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A Project on Duhamel's Principle for Solving One Dimensional Evolution Equations

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A PROJECT

ON

DUHAMEL'S PRINCIPLE FOR SOLVING ONE DIMENSIONAL EVOLUTION EQUATIONS

BY

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Duhamel's Principle for Solving One Dimensional Evolution Equations

A project submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of "Master of Science in Mathematics".

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Approval of the project for defense

I hereby certify that I have supervised, read and evaluated this project entitled "Duhamel's principle for solving one dimensional evolution equations" prepared by Yeshambel Melkamu under my guidance. I recommend the project to be submitted for oral defense.

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Approval of the project for defense result

We hereby certify that we have examined this project entitled “Duhamel's principle for solving one dimensional evolution equation” by Yeshambel Melkamu. We recommend this project to be approved for the degree of “Master of science in mathematics”.

Board of Examiners

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Abstract

The aim of this project is to study Duhamel's principle for solving one dimensional evolution equations (heat and wave). First, find the formula in which we able to obtain its solution we derived by using the D' Alembert's formula, Fourier transform and method of separation variables. Finally, by applied Duhamel's principle on one dimensional non-homogeneous evolution equations and we can get the solution.

CHAPTER ONE

INTRODUCTION AND PRELIMINARY

1.1 Introduction

Duhamel's principle was introduced by Jean-Marie-Duhamel in 1830.He is a French scientist in mathematics and physics. It is named after Jean-Marie-Duhamel who first applied the principle to the inhomogeneous heat equation that models the distribution of heat in a thin plate which is heat from beneath[1]. Duhamel's principle is a general method for obtaining solution to inhomogeneous linear evolution equations like the heat equations, wave equations and vibrating plate equations. The philosophy underlying Duhamel's principle is that it is possible to go from solutions of the Cauchy problem (or initial value problem) to solutions of the inhomogeneous problem[1]. The well-known Duhamel's principle allows reducing the Cauchy problem for linear inhomogeneous partial differential equations to the Cauchy problem for the corresponding homogeneous equations, which are more easier to handle[1].

Duhamel's principle is the technique that the solution to an inhomogeneous heat and wave equations can be solved by first finding the solution for a step input and then superposing using Duhamel's integral.

The heat equation is an important partial differential equation which describes the distribution of heat(or variation in temperature) in a given region over time. It is also highly practical engineers have to make sure engines do not melt and computer chips do not over heat. Heat is flow of thermal energy from a warmer place to a cooler place.

The theory of heat equation was first developed and solved by Joseph Fourier in 1822 for the purpose of modeling how a quantity such as heat diffuses through a given region[2]. In physics and mathematics, the heat equation is a partial differential equation describes how the distribution of some quantity (such as heat) evolves over time in a solid medium as it spontaneously flows from places where it is higher towards places where it is lower. It is a special case of the diffusion equation. It has a fundamental importance in diverse of scientific field. In mathematics, it is prototypical parabolic partial differential equation (Crank.;Nicolson,P.1947).In statistics, the heat equation is connected with the study of Brownian motion via the Fokker-Planck equation[2].

The wave equation is an important second order linear partial differential equation for the description of waves that occur in classical physics such as mechanical wave (water waves, sound waves),electromagnetic waves(radio waves, light waves).Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean Le Rond D' Alembert's, Leonhard Euler, Daniel Bernoulli and Joseph-Louis Lagrange[3].In 1746,D'Alembert's discovered the one-dimensional wave equation and within ten years Euler discovered the three-dimensional wave equation[4].

In this project, we apply Duhamel's principle for solving one dimensional evolution equation (particularly, non-homogeneous heat and wave equations) with non-homogeneous initial and initial boundary conditions.

1.2. Evolution Equation

An equation that can be interpreted as the differential law of the development (evolution) in time of system is termed as evolution equation. For examples:

a)Heat equation is a parabolic type of a partial differential equation that describes how the temperature varies in space over time. The heat equation arises in with the study of chemical diffusion and other related processes.

The partial differential equation

$$u_t = ku_{xx}$$

is used to model one dimensional heat equation, where k represents the thermal conductivity of the material.

b)Wave equation is one of the most important equations in mechanics. It is a hyperbolic partial differential equation. It describes not only the movement of strings and wires, sound waves, or electric current along a wire, but also the movement of fluid surfaces, example: water waves. Wave equation can calculate the displacement of a wave in time at any point. By this equation one can know the angle made by the wave at any point of instant, velocity, acceleration, initial phase, angular frequency, such important parameters which can give every type of information about the wave and its nature. Many wave motion problems in physics can be modeled by the standard linear one-dimensional wave equation of the form

$$u_{tt} = c^2 u_{tt},$$

where c is the speed of the material.

Definition1.1: The term evolution equation refers to a dynamical partial differential equation that involves both time and space as independent variables, unknown function and its space and /or time derivatives.

Definition 1.2: The evolution equation in one space dimension with a source function $h(x,t)$ can be formulated as:

$$\frac{\partial^i u}{\partial t^i} = c^2 \frac{\partial^2 u}{\partial x^2} + h(x,t), \quad x \in \mathbb{R}, \quad t \geq 0, \quad i = 1,2,$$

where $u = u(x,t),\ x$ is aspace variable, t is a time variable and c is a positive constant called the heat conduction (or the speed) constant. For $i = 1$, the equation is one-dimensional heat equation and for $i = 2$, the equation is one-dimensional wave equation.

Definition 1.3: A problem which we are looking for unknown function of a differential equation whose values of the unknown function and /or derivative at a single point are known, then the problem is said to be initial value problem.

For examples:

An initial value problem of wave equation is given by

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), & -\infty < x < \infty, \ \ t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty, \\ u_t(x,0) = g(x), & -\infty < x < \infty. \end{cases}$$

An initial value problem of heat equation is given by

$$\begin{cases} u_t - k u_{xx} = h(x,t), & -\infty < x < \infty, \ \ t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty. \end{cases}$$

Definition1.4: If the values of the unknown function and/or its derivatives are known at boundary points, then the problem is called boundary value problem. If it involves both initial and boundary conditions, then it is an initial – boundary value problem.

Much theoretical work in the field of partial differential equations is devoted to solve boundary value problems arising from scientific and engineering applications are in fact well posed. For instance:

An initial - boundary value problem of heat equation is given by

$$\begin{cases} u_t - ku_{xx} = h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) = T_0(t), \quad t > 0, \\ u(L,t) = T_1(t), \quad t > 0, \\ u(x,0) = f(x), \;\; 0 \le x \le L. \end{cases}$$

An initial - boundary value problem of wave equation is given by

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), \quad 0 \le x \le L, \;\; t > 0, \\ u(0,t) = T_0(t), \;\; t > 0, \\ u(L,t) = T_1(t), \;\; t > 0, \\ u(x,0) = f(x), \;\; 0 \le x \le L, \\ u_t(x,0) = g(x), \;\; 0 \le x \le L. \end{cases}$$

1.3 Method of Solutions

In this subsection, we will discuss about Fourier transform, properties of Fourier transforms and method of separation of variables. These are used to solve problems in the subsequent chapter.

1.3.1 Fourier transforms

We obtain Fourier Transform by a limiting process of Fourier series. Since it was first used by French Mathematician Jean Baptiste Fourier (1768-1830) in a manuscript submitted to the Institute of France in 1807, he said that Fourier Transform is a mathematical procedure which transforms a function from time domain to frequency domain. Fourier Transform is useful in the study of solution of partial differential equation to solve initial value problems. A Fourier Transform when applied to partial differential equation reduces the number of independent variables by one.

Definition 1.5[6]: The Fourier transform of absolutely integrable function $f(x)$ defined on $(-\infty, \infty)$ is given by

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = \mathrm{F}(\omega).$$

The inverse of the Fourier transforms of a function $\mathrm{F}(\omega)$ is given by

$$f(x) = \mathcal{F}^{-1}\{\mathrm{F}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{F}(\omega)e^{i\omega x}d\omega,$$

Where ω is angular frequency parameter and x is space variable

But existence of inverse of Fourier transform of a function is guaranteed by the following conditions.

a) It must be absolutely integrable on $(-\infty, \infty)$.

b) It has a finite number of discontinuities in $(-\infty, \infty)$.

c) It must have a finite number of extrema (maxima and minima) in$(-\infty, \infty)$.

Similarly, we can define Fourier transforms for function of two variables by taking one as a constant. The Fourier transforms of a function $u(x, t)$ defined on $(-\infty, \infty) \times (0, \infty)$ is given by

$$\mathcal{F}\{u(x,t)\} = U(\omega, t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx \tag{1.1}$$

where t as fixed with $t > 0$.

