompose the nonlinear part and solve it.

Step 5: Set series solution of given differential equation.

2.2 Description of the Method

In this subtopic, we will describe the idea of Elzaki transform-homotopy perturbation method and formulate the desired solution.

Consider a general nonlinear fractional differential equation with initial conditions of the form:

$$cD_t^{\alpha}u(x,t) = L_x u(x,t) + N_x u(x,t), \qquad n-1 < \alpha < n, n \in \mathbb{N} \tag{2.1}$$

with initial conditions

$$cD_0^k u(x,0) = g_k(x),\ \ k = 0,1,2,\dots n-1,\ \ cD_0^m u(x,0) = 0\ ,\ \ m = [\alpha] \tag{2.2}$$

where cD_t^α denotes the Caputo fractional derivative operator, N_x is nonlinear fractional differential operator, and L_x is linear fractional differential operator.

Taking Elzaki transform on both sides of equation (2.1) gives

$$E_t[cD_t^\alpha u(x,t)] = E_t[L_x(u(x,t))] + E_t[N_x(u(x,t))]\ \ . \tag{2.3}$$

Using the differential property of Elzaki transform and initial conditions (2.2), we have

$$E_t[u(x,t)] = \sum_{k=0}^{m-1} v^{2+k} g_k(x) + v^\alpha E_t[L_x(u(x,t))] + v^\alpha E_t[N_x(u(x,t))]\ . \tag{2.4}$$

Applying the inverse of Elzaki transform on both side of the equation (2.4) gives

$$u(x,t) = G(x,t) + E_t^{-1}\big[v^\alpha E_t[L_x(u(x,t))] + v^\alpha E_t[N_x(u(x,t))]\big]\,, \tag{2.5}$$

where G(x, t) represents the term arising from the inverse of prescribed initial condition.

Next, by the Homotopy perturbation method, we have

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)\ \ . \tag{2.6}$$

Then, the nonlinear term can be decomposed as:

$$N_x[u(x,t)] = \sum_{n=0}^{\infty} p^n H_n\,(u), \tag{2.7}$$

where H_n(u) are He's polynomial and given by

$$H_n(u_0,\ u_1, u_2, .\ .\ .) = \left[\frac{1}{n!} cD_p^n N_x\left(\sum_{i=0}^{\infty} p^i\, u_i(x,t)\right)\right]_{p=0}\ ,\ \ n = 0,1,2,\dots\ \ . \tag{2.8}$$

Substitute equations (2.6) & (2.7) into equation (2.5), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) =\ G(x,t) + p\left[E_t^{-1}\left[v^\alpha E_t\left[L_x\left(\sum_{n=0}^{\infty} p^n u_n\right)\right] + v^\alpha E_t\left[N_x\left(\sum_{n=0}^{\infty} p^n H_n(u)\right)\right]\right]\right] \tag{2.9}$$

which is coupling of the Elzaki Transform and Homotopy perturbation method using He's polynomial. Comparing the coefficient like powers of p, the following approximations are obtained.

$$p^0: u_0(x,t) = G(x,t) .$$

$$p^1: u_1(x,t) = E_t^{-1}\left[v^\alpha E_t\left[L_x\left(u_0(x,t)\right) + H_0(u)\right]\right].$$

$$p^2: u_2(x,t) = E_t^{-1}\left[v^\alpha E_t\left[L_x\left(u_1(x,t)\right) + H_1(u)\right]\right].$$

$$p^3: u_3(x,t) = E_t^{-1}\left[v^\alpha E_t\left[L_x\left(u_2(x,t)\right) + H_2(u)\right]\right].$$

Continuing the same way leads to

$$p^n: u_n(x,t) = E_t^{-1}\left[v^\alpha E_t\left[L_x\left(u_{n-1}(x,t)\right) + H_{n-1}(u)\right]\right]. \quad (2.10)$$

Then the solution of (2.1) is given by;

$$u(x,t) = \lim_{p\to 1}\sum_{i=0}^{\infty} p^i u_i(x,t) \quad = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots$$

$$= \sum_{i=0}^{\infty} u_i(x,t) \quad (2.11)$$

which is a series solution and converges very rapidly.

2.3 Application of the Method

In this section, we consider some examples on single nonlinear fractional heat-like equation and system of nonlinear fractional heat-like equations.

