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# **BAHIR DAR UNIVERSITY**

# **COLLEGE OF SCIENCE**

# **DEPARTMENT OF MATHEMATICS**

# A PROJECT ON

# **MS-ALMOST DISRIBUTIVE LATTICES (MS-ADL)**

# BY: ARAGAW ALIE DAMTIE

DECEMBER, 2022

BAHIR DAR, ETHIOPIA

**Bahir Dar University** 

# **College of Science**

### **Department of Mathematics**

## A project on

### MS –Almost Distributive Lattice (MS-ADL)

A project submitted to the department of mathematics in the Partial fulfillment of the requirements for the degree of "Master of Science in Mathematics".

By

**Aragaw Alie Damtie** 

Advisor: Berhanu Assaye (Prof., PhD)

December, 2022

**Bahir Dar, Ethiopia** 

### **Bair Dar University**

# **College of Science**

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I here by certify that I have supervised, read and evaluated this project entitled "MS-Almost Distributive Lattice" by Aragaw Alie prepared under my guidance. I recommend that the project is submitted to oral defense.

Advisor's name: Berhanu Assaye (Prof., PhD)

Signature:	
0	

Date:	

### **Bahir Dar University**

# **College of Science**

### **Department of Mathematics**

We here by certify that we have examined this project entitled "MS– Almost Distributive Lattice(MS-ADL)" by Aragaw Alie and We Recommend that Aragaw Alie is Approved for the Degree of "Master of Science in Mathematics".

**Board of Examiners:** 

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External examiner:		

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### Abstract

In this poject, we understand the new equational class of algebra which we call MS-almost distributive lattice (MS-ADL) as a common abstraction of De Morgan ADLs and Stone ADLs. We observed that the class of MS-ADLs properly contain the class of MS-algebras and most of the properties of MS-algebras are extended to the class of MS-ADL. The main objective of this project is to develop a better understanding of the concept of MS-ADLs. Moreover, in this project we observed some basic properties, state and prove basic theorems, lemmas and corollaries related to MS-ADL.

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# CHAPTER-ONE INTRODUCTION AND PRELIMINARIES 1.1 Introduction

The term lattice is one of the fundamental algebraic structures used in an abstract algebra as mathematical disciplines of order theory [4]. It consists of a partially order set P together with a binary relation " $\leq$ " in which every two elements have a unique supremum or a least upper bound called ioin(V) and a unique infimum or a greatest lower bound called meet( $\Lambda$ ). The lattice structure  $(L, \vee, \wedge)$  and the operation called complementation (\*) together with the nullary operations 0 and 1 gives another algebraic structure  $\{B, \lor, \land, \ast, 0, 1\}$  is called Boolean algebra. In  $\{B, \vee, \wedge, *, 0, 1\}$ ,  $\vee$  and  $\wedge$  are binary operations and \* is a unary operation. The general lattice theory was developed into another abstract structure called Almost Distributive Lattice (ADL) [13]. The concept of an almost distributive lattice (abbreviated as ADL) was introduced by U.M Swamy and G.C Rao [13] as a common abstraction of most of the existing ring theoretic and lattice theoretic generalization of a Boolean algebra and Boolean rings. An ADL is an algebra with two binary operations "V" and "A" which satisfies most of the properties of a distributive lattice with smallest element 0, except possibly the commutativity of the binary operations "V" and " $\Lambda$ ", and the right distributivity of "V" over "  $\wedge$  ". The class of ADLs with pseudo-complementation was introduced in[16]. Later on, Swamy et. al.[17] introduced a more general class of ADLs called a Stone ADLs, which properly contains the class of pseudo-complemented ADLs. An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contain the well-known classes of algebras; for example Boolean algebras, De Morgan algebras, Kleene algebras and stone algebras [10]. Blyth and Varlet[11] defined a subclass of Ockham algebras so called MS-algebras which generalizes both De Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by Berman[8]. The classes of MS-algebras form an equational class. Blyth and Varlet characterized the sub-varieties of MS-algebras in [12]. More recently, in the paper [6], the author defines De Morgan ADLs as a generalization of De Morgan algebras. In this project, we define a new equational class of algebras called MS-ADL as a common abstraction of De Morgan ADL and Stone ADL. The class of MS-ADL properly contains the class of MS-algebras and most of the properties of MS-algebras are extended to the class of MS-ADLs.

#### **1.2 preliminaries**

This section contains some necessary definitions and results which will be used in the project.

#### **1.2.1 Partially Ordered Sets and Lattice Theory**

**Definition 1.2.1.1[4, 5]** A partially ordered set (abbreviated as Poset) or simply an ordered set is an algebraic system ( $P, \leq$ ) where P is a non-empty set with a binary relation  $\leq$  on P which satisfies the following set of axioms:

[P1]: Reflexive law:  $a \le a$ , for all  $a \in P$ .

[P2]: Anti-symmetric law:  $a \le b$  and  $b \le a$  implies that a = b for all  $a, b \in P$ .

[P3]: Transitive law:  $a \le b$  and  $b \le c$  implies that  $a \le c$  for all  $a, b, c \in P$ .

If  $(P, \le)$  is a Poset and every two elements of P are comparable (in the sense that either  $x \le y$  or  $y \le x$ , for all  $x, y \in P$ ), then P is called a totally ordered set (also called a chain). The binary relation  $\le$  is called a total order or a linear order.

**Example 1.2.1.2** Let S be a non-empty set. Then,  $(P(S), \subseteq)$  is a Poset. If the quotient relation a|b defines a divides b, for all  $a, b \in P$ , then  $(\mathbb{Z}, |)$  defines a Poset. Also, since every two pairs of integers are comparable with respect to  $\leq$ , the algebraic system  $(\mathbb{Z}, \leq)$  is a totally ordered set or a chain.

**Definition 1.2.1.3[4,7]** A lattice is an algebra  $(L, \vee, \wedge)$  of type (2, 2) where L is a nonempty set with two binary operations join( $\vee$ ) and meet( $\wedge$ ), satisfying the following axioms:

[L1]: Commutative law:  $a \lor b = b \lor a$  and  $a \land b = b \land a$  for all  $a, b \in L$ .

[L2]: Associative law:  $a \lor (b \lor c) = (a \lor b) \lor c$  and  $a \land (b \land c) = (a \land b) \land c$  for all  $a, b, c \in L$ .

[L3]: Idempotent law:  $a \lor a = a$  and  $a \land a = a$  for all  $a \in L$ .

[L4]: Absorption law:  $a \lor (a \land b) = a$  and  $a \land (a \lor b) = a$  for all  $a, b \in L$ .

A lattice L is a special type of Poset ( $P, \leq$ ) in which every pair of elements in L has the least upper bound or a unique suppremum called join (V) and the greatest lower bound or a unique infimum called meet ( $\Lambda$ ). Let L be a lattice under the ordering relation  $\leq$ . Then, we define  $a \leq b$  if and only if  $a \wedge b = a$  (or equivalently;  $a \vee b = b$ ) for all  $a, b \in L$ . **Note that:** The lattice operations meet ( $\Lambda$ ) and join (V) are binary operations on L; which means they can be applied on any two pairs of elements in a lattice L.

**Example 1.2.1.4** The natural example of lattice is the power set P(X) of a non-empty set X with set theoretic operations union and intersection. That is,  $(P(X), \cup, \cap)$  is a lattice.

**Example 1.2.1.5** Let  $L = \{1, 2, 3, 5, 30\}$  be a set. Define the binary operations  $\lor$  and  $\land$  on L by;  $a \lor b = LCM(a, b)$  and  $a \land b = GCD(a, b)$  for all  $a, b, \in L$ . Then,  $(L, \lor, \land)$  is a lattice.

**Definition 1.2.1.6 [7]** A lattice  $(L, \vee, \wedge)$  is said to be a distributive lattice if join  $(\vee)$  and meet  $(\wedge)$  are distributive over each other. That is;

(DL1):  $a \lor (b \land c) = (a \lor b) \land (a \lor c)$  for all  $a, b, c \in L$ .

(DL2):  $a \land (b \lor c) = (a \land b) \lor (a \land c)$  for all  $a, b, c \in L$ .

Lemma 1.2.1.7[7] The axioms (DL1) and (DL2) of a distributive lattice are equivalent.

**Definition 1.2.1.8 [13]** A lattice L is said to be bounded if it has least element 0 and greatest element 1. That is, a lattice  $(L, \vee, \wedge)$  is said to be bounded if  $0 \le x$  and  $x \le 1$ , for all  $x \in L$ .

**Definition 1.2.1.9 [13]** A bounded lattice  $(L, \vee, \land, 0, 1)$  in which  $(L, \vee, \land)$  is a distributive lattice is called a bounded distributive lattice. Let  $(L, \vee, \land, 0, 1)$  be a bounded distributive lattice and  $a \in L$ . Then, the complement of a is defined to be an element  $a' \in L$  such that  $a \land a' = 0$  and  $a \lor a' = 1$ .

**Example 1.2.1.10** Let P(X) be the power set of a non-empty set X. Then,  $(P(X), \cup, \cap, \phi, X)$  is a bounded distributive lattice.

**Definition 1.2.1.11[4, 5]** A Boolean algebra is an algebra (B,  $\vee$ ,  $\wedge$ , ', 0, 1) where B is a nonempty set with two binary operations ( $\vee$  and  $\wedge$ ), one unary operation (complementation) and two nullary operations (0 and 1) which satisfies the following set of axioms:

[B5]:  $(a \lor b)' = a' \land b'$  and  $(a \land b)' = a' \lor b'$  ...... [De Morgan's law]

Note that: By a Boolean algebra we mean complemented distributive lattice.

**Example1.2.1.12** The algebraic system (P(X),  $\cup$ ,  $\cap$ , ',  $\phi$ , X) is a Boolean algebra.

#### **1.2.2 Almost Distributive Lattice (ADL)**

**Definition 1.2.2.1[13]** An almost distributive lattice with zero or simply an ADL is an algebra (L, V,  $\wedge$ , 0) of type (2, 2, 0) which satisfies the following axioms; for all *a*, *b*, *c*  $\in$  L.

**Definition 1.2.2.2[13]** Let  $(L, \vee, \wedge, 0)$  be an ADL. For any  $a, b \in L$ , we say that a is less than or equals to b written as  $a \le b$  and define  $a \le b$  if and only if  $a \wedge b = a$  (or equivalently  $a \vee b = b$ ). Then, the binary relation  $\le$  is called a partial ordering on L. Throughout this paper L denotes an ADL unless otherwise stated.

**Definition 1.2.2.3 [13]** An ADL  $(L, \lor, \land, 0)$  is said to be discrete if every non-zero element is maximal. That is, an ADL =  $(L, \lor, \land, 0)$  is called discrete if and only if  $a \land b = b$  or  $a \lor b = a$  for all  $0 \neq a \in L$ . Moreover, every discrete ADL is an associative.

**Example 1.2.2.4[13]** Let X be a non-empty set with a fixed arbitrarily chosen element  $0 \in X$ . If for all  $a, b \in X$ , we define the binary operations  $\land$  and  $\lor$  on X as follows:

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases} \text{ and } a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0 \end{cases}$$

Then,  $(X, \vee, \wedge, 0)$  is an ADL with 0 as its zero element. This is also called a discrete ADL.

**Example 1.2.2.5** [13] Every distributive lattice with zero (0) is an ADL.

**Lemma 1.2.2.6** [13] Let  $(L, \vee, \wedge, 0)$  be an ADL. Then, the following hold for all  $a, b, c \in L$ :

- (1).  $a \wedge a = a$  and  $a \vee a = a$ (2).  $a \wedge 0 = 0$  and  $0 \vee a = a$ (3).  $a \wedge (a \vee b) = a = (a \wedge b) \vee a$ ,  $(a \wedge b) \vee b = b$  and  $(b \vee a) \wedge b = b$ (4).  $(a \vee b) \wedge a = a$  and  $a \vee (b \wedge a) = a$ (5).  $a \wedge b = a \Leftrightarrow a \vee b = b$  and  $a \wedge b = b \Leftrightarrow a \vee b = a$ (6).  $(a \wedge b) \wedge c = a \wedge (b \wedge c) \dots \dots (\wedge \text{ is associative in L})$ (7).  $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$ (8).  $(a \vee b) \vee a = a \vee b = (b \vee a) \vee a$ (9).  $a \wedge b \leq a$ ,  $a \wedge b \leq b$ ,  $a \leq a \vee b$  and  $b \leq a \vee b$ (10).  $a \vee (a \wedge b) = a$  and  $(a \vee b) \wedge b = b$ (11). If  $a \leq b$ , then  $a \wedge b = a = b \wedge a$  and  $a \vee b = b = b \vee a$
- (12).  $a \land b = \inf\{a, b\} \Leftrightarrow a \land b = b \land a \Leftrightarrow a \lor b = sup\{a, b\}$

(13).  $(a \land b) \land c = (b \land a) \land c$  and  $(a \lor b) \land c = (b \lor a) \land c$ 

(14). If  $a \leq c$  and  $b \leq c$ , then  $a \wedge b = b \wedge a$  and  $a \vee b = b \vee a$ .

**Definition 1.2.2.7 [13]** An ADL, L with 0 is said to be directed above (also called bounded above) if L has an upper bound. More precisely, an ADL  $(L, \vee, \wedge, 0)$  is called directed above if for all  $a \in L$ , there exists an element  $m \in L$  such that  $a \leq m$ .

**Theorem 1.2.2.8** [13] Let  $(L, \vee, \wedge, 0)$  be an ADL. Then, the following are equivalent.

- (1). (L,  $\vee$ ,  $\wedge$ , 0) is a distributive lattice.
- (2). V is commutative.
- (3).  $\Lambda$  is commutative.

(4). V is right distributive over  $\wedge$  in L.