The inverse of Fourier transforms of $U(\omega, t)$is given by

$$\mathcal{F}^{-1}\{U(\omega,t)\} = u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)\, e^{i\omega x}d\omega \tag{1.2}$$

where t as fixed with $t > 0$.

The following properties of Fourier transform are important to apply it as a method of solving differential equations.

Property 1 (Linearity property): If $u_1(x,t)$ and$u_2(x,t)$ are transformable functions and $a, b \in \mathbb{R}$, then

$$a)\quad \mathcal{F}\{au_1(x,t) + bu_2(x,t)\} = a\mathcal{F}\{u_1(x,t)\} + b\mathcal{F}\{u_2(x,t)\}$$

$$b)\quad \mathcal{F}^{-1}\{aU_1(\omega,t) + bU_2(\omega,t)\} = a\mathcal{F}^{-1}\{U_1(\omega,t)\} + b\mathcal{F}^{-1}\{U_2(\omega,t)\}$$

Proof: a) It is given that$\mathcal{F}\{u_1(x,t)\} = U_1(\omega,t)$, $\mathcal{F}\{u_2(x,t)\} = U_2(\omega,t)$.Then because integration is linear, we have

$$\mathcal{F}\{au_1(x,t)+bu_2(x,t)\} = \int_{-\infty}^{\infty}[au_1(x,t)+bu_2(x,t)]e^{-i\omega x}\,dx$$

$$= a\int_{-\infty}^{\infty}u_1\,(x,t)e^{-i\omega x}dx + b\int_{-\infty}^{\infty}u_2\,(x,t)e^{-i\omega x}dx$$

$$= aU_1(\omega,t)+bU_2(\omega,t) \;= a\mathcal{F}\{u_1(x,t)\}+b\mathcal{F}\{u_2(x,t)\}$$

This implies that,

$$\mathcal{F}\{au_1(x,t)+bu_2(x,t)\} = \; a\mathcal{F}\{u_1(x,t)\}+b\mathcal{F}\{u_2(x,t)\}.$$

b) By definition

$$\mathcal{F}^{-1}\{aU_1(\omega,t)+bU_2(\omega,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[aU_1(\omega,t)+bU_2(\omega,t)]\,e^{i\omega x}d\omega$$

$$= \frac{a}{2\pi}\int_{-\infty}^{\infty}U_1(\omega,t)e^{i\omega x}d\omega + \frac{b}{2\pi}\int_{-2\pi}^{\infty}U_2\,(\omega,t)e^{i\omega x}d\omega$$

$$= a\mathcal{F}^{-1}\{U_1(\omega,t)\}+b\mathcal{F}^{-1}\{U_2(\omega,t)\}$$

Hence,

$$\mathcal{F}^{-1}\{aU_1(\omega,t)+bU_2(\omega,t)\} = a\mathcal{F}^{-1}\{U_1(\omega,t)\}+b\mathcal{F}^{-1}\{U_2(\omega,t)\}$$

Property 2: (Fourier transforms of derivatives): Let $u(x,t)$be piecewise continuous function on$(-\infty,\infty)\times(0,\infty)$,$\mathcal{F}\{u(x,t)\} = U(\omega,t)$and $u_x(x,t)\to 0,\;u(x,t)\to 0$ as $|x|\to\infty$.

Suppose$u_x(x,t)$ and $u_{xx}(x,t)$ be absolutely integrable, then

a) $\mathcal{F}\{u_t(x,t)\} = \dfrac{\partial}{\partial t}U(\omega,t)$

b) $\mathcal{F}\{u_x(x,t)\} = (i\omega)U(\omega,t)$

c) $\mathcal{F}\{u_{xx}(x,t)\} = (i\omega)^2U(\omega,t) = -\omega^2U(\omega,t).$

Proof:a) By definition

$$\mathcal{F}\{u_t(x,t)\} = \int_{-\infty}^{\infty}u_t(x,t)e^{-i\omega x}dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}u(x,t)e^{-i\omega x}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx$$

$$= \frac{\partial}{\partial t}U(\omega,t)$$

Therefore,

$$\mathcal{F}\{u_t(x,t)\} = \frac{\partial}{\partial t}U(\omega,t)$$

b) Using integration by part once, we can transform the given function with respect to x:

$$\mathcal{F}\{u_x(x,t)\} = \int_{-\infty}^{\infty} u_x(x,t)e^{-i\omega x}\,dx$$

$$= e^{-i\omega x}u(x,t)|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx = i\omega U(\omega,t).$$

which follows from the assumptions on $u(x,t)$ approaches to zero as$|x|$ approaches to infinity.

Therefore,

$$\mathcal{F}\{u_x(x,t)\} = i\omega U(\omega,t)$$

c) Using integration by part twice, we can transform the given function with respect to$x$:

$$\mathcal{F}\{u_{xx}(x,t)\} = \int_{-\infty}^{\infty} u_{xx}(x,t)e^{-i\omega x}dx$$

$$= e^{-i\omega x}u_x(x,t)|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} u_x(x,t)e^{-i\omega x}dx$$

$$= i\omega\int_{-\infty}^{\infty} u_x(x,t)e^{-i\omega x}dx$$

which follows from the assumptions on $u_x(x,t)$ approaches to zero as $|x|$ approaches to infinity.

Then, also apply integration by part

$$\mathcal{F}\{u_{xx}(x,t)\} = i\omega\int_{-\infty}^{\infty} u_x(x,t)e^{-i\omega x}dx$$

$$= i\omega\left[e^{-i\omega x}u(x,t)|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx\right]$$

$$= i\omega\left[i\omega\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx\right]$$

$$= (i\omega)^2 U(\omega, t)$$

by assumptions on $u(x,t)$ and definition of Fourier transform.

Therefore,

$$\mathcal{F}\{u_{xx}(x,t)\} = (i\omega)^2 U(\omega, t) = -\omega^2 U(\omega, t).$$

The following lemmas are used to apply it as a method of solving differential equations.

Lemma 1.1[5]:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Proof: Let

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

So we can also rewrite the integral as

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right),$$

which can be rewritten as

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy.$$

Now, we make a change of variables by introducing polar coordinates. Let $x = rcos\theta$,

$y = rsin\theta$ and $dxdy = rdrd\theta$ with $r^2 = x^2 + y^2$, where $0 < r < \infty,\ 0 \le \theta \le 2\pi$. Then,

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy = \int_{0}^{2\pi}\left(\int_{0}^{\infty} e^{-r^2} rdr\right) d\theta,$$

So, by substitution, the integral becomes:

$$I^2 = \pi.$$

Thus,

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Lemma 1.2**:** The Fourier transform of $f(x) = e^{-kx^2}$isgiven by

$$F(\omega) = \sqrt{\frac{\pi}{k}} e^{\frac{-\omega^2}{4k}}, \quad k > 0.$$

Proof: From the definition of Fourier transform, we have

$$\mathcal{F}\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{-\infty}^{\infty} e^{-kx^2}\, e^{-i\omega x} dx$$

By completing the square method,

$$\int_{-\infty}^{\infty} e^{-kx^2}\, e^{-i\omega x} dx = \int_{-\infty}^{\infty} e^{-k(x+\frac{i\omega}{2k})^2 - \frac{\omega^2}{4k}}\, dx = e^{-\frac{\omega^2}{4k}} \int_{-\infty}^{\infty} e^{-k(x+\frac{i\omega}{2k})^2}\, dx$$

$$= e^{-\frac{\omega^2}{4k}} \int_{-\infty}^{\infty} e^{-\left[\sqrt{k}\left(x+\frac{i\omega}{2k}\right)\right]^2}\, dx.$$

Now let $u = \sqrt{k}\left(x + \frac{i\omega}{2k}\right)$, $du = \sqrt{k} dx$

$$\int_{-\infty}^{\infty} e^{-kx^2}\, e^{-i\omega x} dx = \frac{e^{\frac{-\omega^2}{4k}}}{\sqrt{k}} \int_{-\infty}^{\infty} e^{-u^2} du.$$

By Lemma 1.1, this gives as

$$F(\omega) = \sqrt{\frac{\pi}{k}} e^{-\frac{\omega^2}{4k}}$$

Lemma 1.3: The inverse Fourier transform of $F(\omega) = e^{-kt\omega^2}$is given by

$$\mathcal{F}^{-1}\left\{e^{-kt\omega^2}\right\} = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}},$$

where $k, t > 0$.

