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Solving Some Families of Fractional Order Partial Differential Equations by Using Laplace Transform Homotopy Perturbation methods

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MSC PROJECT REPORT

ON

Solving Some Families of Fractional Order Partial Differential Equations by Using Laplace Transform Homotopy Perturbation methods

BY

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BAHIR DAR, ETHIOPIA

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A project submitted to the department of mathematics, in partial fulfilment of the requirement for the degree of masters of Science in mathematics

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Declaration

This is to certify that the project entitled "**Solving Some Families of Fractional Order Partial Differential Equations by Using Laplace Transform Homotopy Perturbation methods**" Submitted in partial fulfilment of the requirements for the Degree of Master of science in Mathematics to the department of Mathematics, Bahir Dar University, is a record of original work carried out by me and has never been submitted to this or any other institution/s to any other degree or certificate. The assistance and help I received during the course of this investigation have been duly acknowledged.

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Approval of the project for defense

I hereby certify that I have supervised, read and evaluated this project entitled "**solving some families of fractional order partial differential equations by using Laplace transform Homotopy Perturbation methods**" prepared by Awoke Wolledie Alem under my guidance. I submitted for oral defense.

Dr. Eshetu Hailie	______________	_____________
Advisor's Name	signature	date

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Approval of the project for defense

We here by certify that we have examined this project entitled "**Solving Some Families of Fractional Order Partial Differential Equations Using Laplace Transform Homotopy Perturbation methods**" prepared and presented by Awoke Wolledie Alem. We recommend that this project is approved for the Degree of Master of Science in Mathematics.

Board of Examiners	Name	Signature	Date
External Examiner	____________________________	__________	_________
Internal Examiner I	____________________________	_________	_________
Internal Examiner II	____________________________	__________	_________

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ABBREVIATIONS

DEs	Differential Equations
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
FDEs	Fractional Differential Equations
FODEs	Fractional Ordinary Differential Equations
FOPDEs	Fractional Order Partial Differential Equations
IVPs	Initial Value Problems
BVPs	Boundary Value Problems
ℓT	Laplace Transform
HPM	Homotopy Perturbation Method

ABSTRACT

In this project we solve solving some families of fractional order partial differential equations using Laplace transform Homotopy Perturbation methods. The aim of the methods is to find series analytic approximate solution by considering small parameter of differential equations. The method is used to find solutions of both fractional ordinary and fractional partial differential equations.

Perturbation methods are based on an assumption that a small parameter must exist in the equation. Determination of small parameter required special art of techniques. An appropriate choice of small parameters leads to ideal results. However, unsuitable choice of small parameter results in bad effects

Using ideas of ordinary calculus, we can differentiate a function $f(x) = x$ to the first or second order. We can also establish a meaning or some potential applications of the results. However, can we differentiate the same function, to say, the halves order? Can we establish a meaning or some potential applications of the results? We may not achieve that through ordinary calculus. But we can achieve through fractional calculus, which is a more generalized form of calculus. It is not mean calculus of fractions, rather is the name for the theory of derivatives and integrals of arbitrary order.

Fractional derivatives have proven their capability to describe several phenomena associated with memory effects [2]. Their non-locality property is common in physical processes and cosmological problems. They are described by fractional derivatives. Thus fractional calculus is needed.

Fractional partial differential equations (FPDEs) have been developed in many different fields of science. They are used to simulating natural physical process and dynamic systems [9]. Solutions of most fractional differential equations are usually nonlinear partial differential equations of science and engineering.

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Chapter One

Introduction and Preliminary Concepts

1.1 General Introduction

The concept of derivative is the main idea of calculus. It shows sensitivity to change of functions, which is rate or slope of quantity.

The intuition of researchers from derivatives and integrals is based on their geometrical or physical meanings. Example of first and second order derivative of displacement with respect to time is called velocity and acceleration respectively. This form of classical calculus was developed extensively over centuries[4].

Many of the general laws of nature find their natural expressions in the language of differential equations. Differential equations allow us to study all kinds of evolutionary processes with the properties of finite dimensionality and differentiability.

The study of differential equations started very soon after the invention of differential and integral calculus. In1671G.C, Newton had laid the foundation to the study of differential equations [11].

Mathematics is the art of giving meaning to things having misleading names. The beautiful and at first look mysterious is name of the fractional calculus [4]. Fractional calculus is not mean calculus of fractions, rather is the name for theory of derivatives and integrals of arbitrary order. For many years, the subject of fractional calculus has been studied by many research scholars. This is an ongoing process and one can recognizes that within the study of fractional calculus, new techniques and mechanisms show up to find important challenging insights and unknown correlations between many areas of science [5].

Fractional calculus owes its origin to a question of whether the meaning of a derivative to an integer order n could be extended to be valid when $n \notin Z$.

The history of fractional calculus started almost at the same time when classical calculus was established. The concept of differentiation is familiar to all who have studied elementary calculus. For instance, if $f(x)=x^n$ for all $n \in Z^+$, then finding $\frac{d^n f(x)}{dx^n}$ is simple. Is the derivative valid for n is not an integer? In 1819G.C, Lacroix, the first Mathematician published the paper 'fractional derivatives' contributed a lot to theory of FDs. Start with $y = x^m$, $m \in Z^+$, he found n^{th} order derivative of $y = x^m$ is:

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}; m \geq n, m, n \in Z^+ \text{ [10].} \tag{1.1}$$

By letting $m = 1$ and $n = \frac{1}{2}$, Lacroix found,

$$\forall x \in R^+ \ , \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} y = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{1.2}$$

1.2 Gamma function and its properties

One of the basic functions of the fractional calculus is Euler's Gamma function[17], which generalizes the factorial of n such that $n\epsilon z^+, (n! = n(n-1)(n-2)\ldots)$, and allows n to take also non-integer and even complex values. We recall some results on the gamma function which are important for the subsequent sections and chapter two.

Definition 1.1: The Gamma function $\Gamma(z)$ is defined by the integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{1.3}$$

it converges in the right half of the complex plane $Re(z) > 0$.

