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A Project Report On Stability Analysis of Systems of First Order Ordinary Differential Equations

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On

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Bahir Dar, Ethiopia

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College of Sciences

Department of Mathematics

Stability Analysis of Systems of First Order Ordinary Differential Equations

A project Submitted to The Department of Mathematics, in Partial Fulfillment of The Requirements for the Degree of Masters of Science in Mathematics

By

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Declaration

I hereby declare that, this project is done by me under the supervision of Dr. Eshetu Haile Department of mathematics, Bahir Dar University, in Partial fulfillment of the requirements for the degree of Master of Science in Mathematics. I am declaring that this project is my original work. I also declare that neither of this project nor any of its parts has been submitted to elsewhere for the award of any other degrees or certificates.

Demamit Setegn ____________ ____________

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Approval of the Project for Oral Defense

I hereby certify that I have supervised, read and evaluated this project entitled **"Stability Analysis of Systems of First Order Ordinary Differential Equations"** by Demamit Setegn prepared under my guidance. I recommend that the project is submitted for oral defense.

Dr. Eshetu Haile	_____________	_____________
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Department Head's name	Signature	Date

Bahir Dar University

College of Science

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Approval of the Project for Defense Result

We hereby certify that we have examined this project entitled **"Stability Analysis of Systems of First Order Ordinary Differential Equations"** by Demamit Setegn. We recommend that this project is approved for the degree of Master of Science in mathematics.

Board of Examiners

____________________	______________	_________________
External examiner's name	Signature	Date
____________________	_____________	______________
Internal examiner's name	Signature Date	
___________________	_______________	______________
Chair person's name	Signature	Date

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Abstract

In this project, we discussed stability, where the stability of the state trajectory or equilibrium state is examined, stability is applied to obtain the behavior of systems of first-order ordinary differential equation and we present two techniques for examining stability: (1) Lyapunov functions, (2) finding the eigenvalues for fundamental matrix. Stability was proposed in 1892 by Russian mathematician A.M.Lyapunov. It is examined with some examples which are presented to show the effectiveness of it for linear and nonlinear systems and also for any higher order differential equation by reducing it into a system of the first order.

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Chapter One

1. Introduction and Preliminary Concepts

1.1. Introduction

Differential equations arise in many areas of science and engineering, specifically whenever deterministic relations involve some continuously varying quantities and their rates of change in space and/or time. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity at the time value varies. Newton's law allows one to express these variables dynamically as a differential equation for the unknown position of the body as a function of time. The study of differential equations is a wide field in pure and applied mathematics, physics, metrology, and engineering [9]. All of these disciplines are concerned with the properties of differential equations of various types; pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justifications of the methods for approximating solutions.

The rate of change of any quantity with respect to other quantity (quantities) is expressed by a differential equation. Differential equations play an important role in mathematical modeling of physical phenomena occurring in science, economics, engineering, medicine, etc. [2]. Solving various problems in science and engineering requires differential equations. Many physical, chemical, mathematical models, biological phenomena, economics, financial forecasting, image processing and other fields are based on differential equations [2].

The general first order ODE can be written as

$$\frac{dy}{dx} = f(x, y) \text{ or using prime notation } y' = f(x, y),$$

where f is a continuous function on a common region.

An ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation that can be expressed in the form of

$$f(x, y, y', y'', \cdots, y^{(n)}) = 0.$$

If f is linear with respect to y and its derivative, we have

$$\sum_{j=0}^{n} a_j y^{(j)} = g(x),$$

which is an n^{th} order linear ordinary differential equation.

A linear first order ordinary differential equation can be written as in the form

$$\frac{dy}{dx} + p(x)y + q(x) = 0,$$

where $p(x)$ and $q(x)$ are continuous function defined in a common interval.

Stability is a quantity, state or degree of being stable: such that, it is the strength to stand or endure, it is the property of a body that causes it when disturbed from a condition of equilibrium or steady or steady state to develop forces or moments that restore the original condition. In an unstable system, the state can have large variations, and small inputs or small changes in the initial state may produce large variations in the output. A common example of an unstable system is illustrated by someone pointing the microphone of a public address (PA) system at a speaker; a loud high-pitched tone results. Often instabilities are caused by too much gain; so to quiet the PA system, decrease the gain by pointing the microphone away from the speaker. Discrete systems can also be unstable. Someone gave the illustration of a person who was reading in a chair and became cold. She/he then went over and increased the temperature of the air conditioning. She/he then went over and raised the heater's thermostat. The home grew warmer. She or he decided to stand up and lowered the thermostat after becoming overheated. The home began to chill. She or he raised the thermostat after becoming cold. This process kept going until someone eventually suggested that she/he put on a sweatshirt to stop their body from losing heat. She/He did and felt a lot more at ease. Because she/he appeared to sample the environment and provide outputs at distinct intervals spaced about 15 minutes, we called this a discrete system.

Stability was probably the first question in a classical dynamical system which was dealt with in a satisfactory way. Stability questions motivated the introduction of new mathematical concepts (tools) in engineering, particularly in control engineering. Stability theory has been of interest to mathematicians and astronomers for a long time and has had a stimulating impact on these fields. The specific problem of attempting to prove that the solar system is stable accounted for the introduction of many new methods. Our treatment of stability will apply to (control) systems described by sets of linear or nonlinear equations. As is to be expected, however, our most explicit results will be obtained for linear systems [15].

The general objective of this proposal is to assess the stability analysis of systems of first order ordinary differential equations by using eigenvalue technique, Routh Hurwitz criterion and Lyapunov function.

The specific objectives this project is

- To identify the stability of systems of first order ODEs.
- To determine the region at which the system is stable.
- To gain an understanding of the behaviors of the solutions to the systems.

1.2. Existence and Uniqueness Theorems of Solutions

Existence and uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to a first order differential equation which satisfies a given initial condition. The existence theorem is used to check whether there exists a solution for an ODE, while the uniqueness theorem is used to check whether there is one solution or infinitely many solutions. The existence and uniqueness theorem are also valid for a certain system of first order differential equations. These theorems are also applicable to a certain higher order ODE since a higher order ODE can be reduced to a system of first order ODE.

1.2.1. Lipschitz Condition and Gronwall Inequality

If the functions satisfy the Lipschitz condition, successive approximations can yield a unique solution to the initial value problem

$$x'(t) = f(t, x), x(t_0) = x_0.$$

Gronwall's inequality, a kind of inequality, is used to prove the solution's uniqueness. We will therefore introduce a class of functions satisfying the Lipschitz condition as a preliminary to the Picard's theorem before discussing Gronwall's inequality.

Definition 1.2.1.1: If there is a positive constant K such that $|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|$ forevery (t, x_1) and (t, x_2) belonging to D, then the function $f(t, x)$ defined in the region $D \subset \mathbb{R}^2$ is said to satisfy the Lipschitz condition with respect to x in D. For the function f in D, the constant $K > 0$ is known as the Lipschitz constant. The class of all functions satisfying the Lipschitz condition with the Lipschitz constant K in a domain $D \subset \mathbb{R}^2$ is denoted by Lip (D, K). From the very definition, we note that if $f \in \text{Lip}(D, K)$, then

$$\frac{|f(t, x_1) - f(t, x_2)|}{|x_1 - x_2|} \leq K.$$

Hence, to show that $f(t,x)$ satisfies Lipschitz condition with respect to x in $D \subset \mathbb{R}^2$, it is enough if we prove that

$$\frac{|f(t,x_1) - f(t,x_2)|}{|x_1 - x_2|} \leq K$$

is bounded for all $(t,x) \in D \subset \mathbb{R}^2$.

Theorem 1.2.1.1: Let $x' = f(t,x)$ be a real valued function which is continuous on the rectangle $R = \{(t,x): |t - t_0| \leq a, |x - x_0| \leq b\}$, where $a, b > 0$. If $\frac{\partial f}{\partial x}$ exists and is continuous on the rectangle R, then $f(x,t)$ satisfies Lipschitz condition with respect to x in R and Lipschitz constant K is given by

$$K = lub f_x(x,t).$$

Proof: Since $\frac{\partial f}{\partial x}$ is continuous in a closed rectangle R, it is bounded in R so that its least upper bound exists in R.

Let $K = lub\left(\frac{\partial f(t,x)}{\partial x}\right)$. (1.1)

Let (t,x_1) and (t,x_2) be any two points of R. Then by the mean value theorem of differential calculus, there exists a point $\varepsilon \in [x_{1,}x_2]$ such that

$$f(t,x_1) - f(t,x_2) = \left[\frac{\partial f(t,\varepsilon)}{\partial x}\right](x_1 - x_2), (t,\varepsilon) \in R. \quad (1.2)$$

Using (1.1) and (1.2), we obtain

$|f(t,x_1) - f(t,x_2)| = K|(x_1 - x_2)|$ for all (t,x_1) and (t,x_2) in R. This proves that $f(x,t)$ satisfies Lipschitz condition with Lipschitz constant K in R.

