2022-09-12

A Project on Normal filter in Almost Distributive Lattices

Getachew Asres

Bahir Dar University

College of Science

Department of Mathematics

A Project on

Normal filter in Almost Distributive Lattices

By

Getachew Asres

September, 2022
Bihar Dar, Ethiopia

BiharDar University

College of Science

Department of Mathematics

A project on

Normal filter in Almost Distributive Lattices

A Project submitted to the department of mathematics in partial fulfillment of the requirements for the degree of "Master of Science in Mathematics".

By

Getachew Asres Kidanu

Advisor: Mihret Alamneh (PhD)

September, 2022

Bihar Dar, Ethiopia

Bihar Dar   University

College of Science

Department of Mathematics

I here by certify that I have supervised, read and evaluated this project entitled "Normal Filter in Almost Distributive Lattices" by Getachew Asres prepared under my guidance. I recommend that the project is submitted for oral defense.

Advisor name: Mihret Alamneh (PhD)

Signature ----------------------

Date----------------------

Bahir dar Univeristy

College of Science

Department of Mathematics

We here by certify that we have examined this project entitled “Normal Filter in Almost Distributive Lattices” by Getachew Asres. We recommend that Getachew Asres is approved for the degree of “Master of Science in Mathematics”.

Board of Examiners

	Name	Sign	Date
External examiner:	-----------------------	--------------	--------------
Internal examiner I:	-----------------------	---------------	------------
Internal examiner II:	----------------------	------------	------------

Acknowledgement

First of all I would like to thank almighty God, the compassionate, the most merciful and source of knowledge and wisdom, who bestowed upon me the health, the power of communication and the audacity to accomplish this project. Second I would like to express my special appreciation to my advisor, Mihret Alamenh (PhD, Associate professor) for his guidance, encouragement and constructive comments until the last submission of the paper. Next I would like to thank my friends, my classmates and everyone around me who were there for me. Finally I would like to express my appreciation to my wife Tadefie Baye, children's Hana Getachew and Nohamin Getachew and my best friend to support by computer service Lingrew Molla.

Abstract

In this paper we introduce normal filters and normlets in an almost distributive lattice with dense elements and reinforce them in both algebraical and topological aspects.

Content

Contents page

Acknowledgement i

Abstract ii

Chapter One --1

Introduction and Preliminary --1

1.1 Introduction--1

1.2 Preliminaries---2

Chapter Two --7

Normal Filter in Almost Distributive Lattice --7

Normal Filters --7

2.2 Normlets---12

2.3 Normal Prime Filters 20

2.4 The Space of Normal Prime Filters in Almost Distributive Lattice 24

Conclusion 27

References 28

Chapter one

Introduction and preliminaries

1.1 Introduction

The structure of distributive lattice is exponentially enrich and has smooth nature. A vast number of researchers broadly studied the class of distributive lattice in different aspects. Some of the authors take a broad view of the structure of distributive lattice in different aspects. In this context, U.M. Swamy and G.C.Rao[15] generalized the structure of distributive lattice as a common abstraction of lattice theoretic and ring theoretic aspects called an almost distributive lattice in 1981. Later the authors [2,3,4,5,6,11,12,13,18,19] analogously extended some concepts to almost distributive lattices which are in distributive lattices. In [7, 8, 9, 14, 15, 16, 17], the authors initiated the ideal (filter) congruence theory in a distributive lattice and they have showed some special class of distributive lattices like normal lattices, quasi complemented distributive lattices etc.

The concept of an Almost Distributive Lattice was introduced by U.M.Swamy and G.C.Rao [15] as a common abstraction to most of the existing ring theoretic generalization of Boolean algebra and distributive lattices. M.Sambasiva Rao in [9] introduced the concept normal filters and normelets are introduced in a distributive lattice in terms of annihilators and proved that the set of all normal filters forms a distributive lattice and the class of normlets is a sub lattice of the lattice of normal filters. In this paper we mainly concentrate on normal filters in almost distributive lattice with dense elements. It has also two chapters and five sections. In this first section, we collect some preliminary results on almost distributive lattices which are useful in the sequent sections. In second section, we introduce normal filters in an almost distributive lattice and certain examples are given and drive some properties on the class of normal filters. In the third section, we study the class of normelets in an almost distributive lattice and obtain several equivalent conditions for a filter to become a normlet. In fourth section we discuss the class of normal prime filter and obtain certain results on them. In last section, we deliberate the space of normal prime filters with hull-kernel topology and obtain a good number of equivalent conditions for the space of normal prime filters to become Hausdorff.

1.2. Preliminaries

This section is consisting of some definitions and results that will be used in the next chapter we simply list these in the form of lemma and theorems without their proofs

Definition 1.2.1 ([7]). A non empty set L together with two binary operations $\wedge$ *and* $\vee$ (*meet and join*) on a set L is called a lattice if it satisfies the following algebraic properties

1. Idempotent; $a \wedge a = a$ *and* $a \vee a = a$.

2. Commutativity; $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

3. Absorption; $a \wedge (a \vee b) = a$ *and* $a \vee (a \wedge b) = a$.

4. Associativity; $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *and* $(a \vee b) \vee c = a \vee (b \vee c)$. For any $a, b, c \in L$.

In any lattice $(L, \vee, \wedge)$ the following identities are equivalent;

•$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

•$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

•$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

•$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$.

Definition 1.2.2([7]) A lattice $(L, \vee, \wedge)$ satisfying any one of the above four identities is called a Distributive Lattice.

Definition 1.2.3 ([11]). An algebra $(L, \vee, \wedge, 0)$ 0f type (2,2,0) is called an Almost Distributive Lattice (ADL) with 0 if it satisfies the following axioms ; for all $a, b, c \in L$

(1) $a \vee 0 = a$

(2) $0 \wedge a = 0$

(3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(6) $(a \vee b) \wedge b = b$.

Definition 1.2.4 ([10)] A binary relation $\leq$ defined on a set A is a partial order on the set A if the following conditions holds identically in A:

(i). $a \leq a$ (reflexivity)

(ii). $a \leq b$ and $b \leq a$ imply $a = b$ (ant symmetry)

(iii). $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity)

If, in addition, for every a, b in A

(iv) $a \leq b$ or $b \leq a$.

.**Definition 1.2.5** A non empty subset F of an ADL L is said to be a filter of L if the following axioms hold true:

1 $a, b \in L \Longrightarrow a \wedge b \in F$.

2 $x \in L$ and $a \in F \Longrightarrow a \vee x \in F$.

Definition1.2.6 For any non–empty subset S of an ADL L $[S) = \{x \vee (\wedge_{i=1}^{n} s_i)\ x \in L$, n is a positive integer$\}$ is said to be the smallest filter of L containing of S.

Definition 1.2.7 For any $a \in L, [a) = \{x \vee a\ |x \in L\}$ is said to be the principal filter generated by a. where $[a) \wedge [b) = [a \vee b)$ and $[a) \vee [b) = [a \wedge b)$, for any $a, b \in L$

Definition 1.2.8 A non empty sub set I of an ADL L is said to be an ideal of L if the following axioms hold true:

1. $a, b \in L \Longrightarrow a \vee b \in I$
2. $x \in L$ and $a \in I \Longrightarrow x \wedge a \in I$.

Definition 1.2.9 For any $a \in L, (a] = \{a \wedge x, x \in L\}$ is said to be the principal ideal generated by a. Where $(a] \wedge (b] = (a \wedge b]$ and $(a] \vee (b] = (a \vee b]$, for any $a, b \in L$.

For any non-empty subset A of L, the set $A^* = \{x \in L, a \wedge x = 0,$ for all $a \in A\}$ is an ideal of L. In particular, for any $a \in L$, $\{a\}^* = (a)^*$, where $(a) = (a]$ is the principal ideal generated by a.

Definition 1.2.10 An element $d \in L$ is said to be dense, if $(d)^* = \{0\}$. The set D denotes the set of dense elements of an ADL of L. It is filter of an ADL L, provided D is non-empty.

Definition 1.2.11([11]) An almost distributive lattice L with 0 is called quasi-complemented ADL if for each $x \in L$, there is $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is a maximal. Here y is called a quasi-complement of x.

Definition 1.2.12 An element $m \in L$ is said to be a maximal element if for any $a \in L, m \leq a$ implies $m = a$.