**Definition 1.2.2.9[13]** Let L be an ADL. Then, an element m of L is called a maximal element if it is maximal in the partially ordered set  $(L, \leq)$ . That is, an element  $m \in L$  (of an ADL) is called a maximal element in  $(L, \leq)$  if for any  $x \in L$ ,  $m \leq x \implies m = x$ .

**Notation:** For a maximal element m in L, we write  $L_m$  to denote the closed interval [0, m]. It was observed in [13] that  $L_m$  is a bounded distributive lattice. Moreover, the members of  $L_m$  are characterized as follows:  $L_m = \{a \land m: a \in L\}$ .

**Theorem 1.2.2.10** [13] Let L be an ADL and  $m \in L$ . Then, the following are equivalent:

(1). m is maximal with respect to  $\leq$ . (3).  $m \land x = x$ , for all  $x \in L$ .

(2).  $m \lor x = m$ , for all  $x \in L$ . (4).  $x \lor m$  is maximal for all  $x \in L$ .

**Definition 1.2.2.11[13]** Let L be an ADL. Then, A non-empty subset I of L is said to be an ideal of L if  $a \lor b \in I$  and  $a \land x \in I$ , for all  $a, b \in I$  and for all  $x \in L$ . A non-empty subset F of L is said to be a filter of L if  $a \land b \in F$  and  $x \lor a \in F$  for all  $a, b \in F$  and for all  $x \in L$ .

**Definition 1.2.2.12[13]** Let *L* and *L'* be any two ADLs. Then, a mapping  $f: L \mapsto L'$  is called an ADL homomorphism if  $f(a \lor b) = f(a) \lor f(b)$ ,  $f(a \land b) = f(a) \land f(b)$  and f(0) = 0'for all  $a, b \in L$ . If  $f: L \mapsto L'$  is an isomorphism, then *L* and *L'* are called isomorphic. The notation  $L \cong L'$  can be read as *L* is isomorphic to *L'*.

**Definition 1.2.2.13[13]** An ADL (L,  $\lor$ ,  $\land$ , 0) is called relatively complemented if the interval [0, b] is a Boolean algebra for all  $b \in L$ . A Boolean algebra is the algebra of a relatively complemented ADL with maximal elements.

**Theorem 1.2.2.14[13]** An ADL =  $(L, \lor, \land, 0)$  is said to be a relatively complemented if and only if for all  $a, b \in L$  there exists a unique element in L denoted by  $a^b$  such that  $a \land a^b = 0$  and  $a \lor a^b = a \lor b$ .

**Definition 1.2.2.15** [16] Let  $(L, \vee, \wedge, 0)$  be an ADL. Then, a unary operation  $a \mapsto a^*$  on L is called a pseudo-complementation, if for any  $a, b \in L$ , it satisfies the following conditions:

 $[PC-1]: a \land b = 0 \Longrightarrow a^* \land b = b$ 

[PC-2]:  $a \wedge a^* = 0$ 

[PC-3]:  $(a \lor b)^* = a^* \land b^*$ 

Then, the algebraic system  $(L, \vee, \wedge, *, 0)$  in which every element has a pseudo-complement is called a pseudo-complemented ADL. By this definition, the unary operation \* is called a pseudo-complementation on L and  $a^*$  is called a pseudo-complementation of a in L. An element a of a pseudo-complemented ADL, L is called a dense element of L if  $a^* = 0$ .

**Theorem 1.2.2.16[16]** Let L be an ADL and \* be a pseudo-complementation on L. Then, the following conditions hold for all  $a, b \in L$ .

(1). $0^*$ is maximal.	$(9). a^* \wedge b^* = b^* \wedge a^*$
(2). If <i>a</i> is maximal, then $a^* = 0$ .	(10). $(a \lor b)^* = (b \lor a)^*$
(3). $0^{**} = 0$ (4). $a^* \wedge a = 0$ (5). $a^{**} \wedge a = a$	(11). $a^* \le (a \land b)^*$ and $b^* \le (a \land b)^*$ (12). $(a \lor b)^* \le a^*$ and $(a \lor b)^* \le b^*$ (13). $0^* \land a = a$
(6). $a^{***} = a^*$	(14). $a^* = 0 \Leftrightarrow a^{**}$ is maximal
(7). $a \le b \Longrightarrow b^* \le a^*$	(15). $a^* \leq b^* \Leftrightarrow b^{**} \leq a^{**}$
(8). $a^* \leq 0^*$	(16). $a = 0 \Leftrightarrow a^{**} = 0$

**Definition1.2.2.17[15, 17]** Let L be an ADL and \* be a pseudo-complementation on L. Then, L is said to be a Stone ADL if L is a pseudo-complemented ADL (L,  $\vee$ ,  $\wedge$ , \*, 0) with a maximal element m which satisfies the condition;  $a^* \vee a^{**} = 0^*$  for all  $a \in L$ .

Lemma1.2.2.18 [17] Let L be a Stone ADL. Then, the following conditions hold:

(1).  $0^* \wedge a = a$  (3).  $(a \wedge b)^* = a^* \vee b^*$ 

(2). 
$$0^* \vee a = 0^*$$
 (4).  $(a \wedge b)^{**} = a^{**} \wedge b^{**}$ , for all  $a, b \in L$ .

Proof: (1). Let L be a stone ADL. Then by definition 1.2.2.1 of [ADL3] it follows that

$$0^* \wedge a = (a^* \vee a^{**}) \wedge a = (a^* \wedge a) \vee (a^{**} \wedge a) = 0 \vee a = a$$
 for all  $a \in L$ .

(2). Since 0 is the minimal element of L, it follows that  $0 \le a$  for all  $a \in L$ .

 $\Rightarrow 0 \land a = 0$  ..... by definition 1.2.2.1 of [ADL2]

 $\Rightarrow 0^* \land a = a$  ..... by definition 1.2.2.15 of [PC-1]

 $\Rightarrow 0^* \lor a = 0^*$ ..... by [lemma1.2.2.6 (5)]

- (3). It is known that  $(a \land b) \land (a \land b)^* = 0$  for all  $a, b \in L$ ... by definition 1.2.2.15 [PC-2]
- Let  $(a \land b)^* = x$ . Then,  $a \land b \land x = 0$ ...... [Since  $x^* \land x = 0$ , by theorem 1.2.2.16 (4)]
- $\Rightarrow a^* \land b \land x = b \land x$  ..... by definition 1.2.2.15 of [PC-1]

Also ,  $a^{**} \land b \land x = b \land x$  ...... by [theorem 1.2.2.16 of (5)]

 $\Rightarrow a^* \land b \land x = 0 = b \land x$  ..... by definition 1.2.2.15 of [PC-1]

 $\Rightarrow a^* \land b \land x = a^{**} \land b \land x = 0$  ..... by [step (4) and (5) above]

But,  $(b \land a^{**}) \land x = (a^{**} \land b) \land x = 0$  ...... by [lemma1.2.2.6 (13)]

 $\Rightarrow b^* \land a^{**} \land x = a^{**} \land x$  ..... by definition 1.2.2.15 of [PC-1]

Now,  $\boldsymbol{a}^* \vee \boldsymbol{b}^* = \boldsymbol{a}^* \vee [\boldsymbol{b}^* \vee (\boldsymbol{a}^{**} \wedge \boldsymbol{x})]$ 

$$= a^* \vee [(b^* \vee a^{**}) \wedge (b^* \vee x)]$$

$$= a^* \vee (b^* \vee a^{**}) \wedge a^* \vee (b^* \vee x)$$

- =  $[b^* \lor (a^* \lor a^{**})] \land a^* \lor (b^* \lor x)$  ..... by [lemma1.2.2.6 (8)]
- =  $(b^* \vee 0^*) \wedge a^* \vee (b^* \vee x)$ ..... [Since L is a Stone ADL]
- =  $0^* \wedge [a^* \vee (b^* \vee x)]$ ..... [Since  $0^*$  is maximal]
- $= a^* \vee (b^* \vee x)$  ..... [Since  $0^*$  is maximal]

Thus,  $(\boldsymbol{a}^* \lor \boldsymbol{b}^*) \land \boldsymbol{x} = [\boldsymbol{a}^* \lor (\boldsymbol{b}^* \lor \boldsymbol{x})] \land \boldsymbol{x}$ 

$$= (a^* \land x) \lor [(b^* \lor x) \land x]$$
..... by definition 1.2.2.1 of [ADL3]

- =  $[a^* \land x] \lor x$ ..... by definition 1.2.2.1 of [ADL6]
- = *x*...... by [lemma1.2.2.6 (3)]

So that  $(\boldsymbol{a} \wedge \boldsymbol{b})^* = \boldsymbol{x} = (a^* \vee b^*) \wedge \boldsymbol{x}$ 

$$= (a^* \land x) \lor (b^* \land x)..... \text{ by definition 1.2.2.1 of [ADL3]}$$
  

$$= [a^* \lor (a \land b)^*] \lor [b^* \lor (a \land b)^*]..... \text{ [Since } x = (a \land b)^*]$$
  

$$= [a \lor (a \land b)]^* \lor [b \lor (a \land b)]^*.... \text{ by definition 1.2.2.15 of [PC-3]}$$
  

$$= [(a \lor a) \land (a \lor b)]^* \lor [(b \lor a) \land (b \lor b)]^*.... \text{ by axiom [ADL5]}$$
  

$$= [a \land (a \lor b)]^* \lor [(b \lor a) \land b]^*.... \text{ by [lemma 1.2.2.6 (1)]}$$
  

$$= a^* \lor b^*..... \text{ by [lemma 1.2.2.6 (3)]}$$

Hence,  $(a \land b)^* = a^* \lor b^*$ , for all  $a, b \in L$ .

(4). 
$$(a \land b)^{**} = (a^* \lor b^*)^* = a^{**} \land b^{**}$$
..... by (3) above and definition 1.2.2.15 of [PC-3]

**Definition 1.2.2.19 [10-11]** An MS-algebra is an algebra  $(L, \vee, \wedge, \circ, 0, 1)$  of type (2, 2, 1, 0, 0) such that  $(L, \vee, \wedge, 0, 1)$  is a bounded distributive lattice and the unary operation  $x \mapsto x^\circ$  on L satisfying the following set of axioms:

[M1]:  $x \le x^{\circ\circ}$ [M3]:  $(x \lor y)^{\circ} = x^{\circ} \land y^{\circ}$ [M2]:  $(x \land y)^{\circ} = x^{\circ} \lor y^{\circ}$ [M4]:  $1^{0} = 0$ , for all  $x, y \in L$ .

**Definition 1.2.2.20** [4, 5, 7, 9] A De Morgan algebra is an MS-algebra ( $L, \lor, \land, \circ, 0, 1$ ) of type (2, 2, 1, 0, 0) such that the unary operation  $x \mapsto x^\circ$  satisfying the following condition;

[M5]:  $x^{\circ\circ} = x$ , for all  $x \in L$ , which is called the involution law.

**Lemma 1.2.2.21** [10,11] Let L be an MS-algebra. Then, the following hold for all  $a, b \in L$ :

(1).  $0^{\circ} = 1$ (2).  $a \le b \Longrightarrow b^{\circ} \le a^{\circ}$ (3).  $a^{\circ \circ \circ} = a^{\circ}$ (4).  $(a \lor b)^{\circ \circ} = a^{\circ \circ} \lor b^{\circ \circ}$ (5).  $(a \land b)^{\circ \circ} = a^{\circ \circ} \land b^{\circ \circ}$ 

**Proof:** (1). Let L be an MS-algebra. This implies that  $(L, \vee, \Lambda, 0, 1)$  is a bounded distributive lattice with least element 0 and greatest element 1..... by definition 1.2.2.19.

This implies that  $1^{\circ} = 0$  and  $0^{\circ} = 1$ ..... by definition 1.2.2.19 of [M4]

Hence,  $0^{\circ} = 1$ .

(2). Suppose that  $a \le b$ . This implies that  $a \land b = a$  (or equivalently;  $a \lor b = b$ ).

Then, by definition 1.2.2.19 of [M2] it follows that  $a^\circ = (a \land b)^\circ = a^\circ \lor b^\circ$ .

Hence,  $b^{\circ} \leq a^{\circ}$  for all  $a, b \in L$ ..... by definition 1.2.2.2

(3). Let L be an MS-algebra. Then,  $a \le a^{\circ\circ}$  for all  $a \in L$ ..... by definition 1.2.2.19 of [M1].

Hence, (\*) and (\*\*) imply that  $a^{\circ\circ\circ} = a^{\circ}$ , for all  $a \in L$ .

(4). 
$$(a \lor b)^{\circ\circ} = (a^{\circ} \land b^{\circ})^{\circ} = a^{\circ\circ} \lor b^{\circ\circ}$$
..... by definition 1.2.2.19 of [M3] and [M2]

(5).  $(a \land b)^{\circ\circ} = (a^{\circ} \lor b^{\circ})^{\circ} = a^{\circ\circ} \land b^{\circ\circ}$ ..... by definition 1.2.2.19 of [M2] and [M3]

**Definition 1.2.2.22[13].** Let A be a non-empty set and  $\theta$  be a binary relation on A ( $\theta \subseteq A \times A$ ). Then,  $\theta$  is said to be an equivalence relation on A if  $\theta$  satisfies the following axioms:

[1]. Reflexive law:  $(a, a) \in \theta$  for all  $a \in A$ .

[2]. Symmetric law:  $(a, b) \in \theta$  implies that  $(b, a) \in \theta$  for all  $a \in A$ .

[3]. Transitive law:  $(a, b) \in \theta$  and  $(b, c) \in \theta$  implies that  $(a, c) \in \theta$  for all  $a, b, c \in A$ .

**Definition 1.2.2.23[13]** Let L be an ADL. Then, an equivalence relation  $\theta$  on L is said to be a congruence relation on L if  $(a, b), (x, y) \in \theta \implies (a \land x, b \land y), (a \lor x, b \lor y) \in \theta$ , for all  $a, b, x, y \in L$ .

