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þÿMS Almost Distributive Lattic

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BAHIR DAR UNIVERSITY

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DEPARTMENT OF MATHEMATICS

A PROJECT ON

MS-ALMOST DISRIBUTIVE LATTICES (MS-ADL)

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A project on

MS –Almost Distributive Lattice (MS-ADL)

A project submitted to the department of mathematics in the Partial fulfillment of the requirements for the degree of "Master of Science in Mathematics".

By

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I here by certify that I have supervised, read and evaluated this project entitled "MS-Almost Distributive Lattice" by Aragaw Alie prepared under my guidance. I recommend that the project is submitted to oral defense.

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We here by certify that we have examined this project entitled "MS–Almost Distributive Lattice(MS-ADL)" by Aragaw Alie and We Recommend that Aragaw Alie is Approved for the Degree of "Master of Science in Mathematics".

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Abstract

In this poject, we understand the new equational class of algebra which we call MS-almost distributive lattice (MS-ADL) as a common abstraction of De Morgan ADLs and Stone ADLs. We observed that the class of MS-ADLs properly contain the class of MS-algebras and most of the properties of MS-algebras are extended to the class of MS-ADL. The main objective of this project is to develop a better understanding of the concept of MS-ADLs. Moreover, in this project we observed some basic properties, state and prove basic theorems, lemmas and corollaries related to MS-ADL.

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CHAPTER–ONE

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

The term lattice is one of the fundamental algebraic structures used in an abstract algebra as mathematical disciplines of order theory [4]. It consists of a partially order set P together with a binary relation " $\leq$ " in which every two elements have a unique supremum or a least upper bound called join($\vee$) and a unique infimum or a greatest lower bound called meet($\wedge$). The lattice structure $(L, \vee, \wedge)$ and the operation called complementation (*) together with the nullary operations 0 and 1 gives another algebraic structure $\{B, \vee, \wedge, *, 0, 1\}$ is called Boolean algebra. In $\{B, \vee, \wedge, *, 0, 1\}$, $\vee$ *and* $\wedge$ are binary operations and * is a unary operation. The general lattice theory was developed into another abstract structure called Almost Distributive Lattice (ADL) [13]. The concept of an almost distributive lattice (abbreviated as ADL) was introduced by U.M Swamy and G.C Rao [13] as a common abstraction of most of the existing ring theoretic and lattice theoretic generalization of a Boolean algebra and Boolean rings. An ADL is an algebra with two binary operations "$\vee$" and "$\wedge$" which satisfies most of the properties of a distributive lattice with smallest element 0, except possibly the commutativity of the binary operations "$\vee$" and "$\wedge$" , and the right distributivity of " $\vee$ " over " $\wedge$ ". The class of ADLs with pseudo-complementation was introduced in[16]. Later on, Swamy et. al.[17] introduced a more general class of ADLs called a Stone ADLs, which properly contains the class of pseudo-complemented ADLs. An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contain the well-known classes of algebras; for example Boolean algebras, De Morgan algebras, Kleene algebras and stone algebras [10]. Blyth and Varlet[11] defined a subclass of Ockham algebras so called MS-algebras which generalizes both De Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by Berman[8]. The classes of MS-algebras form an equational class. Blyth and Varlet characterized the sub-varieties of MS-algebras in [12]. More recently, in the paper [6], the author defines De Morgan ADLs as a generalization of De Morgan algebras. In this project, we define a new equational class of algebras called MS-ADL as a common abstraction of De Morgan ADL and Stone ADL. The class of MS-ADL properly contains the class of MS-algebras and most of the properties of MS-algebras are extended to the class of MS-ADLs.

1.2 preliminaries

This section contains some necessary definitions and results which will be used in the project.

1.2.1 Partially Ordered Sets and Lattice Theory

Definition 1.2.1.1[4, 5] A partially ordered set (abbreviated as Poset) or simply an ordered set is an algebraic system (P, ≤) where P is a non-empty set with a binary relation ≤ on P which satisfies the following set of axioms:

[P1]: Reflexive law: $a \leq a$, for all $a \in P$.

[P2]: Anti-symmetric law: $a \leq b$ and $b \leq a$ implies that $a = b$ for all $a , b \in P$.

[P3]: Transitive law: $a \leq b$ and $b \leq c$ implies that $a \leq c$ for all $a , b , c \in P$.

If (P, ≤) is a Poset and every two elements of P are comparable (in the sense that either $x \leq y$ or $y \leq x$, for all $x, y \in P$), then P is called a totally ordered set (also called a chain). The binary relation ≤ is called a total order or a linear order.

Example 1.2.1.2 Let S be a non-empty set. Then, $(P(S) , \subseteq)$ is a Poset. If the quotient relation $a|b$ defines a divides b, for all $a, b \in P$), then $(\mathbb{Z} , |)$ defines a Poset. Also, since every two pairs of integers are comparable with respect to ≤ , the algebraic system $(\mathbb{Z} , \leq)$ is a totally ordered set or a chain.

Definition 1.2.1.3[4,7] A lattice is an algebra $(L, \vee, \wedge)$ of type (2, 2) where L is a non-empty set with two binary operations join(∨) and meet(∧), satisfying the following axioms:

[L1]: Commutative law: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for all $a , b \in$ L.

[L2]: Associative law: $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ for all $a, b, c \in$ L.

[L3]: Idempotent law: $a \vee a = a$ and $a \wedge a = a$ for all $a \in$ L.

[L4]: Absorption law: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ for all $a , b \in$ L.

A lattice L is a special type of Poset $(P, \leq)$ in which every pair of elements in L has the least upper bound or a unique suppremum called join (∨) and the greatest lower bound or a unique infimum called meet (∧). Let L be a lattice under the ordering relation ≤. Then, we define $a \leq b$ if and only if $a \wedge b = a$ (or equivalently; $a \vee b = b$) for all $a, b \in L$.

Note that: The lattice operations meet (∧) and join (∨) are binary operations on L; which means they can be applied on any two pairs of elements in a lattice L.

Example 1.2.1.4 The natural example of lattice is the power set P(X) of a non-empty set X with set theoretic operations union and intersection. That is, (P(X), ∪, ∩) is a lattice.

Example 1.2.1.5 Let $L = \{1,2,3,5,30\}$ be a set. Define the binary operations ∨ and ∧ on L by; $a \vee b = LCM(a\,,b)$ and $a \wedge b = GCD(a\,,b)$ for all $a, b, \in L$. Then, (L ,∨, ∧) is a lattice.

Definition 1.2.1.6 [7] A lattice $(L, \vee, \wedge)$ is said to be a distributive lattice if join (∨) and meet (∧) are distributive over each other. That is;

(DL1): $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.

(DL2): $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.

Lemma 1.2.1.7[7] The axioms (DL1) and (DL2) of a distributive lattice are equivalent.

Definition 1.2.1.8 [13] A lattice L is said to be bounded if it has least element 0 and greatest element 1. That is, a lattice $(L,\vee, \wedge)$ is said to be bounded if $0 \leq x$ and $x \leq 1$, for all $x \in L$.

Definition 1.2.1.9 [13] A bounded lattice $(L,\vee, \wedge, 0, 1)$ in which (L, ∨, ∧) is a distributive lattice is called a bounded distributive lattice. Let $(L,\vee, \wedge, 0, 1)$ be a bounded distributive lattice and $a \in L$. Then, the complement of a is defined to be an element $a' \in L$ such that $a \wedge a' = 0$ and $a \vee a' = 1$.

Example 1.2.1.10 Let P(X) be the power set of a non-empty set X. Then, (P(X), ∪ ,∩ , ϕ , X) is a bounded distributive lattice.

Definition 1.2.1.11[4, 5] A Boolean algebra is an algebra (B, ∨, ∧ , ′, 0 , 1) where B is a non-empty set with two binary operations (∨ and ∧), one unary operation (complementation) and two nullary operations (0 and 1) which satisfies the following set of axioms:

[B1]: $(B, \vee, \wedge)$ is a distributive lattice……………….….. [Distributive law]

[B2]: $a \vee 0 = a = 0 \vee a$ and $a \wedge 1 = a = 1 \wedge a$…………. [Identity law]

[B3]: $a \vee a' = 1$ and $a \wedge a' = 0$………………………….. [Complement law]

[B4]: $a \vee 1 = 1 = 1 \vee a$ and $a \wedge 0 = 0 = 0 \wedge a$………… [Null law]

[B5]: $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$ …………. [De Morgan's law]

[B6]: $(a')' = a$……………………………………………. [Involution law]

Note that: By a Boolean algebra we mean complemented distributive lattice.

Example1.2.1.12 The algebraic system (P(X) ,∪ , ∩ , ' , ϕ , X) is a Boolean algebra.

1.2.2 Almost Distributive Lattice (ADL)

Definition 1.2.2.1[13] An almost distributive lattice with zero or simply an ADL is an algebra (L, ∨, ∧, 0) of type (2, 2, 0) which satisfies the following axioms; for all $a , b, c \in$ L.

[ADL1]: $a \vee 0 = a$…………………………….. (a join 0)

[ADL2]: $0 \wedge a = 0$…………………………….. (0 meet a)

[ADL3]: $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ ……….. (right distributivity of meet over join)

[ADL4]: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ ……….. (left distributivity of meet over join)

[ADL5]: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ ……..… (left distributivity of join over meet)

(ADL6): $(a \vee b) \wedge b = b$…………………….…. (absorption law)

Definition 1.2.2.2[13] Let (L, ∨, ∧, 0) be an ADL. For any $a, b \in$ L, we say that a is less than or equals to b written as a ≤ b and define a ≤ b if and only if $a \wedge b = a$ (or equivalently $a \vee b = b$). Then, the binary relation ≤ is called a partial ordering on L. Throughout this paper L denotes an ADL unless otherwise stated.

Definition 1.2.2.3 [13] An ADL $(L ,\vee, \wedge , 0)$ is said to be discrete if every non-zero element is maximal. That is, an ADL $= (L ,\vee, \wedge , 0)$ is called discrete if and only if $a \wedge b = b$ or $a \vee b = a$ for all $0 \neq a \in L$. Moreover, every discrete ADL is an associative.

Example 1.2.2.4[13] Let X be a non-empty set with a fixed arbitrarily chosen element 0 ∈ X. If for all $a, b \in$ X, we define the binary operations $\wedge$ *and* $\vee$ on X as follows:

$$a \wedge b = \begin{cases} 0 , & if\ a = 0 \\ b , & if\ a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b , & if\ a = 0 \\ a , & if\ a \neq 0 \end{cases}$$

Then, (X, ∨, ∧, 0) is an ADL with 0 as its zero element. This is also called a discrete ADL.

Example 1.2.2.5 [13] Every distributive lattice with zero (0) is an ADL.

Lemma 1.2.2.6 [13] Let $(L, \vee, \wedge, 0)$ be an ADL. Then, the following hold for all $a, b, c \in L$:

(1). $a \wedge a = a \; and \; a \vee a = a$

(2). $a \wedge 0 = 0 \; and \; 0 \vee a = a$

(3). $a \wedge (a \vee b) = a = (a \wedge b) \vee a$, $(a \wedge b) \vee b = b \; and \; (b \vee a) \wedge b = b$

(4). $(a \vee b) \wedge a = a \; and \; a \vee (b \wedge a) = a$

(5). $a \wedge b = a \Leftrightarrow a \vee b = b$ and $a \wedge b = b \Leftrightarrow a \vee b = a$

(6). $(a \wedge b) \wedge c = a \wedge (b \wedge c) \ldots \ldots$ ($\wedge$ is associative in L)

(7). $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

(8). $(a \vee b) \vee a = a \vee b = (b \vee a) \vee a$

(9). $a \wedge b \leq a, \; a \wedge b \leq b, \; a \leq a \vee b \; and \; b \leq a \vee b$

(10). $a \vee (a \wedge b) = a \; and \; (a \vee b) \wedge b = b$

(11). If $a \leq b, then \; a \wedge b = a = b \wedge a \; and \; a \vee b = b = b \vee a$

(12). $a \wedge b = \inf\{a, b\} \Leftrightarrow a \wedge b = b \wedge a \Leftrightarrow a \vee b = sup\{a, b\}$

(13). $(a \wedge b) \wedge c = (b \wedge a) \wedge c$ and $(a \vee b) \wedge c = (b \vee a) \wedge c$

(14). $If \; a \leq c \; and \; b \leq c$, $then \; a \wedge b = b \wedge a \; and \; a \vee b = b \vee a$.

Definition 1.2.2.7 [13] An ADL, L with 0 is said to be directed above (also called bounded above) if L has an upper bound. More precisely, an ADL $(L, \vee, \wedge, 0)$ is called directed above if for all $a \in L$, there exists an element $m \in L$ such that $a \leq m$.

Theorem 1.2.2.8 [13] Let $(L, \vee, \wedge, 0)$ be an ADL. Then, the following are equivalent.

(1). $(L, \vee, \wedge, 0)$ is a distributive lattice.

(2). $\vee$ is commutative.

(3). $\wedge$ is commutative.

(4). ∨ is right distributive over ∧ in L.

Definition 1.2.2.9[13] Let L be an ADL. Then, an element m of L is called a maximal element if it is maximal in the partially ordered set (L, ≤). That is, an element $m \in L$ (of an ADL) is called a maximal element in $(L, \leq)$ if for any $x \in L,\ m \leq x \Longrightarrow m = x$.

Notation: For a maximal element m in L, we write L_m to denote the closed interval [0, m]. It was observed in [13] that L_m is a bounded distributive lattice. Moreover, the members of L_m are characterized as follows: $L_m = \{a \wedge m : a \in L\}$.

Theorem 1.2.2.10 [13] Let L be an ADL and $m \in$ L. Then, the following are equivalent:

(1). m is maximal with respect to ≤ .

(2). $m \vee x = m$, for all $x \in$ L.

(3). $m \wedge x = x$, for all $x \in$ L.