2.3.1 Solving Nonlinear Single Fractional Heat-Like Equation

In this subsection, we will consider single nonlinear fractional heat- like equation, specifically equations of the form:

$$cD_t^\alpha u(x,t) = k(x)u_{xx}(x,t) + f(u,u_x), 0 < \alpha \leq 1, t \geq 0$$

with initial condition

$$u(x,0) = g\ (x).$$

Example 2.1. Consider the fractional heat-like equation of the form:

$$cD_t^{\alpha}u(x,t) = 2x^3u_{xx} - u_xu\ , \quad 0 < x < 1, \quad 0 < \alpha \le 1, \qquad t > 0 \tag{2.12}$$

with initial condition

$$u(x,0) = \ x\ . \tag{2.13}$$

Appling the Elzaki transform on both sides of equation (2.12) gives

$$E_t[\ cD_t^{\alpha}u(x,t)] = E_t[2x^3u_{xx} - u_xu]\ . \tag{2.14}$$

Using differential property of Elzaki transform, (2.14) can be written as:

$$v^{-\alpha}[E_tu(x,t) - v^2u(x,0)] = E_t[2x^3u_{xx} - u_xu]\ . \tag{2.15}$$

By initial condition (2.13), (2.15) becomes:

$$E_t[u(x,t)] = v^2x + v^{\alpha}E_t[2x^3u_{xx} - u_xu]\ . \tag{2.16}$$

By inverse of Elzaki transform, (2.16) becomes

$$u(x,t) = E_t^{-1}(xv^2) + E_t^{-1}\left[v^{\alpha}E_t[2x^3u_{xx} - u_xu]\right],$$

$$= x + E_t^{-1}\left[v^{\alpha}E_t[2x^3u_{xx} - u_xu]\right]. \tag{2.17}$$

Next, by applying homotopy perturbation method, from (2.17) we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \ x + pE_t^{-1}\left[v^{\alpha}E_t\left[2x^3\sum_{n=0}^{\infty} p^n\frac{\partial^2 u_n}{\partial x^2} - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right], \tag{2.18}$$

where $H_n(u)$ is He's polynomial that represents the nonlinear terms. The first few terms of $H_n(u)$are given by

$$H_0(u) = \frac{\partial u_0}{\partial x}u_0\,.$$

$$H_1(u) = \frac{\partial u_0}{\partial x}u_1 + \frac{\partial u_1}{\partial x}u_0\,.$$

$$H_2(u) = \frac{\partial u_0}{\partial x}u_2 + \frac{\partial u_1}{\partial x}u_1 + \frac{\partial u_2}{\partial x}u_0.$$

$$H_3(u) = \frac{\partial u_0}{\partial x}u_3 + \frac{\partial u_1}{\partial x}u_2 + \frac{\partial u_2}{\partial x}u_1 + \frac{\partial u_3}{\partial x}u_0 \ .$$

By the similar way

$$H_n(u) = \sum_{j=0}^{n} \frac{\partial u_j}{\partial x} u_{n-j} \ \ n = 0, 1, 2, 3,. \ . \ .$$

By comparing the coefficient of like powers of p in the equation (2.18), we have:

$$p^0: u_0(x,t) = x, \qquad \frac{\partial^2 u_0}{\partial x^2} = 0\,, \qquad H_0(u) = \frac{\partial u_0}{\partial x}u_0 = x \ \ .$$

$$p^1: u_1(x,t) = E_t^{-1}\left[v^\alpha E_t\left[2x^3\ \frac{\partial^2 u_0}{\partial x^2} - H_0\right]\right] = E_t^{-1}\left[v^\alpha E_t[-x]\right] = E_t^{-1}[-xv^{\alpha+2}] = -x\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

$$\frac{\partial^2 u_1}{\partial x^2} = 0, \qquad H_1(u) = \frac{\partial u_0}{\partial x}u_1 + \frac{\partial u_1}{\partial x}u_0 = -2x\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$p^2: u_2(x,t) = E_t^{-1}\left[v^\alpha E_t\left[2x^3\ \frac{\partial^2 u_1}{\partial x^2} - H_1(u)\right]\right] = E_t^{-1}\left[v^\alpha E_t\left[2x\frac{t^\alpha}{\Gamma(\alpha+1)}\right]\right] = E_t^{-1}[2xv^{2\alpha+1}]$$

$$= 2x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \ \ .$$

$$\frac{\partial^2 u_2}{\partial x^2} = 0, \qquad H_2(u) = \frac{\partial u_0}{\partial x}u_2 + \frac{\partial u_1}{\partial x}u_1 + \frac{\partial u_2}{\partial x}u_0 = 4x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 4x\frac{t^{2\alpha}}{[\Gamma(\alpha+1)]^2}$$

$$p^3: u_3(x,t) = E_t^{-1}\left[v^\alpha E_t\left[2x^3\ \frac{\partial^2 u_2}{\partial x^2} - H_2(u)\right]\right] = E_t^{-1}\left[v^\alpha E_t\left[-4x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 4x\frac{t^{2\alpha}}{[\Gamma(\alpha+1)]^2}\right]\right]$$

$$= E_t^{-1}\left[-4xv^{3\alpha+2} - 4x\frac{\Gamma(2\alpha+1)v^{3\alpha+2}}{[\Gamma(\alpha+1)]^2}\right] = -4x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 4x\frac{\Gamma(2\alpha+1)t^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)}$$

$$= \frac{-4xt^{3\alpha}}{\Gamma(3\alpha+1)}\left[\frac{\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2} + 1\right].$$