**Remark:** For any congruence relation  $\theta$  on L and  $a \in L$ , we define the congruence class  $[a]_{\theta} = \{b \in L: (a, b) \in \theta\}$  and it is called the congruence class containing *a*.

### **CHAPTER-TWO**

### **MS-ALMOST DISTRIBUTIVE LATTICES (MS-ADL)**

#### 2.1 Definitions, Examples and Theorems on (MS-ADL)

In this section, we define MS-ADLs and investigate some of their properties with examples.

**Definition 2.1.1[6]** An MS-almost distributive lattice (abbreviated as MS-ADL) is an algebra  $(L, \lor, \land, \circ, 0)$  of type (2, 2, 1, 0) such that  $(L, \lor, \land, 0)$  is an ADL with a maximal element m and a unary operation  $x \mapsto x^\circ$  on L which satisfies the following axioms; for all  $x, y \in L$ .

[MS-A1]:  $x^{\circ\circ} \wedge x = x$ 

 $[MS-A2]: (x \lor y)^{\circ} = x^{\circ} \land y^{\circ}$ 

 $[MS-A3]: (x \land y)^{\circ} = x^{\circ} \lor y^{\circ}$ 

[MS-A4]:  $m^{\circ} = 0$ , for all maximal elements m of L.

**Definition 2.1.2[6]** An MS-ADL (L,V,  $\land$ ,  $\circ$ , 0) of type (2, 2, 1, 0) satisfying the condition [MS-A5]:  $x^{\circ\circ} = x \land m$ , is called a De Morgan ADL, for all maximal element m of MS-ADL.

**Example 2.1.3** Let  $(L, \vee, \wedge, 0)$  be a discrete ADL with at least two elements. Choose a nonzero element  $m \in L$  and define a unary operation  $x \mapsto x^\circ$  on L as follows:

$$a^{\circ} = \begin{cases} m, if a = 0\\ 0, otherwise \end{cases}, \text{ for all } a \in L.$$

Then, (L,V,  $\land$ ,  $\circ$ ,0) is an MS-ADL and it is called the discrete MS-ADL.

**Example 2.1.4** Let  $L = \{0, a, b, c\}$ . Define the two binary operations  $\lor$  and  $\land$  on L by the following tables:

V	0	а	b	С
0	0	а	b	С
а	а	а	b	С
b	b	b	b	b
С	С	С	С	С

Λ	0	а	b	С
0	0	0	0	0
а	0	а	а	а
b	0	а	b	С
С	0	а	b	С

Then,  $(L, \lor, \land, 0)$  is an ADL which is neither a distributive lattice nor a discrete ADL. But if we define a unary operation  $x \mapsto x^\circ$  on L as follows:

x	x°
0	b
а	0
b	0
С	0

Then,  $(L, \vee, \wedge, \circ, 0)$  is an MS-ADL, which is also a De Morgan ADL. Moreover, if we define another unary operation  $x \mapsto x^*$  on L as follows:

x	$x^*$
0	b
а	а
b	0
С	0

Then, (L,V,  $\land$ , \*,0) is an MS-ADL but not a De Morgan ADL. Since *b* and *c* are maximal elements of L as given from the table above, by definition2.1.2 of [MS-A5], it follows that  $b^{**} = b \land m = b \land c = c \neq b$ . Hence, L is not a De Morgan ADL.

**Example 2.1.5** Let  $L = \{0, a, b, c, d\}$  and define binary operations  $\lor$  and  $\land$  on L as follows:

V	0	а	b	С	d
0	0	а	b	С	d
а	а	а	а	С	С
b	b	b	b	d	d
С	С	С	С	С	С
d	d	d	d	d	d

Λ	0	а	b	С	d
0	0	0	0	0	0
а	0	а	b	а	b
b	0	а	b	а	b
С	0	а	b	С	d
d	0	а	b	С	d

Then, (*L*,  $\vee$ ,  $\wedge$ , 0) is an ADL with maximal elements c and d, which is neither a distributive lattice nor a discrete ADL. But if we define a unary operation  $x \mapsto x^\circ$  on L as follows:

x	x°
0	С
а	а
b	а
С	0
d	0

Then,  $(L, \vee, \wedge, \circ, 0)$  becomes an MS-ADL.

**Example 2.1.6** Let  $(L, \vee, \wedge, \circ, 0, m)$  be an MS-ADL and X be a non empty set. If  $L^X$  denotes the class of all functions from X to L, then  $(L^X, \vee, \wedge, \circ, 0_x, m_x)$  is an MS-ADL where  $\vee, \wedge, \circ, 0_x$  and  $m_x$  are defined on  $L^X$  as follows:

(1). 
$$(f \lor g)(x) = f(x) \lor g(x)$$

(2). 
$$(f \wedge g)(x) = f(x) \wedge g(x)$$

(3). 
$$f^{\circ}(x) = (f(x))^{\circ}$$
 and  $0_{x}(x) = 0$ ,  $m_{x}(x) = m$  for all  $x \in X$ .

**Proof:** For each functions  $f, g \in L^X$  and each  $x \in X$ , we have the following results:

(i). 
$$f(x) = f(x^{\circ\circ} \wedge x) = f^{\circ\circ}(x) \wedge f(x) = (f(x))^{\circ\circ} \wedge f(x)$$
...by definition2.1.1 of [MS-A1]  
(ii).  $(f \vee g)^{\circ}(x) = [f(x) \vee g(x)]^{\circ} = (f(x))^{\circ} \wedge (g(x))^{\circ} = f^{\circ}(x) \wedge g^{\circ}(x)$   
(iii).  $(f \wedge g)^{\circ}(x) = [f(x) \wedge g(x)]^{\circ} = (f(x))^{\circ} \vee (g(x))^{\circ} = f^{\circ}(x) \vee g^{\circ}(x)$   
(iv).  $m_x^{\circ}(x) = m^{\circ} = 0 = 0_x(x)$ ...... by definition2.1.1 of [MS-A4]  
Hence,  $(L^X, \vee, \wedge, \circ, 0_x, m_x)$  is an MS-ADL whenever  $(L, \vee, \wedge, \circ, 0, m)$  is an MS-ADL  
Lemma 2.1.7 [6] The following conditions hold in an MS-ADL with maximal element m.

- (1). 0° is maximal. (5).  $(a \lor b)^{\circ\circ} = a^{\circ\circ} \lor b^{\circ\circ}$
- (2).  $a \le b \Longrightarrow b^{\circ} \le a^{\circ}$  (6).  $(a \land m)^{\circ} = a^{\circ}$

(3).  $a^{\circ\circ\circ} = a^{\circ}$  (7).  $(a \wedge b)^{\circ} = (b \wedge a)^{\circ}$ , for all  $a, b \in L$ .

(4).  $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$  (8).  $n^{\circ} = 0$ , for all maximal elements n of L.

**Proof:** (1). Let L be an MS-ADL with a maximal element m. Since 0 is the minimal element in L, this gives  $0 \le m$ . This implies  $0 = 0 \land m$ . Put  $m^\circ = 0$ ... by definition2.1.1 of [MS-A4] So that  $0^\circ = (0 \land m)^\circ = 0^\circ \lor m^\circ$  ..... by definition2.1.1 of [MS-A3] This implies that  $m^\circ \le 0^\circ$ ..... by definition1.2.2.2

Hence, 0° is maximal.

$= (m \land m) \lor 0^{\circ} \dots$	by definition2.1.2 of [MS-A5]
$= m \vee 0^{\circ}$	[Since $m \land m = m$ ]
= <i>m</i>	[Since m is maximal].

Hence, 0° is also maximal whenever L is a De Morgan ADL.

(2). Suppose that  $a \le b$ . This implies that  $a \land b = a$  (or equivalently;  $a \lor b = b$ ). Then, by definition 2.1.1 of [MS-A3] it follows that  $a^\circ = (a \land b)^\circ = a^\circ \lor b^\circ$ . So that  $b^\circ \le a^\circ$ .

Hence,  $a \le b$  implies  $b^{\circ} \le a^{\circ}$  for all  $a, b \in L$ .

(3). Let L be an MS-ADL. Then,  $a^{\circ\circ} \wedge a = a$  for all  $a \in L$ .... by definition2.1.1 of [MS-A1] This implies that  $(a^{\circ\circ} \wedge a)^{\circ} = a^{\circ}$ . This gives  $a^{\circ\circ\circ} \vee a^{\circ} = a^{\circ}$ .... by definition2.1.1 of [MS-A3]

So that by definition 1.2.2.2, This shows that  $a^{\circ\circ\circ} \leq a^{\circ}$  for all  $a \in L$ .....(1)

Also, since L is an MS-ADL with maximal element m, by a similar argument of definition 2.1.1 [MS-A1] this gives that  $a^{\circ\circ\circ} \wedge a^{\circ} = a^{\circ}$  for all  $a \in L$ . This implies that  $a^{\circ} = a^{\circ\circ\circ} \wedge a^{\circ} = a^{\circ} \wedge a^{\circ\circ\circ} \wedge m = a^{\circ} \wedge a^{\circ\circ\circ} \wedge m = a^{\circ} \wedge a^{\circ\circ\circ}$ ..... by [lemma1.2.2.6 (13)]

So that by definition 1.2.2.2, this gives that  $a^{\circ} \leq a^{\circ \circ \circ}$  for all  $a \in L$ .....(2)

Hence, (1) and (2) imply that  $a^{\circ \circ \circ} = a^{\circ}$ , for all  $a \in L$ .

Moreover, if L is a De Morgan ADL, then  $a^{\circ\circ} = a \wedge m$ .... by definition2.1.2 of [MS-A5]. So that  $a^{\circ\circ\circ} = (a \wedge m)^{\circ} = a^{\circ} \vee m^{\circ} = a^{\circ} \vee 0$ ..... by definition2.1.1 of [MS-3] and [MS-A4]  $= a^{\circ}$  ...... by definition1.2.2.1 of [ADL1] (4).  $(a \wedge b)^{\circ\circ} = (a^{\circ} \vee b^{\circ})^{\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$  ..... by definition2.1.1 of [MS-A3] and [MS-A2]. (5).  $(a \vee b)^{\circ\circ} = (a^{\circ} \wedge b^{\circ})^{\circ} = a^{\circ\circ} \vee b^{\circ\circ}$  ..... by definition2.1.1 of [MS-A3] and [MS-A3]. (6).  $(a \wedge m)^{\circ} = a^{\circ} \vee m^{\circ} = a^{\circ} \vee 0$ ..... by definition2.1.1 of [MS-A3] and [MS-A4]  $= a^{\circ}$  ...... by definition1.2.2.1 of [MS-A3] and [MS-A4] (7).  $(a \wedge b)^{\circ} = (a \wedge b \wedge m)^{\circ}$  ..... by [lemma2.1.7 (6)]

 $= (b \land a \land m)^{\circ}$ ..... by [lemma1.2.2.6 (13)]

 $= (b \wedge a)^{\circ}$ ..... by [lemma2.1.7 (6)]

(8). Let n be the maximal element of L. Clearly, we have  $n = n \lor a$  for all  $a \in L$ . So that  $n^{\circ} = (n \lor 0^{\circ})^{\circ} = n^{\circ} \land 0^{\circ \circ} = n^{\circ} \land 0 = 0$ ..... by definition2.1.1 of [MS-A2] and [MS-A4] Hence,  $n^{\circ} = 0$  for all maximal elements n of L.

**Corollary 2.1.8** Let L be an MS-ADL. Then,  $a^{\circ} = b^{\circ}$  if and only if  $(a \wedge m)^{\circ} = (b \wedge m)^{\circ}$ . But if L is a De Morgan ADL, then  $a^{\circ} = b^{\circ}$  if and only if  $a \wedge m = b \wedge m$  for all  $a, b \in L$ .

**Proof:** let L be an MS-ADL with maximal element m such that  $a^{\circ} = b^{\circ}$ . Then, for any  $a, b \in L$  we have  $(a \land m)^{\circ} = a^{\circ} = b^{\circ} = (b \land m)^{\circ}$  ..... by [lemma2.1.7 (6)]

**Conversely,** suppose that  $(a \land m)^\circ = (b \land m)^\circ$ . This is equivalent to  $a^\circ \lor m^\circ = b^\circ \lor m^\circ$ . Since  $m^\circ = 0$ , this implies that  $a^\circ \lor 0 = b^\circ \lor 0$ . This gives that  $a^\circ = b^\circ$  for all  $a, b \in L$ .

To prove the second part, let L be a De Morgan ADL such that  $a^{\circ} = b^{\circ}$ . This implies that  $a^{\circ\circ} = b^{\circ\circ}$ . So that  $a \wedge m = a^{\circ\circ} = b^{\circ\circ} = b \wedge m$ ..... by definition2.1.2 of [MS-A5]

**Conversely,** suppose  $a \wedge m = b \wedge m$ . Then,  $a^{\circ} = a^{\circ \circ \circ} = (a \wedge m)^{\circ} = (b \wedge m)^{\circ} = b^{\circ \circ \circ} = b^{\circ}$ . Hence, the equivalences hold for all  $a, b \in L$ . **Theorem 2.1.9** Any relatively complemented ADL with a fixed maximal element m of L can be made into an MS-ADL by defining the unary operation  $x \mapsto x^\circ$  on L as follows:

$$a^{\circ} = a^{m}$$
, for all  $a \in L$ .

**Proof:** Suppose that L is a relatively complemented ADL with maximal elements m. Then, for each  $a, b \in L$  there exists a unique element in L denoted by  $a^b$  such that  $a \wedge a^b = 0$  and  $a \vee a^b = a \vee b$ . Choose a maximal element  $m \in L$  and define a unary operation  $a \mapsto a^\circ$  on L by;  $a^\circ = a^m$  for all  $a \in L$ . Then, by definition 1.2.2.8 and lemma 1.2.2.6 (13) it follows that  $a^m \wedge (a \wedge m) = a^\circ \wedge (a \wedge m) = (a^\circ \wedge a) \wedge m = (a \wedge a^\circ) \wedge m = 0 \wedge m = 0$ .