If $x = \frac{1}{2}$, then $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \int_0^\infty e^{-t} t^{\frac{-1}{2}} dt = \int_0^\infty \frac{1}{e^t t^{\frac{1}{2}}} dt$ (1.4)

let $t = y^2 \Rightarrow dt = 2ydy$ (1.5)

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} y^{-1} 2ydy = 2\int_0^\infty e^{-y^2} dy. \tag{1.6}$$

$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, x is real or complex

Let $t = x^2 \Rightarrow \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx.$ (1.7)

$$\Rightarrow \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = 4\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dxdy \tag{1.8}$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2} r dr d\theta = \pi$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

One of the basic properties of the Gamma function is that, it satisfies the following functional equation.

$$\Gamma(z+1) = z\Gamma(z), \tag{1.9}$$

the integral in (1.9) can be easily proved by integrating by parts:

$$\Gamma(z + 1) = \int_0^\infty e^{-t}t^z dt = \left[-e^{-t}t^z\right]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}dt = z\Gamma(z).$$

Obviously, $\Gamma(l) = 1$, and using (1.9) we obtain for z = 2, 3, 4, …

$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$

$\Gamma(3) = \Gamma(2+1) = 2\,\Gamma(2) = 2(1) = 2!$

$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\,(2!) = 3!$

Continuing in the same way leads to

$$\Gamma(n+1) = n\,\Gamma(n) = n\,(n-1)! = n! \tag{1.10}$$

In addition, we have the following properties of Gamma function.

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
2. $\Gamma(\alpha) = (\alpha+1)\Gamma(\alpha-1) = \frac{\Gamma(\alpha+1)}{\alpha}, \alpha > 0$
3. $\Gamma\left(k + \frac{1}{2} + 1\right) = \frac{(2k+1)!}{2^{2k+1}k!}\sqrt{\pi}$, $k = 0, 1, 2, \ldots$

Gamma function is equal to the generalization of factorial of the integer numbers. As a result, it could be considered as an extension of factorial function to real numbers.

Another important property of the Gamma function is that, it has simple poles at the points, z = -n, (n = 0, 1, 2 . . .). To demonstrate this, let us rewrite (1.3) in the form:

$$\Gamma(z) = \int_0^1 e^{-t}t^{z-1}dt + \int_1^\infty e^{-t}t^{z-1}dt\ . \tag{1.11}$$

The first integral in (1.11) can be evaluated by using the series expansion for the exponential function.

$$\int_0^1 e^{-t} t^{z-1} dt \; = \int_0^1 \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} t^{z-1} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 t^{k+z-1} dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)} \,. \tag{1.12}$$

The second integral in (1.11) defines an entire function of the complex variable z. Indeed, let us rewrite the second integral of (1.11) as:

$$\int_1^{\infty} e^{-t} t^{z-1} dt = \int_1^{\infty} e^{-t+(z-1)\ln t} dt \,. \tag{1.13}$$

The function $e^{-t+(z-1)\ln t}$ is a continuous function of z and t for arbitrary z & t ≥ 1. Moreover, if $t \geq 1$ and therefore $\ln t \geq 0$, then it is an entire function of z.

Bringing together (1.12) &(1.13) we obtained:

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)} + \int_1^{\infty} e^{-t} t^{z-1} dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)} + \text{ entire function.}$$

Thus, $\Gamma(z)$ has only simple poles at the points z = - k, k = 0, 1, 2, …

1.3 Mittage-Leffler function

Mittag-Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations. In 1903, the Swedish mathematician Gosta Mittag-Leffler (Mittag-Leffler, 1903) introduced the function $E_\alpha(z)$ defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \,, \quad \alpha \geq 0, \tag{1.14}$$

where z is a complex variable and $\Gamma(z)$ is a Gamma function. The Mittag-Leffler function is a direct generalization of the exponential function to which it reduces for α = 1. For 0 < α <1, it interpolates between the pure exponential and a hypergeometric function $\frac{1}{1-z}$.

The generalizations of $E_\alpha(z)$ was studied by Wiman in 1905 and he defined the function as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.15}$$

$$(\alpha, \beta \in \mathbb{C}; \ Re(\alpha) > 0, \quad Re(\beta) > 0),$$

is known as Wiman's function or generalized Mittag-Leffler function.

For particular values of parameters, the Mittag-Leffler function coincides with some elementary functions and simple special functions. For instance,

1. $E_0(z) = \frac{1}{z}, \ \ |z| < 1$
2. $E_{1,1}(z) = E_1(z) = e^z$
3. $E_{1,2}(z) = \frac{e^z - 1}{z}$
4. $E_{2,1}(z^2) = E_2(z^2) = coshz$
5. $E_{2,1}(-z^2) = E_2(-z^2) = cosz$
6. $E_{\alpha,1}(z) = E_\alpha(z), \forall \alpha > 0$

1.4 Fractional Partial Differential Equations

Definition 1.2: For a continuous function the Caputo (was introduced by Caputo in 1960s,[12]) fractional order partial derivative of order $\alpha, n-1 < \alpha < n$, is defined,

$$\frac{\partial^\alpha f(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{\partial^n f(x,t)}{\partial t^n} dt. \tag{1.16}$$

Definition 1.3: For m to be the smallest positive integer that exceeds α, the Caputo time fractional partial derivative of $u(x,t), \alpha > 0$ is defined as:

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$$

$$= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,t)}{\partial t^m} dt, m-1<\alpha<m, m\in N \\ \frac{\partial^m u(x,t)}{\partial t^m}, \alpha = m \in N \end{cases} \tag{1.17}$$

Partial differential equations, like other one variable counterpart ordinary differential equations, are important throughout the scientific spectrum. However, they are more difficult to solve. We apply the Laplace transform method to solve PDEs by reducing the initial problem to a simpler ODE.

$$F\left(t, u(t), u^{(1)}(t), u^{(2)}(t), \ldots u^{(m)}(t)\right) = 0$$

is an example of ODE of order m, and

$$F\left(x, y, u(x,y), u_x(x,y), u_y(x,y), u_{xx}(x,y), u_{xy}(x,y), u_{yx}(x,y), u_{yy}(x,y)\right) = 0$$

is PDE of order two. ODE is a special case of a partial differential equation. But the behavior of solutions is quite different in general.

Solutions to PDEs typically depend not on several arbitrary constants but also on one or several arbitrary functions, initial and boundary conditions.

1.5 Fractional Order Derivatives

Fractional order derivative of a constant function, unlike the ordinary derivative, is not always zero. It is sought to answer the aforementioned questions and to construct a comprehensive picture of what fractional calculus is, and how it can be utilized for different purposes.

The concept of FD was introduced after 1695 as a simple academic generalization of integer derivative. It generalizes the order of differentiation from positive integers to set of real numbers, or even to set of complex numbers[4].
Fractional differential equations are equations that relate some functions with their fractional derivatives. They involve fractional derivatives of the form $\frac{d^{\alpha} f(x)}{dx^{\alpha}}$ which are defined for $\alpha > 0$, where α is not an integer. Fractional differential equations of order α are given by:

$$D_t^{\alpha} u(x,t) = f(x, t, u(x,t), u_x^{\alpha}(x,t), \ . \ . \ . \ ), \quad n-1 < \alpha < n, \tag{1.18}$$

Where n is a positive integer.
Most fractional differential equations do not have exact solutions. Approximation techniques, therefore, are used extensively. Recently, the Adomian decomposition method and variational iteration method have been used to solve fractional differential equations [6,9]. we adopt Caputo's definition of fractional derivatives which has advantage of dealing properly with initial value problems. It is also bounded, meaning the derivative of a constant is equal to 0.