Proof: By definition of inverse of Fourier transform and completing the square method, we obtain

$$\mathcal{F}^{-1}\{e^{-kt\omega^2}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-kt\omega^2}e^{i\omega x}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-kt\left[\left(\omega-\frac{ix}{2kt}\right)^2+\frac{x^2}{(2kt)^2}\right]}\, d\omega$$

$$= \frac{1}{2\pi}e^{\frac{-x^2}{4kt}}\int_{-\infty}^{\infty} e^{-\left[\sqrt{kt}\left(\omega-\frac{ix}{2kt}\right)\right]^2}\, d\omega$$

Use substitution, $u = \sqrt{kt}\left(\omega - \frac{ix}{2kt}\right)$, we have

$$\frac{1}{\sqrt{kt}}du = d\omega.$$

Hence, by Lemma1.1

$$\mathcal{F}^{-1}\{e^{-kt\omega^2}\} = \frac{1}{2\pi\sqrt{kt}}e^{\frac{-x^2}{4kt}}\int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{1}{\sqrt{4\pi kt}}e^{\frac{-x^2}{4kt}}$$

Definition1.6: (Convolution): Suppose f and $g$are two transformable functions. Then the convolution of f and g denoted by $f * g$ and is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt = \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

Theorem 1.1: If $F(\omega)$ and $G(\omega)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then the Fourier transforms of convolution of f and g is given by

$$\mathcal{F}\{f(x) * g(x)\} = \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\} = F(\omega)G(\omega)$$

Proof: By definition of Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$G(\omega) = \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx$$

and

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

Then, we have

$$\mathcal{F}\{f(x) * g(x)\} = \int_{-\infty}^{\infty} e^{-i\omega x}\left[\int_{-\infty}^{\infty} f(t)g(x-t)dt\right]dx.$$

By changing the order of integration, we get

$$\mathcal{F}\{f(x) * g(x)\} = \int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} e^{-i\omega x} g(x-t)dx\right]dt.$$

Putting $x - t = u \Longrightarrow dx = du$ in the inner integration, we obtain

$$\mathcal{F}\{f(x) * g(x)\} = \int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} e^{-i\omega(t+u)} g(u)du\right]dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\left[\int_{-\infty}^{\infty} e^{-i\omega u} g(u)du\right]dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)G(\omega)dt = G(\omega)\int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt$$

$$= G(\omega)F(\omega) = F(\omega)G(\omega).$$

Hence,

$$\mathcal{F}\{f(x) * g(x)\} = F(\omega)G(\omega).$$

Example1.1. Given an initial value problem of one dimensional heat equation.

$$\begin{cases} u_t - ku_{xx} = h(x,t), \quad t > 0, \quad x \in (-\infty, \infty), \\ u(x,0) = f(x), \quad x \in (-\infty, \infty). \end{cases} \tag{1.3}$$

Assume that both the unknown function and its derivative vanish when x approaching to $\pm\infty$. That is

$$u(x,t)|_{x\to\pm\infty} = 0, \qquad \frac{\partial}{\partial x}u(x,t)|_{x\to\pm\infty} = 0.$$

Take the Fourier transform of the given equation with respect to x and by applying properties of derivatives. Then the given equation becomes

$$\frac{\partial}{\partial t}U(\omega,t) + k\omega^2 U(\omega,t) = \hbar(\omega,t)$$

with the transformed initial condition , $U(\omega,0) = F(\omega)$, where the following notations for the transformed functions are used

$$\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx = U(\omega,t)$$

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}dx = F(\omega)$$

$$\mathcal{F}\{h(x,t)\} = \int_{-\infty}^{\infty} h(x,t)e^{-i\omega x}\, dx = \hbar(\omega,t).$$

The transformed equation is linear ordinary differential equation in variable t and takes ω as a parameter:

$$\frac{\partial}{\partial t}U(\omega,t) + k\omega^2 U(\omega,t) = \hbar(\omega,t) \tag{1.4}$$

$$U(\omega,0) = F(\omega).$$

Then, the solution of equation (1.4) found by taking an integrating factor$e^{k\omega^2 t}$.Hence,

$$U(\omega,t) = e^{-k\omega^2 t}\int_0^t e^{k\omega^2 s}\hbar(\omega,s)ds + e^{-k\omega^2 t}A(\omega).$$

From the initial condition $U(\omega,0) = F(\omega)$, we have

$$U(\omega,0) = F(\omega) = A(\omega).$$

Therefore, the general solution is

$$U(\omega,t) = e^{-k\omega^2 t}\int_0^t e^{k\omega^2 s}\hbar(\omega,s)ds + e^{-k\omega^2 t}F(\omega).$$

$$= e^{-k\omega^2 t}F(\omega) + \int_0^t e^{-k\omega^2 (t-s)}\hbar(\omega,s)ds \tag{1.5}$$

Since equation (1.5) is the solution of equation (1.3),so to find the solution in variable $x$take the inverse Fourier transform of equation(1.5).That is,

$$u(x,t) = \mathcal{F}^{-1}\{U(\omega,t)\} = \mathcal{F}^{-1}\left\{e^{-k\omega^2 t}F(\omega)\right\} + \mathcal{F}^{-1}\left\{\int_0^t e^{-k\omega^2(t-s)}\hbar(\omega,s)ds\right\}$$

$$= \mathcal{F}^{-1}\{F(\omega)G(\omega,t)\} + \mathcal{F}^{-1}\left\{\int_0^t \hbar(\omega,s)G(\omega,t-s)ds\right\}$$

where,

$$G(\omega,t)=e^{-k\omega^2 t},\qquad G(\omega,t-s)=e^{-k\omega^2(t-s)}$$

Then, the first term in the solution is the inverse Fourier transform of the product of two transformed functions $F(\omega)G(\omega,t)$ which according to a convolution of these two functions:

$$\mathcal{F}^{-1}\{F(\omega)G(\omega,t)\}=f*g=\int_{-\infty}^{\infty}f(\xi)g(x-\xi,t)d\xi=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}e^{\frac{-(x-\xi)^2}{4kt}}f(\xi)d\xi,$$

where from Lemma 1.3,

$$G(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{\frac{-x^2}{4kt}}$$

is the Green's function (heat kernel).

Consider the second term in the solution leads to the convolution:

$$\mathcal{F}^{-1}\{G(\omega,t-s)\hbar(\omega,s)\}=g*f=\int_0^t\int_{-\infty}^{\infty}g(x-\xi,t-s)\,h(\xi,s)d\xi ds$$
$$=\int_0^t\frac{1}{\sqrt{4\pi k(t-s)}}\int_{-\infty}^{\infty}e^{\frac{-(x-\xi)^2}{4k(t-s)}}\,h(\xi,s)d\xi ds.$$

Therefore, the solution of the given initial value problem is given by

$$u(x,t)=v(x,t)+\int_0^t\mathbb{W}(x,t-s)\,ds \tag{1.6}$$

where

$$v(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}e^{\frac{-(x-\xi)^2}{4kt}}f(\xi)d\xi$$

is the solution of

$$\begin{cases}u_t-ku_{xx}=0, & -\infty<x<\infty,\ t>0,\\ u(x,0)=f(x), & -\infty<x<\infty\end{cases}$$

and

$$\mathbb{W}(x,t-s)=\frac{1}{\sqrt{4\pi k(t-s)}}\int_{-\infty}^{\infty}e^{\frac{-(x-\xi)^2}{4k(t-s)}}\,h(\xi,s)d\xi$$

is the solution of

$$\begin{cases} u_t - ku_{xx} = h(x,t), & -\infty < x < \infty, \ \ t > 0, \\ u(x,0) = 0, & -\infty < x < \infty. \end{cases}$$

Example1.2: Consider the one-dimensional wave equations with initial conditions

$$\begin{cases} u_{tt} = c^2 u_{xx} + h(x,t) & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty, \\ u_t(x,0) = g(x), & -\infty < x < \infty. \end{cases} \tag{1.7}$$

To solve equation (1.7), we consider the Fourier transforms:

$$U(\omega,t) = \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx \,,$$

$$\hbar(\omega,t) = \int_{-\infty}^{\infty} h(x,t) e^{-i\omega x} dx$$

and the corresponding inverse of Fourier transforms:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega,t) e^{i\omega x} \, d\omega.$$

$$h(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hbar(\omega,t) e^{i\omega x} d\omega$$

By applying the Fourier transform for equation (1.7) and using the properties of the transforms of a function and its derivatives, we obtain a second order ordinary differential equation with respect to t:

$$\frac{\partial^2 U(\omega,t)}{\partial t^2} = -c^2 \omega^2 U(\omega,t) \ + \hbar(\omega,\mathrm{t}) \tag{1.8}$$

with initial conditions

$$U(\omega,0) = \mathcal{F}\{u(x,0)\} = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$\frac{\partial U(\omega,0)}{\partial t} = G(\omega) = \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \,.$$

But to solve equation (1.8), first find the solution of the homogeneous part(complementary solution) and the solution is given by

$$U_c(\omega,t) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t).$$

Next, by variation of parameter the particular solution of equation (1.8) is given

$$U_p(\omega,t) = -\cos(c\omega t)\int_0^t \frac{sin(c\omega s)\hbar(\omega,s)}{c\omega}ds \ +\sin(c\omega t)\int_0^t \frac{\cos(c\omega s)\hbar(\omega,s)}{c\omega}ds.$$