It is proceeding in a similar manner.

Therefore, the series solution is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + . \quad . \quad . \quad ,$$

$$= x - \frac{xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{2xt^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{4xt^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4x\Gamma(2\alpha+1)t^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} + \cdots \quad .$$

In a particular for $\alpha = 1$, we have

$$u(x,t) = x - \frac{xt}{1!} + \frac{2xt^2}{2!} - \frac{12xt^3}{3!} + \cdots \quad ,$$

$$= x(1 - t + t^2 - 2t^3 + . \, . \, .).$$

Example 2.2. Consider a nonlinear fractional heat-like equation of the form :

$$cD_t^{\alpha}u(x,t) = u_{xx} - u_x + uu_x - u^2 + u\,, \quad 0 < x < 1, \;\; 0 < \alpha \leq 1, \qquad t > 0 \tag{2.19}$$

subject to initial condition

$$u(x,0) = \; e^x \;. \tag{2.20}$$

Appling the Elzaki transform on both sides of equation (2.19) gives

$$E_t[\; cD_t^{\alpha}u(x,t)] = E_t[u_{xx} - u_x + uu_x - u^2 + u] \;. \tag{2.21}$$

By differential property of Elzaki transform, (2.21) becomes

$$v^{-\alpha}[E_t u(x,t) - v^2 u(x,0)] = E_t[u_{xx} - u_x + uu_x - u^2 + u] \;. \tag{2.22}$$

Using initial condition (2.20), from (2.22) we obtain

$$E_t[u(x,t)] = v^2 e^x + v^{\alpha}E_t[u_{xx} - u_x + uu_x - u^2 + u] \quad . \tag{2.23}$$

Then applying inverse of Elzaki transform to (2.23) gives

$$u(x,t) = E_t^{-1}(e^x v^2) + E_t^{-1}\left[v^{\alpha}E_t[u_{xx} - u_x + uu_x - u^2 + u]\right],$$

$$= e^x + E_t^{-1}\left[v^{\alpha}E_t[(u_{xx} - u_x + u) + (uu_x - u^2)]\right]. \tag{2.24}$$

Next, by homotopy perturbation method from (2.24), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \; e^x + pE_t^{-1}\left[v^{\alpha}E_t\left[\sum_{n=0}^{\infty} p^n\left(\frac{\partial^2 u_n}{\partial x^2} - \frac{\partial u_n}{\partial x} + u_n\right) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{2.25}$$

where $H_n(u)$is He's polynomial that represents the nonlinear terms. The first few terms of $H_n(u)$are given by

$$H_0(u) = u_0 \frac{\partial u_0}{\partial x} - {u_0}^2.$$

$$H_1(u) = u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} - 2u_0u_1.$$

$$H_2(u) = u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} - 2u_0u_2 - {u_1}^2.$$

By continuing the same way

$$H_n(u) = \sum_{j=0}^{n} u_j \frac{\partial u_{n-j}}{\partial x} - u_j u_{n-j} \ , n = 0,1,2,3, \ . \ . \ .$$

By comparing the coefficient of like powers of p in the equation (2.25), we have:

$$p^0: u_0(x,t) = e^x, \qquad \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial u_0}{\partial x} + u_0 = e^x \qquad H_0(u) = u_0 \frac{\partial u_0}{\partial x} - {u_0}^2 = 0 \ .$$

$$p^1: u_1(x,t) = E_t^{-1}\left[v^\alpha E_t\left[\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial u_0}{\partial x} + u_0 - H_0(u)\right]\right] = E_t^{-1}\left[v^\alpha E_t[e^x]\right] = E_t^{-1}[e^x v^{\alpha+2}]$$

$$= e^x \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

$$\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_1}{\partial x} + u_1 = e^x \frac{t^\alpha}{\Gamma(\alpha+1)}, \qquad H_1(u) = u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} - 2u_0u_1 = 0 \, .$$