Note: every maximal element is dense. The set M denotes the set of maximal elements of L. It is also a filter of L, provides M is non-empty.

Definition 1.2.13 A proper filter (ideal) F (I) of an ADL of L is said to be a prime, if for any $a, b \in L, (a \vee b), (a \wedge b) \in F(I)$, then $a \in F(I)$ or $b \in F(I)$.

Definition 1.2.14. . A prime ideal(filter) P of ADL L is said to be a minimal prime ideal(filter) if there is no prime ideal(filter) which is properly contained in P.

2). A prime filter(ideal) P of ADL L is said to be a maximal prime filter(ideal) if there is no prime filter(ideal) which properly contains the filter(ideal) P.

Definition 1.2.15 ([13]) An ADL L with 0 is called normal ADL if and only if for all $a, b \in L$

$$(a]^* \vee (b]^* = (a \wedge b]^*.$$

Lemma 1.2.1 ([16]) for any $a, b, c \in L$,we have

(i) $a \wedge 0 = 0$ and $a \vee 0 = a$

(ii) $a \vee a = a \wedge a = a$

(iii) $a \vee (b \vee a) = a \vee b$

(iv) $\wedge$ is associative

(v) $a \wedge b \wedge c = b \wedge a \wedge c$

(vi) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$.

(vii) $a \wedge b \leq b$ and $a \leq a \vee b$

(viii) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(ix) $a \vee b = b \vee a \Leftrightarrow a \wedge b = b \wedge a$.

Lemma 1.2.2 (11). For any a, b ∈ L, we have;

(i) $a \leq b \Rightarrow (b)^* \subseteq (a)^*$
(ii) $(a)^{***} = (a)^*$
(iii)$(a \vee b)^* = (a)^* \cap (b)^*$
(iv)$(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$
(v) $(a)^* \subseteq (b)^* \Leftrightarrow (b)^{**} \subseteq (a)^{**}$
(vi)$a \in (a)^{**}$
(vii) $(a \vee b)^* = (b \vee a)^*$
(viii) $(a \wedge b)^* = (b \wedge a)^*$
(ix)$(a)^* = L \Leftrightarrow a = 0$.

LEMMA 1.2.3 ([19]) Every maximal ideal is prime.

LEMMA1.2.*4 ([11])* P is a prime filter (ideal) of L if and only if L\P is a prime ideal (filter) of L.

LEMMA 1.2.5 ([11]) P is a minimal (maximal) prime filter (ideal) of L if and only if L\P is a maximal (minimal) ideal (filter) of L.

LEMMA 1.2.6 ([11]) Let I be an ideal and F be a filter of L such that $I \cap F = \phi$. Then there exists a prime filters P such that $F \subseteq P$ and $P \cap I = \phi$.

LEMMA1.2.7 ([11]) A prime ideal P of L is minimal prime ideal if and only if for each $x \in P$. There exists $y \notin P$ such that $x \wedge y = 0$.

LEMMA 1.2.8 ([18]) Let L be an ADL with maximal element. Then every prime ideal is minimal if and only if every prime ideal is maximal.

DEFINATION 1.2.16 ([11]) L is said to be weak relatively complemented if for any a, b $\in$L, there exist x$\in$L such that a$\wedge$x=0 and $(a \vee x)^{*}$ =$(a \vee b)^{*}$

UNIT TWO

NORMAL FILTERS IN ALMOST DISTRIBUTIVE LATTICE

2.1 NORMAL FILTERS

In this section we define a normal filter and provide certain examples for it. We observe that the set of normal filters forms a distributive lattice which is not a sub distributive lattice of the set of filters in an almost distributive lattice.

Definition 2.1.1 ([9]) For any filter F of an almost distributive lattice L with 0, the set F^+ is defined as $F^+ = \{x \in L : (x)^* \subseteq (a)^* \text{ for some } a \in F\}$. A filter F of a lattice L is called a normal filter if $F = F^+$.

In particular, for any $a \in L$, $[a)^+ = \{x \in L: (x)^* \subseteq (a)^*\}$, where [a) is a principal filter of L.

Lemma 2.1.1([11]). For any filter F of an ADL L, we have;

(i) $F \subseteq F^+$

(ii) $D \subseteq F^+$. Where D is the set of dense elements

Proof. (i) Let $a \in F$. Then $(a)^* \subseteq (a)^*$. Therefore $a \in F^+$. Hence $F \subseteq F^+$.

(ii). Let $d \in D$. Then $(d)^* = \{0\} \subseteq (a)^*$, for all $a \in F$. Therefore $d \in F^+$. Hence $D \subseteq F^+$.

Lemma 2.1.2. For any filters F, G of L, we have

(i). $F \subseteq G$ implies $F^+ \subseteq G^+$

(ii). $F^{++} = F^+$

(iii). $(F \cap G)^+ = F^+ \cap G^+$

(iv). $(F \vee G)^+ = (F^+ \vee G^+)^+$.

Proof :-(i) Let $x \in F^+$. Then $(x)^* \subseteq (a)^*$ for some $a \in F \subseteq G$. Therefore $x \in G^+$ and hence $F^+ \subseteq G^+$

(ii). (By Lemma 2.1.1 (i)), $F \subseteq F^+$. Then $F^+ \subseteq F^{++}$ (by (i)). Let $x \in F^{++}$.

Then $(x)^* \subseteq (a)^*$. For some $a \in F^+$. For this $a \in F^+$, $(a)^* \subseteq (b)^*$ for some $b \in F$. Therefore $(x)^* \subseteq (b)^*$, for some $b \in F$. Hence $x \in F^+$. Therefore $F^{++} \subseteq F^+$.

Hence $F^{++} = F^+$.

(iii). Since $(F\cap G)^{+}\subseteq F^{+}, G^{+}$ and $(F\cap G)^{+}\subseteq F^{+}\cap G^{+}$. Let $x\in F^{+}\cap G^{+}$. Then $(x)^*\subseteq(a)^*$ and $(x)^*\subseteq(b)^*$. For some $a\in F$ and $b\in G$. Therefore $(x)^*\subseteq(a)^*\cap(b)^*= (a\vee b)^*$, where $a\vee b\in F\cap G$. Hence $x \in(F\cap G)^{+}$. Thus $(F\cap G)^{+} =F^{+}\cap G^{+}$.

(iv). (By Lemma 2.1.1 (i)), $F \subseteq F^{+}$ and $G \subseteq G^{+}$. Therefore $F \vee G \subseteq F^{+}\vee G^{+}$ and

$(F\vee G)^{+} \subseteq(F^{+}\vee G^{+})^{+}$. On the other hand, Let $x \in(F^{+}\vee G^{+})^{+}$, Then there exists $a\wedge b \in F^{+}\vee G^{+}$ such that $(x)^*\subseteq(a\wedge b)^*$, where $a\in F^{+}$ and $b\in G^{+}$. For $a\in F^{+}$ and $b\in G^{+}$, there exist $f\in F$ and $g\in G$ such that $(a)^*\subseteq(f)^*$ and $(b)^* \subseteq(g)^*$.Therefore $(f)^{**} \subseteq(a)^{**},(g)^{**} \subseteq(b)^{**}$and $(f)^{**}\cap(g)^{**}\subseteq(a)^{**},(b)^{**}$. We get $(f\wedge g)^{**} \subseteq(a\wedge b)^{**}$. So that $(x)^*\subseteq(a\wedge b)^* \subseteq (f\wedge g)^*$.Where $f\wedge g \in F\vee G$. Hence $x\in(F\vee G)^{+}$. Thus $(F^{+}\vee G^{+})^{+}= (F\vee G)^{+}$

Lemma 2.1.3 ([9]) We have

(i). $L^{+}=L= [0)^{+}$

(ii). $D^{+} = D = M^{+}$

(iii). For any $d \in D$, $[d)^{+} = D$

(iv). For any filter F of L, $F\subseteq F^{+}$ and $D\subseteq F^{+}$

Proof. (i) Let L be an ADL. Then $L \subseteq L^{+}$ (by Lemma 2.1.1 (i)). Hence $L^{+} = L$. Let $0 \in L$. Then $[0) = L$. Therefore $[0)^{+} = L^{+}$. Hence $L^{+} = L = [0)^{+}$.