Finally, we obtain the general solution of equation (1.8):

$$U(\omega,t) = U_c(\omega,t) + U_p(\omega,t)$$

$$= A(\omega)\cos(c\omega t)\ + B(\omega)sin(c\omega t) - \cos(c\omega t)\int_0^t \frac{sin(c\omega s)\hbar(\omega,s)}{c\omega}ds$$
$$+\sin(c\omega t)\int_0^t \frac{\cos(c\omega s)\hbar(\omega,s)}{c\omega}ds.$$

And from the initial conditions, we get

$$U(\omega,0) = A(\omega), \qquad B(\omega) = \frac{G(\omega)}{c\omega}.$$

Therefore,

$$U(\omega,t) = U(\omega,0)\cos(c\omega t)$$
$$+\frac{G(\omega)}{c\omega}\sin(c\omega t) - \cos(c\omega t)\int_0^t \frac{sin(c\omega s)\hbar(\omega,s)}{c\omega}ds$$
$$+\sin(c\omega t)\int_0^t \frac{\cos(c\omega s)\hbar(\omega,s)}{c\omega}ds$$

is the solution of equation (1.8).

By finding the inverse of Fourier transform the last equation, we obtain the solution of equation (1.7)

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Bigg[U(\omega,0)\cos(c\omega t) + \frac{G(\omega)}{c\omega}\sin(c\omega t) - \cos(c\omega t)\int_0^t \frac{sin(c\omega s)\hbar(\omega,s)}{c\omega}ds$$
$$+\sin(c\omega t)\int_0^t \frac{\cos(c\omega s)\hbar(\omega,s)}{c\omega}ds\Bigg]e^{i\omega x}d\omega$$

To simplify this expression, we use Euler's formula

$$\cos(c\omega t) = \frac{e^{ic\omega t} + e^{-ic\omega t}}{2}$$

Then, the solution becomes

$$u(x,t) = \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,0)\left(e^{i\omega(x+ct)} + e^{i\omega(x-ct)}\right)d\omega\right] + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(\omega)}{c}\frac{\sin(c\omega t)}{\omega}e^{i\omega x}d\omega$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-\cos(c\omega t)\int_0^t \frac{sin(c\omega s)\hbar(\omega,s)}{c\omega}ds \ +\sin(c\omega t)\int_0^t \frac{\cos(c\omega s)\hbar(\omega,s)}{c\omega}ds\right]e^{i\omega x}d\omega.$$

But from the first integral, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}U(\omega,0)\left(e^{i\omega(x+ct)} + e^{i\omega(x-ct)}\right)d\omega = \frac{1}{2}[f(x-ct) + f(x+ct)].$$

And from the second integral, we obtain

$$\mathcal{F}^{-1}\left\{2\frac{\sin(c\omega t)}{\omega}\right\} = H(x+ct) - H(x-ct)$$

$$\mathcal{F}^{-1}\left\{\frac{\sin(c\omega t)}{\omega}\right\} = \frac{1}{2}\left(H(x+ct) - H(x-ct)\right),$$

where $\left(H(x+ct) - H(x-ct)\right)$ is unit pulse function, which is defined by

$$\left(H(x+ct) - H(x-ct)\right) = \begin{cases} 0\,, x < -ct \\ 1, |x| < ct \\ 0, x > ct. \end{cases}$$

By using convolution theorem,

$$\mathcal{F}^{-1}\left\{\frac{G(\omega)}{c}\frac{\sin(c\omega t)}{\omega}\right\} = \frac{1}{2c}\int_{-\infty}^{\infty} g(x-\xi)[H(\xi+ct) - H(\xi-ct)]\,d\xi$$

$$= \frac{1}{2c}\int_{-\infty}^{\infty} g(x-\xi)H(\xi+ct)d\xi - \frac{1}{2c}\int_{-\infty}^{\infty} g(x-\xi)H(\xi-ct)d\xi$$

$$= \frac{1}{2c}\int_{-ct}^{\infty} g(x-\xi)d\xi - \frac{1}{2c}\int_{ct}^{\infty} g(x-\xi)d\xi = \frac{1}{2c}\int_{-ct}^{ct} g(x-\xi)d\xi$$

By applying integration by substitution, we get

$$\mathcal{F}^{-1}\left\{\frac{G(\omega)}{c}\frac{\sin(\mathrm{c\omega t})}{\omega}\right\} = \frac{1}{2\mathrm{c}}\int_{x-ct}^{x+ct} g(z)dz.$$

From the third and fourth integral with convolution theorem, we have

$$\mathcal{F}^{-1}\left\{\left[-\cos(c\omega t)\int_0^t \frac{sin(c\omega s)\hbar(\omega,s)}{c\omega}ds + \sin(c\omega t)\int_0^t \frac{\cos(c\omega s)\hbar(\omega,s)}{c\omega}ds\right]\right\}$$
$$= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(\xi,s)d\xi ds$$

Therefore, the solution of equation (1.7) is given by

$$u(x,t) = v(x,t) + \int_0^t \mathbb{W}(x,t-s)\,ds \qquad (1.9).$$

where

$$v(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz$$

which is the solution of the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, -\infty < x < \infty, t > 0, \\ u(x,0) = f(x), -\infty < x < \infty, \\ u_t(x,0) = g(x), -\infty < x < \infty. \end{cases}$$

and

$$\mathbb{W}(x,t-s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} h(\xi,s)d\xi$$

is the solution of the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), -\infty < x < \infty, t > 0, \\ u(x,0) = 0, -\infty < x < \infty, \\ u_t(x,0) = 0, -\infty < x < \infty\,. \end{cases}$$

1.3.2 The method of separation of variables

Separation of variables (also known as the Fourier method) is a method for solving ordinary and partial differential equations in which algebra allows one to redraft an equation so that each of two variables occurs on a different side of the equation. It is also used to solve a wide range of linear partial differential equations with boundary and initial conditions, such as heat and wave equations. Such a method consists in the following three main steps.

Step1: One searches for solutions of the homogeneous problem, which are called product solutions or divided solutions. These solutions have the following form

$$u(x,t) = X(x)T(t) \tag{1.10}$$

we notice that X is a function of x only and T is a function of t.

In general such solutions should satisfy certain additional conditions. It turns out that X and T satisfy suitable linear ordinary differential equations which are easily derived from the given partial differential equation.

Step2: Use a generalization of the superposition principle to generate out of the separated solutions a more general solution of the homogeneous partial differential equation, in the form of an infinite series of product solutions.

Step3: We compute the coefficients of the series to satisfy the initial conditions.

Example1.3.Consider the following initial-boundary value problem associated to the heat equation:

$$\begin{cases} u_t - ku_{xx} = h(x,t), 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L. \end{cases} \tag{1.11}$$

The problem (1.11) corresponds to the evolution of the temperature$u(x,t)$ in a non homogeneous one-dimensional heat conducting rod of length L, whose initial temperature (at time t = 0) is known and its two ends are immersed in a zero temperature bath.

The problem (1.11) is an initial boundary value problem that is linear and non homogeneous. So, we can apply the method of separation of variables described above. We start by looking for solutions of the homogeneous equation that satisfy the boundary conditions that have the special form

$$u(x,t) = X(x)T(t)$$

At this step, we do not take into account the initial condition $u(x,0) = f(x)$. Obviously, we are not interested in the zero solution $u(x,t) = 0$.

Therefore, we seek functions X and T that do not vanish identically. By working the solution into the partial differential equation, we get

$$XT' = kX''T.$$

Now, we move to one side of the partial differential equation all the functions that depend only on x and to the other side the functions that depend only on t. We get

$$\frac{T'}{kT} = \frac{X''}{X}$$

Since x and t are independent variables, there exists a constant denoted by $-\lambda$ (which is called the separation constant) such that

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda\,.$$

This leads to the following ordinary differential equations:

$$\begin{cases} X'' = -\lambda X, & 0 < x < L, \\ T' = -\lambda kT, & t > 0 \end{cases}$$

The above ordinary differential equations are coupled only by the separation constant$-\lambda$.Thefunction u satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ if andonly if

$$u(0,t) = X(0)T(t) = 0\,, \qquad \forall t > 0$$

$$u(L,t) = X(L)T(t) = 0\,, \qquad \forall t > 0.$$

The above two conditions are satisfied if and only if either T(t) = 0 for all t > 0 (which gives the trivial solution) or X(0) = X(L) = 0 (which represents the interesting case).Now, we are going to see non trivial solution for ordinary differential equations.