(4). $x \vee m$ is maximal for all $x \in$ L.

Definition 1.2.2.11[13] Let L be an ADL. Then, A non-empty subset I of L is said to be an ideal of L if $a \vee b \in I \ and \ a \wedge x \in I$, for all $a , b \in I$ and for all $x \in L$. A non-empty subset F of L is said to be a filter of L if $a \wedge b \in F \ and \ x \vee a \in F$ for all $a , b \in F$ and for all $x \in L$.

Definition 1.2.2.12[13] Let $L \ and \ L'$ be any two ADLs. Then, a mapping $f: L \longmapsto L'$ is called an ADL homomorphism if $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$ and $f(0) = 0'$ for all $a , b \in L$. If $f: L \longmapsto L'$ is an isomorphism, then $L \ and \ L'$ are called isomorphic. The notation $L \cong L'$ can be read as L is isomorphic to L'.

Definition 1.2.2.13[13] An ADL $(L ,\vee, \wedge ,0)$ is called relatively complemented if the interval [0, b] is a Boolean algebra for all $b \in L$. A Boolean algebra is the algebra of a relatively complemented ADL with maximal elements.

Theorem 1.2.2.14[13] An ADL = $(L ,\vee, \wedge ,0)$ is said to be a relatively complemented if and only if for all $a , b \in L$ there exists a unique element in L denoted by a^b such that $a \wedge a^b = 0$ and $a \vee a^b = a \vee b$.

Definition 1.2.2.15 [16] Let $(L,\vee, \wedge, 0)$ be an ADL. Then, a unary operation $a \longmapsto a^*$ on L is called a pseudo-complementation, if for any $a , b \in$ L, it satisfies the following conditions:

[PC-1]: $a \wedge$ b = 0 $\Longrightarrow a^* \wedge b = b$

[PC-2]: $a \wedge a^*$ = 0

[PC-3]: $(a \vee b)^* = a^* \wedge b^*$

Then, the algebraic system $(L, \vee, \wedge, *, 0)$ in which every element has a pseudo-complement is called a pseudo-complemented ADL. By this definition, the unary operation $*$ is called a pseudo-complementation on L and a^* is called a pseudo-complementation of a in L. An element a of a pseudo-complemented ADL, L is called a dense element of L if $a^* = 0$.

Theorem 1.2.2.16[16] Let L be an ADL and $*$ be a pseudo-complementation on L. Then, the following conditions hold for all $a, b \in L$.

(1). 0^* is maximal.

(2). If a is maximal, then $a^* = 0$.

(3). $0^{**} = 0$

(4). $a^* \wedge a = 0$

(5). $a^{**} \wedge a = a$

(6). $a^{***} = a^*$

(7). $a \leq b \Longrightarrow b^* \leq a^*$

(8). $a^* \leq 0^*$

(9). $a^* \wedge b^* = b^* \wedge a^*$

(10). $(a \vee b)^* = (b \vee a)^*$

(11). $a^* \leq (a \wedge b)^*$ and $b^* \leq (a \wedge b)^*$

(12). $(a \vee b)^* \leq a^*$ and $(a \vee b)^* \leq b^*$

(13). $0^* \wedge a = a$

(14). $a^* = 0 \Longleftrightarrow a^{**}$ is maximal

(15). $a^* \leq b^* \Longleftrightarrow b^{**} \leq a^{**}$

(16). $a = 0 \Longleftrightarrow a^{**} = 0$

Definition1.2.2.17[15, 17] Let L be an ADL and $*$ be a pseudo-complementation on L. Then, L is said to be a Stone ADL if L is a pseudo-complemented ADL $(L, \vee, \wedge, *, 0)$ with a maximal element m which satisfies the condition; $a^* \vee a^{**} = 0^*$ for all $a \in L$.

Lemma1.2.2.18 [17] Let L be a Stone ADL. Then, the following conditions hold:

(1). $0^* \wedge a = a$

(2). $0^* \vee a = 0^*$

(3). $(a \wedge b)^* = a^* \vee b^*$

(4). $(a \wedge b)^{**} = a^{**} \wedge b^{**}$, for all $a, b \in L$.

Proof: (1). Let L be a stone ADL. Then by definition1.2.2.1 of [ADL3] it follows that

$$0^* \wedge a = (a^* \vee a^{**}) \wedge a = (a^* \wedge a) \vee (a^{**} \wedge a) = 0 \vee a = a \text{ for all } a \in L.$$

(2). Since 0 is the minimal element of L, it follows that $0 \leq a$ for all $a \in L$.

$\Longrightarrow 0 \wedge a = 0$ ……………………… by definition1.2.2.1 of [ADL2]

$\Longrightarrow\ 0^* \wedge a = a$ ……………………. by definition1.2.2.15 of [PC-1]

$\Longrightarrow 0^* \vee a = 0^*$…………………….. by [lemma1.2.2.6 (5)]

(3). It is known that $(a \wedge b) \wedge (a \wedge b)^* = 0$ for all $a , b \in L$ … by definition1.2.2.15 [PC-2]

Let $(a \wedge b)^* = x$. Then, $a \wedge b \wedge x = 0$…….. [Since $x^* \wedge x = 0$, by theorem1.2.2.16 (4)]

$\Longrightarrow a^* \wedge b \wedge x = b \wedge x$ …………………....... by definition1.2.2.15 of [PC-1]

Also , $a^{**} \wedge b \wedge x = b \wedge x$ …………………… by [theorem1.2.2.16 of (5)]

$\Longrightarrow a^* \wedge b \wedge x = 0 = b \wedge x$ ………………… by definition1.2.2.15 of [PC-1]

$\Longrightarrow\ a^* \wedge b \wedge x = a^{**} \wedge b \wedge x = 0$ ...………… by [step (4) and (5) above]

But , $(b \wedge a^{**}) \wedge x = (a^{**} \wedge b) \wedge x = 0$ …….. by [lemma1.2.2.6 (13)]

$\Longrightarrow\ b^* \wedge a^{**} \wedge x = a^{**} \wedge x$ ………………….. by definition1.2.2.15 of [PC-1]

$\Longrightarrow\ b^* \vee (a^{**} \wedge x) = b^*$……………………… by [lemma1.2.2.6 (5)]

Now , $\boldsymbol{a}^* \vee \boldsymbol{b}^* = a^* \vee [b^* \vee (a^{**} \wedge x)]$

$= a^* \vee [(b^* \vee a^{**}) \wedge (b^* \vee x)]$

$= a^* \vee (b^* \vee a^{**}) \wedge a^* \vee (b^* \vee x)$

$= [b^* \vee (a^* \vee a^{**})] \wedge\ a^* \vee (b^* \vee x)$ ……… by [lemma1.2.2.6 (8)]

$= (b^* \vee 0^*) \wedge\ a^* \vee (b^* \vee x)$……………… [Since L is a Stone ADL]

$= 0^* \wedge [\, a^* \vee (b^* \vee x)]$…………….......... [Since 0^* is maximal]

$= a^* \vee (b^* \vee x)$ ………………………….. [Since 0^* is maximal]

Thus , $(\boldsymbol{a}^* \vee \boldsymbol{b}^*) \wedge \boldsymbol{x} = [a^* \vee (b^* \vee x)] \wedge x$

$= (a^* \wedge x) \vee [(b^* \vee x) \wedge x]$………… by definition1.2.2.1 of [ADL3]

$= [a^* \wedge x] \vee x$……………………….. by definition1.2.2.1 of [ADL6]

$= x$………………………………... by [lemma1.2.2.6 (3)]

So that $(\boldsymbol{a} \wedge \boldsymbol{b})^* = \boldsymbol{x} = (a^* \vee b^*) \wedge x$

$= (a^* \wedge x) \vee (b^* \wedge x)$…….. by definition1.2.2.1 of [ADL3]

$= [a^* \vee (a \wedge b)^*] \vee [b^* \vee (a \wedge b)^*]$.............. [Since $x = (a \wedge b)^*$]

$= [a \vee (a \wedge b)]^* \vee [b \vee (a \wedge b)]^*$…… by definition1.2.2.15 of [PC-3]

$= [(a \vee a) \wedge (a \vee b)]^* \vee [(b \vee a) \wedge (b \vee b)]^*$….. by axiom [ADL5]

$= [a \wedge (a \vee b)]^* \vee [(b \vee a) \wedge b]^*$………………. by [lemma1.2.2.6 (1)]

$= a^* \vee b^*$……………………………………... by [lemma1.2.2.6 (3)]

Hence, $(a \wedge b)^* = a^* \vee b^*$, for all $a, b \in L$.

(4). $(a \wedge b)^{**} = (a^* \vee b^*)^* = a^{**} \wedge b^{**}$…….. by (3) above and definition1.2.2.15 of [PC-3]

Definition 1.2.2.19 [10-11] An MS-algebra is an algebra $(L, \vee, \wedge, \circ, 0, 1)$ of type (2, 2, 1, 0, 0) such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $x \longmapsto x^\circ$ on L satisfying the following set of axioms:

[M1]: $x \leq x^{\circ\circ}$ [M3]: $(x \vee y)^\circ = x^\circ \wedge y^\circ$

[M2]: $(x \wedge y)^\circ = x^\circ \vee y^\circ$ [M4]: $1^0 = 0$, for all $x, y \in L$.

Definition 1.2.2.20 [4, 5, 7, 9] A De Morgan algebra is an MS-algebra $(L, \vee, \wedge, \circ, 0, 1)$ of type (2, 2, 1, 0, 0) such that the unary operation $x \longmapsto x^\circ$ satisfying the following condition;

[M5]: $x^{\circ\circ} = x$, for all $x \in L$, which is called the involution law.

Lemma 1.2.2.21 [10,11] Let L be an MS-algebra. Then, the following hold for all $a, b \in L$:

(1). $0^\circ = 1$ (4). $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$

(2). $a \leq b \Longrightarrow b^\circ \leq a^\circ$ (5). $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$

(3). $a^{\circ\circ\circ} = a^\circ$

Proof: (1). Let L be an MS-algebra. This implies that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice with least element 0 and greatest element 1…………… by definition1.2.2.19.

This implies that $1^\circ = 0$ and $0^\circ = 1$………………………… by definition1.2.2.19 of [M4]

Hence, $0^\circ = 1$.

(2). Suppose that $a \le b$. This implies that $a \wedge b = a$ (or equivalently; $a \vee b = b$).

Then, by definition1.2.2.19 of [M2] it follows that $a^\circ = (a \wedge b)^\circ = a^\circ \vee b^\circ$.

Hence, $b^\circ \le a^\circ$ for all $a, b \in L$………………………………………… by definition1.2.2.2

(3). Let L be an MS-algebra. Then, $a \le a^{\circ\circ}$ for all $a \in L$……. by definition1.2.2.19 of [M1].

So that by lemma1.2.2.21 of (2) above, this implies that $a^{\circ\circ\circ} \le a^\circ$…………...….. $(*)$

Also, since L is an MS-algebra, by a similar argument of definition1.2.2.19 of [M1], for any $a \in L$, it follows that $a^\circ \le a^{\circ\circ\circ}$ …….……………………….....………………….. $(**)$

Hence, $(*)$ and $(**)$ imply that $a^{\circ\circ\circ} = a^\circ$, for all $a \in L$.

(4). $(a \vee b)^{\circ\circ} = (a^\circ \wedge b^\circ)^\circ = a^{\circ\circ} \vee b^{\circ\circ}$………… by definition1.2.2.19 of [M3] and [M2]

(5). $(a \wedge b)^{\circ\circ} = (a^\circ \vee b^\circ)^\circ = a^{\circ\circ} \wedge b^{\circ\circ}$…………. by definition1.2.2.19 of [M2] and [M3]

Definition 1.2.2.22[13]. Let A be a non-empty set and θ be a binary relation on A ($\theta \subseteq A \times A$). Then, θ is said to be an equivalence relation on A if θ satisfies the following axioms:

[1]. Reflexive law: $(a, a) \in \theta$ for all $a \in$ A.

[2]. Symmetric law: $(a, b) \in \theta$ implies that $(b, a) \in \theta$ for all $a \in$ A.

[3]. Transitive law: $(a, b) \in \theta$ and $(b, c) \in \theta$ implies that $(a, c) \in \theta$ for all $a, b, c \in$ A.

Definition 1.2.2.23[13] Let L be an ADL. Then, an equivalence relation θ on L is said to be a congruence relation on L if $(a, b), (x, y) \in \theta \Longrightarrow (a \wedge x, b \wedge y), (a \vee x, b \vee y) \in \theta$, for all $a, b, x, y \in L$.

Remark: For any congruence relation θ on L and $a \in L$, we define the congruence class $[a]_\theta = \{b \in L : (a, b) \in \theta\}$ and it is called the congruence class containing a.

CHAPTER-TWO

MS-ALMOST DISTRIBUTIVE LATTICES (MS-ADL)

2.1 Definitions, Examples and Theorems on (MS-ADL)

In this section, we define MS-ADLs and investigate some of their properties with examples.

Definition 2.1.1[6] An MS-almost distributive lattice (abbreviated as MS-ADL) is an algebra $(L, \vee, \wedge, \circ, 0)$ of type (2, 2, 1, 0) such that $(L, \vee, \wedge, 0)$ is an ADL with a maximal element m and a unary operation $x \longmapsto x^\circ$ on L which satisfies the following axioms; for all $x, y \in$ L.

[MS-A1]: $x^{\circ\circ} \wedge x = x$

[MS-A2]: $(x \vee y)^\circ = x^\circ \wedge y^\circ$

[MS-A3]: $(x \wedge y)^\circ = x^\circ \vee y^\circ$

[MS-A4]: $m^\circ = 0$, for all maximal elements m of L.

Definition 2.1.2[6] An MS-ADL $(L, \vee, \wedge, \circ, 0)$ of type (2, 2, 1, 0) satisfying the condition [MS-A5]: $x^{\circ\circ} = x \wedge m$, is called a De Morgan ADL, for all maximal element m of MS-ADL.