Definition 1.4: A function is said to be piecewise continuous on a closed interval $[a, b]$ if this closed interval can be divided into a finite number of subintervals in each of which a function is continuous and has finite left hand and right hand limits.

Definition 1.5: A function $f(t)$ is of exponential order of α if there exist constants $M > 0$ and $\alpha > 0$ such that for some $t_0 > 0$ such that $|f(t)| \leq Me^{\alpha t}, t > t_0$.

Definition 1.6: Fractional calculus is a generalization of integration and differentiation to non-integer order operators.

Definition 1.7: For a continuous function $f(x)$, the fractional derivative of $f(x)$ in the Caputo sense is defined as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(x) dt, m-1 < \alpha \leq m, m \in N, x > 0.$$

1.6 Laplace Transform and its Properties

The Laplace transform (ℓT) is a widely used integral transform and named for Pierre-Simon Laplace (1749- 1827) who introduced the transform in his work on probability theory[7]. It works efficiently for relatively simple equations, because of difficulty of calculating inverse of Laplace transform. It converts integral and differential equations into algebraic equations.

To solve differential equations using (ℓT), each term in the differential equation is taken. If the unknown function is $y(x,t)$, then on taking the Laplace transform, an algebraic equation involving $Y(x,s) = \ell\{y(x,t)\}$ is obtained. This equation is solved for $Y(x,s)$ which is then inverted to produce the required solution, $y(x,t) = \ell^{-1}\{Y(x,s)\}$

Definition 1.8: For a given function $f(t)$, the Laplace transform of $f(t)$ is defined,

$$\ell\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt = \lim_{t_0 \to \infty} \int_0^{t_0} e^{-st} f(t) dt. \tag{1.19}$$

s is a parameter, may be real or complex number and ℓ is the symbol to Laplace transform.

Definition 1.9: The Laplace transform of the n^{th} order derivative of $f(x) = y^n$

is given by: $\ell\{y^{(n)}\}, n \in N$

$$\Rightarrow \ell\{y^{(n)}\} = s^n \ell(y) - s^{(n-1)}y(0) - s^{(n-2)}y'(0) - s^{(n-3)}y''(0) - ,..., - sy^{(n-2)}(0) - y^{(n-1)}(0)$$

$$= s^n \ell(y) - \sum_{k=0}^{n-1} s^{n-k-1} y^{(k)}(0) \tag{1.20}$$

Definition 1.10: The Laplace transform of Caputo fractional partial derivative of $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$,

is given by:

$$\ell\left\{\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}; s\right\} = s^\alpha F^*(x,s) - \sum_{K=0}^{n-1} s^{\alpha-k-1} \frac{\partial^k u(x,0)}{\partial t^k}, \tag{1.21}$$

$F^*(x,s) = \ell\{u(x,t); s)\}$, $\alpha > 0$.

Let $f(x,t)$ be a function of two independent variables x and t, then

$$\ell_x\{f(x,t)\} = \int_0^\infty e^{-sx} f(x,t)dx = F^*(s,t), \; 0 < x < \infty \tag{1.22}$$

$$\ell_t\{f(x,t)\} = \int_0^\infty e^{-st} f(x,t)dt = F^*(x,s), \; 0 < t < \infty \tag{1.23}$$

Subscripts in the transform indicate the variable to be transformed. It must satisfy the condition that $0 \leq x < \infty$, $0 \leq t < \infty$.Among x and t, only t satisfies the condition. Because x can be real. Thus we use the following.

$$\ell_t\left\{\frac{\partial f(x,t)}{\partial t}\right\} = \int_0^\infty e^{-st}\left\{\frac{\partial f(x,t)}{\partial t}\right\}dt$$

Using integration by parts,

$$\ell_t\left\{\frac{\partial f(x,t)}{\partial t}\right\} = e^{-st} f(x,t)\Big|_0^\infty + \int_0^\infty se^{-st} f(x,t)dt$$

$$= s\int_0^\infty e^{-st} f(x,t)dt - f(x,t)$$

$$= sF^*(x,s) - f(x,0). \tag{1.24}$$

$F^*(x,s) = \ell\{f(x,t)\}$

Likewise we have:

$$\ell_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = s^2F^*(x,s) - sf(x,0) - f'(x,0) \tag{1.25}$$

Definition 1.11: If $F(s)$ is the Laplace transform of $f(t)$, then the Laplace transform of its integral is given by:

$$\ell\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}F(s) \quad , \ell\left\{\int f(t)dt\right\} = \frac{1}{s}F(s) + \frac{1}{s}\left[\int f(t)dt\right].$$

Laplace transform constitutes an important tool in solving linear ordinary and partial differential equations with constant coefficients under suitable initial and boundary conditions. We first find the general solution and then evaluating it from the arbitrary constants. It is the powerful tool in applied mathematics and engineering. The technique is considered as an efficient way in solving differential equations with integer and fractional orders[7].

When Laplace transform is applied to any differential equations, it converts differential equations into algebraic manipulation. In case of partial differential equations involving two independent variables, Laplace transform is applied to one of the variables and the resulting differential equation in the second variable is then solved by the usual method of ordinary differential equations. Therefore, inverse Laplace transform of the resulting equation is the solution of the given PDE [1] .

Laplace transform does not exist for all functions. If it exists, it is uniquely determined. For existence of Laplace transform, the given function has to be continuous on every finite interval and of exponential order. If these conditions are not satisfied, the Laplace transform may or may not exist.

Example 1.1: let $f(t) = \dfrac{1}{\sqrt{t}}$. (1.26)

as $t \to 0^+, f(t) \to \infty$, as $t \to \infty, f(t) \to 0.$ precisely means, $f(t) = \dfrac{1}{\sqrt{t}}$ is not continuous on every finite interval in the domain $t \geq 0$. But, $f(t)$ is integrable from 0 to any positive values, say, t_0 and also $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 1$ with $M = 1$ and $\alpha = 0$ Thus: $\ell\{f(t)\} = \sqrt{\dfrac{\pi}{s}}, s \geq 0$ exists even if $f(t)$ is not continuous in the given domain. Based on the parameter, there are two types of Laplace transform.

1. $\ell\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{t_0 \to \infty} \int_0^{t_0} e^{-st} f(t)dt$,

   where s is real is unilateral or one sided transform.