Differential equation for X:

The function X should be a solution of the boundary value problem

$$\begin{cases} X'' = -\lambda X, \quad 0 < x < L, \\ X(0) = X(L) = 0, \quad t > 0. \end{cases} \tag{1.12}$$

This problem is called eigenvalue problem. A nontrivial solution of the problem is called an Eigen-function with an eigenvalue$-\lambda$.It is known that the general solution of the second order linear ordinary differential equation is of the form

$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ with $c_1, c_2 \in \mathbb{R}$,if $\lambda < 0$.

$X(x) = c_1 + c_2 x$,with $c_1, c_2 \in \mathbb{R}$,if $\lambda = 0$.

$X(x) = \ c_1 cos(\sqrt{\lambda}x) + \ c_2 sin(\sqrt{\lambda}x)$,with $c_1, c_2 \in \mathbb{R}$,if $\lambda > 0$.

Next we will see separately the three cases.

Negative eigenvalue: λ< 0.

The solution is $X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$. By applying boundary conditions $X(0) = X(L) = 0$, we get c_1= c_2= 0, which implies$X(x) \equiv 0$. Thus, the system (1.12) does not admit negative eigenvalues.

Zero eigenvalue:λ = 0.

The solution is$X(x) = c_1 + c_2 x$. By imposing boundary conditions $X(0) = X(L) = 0$, we getc_1= c_2= 0. Similarly, the system (1.12) does not admit zero eigenvalues.

Positive eigenvalue:λ> 0.

The solution is $X(x) = \ c_1 cos(\sqrt{\lambda}x) + \ c_2 sin(\sqrt{\lambda}x)$. The condition X(0) = 0 gives c_1 = 0. The boundary condition X(L) = 0 implies either c_2= 0 orsin($\sqrt{\lambda}$L) = 0, which is the interesting case. Therefore$\sqrt{\lambda}$ L =$n\pi$, n> 0.

Hence $\lambda$is an eigenvalue if and only if

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3, \dots \tag{1.13}$$

The corresponding Eigen functions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{1.14}$$

Thus, the set of all solutions of (1.12) is an infinite sequence of Eigen functions, each associated with a positive eigenvalue. We use the notation

$$X_n(x) = \sin(\lambda_n x)$$

where for all $n > 0$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$.

Differential equation for T:

The general solution of $\frac{dT}{dt} = -\lambda k T$ is $T(t) = Be^{-k\lambda t}$. Substitutingλ_n, we obtain the sequence $T_n(t) = B_n e^{-k\lambda_n t}$.

The sequence of separated solution is given by

$$u_n(x,t) = X_n(x)T_n(t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\lambda_n t}.$$

The superposition principle implies that any finite linear combination of the separated solutions is still a solution of the heat equation. That is

$$u(x,t) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\lambda_n t} \tag{1.15}$$

Now, we are able to solve the problem for a certain family of initial conditions. Suppose that $f(x)$ admits the following Fourier sine series. From the initial condition, we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \tag{1.16}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \tag{1.17}$$

Therefore, a solution of a homogeneous equation in (1.11) is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right)\sin\left(\frac{n\pi}{L}x\right)e^{-k\lambda_n t}$$

Next to solve a non- homogeneous equation, we use a solution formula

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right).$$

Formally computing u_t andu_{xx} and substituting to (1.11), we get

$$u_t - ku_{xx} = \sum_{n=1}^{\infty} \left[T_n{}'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{L}x\right).$$

Hence expanding h and f into the Fourier series

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi}{L}x\right)$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{L}x\right)$$

where

$$h_n(t) = \frac{2}{L}\int_0^L h(x,t) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$f_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

Substituting into the equation gives

$$\sum_{n=1}^{\infty} \left[T_n{}'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} h_n(\mathrm{t})\sin\left(\frac{n\pi}{L}x\right).$$

The uniqueness of the Fourier expansion leads to the family of ordinary differential equations

$$T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t) = h_n(\mathrm{t}) \tag{1.18}$$

In addition,

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

So that

$$T_n(0) = f_n\,, \qquad\qquad n = 1,2,3,\ldots \tag{1.19}$$

Solving (1.18) and (1.19), we obtain

$$T_n(t) = f_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)} h_n(s)ds$$

Therefore, the solution of equation (1.11) is given by

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)} h_n(s)ds$$

which is rewritten as

$$u(x,t) = v(x,t) + \int_0^t \mathbb{W}(x, t-s)ds$$

where

$$v(x,t) = \sum_{n=1}^{\infty} f_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \qquad f_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

is the solution of the problem

$$\begin{cases} u_t - ku_{xx} = 0, \quad 0 < x < L, \;\; t > 0, \\ u(0,L) = u(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L. \end{cases}$$

and

$$\mathbb{W}(x,t-s) = \sum_{n=1}^{\infty} h_n(s)\sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)}, \;\; h_n(s) = \frac{2}{L}\int_0^L h(x,s)\sin\left(\frac{n\pi x}{L}\right)dx$$

is the solution of the problem

$$\begin{cases} u_t - ku_{xx} = h(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,L) = u(L,t) = 0, \quad t > 0, \\ u(x,0) = 0, \quad 0 \le x \le L. \end{cases}$$

Example1.4. Consider an initial boundary value problem for wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), \ \ 0 \le x \le L, \ \ t > 0, \\ u(0,t) = 0 = u(L,t), \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L, \\ u_t(x,0) = g(x), \quad 0 \le x \le L. \end{cases} \tag{1.20}$$

First, we need to find solution of homogeneous equation.

Let the solution be the form

$$u(x,t) = X(x)T(t) \tag{1.21}$$

Substituting equation (1.21) into the differential equation (1.20), we get

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda^2 \tag{1.22}$$

where λ is constant.

The solution of homogeneous equation in (1.20) is the solution of the ordinary differential equations

$$T''(t) - \lambda^2 c^2 T(t) = 0,$$

$$X''(x) - \lambda^2 X(x) = 0.$$

For $\lambda \geq 0$,the solution is identically zero.

For $\lambda < 0$, the solution of the above equation are given by

$$T(t) = c_1 \cos(\lambda ct) + c_2 \sin(\lambda ct)$$

$$X(x) = c_3 \cos(\lambda x) + c_4 \sin(\lambda x).$$

But from the given initial condition, we get

$$\lambda_n = \frac{n\pi}{L}, \quad X_n(x) = c_n \sin\left(\frac{n\pi}{L}x\right), \qquad n = 1,2,3,\ldots$$

For each value of n, the solution of homogeneous equation in (1.20) becomes

$$u_n(x,t) = \left[\mathrm{A_n} \cos\left(\frac{n\pi c}{L}\mathrm{t}\right) + B_n \sin\left(\frac{n\pi c}{L}\mathrm{t}\right)\right] \sin\left(\frac{n\pi}{L}x\right) \tag{1.23}$$

By superposition principle, the sum of all these solutions is also a solution. Therefore, we have

$$u(x,t) = \sum_{n=1}^{\infty} \left[\mathrm{A_n} \cos\left(\frac{n\pi c}{L}\mathrm{t}\right) + B_n \sin\left(\frac{n\pi c}{L}\mathrm{t}\right)\right] \sin\left(\frac{n\pi}{L}x\right) \tag{1.24}$$

By applying the initial conditions, we get

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi}{L}x\right)$$

These are the sine series expansion of $f(x)$ and $g(x)$ respectively.Multiplying both sides with $\sin\left(\frac{n\pi}{L}x\right)$and integrating over $[0,L]$ with respect to x, then the result becomes

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \tag{1.25}$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx \tag{1.26}$$

Substituting equation (1.25) and equation (1.26) in equation(1.24),we get the required solution.