Also,  $a^m \lor (a \land m) = (a^m \lor a) \land (a^m \lor m) = (a^m \lor a) \land m = m = a^m \lor m$ .

Moreover, by definition 2.1.1 of [MS-A2] and [MS-A3] we have  $(a \lor b)^m = (a \lor b)^\circ = a^\circ \land b^\circ = a^m \land b^m$  and  $(a \land b)^m = (a \land b)^\circ = a^\circ \lor b^\circ = a^m \lor b^m$  for all  $a, b \in L$ .

Hence, any relatively complemented ADL together with maximal element m and the unary operation  $x \mapsto x^\circ$  is an MS-ADL.

**Similarly,** one can easily verify that the unary operation  $x \mapsto x^*$  which makes an ADL, L a Stone ADL that respects all the axioms of MS-ADL. So that every Stone ADL is an MS-ADL. The following example presents a natural way to obtain an MS-ADL from De Morgan ADL and Stone ADL.

**Example 2.1.10** If D is a De Morgan ADL and S is a Stone ADL, then D×S is an MS-ADL such that the unary operation  $\circ$  on D×S is defined by;  $(x, y)^{\circ} = (\bar{x}, y^{*})$  for all  $x \in D$  and for all  $y \in S$ , where  $x \mapsto \bar{x}$  is the unary operation on D and  $y \mapsto y^{*}$  is the unary operation on S.

Theorem 2.1.11 Let D be a De Morgan ADL and S be a Stone ADL. Then,

(1).  $D \times S$  is a De Morgan ADL if and only if S is relatively complemented.

(2).  $D \times S$  is a Stone ADL if and only if D is a relatively complemented.

**Proof:** (1). ( $\Rightarrow$ ). Suppose D×S is a De Morgan ADL. Then, by definition2.1.2 of [MS-A5] for every  $x \in D$  and  $y \in S$  we have  $(x, y)^{\circ\circ} = (x, y) \land (m, n)$ , where m and n are maximal elements in D and S respectively. So that by definition2.1.2 of axiom [MS-A5] this gives that  $y^{**} = y \land n$ , for all  $y \in S$ . In this case, the maximal element n of S is precisely 0<sup>\*</sup>.

**Claim:** S is relatively complemented.

Let *a*, *b*  $\in$  S. *Put*  $x = a^* \wedge b$ . Then, we have the following results:

(i). 
$$a \wedge x = a \wedge (a^* \wedge b) = (a \wedge a^*) \wedge b = 0 \wedge b = 0.$$
  
(ii).  $a \vee x = a \vee (a^* \wedge b) = (a \vee a^*) \wedge (a \vee b)....$  by definition 1.2.2.1 [ADL5]  
 $= 0^* \wedge (a \vee a^*) \wedge (a \vee b)...$  [Since 0\* is maximal in S]  
 $= (a \vee a^*) \wedge 0^* \wedge (a \vee b)...$  by [lemma 1.2.2.6 (13)]  
 $= [(a \wedge 0^*) \vee (a^* \wedge 0^*)] \wedge (a \vee b)...$  by definition 1.2.2.1 [ADL3]  
 $= (a^{**} \vee a^*) \wedge (a \vee b)$   
 $= 0^* \wedge (a \vee b)...$  [Since S is Stone ADL]  
 $= a \vee b...$  [Since 0\* is maximal]

Hence, S is relatively complemented.

**Conversely,** suppose that S is relatively complemented. Then, for any  $a, b \in S$ , there exists a unique element in S denoted by  $a^b$  such that  $a \wedge a^b = 0$  and  $a \vee a^b = a \vee b$ . Hence, the maximal element  $n = 0^*$  exists in S. Also, since S is a Stone ADL, there is a unary operation  $y \mapsto y^*$  on S defined by;  $y^* \vee y^{**} = 0^*$  for all  $y \in S$ . This is equivalent to that  $y^* \wedge y = 0$ .

Again, since D is a De Morgan ADL, there is a unary operation  $x \mapsto \bar{x}$  on D such that  $\overline{x} = x \land m$ , for all  $x \in D$  and for all maximal element m of D. So that  $D \times S = (x, y)^{\circ\circ} = (x, y) \land (m, n)$ , which defines a De Morgan ADL..... by definition2.1.2 of [MS-A5].

Therefore,  $D \times S$  is a De Morgan ADL, for all  $x \in D$  and for all  $y \in S$ .

Claim: D is relatively complemented

Let *a*, *b*  $\in$  D. *Put*  $x = \overline{a} \land b$ . Then, we have the following results:

(i).  $a \wedge x = a \wedge (\overline{a} \wedge b) = (a \wedge \overline{a}) \wedge b = 0 \wedge b = 0$ .

(ii). 
$$a \lor x = a \lor (\overline{a} \land b) = (a \lor \overline{a}) \land (a \lor b) = \overline{0} \land (a \lor \overline{a}) \land (a \lor b)$$
  
 $= (a \lor \overline{a}) \land \overline{0} \land (a \lor b)$   
 $= (a \land \overline{0}) \lor (\overline{a} \land \overline{0}) \land (a \lor b)$   
 $= (\overline{a} \lor \overline{a}) \land (a \lor b)$   
 $= \overline{0} \land (a \lor b)$   
 $= (a \lor b)$ 

Therefore, D is relatively complemented.

**Conversely;** Suppose D is relatively complemented. Then, for any  $a, b \in D$  there exists a unique element in D denoted by  $a^b$  such that  $a \wedge a^b = 0$  and  $a \vee a^b = a \vee b$ . Also, since S is a Stone ADL, there is a unary operation  $y \mapsto y^*$  on S defined by;  $y^* \vee y^{**} = 0^*$ , for all  $y \in S$ . This is equivalent to  $y^* \wedge y = 0$  ...... by definition2.1.1 of [MS-A3] So that  $D \times S = (x, y)^\circ \wedge (x, y) = (0, 0)$ ...... by definition1.2.2.15 of [PC-2].

Therefore,  $D \times S$  is a Stone ADL, for all  $x \in D$  and for all  $y \in S$ .

**Note that:** This theorem confirms that the class of De Morgan ADLs and the class of Stone ADLs are proper subclasses of the class of MS-ADLs.

The following theorem shows that a set of necessary and sufficient conditions for which an MS-ADL to be an MS-algebra.

**Theorem 2.1.12** Let  $(L, \vee, \wedge, \circ, 0)$  be an MS-ADL. Then the following are equivalent:

- (1). L is an MS-algebra.
- (2). The Poset (L,  $\leq$ ) is directed above (bounded above).
- (3). (L, V,  $\land$ , 0) is a distributive lattice.
- (4). V is commutative.
- (5).  $\Lambda$  is commutative.
- (6). V is right distributive over meet  $\wedge$  in L.

(7). The relation  $\theta = \{(a, b) \in L \times L: b \land a = a\}$  is anti-symmetric.

(8). For each  $a \in L$ , the relation  $\phi_a$  given by;

 $(x, y) \in \phi_a$  if and only if  $x \lor a = y \lor a$  and  $x^\circ \lor a = y^\circ \lor a$  is a congruence relation on L.

**Proof:** (1)  $\Rightarrow$  (2). Suppose L is an MS-algebra. Since L is an MS-ADL with maximal element m, for any  $a, b \in L$  we have  $a \leq m$  and  $b \leq m$ . This implies L has an upper bound m with respect to the partial ordering  $\leq$ . Hence, the Poset (L,  $\leq$ ) is directed above.

(2)  $\Rightarrow$  (3). Suppose the Poset (L,  $\leq$ ) is directed above (bounded above by m). Since L is an MS-ADL, we have a maximal element  $m \in L$  such that  $a \leq m$  and  $b \leq m$  for all  $a, b \in L$ .

**Claim:** (*L*,  $\lor$ ,  $\land$ , 0) is a distributive lattice.

(i). 
$$a \land (b \lor c) = [a \land (b \lor c)] \land m = [(a \land b) \lor (a \land c)] \land m = (a \land b) \lor (a \land c)$$

(ii). 
$$a \lor (b \land c) = [a \lor (b \land c)] \land m = [(a \lor b) \land (a \lor c)] \land m = (a \lor b) \land (a \lor c).$$

Therefore,  $(L, \vee, \wedge, 0)$  is a distributive lattice.

(3)  $\Rightarrow$  (4). Suppose (L, V,  $\land$  ,0) is a distributive lattice.

**Claim:** V is commutative.

$$a \lor b = [a \land (b \lor a)] \lor [(b \land (b \lor a)]...$$
 by [lemma1.2.2.6 (3)]  
$$= (a \lor b) \land (b \lor a)...$$
 by definition1.2.2.1 of [ADL3]  
$$= [(a \lor b) \land b] \lor [(a \lor b) \land a]...$$
 by definition1.2.2.1 of [ADL4]  
$$= b \lor a...$$
 by definition1.2.2.1 of [ADL6]

Hence,  $\lor$  is commutative for all a,  $b \in L$ .

(4)  $\Rightarrow$  (5). Suppose V is commutative.

**Claim:**  $\land$  is commutative.

$$a \wedge b = (a \wedge b) \vee (a \wedge b) = [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee b].... \text{ by definition 1.2.2.1 [ADL5]}$$
$$= [a \vee (a \wedge b)] \wedge [b \vee (a \wedge b)]..... \text{ [Since \lor is commutative]}$$

 $= (a \lor b) \land (a \land b)...$  by definition 1.2.2.1 of [ADL5]  $= [(b \lor a) \land b] \land (a \land b) \land m...$  [Since m is maximal]  $= [(a \lor b) \land b] \land (a \land b) \land m...$  [Since  $\lor$  is commutative]  $= b \land a \land b \land m...$  by definition 1.2.2.1 of [ADL6]  $= b \land a \land m$  $= b \land a$ 

Hence,  $\Lambda$  is commutative.

(5)  $\Rightarrow$  (6). Suppose  $\land$  is commutative.

**Claim:**  $\lor$  is right distributive over  $\land$ . That is,  $(a \land b) \lor c = (a \lor c) \land (b \lor c)$ .

$$(a \land b) \lor c = [(b \land a) \lor c] = [c \lor (b \land a)] \land m.....$$
 [Since m is maximal]  
$$= [(c \lor b) \land (c \lor a)] \land m.....$$
 by definition 1.2.2.1 [ADL5]  
$$= (c \lor b) \land m \land (c \lor a) \land m$$
  
$$= (b \lor c) \land (a \lor c).....$$
 by [lemma 1.2.2.6 (13)]  
$$= (a \lor c) \land (b \lor c).....$$
 [Since  $\land$  is commutative]

Therefore,  $\lor$  is right distributive over  $\land$  in L.

(6)  $\Rightarrow$  (7). Suppose  $\lor$  is right distributive over  $\land$ . That is,  $(a \land b) \lor c = (a \lor c) \land (b \lor c)$ .

**Claim:** The relation defined by  $\theta = \{(a, b) \in L \times L : b \land a = a\}$  is anti-symmetric.

By a relation  $\theta$  on L is anti-symmetric, we mean  $(a, b) \in \theta$  and  $(b, a) \in \theta \implies a = b$ .

Assume that  $(a, b) \in \theta \implies b \land a = a$  and  $(b, a) \in \theta \implies a \land b = b$ . This is equivalent to  $a \lor b = a$  and  $b \lor a = b$ . Then, since the relation  $\theta$  on L is defined by;  $b \land a = a$ , we have  $a = b \land a = b \land a \land m = a \land b \land m = b \land m = b$ ..... [Since m is maximal in L].

Therefore, the relation  $\theta = \{(a, b) \in L \times L : b \land a = a\}$  is anti-symmetric.

(7)  $\Rightarrow$  (8). Suppose  $\theta = \{(a, b) \in L \times L : b \land a = a\}$  is anti-symmetric.

**Claim:** For each  $a \in L$ , the relation  $\phi_a$  given by;

 $(x, y) \in \phi_a$  if and only if  $x \lor a = y \lor a$  and  $x^\circ \lor a = y^\circ \lor a$  is a congruence relation on L.

Let L be an MS-ADL with maximal element m. A relation  $\phi_a$  is called a congruence relation on L if  $\phi_a$  is a congruence relation on the ADL, (L,  $\lor$ ,  $\land$ , 0), that is  $(a, b), (x, y) \in \phi_a \Rightarrow$  $(a \land x, b \land y), (a \lor x, b \lor y) \in \phi_a$  and  $(x, y) \in \phi_a \Rightarrow (x^\circ, y^\circ) \in \phi_a$ , for all  $a, b, x, y \in L$ .

Given that the relation  $\theta = \{(a, b) \in L \times L : b \land a = a\}$  is anti-symmetric.

Equivalently; this gives that  $b \lor a = b$  for all  $a, b \in L$ .

Assume that  $(x, y) \in \phi_a \Rightarrow x \lor a = y \lor a$  and  $x^\circ \lor a = y^\circ \lor a$ . By the same argument assume that  $(c, d) \in \phi_a \Rightarrow c \lor a = d \lor a$  and  $c^\circ \lor a = d^\circ \lor a$  for all  $x, y, c, d \in L$ .

Then, this gives that  $(x \land c, y \land d) = (c, d) \in \phi_a$  and  $(x \lor c, y \lor d) = (x, y) \in \phi_a$ .

Hence,  $\phi_a$  is a congruence relation on the ADL (*L*,  $\vee$ ,  $\wedge$ , 0).

Also, if  $(x, y) \in \theta$ , then  $y \land x = x$ , for all  $x, y \in L$ ..... by [theorem2.1.12 (7)]

Moreover,  $x^{\circ} = y^{\circ}$  implies  $x^{\circ\circ} = y^{\circ\circ}$ . So that for each  $a \in L$ , we have  $x^{\circ} \lor a = y^{\circ} \lor a$  and  $x^{\circ\circ} \lor a = y^{\circ\circ} \lor a$ . That is,  $(x^{\circ}, y^{\circ}) \in \phi_a$ .