Theorem 1.2.1.2: (Gronwall Inequality) Let $f(t)$ an $g(t)$ be two non-negative continuous functions for $t \geq t_0$. Let K be any positive constant. Then, the inequality

$$f(t) \leq K + \int_{t_0}^{t}\big(g(s) * f(s)\big)d(s), \;\; t \geq t_0 \quad (1.3)$$

Implies the inequality

$$f(t) \leq K exp\left[\int_{t_0}^{t} g(s)d(s)\right], t \geq t_0. \quad (1.4)$$

Proof: Let $F(t) = K + \int_{t_0}^{t}\big(f(s) * g(s)\big)d(s)$. (1.5)

First note that $F(t) \neq 0$ for any $t \geq t_0$ and $F'(t) = g(t) * f(t)$ and $F(t_0) = K$ (1.6)

Since $F(t) \neq 0$ for any $t > t_0$, from the hypothesis, we get

$$\frac{f(t)}{K+\int_{t_0}^{t}(g(s)*f(s))d(s)} = \frac{f(t)}{F(t)} \leq 1.$$

Since $g(t)$ is non-negative, we obtain from the above inequality

$$\frac{g(t)*f(t)}{F(t)} \leq g(t), \text{ for any } t \geq t_0.$$

Using (1.6) in the above inequality, we get

$$\frac{F'(t)}{F(t)} \leq g(t). \tag{1.7}$$

Integrating (1.7) from t_0 to t, which gives

$$\log F(t) - \log F(t_0) \leq \int_{t_0}^{t} g(s)d(s).$$

Using $F(t_0) = K$ and $F(t)$, we have

$$\log\left[K + \int_{t_0}^{t}\big(f(s)*g(s)\big)d(s)\right] - \log K \leq \int_{t_0}^{t} g(s)d(s).$$

Taking exponential on both sides of the above, we get

$$F(t) = \quad K + \int_{t_0}^{t}\big(f(s)*g(s)\big)d(s) \leq Kexp\left[\int_{t_0}^{t} g(s)d(s)\right]. \tag{1.8}$$

Replacing the left hand side of (1.8) by lesser term given in the hypothesis, we get Gronwall's inequality

$$f(t) \leq Kexp\int_{t_0}^{t} g(s)d(s).$$

1.2.2. Successive Approximations and Picard’s Theorem

It was Emile Picard (1856–1941) who developed the method of successive approximations to show the existence of solutions to ordinary differential equations. He proved that it is possible to construct a sequence of functions that converges to a solution of the differential equation. One of the first steps towards understanding Picard’s iteration is to realize that an initial value problem can be reconstructed in terms of an integral equation. The Picard's theorem gives the unique solution of the equation $x'(t) = f(t,x),\ x(t_0) = x_0$, where $f(t,x)$ is some arbitrary function defined and continuous in some neighborhood of (x_0, t_0) by the method of successive approximations using the integral equation equivalent to the given differential equation. The main emphasis of the theorem is that it asserts the existence and uniqueness of the solution of an initial value problem under very general conditions. In theory of differential equations, such theorems are called the existence and uniqueness theorems. Thus, the theorem is more of theoretical importance than practical utility in solving initial value problems.

Theorem 1.2.2.1: *(Picard's Iteration Theorem)*

The function $x(t)$ is a solution to the initial value problem

$$x' = f(t,x), x(t_0) = x_0 \tag{1.9}$$

If and only if $x(t)$ is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\big(s, x(s)\big)ds. \tag{1.10}$$

Proof: Let x(t) be a solution of the initial value problem (1.9). Then we have from (1.9)

$$x^{'}(t) = f(t, x(t)), t \in I = [t_0, t]. \tag{1.11}$$

If x(t) is a solution of (1.9), it is a continuous function on I because it is differentiable on I. Since x(t) is continuous on I and f is continuous on D, the function $F(t) = f(t, x(t))$ is continuous on D so that it is integrable on I.

Integrating (1.11) from t_0 to t, we get

$$x(t)\text{-}x(t_0) = \int_{t_0}^{t} f(s, x(s))d(s)$$

Since we take $x(t_0) = x_0$, we get

$$x(t) = x_0 + \int_{t_0}^{t} f\big(s, x(s)\big)d(s)$$

Conversely, if $x(t)$ satisfies (1.10) on I, using the fundamental theorem of integral calculus, we obtain its derivative $x'(t)$as

$$x'(t) = f(t, x(t)), \text{ for all } t \in I.$$

Thus, $x(t)$ is a solution of the initial value problem.

Theorem 1.2.2.1: [5] Let $x' = f(t,x)$ be a real valued function which is continuous on the rectangle $R = \{(t,x): |t - t_0| \leq a, |x - x_0| \leq b\}$. Assume f has a partial derivative with respect to x and that $\frac{\partial f}{\partial x}$ is also continuous on the rectangle R. Then there exists an interval $I = [t_0 - h, t_0 + h]$ (with $h < a$) such that the initial value problem

$$x' = f(t,x), x(t_0) = x_0$$

has a unique solution $x(t)$ defined on the interval I.

Definition 1.2.2.1: Equilibrium solution is a solution to a differential equation whose derivative is zero everywhere.

Example 1: Suppose that $x' = x^2$ with$x(0) = 1$. Since $f(t, x) = x^2$ and $\frac{\partial f}{\partial x} = 2x$ are continuous everywhere, a unique solution exists near $t = 0$. Separating the variables,

$$\frac{1}{x^2}dx = dt.$$

We see that

$$x = -\frac{1}{t+c}$$

Or

$$x = \frac{1}{1-t}$$

Therefore, a solution also exists on $(-\infty, 1)$ if $x(0) = -1$. In the case that $x(0) = -1$ and the solution is

$$x = -\frac{1}{t+1}$$

and a solution exists on $(-1, \infty)$. Solutions are only guaranteed to exist on an open interval containing the initial value and are very dependent on the initial condition.

Example 2: Consider the initial value problem $x' = x^{\frac{1}{3}}$ with $x(0) = 0$ and $t \geq 0$. Separating the variables,

$$x^{-\frac{1}{3}}dx = dt.$$

Thus, by integrating both sides

$$\frac{3}{2}x^{\frac{2}{3}} = t + c.$$

Or

$$x = \left(\frac{2}{3}(t+c)\right)^{\frac{3}{2}}.$$

If $c = 0$, the initial condition is satisfied and

$$x = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$$

is a solution for $t \geq 0$. However, we can find two additional solutions for $t \geq 0$:

$$x = \left(\frac{2}{3}t\right)^{\frac{3}{2}}, \text{ and}$$

$$x \equiv 0$$

This is especially troubling if we are looking for equilibrium solutions. Although $x' = x^{\frac{1}{3}}$ is an autonomous differential equation, there is no equilibrium solution at $x = 0$. The problem is that

$$\frac{\partial}{\partial x} x^{\frac{1}{3}} = \frac{1}{3} x^{\frac{-2}{3}}$$

is not defined at $x = 0$.

1.3. Systems of Ordinary Differential Equations

Definition 1.3.1: A system of first order ordinary differential equation is an equation that can be written in the form:

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \cdots, x_n),\ i = 1,2,\cdots,n.$$

If each f_i, $i = 1,2,\cdots,n$ is linear with respect to x_i, $i = 1,2,\cdots,n$, then the system is linear and can be expressed as

$$\begin{cases} \frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ \frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{cases}, \tag{1.12}$$

where the coefficients a_{ij} and f_i are continuous on a common interval I.

When $f_i(t) = 0,\ \forall i = 1,2,\dots,n,$ the system is homogeneous. Otherwise, it is nonhomogeneous. We can write this system in matric form if X, $A(t)$ and $F(t)$ denote the respective matrices

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

Then the linear system of a first order ordinary differential equation can be written as:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}. \tag{1.13}$$

Or simply

$$X' = AX + F(t).$$

If the system is homogeneous, its matric form becomes

$$X' = AX.$$

Definition 1.3.2: An Autonomous System is a system of DEs which does not depend on independent variables. It is of the form

$$\frac{dx_i}{dt} = f_i(x_1,\ x_2,\ x_3,\ \cdots,\ x_n), i = 1, 2,3\ \cdots n.).$$

Definition 1.3.3: Equilibrium point (fixed point) is the set of intersection of the set of point which satisfy

$$f_i(x_1,\ x_2,\ x_3,\ \cdots,\ x_n) = 0.$$

Definition 1.3.4: Given n vectors $v_1 = \begin{pmatrix} y_{1,1(x)} \\ \vdots \\ y_{1,n(x)} \end{pmatrix}, \cdots,\ v_n = \begin{pmatrix} y_{n,1(x)} \\ \vdots \\ y_{n,n(x)} \end{pmatrix}$ of length n with functions as entries, the Wronskian is defined as the determinant of $W = \begin{bmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{bmatrix}$.