(ii). Let $a \in D^{+}$. Then there exists $d \in D$ such that $(a)^* \subseteq (d)^* = \{0\}$ (since d is dense). Therefore $(a)^* = \{0\}$. So that $a \in D$ and hence $D^{+} \subseteq D$. (By Lemma 2.1.1 (i)), $D\subseteq D^{+}$ and hence $D^{+} = D$. Let $d \in D$. Then $(d)^* = \{0\} = (m)^*$, for all $m \in M$ (since every maximal element is dense). Therefore $d \in M^{+}$. Hence $D \subseteq M^{+}$. (By Lemma 2.1.2 (i) (ii)), $D^{+} \subseteq M^{++} = M^{+}$ and $M^{+} \subseteq D^{+}$ (since $M \subseteq D$). Thus $D^{+} = D = M^{+}$.

(iii) Let $d \in D$. Then $[d) \subseteq D$ (since D is filter). Therefore $[d)^{+} \subseteq D^{+} = D$ (by (ii)). Let $a \in D$. Then $(a)^* = \{0\} = (d)^*$ (since d is dense). Therefore $a \in [d)^{+}$. Hence $D \subseteq [d)^{+}$. Thus $[d)^{+} = D$.

(iv). Let $x, y \in F^{+}$. Then $(x)^* \subseteq (a)^*$ and $(y)^* \subseteq (b)^*$ for some $a, b \in F$. Then $(a\wedge b)^{**} = (a)^{**} \wedge (b)^{**} \subseteq (x)^{**} \cap (y)^{**} = (x\wedge y)^{**}$. Hence $(x\wedge y)^* \subseteq (a\wedge b)^*$ and $a\wedge b \in F$. Therefore $x\wedge y \in F^{+}$. Again, let $x \in F^{+}$ and $x\leq y$, then $(y)^* \subseteq (x)^* \subseteq (a)^*$ for some $a \in F$. Thus we get $y \in F^+$. Therefore F^+ is a filter of L

Such that $F \subseteq F^+$. Let $D = \{x \in L: (x)^* \subseteq (d)^* = \{0\}\}$, since $(d)^* = \{0\} = (a)^*$

$F^+ = \{x \in L: (x)^* \subseteq (a)^*$, for some $a \in F\}$ (by the Definition 1.2.12) m = a

and also every maximal element is dense, since d = a. Therefore $D \subseteq F^+$.

Lemma 2.1.3 For any proper filter F of L, F^+ is always a proper normal filter.

Proof:- Let F be a proper filter of L. Suppose F^+ is not proper normal filter. It means that $F^+=L$.

Since $0 \in L=F^+$, there exists $a \in F$ such that $L= (0)^* \subseteq (a)^*$. Therefore a=0 and $0 \in F$. Hence F=L. Which is a contradiction to our assumption. Thus F^+ is proper normal filter.

Theorem 2.1.1 Every maximal filter is a normal filter.

Proof. Let K be a maximal filter of L. Assume that K is not normal. Then K & K^+.

Therefore $K^+ = L$ (since K^+ is a filter). Let $0 \in L = K^+$. Then $L = (0)^* \subseteq (a)^*$, for some $a \in K$.

Therefore $(a)^* = L$ and hence a = 0 (by Lemma 1.2.2 (ix)). We get $0 \in K$. So that K = L. Which is a contradiction (since K is proper). Thus K is normal.

Example 2.1.1 Let $L=\{0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,d,m\}$ with the operation $\wedge$ and $\vee$ defined as follows.

$\wedge$	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
0	0	0	0	0	0	0	0	0	0	0
b_1	0	b_1	0	b_1	b_1	0	b_1	0	b_1	b_1
b_2	0	0	b_2	b_2	b_2	0	0	b_2	b_2	b_2
b_3	0	b_1	b_2	b_3	b_3	0	b_1	b_2	b_3	b_3
b_4	0	b_1	b_2	b_3	b_4	0	b_1	b_2	b_3	b_4
b_5	0	0	0	0	0	b_5	b_5	b_5	b_5	b_5
b_6	0	b_1	0	b_1	b_1	b_5	b_6	b_5	b_6	b_6
b_7	0	0	b_2	b_2	b_2	b_5	b_5	b_7	b_7	b_7
d	0	b_1	b_2	b_3	b_3	b_5	b_6	b_7	d	d
m	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m

Table 1

$\vee$	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
0	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
b_1	b_1	b_1	b_3	b_3	b_4	b_6	b_6	d	d	m
b_2	b_2	b_3	b_2	b_3	b_4	b_7	d	b_7	d	m
b_3	b_3	b_3	b_3	b_3	b_4	d	d	d	d	m
b_4	b_4	b_4	b_4	b_4	b_4	m	m	m	m	m
b_5	b_5	b_6	b_7	d	m	b_5	b_6	b_7	d	m
b_6	b_6	b_6	d	d	m	b_6	b_6	d	d	m
b_7	b_7	d	b_7	d	m	b_7	d	b_7	d	m
d	d	d	d	d	m	d	d	d	d	m
m	m	m	m	m	m	m	m	m	m	m

Table 2

Then $(L,\wedge,\vee,0)$ is an ADL in which $[b_3)$ is a normal filter but not maximal. From the above example we have the following.

Remark 2.1.1 The converse of above theorem not be is true. For, see the above example.

Remark 2.1.2:- Every minimal filter need not be normal. For, see example 2.1.,

[m) = {m} is a minimal filter but not normal (because $[m)^+ = \{d, m\} \neq [m)$).

Remark 2.1.3:- Every normal filter need not be prime. For, see example 2.1.,

$[b_3) = \{b_3, b_4, d, m)$ is a normal filter but not a prime (because $b_3=b_1 \vee b_2 \in [b_3)$),

but $b_1 \notin [b_3)$ and $b_2 \notin [b_3)$.

Remark 2.1.4:- Every prime filter need not be normal. For, see example 2.1.,

$[b_4)= \{b_4,m\}$is a prime filter but not normal. (Because $[b_4)^+=\{b_3,b_4,d,m\} \neq [b_4)$.

Remark 2.1.5:- Every minimal prime filer need not be normal. For see example 2.1.,

$[b_4)=\{b_4,m\}$ is a minimal prime filter but not normal (because $[b_4)^+=\{b_3,b_4,d,m\} \neq [b_4)$. Let us denote the set o normal filter of $\mathcal{NF}$ (L). It can be observing that $\mathcal{NF}$ (L) is a distributive lattice.

Theorem 2.1.2:- $\mathcal{NF}$ (L) can be a distributive lattice with the operations $F^+ \cap G^+ = (F \cap G)^+$ and $F \cup G=(F \vee G)^+$, for any F,G $\in \mathcal{NF}$(L).

Proof: - Let F, G $\in \mathcal{NF}$(L). By lemma 2.1.2., $(F \cap G)^+$ is the infimum of F and G in $\mathcal{NF}$(L).

Also $(F \vee G)^+$ is an upper bound of F and G. Let H $\in \mathcal{NF}$ (L) such that $F^+ \subseteq H$, $G^+ \subseteq H$

and $x \in (F \vee G)^+$.

Then $(x)^* \subseteq (a)^*$ for some $a \in F \vee G \subseteq H$. Therefore $x \in H^+=H$ (since H$\mathcal{NF}$ (L)).

Thus $(F \vee G)^+=F \cup G$ is the supremum of F and G in $\mathcal{NF}$ (L). Let F, G $\in$ H$\mathcal{NF}$ (L).

then $F \cap (G \sqcup H)=F^+ \cap (G \vee H)^+=(F \cap (G \vee H))^+=\{(F \cap G) \vee (F \cap H)\}^+=(F \cap G) \sqcup (F \cap H)$.

(Since $\mathcal{F}$ (L) is a distributive lattice). Therefore ($\mathcal{NF}$ (L),$\wedge, \sqcup$) is a distributive lattice with

the greatest element $L^+=L=|0)^+$

Theorem 2.1.3:- There is an epimorphism from $\mathcal{F}(L)$ on to $\mathcal{NF}(L)$.

Proof: - Let F, G$\in\mathcal{F}$ (L). Define a map $\Phi:\mathcal{F}(L) \rightarrow \mathcal{NF}(L)$ by $\phi(F)=F^{+}$.

Then $\phi(F\wedge G) =(F\wedge G)^{+}=F^{+}\cap G^{+}=\phi(F) \cap \phi(G)$ and $\phi(F \vee G)= (F\vee G)^{+}= (F^{+}\vee G^{+})^{+}$ (by Lemma 2.1.2 (iv)) = $F^{+}\sqcup G^{+} =\phi(F) \sqcup \phi(G)$. Therefore ϕ is a homomorphism.