$$p^2: u_2(x,t) = E_t^{-1}\left[v^\alpha E_t\left[\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_1}{\partial x} + u_1 - H_1(u)\right]\right] = E_t^{-1}\left[v^\alpha E_t\left[e^x \frac{t^\alpha}{\Gamma(\alpha+1)}\right]\right]$$

$$= E_t^{-1}[e^x v^{2\alpha+2}] = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad .$$

$$\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial u_2}{\partial x} + u_2 = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \qquad H_2(u) = u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} - 2u_0u_2 - {u_1}^2 = 0.$$

$$p^3: u_3(x,t) = E_t^{-1}\left[v^\alpha E_t\left[\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial u_2}{\partial x} + u_2 - H_2(u)\right]\right] = E_t^{-1}\left[v^\alpha E_t\left[e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\right]$$

$$= E_t^{-1}[e^x v^{3\alpha+2}] \quad = e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \ .$$

Proceeding in the similar manner, we obtain

$$p^n: u_n(x,t) = e^x \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \ .$$

Therefore, the series solution is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + . \quad . \quad .$$

$$= e^x + \frac{e^x t^\alpha}{\Gamma(\alpha+1)} + \frac{e^x t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{e^x t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{e^x t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots = e^x \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}$$

$$= e^x E_\alpha(t^\alpha) \, .$$

In particular for $\alpha = 1$, we have the solution in the closed form

$$u(x,t) = \ e^{x+t} \, .$$

2.3.2 Solving System of Nonlinear Fractional Heat-Like Equations

In this subsection, we apply the combined Elzaki transform method with homotopy perturbation method for solving systems of nonlinear fractional heat-like equations. Specifically system of equations of the following forms.

$$\begin{cases} cD_t^\alpha u(x,t) = f_1(w)u_{xx} + g_1(u,w,u_x,w_x) \\ cD_t^\beta w(x,t) = f_2(u)w_{xx} + g_2(w,u,u_x,w_x) \end{cases}, \quad 0 < x < 1, \ \ 0 < \alpha, \beta \le 1, \quad \ t > 0,$$

with initial condition

$$u(x,0) = p(x), \qquad w(x,0) = q(x).$$

Example 2.3. Consider the following system of nonlinear fractional heat-like equations

$$\begin{cases} cD_t^\alpha u(x,t) = w u_{xx} + u \\ cD_t^\beta w(x,t) = u w_{xx} - w \end{cases}, \quad 0 < x < 1, \ \ 0 < \alpha, \beta \le 1, \ \ t > 0 \qquad (2.26)$$

with the initial conditions

$$u(x,0) = x, \qquad w(x,0) = -x \ \ . \qquad (2.27)$$

Applying the Elzaki transform on both sides of (2.26) gives

$$\begin{cases} E_t\big(cD_t^\alpha u(x,t)\big) = E_t(w u_{xx} + u), \\ E_t\left(cD_t^\beta w(x,t)\right) = E_t(u w_{xx} - w). \end{cases} \qquad (2.28)$$

Using differential property of Elzaki transform, (2.28) can be written as

$$\begin{cases} v^{-\alpha}[E_t(u(x,t)) - v^2u(x,0)] = E_t(wu_{xx} + u), \\ v^{-\beta}\left[E_t\big(w(x,t)\big) - v^2w(x,0)\right] = E_t(uw_{xx} - w). \end{cases} \tag{2.29}$$

By the initial condition (2.27), (2.29) becomes

$$\begin{cases} E_T(u(x,t)) = v^2x + v^{\alpha}E_t(wu_{xx} + u), \\ E_T(w(x,t)) = -v^2x + v^{\beta}E_t(uw_{xx} - w). \end{cases} \tag{2.30}$$

Applying the inverse of Elzaki transform, (2.30) becomes

$$\begin{cases} u(x,t) = x + E_t^{-1}[v^{\alpha}E_t(wu_{xx} + u)], \\ w(x,t) = -x + E_t^{-1}\left[v^{\beta}E_t(uw_{xx} - w)\right]. \end{cases} \tag{2.31}$$