Definition 1.3.5: Let $v_1, v_2, \cdots, v_n$ be a set of solution vectors of the homogenous system on an interval I, then the set of solution is linearly dependent on the interval I if there exists non-zero constant $c_1, c_{2,\ldots}, c_k$ such that $c_1v_1 + c_2v_2 + \cdots + c_kv_n = 0$, forevery $k \in I$. Otherwise, it is linearly independent. If the set of solution vectors are linearly independent, then it is a fundamental solution of the homogenous system.

Theorem 1.3.1: [16] Let the vector functions $v_1, v_2, \cdots, v_n$ be n solutions of the homogeneous linear vector differential equation $X' = AX$ on I. Then, the n-solutions are linearly independent on I if and only if $W(v_1(t), v_2(t), \cdots, v_n(t)) \neq 0$ for all $t \in I$.

Definition 1.3.6: Let $A = [a_{ij}]$ denotes an $n\ x\ n$ matrix. Then the norm of the matrix A denoted by $\|A\|$ is defined as $\|A\| = \sum_{i,j=1}^{n}|a_{ij}|$. If $A = [a_{ij}]$ is a continuous matrix on I, then as $\|A\|$ is also continuous on I.

1.3.1. Existence and Uniqueness of Solutions of a System of Equations

Theorem 1.3.1.1[14, 16] Let a system is given by

$$\frac{dx_i}{dt}(t) = f_i\big(t, x_1(t), x_2(t), \cdots, x_n(t)\big), \forall i = 1,2,\cdots,n, \ \ x_i(t_0) = \alpha_i, i = 1,2,\cdots,n.$$

If the functions f_i, $i = 1,2,\cdots,n$ and the partial derivatives $\frac{\partial f_i}{\partial x_j}$, $i,j = 1,2,\cdots,n$ are all continuous on some open region $R \subset \mathbb{R}^{n+1}$ containing the point $(t_{0,}\alpha_{1,}\alpha_2,\cdots,\alpha_n)$, then there exists an open interval in which the system has a unique solution $\big(x_1(t), x_2(t), \cdots, x_n(t)\big)$ which satisfies the initial condition of the system. In other words, if the vector valued functions $A(t)$ and $F(t)$ are continuous over an open interval I containing t_0, then the initial value problem

$$X' = AX + F(t), \quad X(t_0) = x_0$$

has a unique vector valued solution that is defined on the entire interval I for any given initial value x_0. In other words, if the entries of matrices $A(t)$ and $F(t)$ are continuous functions on a common interval I that contains the point t_0, then there exists a unique solution.

1.4. Stability at Equilibrium Point

With the advent of Isaac Newton's second law of motion and the law of universal gravitation, the motions of the planets in the solar system were understood to correspond to the solutions of the Newtonian system of ordinary differential equations that modeled the positions and velocities of the planets and the sun according to their mutual gravitational attractions. Short-term approximate predictions (up to a few years in the future) verified this theory; but, due to the complexity of the differential equations of motion, the problem of long-term prediction was not solved; it still occupies a central place in mathematical research.

During the 18th Century, Pierre Simon Laplace asserted a proof of the stability of the solar system. He considered the changes in the semi-major axes and eccentricities of the elliptical motions of the planets around the sun. Using reasonable approximations of the Newtonian equations of motion, he showed that for his approximate model these orbital elements do not change over long periods of time due to the disturbances caused by the gravitational attractions of the other bodies in the solar system. If it is true for the full Newtonian equations of motion, these assertions would imply the stability of the Newtonian solar system. In fact, no proof of the stability of the solar system is known [11].

Laplace's results merely provide evidence in favor of the stability of the solar system; on the other hand, this work was a primary stimulus for the later development of a general theory of stability.

One of the first rigorous results in stability theory was stated by Joseph-Louis Lagrange and proved by Lejeune Dirichlet; it states that an isolated minimum of the potential energy of a conservative mechanical system is the position of a stable equilibrium point.

Joseph Liouville discussed the problem of the stability of rotating fluid bodies. The further development of this theory was suggested to Aleksandr Mikhailovich Lyapunov as a thesis topic by his advisor Pafnuty Chebyshev. This led to the fundamental and foundational work of Lyapunov on stability theory.

Henri Poincare's introduction of the qualitative theory of differential equations influenced *Lyapunov*'s treatment of stability theory and laid much of the foundation for the modern theory of nonlinear dynamical systems. In addition, Poincare's work on celestial mechanics discusses stability theory [8].

The following is the definition of stability in the sense of Lyapunov after the Russian mathematician Aleksandra M. (Lyapunov 1857).

Definition 1.3.1: An *equilibrium* state $x = 0$, is said to be

(a) *Stable* if for any positive scalar ε there exists a positive scalar δ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.

(b) *Asymptotically stable* if it is stable and if in addition $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

(c) *Unstable* if it is not stable; that is, there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x(t_0)$ with $\|x(t_0)\| < \delta$, $\|x(t_1)\| \geq \varepsilon$ for some $t_1 > t_0$.

Let us consider in more detail the concept of stability introduced by *Lyapunov*.

The solution $\varphi(t)$ of the system of differential equations $X' = f(t, X)$ with initial conditions $X(0) = X_0$ is stable (in the sense of *Lyapunov*). If for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that if $\|X(0) - \varphi(0)\| < \delta$, then $\|X(t) - \varphi(t)\| < \varepsilon$ for all values $t \geq 0$. Otherwise, the solution $\varphi(t)$ is said to be unstable [12]. The majority of our analysis of systems of ODEs will focus on whether or not the systems have stable equilibria. We characterize an equilibrium as stable or unstable based on the behavior of solutions whose initial conditions are in the neighborhood of the equilibrium. If solutions near a critical point of a system stay close to the critical point as time approaches infinity, we think of the critical point as stable; if this condition is not met then the critical point is unstable. Furthermore, we call a stable critical point asymptotically stable if, over time, the solutions approach the critical point as opposed to simply staying within a certain radius [10].

The reason why we choose the equilibrium point at the origin $(x_e = 0)$ is that, we have transformation by introducing new variables $y = x + x_e$, we can arrange for the equilibrium state to be transferred to the origin (of the state space R^n).

Let the system is given by

$$\begin{cases} \dfrac{dx_1}{dt} = f(x_1, x_2) \\ \dfrac{dx_2}{dt} = g(x_1, x_2) \end{cases}$$

Assume the system has equilibrium point at $(a, b), (c, d), \cdots$ which is nonzero, then the new variable $y = x + x_e$ becomes

$$y_1 = x_1 + x_e \Longrightarrow y_1 = x_1 + a$$

$$y_2 = x_2 + x_e \Longrightarrow y_2 = x_2 + b$$

when we derivate, we get

$$\frac{dy_1}{dt} = y_1' = \frac{dx_1}{dt} = f(y_1 - a, y_2 - b)$$

$$\frac{dy_2}{dt} = y_2' = \frac{dx_2}{dt} = g(y_1 - a, y_2 - b)$$

Thus, the new system is

$$\begin{cases} \frac{dy_1}{dt} = f(y_1 - a, y_2 - b) \\ \frac{dy_2}{dt} = g(y_1 - a, y_2 - b) \end{cases},$$

has the equilibrium point at origin.

The procedures to determine stability of a linear system by using eigenvalue technique:

1. Construct the coefficient matrix.
2. Compute eigenvalues.
3. Conclude stability or instability based on the real parts of the eigenvalues.

If the system is nonlinear, then the procedures are given by

1. Compute all partial derivatives of the right-hand side of the original system of differential equations (1.12), and construct the Jacobian matrix.
2. Evaluate the Jacobian matrix at the steady state.
3. Compute eigenvalues
4. Conclude stability or instability based on the real parts of the eigenvalues.

1.4. Test for Stability

Systems of first order ordinary differential equations can be linear or nonlinear. We can transform a nonlinear system into a linear system by using linearization and by constructing the Jacobian matrix, so that we can use the procedures and method for determining the stability of a linear system. Given a system of two equations:

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}. \tag{1.13}$$

Then the coefficient matrix is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let λ_1 and λ_2 be the eigenvalues of the coefficient matrix of the system, which are roots of the characteristic equation

$$det\ (A - \lambda I)\ = 0.$$

Then, depending on this we have the following theorems:

Theorem 1.4.1: If the eigenvalues are real and negative, then the system is asymptotically stable node. All trajectories in the neighborhood of the fixed point will be directed towards the fixed point.