Example 2.1.3 Let $(L, \vee, \wedge, 0)$ be a discrete ADL with at least two elements. Choose a non-zero element m $\in L$ and define a unary operation $x \longmapsto x^\circ$ on L as follows:

$$a^\circ = \begin{cases} m, & if\ a = 0 \\ 0, & otherwise \end{cases}, \text{ for all } a \in L.$$

Then, $(L, \vee, \wedge, \circ, 0)$ is an MS-ADL and it is called the discrete MS-ADL.

Example 2.1.4 Let $L = \{0, a, b, c\}$. Define the two binary operations $\vee$ *and* $\wedge$ on L by the following tables:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	b
c	c	c	c	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c
c	0	a	b	c

Then, $(L, \vee, \wedge, 0)$ is an ADL which is neither a distributive lattice nor a discrete ADL. But if we define a unary operation $x \longmapsto x^{\circ}$ on L as follows:

x	x°
0	b
a	0
b	0
c	0

Then, $(L, \vee, \wedge, \circ, 0)$ is an MS-ADL, which is also a De Morgan ADL. Moreover, if we define another unary operation $x \longmapsto x^{*}$ on L as follows:

x	x^{*}
0	b
a	a
b	0
c	0

Then, (L ,$\vee$, $\wedge$, $*$,0) is an MS-ADL but not a De Morgan ADL. Since b and c are maximal elements of L as given from the table above, by definition2.1.2 of [MS-A5], it follows that $b^{**} = b \wedge m = b \wedge c = c \neq b$. Hence, L is not a De Morgan ADL.

Example 2.1.5 Let $L = \{0, a, b, c, d\}$ and define binary operations $\vee$ and $\wedge$ on L as follows:

$\vee$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	c	c
b	b	b	b	d	d
c	c	c	c	c	c
d	d	d	d	d	d

$\wedge$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	a	b
b	0	a	b	a	b
c	0	a	b	c	d
d	0	a	b	c	d

Then, $(L, \vee, \wedge, 0)$ is an ADL with maximal elements c and d, which is neither a distributive lattice nor a discrete ADL. But if we define a unary operation $x \longmapsto x^{\circ}$ on L as follows:

x	x°
0	c
a	a
b	a
c	0
d	0

Then, $(L, \vee, \wedge, \circ, 0)$ becomes an MS-ADL.

Example 2.1.6 Let $(L, \vee, \wedge, \circ, 0, m)$ be an MS-ADL and X be a non empty set. If L^X denotes the class of all functions from X to L, then $(L^X, \vee, \wedge, \circ, 0_x, m_x)$ is an MS-ADL where $\vee, \wedge, \circ, 0_x$ *and* m_x are defined on L^X as follows:

(1). $(f \vee g)(x) = f(x) \vee g(x)$

(2). $(f \wedge g)(x) = f(x) \wedge g(x)$

(3). $f^{\circ}(x) = (f(x))^{\circ}$ and $0_x(x) = 0$, $m_x(x) = m$ for all $x \in X$.

Proof: For each functions $f, g \in L^X$ *and each* $x \in X$, we have the following results:

(i). $f(x) = f(x^{\circ\circ} \wedge x) = f^{\circ\circ}(x) \wedge f(x) = (f(x))^{\circ\circ} \wedge f(x)$...by definition2.1.1 of [MS-A1]

(ii). $(f \vee g)^{\circ}(x) = [f(x) \vee g(x)]^{\circ} = (f(x))^{\circ} \wedge (g(x))^{\circ} = f^{\circ}(x) \wedge g^{\circ}(x)$

(iii). $(f \wedge g)^{\circ}(x) = [f(x) \wedge g(x)]^{\circ} = \left(f(x)\right)^{\circ} \vee \left(g(x)\right)^{\circ} = f^{\circ}(x) \vee g^{\circ}(x)$

(iv). $m_x{}^{\circ}(x) = m^{\circ} = 0 = 0_x(x)$.................................. by definition2.1.1 of [MS-A4]

Hence, $(L^X, \vee, \wedge, \circ, 0_x, m_x)$ is an MS-ADL whenever $(L, \vee, \wedge, \circ, 0, m)$ is an MS-ADL

Lemma 2.1.7 [6] The following conditions hold in an MS-ADL with maximal element m.

(1). 0° is maximal.

(2). $a \leq b \Longrightarrow b^{\circ} \leq a^{\circ}$

(5). $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$

(6). $(a \wedge m)^{\circ} = a^{\circ}$

(3). $a^{\circ\circ\circ} = a^{\circ}$

(4). $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$

(7). $(a \wedge b)^{\circ} = (b \wedge a)^{\circ}$, for all $a, b \in L$.

(8). $n^{\circ} = 0$, for all maximal elements n of L.

Proof: (1). Let L be an MS-ADL with a maximal element m. Since 0 is the minimal element in L, this gives $0 \leq m$. This implies $0 = 0 \wedge m$. Put $m^{\circ} = 0$… by definition2.1.1 of [MS-A4]

So that $0^{\circ} = (0 \wedge m)^{\circ} = 0^{\circ} \vee m^{\circ}$ ..…………………………. by definition2.1.1 of [MS-A3]

This implies that $m^{\circ} \leq 0^{\circ}$…………………………………... by definition1.2.2.2

Hence, 0° is maximal.

Moreover, if L is a De Morgan ADL with maximal element m of L, then by definition2.1.2 of [MS-A5] it follows that $x^{\circ\circ} = x \wedge m$. Put $m^{\circ} = 0$ and this gives that $0 = 0 \wedge 0 = m^{\circ} \wedge 0$. So that $0^{\circ} = (m^{\circ} \wedge 0)^{\circ} = m^{\circ\circ} \vee 0^{\circ}$………………………… by definition2.1.1 of [MS-A3]

$= (m \wedge m) \vee 0^{\circ}$…………………….. by definition2.1.2 of [MS-A5]

$= m \vee 0^{\circ}$…………………………… [Since $m \wedge m = m$]

$= m$…………….…………………… [Since m is maximal].

Hence, 0° is also maximal whenever L is a De Morgan ADL.

(2). Suppose that $a \leq b$. This implies that $a \wedge b = a$ (or equivalently; $a \vee b = b$). Then, by definition 2.1.1 of [MS-A3] it follows that $a^{\circ} = (a \wedge b)^{\circ} = a^{\circ} \vee b^{\circ}$. So that $b^{\circ} \leq a^{\circ}$.

Hence, $a \leq b$ implies $b^{\circ} \leq a^{\circ}$ for all $a, b \in L$.

(3). Let L be an MS-ADL. Then, $a^{\circ\circ} \wedge a = a$ for all $a \in L$…. by definition2.1.1 of [MS-A1] This implies that $(a^{\circ\circ} \wedge a)^{\circ} = a^{\circ}$. This gives $a^{\circ\circ\circ} \vee a^{\circ} = a^{\circ}$…. by definition2.1.1 of [MS-A3]

So that by definition1.2.2.2, This shows that $a^{\circ\circ\circ} \leq a^{\circ}$ for all $a \in L$.……………………. (1)

Also, since L is an MS-ADL with maximal element m, by a similar argument of definition 2.1.1 [MS-A1] this gives that $a^{\circ\circ\circ} \wedge a^{\circ} = a^{\circ}$ for all $a \in L$. This implies that $a^{\circ} = a^{\circ\circ\circ} \wedge a^{\circ} = a^{\circ} \wedge a^{\circ\circ\circ} \wedge m = a^{\circ} \wedge a^{\circ\circ\circ} \wedge m = a^{\circ} \wedge a^{\circ\circ\circ}$………………………….. by [lemma1.2.2.6 (13)]

So that by definition1.2.2.2, this gives that $a^{\circ} \leq a^{\circ\circ\circ}$ for all $a \in L$…………………….. (2)

Hence, (1) and (2) imply that $a^{\circ\circ\circ} = a^{\circ}$, for all $a \in L$.

Moreover, if L is a De Morgan ADL, then $a^{\circ\circ} = a \wedge m$…. by definition2.1.2 of [MS-A5].

So that $a^{\circ\circ\circ} = (a \wedge m)^{\circ} = a^{\circ} \vee m^{\circ} = a^{\circ} \vee 0$….. by definition2.1.1 of [MS-3] and [MS-A4]

$= a^{\circ}$ …….. by definition1.2.2.1 of [ADL1]

(4). $(a \wedge b)^{\circ\circ} = (a^{\circ} \vee b^{\circ})^{\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$ ………. by definition2.1.1 of [MS-A3] and [MS-A2].

(5). $(a \vee b)^{\circ\circ} = (a^{\circ} \wedge b^{\circ})^{\circ} = a^{\circ\circ} \vee b^{\circ\circ}$…….......... by definition2.1.1 of [MS-A2] and [MS-A3].

(6). $(a \wedge m)^{\circ} = a^{\circ} \vee m^{\circ} = a^{\circ} \vee 0$.................... by definitionn2.1.1 of [MS-A3] and [MS-A4]

$= a^{\circ}$………………… by definition1.2.2.1 of [ADL1]

(7). $(a \wedge b)^{\circ} = (a \wedge b \wedge m)^{\circ}$……………….... by [lemma2.1.7 (6)]

$= (b \wedge a \wedge m)^{\circ}$……………….... by [lemma1.2.2.6 (13)]

$= (b \wedge a)^{\circ}$…………………….... by [lemma2.1.7 (6)]

(8). Let n be the maximal element of L. Clearly, we have $n = n \vee a$ for all $a \in L$. So that

$n^{\circ} = (n \vee 0^{\circ})^{\circ} = n^{\circ} \wedge 0^{\circ\circ} = n^{\circ} \wedge 0 = 0$…… by definition2.1.1 of [MS-A2] and [MS-A4]

Hence, $n^{\circ} = 0$ for all maximal elements n of L.

Corollary 2.1.8 Let L be an MS-ADL. Then, $a^{\circ} = b^{\circ}$ if and only if $(a \wedge m)^{\circ} = (b \wedge m)^{\circ}$. But if L is a De Morgan ADL, then $a^{\circ} = b^{\circ}$ if and only if $a \wedge m = b \wedge m$ for all $a, b \in L$.

Proof: let L be an MS-ADL with maximal element m such that $a^{\circ} = b^{\circ}$**.** Then, for any $a, b \in L$ we have $(a \wedge m)^{\circ} = a^{\circ} = b^{\circ} = (b \wedge m)^{\circ}$ ...…………….. by [lemma2.1.7 (6)]

Conversely, suppose that $(a \wedge m)^{\circ} = (b \wedge m)^{\circ}$. This is equivalent to $a^{\circ} \vee m^{\circ} = b^{\circ} \vee m^{\circ}$. Since $m^{\circ} = 0$, this implies that $a^{\circ} \vee 0 = b^{\circ} \vee 0$. This gives that $a^{\circ} = b^{\circ}$ for all $a, b \in L$.

To prove the second part, let L be a De Morgan ADL such that $a^{\circ} = b^{\circ}$**.** This implies that $a^{\circ\circ} = b^{\circ\circ}$. So that $a \wedge m = a^{\circ\circ} = b^{\circ\circ} = b \wedge m$………….. by definition2.1.2 of [MS-A5]

Conversely, suppose $a \wedge m = b \wedge m$. Then, $a^{\circ} = a^{\circ\circ\circ} = (a \wedge m)^{\circ} = (b \wedge m)^{\circ} = b^{\circ\circ\circ} = b^{\circ}$. Hence, the equivalences hold for all $a, b \in L$.

Theorem 2.1.9 Any relatively complemented ADL with a fixed maximal element m of L can be made into an MS-ADL by defining the unary operation $x \longmapsto x^{\circ}$ on L as follows:

$$a^{\circ} = a^{m}, \text{ for all } a \in L.$$

Proof: Suppose that L is a relatively complemented ADL with maximal elements m. Then, for each $a, b \in L$ there exists a unique element in L denoted by a^{b} such that $a \wedge a^{b} = 0$ and $a \vee a^{b} = a \vee b$. Choose a maximal element $m \in$ L and define a unary operation $a \longmapsto a^{\circ}$ on L by; $a^{\circ} = a^{m}$ for all $a \in$ L. Then, by definition1.2.2.8 and lemma1.2.2.6 (13) it follows that

$$a^{m} \wedge (a \wedge m) = a^{\circ} \wedge (a \wedge m) = (a^{\circ} \wedge a) \wedge m = (a \wedge a^{\circ}) \wedge m = 0 \wedge m = 0.$$

Also, $a^{m} \vee (a \wedge m) = (a^{m} \vee a) \wedge (a^{m} \vee m) = (a^{m} \vee a) \wedge m = m = a^{m} \vee m$.

Moreover, by definition 2.1.1 of [MS-A2] and [MS-A3] we have $(a \vee b)^{m} = (a \vee b)^{\circ} = a^{\circ} \wedge b^{\circ} = a^{m} \wedge b^{m}$ and $(a \wedge b)^{m} = (a \wedge b)^{\circ} = a^{\circ} \vee b^{\circ} = a^{m} \vee b^{m}$ for all $a, b \in L$.

Hence, any relatively complemented ADL together with maximal element m and the unary operation $x \longmapsto x^{\circ}$ is an MS-ADL.

Similarly, one can easily verify that the unary operation $x \longmapsto x^{*}$ which makes an ADL, L a Stone ADL that respects all the axioms of MS-ADL. So that every Stone ADL is an MS-ADL. The following example presents a natural way to obtain an MS-ADL from De Morgan ADL and Stone ADL.

Example 2.1.10 If D is a De Morgan ADL and S is a Stone ADL, then D×S is an MS-ADL such that the unary operation $\circ$ on D×S is defined by; $(x, y)^{\circ} = (\bar{x}, y^{*})$ for all $x \in$ D and for all $y \in$ S, where $x \longmapsto \bar{x}$ is the unary operation on D and $y \longmapsto y^{*}$ is the unary operation on S.