Next we need to find solution of non homogeneous wave equation by using a formula

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right) \tag{1.27}$$

Formally computing u_{tt} and u_{xx} and substituting to (1.27), we get

$$u_{tt} - c^2 u_{xx} = \sum_{n=1}^{\infty} \left[T_n{}''(t) + c^2 \left(\frac{n\pi}{L}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{L}x\right).$$

Hence expanding h, f and g into the Fourier series

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi}{L}x\right),$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{L}x\right),$$

$$g(x) = \sum_{n=1}^{\infty} g_n(\sin\left(\frac{n\pi}{L}x\right)$$

where

$$h_n(t) = \frac{2}{L}\int_0^L h(x,t) \sin\left(\frac{n\pi}{L}x\right) dx,$$

$$f_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx,$$

$$g_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

Substituting into the equation gives

$$\sum_{n=1}^{\infty} \left[T_n{}''(t) + c^2 \left(\frac{n\pi}{L}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} h_n(\mathrm{t})\sin\left(\frac{n\pi}{L}x\right).$$

The uniqueness of the Fourier expansion leads to the family of ordinary differential equations

$$T_n''(t) + c^2 \left(\frac{n\pi}{L}\right)^2 T_n(t) = h_n(\mathrm{t}) \tag{1.28}$$

In addition,

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} T_n{}'(0) \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right) = g(x).$$

So that

$$T_n(0) = f_n$$

$$T_n'(0) = g_n, \qquad n = 1,2,3, \ldots \tag{1.29}$$

Solving (1.28) and (1.29), we obtain

$$T_n(t) = f_n \cos\left(\frac{cn\pi}{L} t\right) + \frac{L}{cn\pi} g_n \sin\left(\frac{cn\pi}{L} t\right) + \frac{L}{cn\pi} \int_0^t h_n(s) \left[\sin\left(\frac{cn\pi}{L}(t-s)\right)\right] ds.$$

Therefore, the solution of (1.20) is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[f_n \cos\left(\frac{cn\pi}{L} t\right) \right.$$
$$\left. + \frac{L}{cn\pi} g_n \sin\left(\frac{cn\pi}{L} t\right) + \frac{L}{cn\pi} \int_0^t h_n(s) \left[\sin\left(\frac{cn\pi}{L}(t-s)\right)\right] ds \right] \sin\left(\frac{n\pi}{L} t\right).$$

This solution can be rewritten as

$$u(x,t) = v(x,t) + \int_0^t \mathbb{W}(x, t-s) ds$$

where

$$v(x,t) = \sum_{n=1}^{\infty} \left[f_n \cos\left(\frac{cn\pi}{L} t\right) + \frac{L}{cn\pi} g_n \sin\left(\frac{cn\pi}{L} t\right)\right] \sin\left(\frac{n\pi}{L} x\right)$$

$$f_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx, g_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

is the solution of the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) = 0 = u(L,t), \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L, \\ u_t(x,0) = g(x), \quad 0 \le x \le L. \end{cases}$$

and

$$\mathbb{W}(x, t-s) = \sum_{n=1}^{\infty} \frac{L}{cn\pi} h_n(s) \sin\left(\frac{cn\pi}{L}(t-s)\right) \sin\left(\frac{n\pi}{L} x\right),$$

$$h_n(s) = \frac{2}{L} \int_0^L h(x,s) \sin\left(\frac{n\pi}{L} x\right) dx$$

is the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) = 0 = u(L,t), \quad t > 0, \\ u(x,0) = 0, \quad 0 \le x \le L, \\ u_t(x,0) = 0, \quad 0 \le x \le L. \end{cases}$$

CHAPTER-TWO

APPLICATION OF DUHAMEL'S PRINCIPLE

2.1. Duhamel's principle

Duhamel's principle is a way to express the solution of a nonhomogeneous partial differential equations as an integral of the solution of a homogeneous equation with appropriate initial or/and boundary conditions. The Duhamel's principle is a general method for obtaining solutions to nonhomogeneous linear evolution equations like the heat equation and wave equation. Duhamel's principle is a method which used to break down the given nonhomogeneous heat and wave equations. After break down the initial or initial-boundary value problems for nonhomogeneous heat/wave equations given in equations (1.3), (1.7), (1.11) and (1.20) into two simpler problems, we obtain

Homogenous equations with nonhomogeneous initial condition(s) and

Nonhomogeneous equations with homogeneous initial condition(s)

whose solutions for the later problems are obtained by the well-known technique called Duhamel's principle as follows.

From Example 1.1, if $f(x) = 0$, then we deduce that the solution of (1.3), that is

$$\begin{cases} u_t - ku_{xx} = h(x,t), \quad t > 0, \quad x \in (-\infty, \infty), \\ u(x,0) = 0, \quad x \in (-\infty, \infty). \end{cases}$$

is given by

$$u(x,t) = \int_0^t \mathbb{W}(x, t-s)ds$$

where

$$\mathbb{W}(x, t-s) = \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-\xi)^2}{4k(t-s)}} h(\xi, s)d\xi,$$

t is time , every fixed $s > 0$ $(t > s), h$ is source function,k is heat conduction constant, x is space variable and ξ is initial horizontal distance.

From Example 1.2, if $f(x) = 0$ and $g(x) = 0$, then we deduce that the solution of (1.7), that is

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), t > 0, x \in (-\infty, \infty), \\ u(x,0) = 0, x \in (-\infty, \infty), \\ u_t(x,0) = 0, x \in (-\infty, \infty). \end{cases}$$

is given by

$$u(x,t) = \int_0^t \mathbb{W}(x, t-s)ds$$

where

$$\mathbb{W}(x, t-s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} h(\xi, s) d\xi.$$

In Example 1.3, if $f(x) = 0$, then we deduce that the solution of (1.11), that is

$$\begin{cases} u_t - k u_{xx} = h(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) = u(L,t) = 0, \qquad t > 0, \\ u(x,0) = 0, \quad 0 \le x \le L. \end{cases}$$

is given by

$$u(x,t) = \int_0^t \mathbb{W}(x, t-s)ds$$

where

$$\mathbb{W}(x, t-s) = \sum_{n=1}^{\infty} h_n(s) \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)}$$

with

$$h_n(s) = \frac{2}{L} \int_0^L h(x,s) \sin\left(\frac{n\pi}{L} x\right) dx.$$

In Example 1.4, if $f(x) = 0$ and $g(x) = 0$, then we deduce that the solution of (1.20), that is

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), & 0 \le x \le L, \quad t > 0, \\ u(0,t) = 0 = u(L,t), & t > 0, \\ u(x,0) = 0, & 0 \le x \le L, \\ u_t(x,0) = 0, & 0 \le x \le L. \end{cases}$$

is given by

$$u(x,t) = \int_0^t \mathbb{W}(x, t-s) ds$$

where

$$\mathbb{W}(x, t-s) = \sum_{n=1}^{\infty} \frac{L}{cn\pi} h_n(s) \sin\left(\frac{cn\pi}{L}(t-s)\right) \sin\left(\frac{n\pi}{L}x\right)$$

with

$$h_n(s) = \frac{2}{L} \int_0^L h(x,s) \sin\left(\frac{n\pi}{L}x\right) dx.$$

2.2. Duhamel's principle for heat equations

In this section, we will solve nonhomogeneous heat equation by using Duhamel's principle. A problem which we are looking for unknown function of a differential equation whose values of the unknown function and /or derivative at a single point are known, then the problem is said to be initial value problem. If the values of the unknown function and/or its derivatives are known at boundary points, then the problem is called boundary value problem. If it involves both initial and boundary conditions, then it is an initial boundary value problem.

2.2.1 Initial value problem for heat equations

Consider the initial value problem for non-homogeneous heat equation:

$$\begin{cases} u_t - k u_{xx} = h(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty. \end{cases} \tag{2.1}$$

Now, breakdown (2.1) into two simpler problems of the form

$$\begin{cases} v_t - k\, v_{xx} = 0, & -\infty < x < \infty, \quad t > 0, \\ v(x,0) = f(x), & -\infty < x < \infty. \end{cases} \tag{2.2}$$

and

$$\begin{cases} w_t - k\, w_{xx} = h(x,t), & -\infty < x < \infty, \quad t > 0, \\ w(x,0) = 0, & -\infty < x < \infty. \end{cases} \tag{2.3}$$

Solving (2.2) and (2.3) gives

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-\xi)^2}{4kt}} f(\xi) d\xi,$$

$$w(x,t) = \int_0^t \left(\frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-\xi)^2}{4k(t-s)}} h(\xi,s) d\xi \right) ds.$$

Note that

$$w(x,t) = \int_0^t \mathbb{W}(x, t-s)\, ds$$

where

$$\mathbb{W}(x,t-s) = \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-\xi)^2}{4k(t-s)}} h(\xi,s) d\xi$$

Therefore, the solution of equation (2.1) is given by

$$u(x,t) = v(x,t) + w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-\xi)^2}{4kt}} f(\xi) d\xi$$

$$+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x-\xi)^2}{4k(t-s)}} h(\xi,s) d\xi ds \tag{2.4}$$

Example2.1.Consider the initial value problem for heat equation:

$$\begin{cases} u_t - u_{xx} = x + t, & -\infty < x < \infty, \quad t > 0. \\ u(x,0) = 3x, & -\infty < x < \infty. \end{cases} \tag{2.5}$$

Solution: Let's breakdown equation (2.5) into two simpler problems:

$$\begin{cases} v_t - v_{xx} = 0, -\infty < x < \infty, t > 0, \\ v(x,0) = 3x, \quad -\infty < x < \infty. \end{cases} \tag{2.6}$$

and

$$\begin{cases} w_t - w_{xx} = x + t, \; -\infty < x < \infty, \;\; t > 0, \\ w(x,0) = 0, \;\; -\infty < x < \infty. \end{cases} \tag{2.7}$$