Proof: Let $\lambda_1 < \lambda_2 < 0$. Then the general solution of the system (1.13) may be written then as

$$\begin{cases} x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t} \\ y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}, \end{cases} \tag{1.14}$$

where A_1, A_2, B_1 and B_2 are definite constant and $A_1 * B_2 \neq A_2 * B_1$, and c_1, c_2 are arbitrary constants. Let us choose $c_2 = 0$, we obtain

$$\begin{cases} x = c_1 A_1 e^{\lambda_1 t} \\ y = c_1 B_1 e^{\lambda_1 t}, \end{cases} \tag{1.15}$$

Similarly choose $c_1 = 0$, we obtain

$$\begin{cases} x = c_2 A_2 e^{\lambda_2 t} \\ y = c_2 B_2 e^{\lambda_2 t}, \end{cases} \tag{1.16}$$

For $c_1 > 0$, the solutions (1.5) represent a path consisting of half of the line of $B_1 x = A_1 y$ with slope $\frac{B_1}{A_1}$. For any $c_1 < 0$, it represent a path consisting of the other half of the line. Since $\lambda_1 < 0$, both of these half line paths approach $(0, 0)$ as $t \to \infty$. Also, since $\frac{y}{x} = \frac{B_1}{A_1}$, these two paths approaches $(0, 0)$ as $t \to \infty$, with slope $\frac{B_1}{A_1}$. In the same way, for any $c_2 > 0$, the solutions (1.16) represent a path consisting of half of the line $B_2 x = A_2 y$, while for any $c_2 < 0$, it represent a path consisting of the other half of the line. These two half line paths approach $(0, 0)$ as $t \to \infty$. Also, since $\frac{y}{x} = \frac{B_1}{A_1} = \frac{B_2}{A_2}$, these two half-line paths approach (0, 0) as $t \to \infty$.

Thus the solutions (1.15) and (1.16) provide us with four half-line paths which all approach $(0, 0)$ as $t \to \infty$.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (1.14) represents nonrectilinear paths. Again, since all of these paths approach $(0, 0)$ as $t \to \infty$. Further, since

$$\frac{y}{x} = \frac{c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}}{c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t}} = \frac{\left(\frac{(c_1 * B_1)}{c_2}\right) e^{(\lambda_1 - \lambda_2)t} + B_2}{\left(\frac{(c_1 * A_1)}{c_2}\right) e^{(\lambda_1 - \lambda_2)t} + A_2}$$

$\lim_{t \to \infty} \frac{y}{x} = \frac{B_2}{A_2}$, approaches (0,0).

Thus all the path approaches (0,0) as $t \to \infty$.According to the definition the critical point (0,0) is node and it is asymptotically stable.

Theorem 1.4.2: If the eigenvalue are real and positive, then the stability of the system is unstable node that means all trajectories in the neighborhood of the fixed point will be directed outside and away from the fixed point

Proof: Let $\lambda_1 > \lambda_2 > 0$, the general solution of (1.13) is still (1.14) and similarly (1.15), (1.16). The proof is the same as the above theorem. But the path approaches to (0,0) as $t \to -\infty$. The direction of the arrow is in opposite sign. Therefore the critical point (0,0) is node and it is unstable.

Theorem 1.4.3: If the eigenvalue are in opposite sign, then the stability of the system is unstable saddle node. In this case trajectories in the general direction of the negative eigenvalues′

eigenvector will initially approach the fixed point but will diverge as they approach a region dominated by the positive eigenvalue.

Proof: Let $\lambda_1 < 0 < \lambda_2$, the general solution of (1.13) is still (1.14) and similarly (1.15), (1.16). For $c_1 > 0$, the solutions (1.15) represent a path consisting of half of the line of $B_1 x = A_1 y$. For any $c_1 < 0$, it represent a path consisting of the other half of the line, also $\lambda_1 < 0$, both of these half line paths approach $(0, 0)$ as $t \to \infty$. For any $c_2 > 0$, the solutions (1.16) represent a path consisting of half of the line $B_2 x = A_2 y$, for any $c_2 < 0$, it represent a path consisting of the other half of the line. But $\lambda_2 > 0$, both of these half line paths approach $(0, 0)$ as $t \to \infty$.

if $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (1.14) represents non-rectilinear paths. But here since $\lambda_1 < 0 < \lambda_2$, none of these paths can approach $(0, 0)$ as $t \to \infty$ or as $t \to -\infty$. From (1.14) that each of these non-rectilinear paths becomes asymptotic to one of the half-line paths defined by (1.16) as $t \to -\infty$. Each of them becomes asymptotic to one of the paths defined by (1.15). Thus, there are two half line paths which approach $(0, 0)$ as $t \to \infty$ and the other two half-line paths which approach $(0, 0)$ as $t \to -\infty$. According to the definition, the critical point (0,0) is a saddle point and it is unstable.

Theorem 1.4.4: If the real parts of eigenvalues are negative, then the stability of system is asymptotically stable spiral (focal). All trajectories in the neighborhood of the fixed point spiral in to the fixed point with ever decreasing radius.

Proof: Since λ_1 and $\lambda_2$are complex conjugate with real part not zero, we may write these roots $\alpha \pm \beta i$, where α and β are both real and unequal to zero. Then the general equation of the system can be written as

$$\begin{cases} x_1 = e^{\alpha t}(c_1(A_1 \cos \beta t - A_2 \sin \beta t) + c_2(A_2 \cos \beta t + A_1 \sin \beta t) \\ x_2 = e^{\alpha t}(c_1(B_1 \cos \beta t - B_2 \sin \beta t) + c_2(B_2 \cos \beta t + B_1 \sin \beta t)' \end{cases} \tag{1.17}$$

where A_1, A_2, B_1 and B_2 are definite constant and c_1, c_2 are arbitrary.

Assume$\alpha < 0$, then from (1.17), $\lim_{t\to\infty} x_1 = 0$,$\lim_{t\to\infty} x_2 = 0$ and the path defined by (1.17) approaches to (0,0) as $t \to \infty$.We can write (1.17) in the form

$$\begin{cases} x_1 = e^{\alpha t}(c_3 \cos \beta t + c_4 \sin \beta t) \\ x_2 = e^{\alpha t}(c_5 \cos \beta t + c_6 \sin \beta t) \end{cases} \tag{1.18}$$

Where $c_3 = c_1 A_1 + c_2 A_2, c_4 = c_2 A_1 + c_1 A_2, c_5 = c_1 B_1 + c_2 B_2,$ and $c_6 = c_2 B_1 + c_1 B_2$.

Assuming c_1 and c_2 are real, the solutions (1.18) represent all paths in the real xy phase plane. We can rewrite the solutions in the form

$$\begin{cases} x_1 = k_1 e^{\alpha t}\cos(\beta t + \phi_1) \\ x_2 = k_2 e^{\alpha t}\cos(\beta t + \phi_2) \end{cases}, \tag{1.19}$$

where $k_1 = \sqrt{{c_3}^2 + {c_4}^2}$, $k_2 = \sqrt{{c_5}^2 + {c_6}^2}$, and ϕ_1 and ϕ_2 is defined by equation $\cos\phi_1 = \frac{c_3}{k_1}$, $\cos\phi_2 = \frac{c_5}{k_2}$, $\sin\phi_1 = -\frac{c_4}{k_1}$ and $\sin\phi_2 = -\frac{c_6}{k_2}$

Let us consider

$$\frac{x_2}{x_1} = \frac{k_2 e^{\alpha t}\cos(\beta t+\phi_2)}{k_1 e^{\alpha t}\cos(\beta t+\phi_1)} \tag{1.20}$$

Let $K = \frac{k_2}{k_1}$ and $\phi_3 = \phi_2 - \phi_1$, equation (1.20) becomes

$$\begin{aligned} \frac{x_2}{x_1} &= K\frac{\cos(\beta t + \phi_1 - \phi_3)}{\cos(\beta t + \phi_1)} \\ &= K\left[\frac{\cos(\beta t+\phi_1)\cos\phi_3 + \sin(\beta t+\phi_1)\sin\phi_3}{\cos(\beta t+\phi_1)}\right] \\ &= K[\cos\phi_3 + \tan(\beta t + \phi_1)\sin\phi_3] \end{aligned}$$

Provided that $\cos(\beta t + \phi_1) \neq 0$. As a result of the periodicity of trigonometric functions we conclude that $\lim_{t\to\infty}\frac{x_2}{x_1}$ does not exist, the path approach (0,0) in a spiral like manner as $t \to \infty$. According to the definition, the critical point (0,0) is spiral point and it is asymptotically stable.

Theorem 1.4.5: If the characteristics of the systems are conjugate complex, then the system is unstable spiral (focal), all trajectories in the neighborhood of the fixed-point spiral away from the fixed point with ever increasing radius.

Proof: If $\alpha > 0$, the proof is the same except the path approach (0,0) as $t \to -\infty$, then the critical point (0,0) is spiral point and it is unstable.

Theorem 1.4.6: If the real part of the eigenvalues are zero, then the system is center, and it is stable but not asymptotically stable. Trajectories circulate about the fixed point in a stable orbit.