2. $\ell\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{t_0 \to \infty} \int_0^{t_0} e^{-st} f(t)dt.$

   s is complex, is bilateral or two sided transform.

Let $\ell\{f(t)\} = F(s)$ and $\ell\{g(t)\} = G(s)$

such that a, b are constants, then the following holds.

1) Linearity: $\ell\{af(t) \pm bg(t)\} = aF(s) \pm bG(s).$
2) $\ell(x^\beta) = \dfrac{\Gamma(\beta+1)}{s^{(\beta+1)}}, \beta > -1$
3) $\ell\{x^n f(t)\} = (-1)^{(n)} F^{(n)}(s).$
4) $\ell\left\{\int_0^x f(t)dt\right\} = \dfrac{F(s)}{s}.$
5) Scaling: $\ell\{f(at)\} = \dfrac{1}{a} F\left(\dfrac{s}{a}\right), a > 0$
6) Initial value: $f(0_+) = \lim_{s\to\infty} sF(s)$, if $\lim_{s\to\infty} sF(s)$ exists.
7) Final value: if $\lim_{s\to 0} sF(s)$ exists ,then $f(\infty) = \lim_{s\to 0} sF(s)$
8) Time shifting: $\ell\{f(t-a)\} = e^{-as} F(s), a > 0$
9) Frequency shift: $\ell\{e^{-at} f(t)\} = F(s+a), a \in R$
10) One to one: if $\ell\{f(t)\} = \ell\{g(t)\}$, then $f(t) = g(t)$
11) $\ell\{\sin(bx)\} = \dfrac{b}{s^2 + b^2}.$

12) $\ell\{\cos(bx)\} = \dfrac{s}{s^2+b^2}$.

Theorem 1.1: If we assume that $f'(x)$ is continuous on $[0,\infty)$ and also of exponential order, then it follows that the same is true of $f(x)$.

Proof: Suppose that $|f'(x)| \le Me^{\alpha t}$ such that $t \ge t_0, \alpha \ne 0$

, Then $f(t) = \int_{t_0}^{t} f'(t)dt + f(t_0)$

$$\Rightarrow |f(t)| \le M\int_{t_0}^{t} e^{\alpha t}dt + |f(t_0)| \le \frac{Me^{\alpha t}}{\alpha} + |f(t_0)| \le ce^{\alpha t}, t \ge t_0, c = \frac{M}{\alpha}$$

Theorem 1.2: If $f(x)$ is piecewise continuous on $[0,\infty)$ and of exponential order α then the Laplace transform $\ell\{f(x)\}$ exists for $Re(s) > \alpha$ and converges absolutely.

Proof: $|f(t)| \le Me^{\alpha t}, t \ge t_0$ for some real α, $f(t)$ is continuous on $[0, t_0]$ and hence bounded. Since $e^{\alpha t}$ has positive minimum on $[0, t_0]$, a constant M can be chosen sufficiently large so that $|f(t)| \le Me^{\alpha t}, t_0 \ge 0$.

$$\Rightarrow \int_0^{t_0} |e^{-st} f(t)|dt \le M\int_0^{t_0} e^{-(s-\alpha)t}dt$$

$$= \frac{Me^{-(x-\alpha)t}}{-(x-a)}\Big|_0^{t_0}$$

$$= \frac{M}{x-\alpha} - \frac{Me^{-(x-\alpha)t_0}}{x-\alpha},$$

As $t_0 \to \infty$, nothing that $\operatorname{Re}(s) = x > a$

$$\Rightarrow \frac{Me^{-(x-\alpha)t_0}}{x-\alpha} \to 0$$

$$\Rightarrow \int_0^{\infty} |e^{-st}| f(t)dt \le \frac{M}{x-\alpha},$$

this shows Laplace integral converges absolutely.

1.7 Perturbation method

Definition 1.12: Perturbation methods are a class of analytical methods used for determining approximate solutions of non-linear equations. It leads to an expression for the desired solution in terms of a formal power series in small parameter (ε), those quantities are deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem and further terms describe the deviation in the solution.

Consider, $\mathrm{x} = \mathrm{x}_0 + \varepsilon \mathrm{x}_1 + \varepsilon^2 \mathrm{x}_2 + \cdots$

Here, x_0 be the known solution to the exactly solvable initial problem and $\mathrm{x}_1, \mathrm{x}_2 \ldots$ are the higher order terms. For small ε these higher order terms are successively smaller. An approximation "perturbation solution" is obtained by truncating the series, usually by keeping only the first two terms. Perturbation methods have their own limitations. At first, almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. Secondly, the determination of small parameter seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to ideal results. However, unsuitable choice of small parameter results in bad effects, sometimes seriously. Furthermore, the approximate solutions are valid, in most cases, only for small values of the parameters. It is obvious that all these limitations come from the small parameter assumption.

1.8 Homotopy Perturbation Method

Two continuous functions from one topological space to another topological space are called Homotopic (Greek, homos = identical, same, similar and topos = place) if one can be continuously deformed into the other and such a deformation is called a Homotopy between the two functions [18].

Definition 1.13: A Homotopy between two continuous functions $f(x)$ and $g(x)$ from a topological space X to topological space Y is formally defined to be a continuous function $H: X \times [0,1] \rightarrow Y$ such that, if $x \in X$, then $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Example 1.2: For continuous real valued functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) \neq g(x)$, define a function $H: \mathbb{R} \times [0,1] \rightarrow \mathbb{R}$ by:

$H(x,p) = (1-p)f(x) + \ pg(x), p \in [0,1]$.

As H is a composite of continuous functions, it is continuous and satisfied:

$H(x,0) = (1-0)f(x) + 0. g(x) = f(x)$,

$H(x,1) = (1-1)f(x) + 1g(x) = g(x)$.

Thus, H is Homotopy between $f(x)$ and $g(x)$.

Definition 1.14: Homotopy perturbation Method (HPM) is the coupling of the perturbation and the Homotopy methods.

Perturbation methods assume a small parameter. Many methods such as Adomian decomposition method, variational iteration method and others are proposed to eliminate the short comings arising in the small parameter assumptions. Recently, the applications of Homotopy perturbation method have appeared in the works of many authors which has become a powerful mathematical tool [18].

Homotopy perturbation method (HPM) is a widely applied techniques[6]. The method has been found to be very efficient for solving nonlinear differential equations with known initial or boundary value problems which are governed by the nonlinear ordinary (partial) differential equations. In this method, the solution is given in an infinite series usually converges to an accurate solution.

