DSpace Institution

DSpace Repository | http://dspace.org

Mathematics | Thesis and Dissertations

2021-11-17

þÿGREEN S FUNCTION METHOD SOLVING NON-HOMOGENOUS HEAT EQUATIO

DESALEGN ASSEFA

http://ir.bdu.edu.et/handle/123456789/12916

Downloaded from DSpace Repository, DSpace Institution's institutional repository

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS

A PROJECT REPORT

ON

GREEN'S FUNCTION METHOD FOR SOLVING NON-HOMOGENOUS HEAT EQUATION

DESALEGN ASSEFA ADDIS

OCTOBER, 2021

BAHIRDAR, ETHIOPIA

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS

GREEN'S FUNCTION METHOD FOR SOLVING NON -HOMOGENOUS HEAT EQUATION

A PROJECT REPORT

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

BY

DESALEGN ASSEFA

BDU09010092PS

ADVISOR'S NAME: GETACHEW ADAMU (Ph.D)

OCTOBER, 2021

BAHIR DAR, ETHIOPIA

Declaration

I hereby declare that, this project is done by me under the supervision of Dr. Getachew Adamu, Department of mathematics, Bahir Dar University, in Partial fulfillment of the requirements for the degree of Master of Science in Mathematics. I am declaring that this project is my original work. I also declare that neither of this project nor any of its parts has been submitted to elsewhere for the award of any other degrees or certificates.

Desalegn Assefa	________________	____________
Name of the candidate	*Signature*	*Date*

BAHIR DAR UNIVERSITY

COLLEGE OF SICENCE

DEPARTMENT OF MATHEMATICS

Approval of the project for oral defense

I hereby certify that I have supervised, read and evaluated this project entitled **"**Green's Function Method for Solving non- homogenous Heat equation**"** by Desalegn Assefa prepared under my guidance. I recommend that the project is submitted for oral defense.

Dr. Getachew Adamu	_____________	____________
Advisor's name	*Signature*	*Date*
Dr. Endalew Getnet	______________	____________
Department Head's name	*Signature*	*Date*

BAHIR DAR UNIVERSITY

COLLEGE OF SICENCE

DEPARTMENT OF MATHEMATICS

Approval of the project for defense result

We hereby certify that we have examined this project entitled **"Green's Function Method for Solving non- homogenous Heat equation"** by Desalegn Assefa. We recommend that this project is approved for the degree of Master of Science in mathematics.

Board of Examiners

____________________ ______________ _________________

External examiner's name *Signature* *Date*

____________________ _____________ ______________

Internal examiner's name *Signature* *Date*

___________________ _______________ ______________

Chair person's name *Signature* *Date*

Acknowledgment

Thanks to God, all praises be to God, the most Beneficent and the most Merciful, the most Gracious, the Absolute Holly, the knower of all who enabled me to undertake and complete my project work and finally write up the outcome report leading towards the fulfillment of degree of Masters.

Also, I wish to express my profound sense of gratitude and appreciation to my most respected advisor Dr. Getachew Adamu, Department of Mathematics, Bahir Dar University, for his inspiration, constant guidance, valuable suggestions, sympathetic advice and unparalleled encouragement made throughout my project work as well as my study.

I sincerely thank all staff members of mathematics department and my classmates for their particular interests they carried in my MSc project.

Abstract

In this project, Green’s function method is applied to obtain the analytic solution of non-homogenous heat equation defined on a given domain with appropriate initial and/or boundary conditions. With this purpose, firstly the theory, how to get Green's function for a heat equation, in n-dimensional infinite space is discussed, and then using method of images, how this infinite domain Green's function, should be modified, so that it can serve for semi- infinite and finite domains is shown. Applications of the method are illustrated by examples for one-dimensional non-homogenous heat equation problems with given domains, namely, infinite, semi-infinite and finite domains.

Table of Contents

CHAPTER ONE

INTRODUCTION AND PRELIMINARY CONCEPTS

1.1. Introduction

Real world problems in any physical situations of different discipline of engineering and science are modeled by differential equations along with the conditions. In other words, boundary value and/or initial value problems are generated. These problems are extremely important as they model a vast amount of phenomena and applications, from solid mechanics to a heat transfer, from fluid mechanics to acoustic diffusion (Stakgold & Holst, 2011).

The heat equation is an important second order partial differential equation, which describe the distribution of heat or variation in temperature in a given region over time. It is a wonderland for mathematical analysis, numerical computations, and experiments. Joseph Fourier first developed the theory of the heat equation in 1822, for modeling how a quantity such as heat diffuses through a given region (Narasimhan, 1999). In mathematics, it arises in connection with the study of chemical diffusion and other related processes. In physics, it describes the macroscopic behavior of many micro-particles, resulting from the random movements and collisions of the particles (Hahn & Özisik, 2012).

In nature, mathematical models describe many physical problems. Some of these models are governed by partial differential equations in general and heat equation, in particular. Their analytical solution has significance in various fields to understand the physical events of the model.

Literatures showed that various problems of partial differential equations, depending on the spatial domain, whether it is infinite or finite, can be solved by different methods such as Eigenfunction expansion method, Laplace transform method, Fourier series and Fourier transform method , Separation of variables method, Green’s function method, etc. (Duffy, 2015).

Analytic solution of the heat conduction problem has been studied by using different methods. For example, (Njogu, 2009) used Fourier series and separation of variables method for solving the one-dimensional heat equation

$$u_t(x,t) - ku_{xx}(x,t) = 0, 0 \leq x \leq L, t > 0,$$

with initial condition

$$u(x,0) = f(x),$$

and boundary condition

$$u(0,t) = u(L,t) = 0,$$

where $u = u(x,t)$ is a continuous function of two variables t and x. Here x is the space variable, L is the length of the rod and t is the time variable.

Fourier transform method (Abdisa, 2021) has been used for solving the analytic solution of the heat equation of the type

$$u_t(x,t) - ku_{xx}(x,t) = 0, -\infty < x < \infty, t > 0,$$

with initial condition

$$u(x,0) = f(x).$$

(Subani, Jamaluddin, Mohamed, & Badrolhisam, 2020) implemented separation of variables method for the one dimensional homogenous heat equation of the form

$$u_t(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0,$$

with the Neumann boundary conditions

$$u_x(0,t) = 0,\ t > 0 \text{ and } u_x(1,t) = 0,\ t > 0,$$

and initial conditions

$$u(x,0) = x,\ 0 < x < 1,$$

where u is defined as temperature, x is space and t is time.

The purpose of this project is to solve the non-homogenous heat equation defined on infinite, semi-infinite and finite domains using Green’s function method.

This project contains two chapters. The first chapter includes introduction and preliminary concepts, which focuses on basic concepts of the heat equation, initial and boundary value problems, method of images, definitions, and properties of Fourier transform and Green's function and its properties. The second chapter deals with applications of Green's function method for solving the non-homogenous heat equation.

1.2. Preliminary concepts

1.2.1. Heat equation

From the very ancient time, the study of heat conduction in a thermal body has remained a point of attraction in Mathematical physics. The heat equation is a second order partial differential equation, which models how a quantity such as heat diffuses through a given region.

The governing partial differential equation for the Heat conduction in terms of the temperature function u is:

$$\frac{\partial u}{\partial t} = k\nabla^2 \mathrm{u} + Q(X,t), \tag{1.1}$$

where $u(X,t)$ is the unknown function of the heat equation that models the heat flow, $Q(X,t)$ is the source position X at time t and the positive constants k represents the thermal diffusivity,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} \text{ or } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ or } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \ldots,$$

depending up on the dimension of the problem under discussion. Accordingly,

$$\mathrm{X} = x \text{ or } X = (x,y) \text{ or } X = (x,y,z,\ldots).$$

The source term Q, usually denoted by $Q(X,t)$ if it is time-dependent, otherwise, denoted by $Q(X)$, if it does not depend upon time and depends upon position X only. It may be simply a constant also.

If $Q(X,t) \neq 0$, equation (1.1), is known as linear non-homogenous heat equation and if $Q(X,t) = 0$ in (1.1), then it is called homogenous. The region, on which the problem is defined, may be finite, semi-infinite, or infinite.

In equation (1.1), if $X = x$ and $\nabla^2 = \frac{\partial^2}{\partial x^2}$, then it is called one dimensional heat equation. That is, for one dimensional uniform rod of length L, the temperature $u(x,t)$ at x at time t satisfies the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

where $Q(x,t)$ is the source at position x at time t.

1.2.2. Initial and Boundary Value Problems

Definition 1.1: An Initial Value Problem is a problem of differential equations with the specified values of the unknown function and/or appropriate number of derivatives at the same point which are called initial conditions.

Example 1.1: Consider the heat equation (1.1) with initial condition:

$$u(x,0) = f(x)\,, -\infty < x < \infty. \tag{1.2}$$

The problem (1.1) with initial condition (1.2) is called an initial value problem for the non-homogenous heat equation.

Definition 1.2: A boundary value problem is a differential equation together with boundary conditions prescribed at two or more different points for the unknown functions and/or its derivatives (Boyce, DiPrima, & Meade, 2017) .

Boundary conditions in heat-equation problems may be of the Dirichlet, Neumann, or mixed type.

Definition 1.3: A boundary condition which specifies the value of the unknown function at the boundary points is called a Dirichlet boundary condition.

Definition 1.4: A boundary condition which specifies the value of the normal derivative of the unknown function at the boundaries is a **Neumann** boundary condition.

Definition 1.5: **Robin or mixed** boundary condition defines the linear combination of Dirichlet boundary conditions and **Neumann** boundary conditions.

Example 1.2: Consider the non-homogenous heat equation in an open bounded region D of $\mathbb{R}^n$ and $k > 0$, such that

$$\frac{\partial u}{\partial t} = k\nabla^2 u + Q(x,t) \ \ in\ D, t > 0,$$

$$u = f(\mathrm{x}) \ \text{ at } t = 0 \text{ in D and}$$

$$\alpha u(\mathrm{x},t) + \beta \frac{\partial u}{\partial n}(\mathrm{x},t) = g(\mathrm{x},t) \text{ on the boundary } \partial \mathrm{D}.$$

Then, $\beta = 0$ gives the Dirichlet problem, $\alpha = 0$ gives the Neumann problem and $\alpha \neq 0, \beta \neq 0$ gives the Robin or mixed problem, where $\frac{\partial u}{\partial n}$ denotes the directional derivative of u in the out ward normal direction. If D is not bounded, then additional behavior-at-infinity condition may be needed.