Hence, for each  $a \in L$ , the relation  $\phi_a$  given by;  $(x, y) \in \phi_a$  if and only if  $x \lor a = y \lor a$  and  $x^\circ \lor a = y^\circ \lor a$  is a congruence relation on L.

(8)  $\Rightarrow$  (1). Suppose that for each  $a \in L$ , the relation  $\phi_a$  given by;

 $(x, y) \in \phi_a$  if and only if  $x \lor a = y \lor a$  and  $x^\circ \lor a = y^\circ \lor a$  is a congruence relation on L.

Claim: L is an MS-algebra.

Since,  $\phi_a$  is a congruence relation on L, for any  $x, y, c, d \in L$  such that  $(x, y) \in \phi_a$  and  $(c, d) \in \phi_a$ , we have  $(x \land c, y \land d) \in \phi_a$ ,  $(x \lor c, y \lor d) \in \phi_a$  and  $(x^\circ, y^\circ) \in \phi_a$ . This gives that  $(x \land c)^\circ \lor a = (y \land d)^\circ \lor a$  and  $(x \lor c)^\circ \lor a = (y \lor d)^\circ \lor a$ . Also,  $x^{\circ\circ} \lor a = y^{\circ\circ} \lor a$ .

This is equivalent to  $(x^{\circ} \lor c^{\circ}) \lor a = (y^{\circ} \lor d^{\circ}) \lor a$  and  $(x^{\circ} \land c^{\circ}) \lor a = (y^{\circ} \land d^{\circ}) \lor a$ .

Hence, L is an MS-algebra.

Therefore,  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (1)$  hold.

**Theorem 2.1.13** Let  $(L, \vee, \land, 0)$  be an ADL. Then, the following conditions are equivalent:

- (1). L is an MS-ADL.
- (2).  $L_n$  is an MS-algebra for all maximal element n of L.

(3).  $L_m$  is an MS-algebra for some maximal element m.

**Proof:** (1)  $\Rightarrow$  (2). Suppose that L is an MS-ADL. Let n be a maximal element of L.

Define a unary operation  $x \mapsto x^*$  on  $L_n$  by;  $(a \land n)^* = a^\circ \land n$ , for all  $a \in L$ , where  $\circ$  is the unary operation on L. Consequently, this gives the following results:

 $(a \wedge n)^* = a^\circ \wedge n = a^\circ \wedge (a \wedge n) = (a^\circ \wedge a) \wedge n = (a \wedge a^\circ) \wedge n = 0 \wedge n = 0$ . This implies that  $a^\circ \wedge n \in L_n$ . Hence,  $(a \wedge n)^* \in L_n$  for all  $a \in L$ .

Also, let *a*, *b*  $\in$  *L* such that  $a \land n = b \land n$ .

Claim:  $(a \wedge n)^* = (b \wedge n)^*$ .

For this we consider the following implications:

$$a \wedge n = b \wedge n \Longrightarrow (a \wedge n)^{\circ} = (b \wedge n)^{\circ}$$
  

$$\Rightarrow a^{\circ} \vee n^{\circ} = b^{\circ} \vee n^{\circ} \dots \text{ by definition 2.1.1 of axiom [MS-A3]}$$
  

$$\Rightarrow a^{\circ} \vee 0 = b^{\circ} \vee 0 \dots \text{ [Since n is maximal]}.$$
  

$$\Rightarrow a^{\circ} = b^{\circ} \dots \text{ by definition 1.2.2.1 of axiom [ADL1]}$$
  

$$\Rightarrow a^{\circ} \wedge n = b^{\circ} \wedge n, \text{ for all maximal element } n \text{ of } L.$$
  

$$\Rightarrow (a \wedge n)^{*} = (b \wedge n)^{*}.$$

**Note that:** This confirms that the unary operation  $x \mapsto x^*$  is well defined as a mapping. Moreover, let  $a, b \in L_n$ . Then, it can be easily verified that  $L_n$  together with this unary operation  $x \mapsto x^*$  is an MS-algebra as follows:

(i). 
$$(a \wedge b)^* = [(a \wedge b) \wedge n]^* = (a \wedge b)^\circ \wedge n = (a^\circ \vee b^\circ) \wedge n.$$
  
=  $(a^\circ \wedge n) \vee (b^\circ \wedge n)$ 

$$= (a \wedge n)^* \vee (b \wedge n)^*.$$
$$= a^* \vee b^*$$

(ii).  $(a \lor b)^* = [(a \lor b) \land n]^* = (a \lor b)^\circ \land n = a^\circ \land b^\circ \land n$  $= (a^\circ \land n) \land (b^\circ \land n)$ 

$$= a^* \wedge b^*$$

 $= (a \wedge n)^* \wedge (b \wedge n)^*.$ 

Particularly, if a = 1, then since the unary operation  $x \mapsto x^*$  on  $L_n$  is defined by;  $(a \land n)^* = a^\circ \land n$ , this gives that  $1^* = (1 \land n)^* = 1^\circ \land n = 0 \land n = 0$ .... by definition 1.2.2.19 of [M4]

Therefore,  $L_n$  is an MS-algebra for all maximal element n of L.

(2)  $\Rightarrow$ (3). Suppose  $L_n$  is an MS-algebra for all maximal element n of L. So that we can generate a unary operation  $x \mapsto x^*$  on  $L_m$  for some fixed maximal element m defined by:  $(a \land m)^* = a^\circ \land m$ , for all  $a \in L$ , where  $\circ$  is the unary operation on L.

Hence, by (2)  $L_m$  becomes an MS-algebra for some fixed maximal element m.

(3)  $\Rightarrow$ (1). Suppose that  $L_m$  is an MS-algebra for some fixed maximal element m.

Define a unary operation  $x \mapsto x^*$  on L by;  $a^* = (a \land m)^\circ$  for all  $a \in L$ , where  $\circ$  is the unary operation on  $L_m$ .

Claim: L is an MS-ADL.

Since  $L_m$  has a fixed maximal element m, we have the following results:

$$a^* \wedge (a \wedge m) = (a \wedge m)^\circ \wedge (a \wedge m) = a^\circ \wedge (a \wedge m)...$$
 by [lemma2.1.7 (6)]  
$$= (a^\circ \wedge a) \wedge m...$$
 [Since  $\wedge$  is associative]  
$$= (a \wedge a^\circ) \wedge m...$$
 by [lemma1.2.2.6 (13)]  
$$= 0 \wedge m...$$
 by definition1.2.2.15 of [PC-2]  
$$= 0...$$
 by definition1.2.2.1 of [ADL2]

Also, for any  $a, b \in L$ , we have the following characterizations and results:

(i).  $(a \land b)^* = [(a \land b) \land m]^\circ = (a \land b)^\circ \lor m^\circ$ ..... by definition2.1.1 of [MS-A3]  $= (a^\circ \lor b^\circ) \lor 0$ ..... by definition2.1.1 of [MS-A4]  $= a^\circ \lor b^\circ$ ..... by definition1.2.2.1 of [ADL1]  $= (a \land m)^\circ \lor (b \land m)^\circ$ ..... by [lemma2.1.7(6)]  $= a^* \lor b^*$ 

(ii).  $(a \lor b)^* = [(a \lor b) \land m]^\circ = [(a \land m) \lor (b \land m)]^\circ$ ..... by definition 1.2.2.1 of [ADL3] =  $(a \land m)^\circ \land (b \land m)^\circ$ ..... by definition 2.1.1 of [MS-A2] =  $a^* \land b^*$ 

Hence, L is an MS-ADL, for some fixed maximal element m.

**Definition 2.1.14 [11-13]** Let L be an MS-ADL. Then, the set of all closed elements of L denoted by  $L^{\circ\circ}$  is called the skeleton of L and it is defined by  $L^{\circ\circ} = \{x^{\circ\circ} : x \in L\}$ .

The skeleton  $L^{\circ\circ}$  of an MS-ADL is a De Morgan algebra under the induced operations on L.

**Theorem 2.1.15** Let L be an MS-ADL. Then, the map  $\phi : L \mapsto L$  defined by:

 $\phi(x) = x^{\circ\circ}$  for all  $x \in L$  forms a closure operator on L.

**Proof:** Let *x*, *y*  $\in$  *L* such that *x*  $\leq$  *y*. This implies *x*  $\land$  *y* = *x* (or equivalently, *x*  $\lor$  *y* = *y*).

Then,  $\phi(x) = \phi(x \land y) = (x \land y)^{\circ\circ} = x^{\circ\circ} \land y^{\circ\circ} = \phi(x) \land \phi(y)$  and  $\phi(y) = \phi(x \lor y) = (x \lor y)^{\circ\circ} = x^{\circ\circ} \lor y^{\circ\circ} = \phi(x) \lor \phi(y)$ . So that  $x \le y$  implies  $\phi(x) \le \phi(y)$  or  $\phi$  is an order preserving on L. Moreover, for any  $x, y \in L$  we have  $\phi(x \land y) = (x \land y)^{\circ\circ} = x^{\circ\circ} \land y^{\circ\circ} = \phi(x) \land \phi(y)$  and  $\phi(x \lor y) = (x \lor y)^{\circ\circ} = x^{\circ\circ} \lor y^{\circ\circ} = \phi(x) \lor \phi(y)$ . This shows that  $\phi$  is a lattice homomorphism and it preserves the unary operation  $\circ$  on L.

Hence, the map  $\phi: L \mapsto L$  defined by  $\phi(x) = x^{\circ\circ}$  for all  $x \in L$  forms a closure operator on L.

**Definition 2.1.16 [13]** Let L be an MS-ADL. An element  $x \in L$  is called dense element if  $x^{\circ} = 0$ . The set of all dense elements is denoted by D(L) such that  $D(L) = \{x \in L : x^{\circ} = 0\}$ .

**Theorem 2.1.17** Let L be an MS-ADL. Then, the set  $D(L) = \{x \in L : x^\circ = 0\}$  is a filer of L.

**Proof:** Let L be an MS-ADL with maximal element m. Then, by definition2.1.1 of [MS-A4] it follows that  $m^{\circ} = 0$ . This implies that  $m \in D(L)$ . Hence, D(L) is a non-empty subset of L.

Also, let  $a, b \in D(L)$  and  $x \in L$ . Then, by definition 2.1.16 we have  $a^{\circ} = 0$  and  $b^{\circ} = 0$ .

**Claim:**  $a \land b \in D(L)$  and  $x \lor a \in D(L)$ ..... by [definition1.2.2.11]

Since L is an MS-ADL with maximal element m, then by definition2.1.1 of [MS-A3] and [MS-A2] we have the following results:

(i).  $(a \wedge b)^\circ = a^\circ \vee b^\circ = 0 \vee 0 = 0$ . This implies that  $a \wedge b \in D(L)$ .

(ii).  $(x \lor a)^\circ = x^\circ \land a^\circ = x^\circ \land 0 = 0$ . This implies that  $x \lor a \in D(L)$ .

Therefore, the set  $D(L) = \{x \in L : x^\circ = 0\}$  is a filer of L.

#### 2.2 Congruence Relations on MS-algebras and MS-ADLs

In this section, we give some important congruence relations which are using to characterize MS-ADLs and MS-algebras.

**Definition 2.2.1[18]** Let  $(L, \lor, \land, \circ, 0, m)$  be an MS-ADL. Then, an equivalence relation  $\theta$  on L is said to be a congruence relation on L if and only if  $\theta$  is a congruence relation on the ADL  $(L, \lor, \land, 0)$ , that is  $(a, b), (x, y) \in \theta \implies (a \land x, b \land y), (a \lor x, b \lor y) \in \theta$  and if it satisfies the substitution property:

 $(x, y) \in \theta \Longrightarrow (x^{\circ}, y^{\circ}) \in \theta$ , for all  $a, b, x, y \in L$ .

**Note that:** An equivalence relation  $\theta$  on the MS-ADL ( $L, \vee, \wedge, \circ, 0, m$ ) is a congruence relation on L if  $\theta$  is closed under the binary operations  $\vee$  and  $\wedge$  and the unary operation  $\circ$ .

A congruence relation on the MS-algebra (L,  $\lor$ ,  $\land$ ,  $\circ$ , 0, 1) is a lattice congruence  $\theta$  such that  $(x, y) \in \theta \implies (x^\circ, y^\circ) \in \theta$  for all  $x, y \in L$ .

**Example 2.2.2** Let L be an MS-ADL and  $\theta$  be a binary relation on L defined by;  $(a, b) \in \theta$  if and only if  $a^\circ = b^\circ$  for all  $a, b \in L$ . Then, show that  $\theta$  is a congruence relation on L.

**Proof:** First let us show that  $\theta$  is an equivalence relation on L.

(1).  $(a, a) \in \theta$ , since  $a^{\circ} = a^{\circ}$  for all  $a \in L$ .

(2).  $(a, b) \in \theta$  implies  $(b, a) \in \theta$  since,  $a^{\circ} = b^{\circ}$  implies  $b^{\circ} = a^{\circ}$  for all  $a, b \in L$ .

(3).  $(a, b) \in \theta$  and  $(b, c) \in \theta$  implies  $(a, c) \in \theta$  since,  $a^{\circ} = b^{\circ}$  and  $b^{\circ} = c^{\circ}$  implies  $a^{\circ} = c^{\circ}$  for all  $a, b, c \in L$ .

Hence,  $\theta$  is an equivalence relation (that is, reflexive, symmetric and transitive) on L.