To describe the Homotopy perturbation method, we consider a general nonlinear differential equation of the type:

$$A(u) - f(x) = 0, x \in X \tag{1.27}$$

with boundary conditions $B\left(u, \frac{\partial u}{\partial x}\right) = 0, x \in \Omega$, A is a general differential operator, B is boundary operator, $f(x)$ is known analytic function, X is domain and Ω is the boundary of the domain.

The operator A can be divided into two parts L and N, where L is linear while N is nonlinear. Thus equation (1.43) can be written as:

$$L(u)+N(u)-f(x)=0. \tag{1.28}$$

Next construct Homotopy

$H(x,p): X\times[0,1]\to\mathbb{R}$, satisfied;

$$H(u,p)=(1-p)[L(u)-L(u_0)]+p[L(u)+N(u)-f(x)]=0.$$

Thus, $H(u,p)=L(u)-L(u_0)+pL(u_0)+p[N(u)-f(x)]=0.$ (1.29)

Where, $p\in[0,1]$ is embedding parameter, u_0 is an initial approximation of (1.28) from (1.29) we have $H(u,0)=L(u)-L(u_0)=0.$ (1.30)

$$H(u,1)=L(u)+N(u)-f(x)=0. \tag{1.31}$$

It is obvious that when $p=0$, equation (1.29) becomes a linear equation (1.30); when $p=1$, it becomes the original nonlinear equation (1.31). So the changing process of p from zero to unity is just that of $L(u)-L(u_0)=0$ to $L(u)+N(u)-f(x)=0$.

The imbedding parameter p monotonically increase from zero to unity as the trivial problem $L(u)-L(u_0)=0$ is continuously deformed to the problem

$L(u)+N(u)-f(x)=0$, is basic idea of Homotopy method which is to continuously deform a simple problem easy to solve difficult problem under the study. According to HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of equation (1.28) can be written as power series in p as:

$$u=\sum_{i=0}^{\infty}p^i u_i=u_0+u_1p+u_2p^2+\ldots., \tag{1.32}$$

as $P\to 1$, the approximate solution of (1.32) is obtained as follows

$$u=\lim_{p\to 1}\sum_{i=0}^{\infty}u_i p^i=\sum_{i=0}^{\infty}u_i \tag{1.33}$$

the series in equation (1.33) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(u)$.

$$L(u)-L(u_0)=p[f(x)-L(u_0)-N(u)]=0\,. \tag{1.34}$$

Substituting equation (1.33) into equation (1.34) leads to:

$$L\left(\sum_{i=0}^{\infty} u_i p^i\right) - L(u_0) = p\left[f(x) - L(u_0) - N\left(\sum_{i=0}^{\infty} u_i p^i\right)\right]. \qquad (1.35)$$

By linearity property of L, it follows that;

$$\sum_{i=0}^{\infty} p^i L(u_i) - L(u_0) = p\left[f(x) - L(u_0) - N\left(\sum_{i=0}^{\infty} u_i p^i\right)\right]. \qquad (1.36)$$

According to McLaurin expansion of $N\left(\sum_{i=0}^{\infty} u_i p^i\right)$ with respect to p we have,

$$N\left(\sum_{I=0}^{\infty} u_i p^i\right) = \sum_{i=0}^{\infty}\left(\frac{1}{n!} D_p^n N\left(\sum_{I=0}^{\infty} u_i p^i\right)\right). \qquad (1.37)$$

Now set $H(u_0, u_1, u_2, ..., u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N(\sum_{i=0}^{n} p^i u_i)\right]_{p=0}$ $, n = 0,1,2,3,...$ (1.38)

where H_n is called He's polynomial, then

$$N[u(x,t)] = \sum_{i=0}^{\infty} p^i H_i. \qquad (1.39)$$

Substituting (1.39) into (1.34),

$$\sum_{i=0}^{\infty} p^i L(u_i) - L(u_0) = p\left[f(x) - L(u_0) - N\left(\sum_{i=0}^{\infty} p^i H_i\right)\right]. \qquad (1.40)$$

Now equating identical powers of p of (1.40) we obtained:

$p^0 : L(u_0) - L(u_0) = 0$, $p^1 : L(u_1) = f(x) - L(u_0) - H_0$, $p^2 : L(u_2) = -H_1$,

Continuing in the same way we obtained; $p^{n+1} : L(u_{n+1}) = -H_n$

Solving for $u_i, i = 0,1,2,3,...$ $u_0 = u_0$, $u_1 = \ell^{-1}[f(x)] - u_0 - \ell^{-1}(H_0)$ $u_2 = -\ell^{-1}(H_1)$,

and continuing the same manner, we obtained: $u_{(n+1)} = -\ell^{-1}(H_n)$ Thus, the series solution of the given FPDE is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_{(n+1)}(x,t). \qquad (1.41)$$

Chapter Two

Applications of Laplace Transform Homotopy Perturbation Methods to Solve FOPDEs

2.1 Introduction

Fractional partial differential equations (FPDEs) have been developed in many different fields of science such as physics, finance, fluid mechanics, engineering and biology. They are used to describe models of different phenomena [5].

Due to frequent appearance of FPDEs in different disciplines of engineering and science, the scholars have added a lot of research contributions to both theory of mathematical science and technology. They are used to simulating natural physical process and dynamic systems. Find general solutions of most fractional differential equations which are usually nonlinear partial differential equations of science and engineering is too difficult.

There are many integral transforms used in solving differential equations and integral equations by converting a problem into a simpler one. Laplace transform is one type of integral transform and it is insufficient to handle the nonlinear equations due to nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities, one of these are combinations of Homotopy perturbation method and Laplace transform. The linear terms in the equation can be solved by using Laplace transform method and the nonlinear terms in the equation can be handled by using Homotopy perturbation method (HPM).

$$F(x_1, x_2, ..., x_n, u, \partial^{\alpha} u_{x_1}, \partial^{\alpha} u_{x_1 x_2}, ..., \partial^{\alpha n} u_{x_1 x_2, ..., x_n}) = 0, \tag{2.1}$$

$0 < \alpha \leq 1, n \in N,$ is FPDE.

FPDE with boundary or initial conditions is well formed, if its solution exists globally, is unique and depends continuously on the assigned domain.

FPDEs are used to simulating natural physical processes and dynamic systems. In the current study, the researcher implemented Laplace transform-Homotopy perturbation methods to find series solutions of some families of FPDEs.

We followed the following steps or procedures to apply Laplace transform-Homotopy perturbation method to solve some families of FPDEs.

Step 1: Apply the Laplace transform to each term of differential equation to get a simpler equation.

Step 2: Solve the transformed differential equation with respect to transformed variable using differential property and given initial condition.