Since $\mathcal{NF}(L)\subseteq\mathcal{F}(L)$, ϕ is an onto homomorphism.

2.2 Normlets

In this section, we define normlets in an almost distributive lattice. We obtain necessary and sufficient conditions for a filter to become normal in terms of normlets. Finally we obtain necessary and sufficient conditions for an almost distributive lattice to become weak relatively complemented.

Definition 2.2.1.([9]) A filter F of L is said to be a normlet, if $F= [a)^{+}$, for some $a\in L$.

Theorem 2.2.1. Every normlet is a normal filter.

Proof. Let $x\in L$ and $t\in[x)^{+}$. Then $(t)^*\subseteq(a)^*$, for some $a\in[x)^*$ and $(a)^*\subseteq (x)^*$.

Therefore $(t)^* \subseteq (x)^*$ and hence $t \in [x)^{+}$. Thus $[x)^{+}$ is a normal filter.

Lemma 2.2.1. For any a, b $\in$ L, we have

(i). $a\leq b$ implies $[b)^{+} \subseteq [a)^{+}$

(ii). $a\in [b)^{+}$ implies $[a)^{+}\subseteq [b)^{+}$

(iii). $[a)^{+} =D$ if and only if $a\in D$

(iv). $[a)^{+}= L$ if and only if $a = 0$

(v). For any normal element m of L, $[m)^{+} = D$

(vi) $[a)^{+} \cap [b)^{+} = [a\vee b)^{+}$

Proof. (i) Suppose that a$\leq$ b. Then $[b) \subseteq [a)$. Therefore $[b)^{+} \subseteq [a)^{+}$ (by Lemma 2.1.2(i)).

(ii) Suppose that a$\in [b)^{+}$. Then $[a) \subseteq [b)^{+}$. Therefore $[a)^{+} \subseteq [b)^{++} = [b)^{+}$

and hence $[a)^{+} \subseteq [b)^{+}$ (Since $[b)^{+}$ is normlet).

(iii) Suppose that $[a)^{+} = D$. Then $a \in [a)^{+} = D$. On the other hand,

Let $d \in D$, then $D^{+} = \{x\in L, (x)^* \subseteq (d)^* = \{0\}\} = D$.

(iv). Suppose that $[a)^{+} = L$. Then $0 \in L = [a)^{+}$. Therefore $L = (0)^* \subseteq (a)^*$.

Hence a = 0. Let a = 0 and $(0)^* \subseteq (a)^*$ and $0 \in L = [a)^{+}$. Therefore $[a)^{+} = L$.

(v). Since every maximal element is dense and from (iii), we have $|m)^{+} = D$.

(vi). $[a)^{+} \cap [b)^{+} = ([a) \cap [b))^{+} = [a \vee b)^{+}$ (by lemma 2.1.2)

Lemma 2.2.2 For any a, b $\in$ L, we have

(i). $a \wedge b = 0$ implies $[a)^{+} \vee [b)^{+} = L$

(ii). $a \vee b \in D$ if and only if $[a)^{+} \cap [b)^{+} = D$

(iii). If $a \neq 0$, then $(a)^* \cap [a)^{+} = \phi$

(iv). $(a)^* = (b)^*$ if and only if $[a)^{+} = [b)^{+}$

(v). $[a)^{+} = [b)^{+}$ implies $[a \wedge c)^{+} = [b \wedge c)^{+}$ for all $c \in L$

(vi). $[a)^{+} = [b)^{+}$ implies $[a \vee c)^{+} = [b \vee c)^{+}$ for all $c \in L$.

Proof. (i) Suppose that $a \wedge b = 0$. Then $L = [0) = [a \wedge b) = [a) \vee [b) \subseteq [a)^{+} \vee [b)^{+} \subseteq L$.

Therefore $[a)^{+} \vee [b)^{+} = L$.

(ii) It can be obtain by Lemma 2.2.1.

(iii). Suppose that $a \neq 0$. Let $x \in (a)^* \cap [a)^{+}$. Then $(x)^* \subseteq (a)^*$ and $a \wedge x = 0$.

Therefore $a \in (x)^* \subseteq (a)^*$. Hence $a \wedge a = 0$. Which is a contradiction Thus $(a)^* \cap [a)^+ = \phi$.

(iv). Suppose that $(a)^* = (b)^*$. Then $a \in [b)^+$ and $b \in [a)^+$. Therefore $[a)^+ \subseteq [b)^+$ and $[b)^+ \subseteq [a)^+$.

Hence $[a)^+ = [b)^+$. On the other hand, suppose that $[a)^+ = [b)^+$. Then $a \in [b)^+$ and $b \in [a)^+$

Therefore $(a)^* \subseteq (b)^*$ and $(b)^* \subseteq (a)^*$ and hence $(a)^* = (b)^*$.

(v). Suppose that $[a)^+ = [b)^+$. For any $t \in L$, $t \in (a \wedge c)^*$

$$\Leftrightarrow t \wedge a \wedge c = 0$$

$$\Leftrightarrow t \wedge c \in (a)^* = (b)^* \text{(from (iv))}$$

$$\Leftrightarrow t \wedge b \wedge c = 0$$

$$\Leftrightarrow t \in (b \wedge c)^*$$

By (iv) we get $[a \wedge c)^+ = [b \wedge c)^+$.

(vi). suppose that $[a)^+ = [b)^+$. For any $t \in L$

$$t \in (a \vee c)^* \Leftrightarrow t \vee a \vee c = 0$$

$$\Leftrightarrow t \vee c \in (a)^* = (b)^* \text{(from (iv))}$$

$$\Leftrightarrow t \vee b \vee c = 0$$

$$\Leftrightarrow t \in (b \vee c)^*$$

By (iv) we get $[a \vee c)^+ = [b \vee c)^+$.

Theorem 2.2.2. For any filter F of L, the following are equivalent;

(i). F is normal

(ii) For $x \in L$, $x \in F$ implies $[x)^+ \subseteq F$

(iii) For any x. y$\in$ L, $(x)^* = (y)^*$ and x$\in$ F implies y$\in$ F

(iv) For x, y$\in$ L, $[x)^+ = [y)^+$ and x$\in$ F and y$\in$ F

(v) $F = \bigcup_{x\in F}[x)^+$

Proof. (i)$\Longrightarrow$ (ii). Assume F is normal. Let x$\in$ F. Then $[x)\subseteq F$. Therefore $[x)^+ \subseteq F^+ = F$.

Thus $[x)^+ \subseteq F$.

(ii)$\Longrightarrow$ (iii). Assume (ii). Let x, y $\in$ L such that $(x)^* = (y)^*$ and x$\in$ F. Then $[y)^+ = [x)^+ \subseteq F$.

(by our assumption). Therefore y $\in$ F.

(iii)$\Longrightarrow$ (iv) Suppose that $(x)^* = (y)^*$. Then $x\in[y)^+$ and $y\in[x)^+$.

Therefore $[x)^+ \subseteq [y)^+$ and $[y)^+ \subseteq [x)$ Hence $[x)^+ = [y)^+$.

(iv)$\Longrightarrow$(v) Assume (iv). Let x$\in$F. Then $x\in [x)^+$. Hence $F\subseteq \bigcup_{x\in F}[x)^+$

On the other hand, let $x \in F$ and $y \in [x)^+$. Then $[y)^+ \subseteq [x)^+$.

Therefore $[y)^+ = [y)^+ \cap [x)^+ = |y \vee x)^+$ and $y \vee x \in F$.

By our assumption, y $\in$ F. Therefore $[x)^+ \subseteq F$ and hence $\bigcup_{x\in F}[x)^+ \subseteq F$

(v)$\Longrightarrow$ (i). Assume (v). Let $x \in F^+$. Then there exists a $\in$ F such that $(x)^* \subseteq (a)^*$.

Therefore $x \in [a)^+$ and hence $x \in \bigcup_{x\in F}[x)^+ = F$ (by assumption) thus F is normal.

Let us denote the set of normlets of L as $\mathcal{N}^+\digamma$ (L). Then we have the following;

Theorem2.2.3. ($\mathcal{N}^+\digamma$ (L),$\cap$,$\sqcup$) is a sub lattice of $\mathcal{N}\digamma$ (L) in which $[0)^+$ is a greatest element in $\mathcal{N}^+\digamma$(L).Moreover $\mathcal{N}^+\digamma$(L) has the smallest element if and only if L has dense element.