Next, applying homotopy perturbation method leads to

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x,t) &= x + p\left[E_t^{-1}\left[v^{\alpha}E_t\left(\sum_{n=0}^{\infty} p^n H_n(u) + \sum_{n=0}^{\infty} p^n u_n\right)\right]\right] \\ \sum_{n=0}^{\infty} p^n w_n(x,t) &= -x + p\left[E_t^{-1}\left[v^{\beta}E_t\left(\sum_{n=0}^{\infty} p^n H_n(w) - \sum_{n=0}^{\infty} p^n w_n\right)\right]\right] \end{aligned} \tag{2.32}$$

where $H_n(u)$ & $H_n(w)$ are He's polynomials. That is

$$H_n(u): p(wu_{xx}) = 0 \ .$$

$$H_n(w): p(uw_{xx}) = 0 \ .$$

The first few components of He's polynomials are given by

$$H_0(u) = w_0\frac{\partial^2 u_0}{\partial x^2}, \qquad H_0(w) = u_0\frac{\partial^2 w_0}{\partial x^2}.$$

$$H_1(u) = w_0\frac{\partial^2 u_1}{\partial x^2} + w_1\frac{\partial^2 u_0}{\partial x^2}, \qquad H_1(w) = u_0\frac{\partial^2 w_1}{\partial x^2} + u_1\frac{\partial^2 w_0}{\partial x^2}.$$

$$H_2(u) = w_0\frac{\partial^2 u_2}{\partial x^2} + w_1\frac{\partial^2 u_1}{\partial x^2} + w_2\frac{\partial^2 u_0}{\partial x^2}, \qquad H_2(w) = u_0\frac{\partial^2 w_2}{\partial x^2} + u_1\frac{\partial^2 w_1}{\partial x^2} + u_2\frac{\partial^2 w_0}{\partial x^2}.$$

By continuing in the same way, we get

$$H_n(u) = \sum_{j=0}^{\infty} w_j\frac{\partial^2 u_{n-j}}{\partial x^2}, \qquad H_n(w) = \sum_{j=0}^{\infty} u_j\frac{\partial^2 w_{n-j}}{\partial x^2}\ , n = 0,1,2,3,\ldots$$

By comparing the coefficient of like powers of p in the equation (2.32), we have

$$p^0: \begin{cases} u_0(x,t) = x \\ w_0(x,t) = -x \end{cases}, \quad H_0(u) = w_0\frac{\partial^2 u_0}{\partial x^2} = 0, \quad H_0(w) = u_0\frac{\partial^2 w_0}{\partial x^2} = 0 \ .$$

$$p^1: \begin{cases} u_1(x,t) = E_t^{-1}[v^{\alpha}E_t(H_0(u) + u_0)] = E_t^{-1}[v^{\alpha}E_t(x)] = E_t^{-1}[xv^{\alpha+2}] = x\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ w_1(x,t) = E_t^{-1}\left[v^{\beta}E_t(H_0(w)) - w_0\right] = E_t^{-1}\left[v^{\beta}E_t(x)\right] = E_t^{-1}\left[xv^{\beta+2}\right] = x\dfrac{t^{\beta}}{\Gamma(\beta+1)} \end{cases}$$

$$H_1(u) = w_0\frac{\partial^2 u_1}{\partial x^2} + w_1\frac{\partial^2 u_0}{\partial x^2} = 0, \qquad H_1(w) = u_0\frac{\partial^2 w_1}{\partial x^2} + u_1\frac{\partial^2 w_0}{\partial x^2} = 0$$

$$p^2: \begin{cases} u_2(x,t) = E_t^{-1}[v^\alpha E_t(H_1(u) + u_1)] = E_t^{-1}\left[v^\alpha E_t\left(x\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\right] \\ \qquad = E_t^{-1}[xv^{2\alpha+2}] = x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ w_2(x,t) = E_t^{-1}\left[v^\beta E_t(H_1(w) - w_1)\right] = E_t^{-1}\left[v^\beta E_t\left(-x\frac{t^\beta}{\Gamma(\alpha+1)}\right)\right] \\ \qquad = E_t^{-1}\left[(-x)v^{2\beta+2}\right] = -x\frac{t^{2\beta}}{\Gamma(2\beta+1)} \end{cases}$$

$$H_2(u) = w_0\frac{\partial^2 u_2}{\partial x^2} + w_1\frac{\partial^2 u_1}{\partial x^2} + w_2\frac{\partial^2 u_0}{\partial x^2} = 0, \qquad H_2(w) = u_0\frac{\partial^2 w_2}{\partial x^2} + u_1\frac{\partial^2 w_1}{\partial x^2} + u_2\frac{\partial^2 w_0}{\partial x^2} = 0$$

$$p^3: \begin{cases} u_3(x,t) = E_t^{-1}[v^\alpha E_t(H_2(u) + u_2)] = E_t^{-1}\left[v^\alpha E_t\left(x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right] \\ \qquad = E_t^{-1}[xv^{3\alpha+2}] = x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ w_3(x,t) = E_t^{-1}\left[v^\beta E_t(H_2(w) - w_2)\right] = E_t^{-1}\left[v^\beta E_t\left(x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right] \\ \qquad = E_t^{-1}\left[xv^{3\beta+2}\right] = x\frac{t^{3\beta}}{\Gamma(3\beta+1)} \end{cases}$$