Also, let  $(a, b) \in \theta$  implies  $a^{\circ} = b^{\circ}$  and  $(c, d) \in \theta$  implies  $c^{\circ} = d^{\circ}$ . Then, by definition 2.2.1 we have  $a^{\circ} \wedge c^{\circ} = b^{\circ} \wedge d^{\circ}$  and  $a^{\circ} \vee c^{\circ} = b^{\circ} \vee d^{\circ}$ . This is equivalent to  $(a \vee c)^{\circ} = (b \vee d)^{\circ}$  and  $(a \wedge c)^{\circ} = (b \wedge d)^{\circ}$ . This implies that  $(a \wedge c, b \wedge d) \in \theta$  and  $(a \vee c, b \vee d) \in \theta$  for all  $a, b, c, d \in L$ . Moreover,  $a^{\circ} = b^{\circ}$  implies that  $a^{\circ\circ} = b^{\circ\circ}$ . So that  $(a^{\circ}, b^{\circ}) \in \theta$ .

Therefore,  $\theta$  is a congruence relation on L.

**Definition 2.2.3 [14]** Let  $(L, \vee, \wedge, \circ, 0, m)$  be an MS-ADL and  $\eta$  be a binary relation on L. Then, for each  $a, b \in L$ , consider that  $(a, b) \in \eta$  if and only if  $a \wedge b = b$  and  $b \wedge a = a$ .

**Lemma 2.2.4** Let L be an MS-ADL. Then,  $\eta$  is a congruence relation on L.

**Proof:** (1).  $(a, a) \in \eta$ , since  $a \land a = a$  and  $a \land a = a$ , for all  $a \in L \dots [\eta$  is reflexive on L].

(2).  $(a, b) \in \eta$  implies  $(b, a) \in \eta$  since  $[a \land b = b \text{ and } b \land a = a]$  implies that

 $[b \land a = a \text{ and } a \land b = b]$  .....  $[\eta \text{ is symmetric on L}].$ 

Hence,  $\eta$  is an equivalence relation on L.

(i). Let  $(a, b) \in \eta$  and  $(c, d) \in \eta$ . Then, by definition 2.2.1 it follows that  $(a \land c, b \land d) = (c, d) \in \eta$  and  $(a \lor c, b \lor d) = (a, b) \in \eta$ .

So that  $\eta$  is a congruence relation on the ADL (L,  $\vee$ ,  $\wedge$ , 0).

(ii). Let  $(a, b) \in \eta$ . Then, by definition 2.2.3 we have  $a \wedge b = b$  and  $b \wedge a = a$ . So that by lemma 2.1.7 (7) it follows that  $a^{\circ} = (b \wedge a)^{\circ} = (a \wedge b)^{\circ} = b^{\circ}$ . This implies that  $(a^{\circ}, b^{\circ}) \in \eta$ .

Hence,  $\eta$  is a congruence relation on L.

**Lemma 2.2.5** [18]  $\eta$  is the smallest congruence on L for which the quotient L| $\eta$  is an MS-algebra.

**Proof:** First let us consider a congruence relation  $\theta$  on L (as given in example 2.2.2). That is,  $(a, b) \in \theta$  if and only if  $a^{\circ} = b^{\circ}$  for all  $a, b \in L$ . Let m be the maximal element of L. Then, by lemma2.1.7 (6) it follows that  $(a \land b)^{\circ} = (a \land b \land m)^{\circ} = (b \land a \land m)^{\circ} = (b \land a)^{\circ}$ . This implies that  $(a \land b, b \land a) \in \theta$  for all  $a, b \in L$ . That is,  $\land$  commutes on L| $\theta$ . Therefore, L| $\theta$  is an MS-algebra.

#### Claim: $\eta \subseteq \theta$ .

For any  $a, b \in L$ , we have  $a^{\circ} = (b \wedge a)^{\circ} = (a \wedge b)^{\circ} = b^{\circ}$ ..... by [lemma2.1.7 (7)]. So that  $(a \wedge b, b \wedge a) \in \eta$ . That is,  $\wedge$  commutes on L| $\eta$ . Hence, L| $\eta$  is also an MS-algebra. Let  $(a, b) \in \eta$ . Then,  $a \wedge b = b$  and  $b \wedge a = a$ ..... by definition2.2.5. Also, by lemma2.1.7 (7) this gives that  $a^{\circ} = (b \wedge a)^{\circ} = (a \wedge b)^{\circ} = b^{\circ}$ . This shows that  $(b \wedge a, a \wedge b) \in \theta$  for all  $a, b \in L$ . Hence,  $\eta \subseteq \theta$ .

Therefore,  $\eta$  is the smallest congruence on L for which the quotient L| $\eta$  is an MS-algebra.

**Definition 2.2.6** [18] Let  $(L, \vee, \wedge, \circ, 0, m)$  be an MS-ADL and  $\Phi$  be a binary relation on L. Then, for each  $x, y \in L$  consider that  $(x, y) \in \Phi$  if and only if  $x^{\circ\circ} = y^{\circ\circ}$ .

**Lemma 2.2.7[18]** Let L be an MS-ADL. Then,  $\Phi$  is a congruence relation on L.

**Proof:** (1).  $(x, x) \in \Phi$ , since  $x^{\circ\circ} = x^{\circ\circ}$  for all  $x \in L$ ..... [ $\Phi$  is reflexive on L].

(2).  $(x, y) \in \Phi \implies (y, x) \in \Phi$ , since  $x^{\circ\circ} = y^{\circ\circ} \implies y^{\circ\circ} = x^{\circ\circ} \dots \dots [\Phi$  is symmetric on L].

(3).  $(x, y) \in \Phi$  and  $(y, z) \in \Phi \implies (x, z) \in \Phi$  since,  $x^{\circ\circ} = y^{\circ\circ}$  and  $y^{\circ\circ} = z^{\circ\circ} \implies x^{\circ\circ} = z^{\circ\circ}$  for all  $x, y, z \in L$  ..... [ $\Phi$  is transitive on L].

Hence,  $\Phi$  is an equivalence relation on L.

(i). Let  $(x, y) \in \Phi$  implies  $x^{\circ\circ} = y^{\circ\circ}$  and  $(a, b) \in \Phi$  implies  $a^{\circ\circ} = b^{\circ\circ}$  for all  $x, y, a, b \in L$ . Then, by definition 1.2.2.23 it follows that  $x^{\circ\circ} \wedge a^{\circ\circ} = y^{\circ\circ} \wedge b^{\circ\circ}$  and  $x^{\circ\circ} \vee a^{\circ\circ} = y^{\circ\circ} \vee b^{\circ\circ}$ . This is equivalent to  $(x \wedge a)^{\circ\circ} = (y \wedge b)^{\circ\circ}$  and  $(x \vee a)^{\circ\circ} = (y \vee b)^{\circ\circ}$ . So that  $(x \wedge a, y \wedge b) \in \Phi$  and  $(x \vee a, y \vee b) \in \Phi$ . Hence,  $\Phi$  is a congruence relation on the ADL  $(L, \vee, \wedge, 0)$ . (ii). Moreover,  $x^{\circ\circ} = y^{\circ\circ}$  implies that  $x^{\circ\circ\circ} = y^{\circ\circ\circ}$ . This shows that  $(x^{\circ}, y^{\circ}) \in \Phi$ .

Therefore,  $\Phi$  is a congruence relation on L.

**Lemma 2.2.8[18]**  $\Phi$  is the smallest congruence relation on L such that its quotient L| $\Phi$  is a De Morgan algebra.

**Proof:** Since  $a \wedge m = a^{\circ\circ} = b^{\circ\circ} = b \wedge m$ ,  $\Phi$  is a De Morgan ADL. Also,  $\Phi \subseteq \Phi$ . This implies that L| $\Phi$  is also a De Morgan ADL.

It suffices to show that one of the binary operations (either V or  $\Lambda$ ) commutes on L| $\Phi$ .

Let  $a, b \in L$  and m be the maximal element of L. Then, by lemma2.1.7 (6) it follows that

$$(a \wedge b)^{\circ} = (a \wedge b \wedge m)^{\circ} = (b \wedge a \wedge m)^{\circ} = (b \wedge a)^{\circ}$$
. That is,  $(a \wedge b)^{\circ} = (b \wedge a)^{\circ}$ .

This implies that  $(a \land b)^{\circ\circ} = (b \land a)^{\circ\circ}$ . So that  $(a \land b , b \land a) \in \Phi$ . That is,  $\land$  commutes on  $L|\Phi$ . Hence,  $L|\Phi$  is a De Morgan algebra.

Suppose  $\theta$  is a congruence relation on L (as given in example 2.2.2) such that L| $\theta$  is a De Morgan algebra. That is,  $(a \land b, b \land a) \in \theta$  for all  $a, b \in L$ .

#### Claim: $\Phi \subseteq \theta$ .

Let  $(a, b) \in \Phi$ . Then, by definition 2.2.6 this gives that  $a^{\circ\circ} = b^{\circ\circ}$ . This implies  $a^{\circ\circ\circ} = b^{\circ\circ\circ}$ . This is equivalent to  $a^{\circ} = b^{\circ}$ ..... by lemma 2.1.7 (3). Also, by lemma 2.1.7 (7) this gives that  $a^{\circ} = (b \land a)^{\circ} = (a \land b)^{\circ} = b^{\circ}$ . That is,  $a \land b = b$  and  $b \land a = a$ . So that  $(a, b) \in \theta$ .

This shows that  $\Phi \subseteq \theta$ .

Therefore,  $\Phi$  is the smallest congruence relation on L such that its quotient L| $\Phi$  is a De Morgan algebra.

The following theorem characterizes MS-ADLs using the congruence relation  $\Phi$ .

**Theorem 2.2.9** Let L be an MS-ADL. Then, L is a De Morgan ADL if and only if  $\eta = \Phi$ .

**Proof:**  $(\Longrightarrow)$ . Suppose that L is a De Morgan ADL.

Claim:  $\eta = \Phi$ .

Let  $(a, b) \in \eta$ . Then,  $a \wedge b = b$  and  $b \wedge a = a$  for all  $a, b \in L$ ..... by [Definition2.2.3]

Since L is a De Morgan ADL with maximal element m of L, this gives that  $a^{\circ\circ} = a \wedge m$ =  $b \wedge a \wedge m = a \wedge b \wedge m = b \wedge m = b^{\circ\circ}$ ..... by definition2.1.2 of [MS-A5].

That is,  $a^{\circ\circ} = b^{\circ\circ}$ . This implies that  $(a, b) \in \Phi$ .

So that  $\eta \subseteq \Phi$ .....(1)

Put m = 0°. Since L is a De Morgan ADL, by definition2.1.2 of [MS-A5] it follows that  $a \wedge m = a^{\circ\circ} = b^{\circ\circ} = b \wedge m$ . That is,  $a \wedge m = b \wedge m$ . This gives  $a \wedge b = b$  and  $b \wedge a = a$ .

This implies that  $(a, b) \in \eta$ . So that  $\Phi \subseteq \eta$ ......(2)

Hence, the two inclusions (1) and (2) imply that  $\Phi = \eta$ .

( $\Leftarrow$ ). Suppose that  $\Phi = \eta$ .

Claim: L is a De Morgan ADL.

This shows that  $a^{\circ\circ} \leq m$  for all  $a \in L$ . Also, since  $a^{\circ\circ\circ\circ} = a^{\circ\circ}$ , we have  $(a^{\circ\circ}, a) \in \Phi = \eta$ . That is,  $(a^{\circ\circ}, a) \in \eta$ . This gives that  $a \wedge a^{\circ\circ} = a^{\circ\circ}$  and  $a^{\circ\circ} \wedge a = a$ .... by [definition 2.2.3]

 $= a \wedge a^{\circ \circ} \wedge m$  ..... by [lemma1.2.2.6 (13)]

 $= a^{\circ\circ} \wedge m$  ..... [Since  $a \wedge a^{\circ\circ} = a^{\circ\circ}$ ]

$=a^{\circ\circ}$	. [Since	$a^{\circ\circ}$	$\leq m$	]
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Therefore, L is a De Morgan ADL.

**Corollary 2.2.10** Let L be an MS-ADL. Then, L is a De Morgan algebra if and only if  $\Phi = \Delta_L$  (the diagonal of L). For an MS-ADL, L to be a Stone ADL it is necessary and sufficient that  $a \wedge a^\circ = 0$ .

**Proof:** ( $\Rightarrow$ ). Suppose that L is a De Morgan algebra. Then, for any  $a, b \in L$ , we have  $a^{\circ\circ} = a$  and  $b^{\circ\circ} = b$ ..... by definiton 1.2.2.20 of [M5].

#### **Claim:** $\Phi = \Delta_L$ .

Claim: L is a De Morgan algebra.

Since L is an MS-ADL, for any  $a \in L$ , we have  $a^{\circ \circ \circ \circ} = a^{\circ \circ}$ . So that  $(a^{\circ \circ}, a) \in \Phi = \Delta_L$ . That is,  $(a^{\circ \circ}, a) \in \Delta_L$ . This implies that  $a^{\circ \circ} = a$ , for all  $a \in L$ .

Therefore, L is a De Morgan algebra.

To prove the second part, it is known that an MS-ADL, L is a Stone ADL if  $a^{\circ} \lor a^{\circ \circ} = 0^{\circ}$ 

for all  $a \in L$ . Given that  $a \wedge a^\circ = 0$ . So that by definition2.1.1 of axiom [MS-A3] it follows that  $0^\circ = (a \wedge a^\circ)^\circ = a^\circ \vee a^{\circ\circ}$ . Hence, L is a Stone ADL whenever  $a \wedge a^\circ = 0$ .

**Definition 2.2.11** let L be an MS-ADL and  $\psi^a$  and  $\phi^a$  be binary relations on L. Then, for each  $a \in L$ , consider the following definitions:

(1).  $(x, y) \in \psi^a$  if and only if  $x \wedge a = y \wedge a$  and  $x^\circ \wedge a = y^\circ \wedge a$ 

(2).  $(x, y) \in \phi^a$  if and only if  $a \wedge x = a \wedge y$  and  $a \wedge x^\circ = a \wedge y^\circ$ 

By these two definitions, we have the following characterizations and results:

**Theorem 2.2.12** For each  $a \in L$ ,  $\psi^a$  and  $\phi^a$  are congruence relations on L.

**Proof:** (1). Let  $(x, y) \in \psi^a$ . Then,  $x \wedge a = y \wedge a$  and  $x^\circ \wedge a = y^\circ \wedge a$ .