Proof. It can be observing that by theorem 2.1.2 ($\mathcal{N}^{+}$Ƒ (L),⊓,⊔) is a sub lattice of a distributive +lattice ($\mathcal{N}$Ƒ (L),⊓,⊔) with the greatest element $[0)^{+}$= L. Now, suppose that L has a dense element, say d and let x∈ $[d)^{+}$, then (x)*⊆ (d)* = {0} ⊆ (a)* for all a∈ L. Therefore d ∈ $[a)^{+}$ for all a ∈ L. Hence $[d)^{+}$⊆ $[a)^{+}$ for all a∈ L. thus $[d)^{+}$ is the smallest element in $\mathcal{N}^{+}$Ƒ (L). Conversely suppose that $\mathcal{N}^{+}$Ƒ (L) has the smallest element, say $[a)^{+}$ for some a ∈ L. Let x ∈ (a)*. Then x ∧ a =0.Therefore $[x\wedge a)^{+} = [x)^{+} \sqcup [a)^{+} = [x)^{+}$ =L. Hence x =0. Thus a is dense in L.

Definition 2.2.2. ([2]): An almost distributive lattice L is said to be a disjunctive, if for any x, y ∈ L, x ≠ y implies (x)* ≠ (y)*.

Theorem 2.2.4. If L is a disjunctive ADL, then every filter is normal.

Proof. Suppose that a filter F of L is not normal. Then there exists x, y ∈ L such that $[x)^{+} = [y)^{+}$, x∈ F and y∉ F. Therefore (x)* = (y)*. Since L is disjunctive, x =y. hence y∈ F. which is a contradiction. Thus F is normal.

Remark 2.2.1. The converse of above theorem need not be true. For, see the following example

Example 2.2.1. Let L= {0, d_1, d_2, d_3, m_1, m_2}

∧	0	d_1	d_2	d_3	m_1	m_2
0	0	0	0	0	0	0
d_1	0	d_1	d_2	0	d_1	d_2
d_2	0	d_1	d_2	0	d_1	d_2
d_3	0	0	0	d_3	d_3	d_3
m_1	0	d_1	d_2	d_3	m_1	m_2
m_2	0	d_1	d_2	d_3	m_1	m_2

Table 1

∨	0	d_1	d_2	d_3	m_1	m_2
0	0	d_1	d_2	d_3	m_1	m_2
d_1	d_1	d_1	d_1	m_1	m_1	m_1
d_2	d_2	d_2	d_2	m_2	m_2	m_2
d_3	d_3	m_1	m_2	d_3	m_1	m_2
m_1	m_1	m_1	m_1	m_1	m_1	m_1
m_2	m_2	m_2	m_2	m_2	m_2	m_2

Table 2

Then $(L, \wedge, \vee, 0)$ is an ADL in which every filter is normal but it is not a disjunctive ADL

(because $(m_1)^* = (m_2)^*$ but $m_1 \neq m_2$).

Define a relation ψ on L by $\psi = \{(x, y) \in L \times L\ [x)^+ = [y)^+\}$. It is easy to observe that ψ is a congruence relation on L (by lemma2.2.2).

Lemma 2.2.3.([11]) For any element $a \in L$, we have;

$a/\psi = \{0\}$ if and only if $a = 0$

(ii) $a/\psi = D$ if and only if $a \in D$.

Proof. (i) Let $a \in L$. Then $a \in a/\psi$. Suppose that $a/\psi = \{0\}$. Then $a = 0$. On the other hand, let $a = 0$, $0/\psi = \{x \in L \mid (x, 0) \in \psi\} = \{x \in L \mid [x)^+ = [0)^+\} = \{x \in L \mid (x)^* = (0)^*\}$ (by Lemma 2.2.2 (iv)) $= \{x \in L \mid x = 0\}$. Therefore $0/\psi = \{0\}$.

(ii) Suppose that $a/\psi = D$. Then $a \in a/\psi = D$. Therefore $a \in D$. On the other hand, let $a \in D$, $a/\psi = \{x \in L \mid (x, a) \in \psi\} = \{x \in L \mid [x)^+ = [a)^+\} = \{x \in L \mid (x)^* = (a)^*\}$ (by Lemma 2.2.2 (iv)) $= \{x \in L \mid (x)^* = \{0\}\}$ (since $a \in D$) $= \{x \in L \mid x \in D\}$. Therefore $a/\psi = D$.

Theorem 2.2.5. The quotient lattice L/ψ forms a distributive lattice with the operations $x/\psi \wedge y/\psi = (x \wedge y)/\psi$ and $x/\psi \vee y/\psi = (x \vee y)/\psi$. More ever the least element is $0/\psi = \{0\}$ and the greatest element is $d/\psi = D$.

Proof: suppose L/ψ is a distributive lattice with the least element $0/\psi = \{0\}$. Suppose that L/ψ has the greatest element say, a/ψ, for some $a \in L$. For any $t \in L$, $t \in (a)^* \Rightarrow t \wedge a = 0 \Rightarrow (t \wedge a)/\psi = 0/\psi \Rightarrow t/\psi \wedge a/\psi = 0/\psi \Rightarrow t/\psi = \{0\}$. (since a/ψ is the greatest element) Therefore $t = 0$ (by Lemma 2.2.3 (i)). So that a is dense. Hence L has dense. On the other hand, suppose that L has a dense element say, d. For any $x \in L$, $x/\psi \vee d/\psi = (x \vee d)/\psi$. By the above lemma $(x \vee d)/\psi = D = d/\psi$ (since $x \vee d$ is dense). Hence d/ψ is the greatest element. Thus L/ψ has the greatest element.

S.Ramesh and G.Jogarao [12] introduced the concept of dense complemented ideal in ADL. An ideal I of L is said to be dense complemented in L, if there exists an ideal J in L such that $I \wedge J = \{0\}$ and $I \wedge J$ is an ideal generated by a dense element in L.

Theorem 2.2.6. The following are equivalent;

(i) L is a weak relatively complemented

(ii) $(\mathcal{N}+\mathcal{F}(L), \cap, \sqcup, D, L)$ is a Boolean algebra

(iii) $(L/\psi, \wedge, \vee, 0/\psi, d/\psi)$ is a Boolean algebra

(iv) Every principal ideal of L is dense complemented.

Proof . (i) ⟹ (ii) Suppose that L is a weak relatively complemented ADL. Let $x \in L$ and d is a dense element in L. Then by our assumption, there exists $y \in L$ such that $x \wedge y = 0$ and

$(x \vee y)^* = (x \vee d)^* = \{0\}$. Therefore $x \vee y$ is dense. Now $[x)^+ \cap [y)^+ = ([x) \cap [y))^+ = [x \vee y)^+ = D$ and $[x)^+ \sqcup [y)^+ = ([x) \vee [y))^+ = [x \wedge y)^+ = [0)^+ = L$. Therefore $\mathcal{N}+\mathcal{F}$ (L) is a Boolean algebra.

(ii)⟹ (iii) Suppose that $\mathcal{N}+\mathcal{F}(L), \cap, \sqcup)$ is a Boolean algebra. Let $x \in L$. Then by our assumption, there exists $y \in L$ such that $[x)^+ \cap [y)^+ = D$ and $[x)^+ \sqcup [y)^+ = L$. That is $[x \vee y)^+ = D$ and $L = [x \wedge y)^+$. Therefore $x \vee y$ is dense and $x \wedge y = 0$ and hence. $x/\psi \wedge y/\psi = (x \wedge y)/\psi = 0/\psi = \{0\}$ and

$x/\psi \vee y/\psi = (x \vee y)/\psi = D$. Thus L/ψ is Boolean algebra

(iii)⟹(iv). Suppose that $(L/\psi, \wedge, \vee)$ is a Boolean algebra. Let $x \in L$. Then by our assumption,

there exists $y \in L$ such that $x/\psi \wedge y/\psi = (x \wedge y)/\psi = 0/\psi$ and $x/\psi \vee y/\psi = (x \vee y)/\psi = d/\psi$. Therefore $x \wedge y = 0$ and $x \vee y$ is a dense and hence $(x] \cap (y] = (x \wedge y] = (0]$ and $(x] \vee (y] = (x \vee y]$ is an ideal generated by a dense element $x \vee y$. Thus (x] is a dense complemented ideal.