Continuing in the same way, we get

$$p^n: \begin{cases} u_n(x,t) = x\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ w_n(x,t) = (-1)^{n+1}x\frac{t^{n\beta}}{\Gamma(n\beta+1)} \end{cases}$$

Then, the series solutions $u_n(x,t)$ and $w_n(x,t)$ are given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$
$$= x\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right)$$

$$= x\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(k\alpha+1)} = xE_\alpha(t^\alpha)$$

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \ldots$$
$$= -x\left(1 - \frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} - \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \ldots\right)$$

$$= -x\sum_{n=0}^{\infty}\frac{(-t)^{n\beta}}{\Gamma(n\beta+1)} = -xE_\beta\left(-t^\beta\right)$$

In particular for $\alpha = \beta = 1$, we have the solution in the closed form

$$u(x,t) = x\sum_{n=0}^{\infty}\frac{t^n}{n!} = xe^t$$

$$w(x,t) = -x\sum_{n=0}^{\infty}\frac{(-t)^n}{n!} = -xe^{-t}$$

Example 2.4. Consider the following system of nonlinear fractional heat-like equations

$$\begin{cases} cD_t^{\alpha}u(x,t) = u_{xx} - (uw)_x \\ cD_t^{\beta}w(x,t) = w_{xx} + (wu)_x \end{cases},\quad 0 < x < 1,\ \ 0 < \alpha,\beta \le 1,\ \ t > 0 \qquad (2.33)$$

with the initial conditions

$$u(x,0) = e^x,\ w(x,0) = e^{-x}\ \ . \qquad (2.34)$$

Applying the Elzaki transform on both sides of equation (2.33) gives

$$\begin{cases} E_t\left(cD_t^{\alpha}u(x,t)\right) = E_t(u_{xx} - (uw)_x), \\ E_t\left(cD_t^{\beta}w(x,t)\right) = E_t(w_{xx} + (wu)_x). \end{cases} \qquad (2.35)$$

Using differential property of Elzaki transform, equation (2.35) can be written as

$$\begin{cases} v^{-\alpha}[E_T(u(x,t)) - v^2u(x,0)] = E_t(u_{xx} - (uw)_x), \\ v^{-\beta}[E_T(w(x,t)) - v^2w(x,0)] = E_t(w_{xx} + (wu)_x). \end{cases} \qquad (2.36)$$

By the initial condition (2.34), (2.36) becomes

$$\begin{cases} E_T(u(x,t)) = v^2e^x + v^{\alpha}E_t(u_{xx} - (uw)_x)\,, \\ E_T(w(x,t)) = v^2e^{-x} + v^{\beta}E_t(w_{xx} + (wu)_x)\,. \end{cases} \qquad (2.37)$$

Applying the inverse of Elzaki transform, (2.37) becomes

$$\begin{cases} u(x,t) = e^x + E_t^{-1}[v^{\alpha}E_t(u_{xx} - uw_x - wu_x)], \\ w(x,t) = e^{-x} + E_t^{-1}[v^{\beta}E_t(w_{xx} + uw_x + wu_x)]. \end{cases} \qquad (2.38)$$

Next, applying homotopy perturbation method leads to

$$\begin{aligned} \sum_{n=0}^{\infty}p^nu_n(x,t) &= e^x + p\left[E_t^{-1}\left[v^{\alpha}E_t\left(\sum_{n=0}^{\infty}p^n\frac{\partial^2u_n}{\partial x^2} - \sum_{n=0}^{\infty}p^nH_n(u)\right)\right]\right] \\ \sum_{n=0}^{\infty}p^nw_n(x,t) &= e^{-x} + p\left[E_t^{-1}\left[v^{\beta}E_t\left(\sum_{n=0}^{\infty}p^n\frac{\partial^2w_n}{\partial x^2} + \sum_{n=0}^{\infty}p^nH_n(w)\right)\right]\right] \end{aligned} \qquad (2.39)$$

where $H_n(u)$ & $H_n(w)$ are He's polynomials. That is

$$H_n(u){:}\,p(uw_x + wu_x) = 0\,.$$

$$H_n(w){:}\,p(uw_x + wu_x) = 0.$$

The first few components of He's polynomials are given by

$$H_0(u) = H_0(w) = u_0\frac{\partial w_0}{\partial x} + w_0\frac{\partial u_0}{\partial x}.$$

$$H_1(u) = H_1(w) = u_0\frac{\partial w_1}{\partial x} + u_1\frac{\partial w_0}{\partial x} + w_0\frac{\partial u_1}{\partial x} + w_1\frac{\partial u_0}{\partial x}\,.$$

$$H_2(u) = \ H_2(w) = u_0 \frac{\partial w_2}{\partial x} + u_1 \frac{\partial w_1}{\partial x} + u_2 \frac{\partial w_0}{\partial x} + w_0 \frac{\partial u_2}{\partial x} + w_1 \frac{\partial u_1}{\partial x} + w_2 \frac{\partial u_0}{\partial x}.$$