Step 3: Take the inverse of the Laplace transform which gives term arising from the known function, the prescribed initial conditions and inverse of the nonlinear part.

Step 4: Apply the Homotopy perturbation method to decompose the nonlinear part and solve it.

Step 5: Set series solution to a given differential equation.

2.2 Description of the Method

To illustrate the method, we considered a general nonlinear nonhomogeneous partial differential equation with initial conditions of the form[12]:

$$Du(x,t)+Ru(x,t)+Nu(x,t)=g(x,t),\ u(x,0)=h(x), u_t(x,0)=f(x) \tag{2.2}$$

$D=\dfrac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}, \alpha>0$, R is the linear differential operator, N is the general nonlinear differential operator and $g(x,t)$ is source term.

Taking Laplace transform on both sides of (2.2) we obtained:

$$\ell\{Du(x,t)\}+\ell\{Ru(x,t)\}+\ell\{Nu(x,t)\}=\ell\{g(x,t)\}.$$

$$\ell u(x,t)=\frac{h(x)}{s}+\frac{f(x)}{s^2}-\frac{\ell}{s^2}\{Ru(x,t)\}+\frac{\ell}{s^2}\{g(x,t)\}-\frac{\ell}{s^2}\{Nu(x,t)\}. \tag{2.3}$$

Operating inverse Laplace transform to (2.3) we have:

$$u(x,t)=A(x,t)-\ell^{-1}\left\{\frac{\ell}{s^2}\left[Ru(x,t)+Nu(x,t)\right]\right\}, \tag{2.4}$$

where, $A(x,t)$ represents the term arising from the source term and prescribed initial conditions. Next by Homotopy perturbation method we have:

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t), \tag{2.5}$$

then the non-linear operator is decomposed as:

$$N[u(x,t)]=\sum_{n=0}^{\infty}p^n H_n(u). \tag{2.6}$$

where $H_n(u)$ are He's polynomial and defined as:

$$H_n(u_0,u_1,u_2,...,u_n)=\left[\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n}p^i u_i(x,t)\right)\right)\right]_{p=0},n=0,1,2,...$$

Substituting (2.6) and (2.5) into (2.4) we obtained:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=A(x,t)-p\left[\ell^{-1}\left\{\frac{\ell}{s^2}\left\{Ru(x,t)+\sum_{n=0}^{\infty}p^n H_n(x,t)\right\}\right\}\right],$$

which is coupling of the Laplace transform and Homotopy perturbation method using He's polynomial. Comparing the coefficients of linear powers of p, the following approximations are obtained.

$$p^0:u_0(x,t)=A(x,t),$$

$$p^1:u_1(x,t)=\ell^{-1}[\ell\{u_0(x,t)\}+H_0(u)],$$

$$p^2:u_2(x,t)=\ell^{-1}[\ell\{u_1(x,t)\}+H_1(u)],$$

$$p^3:u_3(x,t)=\ell^{-1}[\ell\{u_2(x,t)\}+H_2(u)],\ldots$$

$$p^n:u_n(x,t)=\ell^{-1}[\ell\{u_{n-1}(x,t)\}+H_{n-1}(u)].$$

Then the solution of the given DE is given by:

$$u(x,t)=\lim_{p\to1}u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+...$$

$$=\sum_{n=0}^{\infty}u_n(x,t), \tag{2.7}$$

is series solution and converges very rapidly.

2.3 Solving Ordinary Differential Equations Using Laplace Transform

Laplace transform to each term in the given differential equation is taken. If the unknown function is $y(t)$, then we have

$$Y(s) = \ell\{y(t)\}. \tag{2.8}$$

$$\Rightarrow y(t) = \ell^{-1}\{Y(s)\}.$$

Example 2.1: Solve the following ODE.

$$\frac{dy}{dt} + 2y = 12e^{3t}, y(0) = 3 \tag{2.9}$$

Solution: To solve this problem by using Laplace transform, we have the following. Taking the Laplace transform of every term in (2.9),

$$\ell\left\{\frac{dy}{dt}\right\} + \ell\{2y\} = \ell\{12e^{3t}\}$$

$$\Rightarrow \ell\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$

$$\Rightarrow \ell\{2y\} = 2Y(s),$$

$$\Rightarrow \ell\{12e^{3t}\} = \frac{12}{s-3}$$

$$\Rightarrow -3 + sY(s) + 2Y(s) = \frac{12}{s-3}$$

$$\Rightarrow (s+2)Y(s) = \frac{12}{s-3} + 3 = \frac{3+3s}{s-3}.$$

$$\Rightarrow Y(s) = \frac{3(s+1)}{(s+2)(s-3)}$$

$$Y(s) = \frac{3}{5(s+2)} + \frac{12}{5(s-3)}.$$

$$y(t) = \ell^{-1}\{Y(s)\} = \frac{3}{5}\ell^{-1}\left\{\frac{1}{s+2}\right\} + \frac{12}{5}\ell^{-1}\left\{\frac{1}{s-3}\right\} = \frac{3}{5}e^{-2t} + \frac{12}{5}e^{3t},$$

is the solution to the given initial value problem.

2.4 Laplace Transform to Solve Partial Differential Equations

PDEs are used to formulate problems involving functions of several variables. Laplace transform is used to solve PDEs.

Example 2.2: Consider the partial differential equation

$$\frac{\partial u(x,y)}{\partial x \partial y} = e^{-y}\cos x. \tag{2.10}$$

$$u(x,0) = 0, u_y(0,y) = 0$$

Taking Laplace transform on both sides of (2.10) with respect to x, we gate:

$$su(s,y) - u(0,y) = \ell_x\{e^{-y}\cos x\}$$

$$\Rightarrow u(s,y) = e^{-y}\frac{1}{s(1-s^2)}$$

$$\Rightarrow u(x,y) = e^{-y}\sin x$$

$$\Rightarrow \frac{\partial u(x,y)}{\partial y} = -e^{-y}\sin x, \tag{2.11}$$

the equation (2.11) is again the PDE of first order in the variables x and y. Taking Laplae transform of it with `respect to variable y we get

$$su(x,s) - u(x,0) = \sin x\frac{1}{(1+s)}$$

$$\Rightarrow u(x,s) = \sin x\frac{1}{s(1+s)}$$

$\Rightarrow u(x,y) = \sin x(1-e^{-y})$, is the solution to the given DE.

2.5 Illustrative Examples

We have used the following examples to illustrate.