(iv)⟹ (i) Let $a, b \in L$. Then there exist $c, d \in L$ such that $(a] \wedge (c] = \{0\} = (b] \wedge (d]$ and $(a] \vee (c]$ and $(b] \vee (d]$ are the principal ideals generated by dense elements. Thus $a \wedge c = 0 = b \wedge d$ and $a \vee c$, $b \vee d$ are dense elements. Take $x = c \wedge b$. Then $a \wedge x = a \wedge c \wedge b = 0$ (since $a \wedge c = 0$) and $(a \vee x) \wedge (a \vee b) = a \vee (x \wedge b) = a \vee (c \wedge b \wedge b) = a \vee x$. So that $(a \vee b)^* \subseteq (a \vee x)^*$. Now, for $t \in L$.

$$
\begin{aligned}
T \in (a \vee x)^* &\Longrightarrow t \wedge (a \vee x) = 0 \\
&\Longrightarrow t \wedge a = 0 \text{ and } t \wedge c \wedge b = 0 \\
&\Rightarrow t \wedge b \wedge (a \vee c) = 0 \\
&\Rightarrow t \wedge b = 0 \qquad \text{(since } (a \vee c) \text{ is dense)} \\
&\Rightarrow t \wedge (a \vee b) = 0
\end{aligned}
$$

$\Rightarrow t \in (a \vee b)^*$

Therefore $(a \vee x)^* \subseteq (a \vee b)^*$ and hence $(a \vee x)^* = (a \vee b)^*$. Thus L is a weak relatively complemented.

Theorem 2.2.7. If L is an ADL in which every dense element is maximal, then the following are equivalent.

i. L is quasi complemented
ii. L is relatively complemented
iii. $(\mathcal{N}^{+}\mathcal{F}(L), \cap, \sqcup, M, L)$ is Boolean algebra
iv. $(L/\psi, \wedge, \vee, 0/\psi, m/\psi)$ is Boolean algebra
v. Every principal ideal of L is complemented

Proof. Let every dense element is maximal in L.

(i)⇒ (ii) Suppose that L is quasi complemented. Let $a, b \in L$. Then there exist $c, d \in L$ such that $a \wedge c = 0 = b \wedge d$, $a \vee c$ and $b \vee d$ are maximal. Take $x = c \wedge b$, $a \wedge x = a \wedge (c \wedge b)$ (since $x = c \wedge b$) $= (a \wedge c) \wedge b = 0$ (since $a \wedge c = 0$) and $a \vee x = a \vee (c \wedge b)$ (since $x = c \wedge b$) $= (a \vee c) \wedge (a \vee b) = a \vee b$. (since $a \vee c$ is maximal) Therefore L is relatively complemented.

(ii) ⇒ (iii) Suppose that L is relatively complemented. Let $F \in N^{+}F(L)$. Then there exists $a \in L$ such that $F = [a)^{+}$. For any dense element $d \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $a \vee x = a \vee d$. Now, $a \wedge x = 0 \Rightarrow [a \wedge x) = [0)$

$\Rightarrow [a \wedge x)^{+} = [0)+ \Rightarrow ([a)^{+} \vee [x)^{+})^{+} = L$

$\Rightarrow (F \vee G)^{+} = L$, where $G = [x)^{+}$

$\Rightarrow F \sqcup G = L$. And $a \vee x = a \vee d$

$\Rightarrow [a \vee x) = [a \vee d)$

$\Rightarrow [a \vee x)^{+} = [a \vee d)^{+}$

$\Rightarrow ([a) \cap [x))^{+} = D$ (since $a \vee d$ is dense) $\Rightarrow [a)^{+} \cap [x)^{+} = M$ (since every dense element is maximal)

$\Rightarrow F \cap G = M$, where $G = [x)^{+}$. Therefore $N^{+}F(L)$ are a Boolean algebra.

(iii)⇒ (i) Suppose that $N^{+}F(L)$ is a Boolean algebra. Let $a \in L$. Then there exists $c \in L$ such that $[a)^{+} \cap [c)^{+} = M$ and $[a)^{+} \sqcup [c)^{+} = L$.

Now, $[a)^{+} \cap [c)^{+} = M \Rightarrow [a \vee c)^{+} = M$ (by Lemma 2.2.1 (vi))

$\Rightarrow a \vee c \in M$ (since $a \vee c \in [a \vee c)^{+}$) and $[a)^{+} \sqcup [c)^{+} = L$

$\Rightarrow ([a)^{+} \vee [c)^{+})^{+} = L$

$\Rightarrow [a \wedge c)^{+} = L$

$\Rightarrow a \wedge c = 0$. (by Lemma 2.2.1 (iv)) Therefore $a \wedge c = 0$ and $a \vee c$ is maximal (since every dense is maximal). Hence L is quasi complemented. .

(iii)⇒ (iv) Suppose that $\mathcal{N}+\mathcal{F}(L),\cap,\sqcup)$ is a Boolean algebra. Let x ∈ L. Then by our assumption, there exists y ∈ L such that $[x)^+ \cap [y)^+ = D$ and $[x)^+ \sqcup [y)^+ = L$. That is $[x\vee y)^+ = D$ and $L = [x \wedge y)^+$. Therefore x ∨ y is dense and x ∧ y = 0 and hence. $x/\psi \wedge y/\psi = (x\wedge y)/\psi = 0/\psi = \{0\}$ and $x/\psi \vee y/\psi = (x\vee y)/\psi = D$. (by the Definition (1.2.12)). Thus L/ψ is Boolean algebra.

(iv)⇒ (v) Suppose that $(L/\psi, \wedge, \vee, 0/\psi, m/\psi)$ is a Boolean algebra. Let x ∈ L. By the above theorem, there exists y ∈ L such that x∧y = 0 and x∨y is maximal (since every dense element is maximal). Therefore (x] ∩ (y] = (x ∧ y] = (0] and (x] ∨ (y] = (x∨y] = L (since x∨y is maximal). Hence (x] is a complemented ideal

Theorem 2.2.8 :- L is weak relatively complemented almost distributive lattice if and only if for any a ,b ∈ L, there exists x ∈ L such that a ∧ x = 0and $[a\vee x)^+ = [a\vee b)^+$

Proof. Let a, b, x ∈ L. Then a∧x = 0 and $(a\vee x)^* = (a\vee b)^*$ if and only if a∧x = 0 and

$[a \vee x)^+ = [a \vee b)^+$(by Lemma 2.2.2 (iv)). Hence the theorem is proved.

2.3 Normal prime filters

In this section, we study the class of normal prime filters in an almost distributive lattice with dense element. We obtain some properties on them. For any filter F of L, we prove that the intersection of all normal prime filters containing F is the smallest normal filter containing F.

Theorem 2.3.1. Let F be a filter of L and for any chain of filters $C_1, C_2, C_3\ldots$ Of L such that

$F\subseteq C_1\subseteq C_2\subseteq C_3\subseteq \ldots\subseteq F^+$. Then $C_1^+ = C_2^+ = C_3^+ = \ldots = F^+$.

Proof: Suppose that F be a filter of L and for any chain of filters $C_1, C_2, C_3\ldots$of L such that

$F\subseteq C_1\subseteq C_2 \subseteq C_3\subseteq \ldots\subseteq .F^+$. Then $C_1^+ \subseteq C_2^+ \subseteq C_3^+ \subseteq\ldots\subseteq F^{++} = F^+$ (since F^+ is normal. Therefore $C_1^+ = C_2^+ = C_3^+ = \ldots = F^+$

Theorem 2.3.2. Let F be a proper filter of L. Then there exists a normal prime filter containing F.

Proof: - Let F be a proper filter of L. Take ρ = {G: G is a proper normal filter of L and F ⊆G.

(By lemma 2.1.3).; F^+ is a proper normal filter containing F. Therefore $F^+\in \rho$ and ρ satisfies the hypothesis of Zorn`s lemma. Hence ρ has a maximal element, say P. let a, b ∈ L such that a ∉ P and b∉ P. then $P\sqcup [a)^+$ and $P\sqcup [b)^+$ are normal filters, which containing P properly. By the maximality of P, $L= P\sqcup [a)^+ = P\sqcup [b)^+$.Therefore $L = \{ P\sqcup [a)^+\} \cap \{ P\sqcup [b)^+\}= \{(P\sqcup [a)) \cap (P\sqcup$

$[b))\}^{+}=\{P\vee[a\vee b)\}^{+}$. If $a\vee b\in P$, then $L=P^{+}=P$. Which is a contradiction. Hence $a\vee b\notin P$. Thus P is prime.