Also, let  $(c, d) \in \psi^a$ . Then,  $c \wedge a = d \wedge a$  and  $c^{\circ} \wedge a = d^{\circ} \wedge a$ .... by [definition2.2.11 (1)]

So that (i).  $x \wedge c \wedge a = c \wedge x \wedge a = c \wedge y \wedge a = y \wedge c \wedge a = y \wedge d \wedge a$ .

Also, 
$$(x \lor c) \land a = (x \land a) \lor (c \land a) = (y \land a) \lor (d \land a) = (y \lor d) \land a$$
.

This implies that  $(x \land c, y \land d) \in \psi^a$  and  $(x \lor c, y \lor d) \in \psi^a$ .

Hence,  $\psi^a$  is a congruence relation on the ADL (*L*, V,  $\wedge$ ,0).

(ii). Let  $(x, y) \in \psi^a$ . Then,  $x \wedge a = y \wedge a$  and  $x^\circ \wedge a = y^\circ \wedge a$ . Also, Since L is an MS-ADL with maximal element m, we have  $a^{\circ\circ} \wedge a = a$  for all  $a \in L$ ... by definition2.1.1 of [MS-A1]

**Claim:**  $x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$ . For this, we consider the following implications:

$$x \wedge a = y \wedge a \Longrightarrow (x \wedge a)^{\circ\circ} = (y \wedge a)^{\circ\circ}$$
$$\Rightarrow x^{\circ\circ} \wedge a^{\circ\circ} = y^{\circ\circ} \wedge a^{\circ\circ}$$
$$\Rightarrow x^{\circ\circ} \wedge a^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a^{\circ\circ} \wedge a$$
$$\Rightarrow x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a....$$
by definition2.1.1 of [MS-A1].
$$\Rightarrow (x^{\circ}, y^{\circ}) \in \psi^{a}$$

Therefore,  $\psi^a$  is a congruence relation on L.

(2). Let  $(x, y) \in \phi^a$ . Then,  $a \wedge x = a \wedge y$  and  $a \wedge x^\circ = a \wedge y^\circ$ ..... by [definition2.2.11 (2)] Also, let  $(c, d) \in \phi^a$ . Then,  $a \wedge c = a \wedge d$  and  $a \wedge c^\circ = a \wedge d^\circ$ ..... by [definition2.2.11 (2)] So that (i).  $a \wedge x \wedge c = a \wedge y \wedge c = y \wedge a \wedge c = y \wedge a \wedge d = a \wedge y \wedge d$ . Also,  $a \wedge (x \vee c) = (a \wedge x) \vee (a \wedge c) = (a \wedge y) \vee (a \wedge d) = a \wedge (y \vee d)$ .

This implies that  $(x \land c, y \land d) \in \phi^a$  and  $(x \lor c, y \lor d) \in \phi^a$ .

So that  $\phi^a$  is a congruence relation on the ADL (*L*, V,  $\land$  ,0).

(ii). Let  $(x, y) \in \phi^a$ . Then,  $a \wedge x = a \wedge y$  and  $a \wedge x^\circ = a \wedge y^\circ \dots$  by [definition2.2.11 (2)]

Also, since L is an MS-ADL with maximal element m, we have  $a^{\circ\circ} \wedge a = a$ , for all  $a \in L$ .

**Claim:**  $a \wedge x^{\circ\circ} = a \wedge y^{\circ\circ}$ . For this, we consider the following implications:

$$a \wedge x = a \wedge y \Longrightarrow (a \wedge x)^{\circ\circ} = (a \wedge y)^{\circ\circ}$$
  

$$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} = a^{\circ\circ} \wedge y^{\circ\circ}$$
  

$$\Rightarrow a \wedge a^{\circ\circ} \wedge x^{\circ\circ} = a \wedge a^{\circ\circ} \wedge y^{\circ\circ}$$
  

$$\Rightarrow a^{\circ\circ} \wedge a \wedge x^{\circ\circ} = a^{\circ\circ} \wedge a \wedge y^{\circ\circ} \dots \text{ by [lemma1.2.2.6 (13)]}$$
  

$$\Rightarrow a \wedge x^{\circ\circ} = a \wedge y^{\circ\circ} \dots \text{ by definition2.1.1 of [MS-A1]}$$
  

$$\Rightarrow (x^{\circ}, y^{\circ}) \in \phi^{a}.$$

Therefore,  $\phi^a$  is also a congruence relation on L.

**Corollary 2.2.13** If L is a De Morgan ADL, then  $\psi^a$  and  $\phi^a$  are congruence relations on L.

**Proof:** Suppose L is a De Morgan ADL. Put  $x^{\circ\circ} = x \wedge m$  and  $y^{\circ\circ} = y \wedge m$  for all  $x, y \in L$ and for all maximal element m of L..... by definition2.1.2 of [MS-A5] Let  $(x, y) \in \psi^a$ . Then  $x \wedge a = y \wedge a$  and  $x^{\circ} \wedge a = y^{\circ} \wedge a$  ..... by [definition2.2.11 (1)] Also, let  $(x, y) \in \phi^a$ . Then  $a \wedge x = a \wedge y$  and  $a \wedge x^{\circ} = a \wedge y^{\circ}$ .... by [definition2.2.11 (2)] **Claim:** (i).  $x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$ . (ii).  $a \wedge x^{\circ\circ} = a \wedge y^{\circ\circ}$ . Since L is a De Morgan ADL, by definition2.1.2 of [MS-A5] we have the following results:

(i). 
$$x^{\circ\circ} \wedge a = x \wedge m \wedge a = m \wedge x \wedge a = m \wedge y \wedge a = y \wedge m \wedge a = y^{\circ\circ} \wedge a$$
.

Hence,  $(x^\circ, y^\circ) \in \psi^a$ .

(ii).  $a \wedge x^{\circ \circ} = a \wedge x \wedge m = a \wedge y \wedge m = a \wedge y^{\circ \circ}$ .

Hence, 
$$(x^\circ, y^\circ) \in \phi^a$$
.

Therefore, if L is a De Morgan ADL, then  $\psi^a$  and  $\phi^a$  are congruence relations on L.

**Theorem 2.2.14** The quotient  $L|\psi^a$  is an MS-algebra. If *a* is maximal, then  $\psi^a = \eta$ . The converse holds whenever  $a^{\circ} \wedge a = 0$ .

**Proof**: To prove the first part, it suffices to show that  $\eta \subseteq \psi^a$ .

Let  $(x, y) \in \eta$ . Then,  $x \wedge y = y$  and  $y \wedge x = x$ . Equivalently;  $x \vee y = x$  and  $y \vee x = y$ . Also, we have  $x^{\circ} = (y \land x)^{\circ} = (x \land y)^{\circ} = y^{\circ}$  ..... by [lemma 2.1.7 (7)] Hence,  $x \wedge a = y \wedge x \wedge a = x \wedge y \wedge a = y \wedge a$  and  $x^{\circ} \wedge a = y^{\circ} \wedge a$ This implies that  $(x, y) \in \psi^a$ . So that  $\eta \subseteq \psi^a$ . Therefore,  $L|\psi^a$  is an MS-algebra. To prove the second part, suppose that *a* is maximal. Claim:  $\psi^a = \eta$ . Let  $(x, y) \in \psi^a$ . Then,  $x \wedge a = y \wedge a$  and  $x^\circ \wedge a = y^\circ \wedge a$  ..... by [definition2.2.11 (1)] If a is maximal, then  $x \wedge a = y \wedge a$  and  $x^{\circ} \wedge a = y^{\circ} \wedge a$  are reduced to x = y and  $x^{\circ} = y^{\circ}$ . Put  $x \land y = y$  and  $y \land x = x$ . This implies that  $(x, y) \in \eta$ ..... by [definition2.2.3]. This shows that  $\psi^a \subseteq \eta$ .....(1) Similarly, let  $(x, y) \in \eta$ . This gives that  $x \wedge y = y$  and  $y \wedge x = x$ ..... by [definition2.2.3] Then,  $x \wedge a = y \wedge x \wedge a = x \wedge y \wedge a = y \wedge a$  and  $x^{\circ} \wedge a = y^{\circ} \wedge a$ ... [Since a is maximal] This implies that  $(x, y) \in \psi^a$ ..... by [definition2.2.11 (1)] 

So that the two inclusions (1) and (2) yield that  $\psi^a = \eta$ .

Hence, if *a* is maximal, then  $\psi^a = \eta$ .

**Conversely,** suppose that  $\psi^a = \eta$ .

Claim: *a* is maximal.

Given that  $a^{\circ} \wedge a = 0$ . This shows that  $(a^{\circ} \wedge a, 0) \in \psi^{a} = \eta$ .

That is,  $(a^{\circ} \wedge a, 0)$  satisfies both the equalities  $x \wedge a = y \wedge a$  and  $x^{\circ} \wedge a = y^{\circ} \wedge a$ .

So that we have  $(a^{\circ} \wedge a) \wedge a = 0 \wedge a$  and  $(a^{\circ} \wedge a)^{\circ} \wedge a = 0^{\circ} \wedge a$ ..... by definition 2.2.11(1).

But  $(a^{\circ} \wedge a) \wedge a = a^{\circ} \wedge (a \wedge a) = a^{\circ} \wedge a$ .....[Since  $\wedge$  is associative]

So that  $a^{\circ} \wedge a = 0 \wedge a \implies a^{\circ} = 0$ .

 $\Rightarrow$  *a* is maximal..... by definition 2.1.1 of [MS-A4].

Therefore,  $\psi^a = \eta$  implies *a* is maximal whenever  $a^{\circ} \wedge a = 0$ .

**Remark 2.2.15** The quotient  $L|\phi^a$  is an MS-ADL but not necessarily be an MS-algebra. This can be verified by using the following example.

**Example 2.2.16** Let  $L = \{0, a, b\}$  be the discrete MS-ADL (as given in example2.1.3) and consider the congruence relation  $\phi^a$  on L. Then, the quotient  $L|\phi^a$  is an MS-ADL but not an MS-algebra.

**Proof:** Since L is not a distributive lattice, it suffices to show that  $L|\phi^a$  is isomorphic to L. For this, consider the canonical map  $x \mapsto \phi^a[x]$  of L onto  $L|\phi^a$  which is an epimorphism. We show that this map is an injective map.

Now, for any  $x, y \in L$ , we have  $\phi^a[x] = \phi^a[y] \Longrightarrow (x, y) \in \phi^a$ .

$$\Rightarrow a \land x = a \land y$$

Since L is a discrete MS-ADL, every non-zero element in L is maximal. Thus,  $a \wedge x = a \wedge y$ yields that x = y. This shows that  $L \cong L | \phi^a$ .

Therefore,  $L|\phi^a$  is not an MS-algebra.

**Lemma 2.2.17** For each  $a, b \in L$ ,  $\phi^a \cap \phi^b = \phi^{a \lor b}$ .

**Proof:** Let  $(x, y) \in \phi^a \cap \phi^b$ . Then, by definition 2.2.11 (2) we have the following conditions:

$$a \wedge x = a \wedge y$$
,  $a \wedge x^{\circ} = a \wedge y^{\circ}$ ,  $b \wedge x = b \wedge y$  and  $b \wedge x^{\circ} = b \wedge y^{\circ}$ .

So that  $(a \lor b) \land x = (a \land x) \lor (b \land x) = (a \land y) \lor (b \land y) = (a \lor b) \land y$ .

Also,  $(a \lor b) \land x^\circ = (a \land x^\circ) \lor (b \land x^\circ) = (a \land y^\circ) \lor (b \land y^\circ) = (a \lor b) \land y^\circ$ 

Hence,  $\phi^a \cap \phi^b \subseteq \phi^{a \lor b}$ .....(1)

To prove the converse inclusion, let  $(x, y) \in \phi^{a \vee b}$ . Then, by definition2.2.11 (2) this gives the following conditions:

$$(a \lor b) \land x = (a \lor b) \land y \text{ and } (a \lor b) \land x^{\circ} = (a \lor b) \land y^{\circ}.$$
  
So that  $a \land x = (a \land x) \land [(a \land x) \lor (b \land x)] = (a \land x) \land [(a \lor b) \land x]$ 
$$= (a \land x) \land [(a \lor b) \land y]$$
$$= a \land x \land (a \lor b) \land y$$
$$= a \land (a \lor b) \land x \land y$$
$$= a \land x \land y \dots (EQ1)$$
Also,  $a \land y = [(a \land y) \lor (b \land y)] \land (a \land y) = [(a \lor b) \land y] \land (a \land y)$ 
$$= [(a \lor b) \land x] \land (a \land y)$$
$$= (a \lor b) \land x \land a \land y$$
$$= (a \lor b) \land x \land a \land y$$
$$= (a \lor b) \land a \land x \land y$$
$$= a \land x \land y \dots (EQ2)$$

Hence, (EQ1) and (EQ2) imply that  $a \land x = a \land y$ .

 $= a \wedge x^{\circ} \wedge y^{\circ}$ ....(EQ4)

Thus, (EQ3) and (EQ4) imply that  $a \wedge x^\circ = a \wedge y^\circ$ . So that  $(x, y) \in \phi^a$ .

Now, we remain to show that  $(x, y) \in \phi^b$ .

Since  $(a \lor b) \land x = (b \lor a) \land x$ , by interchanging *a* and *b* each other and by repeating the above argument, we can show that  $(x, y) \in \phi^b$  as follows:

Let  $(x, y) \in \phi^{a \lor b}$ . Then, by definition 2.2.11 (2) this gives the following conditions:

 $(b \lor a) \land x = (b \lor a) \land y$  and  $(b \lor a) \land x^{\circ} = (b \lor a) \land y^{\circ}$ .