A non-homogeneous heat equation (1.1) containing a source term, may have homogeneous or non-homogeneous boundary conditions. A heat equation without any sources may have non-homogeneous boundary conditions. Therefore, a non-homogeneous heat problem may be non-homogeneous due to non-zero source term or due to non-zero boundary conditions or due to both.

Example 1.3: For the one dimensional heat equation for the uniform rod of length L,

i. $\frac{\partial u}{\partial t} = k\nabla^2 \mathrm{u} + Q(x,t)$, $0 < x < L$ with $u(0,t) = 0$, $u(L,t) = 0$ is non-homogeneous due to the source $Q(x,t)$.
ii. $\frac{\partial u}{\partial t} = k\nabla^2 \mathrm{u}, 0 < x < L$ with $u(0,t) = A, u(L,t) = B$ is non-homogeneous due to boundary conditions.
iii. $\frac{\partial u}{\partial t} = k\nabla^2 \mathrm{u} + Q(x,t)$, $0 < x < L$ with $u(0,t) = A$, $u(L,t) = B$ is non- homogeneous due to the source term $Q(x,t)$ as well as the boundary conditions.

1.2.3. Method of images

The method of images (or method of mirror images) is a mathematical tool for solving differential equations, in which the domain of the sought function is extended by the addition of its mirror image with respect to a symmetry hyperplane. As a result, certain boundary conditions are satisfied automatically by the presence of a mirror image, greatly facilitating the solution of the original problem.

The domain of the function is not extended. The function is made to satisfy given boundary conditions by placing singularities outside the domain of the function.

The original singularities are inside the domain of interest. The additional (fictitious) singularities are an artifact needed to satisfy the prescribed but yet unsatisfied boundary conditions (Hildebrand & Rüegsegger, 1997). This method is important to modify the infinite domain Green's function so that it can serve for semi-infinite domain heat problems.

1.3. Fourier transform and its inverse

An integral transform method maps a function from its original function space into another function space via integration where some of the properties of the original function might be more easily characterized and manipulated than in the original function space.

For a well-defined function $f(x)$ on (a, b), an integral transform of $f(x)$ is defined by

$$F(\omega) = \int_a^b K(\omega, x) f(x) dx \qquad (1.3)$$

where ω denotes the variable of the transform, $F(\omega)$ is the integral transform of $f(x)$, and $K(\omega, x)$ is kernel of the transform (Watanabe & Watanabe, 2014).

An integral transform is a linear transformation, which, when applied to a linear initial or boundary value problem, reduces the number of independent variables by one for each of its application. Thus, a partial differential equation can be reduced to an algebraic equation by repeated application of integral transforms.

Several transforms are commonly named for the mathematicians who introduced them and defined with taking appropriate kernel and limit of integration (Lange, 1999) . For example, the formula given in (1.3) defines

a) Laplace transform when $K(\omega, x) = e^{-\omega x},\ a = 0$ and $b = +\infty$,
b) Fourier transform when $K(\omega, x) = (2\pi)^{-1}\, e^{i\omega x},\ a = -\infty$ and $b = +\infty$, and others.

The Fourier integral transform method is very useful in the study of linear partial differential equations and determination of Green's functions. With this purpose, we introduce Fourier Transform. We will discuss basic definitions and some important results of the Fourier transform, which are used in the subsequent chapters.

Definition 1.6 (Cochran et al., 1967): The function $\hat{f}$ defined by

$$\mathcal{F}\big(f(x)\big) = \hat{f}(\omega) = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx,$$

is called the Fourier transform or Fourier integral of the function $f(x)$ and

$$\mathcal{F}^{-1}\left(\hat{f}(\omega)\right) = \mathcal{F}^{-1}\big(F(\omega)\big) = f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega,$$

is the inverse Fourier transform of $\hat{f}$.

Theorem 1.1 (Duffy, 2015) : If $f(x)$ is absolutely integrable on the x-axis and piece-wise continuous on every finite interval, then the Fourier transform of $f(x)$ always exists.

1.3.1. Properties of Fourier Transform

The following properties of Fourier transform (Cochran et al., 1967) are derived from the definition and which will be used in the next chapter

1. **Linearity property:** Let $f(x)$ and $g(x)$ be any two functions whose Fourier transforms with respect to x exist. Then for arbitrary constants a and b, we have

$$\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}.$$

2. **Shifting property**: If the Fourier transform of $f(x)$with respect to x is $F(\omega)$, then

   i. $\mathcal{F}\{f(x-a)\} = e^{i\omega a}F(\omega)$.

   ii. $\mathcal{F}\{e^{iax}f(x)\} = F(\omega + a)$.

3. **Fourier transforms of derivatives**: If the Fourier transform of $f(x)$ with respect to x is $g(\omega)$, then

$$\mathcal{F}(f'(x)) = -(i\omega)F(\omega).$$

If f and its first $(n-1)$ derivatives are continuous, and if its n^{th} derivative is piecewise continuous, then,

$$\mathcal{F}\left(f^{(n)}(x)\right) = -(i\omega)^n F(\omega), n = 0,1,2, \ldots$$

Provided f and its derivatives are absolutely integrable. In addition, we assume that and its first $(n-1)$ derivatives tend to zero as $|x|$ tends to infinity.

4. **Fourier transforms of multiplication by x^n:** If the Fourier transform of $f(x)$ with respect to x is $F(\omega)$, then

$$\mathcal{F}\{x^n f(x)\} = (-i)^n F^{(n)}(\omega).$$

5. **(Convolution property):** Let $F(\omega)$ and $G(\omega)$ be the Fourier transform of $f(x)$ and $g(x)$ respectively. Then the Fourier transform of the convolution of f and g is given by

$$\mathcal{F}\{(f * g)(x)\} = F(\omega)G(\omega)$$

where the convolution integral is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

Lemma 1: $\int_{-\infty}^{\infty} e^{-(x)^2} dx = \sqrt{\pi}.$

Proof: Let

$$I = \int_{-\infty}^{\infty} e^{-(x)^2} dx.$$

We can also write the integral as

$$I = \int_{-\infty}^{\infty} e^{-(y)^2} dy$$

Therefore, $$I^2 = (\int_{-\infty}^{\infty} e^{-(x)^2} dx)\,(\int_{-\infty}^{\infty} e^{-(y)^2} dy).$$

Because the variables are not "coupled" here we can combine this into a double integral

$$I^2 = (\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[(x)^2+(y)^2]} dxdy.$$

Now we make a change of variables introducing polar coordinates (r, θ).

let $x = r\cos\theta, y = r\sin\theta$ such that $x^2 + y^2 = r^2$, which is a circle of radius r,then the Jacobian

$$J(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} d(x,y) = |J(r,\theta)| d(r,\theta) = r d(r,\theta)$$

Let $0 < r < \infty$ and $0 < \theta < 2\pi$. Now,

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[(x)^2+(y)^2]}dxdy$$

$$= \int_0^{2\pi}[\int_0^{\infty} e^{-(r)^2} rdr]d\theta.$$

Let $r^2 = t \Rightarrow rdr = \frac{1}{2}dt.$

$$\Rightarrow \int_0^{\infty} e^{-(r)^2} rdr = -\frac{1}{2}e^{-t}\Big|_0^{\infty} = \frac{1}{2}.$$

The double integral then reduces to

$$I^2 = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi.$$

Thus, $$I = \int_{-\infty}^{\infty} e^{-(x)^2}dx = \sqrt{\pi}.$$

Example 1.4: Consider the function

$$f(x) = e^{-kx^2}, k > 0.$$

To find its Fourier transform, from the definition we have

$$\hat{f}(\omega) = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-kx^2}e^{i\omega x}dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-kx^2+i\omega x}dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k\left(x^2-\frac{i\omega x}{k}\right)}dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k\left(x^2-\frac{i\omega x}{k}-\frac{\omega^2}{(4k^2)}+\frac{\omega^2}{(4k^2)}\right)}dx$$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k\left((x-\frac{i\omega}{2k})^2+\frac{\omega^2}{(4k^2)}\right)}dx$$

$$F(\omega)=\frac{1}{2\pi}e^{-\frac{\omega^2}{4k}}\int_{-\infty}^{\infty}e^{-k\left((x-\frac{i\omega}{k})^2\right)}dx$$

$$=\frac{1}{2\pi}e^{-\frac{\omega^2}{4k}}\int_{-\infty}^{\infty}e^{-\left[\sqrt{k}\left(x-\frac{i\omega}{k}\right)\right]^2}dx$$

Let $u=\sqrt{k}\left(x-\frac{i\omega}{k}\right)\Rightarrow du=\sqrt{k}\,dx$, then,

$$F(\omega)=\frac{1}{2\pi}e^{-\frac{\omega^2}{4k}}\int_{-\infty}^{\infty}e^{-u^2}\frac{du}{\sqrt{k}}$$

$$=\frac{1}{2\sqrt{k}\pi}e^{-\frac{\omega^2}{4k}}\int_{-\infty}^{\infty}e^{-u^2}du$$

$$=\frac{1}{2\sqrt{k}\pi}e^{-\frac{\omega^2}{4k}}(\sqrt{\pi}),\qquad \text{(using lemma1.)}$$

$$=\frac{1}{2\sqrt{k\pi}}e^{-\frac{\omega^2}{4k}}.$$

1.4. Green's function and its properties

The Dirac Delta function $\delta(x)$ is a generalized function or a distribution. It was introduced by theoretical physicist Paul Dirac in the 1930's in his study of quantum mechanics. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distributions, only makes sense under an integral.

Definition 1.7: The Dirac Delta function $\delta(x-t)$ is defined as

$$\delta(x-t)=\begin{cases}0, & x\neq t,\\ \infty, & x=t,\end{cases}$$

and satisfies the following identities:

a) $\int_{-\infty}^{\infty}\delta(x-t)dx=1.$

b) $\delta(-x)=\delta(x)$

c) $\int_{-\infty}^{\infty}f(x)\delta(x-t)d\,x=f(t)$, for any continuous function $f(x)$.