In view of (2.4), $k = 1,\; f(\xi) = 3\xi,\; h(\xi, s) = \xi + s$. Then, equation (2.6) is solved by the formula

$$v(x,t) = \frac{3}{\sqrt{4\pi(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-\xi)^2}{4t}} \xi d\xi = 3x$$

And equation (2.7) is solved by Duhamel's principle

$$w(x,t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left(\int_{-\infty}^{\infty} e^{\frac{-(x-\xi)^2}{4(t-s)}} (\xi + s) d\xi \right) ds = \frac{t^2}{2} + xt$$

Therefore, the solution to (2.5) is given by

$$u(x,t) = v(x,t) + w(x,t) = 3x + \frac{t^2}{2} + xt, \quad t > s, -\infty < x < \propto, t > 0.$$

2.2.2 Initial boundary value problem for heat equation

Consider the initial boundary value problem for a nonhomogeneous heat equation.

$$\begin{cases} u_t - ku_{xx} = p(x,t), \;\; 0 \le x \le L, \;\; t > 0, \\ u(0,t) = T_0(t), u(L,t) = T_1(t), \;\; t > 0, \\ u(x,0) = g(x), \;\; 0 \le x \le L. \end{cases} \tag{2.8}$$

First, reduce the boundary conditions to homogeneous. That is, take an arbitrary function $v(x,t)$ satisfying the nonhomogeneous boundary conditions:

$$v(0,t) = T_0(t), \quad v(L,t) = T_1(t), \quad t > 0.$$

For instance, one can take

$$v(x,t) = T_0(t) + \frac{x}{L}[T_1(t) - T_0(t)], \quad 0 < x < L, \quad t > 0.$$

Secondly, consider $w(x,t) = u(x,t) - v(x,t)$, as the new unknown.

Then problem for $w(x,t)$ becomes

$$
\begin{cases}
w_t - kw_{xx} = h(x,t), \;\; 0 \le x \le L, \;\; t > 0\,, \\
w(0,t) = 0, \;\; w(L,t) = 0, \;\; t > 0, \\
w(x,0) = f(x)\,, \;\; 0 \le x \le L,
\end{cases}
\tag{2.9}
$$

where $h(x,t)$and $f(x)$ are given by

$$
h(x,t) = p(x,t) - v_t + kv_{xx} = p(x,t) - T'_0(t) - \frac{x}{L}[T_1{}'(t) - T_0{}'(t)],
$$

$$
f(x) = g(x) - v(x,0) \; = g(x) - T_0(0) - \frac{x}{L}[T_1(0) - T_0(0)].
$$

Now, break down equation (2.9) into two simpler problems:

$$
\begin{cases}
W_t - kW_{xx} \; = 0\,, \;\; 0 \le x \le L\,, \;\; t > 0, \\
W(0,t) = 0, \;\; W(L,t) = 0, \;\; t > 0, \\
W(x,0) = f(x), \;\; 0 \le x \le L,
\end{cases}
\tag{2.10}
$$

and

$$
\begin{cases}
q_t - kq_{xx} = h(x,t), \;\; 0 \le x \le L, \;\; t > 0, \\
q(0,t) = 0, \;\; q(L,t) = 0, \;\; t > 0, \\
q(x,0) = 0\,, \;\; 0 \le x \le L.
\end{cases}
\tag{2.11}
$$

Then, equation (2.10) is solved by

$$
W(x,t) = \sum_{n=1}^{\infty} f_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right),
$$

where

$$
f_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.
$$

Equation (2.11) is solved by Duhamel's principle

$$
q(x,t-s) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)} h_n(s)ds
$$

where

$$h_n(s) = \frac{2}{L}\int_0^L h(x,s)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Hence, equation (2.8) is solved by

$$u(x,t) = v(x,t) + w(x,t) = v(x,t) + W(x,t) + q(x,t-s)$$

$$= T_0(t) + \frac{x}{L}[T_1(t) - T_0(t)] \quad + \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$+ \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[\int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2(t-s)}\left(\int_0^L h(x,s)\sin\left(\frac{n\pi}{L}x\right)dx\right)\right]ds. \tag{2.12}$$

Example2.2. Consider an initial-boundary value problem for heat equation

$$\begin{cases} u_t - u_{xx} = xt\,, \quad 0 \le x \le 1, \quad t > 0, \\ u(0,t) = 3t, \quad u(1,t) = 2t, \quad t > 0, \\ u(x,0) = 4x + 1, \quad 0 \le x \le 1. \end{cases} \tag{2.13}$$

Solution: To solve the initial boundary value problem (2.13) by Duhamel's principle, we use a formula in equation (2.12).

$$u(x,t) = T_0(t) + \frac{x}{L}[T_1(t) - T_0(t)] + \sum_{n=1}^{\infty} f_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)$$

$$+ \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2(t-s)}h_n(s)ds$$

where$k = 1, \;\; L = 1, \;\; T_1(t) = 2t$ and$T_0(t) = \; 3t$. Then

$$v(x,t) = 3t + \frac{x}{1}[2t - 3t] = 3t - xt,$$

$$h(x,t) = xt - 3 - x(2-3) = xt + x - 3,$$

$$f(x) = 4x + 1 - 0 - x(0-0) = 4x + 1,$$

$$h_n(s) = \frac{2}{L}\int_0^L h(x,s)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= 2\int_0^1 (xs + x - 3)\sin(n\pi x)\,dx$$

$$= \frac{2}{n\pi}[s(-1)^{n+1} + (-1)^{n+1} + 3((-1)^n - 1)]$$

$$f_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx = 2\int_0^1 (4x+1)\sin(n\pi x)\,dx$$

$$= \frac{2}{n\pi}[4(-1)^{n+1} - ((-1)^n - 1)].$$

Therefore, the solution of (2.13) is given by

$$u(x,t) = 3t - xt + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[4(-1)^{n+1} - ((-1)^n - 1)]}{n}e^{-(n\pi)^2}\sin(n\pi x)$$

$$+\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}[-e^{-n^2\pi^2 t}]\left[\frac{t(-1)^{n+1}}{1-e^{-n^2\pi^2 t}} - \frac{(-1)^{n+1}}{n^2\pi^2} + (-1)^{n+1} + 3((-1)^n - 1)\right]\sin(n\pi x)$$

2.3. Duhamel’s principle for wave equation

In this section, we will solve nonhomogeneous wave equation by using Duhamel's principle. A problem which we are looking for unknown function of a differential equation whose values of the unknown function and /or derivative at a single point are known, then the problem is said to be initial value problem. If the values of the unknown function and/or its derivatives are known at boundary points, then the problem is called boundary value problem.

2.3.1 Initial value problem for wave equation

Consider the initial value problem for wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty, \\ u_t(x,0) = g(x), & -\infty < x < \infty. \end{cases} \qquad (2.14)$$

To solve equation (2.14), first break down equation (2.14) into two simpler problems as follows:

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & -\infty < x < \infty\,, \ t > 0, \\ v(x,0) = f(x)\,, & -\infty < x < \infty, \\ v_t(x,0) = g(x)\,, & -\infty < x < \infty \end{cases} \tag{2.15}$$

and

$$\begin{cases} w_{tt} - c^2 w_{xx} = h(x,t)\,, & -\infty < x < \infty, \ t > 0, \\ w(x,0) = 0, & -\infty < x < \infty, \\ w_t(x,0) = 0, & -\infty < x < \infty. \end{cases} \tag{2.16}$$

Then, equation (2.15) is solved by

$$v(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz$$

Equation (2.16) is solved by Duhamel's principle

$$w(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(\xi,s)d\xi ds.$$

Therefore, the solution of equation (2.14) is given by

$$u(x,t) = v(x,t) + \int_0^t \mathbb{W}(x,t-s)ds$$

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$$

$$+\frac{1}{2\mathrm{c}}\int_{x-ct}^{x+ct} g(z)dz + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(\xi,s)d\xi ds \tag{2.17}$$

Example2.3. Consider the initial value problem for wave equation:

$$\begin{cases} u_{tt} - u_{xx} = xt\,, & -\infty < x < \infty\,, \ t > 0 \\ u(x,0) = x+1, & -\infty < x < \infty, \\ u_t(x,0) = -3x, & -\infty < x < \infty. \end{cases} \tag{2.18}$$

Solution: Here**,** $c = 1,\ h(\xi,s) = \xi s,\ f(x) = x+1$ and $g(z) = -3z$. To solve equation (2.18), we use a solution formula of equation (2.17)

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(\xi,s)d\xi ds$$

$$= \frac{1}{2}(x - t + 1 + x + t + 1) + \frac{1}{2}\int_{x-t}^{x+t} -3z dz \ + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \xi s d\xi ds$$

$$= \frac{1}{6}xt^3 - 3xt + x + 1.$$

2.3.2 Initial boundary value problem for wave equation

Consider an initial -boundary value problem for wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = p(x,t), \ \ 0 < x < L, \ \ t > 0, \\ u(0,t) = T_0(t), \ \ u(L,t) = T_1(t), \quad t > 0, \\ u(x,0) = r(x), \quad u_t(x,0) = s(x), \ \ 0 < x < L. \end{cases} \tag{2.19}$$

To solve equation (2.19), first convert the given boundary conditions to homogeneous. That is, reduce the boundary conditions to homogeneous: take an arbitrary function $v(x,t)$ satisfying the nonhomogeneous boundary conditions:

$$v(0,t) = T_0(t), \quad v(L,t) = T_1(t), \quad t > 0.$$

For instance, one can take

$$v(x,t) = T_0(t) + \frac{x}{L}[T_1(t) - T_0(t)], \qquad 0 \le x \le L, \qquad t > 0.$$

Consider $w(x,t) = u(x,t) - v(x,t)$, as the new unknown.