Proof: Since λ_1 and λ_2 are pure imaginary we can write the equation as $\alpha \pm \beta i$, where $\alpha = 0, \beta$ is real number different from zero. Then the general solution of the system (1.13) in the form of

$$\begin{cases} x_1 = k_1\cos(\beta t + \phi_1) \\ x_2 = k_2\cos(\beta t + \phi_2) \end{cases}, \tag{1.21}$$

where k_1, k_2, ϕ_1 and ϕ_2 defined as before. The solution oscillate between -1 and 1 as $t \to \infty$ and $t \to -\infty$.

Table 1: Summarization of our result

Nature of roots λ_1 and λ_2 of characteristics equation	**Nature of critical point $(0,0)$ of the linear system**	**Stability of critical point$(0,0)$**
Real and the same sign	Node	Asymptotically stable if the roots are negative and unstable if roots are positive
Real and the opposite sign	Saddle point	Unstable
Conjugate complex but not pure imaginary	Spiral point	Asymptotically stable if the roots are negative and unstable if roots are positive
Pure imaginary	Center	Stable but not asymptotically stable

The above procedures and theorems are used to apply to N dimensional systems.

We return to the general linear system given by

$$X' = AX, X \in R^n, \tag{1.22}$$

where $A \in \mathbb{R}^{n\times n}$ and equation (1.22) may represent the closed or open loop system. Provided the matrix A is nonsingular, the only equilibrium state of equation (1.22) is the origin, so it is meaningful to refer to the stability of the system (1.22). If the system is stable (at the origin) but not asymptotically stable we shall call it neutrally stable.

Suppose we are given an n^{th} order homogeneous system of differential equations with constant coefficients:

$$X'(t) = AX(t),\ X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},\ \ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where $X(t)$ is an n dimensional vector containing the unknown functions and A is a square matrix of size $n \times n$. Then without loss of generality, we may assume that the equilibrium point is at the origin. It is always possible to reach by choosing a suitable coordinate system.

The stability or instability of the equilibrium state is determined by the signs of the real parts of the eigenvalues of A. To find the eigenvalues λ, it is necessary to solve the auxiliary equation

$$det(A - \lambda I) = 0$$

which is reduced to an algebraic equation of the n^{th} degree

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0.$$

The roots of this equation can be easily calculated in the case $n = 2$, and in some cases when $n \geq 3$. In other cases, solving the auxiliary equation can be a difficult problem. Moreover, Norwegian mathematician $(1802 - 1829)$ proved a theorem according to which the general algebraic equation of degree $n \geq 5$ cannot be solved using four basic arithmetical operations, that is there is no formula expressing the roots of the equation through its coefficients in the case$n \geq 5$. In such a situation, methods allowing determining whether all roots have negative real parts and establish the stability of the system without solving the auxiliary equation itself, are of great importance. One of these methods is the Routh-Hurwitz criterion [4], which contains the necessary and sufficient conditions for the stability of the system.

The characteristic equation of n^{th} order can be written as:

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0.$$

The stability criterion is applied using a Routh table which is defined as:

Table 2: Routh- Hurwitz table

λ^n	a_0	a_2	a_4	a_6	$\cdots$
λ^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
λ^{n-2}	b_1	b_2	b_3	$\cdots$	
λ^{n-3}	c_1	c_2	c_3	$\cdots$	
$\vdots$					
λ^0	a_n				

where $b_1 = \frac{(a_1a_2 - a_0a_3)}{a_1}$

$$b_2 = \frac{(a_1a_4 - a_0a_5)}{a_1}$$

$$c_1 = \frac{(b_1a_3 - a_1b_2)}{b_1}$$

$$c_2 = \frac{(b_1a_5 - a_1b_3)}{b_1}$$

Chapter Two

Stability Analysis of Systems of First Order Ordinary Differential Equations

Stability analysis is part of system and control theory which is used to study and predict the stability or instability characteristics of a system and used to indicate how a model reacts to perturbation and change.

Stability analysis of systems of ordinary differential equations is one important problem in the qualitative theory of differential equations. The fundamental method of Lyapunov characteristic exponents permits describing the asymptotic behavior of solutions of a system via these exponents and thereby clarifying the stability properties of the system. However, the application of this method encounters difficulties arising when one tries to compute or estimate the Lyapunov characteristic exponents. Proving stability with Lyapunov functions is very general: it even works for non-linear and time-varying systems. It is also good for doing proofs. However, proving the stability of a system with Lyapunov functions is difficult. And failure to find a Lyapunov function that proves a system is stable does not prove that the system is unstable. The next technique we present, finding the fundamental matrix, requires the solution of systems of differential equations, or in the time invariant case, the computation of the eigenvalues. Determining the eigenvalues or the poles of the transfer function is sometimes difficult because it requires factoring high-order polynomials. However, there is a criterion that is applied to obtain the behavior of systems of first order ordinary differential equation.

2.1. Stability Analysis of Linear Systems of First Order ODEs

The main goal when analyzing systems of ordinary differential equations is to gain an understanding of the behaviors of the solutions to the systems. The natural approach for analyzing a system is to solve it explicitly, and this method works well if the system is linear. If the system is not linear, then solving explicitly can be very complicated (may be impossible). So, we instead linearize the system at its equilibria and gain a qualitative understanding of the solutions by analyzing the linearized system. It turns out that the non-homogeneous linear system is stable with any free term if the zero solution of the associated homogeneous system is stable.

Therefore, when investigating stability in the class of linear systems, it is sufficient to analyze the homogeneous differential systems. In the simplest case, when the coefficient matrix is constant, the stability conditions are formulated in terms of the eigenvalues of the matrix. Therefore, all the linear systems given by $\frac{dy}{dx} + p(x)y = q(x)$, have the same Lyapunov stability property (stable, unstable and asymptotically stable). This is the same as the homogenous system.

We have a rigorous mathematical technique to test the stability of a steady state and the behavior of the system near a steady state. We learn to analyze a linear system of ODEs first and use a similar approach for nonlinear systems [13].

Consider a system of linear ODEs, for two dependent variables:

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

Note that this is an autonomous, homogeneous and linear system of ODEs. In matrix notation we can rewrite this system of ODEs as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We can further represent in vector notation

$$X' = AX.$$

$X'$and $X$are column vectors for the derivatives and the dependent variables, respectively.

A is called the coefficient matrix and it is a square matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Before we proceed with the system of ODEs, it will be better to recapitulate some essential aspects of linear algebra. Suppose we have a system of linear equations, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This system of equations can be solved algebraically by arranging terms,

$$x = -\frac{b}{a} y$$

$$(ad - bc)y = 0$$

Therefore, this system will have a unique solution, $x = 0$ and $y = 0$, iff $(ad - bc) \neq 0$. $(ad - bc)$ is the determinant of the coefficient matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. In general, any linear homogeneous system of equations, of the form $Ax = 0$, will have a unique solution $x = 0$, iff $detA \neq 0$. That is called the trivial solution for the system of equations. The equation $X' = AX$ is an autonomous, homogenous, linear system of ODEs of two dependent variables. We can find its steady state by setting $x' = 0$,

$$\therefore AX = 0.$$

$AX = 0$ will have a unique solution $X = 0$, iff $detA \neq 0$. Therefore, $x = 0, y = 0$ is the only steady state for the system of ODEs, when the determinant of the coefficient matrix is not equal to zero. In general, when the determinant of the coefficient matrix is not equal to zero, an autonomous, homogeneous, linear system of ODE will have only one steady state at $x = 0$. We have identified the steady state of the system. Now we analyze the stability of the steady state. The concepts of eigenvalue will be valuable in this analysis.

Example 1: consider the system given by

$$\begin{cases} x_1' = 2x_1 + 3x_2 \\ x_2' = 2x_1 - x_2 \end{cases}.$$

Rewrite in the form of $X' = AX$ gives

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This system has equilibrium point (steady state) at (0,0), $A = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix}$ is the coefficient matrix of the given system. Then $|A - \lambda I| = 0$, is the characteristics equation of the given system.

$$(2 - \lambda)(-1 - \lambda) - 6 = 0$$

$$\lambda^2 - \lambda - 8 = 0$$

$\lambda_1 = \frac{1+\sqrt{33}}{2}$ and $\lambda_2 = \frac{1-\sqrt{33}}{2}$ are complex roots with positive real parts. Therefore, $(0,0)$ is unstable focal.

Example 2: Consider the system given by $\begin{cases} x_1' = x_2 + x_3 \\ x_2' = x_1 + x_3 \\ x_3' = x_1 + x_2 \end{cases}$

The coefficient matrix is given by $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $|A - \lambda I| = 0$, is the characteristics equation of the given system. This implies $\lambda^3 - 3\lambda - 2 = 0$. Then $\lambda_{1,2} = -1$ and $\lambda_3 = 2$. Therefore the given system is unstable because the roots are in opposite sign.