Alternatively, $\delta(x)$ may be viewed as the derivative of a discontinuous function, known as Heaviside step function.

Definition 1.8: The Heaviside step function, $H(x - t)$ is defined by

$$H(x-t) = \begin{cases} 0, & x < t, \\ 1, & x > t, \end{cases}$$

and undefined or constant at $x = t$.

Now, the derivative of the Heaviside step function is zero for $x \neq t$ and is undefined at $x = t$, (Wolff & Bucher, 2013). Thus,

$$\delta(x-t) = \frac{d}{dx} H(x-t).$$

Some integrals may be impossible to evaluate completely in terms of elementary functions for general initial data. Due to this, the answers for particular problems are usually written in terms of the error function. The name "error function" and its abbreviation erf were proposed by J. W. L. Glaisher in 1871 on account of its connection with "the theory of Probability, and notably the theory of Errors. The error and complementary error functions occur, for example, in solutions of the heat equation when boundary conditions are given by the Heaviside step function.

Definition 1.9: The error function denoted by $\operatorname{erf}(x)$ and is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

and the complementary error function, $\operatorname{erfc}(x)$, is defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-p^2} dp,$$

and satisfies the following identities:

a) $erf(\infty) = 1$
b) $\operatorname{erf}(0) = 0$
c) $\operatorname{erf}(-x) = -\operatorname{erf}(x)$
d) $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$
e) $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$

Most of the boundary value problems involving linear partial differential equations may be solved using integral transforms. The advantage of using integral transform is that it provides explicit representations of the solutions. Most probably the explicit form of the solution contains the integral forms in terms of the non-homogeneous terms together with some auxiliary function or its differential form. Such an auxiliary function is known as Green's function.

The concept of Green's function is first developed in 1830s after the British mathematician George Green. Green's function for a differential equation is its solution when the forcing term is the Dirac delta function due to a unit point source (sink) in a given domain.

Green's function is a basic solution to a linear differential equation, a building block that can be used to construct many useful solutions. In general, the exact form of Green's function depends on the differential equation, the geometry of the domain, and the types of boundary condition present (Qin, 2010).

In mathematics, a Green's function is the impulse response of an inhomogeneous linear differential operator defined on a domain with specified initial conditions or boundary conditions.

Definition 1.10: Consider the differential equation

$$\begin{cases} Lu(x) = Q(x), x \in D \subset \mathbb{R}^n, \\ u(x) = f(x), \text{on } \partial D = S, \end{cases}$$

where L is a differential operator, $u(x)$ is unknown function, $Q(x)$ is the source term, and D a bounded domain with boundary ∂D.

To solve this differential equation, we must find the operator L^{-1} which inverts the problem to $u(x) = L^{-1}Q(x)$. For a differential operator L, it is reasonable to expect that the inverse operator L^{-1} is an integral operator.

$$Lu(x) = Q(x)$$

$$\Rightarrow u(x) = L^{-1}Q(x)$$

$$= \int_D G(x, x_0)Q(x_0)dx_0$$

Here $G(x, x_0)$is an integral kernel.

If

$$\begin{cases} LG = \delta(x - x_0), in\ D \\ G = 0 \text{ on } \partial D, \end{cases}$$

then $G(x, x_0)$ is called Green's function for the operator L, where δ is the Dirac delta function.

If the domain D is the entire space, the solution to this system is known as the fundamental or (singularity) solution for the operator L. The fundamental solution or Green's function G in unbounded domain is a function which satisfies $LG = \delta$, where δ is the Dirac delta function.

Definition 1.11: The differential operator L is said to be self -adjoint, if $\int [uL(v) - vL(u)]d^n x = 0$ where u and v are any two continuous functions satisfying same set of homogeneous boundary conditions.

If L is a linear differential operator with dependent variable u and independent variable x, then an operator L^* which satisfies the relation

$$\int_a^b vL[u]dx = \int_a^b uL^*[v]dx + [M(u, v, u', v', x)]_a^b, \tag{1.4}$$

where $M(u, v, u', v', x)$ represents the boundary terms obtained after integration by parts, is called the adjoint operator of L. An operator L is said to be self-adjoint if $L = L^*$.

Theorem 1.2 (Duffy, 2015) : If $G(x, x_0)$ is Green's function for the linear operator L and $G^*(x, x_0)$ is Green's function for its adjoint operator L^*, then $G(x, x_0) = G^*(x_0, x)$.

To show the application of this theorem, consider the following example.

Example 1.5: Consider a second-order partial differential operator L, and let $u(\mathrm{x})$ and $v(\mathrm{x})$ be two C^2-functions. Then, integrating (1.4) by parts, we get

$$\iiint_D v(\mathrm{x})L[u](\mathrm{x})d\mathrm{x} = \iint_{\partial D} M(u, v)dS + \iiint_D u(\mathrm{x})L^*[v](\mathrm{x})dx \tag{1.5}$$

where $M(u, v)$ is a differential operator of the first order, ∂D is the boundary of the region D, L^* is the adjoint operator of L, and $d\mathrm{x} = dxdydz$ is the volume element.

Consider the following boundary value problem:

$$L[u] = f(\mathrm{x}) \text{ in } D, \text{ such that } Bu = 0 \text{ on } \partial D,$$

where B is a linear partial differential operator of the first order. To find the solution of this problem, we use (1.5) and the solution of the following problem:

$$L^*[v] = \delta(\mathrm{x}, \mathrm{x}_0) \text{ in } D, \text{ such that } B^*v = 0 \text{ on } \partial D, \tag{1.6}$$

where B^* is a linear partial differential operator of the first order such that $M(u, v) = 0$ on ∂D. The boundary condition $B^*v = 0$ on ∂D is known as the adjoint boundary condition for $Bu = 0$ on ∂D. The solution $v(\mathrm{x}, \mathrm{x}_0)$ of the problem (1.6) is denoted by $G^*(x, x_0)$ and is known as Green's function for the problem (1.6). Using (1.5),

we get

$$\iiint_D G^*(x, x_0)L[u](x)dx = \iint_{\partial D} M(u, G^*)dS + \iiint_D u(x)L^*[G^*](x, x_0)dx \tag{1.7}$$

which gives

$$\iiint_D G^*(x, x_0)f(x)dx = \iiint_D u(x)\delta(x - x_0)dx = u(x_0)$$

Since $x_0$is arbitrary, we interchange x and x_0 and obtain

$$u(x) = \iiint_D G^*(x_0, x)f(x_0)dx_0$$

Then using theorem 1.2, we obtain

$$u(x) = \iiint_{D^0} G(x, x_0)f(x_0)dx_0.$$

It turns out that even when the boundary condition on u is non-homogeneous, we can find the solution from (1.7). In the above solution we needed Green's function $G^*(\mathrm{x}, \mathrm{x}_0)$ for the adjoint operator L^* defined in the problem (1.6).

Theorem 1.3: (Duffy, 2015) : the boundary value problem

$$L[u](x) = f(x) \;\; in \;\; D \subset \mathbb{R}^n,$$
$$B[u] = 0 \;\; on \;\; \partial D = S,$$

where $f \in C(D)$, and B[u] represents linear initial and boundary conditions, has a unique solution in terms of Green's function $G(\mathrm{x}, \mathrm{x}_0)$:

$$u(x) = \int_D G(x, x_0) f(x_0) dx_0,$$

where dx_0 denotes integration with respect to the variable x_0 (source point or singularity).

Green's function $G(\mathrm{x}, x_0)$ is singular at the fixed point $x_0 \in D$, and the singular part of $G(\mathrm{x}, x_0)$, denoted by $u^*(\mathrm{x}, x_0)$, is known as the fundamental solution (or singular solution, or 'free-space' Green's function, or Green's function in the large) for the operator L such that

$$L[u^*](x, x_0) = \delta(x, x_0).$$

Physically, the function $u^*$is the response to a concentrated unit source located at $\mathrm{x} = \mathrm{x}_0$. An important physical application of the fundamental solution $u^*(\mathrm{x})$ is that it enables us to solve the nonhomogeneous equation

$$L[u](x) = f(x), \text{ where } f \in C_0^\infty(\mathbb{R}^n).$$

Once Green's function is known, the solution u of the problem can be easily determined. This is known as the Green's function method of finding the solution to boundary value problems.

This property of Green's function can be exploited to the non-homogenous heat equation of the form (1.1) with a given domain.

For the non-homogenous heat equation (1.1), two point Green's function $G(X, X_0)$ which is supposed to represent the temperature response at X at time t due to the source kept at $X = X_0$ acting instantaneously at time $t = t_0$, will be denoted by $G(X, t; X_0, t_0)$.

That is, $G(X, t; X_0, t_0)$ should be the solution of

$$\frac{\partial G}{\partial t} = k\nabla^2 G + \delta(X - X_0)\delta(t - t_0), \tag{1.8}$$

on the same region, but with the related homogeneous boundary conditions, where $\delta(X - X_0)$ is the Dirac delta function of appropriate dimension.

Also, as the Green's function $G(X, t; X_0, t_0)$ denotes the temperature response due to source located at X_0 at starting time $t = t_0$, it should be zero before the source acts. That is,

$$G(X, t; X_0, t_0) = 0 \text{ for } t < t_0.$$

This result is known as the causality principle.

As the Green's function $G(X, t; X_0, t_0)$ only depends on the time after the occurrence of the concentrated source, if we introduce the elapsed time, $T = t - t_0$,

$$\frac{\partial G}{\partial T} = k\nabla^2 G + \delta(X - X_0)\delta(T),$$
$$G = 0,\ T < 0,$$

then G is also seen to be the response due to a concentrated source at $x = x_0$ at $T = 0$.
We call this translation property:

$$G(\mathrm{X}, t; \mathrm{X}_0, t_0) = G(\mathrm{X}, t - t_0; \mathrm{X}_0, 0).$$

1.4.1. Green's formula and adjoint Green's function for the non-homogenous heat equation

Before solving for the Green's function, we will show how the solution of the non-homogeneous heat equation (1.1), with non-homogeneous initial and boundary conditions, is obtained using the Green's function by using appropriate Green's formula.