Theorem 2.1.11 Let D be a De Morgan ADL and S be a Stone ADL. Then,

(1). D×S is a De Morgan ADL if and only if S is relatively complemented.

(2). D×S is a Stone ADL if and only if D is a relatively complemented.

Proof: (1). ($\Longrightarrow$). Suppose D×S is a De Morgan ADL. Then, by definition2.1.2 of [MS-A5] for every $x \in$ D and $y \in$ S we have $(x, y)^{\circ\circ} = (x, y) \wedge (m, n)$, where m and n are maximal elements in D and S respectively. So that by definition2.1.2 of axiom [MS-A5] this gives that $y^{**} = y \wedge n$, for all $y \in$ S. In this case, the maximal element n of S is precisely 0^{*}.

Claim: S is relatively complemented.

Let $a\,,b \in$ S. $Put\ x = \ a^{*} \wedge b$. Then, we have the following results:

(i). $a \wedge x = a \wedge (a^{*} \wedge b) = (a \wedge a^{*}) \wedge b = 0 \wedge b = 0$.

(ii). $a \vee x = a \vee (a^{*} \wedge b) = (a \vee a^{*}) \wedge (a \vee b)$…………….. by definition1.2.2.1 [ADL5]

$= 0^{*} \wedge (a \vee a^{*}) \wedge (a \vee b)$ ………. [Since 0^{*} is maximal in S]

$= (a \vee a^{*}) \wedge 0^{*} \wedge (a \vee b)$ ……….. by [lemma1.2.2.6 (13)]

$= [(a \wedge 0^{*}) \vee (a^{*} \wedge 0^{*})] \wedge (a \vee b)$…. by definition1.2.2.1 [ADL3]

$= (a^{**} \vee a^{*}) \wedge (a \vee b)$

$= 0^{*} \wedge (a \vee b)$……………………….. [Since S is Stone ADL]

$= a \vee b$……………………………….. [Since 0^{*} is maximal]

Hence, S is relatively complemented.

Conversely, suppose that S is relatively complemented. Then, for any $a\,,b \in$ S, there exists a unique element in S denoted by a^{b} such that $a \wedge a^{b} = 0$ and $a \vee a^{b} = a \vee b$. Hence, the maximal element $n = 0^{*}$ exists in S. Also, since S is a Stone ADL, there is a unary operation $y \longmapsto y^{*}$ on S defined by; $y^{*} \vee y^{**} = 0^{*}$ for all $y \in$ S. This is equivalent to that $y^{*} \wedge y = 0$.

Again, since D is a De Morgan ADL, there is a unary operation $x \longmapsto \bar{x}$ on D such that $\bar{\bar{x}} = x \wedge m$, for all $x \in$ D and for all maximal element m of D. So that $D \times S = (x\,,y)^{\circ\circ} = (x\,,y) \wedge (m\,,n)$, which defines a De Morgan ADL…….. by definition2.1.2 of [MS-A5].

Therefore, D×S is a De Morgan ADL, for all $x \in$ D and for all $y \in$ S.

(2). (⟹). Suppose that D×S is a Stone ADL. Then, by definition1.2.2.15 of [PC-2] it follows that $(x\,,y)^{\circ} \wedge (x\,,y) = (0\,,0)$ for all $x \in$ D and $y \in$ S. This gives; $\bar{x} \wedge x = 0$ for all $x \in$ D. This is equivalent to; $\bar{x} \vee \bar{\bar{x}} = \bar{0}$…………………….. by definition2.1.1 of axiom [MS-A3].

Claim: D is relatively complemented

Let $a\,,b \in$ D. $Put\ x = \bar{a} \wedge b$. Then, we have the following results:

(i). $a \wedge x = a \wedge (\bar{a} \wedge b) = (a \wedge \bar{a}) \wedge b = 0 \wedge b = 0$.

(ii). $a \vee x = a \vee (\bar{a} \wedge b) = (a \vee \bar{a}) \wedge (a \vee b) = \bar{0} \wedge (a \vee \bar{a}) \wedge (a \vee b)$

$$= (a \vee \bar{a}) \wedge \bar{0} \wedge (a \vee b)$$

$$= (a \wedge \bar{0}) \vee (\bar{a} \wedge \bar{0}) \wedge (a \vee b)$$

$$= (\bar{\bar{a}} \vee \bar{a}) \wedge (a \vee b)$$

$$= \bar{0} \wedge (a \vee b)$$

$$= (a \vee b)$$

Therefore, D is relatively complemented.

Conversely; Suppose D is relatively complemented. Then, for any $a, b \in$ D there exists a unique element in D denoted by a^b such that $a \wedge a^b = 0$ and $a \vee a^b = a \vee b$. Also, since S is a Stone ADL, there is a unary operation $y \longmapsto y^*$ on S defined by; $y^* \vee y^{**} = 0^*$, for all $y \in$ S. This is equivalent to $y^* \wedge y = 0$ .... by definition2.1.1 of [MS-A3]

So that D × S $= (x, y)^\circ \wedge (x, y) = (0, 0)$…………………. by definition1.2.2.15 of [PC-2].

Therefore, D×S is a Stone ADL, for all $x \in$ D and for all $y \in$ S.

Note that: This theorem confirms that the class of De Morgan ADLs and the class of Stone ADLs are proper subclasses of the class of MS-ADLs.

The following theorem shows that a set of necessary and sufficient conditions for which an MS-ADL to be an MS-algebra.

Theorem 2.1.12 Let $(L, \vee, \wedge, \circ, 0)$ be an MS-ADL. Then the following are equivalent:

(1). L is an MS-algebra.

(2). The Poset (L, ≤) is directed above (bounded above).

(3). (L, ∨, ∧ , 0) is a distributive lattice.

(4). ∨ is commutative.

(5). ∧ is commutative.

(6). ∨ is right distributive over meet ∧ in L.

(7). The relation $\theta = \{(a,b) \in L \times L: b \wedge a = a\}$ is anti-symmetric.

(8). For each $a \in L$, the relation ϕ_a given by;

$(x,y) \in \phi_a$ if and only if $x \vee a = y \vee a$ and $x^\circ \vee a = y^\circ \vee a$ is a congruence relation on L.

Proof: (1) $\Longrightarrow$ **(2)**. Suppose L is an MS-algebra. Since L is an MS-ADL with maximal element m, for any $a, b \in L$ we have $a \leq m$ and $b \leq m$. This implies L has an upper bound m with respect to the partial ordering $\leq$. Hence, the Poset $(L, \leq)$ is directed above.

(2) $\Longrightarrow$ **(3)**. Suppose the Poset (L, ≤) is directed above (bounded above by m). Since L is an MS-ADL, we have a maximal element $m \in L$ such that $a \leq m$ and $b \leq m$ for all $a, b \in L$.

Claim: $(L, \vee, \wedge, 0)$ is a distributive lattice.

(i). $a \wedge (b \vee c) = [a \wedge (b \vee c)] \wedge m = [(a \wedge b) \vee (a \wedge c)] \wedge m = (a \wedge b) \vee (a \wedge c)$

(ii). $a \vee (b \wedge c) = [a \vee (b \wedge c)] \wedge m = [(a \vee b) \wedge (a \vee c)] \wedge m = (a \vee b) \wedge (a \vee c)$.

Therefore, $(L, \vee, \wedge, 0)$ is a distributive lattice.

(3) $\Longrightarrow$ **(4)**. Suppose $(L, \vee, \wedge, 0)$ is a distributive lattice.

Claim: $\vee$ is commutative.

$a \vee b = [a \wedge (b \vee a)] \vee [(b \wedge (b \vee a)]$……………………… by [lemma1.2.2.6 (3)]

$= (a \vee b) \wedge (b \vee a)$….. by definition1.2.2.1 of [ADL3]

$= [(a \vee b) \wedge b] \vee [(a \vee b) \wedge a]$……………………… by definition1.2.2.1 of [ADL4]

$= b \vee a$………………………………………………… by definition1.2.2.1 of [ADL6]

Hence, $\vee$ is commutative for all $a, b \in L$.

(4) $\Longrightarrow$ **(5)**. Suppose $\vee$ is commutative.

Claim: $\wedge$ is commutative.

$a \wedge b = (a \wedge b) \vee (a \wedge b) = [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee b]$….. by definition1.2.2.1 [ADL5]

$= [a \vee (a \wedge b)] \wedge [b \vee (a \wedge b)]$…… [Since $\vee$ is commutative]

$= \ (a \vee b) \wedge (a \wedge b)$…………….. by definition1.2.2.1 of [ADL5]

$= \ [(b \vee a) \wedge b] \wedge (a \wedge b) \wedge m$……… [Since m is maximal]

$= \ [(a \vee b) \wedge b] \wedge (a \wedge b) \wedge m$……… [Since ∨ is commutative]

$= \ b \wedge a \wedge b \wedge m$……………….. by definition1.2.2.1 of [ADL6]

$= \ b \wedge a \wedge m$

$= \ b \wedge a$

Hence, ∧ is commutative.

(5) $\Longrightarrow$ $(\mathbf{6})$. Suppose ∧ is commutative.

Claim: ∨ is right distributive over ∧. That is, $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$.

$(a \wedge b) \vee c$ = $[(b \wedge a) \vee c] = [c \vee (b \wedge a)] \wedge m$…………… [Since m is maximal]

= $[(c \vee b) \wedge (c \vee a)] \wedge m$……… by definition1.2.2.1 [ADL5]

= $(c \vee b) \wedge m \wedge (c \vee a) \wedge m$

= $(b \vee c) \wedge (a \vee c)$……… by [lemma1.2.2.6 (13)]

= $(a \vee c) \wedge (b \vee c)$..…… [Since ∧ is commutative]

Therefore, ∨ is right distributive over ∧ in L.

(6) $\Longrightarrow$ $(\mathbf{7})$. Suppose ∨ is right distributive over ∧. That is, $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$.

Claim: The relation defined by θ $=\{(a, b) \in L \times L\colon b \wedge a = a\}$ is anti-symmetric.

By a relation θ on L is anti-symmetric, we mean $(a\ , b) \in \theta$ and $(b, a) \in \theta \Longrightarrow a = b$.

Assume that $(a, b) \in \theta \Longrightarrow b \wedge a = a$ and $(b\ , a) \in \theta \Longrightarrow a \wedge b = b$. This is equivalent to $a \vee b = a$ and $b \vee a = b$. Then, since the relation θ on L is defined by; $b \wedge a = a$, we have $a = b \wedge a = b \wedge a \wedge m = a \wedge b \wedge m = b \wedge m = b$…… [Since m is maximal in L].

Therefore, the relation θ $=\{(a\ , b) \in L \times L\colon b \wedge a = a\}$ is anti-symmetric.

(7) $\Longrightarrow$ $(\mathbf{8})$. Suppose θ $=\{(a\ , b) \in L \times L\colon b \wedge a = a\}$ is anti-symmetric.

Claim: For each $a \in L$, the relation ϕ_a given by;

$(x, y) \in \phi_a$ if and only if $x \vee a = y \vee a$ and $x^\circ \vee a = y^\circ \vee a$ is a congruence relation on L.

Let L be an MS-ADL with maximal element m. A relation ϕ_a is called a congruence relation on L if ϕ_a is a congruence relation on the ADL, (L, ∨, ∧ ,0), that is $(a, b), (x, y) \in \phi_a \Longrightarrow (a \wedge x , b \wedge y), (a \vee x , b \vee y) \in \phi_a$ and $(x, y) \in \phi_a \Longrightarrow (x^\circ, y^\circ) \in \phi_a$, for all $a, b, x, y \in L$.

Given that the relation $\theta = \{(a , b) \in L \times L: b \wedge a = a\}$ is anti-symmetric.

Equivalently; this gives that $b \vee a = b$ for all $a , b \in L$.

Assume that $(x, y) \in \phi_a \Longrightarrow x \vee a = y \vee a$ and $x^\circ \vee a = y^\circ \vee a$. By the same argument assume that $(c, d) \in \phi_a \Longrightarrow c \vee a = d \vee a$ and $c^\circ \vee a = d^\circ \vee a$ for all $x, y, c, d \in L$.

Then, this gives that $(x \wedge c , y \wedge d) = (c, d) \in \phi_a$ and $(x \vee c , y \vee d) = (x , y) \in \phi_a$.

Hence, ϕ_a is a congruence relation on the ADL $(L,\vee , \wedge , 0)$.

Also, if $(x, y) \in \theta$, then $y \wedge x = x$, for all $x, y \in L$.………………… by [theorem2.1.12 (7)]

So that $x^\circ = (y \wedge x)^\circ = (x \wedge y)^\circ = y^\circ$……………….................... by [lemma2.1.7 (7)]

Moreover, $x^\circ = y^\circ$ implies $x^{\circ\circ} = y^{\circ\circ}$. So that for each $a \in L$, we have $x^\circ \vee a = y^\circ \vee a$ and $x^{\circ\circ} \vee a = y^{\circ\circ} \vee a$. That is, $(x^\circ, y^\circ) \in \phi_a$.

Hence, for each $a \in L$, the relation ϕ_a given by; $(x, y) \in \phi_a$ if and only if $x \vee a = y \vee a$ and $x^\circ \vee a = y^\circ \vee a$ is a congruence relation on L.

(8) $\Longrightarrow$ $(\mathbf{1})$. Suppose that for each $a \in L$, the relation ϕ_a given by;

$(x, y) \in \phi_a$ if and only if $x \vee a = y \vee a$ and $x^\circ \vee a = y^\circ \vee a$ is a congruence relation on L.

Claim: L is an MS-algebra.