Example 3: Determine the stability of the given non- homogenous system

$$\begin{cases} x_1' = -2x_1 + x_2 + e^t \\ x_2' = x_1 - 2x_2 - 2e^t \end{cases}$$

$$\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^t$$

$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ is the coefficient matrix of the given system. And $|A - \lambda I|$ =0 is the characteristics equation.

$$(-2 - \lambda)(-2 - \lambda) - 1 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda + 1)(\lambda + 3) = 0.$$

$\lambda_1 = -1$ and $\lambda_2 = -3$ are the roots of the system. Since both roots are negative then the origin is asymptotically stable.

We can determine the stability of the system of ODEs if the system has characteristics equation with degree less than 4. But if the characteristics equation is higher order, then it is difficult to calculate the roots to determine the stability of the system. Because of this we need other criteria that is Routh-Hurwitz Stability criterion.

2.1.1 Reduction of Higher Order DE in to a System of First Order DEs

Since we are using matrices as a system of equations, sometimes it is easier to study the higher order equations by converting them into a system of equations by suitable substitutions. In this connection we have the following theorem.

Theorem 2.1.1.1: The general n^{th} order initial value problem

$$x^{(n)} = f(t, x, x', \cdots, x^{(n-1)})$$

$$x(t_0) = a_0, x'(t_0) = a_1, \cdots, x^{(n-1)}(t_0) = a_{n-1}, t_0 \in I$$

where $a_0, a_1, \cdots, a_{n-1}$ are constants is equivalent to a system of n- linear differential equations.

Proof: The general n^{th} order equation is given by

$$a_0(t)x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t) = b(t) \;\; t\epsilon I$$

where $a_n(t) \neq 0$, for $t \in I$. Let as make the following substitution

$$x_1 = x, x_2 = x', \cdots, x_{n-1} = x^{(n-2)}, x_n = x^{(n-1)}$$

From these we have the substitution

$$x(t) = x_1(t)$$
$$x_1'(t) = x'(t) = x_2$$
$$x_2'(t) = x''(t) = x_3$$
$$x_3'(t) = x'''(t) = x_4$$
$$\vdots$$
$$x_{n-1}'(t) = x^{(n-1)}(t) = x_n$$
$$x_n'(t) = x^{(n)}(t)$$

Rewrite the equation, we get

$$x^{(n)} = -\frac{a_n(t)}{a_0(t)}x_1 - \frac{a_{n-1}(t)}{a_0(t)}x_2 - \cdots - \frac{a_1(t)}{a_0(t)}x_n + \frac{b(t)}{a_0(t)}$$

Using matrix notation the above system can be rewrite as

$$x_1'(t) = 0x_1 + 1x_2 + \cdots + 0$$
$$x_2'(t) = 0x_1 + 0x_2 + 1x_3 \cdots + 0$$
$$\vdots$$
$$x^{(n)} = -\frac{a_n(t)}{a_0(t)}x_1 - \frac{a_{n-1}(t)}{a_0(t)}x_2 - \cdots - \frac{a_1(t)}{a_0(t)}x_n + \frac{b(t)}{a_0(t)}$$

Then

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_1}{a_0} \cdots & -\frac{a_1}{a_0} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{b(t)}{a_0} \end{pmatrix}.$$

Using vector notation

$$X' = A(t)X + B(t).$$

Therefore the n^{th} order linear equation is equivalent to the linear system of differential equation.

Example 1: Suppose the higher order linear DE is given by

$$x^{(4)} - 7x^{(3)} + 4x'' + 5x' - 2x = 0.$$

First rewrite the equation in to

$$x^{(4)} = 7x^{(3)} - 4x'' - 5x' + 2x.$$

Now substitution yields

$$x_1 = x,$$
$$x_2 = x' = x_1'$$
$$x_3 = x'' = x_2'$$
$$x_4 = x^{(3)}$$
$$x_4' = x^{(4)}$$

This implies

$$x_4' = 7x_4 - 4x_3 - 5x_2 + 2x_1$$

Then

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -5 & -4 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -5 & -4 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \text{ where } X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix}$$

Or $X' = AX,$ where A $= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -5 & -4 & 7 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is a linear system and to determine the stability of this system, from $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -5 & -4 & 7 \end{pmatrix}$, we have

$$\lambda^4 - 7\lambda^3 + 4\lambda^2 + 5\lambda - 2 = 0.$$

It is difficult to factorize. Therefore the interpretation is mentioned in section 2.1.2.

Example2: Consider a damped harmonic oscillator. The dynamics of the system are given by the equation

$$Mq'' + Bq' + Kq = 0,$$

Where M, B and $K$are all positive quantities. Since the equation is second order, we rewrite the equation as:

$$\frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_2 \\ -\frac{K}{M}q_1 - \frac{B}{M}q_2 \end{pmatrix}$$

$$Q' = \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \text{ where } Q' = \begin{pmatrix} q_1' \\ q_2' \end{pmatrix}.$$

which has the characteristics equation

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{pmatrix}.$$

The solution for the characteristics equation is

$$-\lambda\left(-\frac{B}{M} - \lambda\right) + \frac{K}{M} = 0$$

$$\lambda^2 + \frac{B}{M}\lambda + \frac{K}{M} = 0$$

$$M\lambda^2 + B\lambda + K = 0$$

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4MK}}{2M},$$

which always have negative real parts, and hence the system is Asymptotically stable at$(0,0)$.

2.1.2 Routh-Hurwitz Stability Criterion

The character of stability can be determined by using a criterion of stability without solving the system of equations. One of these is the Routh-Hurwitz stability criterion. It allows to judge the stability of a system by knowing only the coefficients of the characteristic equation of the matrix. If any elements of the Routh table have some common factor, then we can divide the row elements with that factor for simplification will be easy.

Theorem 2.2.1: [3] Routh-Hurwitz Stability criterion states that the numbers of roots of the characteristics equation with positive real parts are equal to the number of change of signs of coefficients in the first column of an array.

In order to construct the Routh array

- The first row consists of all the coefficient of even terms (degree) of the characteristic equation. Arrange them from first (even term) to last (even term).

- The second row consists of all the coefficient of odd terms of the characteristic equation. Arrange them from first (odd term) to last (odd term). When characteristics equation of an n^{th} order is given by:

  $a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$, and the highest degree of the characteristics equation n is even.

There are necessary and sufficient conditions for a system to be stable:

- None of the coefficient of the characteristics equation should be missing or zero.
- Every coefficient should be real and have the same sign.
- If the characteristics equation contains only odd or even power of λ, this indicates that the root has no real part and posses only imaginary.
- Each term of the first column of Routh's array should be positive and should have the same sign.

Special cases of Routh Hurwitz criterion:

- If the first term in any row of the Routh array is zero while the rest of the row has at least one none zero term, the first element in the third row is zero. So, we replace it with ε. In this case, we will assume a very small value (ε) which is tending to be zero in place of zero. By replacing zero with (ε) we will calculate all the elements of the Routh array. After calculating all the elements, we will apply the limit at each element containing (ε). On solving the limit at every element if we will get a positive limiting value then we will say the given system is stable otherwise in all the other condition, we will say the given system is not stable.
- Let all elements of any row of the Routh array be zero. In order to find out the stability in this case, we will first find out the auxiliary equation. The auxiliary equation can be formed by using the elements of the row just above the row of zeros in the Routh array. After finding the auxiliary equation we will differentiate the auxiliary equation to obtain elements of the zero row. If there is no sign change in the new Routh array formed by using the auxiliary equation, then in this case we say the given system is stable [1].

Example **1**: From the 2.1.1 we have

$$\lambda^4 - 7\lambda^3 + 4\lambda^2 + 5\lambda - 2 = 0.$$

In order to determine stability of this system, we have to construct Routh array.

Table 3: Routh array for a given example

λ^4	1	4	-2
λ^3	-7	5	0
λ^2	$\frac{33}{7}$	2	0
λ^1	$\left(\frac{\left(\frac{33}{7}*5\right)+14}{\frac{33}{7}}\right)$	0	0
λ^0	2	0	0

There is sign change from positive to negative and negative to positive. Therefor the system is unstable.

Example 2: suppose the characteristics equation is

$$2\lambda^5 + \lambda^4 + 6\lambda^3 + 3\lambda^2 + \lambda + 1 = 0$$

Now construct Routh array

Table 4: Routh array for a given example

λ^5	2	6	1
λ^4	1	3	1
λ^3	$0(\varepsilon)$	-1	0
λ^2	$\frac{(3\varepsilon+1)}{\varepsilon}$	1	0
λ^1	$\left(\frac{-\left(\frac{3\varepsilon+1}{\varepsilon}\right)-\varepsilon}{\frac{(3\varepsilon+1)}{\varepsilon}}\right)$	0	0
λ^0	1	0	0

Since the first element in the third row is zero. So, we replace it with ε.