Theorem 2.3.3. If P is a minimal in the class of prime filter containing a normal filter F, then P is a normal.

Proof. Let F be a normal filter of L and P is a minimal in the class of prime filters of L Containing F. Suppose that P is not a normal. Then there exists x, y $\in$ L such that $[x)^{+}=[y)^{+}$, $x\in P$ and $y\notin P$. Take $I=L-P\vee(x\vee y]$ is an ideal of L. Then $I\cap F=\phi$. If $I\cap F=\phi$, then $a\in I\cap F$. Therefore $a=r\vee s$ for some $r\in L-P$ and $s\in(x\vee y)$.So that
$r\vee s=r\vee\{(x\vee y)\wedge s\}=r$
$\vee\{(y\vee x)\wedge s\}=\{r\vee(y\vee x)\}\wedge(r\vee s)\in F$ (since $r\vee s=a\in F$).So that $r\vee\{(y\vee x)\in F$. We have
$[x)^{+}=[y)^{+}$. Then $[r\vee y\vee x)^{+}=[r\vee y\vee y)^{+}=[r\vee y)^{+}$. Since F is normal, $r\vee y\in P$, which is a Contradiction. Therefore $I\cap F=\phi$. So that there exists a prime filter Q such that $I\cap Q=\phi$, $F\subseteq Q$ and $Q\subseteq P$. Also $x\vee y\notin Q$ and $x\vee y\in P$. We get $Q\subsetneq P$. hence P is not minimal. Which is a contradiction. Thus P is normal prime filter.

Corollary 2.3.1. Every minimal prime filter containing D is normal.

Proof: - Let P be the minimal prime filter of L and $D\subseteq P$. that is P is the minimal is the class of prime filters containing (the normal filter) D. By the above theorem, P is normal.

Theorem 2.3.4. Let F be a normal filter and I is an ideal of L such that $F\cap I=\phi$. Then there exists a normal prime filter P such that $F\subseteq P$ and $P\cap I=\phi$.

Proof: - Let F be a normal filter and I is an ideal of L such that $F\cap I=\phi$. Take P= {G: G is a normal filter, $F\subseteq G$ and $G\cap I=\phi$}. Clearly $F\in P$ and it satisfies the hypothesis of Zorn`s Lemma. Therefore P has a maximal element, say P. Choose x, $y\in L$ such that $x\notin P$ and $y\notin P$. then $P\subseteq P\sqcup[x)^{+}=\{P\vee[x)\}^{+}$ and $P\subseteq P\sqcup[y)^{+}=\{P\vee[y)\}^{+}$. By the maximality of P, $\{P\vee[x)\}^{+}\cap I\neq\phi$ and $\{P\vee[y)\}^{+}\cap I\neq\phi$. Let $a\in\{P\vee[x)\}^{+}\cap I$ and $b\in\{P\vee[y)\}^{+}\cap I$. then $a\vee b\in I$. and $a\vee b\in\{P\vee[x)\}^{+}\cap\{P\vee[y)\}^{+}=\{P\vee[x)\}\cap\{P\vee[y)\}\}^{+}=\{P\vee[x\vee y)\}^{+}$. If $x\vee y\in P$, then $a\vee b\in P^{+}=P$ (Since P is normal) and $a\vee b\in I$. Hence P is prime. Thus P is a normal prime filter of L such that $F\subseteq P$ and $P\cap I=\phi$.

Remark 2.3.1. If F is not a normal filter, then the above theorem need not be true.

For see example 2.1.1. Let F = $[b_4)$ and I = [d], then F∩I = ϕ. But there is no normal prime filter P such that P∩I = ϕ and F ⊆P.

Corollary 2.3.2. Let F be a normal filter of L and x∉F. Then there exists a normal prime filter P of L such that F ⊆ P and x∉P.

Proof: - Let F be a normal filter of L and x ∉ F. then (x] ∩F=ϕ. Therefore (by Theorem 2.3.4)., there exists a normal prime filter P such that F ⊆P and (x] ∩P=ϕ. Thus x ∉ P.

Theorem 2.3.5. For any filter F, F^+= ∩{P: P is a normal prime filter of L and F ⊆ P}

Proof: - Let F be a filter of L and x∉F^+. Put P = {G: G is a normal filter of L and x∉G and

F ⊆G}. clearly F^+∈P and it satisfies the hypothesis of Zorn`s Lemma. Therefore P has a maximal element, say P let a, b∈L such that a∉P and b∉P. Then x∈P⊔|a)$^+$ = {P ∨[a)$^+$}$^+$ = {P ∨[a)}$^+$ and x∈P⊔[b)$^+$ = {P∨[b)$^+$}$^+$ = {P∨[b)}$^+$. Thus x ∈ {P∨ [a)}$^+$ ∩ {P ∨ [b)}$^+$ = {P∨[a∨b)}$^+$. Suppose that a∨b ∈ P. Then x∈ P^+=P (since P is normal. So that x∈P. which is a contradiction. Therefore P is prime. Hence P is normal prime filter containing F and x∉P. Thus F^+ = ∩{P: P is a normal prime filter of L and F ⊆ P}.

Corollary 2.3.3. The intersection of normal prime filter of L is equal to D.

Proof :- We have every normal filter of L containing the filter D. hence (by Lemma 2.1.1) from the above theorem it is obvious.

Theorem 2.3.6. For any filter F of L, F^+∩ F* =ϕ

Proof :- Let F be a filter of L. Suppose that F^+∩ F* ≠ϕ. Let t∈ F^+∩ F*. Then there exist a∈ F such that (t)* ⊆(a)* and t∧f = 0for all f∈F. Therefore a∈(t)* (since t∧a=0). Hence a=0 and a∈F. Which is a contradiction. Thus F^+∩ F* =ϕ.

Corollary 2.3.5. For any filter F of L, there exists a normal prime filter P of L such that F⊆P and P∩F* = ϕ.

Proof :- Let F be a filter of L. Then by the above theorem $F^{+} \cap F^{*} = \phi$. (By Theorem 2.3.6.)., there exists a normal prime filter P of L such that $F^{*} \subseteq P$ and $P \cap F^{*} = \phi$. Thus $F \subseteq P$ and $P \cap F^{*} = \phi$.

2.4 The space of normal prime filters in Almost Distributive Lattice

In this section we discuss the space of normal prime filters in an almost distributive lattice which the hull-kernel topology. Finally we obtain necessary and sufficient conditions for the space of normal prime filter to become Hausdorff.

Let us denote specL the set of normal prime filter of L. For any $A \subseteq L$, K (A) = {$P \in$ specL $A \not\subseteq P$}. In particular, for $a \in L$, K (a) = {$P \in$ specL $a \notin P$}

Theorem 2.4.1. We have the following

(i) For any $a \in L$, K(a) is compact

(ii) If C is a compact open subset of specL, then C=K(a)

Proof :- (i) Let $a \in L$ and B $\subseteq L$ such that K(a) $\subseteq \bigcup_{b \in B} K(b)$ and F=[B) is a normal filter of L Generated by B. If $a \notin F$, by corollary 2.3.2. , there exists a normal prime filter of P such that $F \subseteq P$ and $a \notin P$. Therefore $P \in K(a) \subseteq \bigcup_{b \in B} K(b)$. Hence $b \notin P$ for some $b \in B$ .Which is a contradiction. So that a $\in F=[B)$ and a $= x \vee (\wedge_{i=1}^{n} b_i)$ for some $b_1, b_2, b_3, \ldots, b_n \in B$ and $x \in L$. (by Lemma 2.4.2.) ,K(a) = $K(x \vee (\wedge_{i=1}^{n} b_i))$ = $K(x) \cap K(\wedge_{i=1}^{n} b_i) \subseteq K(\wedge_{i=1}^{n} b_i) = \bigcup_{i=1}^{n} K(b_i)$ and hence K(a) is compact in specL.

(iii) Let C is a compact open subset of specL. Then C =K(A), for some $A \subseteq L$ (since C is open). Therefore C $= \bigcup_{a \in A} K(a)$. Therefore there exist $a_1, a_2, a_3, \ldots a_n \in A$ such that

C = $\bigcup_{i=1}^{n} K(a_i)$ = K(a) for some $a \in L$ (since C is compact).