Theorem 1.4 (Pfeffer, 1986) (Green's second theorem): Let S be a closed surface in the (x, y, z) space and let V be the closed region bounded by S . Then, for the operator ∇^2,the Green's formula is

$$\iiint_V (u\nabla^2 v - v\nabla^2 u)dV = \iint_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS,$$

where dV is an element of volume, dS is an element of surface area and n is the outward drawn normal to the surface S. Here, both u and v must possess continuous second order derivatives.

Now we introduce the following appropriate linear operator L for the non-homogeneous heat equation(1.1):

$$L = \frac{\partial}{\partial t} - k\nabla^2. \tag{1.9}$$

Using this notation, the nonhomogeneous heat equation (1.1) satisfies

$$L(u) = Q(X, t), \tag{1.10}$$

while the Green's function (1.8) satisfies

$$L(G) = \delta(X - X_0)\delta(t - t_0). \tag{1.11}$$

We note that for time-dependent problems L has both space and time variables.

The operator, $L = \frac{\partial}{\partial t} - k\nabla^2$, known as the heat operator, possesses the first order operator $\frac{\partial}{\partial t}$, along with second order operator ∇^2. To simplify $\frac{\partial}{\partial t}$ is difficult, because the operator $\frac{\partial}{\partial t}$ is not self-adjoint.

Now, in particular for the operator $L = \frac{\partial}{\partial t}$, $\int [uL(v) - vL(u)]dt = \int \left[u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right] dt$ cannot be simplified directly, as the operator $L = \frac{\partial}{\partial t}$ is not self-adjoint.

To tackle with this difficulty, it is better to find $\int_a^b [uL(v)]dt$ using integration by parts, that is:

$$\int_a^b [uL(v)]dt = \int_a^b \left[u\frac{\partial v}{\partial t}\right] dt = [uv]_a^b - \int_a^b v\frac{\partial u}{\partial t} dt = [uv]_a^b - \int_a^b vL(u)dt$$

$$\Rightarrow \int_a^b [uL(v) + vL(u)]dt = [uv]_a^b$$

So it is advisable to define an adjoint operator $L^* = -\frac{\partial}{\partial t}$ for the operator $L = \frac{\partial}{\partial t}$, so that

$$\int_a^b [uL^*(v) - vL(u)]dt = \int_a^b \left[-u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right] dt = -[uv]_a^b \tag{1.12}$$

Similarly for the heat operator $L = \frac{\partial}{\partial t} - k\nabla^2$, introducing the adjoint heat operator

$$L^* = -\frac{\partial}{\partial t} - k\nabla^2,$$

then,

$$uL^*(v) - vL(u) = u\left(-\frac{\partial v}{\partial t} - k\nabla^2 v\right) - v\left(\frac{\partial u}{\partial t} - k\nabla^2 u\right)$$

$$= -u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} + k(v\nabla^2 u - u\nabla^2 v).$$

So, $\int_{t_i}^{t_f} \iiint [uL^*(v) - vL(u)]d^3Xdt = \int_{t_i}^{t_f} \iiint \left[-u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} + k(v\nabla^2 u - u\nabla^2 v)\right] d^3Xdt$

$$\Rightarrow \int_{t_i}^{t_f} \iiint [uL^*(v) - vL(u)]d^3Xdt = \int_{t_i}^{t_f} \iiint \left[-u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right] d^3Xdt + k\int_{t_i}^{t_f} (\iiint [(v\nabla^2 u - u\nabla^2 v)]d^3Xdt$$

Now, using (1.12)

$$\Rightarrow \int_{t_i}^{t_f} \iiint [uL^*(v) - vL(u)]d^3Xdt = -\iiint [uv]_{t_i}^{t_f} d^3X + k\int_{t_i}^{t_f} \iiint [(v\nabla^2 u - u\nabla^2 v)]d^3Xdt$$

For the operator ∇^2, we have the Green's formula :

$$\iiint (u\nabla^2 v - v\nabla^2 u)d^3X = \oiint (u\nabla v - v\nabla u) \cdot \hat{n}ds \qquad (1.13)$$

Using Green's formula (1.13), we will get

$$\int_{t_i}^{t_f} \iiint [uL^*(v) - vL(u)]d^3Xdt$$

$$= -\iiint [uv]_{t_i}^{t_f} d^3X + k\int_{t_i}^{t_f} \left(\oiint (v\nabla u - u\nabla v) \cdot \hat{n}dS\right)dt, \qquad (1.14)$$

where $\iiint d^3X$ indicates integration over the three dimensional space, $\oiint dS$ indicates the surface integration over its boundary and $\hat{n}$ denotes the unit outward normal to surface.

The terms on the right-hand side represent contributions from the boundaries: the spatial boundaries for all time and the temporal boundaries ($t = t_i$ and $t = t_f$) for all space.

Result (1.14) provides the Green's formula for the heat equation (1.1).

If both u and v satisfy the homogeneous boundary conditions (in space, for all time), then

$$\oiint (u\nabla v - v\nabla u) \cdot \hat{n} dS = 0, \text{ but}$$

$$\iiint \left(u\frac{\partial v}{dt} - v\frac{\partial u}{dt} \right)\Bigg|_{t_i}^{t_f} d^3X$$

may not equal zero due to contributions from the 'initial' time t_i and 'final' time t_f and hence in that case, equation (1.14) will be simply:

$$\int_{t_i}^{t_f} \iiint [uL^*(v) - vL(u)]d^3Xdt = -\iiint [uv]_{t_i}^{t_f} d^3X.$$

In order to derive a representation formula for $u(X,t)$ in terms of the Green's function $G(X,t;X_0,t_0)$, we needed adjoint Green's function $G^*(X,t;X_0,t_0)$ for the adjoint operator L^* defined in above. That is, it is required to sum up various source times. For that, source-varying Green's function $G(X,t_1;X_1,t)$ obtained by letting the source time t to vary will be utilized.

Using causality principle, $G(X,t_1;X_1,t) = 0$, for $t > t_1$.

Also, using translation property, $G(X,t_1;X_1,t) = G(X,t_1 - t;X_1,0) = G(X,-t;X_1,-t_1)$.

The source varying Green's function $G(X,t_1;X_1,t)$satisfies

$$\left(-\frac{\partial}{\partial t} - k\nabla^2\right) G(X,t_1;X_1,t) = \left(-\frac{\partial}{\partial t} - k\nabla^2\right) G(X,-t;X_1,-t_1) \qquad (1.15)$$

Let $\tau = -t$, then the expression (1.15) implies:

$$\left(-\frac{\partial}{\partial t} - k\nabla^2\right) G(X, t_1; X_1, t) = \left(\frac{\partial}{\partial \tau} - k\nabla^2\right) G(X, \tau; X_1, -t_1)$$

$$= L\big(G(X, \tau; X_1, -t_1)\big)$$

$$= \delta(X - X_1)\delta\big(\tau - (-t_1)\big)$$

$$= \delta(X - X_1)\delta(-t + t_1)$$

$$= \delta(X - X_1)\delta(t - t_1)$$

(as δ is an even function, $\delta(-t + t_1) = \delta(t - t_1)$).

$$\Rightarrow \left(-\frac{\partial}{\partial t} - k\nabla^2\right) G(X, t_1; X_1, t) = \delta(X - X_1)\delta(t - t_1).$$

as $L^* = -\frac{\partial}{\partial t} - k\nabla^2$,

$$\Rightarrow L^*[G(X, t_1; X_1, t)] = \delta(X - X_1)\delta(t - t_1). \tag{1.16}$$

Now, $L^*[G(X, t_1; X_1, t)]$ means the adjoint heat operator L^* is operated on $G(X, t_1; X_1, t)$.

Here, $G(X, t_1; X_1, t)$ is the Green's function for the adjoint heat operator L^*. So it is known as the adjoint Green's function and is denoted by $G^*(X, t; X_1, t_1)$.

That is $G^*(X, t; X_1, t_1) = G(X, t_1; X_1, t)$. These both are zero for $t > t_1$.

The result is known as reciprocity, that is Green's function is symmetric for the self-adjoint operators.

1.4.2. Representation of the Solution of the non-homogenous heat equation in terms of Green's function

To represent the solution of (1.1) with initial condition, $u(X, 0) = f(X)$, the Green's formula, expressed in result (1.14),will be utilized.

A reciprocity property results from Green's formula (1.14) using the Green's functions, with varying source time,

$$v = G(\mathrm{X}, t_0; \mathrm{X}_0, t).$$

That is the adjoint Green's function $G(\mathrm{x}, t_0; \mathrm{x}_0, t)$ will serve as v.

According to (1.14),

$$\int_{t_i}^{t_f} \iiint [uL^*(v) - vL(u)]d^3Xdt = -\iiint [uv]_{t_i}^{t_f} d^3X + k\int_{t_i}^{t_f} (\oiint (v\nabla u - u\nabla v)\cdot \hat{n}dS)dt$$

and also taking initial time $t_i = 0$ and final time $t_f = t_{0^+}$in this formula, one gets:

$$\int_{t_i}^{t_f} \iiint [uL^*(v) - vL(u)]d^3Xdt = -\iiint [uv]_{t_i}^{t_f} d^3X + k\int_{t_i}^{t_f} (\oiint (v\nabla u - u\nabla v)\cdot \hat{n}dS)dt$$

$$\Rightarrow \int_0^{t_0+} \iiint \left[u(X,t)L^*\big(G(X,t_0;X_0,t)\big) - G(X,t_0;X_0,t)L(u(X,t))\right]d^3Xdt$$

$$= -\iiint [u(X,t)G(X,t_0;X_0,t)]_0^{t_0+} d^3X +$$

$$k\int_0^{t_0+} \left(\oiint \big(G(X,t_0;X_0,t)\nabla u - u\nabla G(X,t_0;X_0,t)\big)\cdot \hat{n}dS\right)dt \qquad (1.17)$$

But, at $t = t_{0^+}, G(X, t_0; X_0, t) = 0$ since $t_{0^+} > t_0$ (causality principle).

$$\Rightarrow -\iiint [u(X,t)G(X,t_0;X_0,t)]_0^{t_0+} d^3X = -\iiint 0 - \big(u(X,0)G(X,t_0;X_0,0)\big)d^3X$$

$$= \iiint \big(u(X,0)G(X,t_0;X_0,0)\big)\, d^3X.$$

Using (1.16) in (1.17), we get:

$$\int_0^{t_0+} \iiint [u(X,t)\delta(X-X_0)\delta(t-t_0) - G(X,t_0;X_0,t)Q(X,t)]d^3X dt$$

$$= \iiint u(X,0)G(X,t_0;X_0,0)d^3X + k\int_0^{t_0+} [\oiint \big(G(X,t_0;X_0,t)\nabla u - u\nabla G(X,t_0;X_0,t)\big)\cdot \hat{n}dS]dt$$