To check the sign change the coefficient of first column, we apply limit $\varepsilon \to 0$.

$$\lim_{\varepsilon\to 0}\left(\frac{3\varepsilon+1}{\varepsilon}\right) \to +\infty$$

$$\lim_{\varepsilon\to 0}\frac{-\left(\frac{3\varepsilon+1}{\varepsilon}\right)-\varepsilon}{\left(\frac{3\varepsilon+1}{\varepsilon}\right)} \to -\infty$$

Here, we have two sign changes from positive to negative and from negative to positive. Therefore, the system is unstable and it has two roots (positive real part).

Example 2: Let the characteristics equation is given by

$$\lambda^5 + 2\lambda^4 + 6\lambda^3 + 10\lambda^2 + 8\lambda + 12 = 0$$

The Routh array becomes

Table 5: Routh array for a given example

λ^5	1	6	8
λ^4	2	10	12
λ^3	1	2	0
λ^2	6	12	0
λ^1	0 replace by 2	0	0
λ^0	12		

Since the fourth row has zero elements, we use the derivative of auxiliary equation that appears above zero's row.

Now $6\lambda^2 + 12\lambda^0 = 0 \Longrightarrow 2\lambda^2 + 6\lambda^0 = 0$, differentiation yields $2\lambda + 0$. Then replace all zero row by this. There is no sign change in the first column. Therefore, the system is stable.

2.2 Stability Analysis of Nonlinear Systems of ODEs

A linear system of ODE with a non-zero determinant of the coefficient matrix has only one trivial steady state solution at $(0, 0)$. However, a nonlinear system can have more than one steady state. Therefore, unlike a linear system, we cannot make a generalized statement on the stability of a nonlinear system. We have to check the stability of each of the steady states individually.

To analyze a nonlinear system, we have to identify all possible steady states. That can be done using linearization. Non-linear systems are in general much less amenable to the analytic and algebraic techniques, but we can use the method Linearization technique and Lyapunov stability to understand the behavior of the solution of non-linear systems near their equilibrium points [7].

2.2.1 Linearization

Consider the following autonomous nonlinear system of ODEs:

$$\begin{cases} \frac{dx}{dt} = g(x,y) \\ \frac{dy}{dt} = h(x,y) \end{cases}.$$

Assume (x_0, y_0) is an equilibrium point. So, we would like to find the closest linear system when (x,y) is close to (x_0, y_0). In order to do that we need to approximate the function $g(x,y)$ and $h(x,y)$ when (x,y) is close to (x_0, y_0). This is a similar problem to approximating a real valued function by its tangent (around a point).

$$\begin{cases} g(x,y) \cong g(x_0,y_0) + (x - x_0)\dfrac{\partial g}{\partial x} + (y - y_0)\dfrac{\partial g}{\partial y} \\ h(x,y) \cong h(x_0,y_0) + (x - x_0)\dfrac{\partial h}{\partial x} + (y - y_0)\dfrac{\partial h}{\partial y} \end{cases}.$$

When (x,y) is close to (x_0, y_0) then the nonlinear system may be approximated by

$$\begin{cases} \dfrac{dx}{dt} \cong g(x_0,y_0) + (x - x_0)\dfrac{\partial g}{\partial x} + (y - y_0)\dfrac{\partial g}{\partial y} \\ \dfrac{dy}{dt} \cong h(x_0,y_0) + (x - x_0)\dfrac{\partial h}{\partial x} + (y - y_0)\dfrac{\partial h}{\partial y} \end{cases}$$

As (x_0, y_0) is a steady state, $g(x_0, y_0) = 0$ and $h(x_0, y_0) = 0$. Therefore, we can write these two equations as:

$$\begin{cases} \dfrac{dx}{dt} \cong (x - y_0)\dfrac{\partial g}{\partial x} + (y - y_0)\dfrac{\partial g}{\partial y} \\ \dfrac{dy}{dt} \cong (x - x_0)\dfrac{\partial h}{\partial x} + (y - y_0)\dfrac{\partial h}{\partial y} \end{cases}$$

This is a linear system with coefficient matrix $\begin{bmatrix} \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \\ \frac{\partial h}{\partial x}(x_0,y_0) & \frac{\partial h}{\partial y}(x_0,y_0) \end{bmatrix}$

This matrix is called the Jacobian matrix of the system at the point (x_0, y_0).

Example 1: Consider a simple pendulum of length l and mass m in the presence of viscous friction with coefficient d is given by

$$ml^2\theta'' + d\theta' + mgl\sin\theta = 0.$$

First, we change in to first order system of ODEs.

Setting $x = \theta$ and $x_1 = x, x_2 = x'$

$$x_1' = x' = x_2$$

$$x_2' = x'' = -\frac{g}{l}\sin x_1 - \frac{d}{ml^2}x_2.$$

This becomes

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{g}{l}\sin x_1 - \frac{d}{ml^2}x_2 \end{cases}.$$

This system has a steady states $(n\pi, 0), n = 1,2$.

Let $f(x_1, x_2) = x_2,\ h(x_2, x_2) = -\frac{g}{l}\sin x_1 - \frac{d}{ml^2}x_2$

This implies that $f_{x_1} = 0,\ \ f_{x_2} = 1,$

$$h_{x_1} = -\frac{g}{l}\cos x_1, h_{x_2} = -\frac{d}{ml^2}.$$

The Jacobian matrix $J = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{d}{ml^2} \end{pmatrix},\ J_{(\pi,0)} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{d}{ml^2} \end{pmatrix}.$

$|A - \lambda I| = 0$, is the characteristics equation of the linearize system.

$$-\lambda\left(-\frac{d}{ml^2} - \lambda\right) + \frac{g}{l} = 0.$$

$$\Rightarrow \lambda^2 + \frac{d}{ml^2}\lambda + \frac{g}{l} = 0$$

$$\Rightarrow \lambda = \frac{-\frac{d}{ml^2} \pm \sqrt{\left(\frac{d}{ml^2}\right)^2 - 4\frac{g}{l}}}{2}$$

If $\left(\frac{d}{ml^2}\right)^2 < 4\frac{g}{l}$, then $(\pi, 0)$ is unstable focal, and if $\left(\frac{d}{ml^2}\right)^2 > 4\frac{g}{l}$, then $(\pi, 0)$ is unstable saddle point.

Example 2: Let a nonlinear system is given by

$$\begin{cases} \frac{dx}{dt} = x - xy \\ \frac{dy}{dt} = xy - y \end{cases}.$$

First, identify all possible steady states. This system of ODEs has two steady states (0,0) and $(1, 1)$. Now we have to construct the Jacobian matrix at each of these two steady states. In general, the Jacobian matrix for this system of ODEs is

$$J = \begin{pmatrix} \frac{\partial(x-xy)}{\partial x} & \frac{\partial(x-xy)}{\partial y} \\ \frac{\partial(xy-y)}{\partial x} & \frac{\partial(xy-y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-y & -x \\ y & x-1 \end{pmatrix} J \text{ at (0,0) will be}$$

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For this matrix $|A - \lambda I| = 0$ is the characteristics equation of the linearized system. Then

$$(1-\lambda)(-1-\lambda) = 0$$

$\lambda_1 = -1$ and $\lambda_2 = 1$.

This steady state (0,0) is a saddle point because the eigenvalues are opposite sign. The Jacobian matrix for the other steady $(1, 1)$ is,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Similarly, $|A - \lambda I| = 0$ is the characteristics equation of the linearized system. Then

$$(-\lambda)(-\lambda) + 1 = 0,$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i.$$

Therefore, (1,1) is center type because it has an eigenvalue with real parts zero.

This system has two steady states with different stability and different types of trajectories around them. The steady state (1, 1) is of center type. Therefore, there are closed orbits around it. The other steady state (0, 0) is unstable saddle. If the system starts near to it, it may move closer for some time, but would eventually move away from this steady state.

Example 3: Determine the stability of nonlinear system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x + (1-x^2)y \end{cases}$$

Let $f(x,y) = y$, $g(x,y) = -x + y - x^2 * y$

Now $f_x = 0$, $f_y = 1$ and

$g_x = -1 - 2xy$, $g_y = 1 - x^2$

$$J = \begin{pmatrix} 0 & 1 \\ -1-2xy & 1-x^2 \end{pmatrix}$$

Since (0,0) is equilibrium point, the linearized system at (0,0) is

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and}$$

$$(-\lambda)(1-\lambda)+1=0$$
$$\Longrightarrow \lambda^2-\lambda+1=0$$

$\lambda=\frac{1}{2}\pm\frac{\sqrt{3}}{2}$ these are complex roots with positive real parts. Therefore (0,0) is unstable spiral (focal).

2.2.2 Lyapunov Stability

One of the powerful tools for stability analysis of systems of differential equations including nonlinear systems are Lyapunov functions. It is used to determine the stability without solving the system.