Theorem 2.4.2. Let L be an ADL in which every prime filter is normal. Then L is a distributive lattice if and only if the map $a \mapsto K(a)$ is an injection.

Proof. Suppose that L is a distributive lattice in which every prime filter is normal. Let a ,b ∈L

such that a≠ b. Then there exists a prime filter P such that a∈P and b∉P. Hence K(a)≠K(b). Thus the map is injection. Conversely suppose that the map is injection. Let a, b ∈L. Then K(a∨b) = K(b∨a). Therefore a∨b = b∨a. Hence L is a distributive lattice.

Lemma 2.4.2 Let L be an ADL with maximal elements. Let P be a normal prime filter of L. Then P is a minimal if and only if for each x∈P there exists y∉P such that x∨y is maximal.

Proof :- Let L be an ADL with maximal elements. Let P be a normal prime filter of L. Suppose that P is a minimal prime filter of L. Let x∈P, then L\ P∨{x} =L (since L\P is a maximal ideal). Therefore there exists a maximal element m∈ L such that m = x∨y, x∈P and y∉P. On the other hand, clearly we have L\P is a prime ideal of L. then there exists x∈L such that x∉ L\P. Therefore by our assumption there exists y∉P such that x∨y is maximal. Hence L\P is maximal. Thus P is a minimal prime filter of L. For any A⊆L, denote H (A) = {P∈ specL A⊆P}. Then H (A) = SpcL\K (A). Therefore H (A) is a closed set in SpecL and hence every closed set in SpecL is of the form H (A) for some A⊆L. Thus we have the following.

Theorem 2.4.3. For any Y⊆SpecL, the closure of Y is given by Ȳ =H $\bigcap_{p\in Y} P$.

Proof: - Let Y ⊆SpecL. Let Q ∈Y. Then $\bigcap_{p\in Y} P$ ⊆Q. Therefore Q∈ H (Q)⊆ $H(\bigcap_{p\in Y} P)$.

Hence $H(\bigcap_{p\in Y} P)$ is a closed set containing Y. Let C e a closed set in SpecL containing Y.

Then C =H (A), for some A⊆L. Therefore A⊆ $\bigcap_{p\in Y} P$. Hence $H(\bigcap_{p\in Y} P)$ ⊆ H (A)=C.

Thus Ȳ = H($\bigcap_{p\in Y} P$).

Theorem 2.4.4. Let L be an ADL in which every dense element is maximal. Then the following are equivalent;

(i) Every normal prime filter is maximal

(ii) Every normal filter is minimal

(iii)SpecL is T_1-space

(iv)SpecL is a Hausdorff space

(v) For any x,y∈ L, there exists z∈L such that x∨z is a maximal and K(y)∩{SpecL-K(x)}=K(y∨z).

Proof :- **(i)⟹(ii)** Let P be a normal prime filter of L. If Q is a normal prime filter of L such that Q⊆P, then by our assumption Q=P, Therefore every normal prime filter is minimal.

(ii) ⟹(i) Let P be a normal prime filter of L. If Q is a normal prime filter of L such that P⊆Q. then P =Q (by our assumption). Therefore P is maximal

(ii) ⟹ (iii) Let P and Q are two distinct normal prime filter of L. Then P⊈Q and Q⊈P (since P and Q are minimal). Take x∈P/Q and y∈Q/P. Then Q ∈K(x)/K(y) and P∈K(y)/K(x). Therefore SpecL is T_1-space.

(iii) ⟹(iv) Suppose that SpecL is T_1- space. Let P ∈ SpecL. Then P= {P} = $\overleftarrow{H}$ (P) = {Q∈SpecL P⊆Q}. Therefore P is a maximal. Hence every normal prime filter is maximal. Let P, Q ∈ SpecL such that P≠Q. Choose x∈ P and x∉Q. Then there exists y∉P such that x∨y is maximal. Therefore P∈K(y) and Q∈K(x) and K(x∨y) =K(x)∩K(y)=ϕ. Hence SpecL is Hausdorff space.

(iv)⟹ (v) Let a∈L. Then K(a) is a compact subset of the Hausdorff space SpecL. Then K(a) is clopen subset of SpecL. Let x,y∈L such that x≠ y. Then K(y) ∩{specL\K(x)} is a compact open subset of SpecL. (By lemma 2.4.1)., there exists z∈L such that K (z) = K(y) ∩{SpecL\K(x)}. So that K(y∨x)=K(y) ∩K(z)= K(y)∩{SpecL\K(x)}= K(z) and K(x∨z) = K(x)∩K(z)= ϕ. (By Lemma 2.4.1)., x∨z is dense. Thus x∨z is maximal (since every dense is maximal).

(iv)⟹(ii) Let P is a normal prime filter of L. Let x, y∈L such that x∈P and y∉P. Then by our assumption there exists z∈L such that x∨z maximal and K(y∨z) = K(y)∩{SpecL\K(x)}, P∉K(x) and P∈K(y). Therefore P∈ K(y) ∩ {SpecL\K(x)} = K (x∨z). If z∈P, then y∨z∈P, which is a contradiction. Hence for each $x \in L$, there exist a normal prime filter P and $z \notin P$ such that $x \vee z$ is a maximal. Thus P is minimal (by Lemma 2.4.2).

3. Conclusion

In this project, we have discussed the concept of lattice, distributive lattice and ADL. Moreover we introduced and characterized normal filter, normlet, normal prime filter in almost distributive lattice. And also we have to understand how to proof a theorem, Lemma and corollary related to normal filter in almost distributive lattice.

4. References

[1] G.Birkhoff, Lattice theory, Amer. Math. Soc. Colloquium pob, 1967.

[2] G. C. Rao and G. Nanaji Rao, Dense elements in almost distributive lattices, Southeast Asian Bull. Math., 27(6)(2004), 1081-1088.

[3] G.C. Rao and M. Sambasiva Rao, Annihilator ideal in almost distributive lattices, Int. Math. Forum., (4915)(2009), 733-746.

[4] G.C. Rao and M. Sambasiva Rao, Annulets in almost distributive lattices, Eur. J. Pure. Appl. Math., 2(1)(2009). 58-72.

[5] G.C. Rao and S. Ravi. Kumar, Minimal prime ideal in Almost distributive lattices, Int. J. Contemp. Math. Sci, 4(10)(2009). 475-484.

[6] G.C. Rao, G. Nanaji Rao and A. Lakshman, Quasi complemented almost distributive lattice, Southeast Asian. Bull. Math,39(3)(2015),311-219.

[7] M. H. Stone, Topological representation of distributive lattices and Broouwerian logics, asopis pro psiovn matemtiky a fysiky, 67(1937-38), 1-25.

[8] M. Mandelker, Relative annihilators in lattices. Duke Math. J., 37(2)(1970). 377-386.

[9] M. Sambasiva Rao, Normal filters of distributive lattice; Bull. Sci. Logic. 41(3-4)(2012), 131-143.

[10] Burris and H.P.Sankappanavar, A course in universal algebra, Springer-Verlag,1980.

[11] S. Ramesh and G. Jogarao, Weak relatively complemented almost distributive lattices, (To appear)

[12] S. Ramesh, On almost distributive lattices, Doctoral thesis, Andra University, 2009.

[13] T. P. Speed, some remarks on a class of distributive lattices, J. Australian Math. Soc., 9(3-4)(1969),289-296.

[14] T.S.Blyth, Ideals and filters of pseudo-complemented semi-lattices, proc. End. Math. Soc., 23(3)(1980),301-316.

[15] U.M.Swamy and G.C Rao, Almost distributive lattices, J. Austral. Math. Soc., 31(1981),77-91

[16] W.H.Cornish, Normal lattices, J. Australian Math. Soc., 14(1972), 200-215

[17] W.H.Cornish, Quasi complemented lattices, Comm. Math. Univer. Carolinae, 15(3)(1974) 501-511

[18] Y.S. Pawar and I. A. Shaikh, On prime, minimal prime and annihilator ideal in an almost distributive lattices, European. J. Pure and Applied Maths .,6(1)(2013),107-118.

[19] Y.S. Pawar, The space of maximal ideals in an almost distributive lattice, Int. Math. Forum., 6(28)(2011), 1387-1396.