Now, using the property of Dirac delta function,

$$\Rightarrow u(X_0,t_0) = \int_0^{t_0^+} \iiint [G(X,t_0;X_0,t)Q(X,t)]d^3X dt + \iiint u(X,0)G(X,t_0;X_0,0)d^3X$$
$$+ k\int_0^{t_0+} [\oiint \big(G(X,t_0;X_0,t)\nabla u - u\nabla G(X,t_0;X_0,t)\big)\cdot \hat{n}dS]dt$$

Changing role of X with X_0, as well as of t with t_0, one gets:

$$u(X,t) = \int_0^{t} \iiint G(X,t;X_0,t_0)Q(X_0,t_0)d^3X_0 dt_0 + \iiint u(X_0,0)G(X,t;X_0,0)d^3X_0$$

$$+k\int_0^{t} \left[\oiint \Big(G(X,t;X_0,t_0)\nabla_{X_0}u - u(X_0,t_0)\nabla_{X_0}G(X,t;X_0,t_0)\Big)\cdot \hat{n}dS_0\right]dt_0 \qquad (1.18)$$

Thus (1.18) provides the representation of the solution $u(X,t)$ in terms of the Green's function.

Note that ∇_{X_0} means a derivative with respect to the source position. Equation (1.18) expresses the response due to the three kinds of nonhomogeneous terms: source terms, initial conditions and nonhomogeneous boundary conditions. That is,

1. The term $\int_0^t \iiint G(X,t;X_0,t_0)Q(X_0,t_0)d^3X_0 dt_0$ implies the contribution due to the source term $Q(X_0,t_0)$ in which the Green's function $G(X,t;X_0,t_0)$ acts as the influence function,
2. The term $\iiint u(X_0,0)G(X,t;X_0,0)d^3X_0$ implies the contribution due to initial function $u(X_0,0)$ in which the Green's function $G(X,t;X_0,t_0)$ is the influence function but with $t_0 = 0$ and,
3. The term $k\int_0^t \left[\oiint \Big(G(X,t;X_0,t_0)\nabla_{X_0}u - u\nabla_{X_0}G(X,t;X_0,t_0)\Big)\cdot \hat{n}dS_0\right]dt_0$ implies the contribution due to non-homogeneous boundary conditions.

CHAPTER TWO

GREEN’S FUNCTION METHOD

In chapter one, we considered important theorems and properties related to Greens function method. In this chapter, we will apply Green’s function method to solve non-homogeneous heat equation defined on an infinite domain, semi-infinite domain and finite domain, which arise in many physical problems.

2.1. Introduction

There are different methods to solve the heat equation (1.1) along with given initial and /or boundary conditions. The Fourier transform and the method of Eigen functions expansion are among these methods. In Green’s function method, we have to determine a certain function, which will remain same for all boundary value problems, having same governing equation. That is this method gives explicit solution, means once, corresponding Green's function to a boundary value problem is determined, we simply have to substitute it, along with initial and/or boundary conditions in derived formula, which is most probably in integral form, to find out required solution. Therefore, it is not the case that, for every distinct problem, but with same governing equation, we need to determine distinct Green's functions.

2.2. Solving non-homogenous Heat equation on an Infinite Domain

In some mathematical models of heat conduction, an unbounded region of space is assumed, and there are no boundary conditions.

In this section, we will consider the non-homogenous heat equation with an infinite domain using Green’s function method. To do this we need the Green’s function first.

Suppose we want to find the solution of the heat equation with initial value condition given in the infinite domain for the case of n dimension. That is, consider

$$\frac{\partial u}{\partial t} = k\nabla^2 u + Q(X,t), -\infty < X < \infty, t > 0, \tag{2.1}$$

$$u(X,0) = f(X), -\infty < X < \infty. \tag{2.2}$$

To determine the Greens function $G(X,t;X_0,t_0)$ for the non-homogenous heat equation (2.1), Fourier transform method will be utilized.

In n-dimensional space, the Fourier transform relationship is:

$$f(X) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega\cdot X}d^n\omega, \text{ and}$$

$$F(\omega) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} f(X)e^{i\omega\cdot X}d^nX$$

Where, $F(\boldsymbol{\omega})$ denotes the Fourier transform of $f(X)$, $X = (x_1, x_2, \dots, x_n)$, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ and $\omega \cdot X = \omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n$.

Let $\bar{G}(\omega, t; X_0, t_0)$ denote the Fourier transform of the Greens function $G(X,t;X_0,t_0)$.
By definition,

$$\bar{G}(\omega, t; X_0, t_0) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} G(X,t;X_0,t_0)e^{i\omega\cdot X}d^nX,$$

and

$$G(X,t;X_0,t_0) = \int_{-\infty}^{\infty} \bar{G}(\omega, t; X_0, t_0)e^{-i\omega\cdot X}d^n\omega.$$

Considering heat equation (2.1) - (2.2), on infinite domain, as discussed before, the required Green's function $G(X,t;X_0,t_0)$, should satisfy

$$\frac{\partial \mathrm{G}}{\partial \mathrm{t}} = k\nabla^2 G + \delta(\mathrm{X}-\mathrm{X}_0)\delta(\mathrm{t}-\mathrm{t}_0), \tag{2.3}$$

and subject to the causality principle, $G(X,t;X_0,t_0) = 0$ for $t < \mathrm{t}_0$.

From the property of Fourier transform , $\frac{\partial \bar{G}}{\partial t} = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\frac{\partial G}{\partial t}e^{i\omega X}d^nX$

$$= \frac{\partial}{\partial t}\left(\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} G e^{i\omega X}d^nX\right) = \frac{\partial \bar{G}}{\partial t},$$

and

$$\nabla^2\bar{G} = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\nabla^2 G e^{i\omega X} d^n X = (i\omega)^2\left(\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} G e^{i\omega X} d^n X\right) = -\omega^2\bar{G}.$$

Also, the Fourier transform of Dirac Delta function $\delta(X - X_0)\delta(t - t_0)$ is

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\delta(X - X_0)\delta(t - t_0)e^{i\omega X} d^n x = \delta(t - t_0)\frac{e^{i\omega X_0}}{(2\pi)^n}$$

(as $\int_{-\infty}^{\infty} \mathrm{f}(X)\delta(X)dX = f(0) \Rightarrow \int_{-\infty}^{\infty}\delta(X - X_0)e^{i\omega\cdot X} d^n x = e^{i\omega\cdot X_0}$.)

Now, the Fourier transform of (2.3) implies that

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(G(X,t;X_0,t_0))e^{i\omega\cdot X} d^n X$$

$$= k\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\nabla^2 G(X,t;X_0,t_0)e^{i\omega.X} d^n X + \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\delta(X - X_0)\delta(t - t_0)e^{i\omega\cdot X} d^n X.$$

$$\Rightarrow \frac{\partial\bar{G}}{\partial t} = -k\omega^2\bar{G} + \delta(t - t_0)\frac{e^{i\omega\cdot X_0}}{(2\pi)^n} \qquad (2.4)$$

As $G(X,t;X_0,t_0) = 0$ if $t < t_0$, the expression

$$\bar{G}(\omega,t;X_0,t_0) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} G(X,t;X_0,t_0)e^{i\omega\cdot X} d^n X$$

implies that $\bar{G}(\omega,t;X_0,t_0) = 0$ for $t < t_0$.

That is Fourier Transform $\bar{G}(\omega,t;X_0,t_0)$ of Green's function $G(X,t;X_0,t_0)$, also satisfies causality principle.

As Delta function $\delta(t - t_0) = 0$, for $t > t_0$, (2.4) implies that,

$$\frac{\partial\bar{G}}{\partial t} + k\omega^2\bar{G} = 0, \quad \text{for } t > t_0. \qquad (2.5)$$

Solving (2.5), that is treating it as an ordinary differential equation, we will get

,

$$\bar{G} = C(\omega)e^{-k\omega^2(t-t_0)}, \text{for } t > t_0, \qquad (2.6)$$

where $C(\omega)$, a function of ω, X_0 and t_0, is to be determined, using initial conditions at $t = t_0$.

Thus,

$$\bar{G} = \begin{cases} 0, & \text{for } t < t_0 \\ C(\omega)e^{-k\omega^2(t-t_0)}, & \text{for } t > t_0 \end{cases} \tag{2.7}$$

To determine $C(\omega)$, integrating (2.4) from $t = t_{0}-$to $t = t_{0^+}$, which yields:

$$\int_{t_0-}^{t_0+} \frac{\partial \bar{G}}{\partial t} dt + k\omega^2 \int_{t_0^-}^{t_0+} \bar{G} dt = \int_{t_0^-}^{t_0} \frac{e^{i\omega x_0}}{(2\pi)^n} \delta(t - t_0) dt$$

$$\Rightarrow \bar{G}(t_{0^+}) - \bar{G}(t_{0^-}) = \frac{e^{i\omega X_0}}{(2\pi)^n}$$

(using property of Dirac delta function).

As $\bar{G}(t_0 -) = 0$ for $t < t_0$,

$$\bar{G}(t_{0^+}) = \frac{e^{i\omega x_0}}{(2\pi)^n} \tag{2.8}$$

At $t = t_0$, the expression (2.6) implies

$$\bar{G}(t_{0^+}) = C(\omega) \tag{2.9}$$

So, (2.8) and (2.9) together imply, $C(\omega) = \frac{e^{i\omega X_0}}{(2\pi)^n}$.