Alexsander Mihailov Lyapunove (1857-1987) elaborated an extremely general method for investigating the solution of system of differential equations for stability.

$$\frac{dx_i}{dt}=f_i(t,x_1,x_2,\cdots x_n)\forall i=1,2,\cdots n$$

Definition 2.3.2.1: Let D be open connected subset of $\mathbb{R}^n$ and $V:D\to\mathbb{R}$. Then, $V$is said to be

- Positive definite in D if $V(x_1,x_2,\cdots x_n)>0,\ \forall x\in D\backslash\{0\})$
- Positive semi definite in D if $V(x_1,x_2,\cdots x_n)\geq 0,\ \forall x\in D$ except at one or more points in the state space including the origin $x=0$, where $V(x_1,x_2,\cdots x_n)=0$.
- Negative definite in D if $\mathrm{V}(x_1,x_2,\cdots x_n)<0,\ \forall x\in D\backslash\{0\}$.
- Negative semi definite in D if $\mathrm{V}(x_1,x_2,\cdots x_n)\leq 0,\ \forall x\in D$ except at one or more points in the state space including the origin $x=0$, where $V(x_1,x_2,\cdots x_n)=0$.

Definition 2.3.2.2: A function$V:D\to R$ is said to be Lyapunov function if

- V is differentiable
- V is positive definite
- V is decreasing function along the solution of the system.

Theorem 2.2.2.1: Suppose that there exists a differentiable positive definite function $V:D\to\mathbb{R}$. $\frac{dV}{dt}$ along the trajectories of the system is negative semi definite, then the equilibrium point is stable. That is

$$\frac{dV}{dt}=\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}f_i(t,x_1,x_2,\cdots\ x_n)\leq 0, \qquad (2.1)$$

$\forall x\in D\backslash\{0\},\ V(x)=0$ only if $x=0$.

Proof: Let $\varepsilon > 0$ and $t_0 \in R$ be given. Assume, without loss of generality, that $B(x_0, \varepsilon)$ is contained in D. Assume positive definite function $W{:}D \to R$, such that $V(t,x) \geq W(x)$ for every $(t,x) \in R \times D$. Let $m = min\{V(t_0,x), |x| \leq \varepsilon\}$. Since W is continuous and positive definite, m is well-defined and positive. Given that $\delta > 0$ small enough that $\delta < \varepsilon$ and max$\{\{V(t_0,x), |x| \leq \delta\} < m\}$. Since V is positive definite and continuous function, it is possible, if $x(t)$ solution of $x'(t) = f(x)$ and $|x(t_0)| < \delta$. Then, $V(t_0, x(t_0) < m$ and $\frac{d}{dt}V\big(t, x(t)\big) = V'(t, x(t)) \leq 0$ for all t. So, $V\big(t, x(t)\big) < m$ forevery $t \geq t_0$. Thus, $W(x) < m$ and $|x(t)| \neq \varepsilon$ for every $t \geq t_0$. Since $|x(t)| \leq \varepsilon$, forevery $t \geq t_0$. The solution of $x' = f(x)$ is stable.

Theorem 2.2.2.2: If a differentiable positive definite function $V{:}\,D \to R$. $\frac{dV}{dt}$ along the trajectories of the system is negative definite, then the equilibrium point is a asymptotically stable. i.e.

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x_1, x_2, \cdots\ x_n) < 0 \ \ \forall x \in D\backslash\{0\} \qquad (2.2)$$

Proof: If there is a solution

$$X' = f_i(x) \qquad (*)$$

stable in D, there exist sphere radius $r > 0$ contained in the region D center at origin of the phase plane, such that $V(x) > 0, (x \neq 0)\|x\| \leq r$ and $V'(x) \leq 0, \|x\| \leq r$ .In particular, for $\varepsilon = r$, there exists$\delta > 0$ such that all solution $\phi(t)$of (*) with $\phi(0) = y_0$ and $\|y\| < \delta$ exist on $0 \leq t < \infty$and satisfy$\|\phi(t)\| < r \ \ (t \geq 0)$. We have a non-increasing function $V(\phi(t))$of t which is bounded below. Therefore $\lim_{t\to\infty} V(\phi(t))$ exists. Suppose that for some $0 < \eta < r$, we could have

$$V\big(\phi(t)\big) \geq \eta > 0, \text{ for } t \geq 0 \qquad (**)$$

We will show that (**) is impossible. By continuity, for the η, there exists a δ >0, such that

$$0 \leq V(x) < \eta, \text{ whenever } \|x\| < \delta \qquad (***)$$

Therefore, $\phi(t)$ the solution for which (**) holds must satisfy $\|\phi(t)\| \geq \delta$, for $t \geq 0$. Let S be the set lying between the spheres of radius δ and r, that is, $S = \{x{:}\, 0 < \delta \leq \|x\| \leq r\}$. Consider the function $-V'(x)$ on the closed bounded set S. By hypothesis on f and V, -$V'(x)$ define (2.2) is continuous and positive definite.

Let $\mu = \lim_{t\to\infty}(-V'(t)) > 0$

Since 0 is not a point of S, we have $-\frac{d}{dt}(V\big(\phi(t)\big) = -V'(\phi(t)) \geq \mu$, for $t > 0$.

Integrating, we obtain $V(\phi(t)) \leq V(x_0) - \mu t$ for $t \geq 0$. But clearly for t large enough $V(\phi(t))$ is negative, which is an obvious contradiction. Thus (**) is impossible and we must have $\lim_{t\to\infty} V(\phi(t)) = 0$ which implies that $\lim_{t\to\infty}\phi(t) = 0$. Since this holds for every solution $\phi(t)$ with $\|x\| < \delta$, this completes the proof.

Theorem 2.2.2.3: [6] If a differentiable positive definite function $V: D \to \mathbb{R}$. $\frac{dV}{dt}$ along the trajectories of the system is positive definite, then the equilibrium point is a unstable. i.e.

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x_1, x_2, \cdots\ x_n) > 0 \ \ \forall x \in D\backslash\{0\} \qquad (2.3)$$

Example 1: Check stability of the system given by

$$\begin{cases} x_1{}' = x_2 \\ x_2{}' = -x_1 - x_2 \end{cases}$$

Define $V(x_1, x_2) = \frac{1}{2}[(x_1 + x_2)^2 + 2x_1^2 + x_2^2]$ is positive definite

$$\Longrightarrow V(x_1, x_2) > 0, \forall\ (x_1, x_2) \in \mathbb{R}^2\backslash\{(0,0)\}$$

$$\text{and } \frac{dV}{dt} = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right)\begin{pmatrix} x_2 \\ -x_1 - x_2 \end{pmatrix}$$

$$= \frac{1}{2}[(6x_1 + 2x_2)(x_2) + (2x_1 + 4x_2(-x_1 - x_2).$$

$$= \frac{1}{2}[6x_1 * x_2 + 2x_2{}^2 - 2x_1{}^2 - 2x_1 * x_2 - 4x_1 * x_2 - 4x_2{}^2]$$

$$= -(x_1{}^2 + x_2{}^2) < 0\ , \forall (x_1, x_2) \in \mathbb{R}^2.$$

Since $\frac{dV}{dt} < 0$, it is negative definite. Therefore, $(x_1, x_2) \equiv (0,0)$ is asymptotically stable.

Example 2: Determine the stability of nonlinear system given by

$$\begin{cases} x_1{}' = -x_1 + 2x_1{}^2 * x_2 \\ x_2{}' = -x_2 \end{cases}$$

Define $V(x_1, x_2) = x_1^2 + x_2^2$

$$\frac{dV}{dt} = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right)\begin{pmatrix} -x_1 + 2x_1{}^2 * x_2 \\ -x_2 \end{pmatrix}$$

$$=(2x_1)(-x_1 + 2x_1{}^2 * x_2) + (2x_2)(-x_2)$$

$$=-2x_1{}^2 + 4x_1{}^3 * x_2 - 2x_2{}^2$$

$$= -2x_1{}^2(1 - 2x_1 * x_2) - 2x_2{}^2$$

If $(1 - 2x_1 * x_2) > 0$, then $\frac{dV}{dt}$ is negative definite. This implies the origin of the system is asymptotically stable. But if $(1 - 2x_1 * x_2) < 0$, $\frac{dV}{dt}$ is positive definite. Then the origin of the system is unstable.

Summary

In this project, stability analysis of systems of first order ordinary differential equations are discussed. The stability of linear systems have been determined by using eigenvalue technique and the Routh-Hurwitz Stability criterion if the degree of the characteristic equation is greater than or equal to four and nonlinear system by linearization technique and Lyapunov functions. Generally, stability plays a very important role in system theory and control design and Lyapunov not only gave a formal statement of the problem but also proposed the methods which till today serve as a key instrument for treating the stability problem.

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