The problem for $w(x,t)$ becomes

$$\begin{cases} w_{tt} - c^2 w_{xx} = h(x,t), \ \ 0 \le x \le L, \ \ t > 0 \,, \\ w(0,t) = 0, \ \ w(L,t) = 0, \ \ t > 0, \\ w(x,0) = f(x), \ \ w_t(x,0) = g(x), \ \ 0 \le x \le L. \end{cases} \tag{2.20}$$

where $h(x,t)$, $f(x)$ and $g(x)$ are given by:

$$h(x,t) = p(x,t) - v_t + c^2 v_{xx} = p(x,t) - T'_0(t) - \frac{x}{L}[T'_1(t) - T'_0(t)],$$

$$f(x) = r(x) - v(x,0) = r(x) - T_0(0) - \frac{x}{L}[T_1(0) - T_0(0)],$$

$$g(x) = s(x) - T'_0(0) - \frac{x}{L}[T'_1(0) - T'_0(0)].$$

Secondly, break equation (2.20) into two simpler problems:

$$\begin{cases} W_{tt} - c^2 W_{xx} = 0, \quad 0 \le x \le L \,, \quad t > 0 \,, \\ W(0,t) = 0, \quad W(L,t) = 0, \quad t > 0 \,, \\ W(x,0) = f(x), \quad W(x,0) = g(x), \quad 0 \le x \le L, \end{cases} \tag{2.21}$$

and

$$\begin{cases} q_{tt} - c^2 q_{xx} = h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ q(0,t) = 0, \quad q(L,t) = 0, \quad t > 0, \\ q(x,0) = 0 \,, \quad q_t(x,0) = 0, \quad 0 \le x \le L. \end{cases} \tag{2.22}$$

Now, equation (2.21) is solved by

$$W(x,t) = \sum_{n=1}^{\infty} \left[f_n \cos\left(\frac{cn\pi}{L} t\right) + \frac{L}{cn\pi} g_n \sin\left(\frac{cn\pi}{L} t\right) \right] \sin\left(\frac{n\pi}{L} x\right)$$

where

$$f_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$g_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

and Equation (2.22) is solved by Duhamel's principle

$$q(x,t-s) = \sum_{n=1}^{\infty} \frac{L}{cn\pi} \sin\left(\frac{n\pi}{L} x\right) \int_0^t h_n(s) \left[\sin\left(\frac{cn\pi}{L}(t-s)\right) \right] ds.$$

where

$$h_n(s) = \frac{2}{L} \int_0^L h(x,s) \sin\left(\frac{n\pi}{L} x\right) dx$$

Therefore, the solution of equation (2.19) is given by

$$u(x,t) = v(x,t) + w(x,t) = v(x,t) + W(x,t) + q(x,t-s)$$

$$= v(x,t) + \sum_{n=1}^{\infty}\left[f_n \cos\left(\frac{cn\pi}{L}t\right) + \frac{L}{cn\pi}g_n \sin\left(\frac{\mathrm{cn\pi}}{\mathrm{L}}\mathrm{t}\right) + \frac{L}{cn\pi}\int_0^t h_n(s)\left[\sin\left(\frac{cn\pi}{L}(t - s)\right)\right]ds\right]\sin\left(\frac{n\pi}{L}x\right). \quad (2.23)$$

Example2.4. Consider the initial boundary value problem for wave equation:

$$\begin{cases} u_{tt} - u_{xx} = x + t, \ \ 0 \le x \le 1, \ \ t > 0, \\ u(0,t) = 3t, \ \ u(1,t) = t, \ \ t > 0, \\ u(x,0) = 2x, \ \ u_t(x,0) = -4x + 1, \ \ ,0 \le x \le 1. \end{cases} \quad (2.24)$$

Solution: To solve equation (2.24) by Duhamel's principle, we can use a formula of equation (2.23):

$$u(x,t) = T_0(t) + \frac{x}{L}[T_1(t) - T_0(t)] + \sum_{n=1}^{\infty}\left[f_n \cos\left(\frac{n\pi c}{L}t\right) + \frac{L}{cn\pi}g_n \sin\left(\frac{n\pi c}{L}t\right)\right]\sin\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} sin\left(\frac{n\pi}{L}x\right)\left(\int_0^t h_n(s)\, sin\frac{n\pi c}{L}(t-s)\right)ds,$$

Since, $c = 1, \ L = 1, \ T_0(t) = 3t, \ T_1(t) = t, \ f(x) = 2x, \ g(x) = -2x - 2$ and

$p(x,t) = \mathrm{x} + \mathrm{t}$, we have

$$v(x,t) = 3t + x(t - 3t) = 3t - 2xt.$$

$$h(x,t) = x + t - 3 - x(1-3) = 3x + t - 3,$$

$$f_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{4(-1)^{n+1}}{n\pi},$$

$$g_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{-4[2(-1)^{n+1} + 1]}{(n\pi)^2},$$

$$h_n(s) = \frac{2}{L}\int_0^L h(x,s)\sin\left(\frac{n\pi}{L}x\right)dx \ = \frac{6}{n\pi}\left[(-1)^{n+1} + ((-1)^n - 1)\left(1 - \frac{s}{3}\right)\right].$$

Hence,

$$u(x,t) = 3t - 2xt + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n\pi}\cos(n\pi t) + \frac{-4[2(-1)^{n+1} + 1]}{(n\pi)^3}\sin(n\pi t)\right]\sin(n\pi x)$$

$$+\sum_{n=1}^{\infty} \sin(n\pi x)\left(\int_0^t \frac{6}{n\pi}\left[(-1)^{n+1} \ + ((-1)^n - 1)\left(1 - \frac{s}{3}\right)\right] \sin n\pi(t-s)\, ds\right)$$

$$= 3t - 2xt + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n\pi}\cos(n\pi t) + \frac{-4[2(-1)^{n+1} + 1]}{(n\pi)^3}\sin(n\pi t)\right]$$

$$+\sum_{n=1}^{\infty}\left[\frac{6}{n\pi}(1 - \cos t)[(-1)^{n+1} + (-1)^n - 1]\right.$$

$$\left. + \frac{2}{n^2\pi^2}[(-1)^n - 1]\left(t - \frac{\sin(n\pi t)}{n\pi}\right)\right]\sin(n\pi x).$$

3. Summary

Evolution equation is a type of a differential equation which involves in time. Heat and wave equations are prototypes of equations. Heat equation is a parabolic type of a partial differential equation that describes how the temperature varies in space over time and wave equation is an important hyperbolic type of partial differential equation for the description of waves that occur in classical physics such as mechanical wave (water waves, sound waves), electromagnetic waves (radio waves, light waves).

Fourier Transform is a mathematical tool which transforms a function from time domain to frequency domain. Fourier Transform is useful in the study of solution of partial differential equation to solve initial value problems. A Fourier Transform when applied to partial differential equation reduces the number of independent variables by one. When we applied Fourier Transform to partial differential equation, then the evaluation of integral is very complex.

Separation of variables is a method for solving ordinary and partial differential equations in which algebra allows one to redraft an equation so that each of two variables occurs on a different side of the equation. Separation of variable is a useful method in the study of solution of partial differential equation to solve initial boundary value problems.

Duhamel's principle is the technique that the solution to an inhomogeneous equation can be solved by first finding the solution for a step input and then superposing using Duhamel's integral. Duhamel's principle is a general method for obtaining solution to inhomogeneous linear evolution equations like the heat equations, wave equations and vibrating plate equations. Breaking down the initial or initial-boundary value problems for inhomogeneous equations into simpler problems gives

- Homogenous equations with nonhomogeneous initial condition(s)
- Nonhomogeneous equations with homogeneous initial condition(s)

whose solutions for the later problem is obtained by the well-known technique called Duhamel's principle.

4. References

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