Substituting this value of $C(\omega)$ in (2.7) , the Fourier Transform of the Green's function is obtained as:

$$\bar{G}(\omega, t; X_0, t_0) = \frac{e^{i\omega X_0}}{(2\pi)^n} e^{-k\omega^2(t-t_0)}.$$

Taking inverse Fourier Transform, we will get,

$$G(X, t; X_0, t_0) = \int_{-\infty}^{\infty} \frac{e^{i\omega \cdot X_0}}{(2\pi)^n} e^{-k\omega^2(t-t_0)} e^{-i\omega \cdot X} d^n\omega$$

$$\Rightarrow G(X, t; X_0, t_0) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{-i\omega \cdot (X-X_0)} e^{-k\omega^2(t-t_0)} d^n\omega$$

$$\Rightarrow G(X,t;X_0,t_0)=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\left[e^{-\left(k\omega^2(t-t_0)+i\omega\cdot(X-X_0)+\frac{i^2(X-X_0)^2}{4k(t-t_0)}\right)}e^{-\frac{(X-X_0)^2}{4k(t-t_0)}}\right]d^n\omega$$

$$\Rightarrow G(X,t;X_0,t_0)=\frac{e^{\frac{-(X-X_0)^2}{4k(t-t_0)}}}{(2\pi)^n}\int_{-\infty}^{\infty}e^{-\left(\sqrt{k(t-t_0)}\omega+\frac{i(X-X_0)}{2\sqrt{k(t-t_0)}}\right)^2}d^n\omega$$

Let $\tau=\sqrt{k(t-t_0)}\omega+\frac{i(X-X_0)}{2\sqrt{k(t-t_0)}}$, then $d^n\omega=\left[\frac{1}{\sqrt{k(t-t_0)}}\right]^n d^n\tau$

Hence, $G(X,t;X_0,t_0)=\frac{e^{\frac{-(X-X_0)^2}{4k(t-t_0)}}}{(2\pi)^n}\left[\frac{1}{\sqrt{k(t-t_0)}}\right]^n\int_{-\infty}^{\infty}e^{-\tau^2}d^n\tau$

$$\Rightarrow G(X,t;X_0,t_0)=\frac{e^{\frac{-(X-X_0)^2}{4k(t-t_0)}}}{(2\pi)^n}\left[\frac{1}{\sqrt{k(t-t_0)}}\right]^n(\sqrt{\pi})^n$$

(using lemma1, $\int_{-\infty}^{\infty}e^{-\tau^2}\mathrm{d}\tau=\sqrt{\pi}$.)

$$\Rightarrow G(X,t;X_0,t_0)=\frac{1}{(2\pi)^n}\left[\frac{\pi}{k(t-t_0)}\right]^{\frac{n}{2}}e^{\frac{-(X-X_0)^2}{4k(t-t_0)}}$$

This is the required n-dimensional infinite space Green's function for heat equation (2.1).

Using this Green's function in result (1.18), as there are no boundaries the solution of the heat equation (2.1)-(2.2), with sources on an infinite domain will be:

$u(X,t)=\int_0^t\int_{-\infty}^{\infty}G(X,t;X_0,t_0)Q(X_0,t_0)d^nX_0dt_0+\int_{-\infty}^{\infty}u(X_0,0)G(X,t;X,0)d^nX_0.$

$$u(X,t)=\int_0^t\int_{-\infty}^{\infty}\frac{1}{(2\pi)^n}\left[\frac{\pi}{k(t-t_0)}\right]^{\frac{n}{2}}e^{\frac{-(X-X_0)^2}{4k(t-t_0)}}Q(x_0,t_0)\mathrm{d}^{\mathrm{n}}X_0dt_0$$
$$+\int_{-\infty}^{\infty}\frac{1}{(2\pi)^n}\left[\frac{\pi}{kt}\right]^{\frac{n}{2}}e^{\frac{-(X-X_0)^2}{4kt}}f(X_0)dX_0.$$

For $n = 1$, that is, for one dimensional infinite space problem, Green's function will be:

$$G(x,t;x_0,t_0) = \frac{1}{(2\pi)}\left[\frac{\pi}{k(t-t_0)}\right]^{\frac{1}{2}} e^{\frac{-(x-x_0)^2}{4k(t-t_0)}}$$

$$\Rightarrow G(x,t;x_0,t_0) \qquad = \frac{1}{\sqrt{4\pi k(t-t_0)}} e^{\frac{-(x-x_0)^2}{4k(t-t_0)}}, \qquad (2.10)$$

and the solution will be:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,t;x_0,t_0)Q(x_0,t_0)\mathrm{d}x_0 dt_0 + \int_{-\infty}^{\infty} u(x_0,0)G(x,t;x_0,0)dx_0$$

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-t_0)}} e^{\frac{-(x-x_0)^2}{4k(t-t_0)}} Q(x_0,t_0)\mathrm{d}x_0 dt_0 + \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-x_0)^2}{4k\mathrm{t}}} f(x_0)dx_0. \quad (2.11)$$

(Physically, the temperature function $u(x,t) \to 0$, for all the time as $|x| \to \infty$).

We can consider the following cases.

In the case where $Q(\mathrm{x},t) = 0$, in (2.1), the Green's function for the problem is

$$G(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-\mathrm{x}_0)^2}{4kt}}, \qquad (2.12)$$

and the solution $u(x,t)$ from (2.11) is

$$u(x,t) = \int_{-\infty}^{\infty} f(x_0)\frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-x_0)^2}{4k\mathrm{t}}} dx_0. \qquad (2.13)$$

In other case, when $f(x) = 0$, we will have the non-homogenous heat equation with zero initial condition. Therefore, by using (2.11) the solution is

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,t;x_0,t_0)Q(x_0,t_0)\mathrm{d}x_0 dt_0$$

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-t_0)}} e^{\frac{-(x-x_0)^2}{4k(t-t_0)}} Q(x_0,t_0)\mathrm{d}x_0 dt_0.$$

Example 2.1: Consider the heat equation

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0, -\infty < x < \infty, t > 0,$$

With initial condition,

$$u(x,0) = 5\delta(x-3) - \delta(x-2), -\infty < x < \infty.$$

In this case $Q(x,t) = 0$ and $f(x) = 5\delta(x-5) - \delta(x-2)$.

Applying (2.12)-(2.13), we have

$$u(x,t) = \int_{-\infty}^{\infty} f(x_0) \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-x_0)^2}{4t}} dx_0$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} [5\delta(x-3) - \delta(x-2)] e^{\frac{-(x-x_0)^2}{4t}} dx_0$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} [5\delta(x-3)] e^{\frac{-(x-x_0)^2}{4t}} dx_0 - \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} [\delta(x-2)] e^{\frac{-(x-x_0)^2}{4t}} dx_0$$

Using property of Dirac delta function, which leads to the solution

$$u(x,t) = \frac{5}{2\sqrt{\pi t}} e^{\frac{-(x-3)^2}{4t}} - \frac{1}{2\sqrt{\pi t}} e^{\frac{-(x-2)^2}{4t}}, -\infty < x < \infty, t > 0.$$

Example 2.2: Consider the heat equation

$$\frac{\partial u}{\partial t} - k\nabla^2 u = \delta(x-2)\delta(t-1), -\infty < x < \infty, t > 0,$$

With initial condition,

$$u(x,0) = 0, -\infty < x < \infty.$$

In this case $Q(x,t) = \delta(x-2)\delta(t-1)$, then the solution will be:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,t;x_0,t_0)Q(x_0,t_0)\mathrm{d}x_0 dt_0$$

$$= \int_0^t \int_{-\infty}^{\infty} G(x,t;x_0,t_0)\delta(x_0-2)\delta(t_0-1)\mathrm{d}x_0 dt_0$$

$$= \int_0^t G(x,t;2,t_0)\,\delta(t_0-1)\,dt_0$$

$$= \frac{1}{\sqrt{4\pi k(t-1)}} e^{\frac{-(x-2)^2}{4k(t-1)}}, t > 1.$$

Example 2.3: Consider the heat equation

$$\frac{\partial u}{\partial t} - k\nabla^2 u = 0, -\infty < x < \infty, t > 0,$$

With initial condition,

$$u(x,0) = e^x, -\infty < x < \infty.$$

In this case $Q(x,t) = 0$ and $f(x) = e^x$.

Applying (2.12)-(2.13), we have

$$u(x,t) = \int_{-\infty}^{\infty} f(x_0) \frac{1}{\sqrt{4\mathrm{k}\pi t}} e^{\frac{-(x-x_0)^2}{4\mathrm{kt}}} dx_0$$

$$= \frac{1}{\sqrt{4\mathrm{k}\pi t}} \int_{-\infty}^{\infty} e^{x_0}\, e^{\frac{-(x-x_0)^2}{4kt}} dx_0$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-x_0)^2}{4kt}} e^{x_0} dx_0 = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{\left[-x^2+2xx_0-\mathrm{x}_0^2+4ktx_0\right]}{4kt}} dx_0$$

We can complete the squares in the numerator of the exponent, writing it as

$$\frac{-x^2+2xx_0-x_0^2+4ktx_0}{4kt}=\frac{-x_0{}^2+2(x+2kt)x_0-x^2}{4kt}$$
$$=\frac{-(x_0-2kt-x)^2+4ktx+4k^2t^2}{4kt}=-\left(\frac{x_0-2kt-x}{\sqrt{4kt}}\right)^2+x+kt$$

We then have

$$u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}e^{x+kt}e^{-\left[\frac{x_0-2kt-x}{\sqrt{4kt}}\right]^2}dx_0$$

Let $p=\frac{x_0-2kt-x}{\sqrt{4kt}}$, then $\frac{dp}{dx_0}=\frac{1}{\sqrt{4kt}}$

$$\Rightarrow u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}e^{x+kt}e^{-\left[\frac{x_0-2kt-x}{\sqrt{4kt}}\right]^2}dx_0$$

$$\Rightarrow u(x,t)=\frac{1}{\sqrt{\pi}}e^{x+kt}\int_{-\infty}^{\infty}e^{-p^2}dp$$

$$\Rightarrow u(x,t)=\frac{1}{\sqrt{\pi}}e^{x+kt}(\sqrt{\pi})$$

$$\Rightarrow u(x,t)=e^{kt+x}.$$

2.3. Solving non-homogenous Heat equation on a Semi- Infinite Domain

The theory developed to solve non-homogeneous heat equation on infinite domain can be generalized to solve non-homogeneous heat equation on the semi-infinite domain also. Here the difference is that semi-infinite domains possess one boundary.