Example 2.3: Consider the following initial-boundary linear nonhomogeneous FPDE,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial u(x,t)}{\partial x}=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin x+t\cos x. \tag{2.12}$$

$t>0, 0<x\le 1, 0<\alpha\le 1,\ u(x,0)=0,\ u(0,t)=0.$

Solution: Taking Laplace transform to both sides to (2.12) we have:

$$\ell\left\{\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial u(x,t)}{\partial x}\right\}=\ell\left\{\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin x+t\cos x\right\}.$$

$$\Rightarrow \ell\left\{\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right\}+\ell\left\{\frac{\partial u(x,t)}{\partial x}\right\}=\ell\left\{\frac{t^{1-\alpha}\sin x}{\Gamma(2-\alpha)}\right\}+\ell\{t\cos x\}$$

$$\Rightarrow s^\alpha u(x,s)-s^{\alpha-1}u(x,0)=\ell\left\{\frac{t^{1-\alpha}\sin x}{\Gamma(2-\alpha)}\right\}+\ell\{t\cos x\}-\ell\left\{\frac{\partial u(x,t)}{\partial x}\right\}$$

$$\Rightarrow u(x,s)=\frac{1}{s^\alpha}\left(\ell\left\{\frac{t^{1-\alpha}\sin x}{\Gamma(2-\alpha)}\right\}\right)+\frac{1}{s^\alpha}\left(\ell\{t\cos x\}\right)-\frac{1}{s^\alpha}\left(\ell\left\{\frac{\partial u(x,t)}{\partial x}\right\}\right)$$

$$\Rightarrow u(x,s)=\frac{1}{s^\alpha}\left(\frac{\sin x}{s^{2-\alpha}}+\frac{\cos x}{s^2}\right)-\frac{1}{s^\alpha}\left(\ell\left\{\frac{\partial u(x,t)}{\partial x}\right\}\right)$$

$$\Rightarrow u(x,s)=\frac{\sin x}{s^2}+\frac{\cos x}{s^{\alpha+2}}-\frac{1}{s^\alpha}\left(\ell\left\{\frac{\partial u(x,t)}{\partial x}\right\}\right) \tag{2.13}$$

Take inverse Laplace transform to both sides to (2.13) we obtained:

$$u(x,t)=t\sin x+\frac{t^{\alpha+1}\cos x}{\Gamma(\alpha+2)}-\ell^{-1}\left\{\frac{1}{s^\alpha}\left\{\ell\left\{\frac{\partial u(x,t)}{\partial x}\right\}\right\}\right\}. \tag{2.14}$$

Apply Homotopy perturbation to (2.14) we have the following.

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=t\sin x+\frac{t^{\alpha+1}\cos x}{\Gamma(\alpha+2)}-p\left[\ell^{-1}\left\{\frac{\ell}{s^\alpha}\left\{\sum_{n=0}^{\infty}\frac{p^n u_n(x,t)}{\partial x}\right\}\right\}\right].$$

Comparing the coefficients of linear powers of p, the following approximations are obtained. $p^0 : u_0(x,t) = t\sin x + \frac{t^{\alpha+1}\cos x}{\Gamma(\alpha+2)}$,

$$p^1 : u_1(x,t) = \frac{t^{2\alpha+1}\sin x}{\Gamma(2\alpha+2)} - \frac{t^{\alpha+1}\cos x}{\Gamma(\alpha+2)},$$

$$p^2 : u_2(x,t) = -\frac{t^{3\alpha+1}\cos x}{\Gamma(3\alpha+2)} - \frac{t^{2\alpha+1}\sin x}{\Gamma(2\alpha+2)},$$

$$p^3 : u_3(x,t) = \frac{t^{3\alpha+1}\cos x}{\Gamma(3\alpha+2)} - \frac{t^{4\alpha+1}\sin x}{\Gamma(4\alpha+2)},...$$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + ...$$

$$\Rightarrow u(x,t) = \left(t\sin x + \frac{t^{\alpha+1}\cos x}{\Gamma(\alpha+2)}\right) + \left(\frac{t^{2\alpha+1}\sin x}{\Gamma(2\alpha+2)} - \frac{t^{\alpha+1}\cos x}{\Gamma(\alpha+2)}\right) + \left(-\frac{t^{3\alpha+1}\cos x}{\Gamma(3\alpha+2)} - \frac{t^{2\alpha+1}\sin x}{\Gamma(2\alpha+2)}\right) + \left(\frac{t^{3\alpha+1}\cos x}{\Gamma(3\alpha+2)} - \frac{t^{4\alpha+1}\sin x}{\Gamma(4\alpha+2)}\right) + ...,$$

is the required series solution to the given FPDE.

Example 2.4: Consider the following initial-boundary value FPDE,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{2.15}$$

$0 < t \le 1, x > 0$, $1 < \alpha \le 2$, $u(0,t) = 0$, $u(x,0) = x$.

$$\frac{\partial u(x,0)}{\partial t} = x^2,\ u(1,t) = 1 + \sum_{k=0}^{\infty} \frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}$$

Solution: Take Laplace transform to both sides of (2.15) we obtained:

$$\ell\left\{\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right\} = \ell\left\{\frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\right\}.$$

$$\Rightarrow s^{\alpha}u(x,s)-\left[\sum_{k=0}^{n-1}s^{\alpha-k-1}\frac{\partial^{k}u(x,0)}{\partial x^{k}}\right]=\ell\left\{\frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\right\}$$

$$\Rightarrow s^{\alpha}u(x,s)-\left[s^{\alpha-1}u(x,0)+s^{\alpha-2}\frac{\partial u(x,0)}{\partial t}\right]=\ell\left\{\frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\right\}$$

$$\Rightarrow s^{\alpha}u(x,s)=\frac{x}{s^{1-\alpha}}+\frac{x^2}{s^{2-\alpha}}+\ell\left\{\frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\right\}$$

$$\Rightarrow u(x,s)=\frac{x}{s}+\frac{x^2}{s^2}+\frac{1}{s^{\alpha}}\left(\ell\left\{\frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right). \tag{2.16}$$

Take inverse Laplace transform to (2.16) we obtained:

$$u(x,t)=x+x^2t+\ell^{-1}\left\{\frac{1}{s^{\alpha}}\left\{\ell\left\{\frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right\}\right\}.$$

Using Homotopy perturbation method we obtained:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=x+x^2t+p\left[\ell^{-1}\left\{\frac{1}{s^{\alpha}}\left\{\ell\left\{\sum_{n=0}^{\infty}p^n\left(\frac{x^2}{2}\frac{\partial^2 u_n(x,t)}{\partial x^2}\right)\right\}\right\}\right\}\right]. \tag{2.17}$$

Comparing coefficients of linear powers of p we obtained,

$$p^0:u_0(x,t)=x+x^2t\,,\ p^1:u_1(x,t)=\frac{x^2t^{\alpha+1}}{\Gamma(\alpha+2)}\,,\ p^2:u_2(x,t)=\frac{x^2t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

$$p^3:u_3(x,t)=\frac{x^2t^{3\alpha+1}}{\Gamma(3\alpha+2)},...$$

Thus, the series solution of the given FPDE is given by;

$$u(x,t)=x+x^2t+\frac{x^2t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{x^2t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{x^2t^{3\alpha+1}}{\Gamma(3\alpha+2)}+...$$

$$=x+x^2\left[t+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+...\right].$$

$$=x+x^2\sum_{k=1}^{\infty}\frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)}$$

In particular, if $\alpha = 2$, $u(x,t) = x + x^2\left[\frac{t}{1!} + \frac{t^3}{3!} + \frac{t^5}{5!} + ... + \frac{t^{(2n+1)}}{(2n+1)} + ...\right]$.