Since, ϕ_a is a congruence relation on L, for any $x, y, c, d \in L$ such that $(x, y) \in \phi_a$ and $(c, d) \in \phi_a$, we have $(x \wedge c , y \wedge d) \in \phi_a$, $(x \vee c , y \vee d) \in \phi_a$ and $(x^\circ, y^\circ) \in \phi_a$. This gives that $(x \wedge c)^\circ \vee a = (y \wedge d)^\circ \vee a$ and $(x \vee c)^\circ \vee a = (y \vee d)^\circ \vee a$. Also, $x^{\circ\circ} \vee a = y^{\circ\circ} \vee a$.

This is equivalent to $(x^\circ \vee c^\circ) \vee a = (y^\circ \vee d^\circ) \vee a$ and $(x^\circ \wedge c^\circ) \vee a = (y^\circ \wedge d^\circ) \vee a$.

Hence, L is an MS-algebra.

Therefore, $(\mathbf{1}) \Rightarrow (\mathbf{2}) \Rightarrow (\mathbf{3}) \Rightarrow (\mathbf{4}) \Rightarrow (\mathbf{5}) \Rightarrow (\mathbf{6}) \Rightarrow (\mathbf{7}) \Rightarrow (\mathbf{8}) \Rightarrow (\mathbf{1})$ hold.

Theorem 2.1.13 Let $(L, \vee, \wedge, 0)$ be an ADL. Then, the following conditions are equivalent:

(1). L is an MS-ADL.

(2). L_n is an MS-algebra for all maximal element n of L.

(3). L_m is an MS-algebra for some maximal element m.

Proof: (1) $\Rightarrow (\mathbf{2})$. Suppose that L is an MS-ADL. Let n be a maximal element of L.

Define a unary operation $x \mapsto x^*$ on L_n by; $(a \wedge n)^* = a^\circ \wedge n$, for all $a \in L$, where $\circ$ is the unary operation on L. Consequently, this gives the following results:

$(a \wedge n)^* = a^\circ \wedge n = a^\circ \wedge (a \wedge n) = (a^\circ \wedge a) \wedge n = (a \wedge a^\circ) \wedge n = 0 \wedge n = 0$. This implies that $a^\circ \wedge n \in L_n$. Hence, $(a \wedge n)^* \in L_n$ for all $a \in L$.

Also, let $a, b \in L$ such that $a \wedge n = b \wedge n$.

Claim: $(a \wedge n)^* = (b \wedge n)^*$.

For this we consider the following implications:

$a \wedge n = b \wedge n \Rightarrow (a \wedge n)^\circ = (b \wedge n)^\circ$

$\Rightarrow a^\circ \vee n^\circ = b^\circ \vee n^\circ$……….. by definition2.1.1 of axiom [MS-A3]

$\Rightarrow a^\circ \vee 0 = b^\circ \vee 0$………… [Since n is maximal].

$\Rightarrow a^\circ = b^\circ$…………………. by definition1.2.2.1 of axiom [ADL1]

$\Rightarrow a^\circ \wedge n = b^\circ \wedge n$, for all maximal element n of L.

$\Rightarrow (a \wedge n)^* = (b \wedge n)^*$.

Note that: This confirms that the unary operation $x \mapsto x^*$ is well defined as a mapping. Moreover, let $a, b \in L_n$. Then, it can be easily verified that L_n together with this unary operation $x \mapsto x^*$ is an MS-algebra as follows:

(i). $(a \wedge b)^* = [(a \wedge b) \wedge n]^* = (a \wedge b)^\circ \wedge n = (a^\circ \vee b^\circ) \wedge n.$

$= (a^\circ \wedge n) \vee (b^\circ \wedge n)$

$= (a \wedge n)^* \vee (b \wedge n)^*.$

$= a^* \vee b^*$

(ii). $(a \vee b)^* = [(a \vee b) \wedge n]^* = (a \vee b)^\circ \wedge n = a^\circ \wedge b^\circ \wedge n$

$= (a^\circ \wedge n) \wedge (b^\circ \wedge n)$

$= (a \wedge n)^* \wedge (b \wedge n)^*.$

$= a^* \wedge b^*$

Particularly, if $a = 1$, then since the unary operation $x \longmapsto x^*$ on L_n is defined by; $(a \wedge n)^* = a^\circ \wedge n$, this gives that $1^* = (1 \wedge n)^* = 1^\circ \wedge n = 0 \wedge n = 0$….. by definition1.2.2.19 of [M4]

Therefore, L_n is an MS-algebra for all maximal element n of L.

(2) ⟹(3). Suppose L_n is an MS-algebra for all maximal element n of L. So that we can generate a unary operation $x \longmapsto x^*$ on L_m for some fixed maximal element m defined by: $(a \wedge m)^* = a^\circ \wedge m$, for all $a \in L$, where ∘ is the unary operation on L.

Hence, by (2) L_m becomes an MS-algebra for some fixed maximal element m.

(3) ⟹(1). Suppose that L_m is an MS-algebra for some fixed maximal element m.

Define a unary operation $x \longmapsto x^*$ on L by; $a^* = (a \wedge m)^\circ$ for all $a \in L$, where ∘ is the unary operation on L_m.

Claim: L is an MS-ADL.

Since L_m has a fixed maximal element m, we have the following results:

$a^* \wedge (a \wedge m) = (a \wedge m)^\circ \wedge (a \wedge m) = a^\circ \wedge (a \wedge m)$…….... by [lemma2.1.7 (6)]

$= (a^\circ \wedge a) \wedge m$…..… [Since ∧ is associative]

$= (a \wedge a^\circ) \wedge m$……... by [lemma1.2.2.6 (13)]

$= 0 \wedge m$…………….. by definition1.2.2.15 of [PC-2]

$= 0$……………….…… by definition1.2.2.1 of [ADL2]

Also, for any $a , b \in L$, we have the following characterizations and results:

(i). $(a \wedge b)^* = [(a \wedge b) \wedge m]^\circ = (a \wedge b)^\circ \vee m^\circ$………….. by definition2.1.1 of [MS-A3]

$= (a^\circ \vee b^\circ) \vee 0$…………… by definition2.1.1 of [MS-A4]

$= a^\circ \vee b^\circ$………………….. by definition1.2.2.1 of [ADL1]

$= (a \wedge m)^\circ \vee (b \wedge m)^\circ$……… by [lemma2.1.7(6)]

$= a^* \vee b^*$

(ii). $(a \vee b)^* = [(a \vee b) \wedge m]^\circ = [(a \wedge m) \vee (b \wedge m)]^\circ$……. by definition1.2.2.1 of [ADL3]

$= (a \wedge m)^\circ \wedge (b \wedge m)^\circ$……… by definition2.1.1 of [MS-A2]

$= a^* \wedge b^*$

Hence, L is an MS-ADL, for some fixed maximal element m.

Definition 2.1.14 [11-13] Let L be an MS-ADL. Then, the set of all closed elements of L denoted by $L^{\circ\circ}$ is called the skeleton of L and it is defined by $L^{\circ\circ} = \{x^{\circ\circ} : x \in L\}$.

The skeleton $L^{\circ\circ}$ of an MS-ADL is a De Morgan algebra under the induced operations on L.

Theorem 2.1.15 Let L be an MS-ADL. Then, the map $\phi : L \longmapsto L$ defined by:

$\phi(x) = x^{\circ\circ}$ for all $x \in L$ forms a closure operator on L.

Proof: Let $x, y \in L$ such that $x \leq y$. This implies $x \wedge y = x$ (or equivalently, $x \vee y = y$).

Then, $\phi(x) = \phi(x \wedge y) = (x \wedge y)^{\circ\circ} = x^{\circ\circ} \wedge y^{\circ\circ} = \phi(x) \wedge \phi(y)$ and $\phi(y) = \phi(x \vee y) = (x \vee y)^{\circ\circ} = x^{\circ\circ} \vee y^{\circ\circ} = \phi(x) \vee \phi(y)$. So that $x \leq y$ implies $\phi(x) \leq \phi(y)$ or ϕ is an order preserving on L. Moreover, for any $x, y \in L$ we have $\phi(x \wedge y) = (x \wedge y)^{\circ\circ} = x^{\circ\circ} \wedge y^{\circ\circ} = \phi(x) \wedge \phi(y)$ and $\phi(x \vee y) = (x \vee y)^{\circ\circ} = x^{\circ\circ} \vee y^{\circ\circ} = \phi(x) \vee \phi(y)$. This shows that ϕ is a lattice homomorphism and it preserves the unary operation $\circ$ on L.

Hence, the map $\phi: L \longmapsto L$ defined by $\phi(x) = x^{\circ\circ}$ for all $x \in L$ forms a closure operator on L.

Definition 2.1.16 [13] Let L be an MS-ADL. An element $x \in$ L is called dense element if $x^\circ = 0$. The set of all dense elements is denoted by $D(L)$ such that $D(L) = \{x \in L : x^\circ = 0\}$.

Theorem 2.1.17 Let L be an MS-ADL. Then, the set $D(L) = \{x \in L : x^\circ = 0\}$ is a filer of L.

Proof: Let L be an MS-ADL with maximal element m. Then, by definition2.1.1 of [MS-A4] it follows that $m^{\circ} = 0$. This implies that $m \in D(L)$. Hence, $D(L)$ is a non-empty subset of L.

Also, let $a , b \in D(L)$ and $x \in L$. Then, by definition2.1.16 we have $a^{\circ} = 0$ and $b^{\circ} = 0$.

Claim: $a \wedge b \in D(L)\ and\ x \vee a \in D(L)$……………………… by [definition1.2.2.11]

Since L is an MS-ADL with maximal element m, then by definition2.1.1 of [MS-A3] and [MS-A2] we have the following results:

(i). $(a \wedge b)^{\circ} = a^{\circ} \vee b^{\circ} = 0 \vee 0 = 0$. This implies that $a \wedge b \in D(L)$.

(ii). $(x \vee a)^{\circ} = x^{\circ} \wedge a^{\circ} = x^{\circ} \wedge 0 = 0$. This implies that $x \vee a \in D(L)$.

Therefore, the set $D(L) = \{x \in L : x^{\circ} = 0\}$ is a filer of L.

2.2 Congruence Relations on MS-algebras and MS-ADLs

In this section, we give some important congruence relations which are using to characterize MS-ADLs and MS-algebras.

Definition 2.2.1[18] Let $(L ,\vee, \wedge, \circ, 0, m)$ be an MS-ADL. Then, an equivalence relation θ on L is said to be a congruence relation on L if and only if θ is a congruence relation on the ADL $(L, \vee, \wedge, 0)$, that is $(a , b) , (x, y) \in \theta \Longrightarrow (a \wedge x , b \wedge y), (a \vee x , b \vee y) \in \theta$ and if it satisfies the substitution property:

$(x , y) \in \theta \Longrightarrow (x^{\circ}, y^{\circ}) \in \theta$, for all $a , b, x, y \in L$.

Note that: An equivalence relation θ on the MS-ADL $(L ,\vee, \wedge, \circ, 0, m)$ is a congruence relation on L if θ is closed under the binary operations $\vee$ and $\wedge$ and the unary operation $\circ$.

A congruence relation on the MS-algebra (L, $\vee$, $\wedge$, $\circ$, 0, 1) is a lattice congruence θ such that $(x , y) \in \theta \Longrightarrow (x^{\circ}, y^{\circ}) \in \theta$ for all $x , y \in L$.

Example 2.2.2 Let L be an MS-ADL and θ be a binary relation on L defined by; $(a , b) \in \theta$ if and only if $a^{\circ} = b^{\circ}$ for all $a , b \in L$. Then, show that θ is a congruence relation on L.

Proof: First let us show that θ is an equivalence relation on L.

(1). $(a , a) \in \theta$, since $a^{\circ} = a^{\circ}$ for all $a \in L$.

(2). $(a\,,b) \in \theta$ implies $(b\,,a) \in \theta$ since, $a^\circ = b^\circ$ implies $b^\circ = a^\circ$ for all $a\,,b \in L$.

(3). $(a\,,b) \in \theta$ and $(b\,,c) \in \theta$ implies $(a\,,c) \in \theta$ since, $a^\circ = b^\circ$ and $b^\circ = c^\circ$ implies

$a^\circ = c^\circ$ for all $a\,,b\,,c \in L$.

Hence, θ is an equivalence relation (that is, reflexive, symmetric and transitive) on L.

Also, let $(a\,,b) \in \theta$ implies $a^\circ = b^\circ$ and $(c\,,d) \in \theta$ implies $c^\circ = d^\circ$. Then, by definition 2.2.1 we have $a^\circ \wedge c^\circ = b^\circ \wedge d^\circ$ and $a^\circ \vee c^\circ = b^\circ \vee d^\circ$. This is equivalent to $(a \vee c)^\circ = (b \vee d)^\circ$ and $(a \wedge c)^\circ = (b \wedge d)^\circ$. This implies that $(a \wedge c\,, b \wedge d) \in \theta$ and $(a \vee c\,, b \vee d) \in \theta$ for all $a\,,b,c,d \in L$. Moreover, $a^\circ = b^\circ$ implies that $a^{\circ\circ} = b^{\circ\circ}$. So that $(a^\circ, b^\circ) \in \theta$.

Therefore, θ is a congruence relation on L.

Definition 2.2.3 [14] Let $(L\,,\vee,\ \wedge,\ \circ,0,m)$ be an MS-ADL and η be a binary relation on L. Then, for each $a\,,b \in L$, consider that $(a\,,b) \in \eta$ if and only if $a \wedge b = b\ and\ b \wedge a = a$.

Lemma 2.2.4 Let L be an MS-ADL. Then, η is a congruence relation on L.

Proof: (1). $(a\,,a) \in \eta$, since $a \wedge a = a$ and $a \wedge a = a$, for all $a \in L$ … [η is reflexive on L].

(2). $(a\,,b) \in \eta$ implies $(b\,,a) \in \eta$ since $[a \wedge b = b\ and\ b \wedge a = a]$ implies that

$[b \wedge a = a\ and\ a \wedge b = b]$ ……………………………… [η is symmetric on L].