By continuing in the same way, we get

$$H_n(u) = H_n(w) = \sum_{j=0}^{\infty} \left(u_j \frac{\partial w_{n-j}}{\partial x} + w_j \frac{\partial u_{n-j}}{\partial x} \right) \ .$$

By comparing the coefficient of like powers of p in the equation (2.39), we have

$$p^0: \begin{cases} u_0(x,t) = e^x, \\ w_0(x,t) = e^{-x}, \end{cases} \qquad \frac{\partial^2 u_0}{\partial x^2} = e^x, \qquad \frac{\partial^2 w_0}{\partial x^2} = e^{-x},$$

$$H_0(u) = H_0(w) = u_0 \frac{\partial w_0}{\partial x} + w_0 \frac{\partial u_0}{\partial x} = 0 \quad .$$

$$p^1: \begin{cases} u_1(x,t) = E_t^{-1}\left[v^\alpha E_t\left(\frac{\partial^2 u_0}{\partial x^2} - H_0(u)\right)\right] = E_t^{-1}[v^\alpha E_t(e^x)] \\ \qquad = E_t^{-1}[e^x v^{\alpha+2}] = \frac{e^x t^\alpha}{\Gamma(\alpha+1)}, \\ w_1(x,t) = E_t^{-1}\left[v^\beta E_t\left(\frac{\partial^2 w_0}{\partial x^2} + H_0(w)\right)\right] = E_t^{-1}\left[v^\beta E_t(e^{-x})\right] \\ \qquad = E_t^{-1}\left[e^{-x} v^{\beta+2}\right] = \frac{e^{-x} t^\beta}{\Gamma(\beta+1)}. \end{cases}$$

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{e^x t^\alpha}{\Gamma(\alpha+1)}, \qquad \frac{\partial^2 w_1}{\partial x^2} = \frac{e^{-x} t^\beta}{\Gamma(\beta+1)}$$

$$H_1(u) = H_1(w) = u_0 \frac{\partial w_1}{\partial x} + u_1 \frac{\partial w_0}{\partial x} + w_0 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_0}{\partial x} = 0.$$

$$p^2: \begin{cases} u_2(x,t) = E_t^{-1}\left[v^\alpha E_t\left(\frac{\partial^2 u_1}{\partial x^2} - H_1(u)\right)\right] = E_t^{-1}\left[v^\alpha E_t\left(\frac{e^x t^\alpha}{\Gamma(\alpha+1)}\right)\right] \\ \qquad = E_t^{-1}[e^x v^{2\alpha+2}] = \frac{e^x t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ w_2(x,t) = E_t^{-1}\left[v^\beta E_t\left(\frac{\partial^2 w_1}{\partial x^2} + H_1(w)\right)\right] = E_t^{-1}\left[v^\beta E_t\left(\frac{e^{-x} t^\beta}{\Gamma(\beta+1)}\right)\right] \\ \qquad = E_t^{-1}\left[e^{-x} v^{2\beta+2}\right] = \frac{e^{-x} t^{2\beta}}{\Gamma(2\beta+1)}. \end{cases}$$

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{e^x t^{2\alpha}}{\Gamma(2\alpha+1)}, \qquad \frac{\partial^2 w_2}{\partial x^2} = \frac{e^{-x} t^{2\beta}}{\Gamma(2\beta+1)}$$

$$H_2(u) = H_2(w) = u_0 \frac{\partial w_2}{\partial x} + u_1 \frac{\partial w_1}{\partial x} + u_2 \frac{\partial w_0}{\partial x} + w_0 \frac{\partial u_2}{\partial x} + w_1 \frac{\partial u_1}{\partial x} + w_2 \frac{\partial u_0}{\partial x} = 0 \ .$$