As $n \to \infty, u(x,t) = x$

Example 2.5: Consider the following initial value FPDE problem,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[u(x,t)\frac{\partial u(x,t)}{\partial x}\right]. \tag{2.18}$$

$u(x,0) = x, t > 0, 1 < \alpha \le 2, x \in \mathbb{R}$

Solution: Taking the Laplace transform to both sides to (2.18) we have:

$$\ell\left\{\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right\} = \ell\left\{\frac{\partial}{\partial x}\left[u(x,t)\frac{\partial u(x,t)}{\partial x}\right]\right\}.$$

$$\Rightarrow s^\alpha u(x,s) = s^{\alpha-1}x + \ell\left\{\frac{\partial}{\partial x}\left[u(x,t)\frac{\partial u(x,t)}{\partial x}\right]\right\}$$

$$\Rightarrow u(x,s) = \frac{x}{s} + \frac{1}{s^\alpha}\left[\ell\left\{\frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right\}\right] \tag{2.19}$$

Taking the inverse Laplace transform to (2.19) we obtained,

$$u(x,t) = x + \ell^{-1}\left\{\frac{\ell}{s^\alpha}\left\{\frac{\partial}{\partial x}\left[u(x,t)\frac{\partial u(x,t)}{\partial x}\right]\right\}\right\}.$$

Using Homotopy perturbation method we have the following;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x + p\left[\ell^{-1}\left\{\frac{1}{s^\alpha}\left\{\ell\left\{\sum_{n=0}^{\infty} p^n\left[\frac{\partial}{\partial x}\left[u(x,t)\frac{\partial u_n(x,t)}{\partial x}\right]\right]\right\}\right\}\right\}\right]. \tag{2.20}$$

Comparing the coefficients of linear powers of p of (2.20) the following approximations are obtained.

$$p^0 : u_0(x,t) = x$$

$$p^1 : u_1(x,t) = \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$p^2 : u_2(x,t) = 0$,

$p^3 : u_3(x,t) = 0$, $p^4 : u_4(x,t) = 0,...$

$$\Rightarrow u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$$

$$\Rightarrow u(x,t) = x + \frac{t^\alpha}{\Gamma(\alpha+1)} + 0 + 0 +,...,\qquad (2.21)$$

is the required series solution to the given FOPDEs.

$$\Rightarrow u(x,t) = x + \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Example 2.6: Given that, $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + x\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} = 2(t^\alpha + x^2 + 1).$ (2.22)

$0 < t \le 1, 0 \le x \le 1, 0 < \alpha \le 1, u(x,0) = x^2$

Solution: Taking Laplace transform to both sides to (2.22) we obtained;

$$u(x,s) = \frac{x^2}{s} + \frac{2x^2}{s^{\alpha+1}} + \frac{2}{s^{2\alpha+1}} + \frac{2}{s^{\alpha+1}} - \ell\left\{x\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2}\right\}\qquad (2.23)$$

Using inverse Laplace transform of (2.23) we get:

$$u(x,t) = x^2 + \frac{2x^2t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^\alpha}{\Gamma(\alpha+1)} - \ell^{-1}\left\{\frac{1}{s^\alpha}\left[\ell\left\{x\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right]\right\},$$

using Homotopy perturbation method we obtained:

$$\begin{aligned}\sum_{n=0}^{\infty} p^n u_n(x,t) &= x^2 + \frac{2x^2t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^\alpha}{\Gamma(\alpha+1)} \\ &- p\left[\ell^{-1}\left\{\frac{1}{s^\alpha}\left[\ell\left\{\sum_{n=0}^{\infty} p^n\left(x\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2}\right)\right\}\right]\right\}\right].\end{aligned}\qquad (2.24)$$

Comparing the coefficients of linear powers of p to (2.24) the following approximations are obtained.

$$p^1 : u_1(x,t) = \frac{-2x^2t^\alpha}{\Gamma(\alpha+1)} + \frac{-4x^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{-4t^\alpha}{\Gamma(2\alpha+1)},$$

$$p^2 : u_2(x,t) = \frac{4x^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{8x^2t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{8t^{3\alpha}}{\Gamma(3\alpha+1)},...$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$$

Thus, the series solution to the given FPDE is given by:

$$u(x,t) = x^2 + \frac{2x^2t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{-2x^2t^\alpha}{\Gamma(\alpha+1)} + \frac{-4x^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^\alpha}{\Gamma(\alpha+1)} +$$
$$\frac{-4t^\alpha}{\Gamma(2\alpha+1)} + \frac{4x^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{8x^2t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{-4t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{-8t^{3\alpha}}{\Gamma(3\alpha+1)} + ...$$

$$= x^2 + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Note: $\ell^{-1}\left\{\frac{1}{s^{\alpha+2}}\right\} = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$, $\ell^{-1}\left\{\frac{1}{s^{n\alpha+2}}\right\} = \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}$

$$\ell^{-1}\left\{\frac{1}{s^{1-\alpha}}\right\} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)},...$$

SUMMARY

There are many integral transforms used in solving differential equations and integral equations by converting a problem into a simpler one. Laplace transform is one type of integral transform and it is insufficient to handle the nonlinear equations due to nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities, one of these are combinations of Homotopy perturbation method and Laplace transform. The linear term in the equation can be solved by using Laplace transform method and the nonlinear terms in the equation can be handled by using Homotopy perturbation method (HPM).

Solutions of FPDEs showed to be of exponential order. Based on that, the fractional order and integer order derivatives are all estimated to be of exponential order. Consequently, the Laplace transform is proved to be valid for fractional order partial differential equations under general conditions. Homotopy-Laplace transform is used to solve FPDEs. So the validity of Laplace transform of fractional-order partial differential equations is justified. It is used to find series solutions of FPDEs.

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