So that  $b \wedge x = (b \wedge x) \wedge [(b \wedge x) \vee (a \wedge x)] = (b \wedge x) \wedge [(b \vee a) \wedge x]$ 

$$= (b \land x) \land [(b \lor a) \land y]$$
$$= b \land x \land (b \lor a) \land y$$

$$= b \wedge (b \vee a) \wedge x \wedge y$$

 $= b \wedge x \wedge y \dots (EQ5)$ 

Also, 
$$b \wedge y = [(b \wedge y) \vee (a \wedge y)] \wedge (b \wedge y) = [(b \vee a) \wedge y] \wedge (b \wedge y)$$
  

$$= [(b \vee a) \wedge x] \wedge (b \wedge y)$$

$$= (b \vee a) \wedge x \wedge b \wedge y$$

$$= (b \vee a) \wedge b \wedge x \wedge y$$

$$= b \wedge x \wedge y......(EQ6)$$

Hence, (EQ5) and (EQ6) show that  $b \land x = b \land y$ .

Similarly, 
$$b \wedge x^{\circ} = (b \wedge x^{\circ}) \wedge [(b \wedge x^{\circ}) \vee (a \wedge x^{\circ})] = (b \wedge x^{\circ}) \wedge [(b \vee a) \wedge x^{\circ}]$$
  

$$= (b \wedge x^{\circ}) \wedge [(b \vee a) \wedge y^{\circ}]$$

$$= b \wedge x^{\circ} \wedge (b \vee a) \wedge y^{\circ}$$

$$= b \wedge (b \vee a) \wedge x^{\circ} \wedge y^{\circ}$$
(EQ7)

Thus, (EQ7) and (EQ8) show that  $b \wedge x^\circ = b \wedge y^\circ$ . So that  $(x, y) \in \phi^b$ .

Hence, we have  $\phi^{a \lor b} \subseteq \phi^a \cap \phi^b$  .....(2).

Therefore, the two inclusions (1) and (2) imply that  $\phi^a \cap \phi^b = \phi^{a \vee b}$ .

**Lemma 2.2.18** In an MS-ADL L, if *a* is maximal, then  $\phi^a = \Delta_L$  (the diagonal of L).

**Proof:**  $(x, y) \in \phi^a$ . Then,  $a \wedge x = a \wedge y$  and  $a \wedge x^\circ = a \wedge y^\circ$ .... by [Definition 2.2.11 (2)]. If *a* is maximal, then  $a \wedge x = a \wedge y$  and  $a \wedge x^\circ = a \wedge y^\circ$  are reduced to x = y and  $x^\circ = y^\circ$ .

This implies  $(x, y) \in \Delta_L$ . That is,  $\phi^a \subseteq \Delta_L$ .....(1)

Conversely, let  $(x, y) \in \Delta_L$ . Then, for any  $x, y \in L$ , we have x = y and  $x^\circ = y^\circ$ .

This implies that  $(x, y) \in \phi^a$ . So that  $\Delta_L \subseteq \phi^a$ .....(2)

Hence, (1) and (2) imply that  $\phi^a = \Delta_L$ .

Therefore, if *a* is maximal, then  $\phi^a = \Delta_L$ .

**Note that:** The converse of this lemma need not necessarily be true. This can be verified by using the following example:

**Example 2.2.19** In an MS-ADL given in example 2.1.5,  $\phi^a = \Delta_L$  (the diagonal of L) but *a* is not maximal. Because from the given table it can be observed that  $a^\circ = a$ . That is;  $a^\circ \neq 0$ . Hence, *a* is not maximal.

**Lemma 2.2.20** If  $\phi^a = \Delta_L$ , then  $a \wedge a^\circ = a^\circ$ . That is,  $\phi^a = \Delta_L$  implies  $a^\circ \leq a$ .

**Proof:** It suffices to show that  $(a \land a^{\circ}, a^{\circ}) \in \phi^{a}$ . Since L is an MS-ADL with maximal element m, we have  $a^{\circ\circ} \land a = a$ , for all  $\in L$ ..... by definition2.1.1 of [MS-A1]. Let  $(x, y) \in \phi^{a}$ . Then  $a \land x = a \land y$  and  $a \land x^{\circ} = a \land y^{\circ}$ ..... by [Definition2.2.11 (2)]. So that by definition2.2.11 (2), we have the following results: (i).  $a \land (a \land a^{\circ}) = (a \land a) \land a^{\circ} = a \land a^{\circ}$ ...... [Since  $\land$  is associative]. (ii).  $a \land (a \land a^{\circ}) = a \land (a^{\circ} \land a)^{\circ}$ ...... by [lemma2.1.7 (7)]  $= a \land (a^{\circ\circ} \lor a^{\circ})$ ...... by definition2.1.1 of [MS-A3]  $= a^{\circ\circ} \land a \land (a^{\circ\circ} \lor a^{\circ})$ ...... by definition2.1.1 of [MS-A1]  $= a \land a^{\circ\circ} \land (a^{\circ\circ} \lor a^{\circ})$ ...... by [lemma1.2.2.6 (13)]  $= a \land a^{\circ\circ}$ ...... by [lemma1.2.2.6 (3)]

Hence,  $(a \wedge a^\circ, a^\circ) \in \phi^a$ .

Therefore,  $a \wedge a^{\circ} = a^{\circ}$  whenever  $\phi^{a} = \Delta_{L}$ .

**Definition 2.2.21** Let L be an MS-ADL and consider  $\theta_a$  be the binary relation on L. Then, for each  $a \in L$ , consider that  $(x, y) \in \theta_a$  if and only if  $a \lor x = a \lor y$  and  $a \lor x^\circ = a \lor y^\circ$ .

By this definition, we have the following characterization and results:

**Theorem 2.2.22** Let L be an MS-ADL. If L is V–associative, then the binary relation  $\theta_a$  is a congruence relation on L.

**Proof:** Suppose that L is V-associative. Since L is an MS-ADL with maximal element m, by definition2.1.1 of [MS-A1] we have  $a^{\circ\circ} \wedge a = a$ , for all  $a \in L$ . Equivalently;  $a^{\circ\circ} \vee a = a^{\circ\circ}$ . Let  $(x, y) \in \theta_a$ . Then,  $a \vee x = a \vee y$  and  $a \vee x^\circ = a \vee y^\circ$ ...... by [definition2.2.21] Also, let  $(c, d) \in \theta_a$ . Then,  $a \vee c = a \vee d$  and  $a \vee c^\circ = a \vee d^\circ$ ...... by [definition2.2.21] So that (i).  $a \vee (x \wedge c) = (a \vee x) \wedge (a \vee c) = (a \vee y) \wedge (a \vee d) = a \vee (y \wedge d)$ . Also,  $a \vee (x \vee c) = (a \vee x) \vee c = (a \vee y) \vee c = y \vee (a \vee c) = y \vee (a \vee d) = a \vee (y \vee d)$ . This implies that  $(x \wedge c, y \wedge d) \in \theta_a$  and  $(x \vee c, y \vee d) \in \theta_a$  for all  $x, y, c, d \in L$ . Hence,  $\theta_a$  is a congruence relation on the ADL (*L*, V,  $\wedge$ , 0).

(ii). Let  $(x, y) \in \theta_a$ . Then,  $a \lor x = a \lor y$  and  $a \lor x^\circ = a \lor y^\circ$  ...... by [definition2.2.21]

**Claim:**  $a \lor x^{\circ\circ} = a \lor y^{\circ\circ}$ . For this, we consider the following implications:

$$a \lor x = a \lor y \Longrightarrow (a \lor x)^{\circ\circ} = (a \lor y)^{\circ\circ}$$
  

$$\Rightarrow a^{\circ\circ} \lor x^{\circ\circ} = a^{\circ\circ} \lor y^{\circ\circ} \dots \text{ by [lemma2.1.7 (5)]}$$
  

$$\Rightarrow (a^{\circ\circ} \lor a) \lor x^{\circ\circ} = (a^{\circ\circ} \lor a) \lor y^{\circ\circ} \dots \text{ [Since } a^{\circ\circ} = a^{\circ\circ} \lor a]$$
  

$$\Rightarrow a^{\circ\circ} \lor (a \lor x^{\circ\circ}) = a^{\circ\circ} \lor (a \lor y^{\circ\circ}) \dots \text{ [Since, L is V-associative]}$$
  

$$\Rightarrow a \lor x^{\circ\circ} = a \lor y^{\circ\circ}$$
  

$$\Rightarrow (x^{\circ}, y^{\circ}) \in \theta_{a}$$

Hence,  $\theta_a$  is a congruence relation on L.

**Lemma 2.2.23** Let L be an associative MS-ADL. If *a* is zero, then  $\theta_a = \Delta_L$ . But  $\theta_a = \Delta_L$  implies  $a \le a^\circ$ . That is,  $\theta_a = \Delta_L$  implies  $a \lor a^\circ = a^\circ$ .

**Proof:** Let  $(x, y) \in \theta_a$ . Then,  $a \lor x = a \lor y$  and  $a \lor x^\circ = a \lor y^\circ$ ..... by [definition2.2.21]

If a = 0, then  $a \lor x = a \lor y$  and  $a \lor x^\circ = a \lor y^\circ$  are reduced to x = y and  $x^\circ = y^\circ$ .

So that  $(x, y) \in \Delta_L$ . Hence, if a = 0, then  $\theta_a = \Delta_L$ .

To prove the second part, suppose that  $\theta_a = \Delta_L$ .

**Claim:**  $(a \lor a^\circ, a^\circ) \in \theta_a$ .

Let L be an MS-ADL with maximal element m such that L is V- associative. So that we have the following results:

(i).  $a \lor (a \lor a^\circ) = (a \lor a) \lor a^\circ = a \lor a^\circ$  ..... [Since L is V-associative MS-ADL].

(ii).  $a \lor (a \lor a^\circ)^\circ = a \lor (a^\circ \land a^{\circ\circ}) = (a \lor a^\circ) \land (a \lor a^{\circ\circ})$ 

$$= m \wedge (a \vee a^{\circ}) \wedge (a \vee a^{\circ \circ})$$
$$= (a \vee a^{\circ}) \wedge m \wedge (a \vee a^{\circ \circ})$$
$$= [(a \wedge m) \vee (a^{\circ} \wedge m)] \wedge (a \vee a^{\circ \circ})$$

$$= (a^{\circ\circ} \lor a^{\circ}) \land (a \lor a^{\circ\circ})$$
$$= m \land (a \lor a^{\circ\circ})$$
$$= a \lor a^{\circ\circ}$$

So that  $(a \lor a^\circ, a^\circ) \in \theta_a$ .

Hence,  $\theta_a = \Delta_L$  implies  $a \le a^\circ$ . That is,  $\theta_a = \Delta_L$  implies  $a \lor a^\circ = a^\circ$ .

### Conclusion

In this project, we discussed the concept of an almost distributive lattice (ADL) which is a generalization of posets and lattice theory. Consequently, a new equational class of algebras called MS-ADL is understand as a common abstraction of De Morgan ADLs and Stone ADLs. It can also be observed that the class of MS-algebras and most of the properties of MS-algebras are extended to the class of MS-ADLs. Moreover, we discussed the congruence relations which characterize MS-algebras and MS-ADLs.

#### References

- G. C. Rao and V. Undurthi, Closure operators on complete almost distributive lattices-I, Int. J. Math. Arch. 5(6) (2014) 119–124.
- G. C. Rao and V. Undurthi, Closure operators on complete almost distributive lattices-II, Southeast Asian Bull. Math. 41 (2017) 91–100.
- G. C. Rao and V. Undurthi, Closure operators on complete almost distributive lattices-III, Bull. Sec. Logic 44 (2015) 81–93.
- 4. G. Gratzer, General Lattice Theory (Academic Press, New York, 1978).
- 5. G. Gratzer, Universal Algebras (Van Nostrand, Princeton, N. J., 1968).
- 6. G. M. Addis, De Morgan Almost Distributive Lattices, to appear. (The Recent Year).
- G.Birkhoff, Lattice Theory (American Mathematical Society Colloque Publications, 1967).
- J. Berman, Distributive lattices with an additional unary operation, an equation Math. 16 (1977) 165–171.
- R. Balbes and P. Dwinger, Distributive Lattices (University of Missouri Press, Columbia, Missouri, 1974).
- T. S. Blyth and J. C. Varlet, Ockham Algebras (Oxford University Press, London, UK, 1994).
- T. S. Blyth and J. C. Varlet, On a common abstraction of De Morgan algebras and Stone algebras, Proc. Roy. Soc. Edinburgh 94 (1983) 301–308.
- T. S. Blyth and J. C. Varlet, Subvarieties of the class of MS-algebras, Proc. Roy. Soc. Edinburgh 95 (1983) 157–169.
- 13. U. M. Swamy and G. C. Rao, ADL, J. Aust. Math. Soc.5 (section -A) 31 (1981) 77-91.
- U. M. Swamy and S. Ramesh, Birkhoff center of an almost distributive lattice, Int. J. Algebra 11 (2009) 539–346.
- 15. U. M. Swamy, G. C. Rao and G. N. Rao, On characterizations of stone almost distributive lattices, Southeast Asian Bull. Math. 32 (2008) 1167–1176.
- U. M. Swamy, G. C. Rao and G. N. Rao, Pseudo-complementation on almost distributive lattices, Southeast Asian Bull. Math. 24 (2000) 95–104.
- U. M. Swamy, G. C. Rao and G. N. Rao, Stone almost distributive lattices, Southeast Asian Bull. Math. 27 (2003) 513–526.
- V. Glivenko, Sur quelques points de la logique de Brouwer, Bull. Acad. Sci. Belg. (1929) 183–188.