$$p^3:\begin{cases} u_3(x,t) = E_t^{-1}\left[v^\alpha E_t\left(\frac{\partial^2 u_2}{\partial x^2} - H_2(u)\right)\right] = E_t^{-1}\left[v^\alpha E_t\left(\frac{e^x t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right] \\ \qquad = E_t^{-1}[e^x v^{3\alpha+2}] = \frac{e^x t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ w_3(x,t) = E_t^{-1}\left[v^\beta E_t\left(\frac{\partial^2 w_2}{\partial x^2} + H_2(w)\right)\right] = E_t^{-1}\left[v^\beta E_t\left(\frac{e^{-x} t^{2\beta}}{\Gamma(2\beta+1)}\right)\right] \\ \qquad = E_t^{-1}\left[e^{-x} v^{3\beta+2}\right] = \frac{e^{-x} t^{3\beta}}{\Gamma(3\beta+1)}. \end{cases}$$

Continuing in the same way, we get

$$p^n:\begin{cases} u_n(x,t) = \frac{e^x t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ w_n(x,t) = \frac{e^{-x} t^{n\beta}}{\Gamma(n\beta+1)}. \end{cases}$$

Then, the series solutions $u_n(x,t)$ and $w_n(x,t)$ are given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

$$= e^x\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right)$$

$$= e^x \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(k\alpha+1)} = e^x E_\alpha(t^\alpha).$$

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \ldots$$

$$= e^{-x}\left(1 + \frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \ldots\right)$$

$$= e^{-x} \sum_{n=0}^{\infty} \frac{t^{n\beta}}{\Gamma(k\beta+1)} = e^{-x} E_\beta\left(t^\beta\right).$$

In particular for $\alpha = \beta = 1$, we have the solution in the closed form

$$u(x,t) = e^x \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^{x+t}.$$

$$w(x,t) = e^{-x} \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^{-x+t}.$$

SUMMARY

The main goal of this project is to show the application the combined method (ELzaki transform with the homotopy perturbation method (ETHPM)) to construct an analytical solution for nonlinear fractional heat-like Equations and corresponding systems, the properties of Elzaki transform simplify its computation. The linear term in the equation can be solved by using Elazaki transform method (ETM) and the nonlinear terms can be handled by using homotopy perturbation method (HPM).The combination of the two methods successfully applied to obtain series solution to nonlinear fractional heat-like differential equations and corresponding system. The solution to nonlinear fractional heat-like differential equation and systems with specified initial conditions, involves Mettag-Liffer functions and Gamma functions.

REFERENCES

Abdelilah, K. (2016). A Coupling Elzaki Transform and Homotopy Perturbation Method for Solving Nonlinear Fractional Heat-Like Equations. *American Journal of Mathematical and Computer Modelling, 1*(1), 15-20.

Afrouzi, G., Ganji, D., Hosseinzadeh, H., & Talarposhti, R. (2011). Fourth order Volterra integro-differential equations using modified homotopy-perturbation method. *The Journal of Mathematics and Computer Science, 3*(2), 179-191.

Chang, S., & Chang, I. (2008). A new algorithm for calculating onedimensional differential transform of nonlinear functions. *Appl. Math. & Compu, 195*(2008), 799-808.

Duan, J., Rach, R., Buleanu, D., & Wazwaz, M. (2012). A review of the Adomian decomposition method and its applications to fractional differential equations. *Communications in FractionalCalculus, 3*(2), 73–99.

Elzaki, T. (2011). The new integral transform Elzaki Transform. *Global Journal of Pure and Applied Mathematics, 7*(1), 57–64.

Elzaki, T., & Hilal, E. (2012). Solution of linear and nonlinear partial differential equations using mixture of Elzaki transform and the projected differential transform method. *Math. Theo. & Model, 2*, 50-59.

Hesameddini, E., Raihi, M., & Latifizadeh, H. (2012). A coupling method of Homotopy technique and Laplace transform for nonlinear fractional differential equations. *International Journal of Advances in Applied Sciences, 1*(4), 159-170.

Mittag-Leffler, G. (1903). Sur la nouvelle fonction Eα (x). *CR Acad. Sci. Paris, 137*(2), 554-558.

Podlubuny, I. (1999). *Fractional Differential Equations* (Vol. 198). NewYork, NY, USA: Academic Press.

Salman, J. H. (2022). An application of the Elzaki homotopy perturbation method for solving fractional Burger's equations. *Int. J. Nonlinear Anal. Appl., 13*(2), 21-30.

Wu, G. (2011). New trends in the variational iteration method. *Communications in Fractional Calculus, 2*(2), 59–75.

Yousif, E., & Hamed, S. (2014). Solution of nonlinear fractional differential equations using the homotopy perturbation Sumudu transform method. *Applied Mathematical Sciences,, 8*(44), 2195-2210.