(3). $(a\,,b) \in \eta$ and $(b\,,c) \in \eta$ implies $(a\,,c) \in \eta$, since $[a \wedge b = b\ and\ b \wedge a = a]$ and $[b \wedge c = c\ and\ c \wedge b = b]$ implies $[a \wedge c = c\ and\ c \wedge a = a]$……….. [$\eta$ is transitive on L].

Hence, η is an equivalence relation on L.

(i). Let $(a\,,b) \in \eta$ and $(c\,,d) \in \eta$. Then, by definition2.2.1 it follows that $(a \wedge c,\ b \wedge d) = (c\,,d) \in \eta$ and $(a \vee c,\ b \vee d) = (a\,,b) \in \eta$.

So that η is a congruence relation on the ADL $(L,\ \vee,\ \wedge\,,0)$.

(ii). Let $(a\,,b) \in \eta$. Then, by definition2.2.3 we have $a \wedge b = b$ and $b \wedge a = a$. So that by lemma2.1.7 (7) it follows that $a^\circ = (b \wedge a)^\circ = (a \wedge b)^\circ = b^\circ$. This implies that $(a^\circ,\ b^\circ) \in \eta$.

Hence, η is a congruence relation on L.

Lemma 2.2.5 [18] η is the smallest congruence on L for which the quotient L$|\eta$ is an MS-algebra.

Proof: First let us consider a congruence relation θ on L (as given in example 2.2.2). That is, $(a\ ,b) \in \theta$ if and only if $a^{\circ} = b^{\circ}$ for all $a\ ,b \in L$. Let m be the maximal element of L. Then, by lemma2.1.7 (6) it follows that $(a \wedge b)^{\circ} = (a \wedge b \wedge m)^{\circ} = (b \wedge a \wedge m)^{\circ} = (b \wedge a)^{\circ}$. This implies that $(a \wedge b, b \wedge a) \in \theta$ for all $a, b \in L$. That is, $\wedge$ commutes on L$|\theta$. Therefore, L$|\theta$ is an MS-algebra.

Claim: $\eta \subseteq \theta$.

For any $a, b \in L$, we have $a^{\circ} = (b \wedge a)^{\circ} = (a \wedge b)^{\circ} = b^{\circ}$. …………. by [lemma2.1.7 (7)].

So that $(a \wedge b, b \wedge a) \in \eta$. That is, $\wedge$ commutes on L$|\eta$. Hence, L$|\eta$ is also an MS-algebra.

Let $(a\ ,b) \in \eta$. Then, $a \wedge b = b$ and $b \wedge a = a$……………………… by definition2.2.5. Also, by lemma2.1.7 (7) this gives that $a^{\circ} = (b \wedge a)^{\circ} = (a \wedge b)^{\circ} = b^{\circ}$. This shows that $(b \wedge a\ , a \wedge b) \in \theta$ for all $a\ ,b \in L$. Hence, $\eta \subseteq \theta$.

Therefore, η is the smallest congruence on L for which the quotient L$|\eta$ is an MS-algebra.

Definition 2.2.6 [18] Let $(L\ ,\vee,\ \wedge,\ \circ, 0, m)$ be an MS-ADL and Φ be a binary relation on L. Then, for each $x, y \in L$ consider that $(x, y) \in \Phi$ if and only if $x^{\circ\circ} = y^{\circ\circ}$.

Lemma 2.2.7[18] Let L be an MS-ADL. Then, Φ is a congruence relation on L.

Proof: (1). $(x, x) \in \Phi$, since $x^{\circ\circ} = x^{\circ\circ}$ for all $x \in L$…………….... [Φ is reflexive on L].

(2). $(x\ ,y) \in \Phi \Longrightarrow (y\ ,x) \in \Phi$, since $x^{\circ\circ} = y^{\circ\circ} \Longrightarrow y^{\circ\circ} = x^{\circ\circ}$……..[$\Phi$ is symmetric on L].

(3). $(x\ ,y) \in \Phi$ and $(y\ ,z) \in \Phi \Longrightarrow (x\ ,z) \in \Phi$ since, $x^{\circ\circ} = y^{\circ\circ}$ and $y^{\circ\circ} = z^{\circ\circ} \Longrightarrow x^{\circ\circ} = z^{\circ\circ}$ for all $x, y, z \in L$ ...…………………………………………………. [Φ is transitive on L].

Hence, Φ is an equivalence relation on L.

(i). Let $(x\ ,y) \in \Phi$ implies $x^{\circ\circ} = y^{\circ\circ}$ and $(a\ ,b) \in \Phi$ implies $a^{\circ\circ} = b^{\circ\circ}$ for all $x, y, a, b \in L$. Then, by definition1.2.2.23 it follows that $x^{\circ\circ} \wedge a^{\circ\circ} = y^{\circ\circ} \wedge b^{\circ\circ}$ and $x^{\circ\circ} \vee a^{\circ\circ} = y^{\circ\circ} \vee b^{\circ\circ}$. This is equivalent to $(x \wedge a)^{\circ\circ} = (y \wedge b)^{\circ\circ}$ and $(x \vee a)^{\circ\circ} = (y \vee b)^{\circ\circ}$. So that $(x \wedge a\ , y \wedge b) \in \Phi$ and $(x \vee a\ , y \vee b) \in \Phi$. Hence, Φ is a congruence relation on the ADL $(L\ ,\vee,\ \wedge, 0)$.

(ii). Moreover, $x^{\circ\circ} = y^{\circ\circ}$ implies that $x^{\circ\circ\circ} = y^{\circ\circ\circ}$. This shows that $(x^{\circ}, y^{\circ}) \in \Phi$.

Therefore, Φ is a congruence relation on L.

Lemma 2.2.8[18] Φ is the smallest congruence relation on L such that its quotient L|Φ is a De Morgan algebra.

Proof: Since $a \wedge m = a^{\circ\circ} = b^{\circ\circ} = b \wedge m$, Φ is a De Morgan ADL. Also, Φ ⊆ Φ. This implies that L|Φ is also a De Morgan ADL.

It suffices to show that one of the binary operations (either ∨ or ∧) commutes on L|Φ.

Let $a, b \in L$ and m be the maximal element of L. Then, by lemma2.1.7 (6) it follows that

$(a \wedge b)^{\circ} = (a \wedge b \wedge m)^{\circ} = (b \wedge a \wedge m)^{\circ} = (b \wedge a)^{\circ}$. That is, $(a \wedge b)^{\circ} = (b \wedge a)^{\circ}$.

This implies that $(a \wedge b)^{\circ\circ} = (b \wedge a)^{\circ\circ}$. So that $(a \wedge b\ , b \wedge a) \in \Phi$. That is, ∧ commutes on L|Φ. Hence, L|Φ is a De Morgan algebra.

Suppose θ is a congruence relation on L (as given in example 2.2.2) such that L|θ is a De Morgan algebra. That is, $(a \wedge b, b \wedge a) \in \theta$ for all $a\ , b \in L$.

Claim: $\Phi \subseteq \theta$.

Let $(a\ , b) \in \Phi$. Then, by definition2.2.6 this gives that $a^{\circ\circ} = b^{\circ\circ}$. This implies $a^{\circ\circ\circ} = b^{\circ\circ\circ}$. This is equivalent to $a^{\circ} = b^{\circ}$……... by lemma2.1.7 (3). Also, by lemma2.1.7 (7) this gives that $a^{\circ} = (b \wedge a)^{\circ} = (a \wedge b)^{\circ} = b^{\circ}$. That is, $a \wedge b = b$ and $b \wedge a = a$. So that $(a\ , b) \in \theta$.

This shows that $\Phi \subseteq \theta$.

Therefore, Φ is the smallest congruence relation on L such that its quotient L|Φ is a De Morgan algebra.

The following theorem characterizes MS-ADLs using the congruence relation Φ.

Theorem 2.2.9 Let L be an MS-ADL. Then, L is a De Morgan ADL if and only if $\eta = \Phi$.

Proof: (⟹). Suppose that L is a De Morgan ADL.

Claim: $\eta = \Phi$.

Let $(a\ , b) \in \eta$. Then, $a \wedge b = b\ and\ b \wedge a = a$ for all $a\ , b \in L$…… by [Definition2.2.3]

Since L is a De Morgan ADL with maximal element m of L, this gives that $a^{\circ\circ} = a \wedge m$ $= b \wedge a \wedge m = a \wedge b \wedge m = b \wedge m = b^{\circ\circ}$………………. by definition2.1.2 of [MS-A5].

That is, $a^{\circ\circ} = b^{\circ\circ}$. This implies that $(a\, , b) \in \Phi$.

So that $\eta \subseteq \Phi$……………………………………………………………… (1)

To prove the converse inclusion, let $(a\, , b) \in \Phi$. Then, $a^{\circ\circ} = b^{\circ\circ}$........... by definition2.2.8

Put m $= 0^{\circ}$. Since L is a De Morgan ADL, by definition2.1.2 of [MS-A5] it follows that $a \wedge m = a^{\circ\circ} = b^{\circ\circ} = b \wedge m$. That is, $a \wedge m = b \wedge m$. This gives $a \wedge b = b$ *and* $b \wedge a = a$.

This implies that $(a, b) \in \eta$. So that $\Phi \subseteq \eta$…………………………………. (2)

Hence, the two inclusions (1) and (2) imply that $\Phi = \eta$.

($\Leftarrow$). Suppose that $\Phi = \eta$.

Claim: L is a De Morgan ADL.

Since L is an MS-ADL, it suffices to show that L satisfies the property $a^{\circ\circ} = a \wedge m$, for some maximal element m of L. *Put* m $= 0^{\circ}$. Then by lemma2.1.7 (6) it follows that $a^{\circ} = (a \wedge m)^{\circ}$. This implies $a^{\circ\circ} = (a \wedge m)^{\circ\circ} = a^{\circ\circ} \wedge m^{\circ\circ} = a^{\circ\circ} \wedge m$ ...…………….. [Since $m^{\circ\circ} = m = 0^{\circ}$].

This shows that $a^{\circ\circ} \leq$ m for all $a \in L$. Also, since $a^{\circ\circ\circ\circ} = a^{\circ\circ}$, we have $(a^{\circ\circ},\ a) \in \Phi = \eta$. That is, $(a^{\circ\circ},\ a) \in \eta$. This gives that $a \wedge a^{\circ\circ} = a^{\circ\circ}$ and $a^{\circ\circ} \wedge a = a$.… by [definition 2.2.3]

So that $a \wedge m = a^{\circ\circ} \wedge a \wedge$ m ……………………………… [Since $a^{\circ\circ} \wedge a = a$]

$= a \wedge a^{\circ\circ} \wedge$ m ……………………………… by [lemma1.2.2.6 (13)]

$= a^{\circ\circ} \wedge$ m …………………………………… [Since $a \wedge a^{\circ\circ} = a^{\circ\circ}$]

$= a^{\circ\circ}$ ……………………………………….. [Since $a^{\circ\circ} \leq$ m]

Therefore, L is a De Morgan ADL.

Corollary 2.2.10 Let L be an MS-ADL. Then, L is a De Morgan algebra if and only if $\Phi = \Delta_L$ (the diagonal of L). For an MS-ADL, L to be a Stone ADL it is necessary and sufficient that $a \wedge a^{\circ} = 0$.

Proof: ($\Longrightarrow$). Suppose that L is a De Morgan algebra. Then, for any $a, b \in L$, we have $a^{\circ\circ} = a$ and $b^{\circ\circ} = b$……………………………….…….. by definiton1.2.2.20 of [M5].

Claim: $\Phi = \Delta_L$.

Let $(a, b) \in \Phi$. This implies that $a^{\circ\circ} = b^{\circ\circ}$……………..………… by [definition2.2.6]

This implies that $a = a^{\circ\circ} = b^{\circ\circ} = b$………………..... [Since L is a De Morgan algebra].

This gives that $a = b$ and $a^{\circ} = b^{\circ}$. So that $(a, b) \in \Delta_L$.

That is, $\Phi \subseteq \Delta_L$………………………………………………………… (1)

Similarly, let $(a, b) \in \Delta_L$. Then, $a = b$ and $a^{\circ} = b^{\circ}$. Also, $a^{\circ} = b^{\circ}$ implies $a^{\circ\circ} = b^{\circ\circ}$.

So that $(a, b) \in \Phi$. That is, $\Delta_L \subseteq \Phi$ ...……………………………………… (2)

Hence, the two inclusions (1) and (2) imply that $\Phi = \Delta_L$.

($\Longleftarrow$). Suppose that $\Phi = \Delta_L$.

Claim: L is a De Morgan algebra.

Since L is an MS-ADL, for any $a \in$ L, we have $a^{\circ\circ\circ\circ} = a^{\circ\circ}$. So that $(a^{\circ\circ}, a) \in \Phi = \Delta_L$. That is, $(a^{\circ\circ}, a) \in \Delta_L$. This implies that $a^{\circ\circ} = a$, for all $a \in$ L.

Therefore, L is a De Morgan algebra.

To prove the second part, it is known that an MS-ADL, L is a Stone ADL if $a^{\circ} \vee a^{\circ\circ} = 0^{\circ}$

for all $a \in L$. Given that $a \wedge a^{\circ} = 0$. So that by definition2.1.1 of axiom [MS-A3] it follows that $0^{\circ} = (a \wedge a^{\circ})^{\circ} = a^{\circ} \vee a^{\circ\circ}$. Hence, L is a Stone ADL whenever $a \wedge a^{\circ} = 0$.

Definition 2.2.11 let L be an MS-ADL and ψ^a *and* ϕ^a be binary relations on L. Then, for each $a \in L$, consider the following definitions:

(1). $(x, y) \in \psi^a$ if and only if $x \wedge a = y \wedge a$ and $x^{\circ} \wedge a = y^{\circ} \wedge a$

(2). $(x, y) \in \phi^a$ if and only if $a \wedge x = a \wedge y$ and $a \wedge x^{\circ} = a \wedge y^{\circ}$

By these two definitions, we have the following characterizations and results:

Theorem 2.2.12 For each $a \in L$, ψ^a *and* ϕ^a are congruence relations on L.