To discuss the solution of non-homogeneous heat equation, in one dimension, on semi-infinite domain, one requires having a boundary condition, defined.

a. Suppose we want find the solution of the heat equation with a Dirichlet boundary condition and an initial condition on a semi-infinite straight line. That is, we want to solve the problem

$$\frac{\partial u}{\partial t} = k\nabla^2 u + Q(x,t), 0 < x < \infty, \tag{2.14}$$

With initial condition $\quad u(x,0) = f(x), 0 < x < \infty, \quad$ (2.15)

with boundary condition $\quad u(0,t) = h(t), t > 0, \quad$ (2.16)

To find the solution of (2.14), satisfying (2.15) - (2.16), one needs to find appropriate Green's function.

Here for semi-infinite domain, $x = 0$ is its boundary, and the required Green's function has always, to satisfy corresponding homogeneous boundary condition. Therefore, here $G(0,t;x_0,t_0)$ must be zero. But $G(0,t;x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}} e^{\frac{-(x_0)^2}{4k(t-t_0)}} \neq 0$. So infinite domain Green's function will not work for one dimensional semi-infinite domain problem.

To make Green's function zero at $x = 0$, Some modification is required. For that, the method of images will be utilized. In this method, it is imagined that, to one actual source concentrated at $x = x_0$ in the region $0 < x < \infty$, a fictitious, negative source (image source) located as $x = -x_0$ is being added. That is, now Green's function for this semi-infinite domain is:

$$W(x,t;x_0,t_0) = G(x,t;x_0,t_0) - G(-x,t;x_0,t_0)$$

$$W(x,\mathrm{t};x_0,\mathrm{t}_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(e^{\frac{-(x-x_0)^2}{4k(t-t_0)}}\right) - \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(e^{\frac{-(x+x_0)^2}{4k(t-t_0)}}\right)$$

$$= \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(e^{\frac{-(x-x_0)^2}{4k(t-t_0)}} - e^{\frac{-(x+x_0)^2}{4k(t-t_0)}}\right) \tag{2.17}$$

Substituting $x = 0$, in (2.17), $W(0,t;x_0,t_0) = 0, \forall t$.

Thus the Green's function given in (2.17) is the required Green's function for a problem on the semi-finite domain. Here it is taken as positive X-axis.

Using (1.18), the solution of the equation (2.14) is:

$$u(x,t) = \int_0^t \int_0^\infty W(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^\infty u(x_0,0)W(x,t;x_0,0)dx_0$$
$$+ k\int_0^t \left[\oint\!\!\oint \left(W(x,t;x_0,t_0)\nabla_{X_0}u - u\nabla_{X_0}W(x,t;x_0,t_0)\right)\cdot \hat{n}dS_0\right]dt_0$$

Using the boundary condition, $W(0,t;x_0,t_0) = 0, \forall t,$

$$\Rightarrow u(x,t) = \int_0^t \int_0^\infty W(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^\infty u(x_0,0)W(x,t;x_0,0)dx_0$$
$$+ k\int_0^t \left[-u(0,t_0)W_{x_0}(x,t;0,t_0)\cdot(-\hat{\imath})\right]dt_0$$

(for positive $X-$ axis, $\hat{n} = -\hat{\imath}$).

$$\Rightarrow u(x,t) = \int_0^t \int_0^\infty W(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^\infty f(x_0)W(x,t;x_0,0)dx_0$$
$$+ k\int_0^t \left[h(t)W_{x_0}(x,t;0,t_0)\right]dt_0 \qquad (2.18)$$

Differentiating $W(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(e^{\frac{-(x-x_0)^2}{4k(t-t_0)}} - e^{\frac{-(x+x_0)^2}{4k(t-t_0)}}\right)$ partially, with respect to x_0, results

$$\frac{\partial}{\partial x_0}W(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(-\frac{(x_0-x)}{2k(t-t_0)}e^{\frac{-(x_0-x)^2}{4k(t-t_0)}} + \frac{(x+x_0)}{2k(t-t_0)}e^{\frac{-(x+x_0)^2}{4k(t-t_0)}}\right)$$

$$\Rightarrow \left[\frac{\partial}{\partial x_0}W(x,t;x_0,t_0)\right]_{x_0=0} = \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\frac{x}{k(t-t_0)}e^{\frac{-x^2}{4k(t-t_0)}}\right) \qquad (2.19)$$

Substituting, (2.18) and (2.20) in (2.19), we will get

$$u(x,t) = \int_0^t \int_0^\infty \frac{1}{\sqrt{4\pi k(t-t_0)}} \left(e^{\frac{-(x-x_0)^2}{4k(t-t_0)}} - e^{\frac{-(x+x_0)^2}{4k(t-t_0)}} \right) Q(x_0,t_0)dx_0dt_0$$

$$+ \int_0^\infty f(x_0) \frac{1}{\sqrt{4\pi kt}} \left(e^{\frac{-(x-x_0)^2}{4kt}} - e^{\frac{-(x+x_0)^2}{4kt}} \right) dx_0$$

$$+k \int_0^t \left[h(t) \frac{1}{\sqrt{4\pi k(t-t_0)}} \left(\frac{x}{k(t-t_0)} e^{\frac{-x^2}{4k(t-t_0)}} \right) dt_0 \right. \quad (2.20)$$

Result (2.20) provides the required solution to (2.14) – (2.16).

(b). Consider the one dimensional heat equation (2.14) with initial condition (2.15) and with Neumann boundary condition,

$$\frac{\partial}{\partial n} u(0,t) = h(t), t > 0. \quad (2.21)$$

In this case the Green's function $W(x,t;x_0,t_0)$ satisfies the boundary condition, that is,

$$\frac{\partial}{\partial n} W(0,t;x_0\,,t_0) = W_{x_0}(0,t;x_0,t_0) = 0.$$

In this case define

$$W(x,\mathrm{t};x_0,\mathrm{t}_0) = G(x,\mathrm{t};x_0,\mathrm{t}_0) + G(-x,\mathrm{t};x_0,\mathrm{t}_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}} \left(e^{\frac{-(x-x_0)^2}{4k(t-t_0)}} + e^{\frac{-(x+x_0)^2}{4k(t-t_0)}} \right).$$

Using (1.18), the solution of the equation (2.14) with Neumann boundary condition (2.21) is:

$$u(x,t) = \int_0^t \int_0^\infty \mathrm{W}(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^\infty u(x_0,0)W(x,t;x_0,0)dx_0$$

$$+ k \int_0^t \left[\oiint \left(W(x,t;x_0,t_0)\nabla_{X_0} u - u\nabla_{X_0} W(x,t;x_0,t_0) \right) \cdot \hat{n} dS_0 \right] dt_0$$

$$\Rightarrow u(x,t) = \int_0^t \int_0^\infty W(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^\infty u(x_0,0)W(x,t;x_0,0)dx_0$$
$$+ k\int_0^t \left[W(x,t;0,t_0)u_{x_0}(0,t) - u(0,t)W_{x_0}(x,t;0,t_0)\cdot((-\hat{\imath}))\right]dt_0$$

(for positive $X-$ axis, $\hat{n} = -\hat{\imath}$).

$$\Rightarrow u(x,t) = \int_0^t \int_0^\infty \mathrm{W}(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^\infty f(x_0)W(x,t;x_0,0)dx_0$$
$$- k\int_0^t h(t)W(x,t;0,t_0)\,dt_0 \qquad (2.22)$$

Result (2.22) provides the required solution to (2.14) along with (2.15) and (2.21).

We will see that the solution (2.20) and (2.22) can be applied to the problem, which has, $Q(x,t) = 0$ or $h(t) = 0$ or $f(x) = 0$ but not all of them are zero.

Example 2.4: Consider the heat equation

$$\frac{\partial u}{\partial t} - k\Delta^2 u = 0, 0 < x < \infty, t > 0,$$

with initial condition, $u(x,0) = u_0(u_0$ is a positive constant).

In this case $Q(x,t) = 0$, $h(t) = 0$ and $f(x) = u_0$.

Applying (2.20), we have

$$u(x,t) = \int_0^\infty f(x_0)\frac{1}{\sqrt{4\pi kt}}\left(e^{\frac{-(x-x_0)^2}{4kt}} - e^{\frac{-(x+x_0)^2}{4kt}}\right)dx_0$$

$$\Rightarrow u(x,t) = u_0\int_0^\infty \frac{1}{\sqrt{4\pi kt}}\left(e^{\frac{-(x-x_0)^2}{4kt}} - e^{\frac{-(x+x_0)^2}{4kt}}\right)dx_0$$

$$\Rightarrow u(x,t) = \frac{u_0}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{4kt}}\left(e^{\frac{-(x-x_0)^2}{4kt}} - e^{\frac{-(x+x_0)^2}{4kt}}\right)dx_0$$

Let $\eta = \frac{(x_0-x)}{\sqrt{4kt}}$ and $\beta = \frac{(x_0+x)}{\sqrt{4kt}}$, Then , then $d\eta = \frac{1}{2\sqrt{kt}}\,dx_0$ and $d\beta = \frac{1}{2\sqrt{kt}}\,dx_0$

Also, $x_0 = 0 \Rightarrow \eta = \frac{-x}{\sqrt{4kt}}$ and $\beta = \frac{x}{\sqrt{4kt}}$, $x_0 = \infty \Rightarrow \eta = \infty, \beta = \infty$. Hence,

$$u(x,t) = \frac{u_0}{\sqrt{\pi}}\left[\int_{\frac{-x}{\sqrt{4Kt}}}^{\infty} e^{-\eta^2} d\eta - \int_{\frac{x}{\sqrt{4Kt}}}^{\infty} e^{-\beta^2} d\beta\right]$$

Using the property

$$\text{erfc(x)} = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\eta^2} d\eta = 1 - \text{erf}(x),$$

where the error function is defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} d\eta, 0 \le x < \infty,$$

we have

$$\int_{\frac{-x}{\sqrt{4Kt}}}^{\infty} e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}\left(1 - \text{erf}\left(\frac{-x}{\sqrt{4kt}}\right)\right) \text{ and}$$

$$\int_{\frac{x}{\sqrt{4Kt}}}^{\infty} e^{-\beta^2} = \frac{\sqrt{\pi}}{2}\left(1 - \text{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right)$$

$$\Rightarrow u(x,t) = \frac{u_0}{2}\left[\text{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right) - \text{erf}\left(\frac{-x}{\sqrt{4kt}}\right)\right]$$

In addition, we have that $\text{erf}(-x) = -\text{erf}(x)$, hence the solution is

$$u(x,t) = u_0 \text{erf}\left(\frac{x}{\sqrt{4kt}}\right).$$

We shall now apply (2.20) to the homogeneous heat equation with a nonzero Dirichlet boundary condition and a zero initial condition.