Proof: (1). Let $(x, y) \in \psi^a$. Then, $x \wedge a = y \wedge a$ and $x^\circ \wedge a = y^\circ \wedge a$.

Also, let $(c, d) \in \psi^a$. Then, $c \wedge a = d \wedge a$ and $c^\circ \wedge a = d^\circ \wedge a$….. by [definition2.2.11 (1)]

So that **(i).** $x \wedge c \wedge a = c \wedge x \wedge a = c \wedge y \wedge a = y \wedge c \wedge a = y \wedge d \wedge a$.

Also, $(x \vee c) \wedge a = (x \wedge a) \vee (c \wedge a) = (y \wedge a) \vee (d \wedge a) = (y \vee d) \wedge a$.

This implies that $(x \wedge c,\ y \wedge d) \in \psi^a$ and $(x \vee c,\ y \vee d) \in \psi^a$.

Hence, ψ^a is a congruence relation on the ADL $(L,\ \vee,\ \wedge\ ,0)$.

(ii). Let $(x, y) \in \psi^a$. Then, $x \wedge a = y \wedge a$ and $x^\circ \wedge a = y^\circ \wedge a$. Also, Since L is an MS-ADL with maximal element m, we have $a^{\circ\circ} \wedge a = a$ for all $a \in L$… by definition2.1.1 of [MS-A1]

Claim: $x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$. For this, we consider the following implications:

$$x \wedge a = y \wedge a \Longrightarrow (x \wedge a)^{\circ\circ} = (y \wedge a)^{\circ\circ}$$

$$\Longrightarrow x^{\circ\circ} \wedge a^{\circ\circ} = y^{\circ\circ} \wedge a^{\circ\circ}$$

$$\Longrightarrow x^{\circ\circ} \wedge a^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a^{\circ\circ} \wedge a$$

$\Longrightarrow x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$……………….. by definition2.1.1 of [MS-A1].

$$\Longrightarrow (x^\circ, y^\circ) \in \psi^a$$

Therefore, ψ^a is a congruence relation on L.

(2). Let $(x, y) \in \phi^a$. Then, $a \wedge x = a \wedge y$ and $a \wedge x^\circ = a \wedge y^\circ$….. by [definition2.2.11 (2)]

Also, let $(c, d) \in \phi^a$. Then, $a \wedge c = a \wedge d$ and $a \wedge c^\circ = a \wedge d^\circ$..… by [definition2.2.11 (2)]

So that **(i).** $a \wedge x \wedge c = a \wedge y \wedge c = y \wedge a \wedge c = y \wedge a \wedge d = a \wedge y \wedge d$.

Also, $a \wedge (x \vee c) = (a \wedge x) \vee (a \wedge c) = (a \wedge y) \vee (a \wedge d) = a \wedge (y \vee d)$.

This implies that $(x \wedge c,\ y \wedge d) \in \phi^a$ and $(x \vee c,\ y \vee d) \in \phi^a$.

So that ϕ^a is a congruence relation on the ADL $(L,\ \vee,\ \wedge\ ,0)$.

(ii). Let $(x, y) \in \phi^a$. Then, $a \wedge x = a \wedge y\ and\ a \wedge x^\circ = a \wedge y^\circ$…. by [definition2.2.11 (2)]

Also, since L is an MS-ADL with maximal element m, we have $a^{\circ\circ} \wedge a = a$, for all $a \in L$.

Claim: $a \wedge x^{\circ\circ} = a \wedge y^{\circ\circ}$. For this, we consider the following implications:

$a \wedge x = a \wedge y \Longrightarrow (a \wedge x)^{\circ\circ} = (a \wedge y)^{\circ\circ}$

$\Longrightarrow a^{\circ\circ} \wedge x^{\circ\circ} = a^{\circ\circ} \wedge y^{\circ\circ}$

$\Longrightarrow a \wedge a^{\circ\circ} \wedge x^{\circ\circ} = a \wedge a^{\circ\circ} \wedge y^{\circ\circ}$

$\Longrightarrow a^{\circ\circ} \wedge a \wedge x^{\circ\circ} = a^{\circ\circ} \wedge a \wedge y^{\circ\circ}$……… by [lemma1.2.2.6 (13)]

$\Longrightarrow a \wedge x^{\circ\circ} = a \wedge y^{\circ\circ}$ …………………. by definition2.1.1 of [MS-A1]

$\Longrightarrow (x^{\circ}, y^{\circ}) \in \phi^{a}$.

Therefore, ϕ^{a} is also a congruence relation on L.

Corollary 2.2.13 If L is a De Morgan ADL, then ψ^{a} and ϕ^{a} are congruence relations on L.

Proof: Suppose L is a De Morgan ADL. Put $x^{\circ\circ} = x \wedge m$ and $y^{\circ\circ} = y \wedge m$ for all $x, y \in L$ and for all maximal element m of L.………………………... by definition2.1.2 of [MS-A5]

Let $(x, y) \in \psi^{a}$. Then $x \wedge a = y \wedge a$ and $x^{\circ} \wedge a = y^{\circ} \wedge a$ ....……… by [definition2.2.11 (1)]

Also, let $(x, y) \in \phi^{a}$. Then $a \wedge x = a \wedge y \ and \ a \wedge x^{\circ} = a \wedge y^{\circ}$….. by [definition2.2.11 (2)]

Claim: (i). $x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$. **(ii).** $a \wedge x^{\circ\circ} = a \wedge y^{\circ\circ}$.

Since L is a De Morgan ADL, by definition2.1.2 of [MS-A5] we have the following results:

(i). $x^{\circ\circ} \wedge a = x \wedge m \wedge a = m \wedge x \wedge a = m \wedge y \wedge a = y \wedge m \wedge a = y^{\circ\circ} \wedge a$.

Hence, $(x^{\circ}, y^{\circ}) \in \psi^{a}$.

(ii). $a \wedge x^{\circ\circ} = a \wedge x \wedge m = a \wedge y \wedge m = a \wedge y^{\circ\circ}$.

Hence, $(x^{\circ}, y^{\circ}) \in \phi^{a}$.

Therefore, if L is a De Morgan ADL, then ψ^{a} and ϕ^{a} are congruence relations on L.

Theorem 2.2.14 The quotient L$|\psi^{a}$ is an MS-algebra. If a is maximal, then $\psi^{a} = \eta$. The converse holds whenever $a^{\circ} \wedge a = 0$.

Proof: To prove the first part, it suffices to show that $\eta \subseteq \psi^{a}$.

Let $(x, y) \in \eta$. Then , $x \wedge y = y$ *and* $y \wedge x = x$. Equivalently; $x \vee y = x$ and $y \vee x = y$.

Also, we have $x^{\circ} = (y \wedge x)^{\circ} = (x \wedge y)^{\circ} = y^{\circ}$ ………………….. by [lemma 2.1.7 (7)]

Hence, $x \wedge a = y \wedge x \wedge a = x \wedge y \wedge a = y \wedge a$ and $x^{\circ} \wedge a = y^{\circ} \wedge a$

This implies that $(x, y) \in \psi^{a}$. So that $\eta \subseteq \psi^{a}$.

Therefore, $L|\psi^{a}$ is an MS-algebra.

To prove the second part, suppose that a is maximal.

Claim: $\psi^{a} = \eta$.

Let $(x, y) \in \psi^{a}$. Then, $x \wedge a = y \wedge a$ and $x^{\circ} \wedge a = y^{\circ} \wedge a$ ...…. by [definition2.2.11 (1)]

If a is maximal, then $x \wedge a = y \wedge a$ and $x^{\circ} \wedge a = y^{\circ} \wedge a$ are reduced to $x = y$ and $x^{\circ} = y^{\circ}$.

Put $x \wedge y = y$ and $y \wedge x = x$. This implies that $(x, y) \in \eta$…………… by [definition2.2.3].

This shows that $\psi^{a} \subseteq \eta$……………………………………………………………… (1)

Similarly, let $(x, y) \in \eta$. This gives that $x \wedge y = y$ and $y \wedge x = x$…… by [definition2.2.3]

Also, $x^{\circ} = (y \wedge x)^{\circ} = (x \wedge y)^{\circ} = y^{\circ}$………………………………… by [lemma2.1.7 (7)]

Then, $x \wedge a = y \wedge x \wedge a = x \wedge y \wedge a = y \wedge a$ and $x^{\circ} \wedge a = y^{\circ} \wedge a$… [Since a is maximal]

This implies that $(x, y) \in \psi^{a}$…………………………….......... by [definition2.2.11 (1)]

This shows that $\eta \subseteq \psi^{a}$……………………………………………………………… (2)

So that the two inclusions (1) and (2) yield that $\psi^{a} = \eta$.

Hence, if a is maximal, then $\psi^{a} = \eta$.

Conversely, suppose that $\psi^{a} = \eta$.

Claim: a is maximal.

Given that $a^{\circ} \wedge a = 0$. This shows that $(a^{\circ} \wedge a , 0) \in \psi^{a} = \eta$.

That is, $(a^{\circ} \wedge a , 0)$ satisfies both the equalities $x \wedge a = y \wedge a$ and $x^{\circ} \wedge a = y^{\circ} \wedge a$.

So that we have $(a^{\circ} \wedge a) \wedge a = 0 \wedge a$ and $(a^{\circ} \wedge a)^{\circ} \wedge a = 0^{\circ} \wedge a$…… by definition2.2.11(1).

But $(a^{\circ} \wedge a) \wedge a = a^{\circ} \wedge (a \wedge a) = a^{\circ} \wedge a$………………. [Since $\wedge$ is associative]

So that $a^{\circ} \wedge a = 0 \wedge a \implies a^{\circ} = 0$.

$\implies a$ is maximal……… by definition 2.1.1 of [MS-A4].

Therefore, $\psi^{a} = \eta$ implies a is maximal whenever $a^{\circ} \wedge a = 0$.

Remark 2.2.15 The quotient L$|\phi^{a}$ is an MS-ADL but not necessarily be an MS-algebra. This can be verified by using the following example.

Example 2.2.16 Let $L = \{0, a, b\}$ be the discrete MS-ADL (as given in example2.1.3) and consider the congruence relation ϕ^{a} on L. Then, the quotient L$|\phi^{a}$ is an MS-ADL but not an MS-algebra.

Proof: Since L is not a distributive lattice, it suffices to show that L$|\phi^{a}$ is isomorphic to L. For this, consider the canonical map $x \longmapsto \phi^{a}[x]$ of L onto L$|\phi^{a}$ which is an epimorphism. We show that this map is an injective map.

Now, for any $x, y \in L$, we have $\phi^{a}[x] = \phi^{a}[y] \implies (x, y) \in \phi^{a}$.

$\implies a \wedge x = a \wedge y$.

Since L is a discrete MS-ADL, every non-zero element in L is maximal. Thus, $a \wedge x = a \wedge y$ yields that $x = y$. This shows that $L \cong$ L$|\phi^{a}$.

Therefore, L$|\phi^{a}$ is not an MS-algebra.

Lemma 2.2.17 For each $a, b \in L$, $\phi^{a} \cap \phi^{b} = \phi^{a \vee b}$.

Proof: Let $(x, y) \in \phi^{a} \cap \phi^{b}$. Then, by definition2.2.11 (2) we have the following conditions:

$a \wedge x = a \wedge y, \quad a \wedge x^{\circ} = a \wedge y^{\circ}, \quad b \wedge x = b \wedge y \ \ and \ \ b \wedge x^{\circ} = b \wedge y^{\circ}$.

So that $(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = (a \wedge y) \vee (b \wedge y) = (a \vee b) \wedge y$.

Also, $(a \vee b) \wedge x^{\circ} = (a \wedge x^{\circ}) \vee (b \wedge x^{\circ}) = (a \wedge y^{\circ}) \vee (b \wedge y^{\circ}) = (a \vee b) \wedge y^{\circ}$

Hence, $\phi^{a} \cap \phi^{b} \subseteq \phi^{a \vee b}$……………………………….......................... (1)

To prove the converse inclusion, let $(x, y) \in \phi^{a \vee b}$. Then, by definition2.2.11 (2) this gives the following conditions:

$(a \vee b) \wedge x = (a \vee b) \wedge y \ \ and \ \ (a \vee b) \wedge x^{\circ} = (a \vee b) \wedge y^{\circ}$.

So that $a \wedge x \ = (a \wedge x) \wedge [(a \wedge x) \vee (b \wedge x)] \ = (a \wedge x) \wedge [(a \vee b) \wedge x]$

$$= \ (a \wedge x) \wedge [(a \vee b) \wedge y]$$

$$= a \wedge x \wedge (a \vee b) \wedge y$$

$$= a \wedge (a \vee b) \wedge x \wedge y$$

$$= a \wedge x \wedge y \text{ .... (EQ1)}$$

Also, $a \wedge y = [(a \wedge y) \vee (b \wedge y)] \wedge (a \wedge y) = \ [(a \vee b) \wedge y] \wedge (a \wedge y)$

$$= [(a \vee b) \wedge x] \wedge (a \wedge y)$$

$$= (a \vee b) \wedge x \wedge a \wedge y$$

$$= \ (a \vee b) \wedge a \wedge x \wedge y$$

$$= a \wedge x \wedge y \text{..................... (EQ2)}$$

Hence, (EQ1) and (EQ2) imply that $a \wedge x = a \wedge y$.

Similarly, $a \wedge x^{\circ} = (a \wedge x^{\circ}) \wedge [(a \wedge x^{\circ}) \vee (b \wedge x^{\circ})] = (a \wedge x^{\circ}) \wedge [(a \vee b) \wedge x^{\circ}]$

$$= (a \wedge x^{\circ}) \wedge [(a \vee b) \wedge y^{\circ}]$$

$$= a \wedge x^{\circ} \wedge (a \vee b) \wedge y^{\circ}$$

$$= a \wedge (a \vee b) \wedge x^{\circ} \wedge y^{\circ}$$

$$= a \wedge x^{\circ} \wedge y^{\circ} \text{ .... (EQ3)}$$

Also, $a \wedge y^{\circ} = [(a \wedge y^{\circ}) \vee (b \wedge y^{\circ})] \wedge (a \wedge y^{\circ}) = \ [(a \vee b) \wedge y^{\circ}] \wedge (a \wedge y^{\circ})$

$$= [(a \vee b) \wedge x^{\circ}] \wedge (a \wedge y^{\circ})$$

$$= (a \vee b) \wedge x^{\circ} \wedge a \wedge y^{\circ}$$

$$= (a \vee b) \wedge a \wedge x^{\circ} \wedge y^{\circ}$$

$$= a \wedge x^\circ \wedge y^\circ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(EQ4)}$$

Thus, (EQ3) and (EQ4) imply that $a \wedge x^\circ = a \wedge y^\circ$. So that $(x, y) \in \phi^a$.

Now, we remain to show that $(x, y) \in \phi^b$.

Since $(a \vee b) \wedge x = (b \vee a) \wedge x$, by interchanging a and b each other and by repeating the above argument, we can show that $(x, y) \in \phi^b$ as follows:

Let $(x, y) \in \phi^{a \vee b}$. Then, by definition2.2.11 (2) this gives the following conditions:

$(b \vee a) \wedge x = (b \vee a) \wedge y$ and $(b \vee a) \wedge x^\circ = (b \vee a) \wedge y^\circ$.

So that $b \wedge x = (b \wedge x) \wedge [(b \wedge x) \vee (a \wedge x)] = (b \wedge x) \wedge [(b \vee a) \wedge x]$

$$= (b \wedge x) \wedge [(b \vee a) \wedge y]$$

$$= b \wedge x \wedge (b \vee a) \wedge y$$

$$= b \wedge (b \vee a) \wedge x \wedge y$$

$$= b \wedge x \wedge y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(EQ5)}$$

Also, $b \wedge y = [(b \wedge y) \vee (a \wedge y)] \wedge (b \wedge y) = [(b \vee a) \wedge y] \wedge (b \wedge y)$

$$= [(b \vee a) \wedge x] \wedge (b \wedge y)$$

$$= (b \vee a) \wedge x \wedge b \wedge y$$

$$= (b \vee a) \wedge b \wedge x \wedge y$$

$$= b \wedge x \wedge y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(EQ6)}$$

Hence, (EQ5) and (EQ6) show that $b \wedge x = b \wedge y$.

Similarly, $b \wedge x^\circ = (b \wedge x^\circ) \wedge [(b \wedge x^\circ) \vee (a \wedge x^\circ)] = (b \wedge x^\circ) \wedge [(b \vee a) \wedge x^\circ]$

$$= (b \wedge x^\circ) \wedge [(b \vee a) \wedge y^\circ]$$

$$= b \wedge x^\circ \wedge (b \vee a) \wedge y^\circ$$

$$= b \wedge (b \vee a) \wedge x^\circ \wedge y^\circ$$

$$= b \wedge x^\circ \wedge y^\circ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(EQ7)}$$

Also, $b \wedge y^{\circ} = [(b \wedge y^{\circ}) \vee (a \wedge y^{\circ})] \wedge (b \wedge y^{\circ}) = [(b \vee a) \wedge y^{\circ}] \wedge (b \wedge y^{\circ})$

$$= [(b \vee a) \wedge x^{\circ}] \wedge (b \wedge y^{\circ})$$

$$= (b \vee a) \wedge x^{\circ} \wedge b \wedge y^{\circ}$$

$$= (b \vee a) \wedge b \wedge x^{\circ} \wedge y^{\circ}$$

$$= b \wedge x^{\circ} \wedge y^{\circ} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(EQ8)}$$

Thus, (EQ7) and (EQ8) show that $b \wedge x^{\circ} = b \wedge y^{\circ}$. So that $(x, y) \in \phi^{b}$.

Hence, we have $\phi^{a \vee b} \subseteq \phi^{a} \cap \phi^{b}$ .... (2).

Therefore, the two inclusions (1) and (2) imply that $\phi^{a} \cap \phi^{b} = \phi^{a \vee b}$.

Lemma 2.2.18 In an MS-ADL L, if a is maximal, then $\phi^{a} = \Delta_{L}$(the diagonal of L).

Proof: $(x, y) \in \phi^{a}$. Then, $a \wedge x = a \wedge y$ and $a \wedge x^{\circ} = a \wedge y^{\circ}$….. by [Definition 2.2.11 (2)].

If a is maximal, then $a \wedge x = a \wedge y$ and $a \wedge x^{\circ} = a \wedge y^{\circ}$ are reduced to $x = y$ and $x^{\circ} = y^{\circ}$.

This implies $(x, y) \in \Delta_{L}$. That is, $\phi^{a} \subseteq \Delta_{L}$…………………………………………..(1)

Conversely, let $(x, y) \in \Delta_{L}$. Then, for any $x, y \in L$, we have $x = y$ and $x^{\circ} = y^{\circ}$.

This gives $a \wedge x = a \wedge y$ and $a \wedge x^{\circ} = a \wedge y^{\circ}$……………. [Since a is maximal]

This implies that $(x, y) \in \phi^{a}$. So that $\Delta_{L} \subseteq \phi^{a}$…………………......................(2)

Hence, (1) and (2) imply that $\phi^{a} = \Delta_{L}$.

Therefore, if a is maximal, then $\phi^{a} = \Delta_{L}$.

Note that: The converse of this lemma need not necessarily be true. This can be verified by using the following example:

Example 2.2.19 In an MS-ADL given in example2.1.5, $\phi^{a} = \Delta_{L}$(the diagonal of L) but a is not maximal. Because from the given table it can be observed that $a^{\circ} = a$. That is; $a^{\circ} \neq 0$. Hence, a is not maximal.

Lemma 2.2.20 If $\phi^{a} = \Delta_{L}$, then $a \wedge a^{\circ} = a^{\circ}$. That is, $\phi^{a} = \Delta_{L}$ implies $a^{\circ} \leq a$.

Proof: It suffices to show that $(a \wedge a^{\circ}, a^{\circ}) \in \phi^{a}$. Since L is an MS-ADL with maximal element m, we have $a^{\circ\circ} \wedge a = a,$ for all $\in L$…………… by definition2.1.1 of [MS-A1].

Let $(x, y) \in \phi^{a}$. Then $a \wedge x = a \wedge y$ and $a \wedge x^{\circ} = a \wedge y^{\circ}$……. by [Definition2.2.11 (2)].

So that by definition2.2.11 (2), we have the following results:

(i). $a \wedge (a \wedge a^{\circ}) = (a \wedge a) \wedge a^{\circ} = a \wedge a^{\circ}$……………..…… [Since $\wedge$ is associative].

(ii). $a \wedge (a \wedge a^{\circ})^{\circ} = a \wedge (a^{\circ} \wedge a)^{\circ}$……………………….. by [lemma2.1.7 (7)]

$= a \wedge (a^{\circ\circ} \vee a^{\circ})$…………………….... by defintion2.1.1 of [MS-A3]

$= a^{\circ\circ} \wedge a \wedge (a^{\circ\circ} \vee a^{\circ})$……..………… by definition2.1.1 of [MS-A1]

$= a \wedge a^{\circ\circ} \wedge (a^{\circ\circ} \vee a^{\circ})$………………. by [lemma1.2.2.6 (13)]

$= a \wedge a^{\circ\circ}$…………………………….. by [lemma1.2.2.6 (3)]

Hence, $(a \wedge a^{\circ}, a^{\circ}) \in \phi^{a}$.

Therefore, $a \wedge a^{\circ} = a^{\circ}$ whenever $\phi^{a} = \Delta_{L}$.

Definition 2.2.21 Let L be an MS-ADL and consider θ_{a} be the binary relation on L. Then, for each $a \in L$, consider that $(x, y) \in \theta_{a}$ if and only if $a \vee x = a \vee y \ \ and \ a \vee x^{\circ} = a \vee y^{\circ}$.

By this definition, we have the following characterization and results:

Theorem 2.2.22 Let L be an MS-ADL. If L is $\vee$–associative, then the binary relation θ_{a} is a congruence relation on L.

Proof: Suppose that L is $\vee$–associative. Since L is an MS-ADL with maximal element m, by definition2.1.1 of [MS-A1] we have $a^{\circ\circ} \wedge a = a,$ for all $a \in L$. Equivalently; $a^{\circ\circ} \vee a = a^{\circ\circ}$.

Let $(x, y) \in \theta_{a}$. Then, $a \vee x = a \vee y \ \ and \ a \vee x^{\circ} = a \vee y^{\circ}$……….... by [definition2.2.21]

Also, let $(c, d) \in \theta_{a}$. Then, $a \vee c = a \vee d \ \ and \ a \vee c^{\circ} = a \vee d^{\circ}$……. by [definition2.2.21]

So that **(i).** $a \vee (x \wedge c) = (a \vee x) \wedge (a \vee c) = (a \vee y) \wedge (a \vee d) = a \vee (y \wedge d)$.

Also, $a \vee (x \vee c) = (a \vee x) \vee c = (a \vee y) \vee c = y \vee (a \vee c) = y \vee (a \vee d) = a \vee (y \vee d)$.

This implies that $(x \wedge c \ , y \wedge d) \in \theta_{a}$ and $(x \vee c \ , y \vee d) \in \theta_{a}$ for all $x, y, c, d \in L$.

Hence, θ_a is a congruence relation on the ADL $(L, \vee, \wedge, 0)$.

(ii). Let $(x, y) \in \theta_a$. Then, $a \vee x = a \vee y$ and $a \vee x^{\circ} = a \vee y^{\circ}$ ……. by [definition2.2.21]

Claim: $a \vee x^{\circ\circ} = a \vee y^{\circ\circ}$. For this, we consider the following implications:

$a \vee x = a \vee y \Longrightarrow (a \vee x)^{\circ\circ} = (a \vee y)^{\circ\circ}$

$\Longrightarrow a^{\circ\circ} \vee x^{\circ\circ} = a^{\circ\circ} \vee y^{\circ\circ}$……………………. by [lemma2.1.7 (5)]

$\Longrightarrow (a^{\circ\circ} \vee a) \vee x^{\circ\circ} = (a^{\circ\circ} \vee a) \vee y^{\circ\circ}$……….. [Since $a^{\circ\circ} = a^{\circ\circ} \vee a$]

$\Longrightarrow a^{\circ\circ} \vee (a \vee x^{\circ\circ}) = a^{\circ\circ} \vee (a \vee y^{\circ\circ})$……….. [Since, L is ∨–associative]

$\Longrightarrow a \vee x^{\circ\circ} = a \vee y^{\circ\circ}$

$\Longrightarrow (x^{\circ}, y^{\circ}) \in \theta_a$

Hence, θ_a is a congruence relation on L.

Lemma 2.2.23 Let L be an associative MS-ADL. If a is zero, then $\theta_a = \Delta_L$. But $\theta_a = \Delta_L$ implies $a \leq a^{\circ}$. That is, $\theta_a = \Delta_L$ implies $a \vee a^{\circ} = a^{\circ}$.

Proof: Let $(x, y) \in \theta_a$. Then, $a \vee x = a \vee y$ and $a \vee x^{\circ} = a \vee y^{\circ}$…… by [definition2.2.21]

If $a = 0$, then $a \vee x = a \vee y$ and $a \vee x^{\circ} = a \vee y^{\circ}$ are reduced to $x = y$ and $x^{\circ} = y^{\circ}$.

So that $(x, y) \in \Delta_L$. Hence, if $a = 0$, then $\theta_a = \Delta_L$.

To prove the second part, suppose that $\theta_a = \Delta_L$.

Claim: $(a \vee a^{\circ}, a^{\circ}) \in \theta_a$.

Let L be an MS-ADL with maximal element m such that L is ∨- associative. So that we have the following results:

(i). $a \vee (a \vee a^{\circ}) = (a \vee a) \vee a^{\circ} = a \vee a^{\circ}$ ....……… [Since L is ∨-associative MS-ADL].

(ii). $a \vee (a \vee a^{\circ})^{\circ} = a \vee (a^{\circ} \wedge a^{\circ\circ}) = (a \vee a^{\circ}) \wedge (a \vee a^{\circ\circ})$

$= m \wedge (a \vee a^{\circ}) \wedge (a \vee a^{\circ\circ})$

$= (a \vee a^{\circ}) \wedge m \wedge (a \vee a^{\circ\circ})$

$= [(a \wedge m) \vee (a^{\circ} \wedge m)] \wedge (a \vee a^{\circ\circ})$.

$$= (a^{\circ\circ} \vee a^{\circ}) \wedge (a \vee a^{\circ\circ})$$

$$= m \wedge (a \vee a^{\circ\circ})$$

$$= a \vee a^{\circ\circ}$$

So that $(a \vee a^{\circ}, a^{\circ}) \in \theta_a$.

Hence, $\theta_a = \Delta_L$ implies $a \leq a^{\circ}$. That is, $\theta_a = \Delta_L$ implies $a \vee a^{\circ} = a^{\circ}$.

Conclusion

In this project, we discussed the concept of an almost distributive lattice (ADL) which is a generalization of posets and lattice theory. Consequently, a new equational class of algebras called MS-ADL is understand as a common abstraction of De Morgan ADLs and Stone ADLs. It can also be observed that the class of MS-algebras and most of the properties of MS-algebras are extended to the class of MS-ADLs. Moreover, we discussed the congruence relations which characterize MS-algebras and MS-ADLs.

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