Example 2.5: Consider the heat equation

$$\frac{\partial u}{\partial t} - k\Delta^2 u = 0, 0 < x < \infty, t > 0,$$

with initial condition, $u(x,0) = 0,$

with boundary condition $u(0,t) = u_1(u_1$ is a positive constant).

In this case $Q(x,t) = 0$, $h(t) = u_1$ and $f(x) = 0$.

Applying(2.20), we have

$$u(x,t) = k\int_0^t h(t)\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\frac{x}{k(t-t_0)}e^{\frac{-x^2}{4k(t-t_0)}}\right)dt_0$$

$$\Rightarrow u(x,t) = ku_1\int_0^t \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\frac{x}{k(t-t_0)}e^{\frac{-x^2}{4k(t-t_0)}}\right)dt_0$$

$$\Rightarrow u(x,t) = \frac{u_1}{\sqrt{4\pi k}}\int_0^t \frac{x}{(t-t_0)^{\frac{3}{2}}}e^{\frac{-x^2}{4k(t-t_0)}}\,dt_0$$

Let $\eta = \frac{x}{\sqrt{4k(t-t_0)}}$, then $d\eta = \frac{x}{4\sqrt{k}}(t-t_0)^{-\frac{3}{2}}\,dt_0$.

Also, $t_0 = 0 \Rightarrow \eta = \frac{x}{\sqrt{4kt}}$ and $t = t_0 \Rightarrow \eta = \infty$.

Hence, $u(x,t) = \frac{u_1}{\sqrt{4\pi k}}\int_0^t \left(\frac{x}{(t-t_0)^{\frac{3}{2}}}e^{\frac{-x^2}{4k(t-t_0)}}\right)dt_0$

$$\Rightarrow u(x,t) = \frac{2u_1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-\eta^2}\,d\eta$$

$$\Rightarrow u(x,t) = \frac{2u_1}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right)\left[1-\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right]$$

Hence the solution is

$$u(x,t) = u_1\left[1-\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right].$$

Example 2.6: Consider the heat equation

$$\frac{\partial u}{\partial t} - k\Delta^2 u = 0, 0 < x < \infty, t > 0,$$

satisfying the initial condition, $u(x, 0) = 0,$

and the Neumann boundary condition $\frac{\partial}{\partial n}u(x,t)\Big|_{x=0} = u_0(u_0$ a nonzero constant) .

We have $Q(x, t) = 0, f(x) = 0$ and $h(t) = u_0 \cdot$

Applying(2.22), gives

$$u(x,t) = -k\int_0^t h(t)w(x,t;0,t_0)\,dt_0$$

$$\Rightarrow u(x,t) = -k\int_0^t u_0 w(x,t;0,t_0)\,dt_0$$

$$\Rightarrow u(x,t) = -ku_0\int_0^t \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(e^{\frac{-(x)^2}{4k(t-t_0)}} + e^{\frac{-(x)^2}{4k(t-t_0)}}\right)dt_0$$

Then,

$$u(x,t) = -\frac{2ku_0}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{4k(t-\mathrm{t}_0)}}e^{-\frac{x^2}{4\kappa(t-\mathrm{t}_0)}}d\mathrm{t}_0$$

Let $\eta = \frac{x}{\sqrt{4k(t-t_0)}}$, then $d\eta = \frac{x}{4\sqrt{k}}(t-t_0)^{-\frac{3}{2}}\,dt_0$.

Also, $t_0 = 0 \Rightarrow \eta = \frac{x}{\sqrt{4kt}}$ and $t = t_0 \Rightarrow \eta = \infty$.

$$\Rightarrow u(x,t) = -\frac{ku_0}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-\eta^2}\frac{4(t-t_0)}{x}\;d\eta$$

$$\Rightarrow u(x,t) = -\frac{u_0 x}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-\eta^2}\frac{4k(t-t_0)}{x^2}\;d\eta$$

$$\Rightarrow u(x,t) = -\frac{u_0 x}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} \frac{1}{\eta^2} e^{-\eta^2} d\eta.$$

Integration by parts gives

$$u(x,t) = \frac{u_0 x}{\sqrt{\pi}} \left[\frac{e^{-\eta^2}}{\eta} \Bigg|_{\eta = \frac{x}{\sqrt{4kt}}}^{\eta = \infty} + 2\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-\eta^2} d\eta \right]$$

$$\Rightarrow u(x,t) = \frac{u_0 x}{\sqrt{\pi}} \left[-\frac{\sqrt{4kt}}{x} e^{-\frac{x^2}{4kt}} + 2\left(\frac{\sqrt{\pi}}{2}\right)\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right] \right]$$

$$\Rightarrow u(x,t) = \frac{u_0 x}{\sqrt{\pi}} \left[-\frac{\sqrt{4Kt}}{x} e^{-\frac{x^2}{4kt}} + \sqrt{\pi} - \sqrt{\pi}\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) \right].$$

2.4. Solving non-homogenous Heat equation on a Finite Domain

Using infinite space Green's function and the method of images, one can solve the heat equation on a finite region also.

Example 2.7: Consider the one- dimensional heat equation

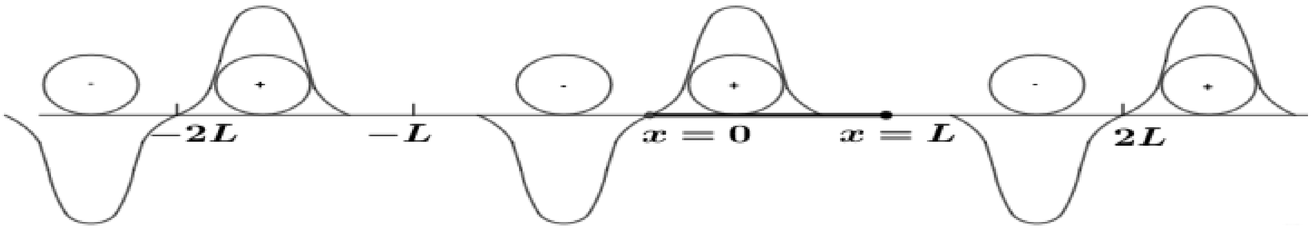
$$\frac{\partial u}{\partial t} = k\nabla^2 u + Q(x,t), 0 < x < L, t > 0, \tag{2.23}$$

With initial condition: $u(x,0) = f(x), 0 < x < L,$ (2.24)

and boundary conditions: $u(0,t) = h_1(t),\ u(L,t) = h_2(t).$ (2.25)

Again, the infinite space Green's function $G(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}} e^{\frac{-(x-x_0)^2}{4k(t-t_0)}}$ will not be zero at boundary points $x = 0$ and at $x = L$, that is, it will not satisfy homogeneous boundary conditions.

To make it zero at boundary points $x = 0$ and $x = L$, the method of image will be utilized. Due to symmetry, the homogeneous boundary conditions at $x = 0$ and at $x = L$ will be satisfied if positive concentrated sources, located at $x = x_0 + (2L)n$ and negative concentrated sources, located at $x = -x_0 + (2L)n$, for all integers $n, -\infty < n < \infty$, are added, as shown in figure below.



Multiple image sources for the Green's function for the Heat equation for a finite one-dimensional rod. Hence, now the resultant Green's function for a finite domain problem will be, as: (for a one-dimensional rod of length L)

$$W(x,t;x_0,t_0) = \sum_{n=-\infty}^{\infty} [G(2nL + x,t;x_0,t_0) - G(2nL - x,t;x_0,t_0)]$$

where G is

$$G(x,t;\ x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}} e^{-\frac{(x-x_0)^2}{4k(t-t_0)}}.$$

$$\Rightarrow W(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}} \sum_{n=-\infty}^{\infty} \left\{ e^{\frac{-(x-x_0-2Ln)^2}{4k(t-t_0)}} - e^{\frac{-(x+x_0-2Ln)^2}{4k(t-t_0)}} \right\}$$

So, now according to the expression (1.18),the solution of (2.23) will be

$$u(x,t) = \int_0^t \int_0^\infty W(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^\infty u(x_0,0)W(x,t;x_0,0)dx_0$$
$$+ k\int_0^t W(x,t;x_0,t_0)\frac{\partial u}{\partial x_0} - u\frac{\partial}{\partial x_0}W(x,t;x_0,t_0)\Big|_{x=0}^{x=L} dt_0$$

$$\Rightarrow u(x,t) = \int_0^t \int_0^L W(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^L f(x_0)W(x,t;x_0,0)dx_0$$
$$- k\int_0^t \left(u(L,t_0)\frac{\partial}{\partial x_0}W(L,t;x_0,t_0) - u(0,t_0)\frac{\partial}{\partial x_0}W(0,t;x_0,t_0) \right) dt_0$$

$$\Rightarrow u(x,t) = \int_0^t \int_0^L W(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 + \int_0^L f(x_0)W(x,t;x_0,0)dx_0$$
$$- k\int_0^t \left(h_2(t_0)\frac{\partial}{\partial x_0}W(L,t;x_0,t_0) - h_1(t_0)\frac{\partial}{\partial x_0}W(0,t;x_0,t_0) \right) dt_0 \qquad (2.26)$$

Substituting, $W(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}}\sum_{n=-\infty}^{\infty}\left\{e^{\frac{-(x-x_0-2Ln)^2}{4k(t-t_0)}} - e^{\frac{-(x+x_0-2Ln)^2}{4k(t-t_0)}}\right\}$, in (2.26), the required solution can be obtained.

CONCLUSION

In this project, we have shown applications of the Green’s function method for solving non-homogenous heat equation. The solution of heat equation is dependent not only on the equation, but also on the boundary conditions or initial conditions. We have solved a non-homogeneous heat equation defined on an infinite domain, a semi-infinite domain and finite domains using Green's function Method. The Green's function always satisfies related homogeneous boundary conditions. To deal with semi-infinite and finite domain problems with non-homogeneous boundary condition, using this method, it is required only to obtain a Green's function, which satisfies corresponding homogenous boundary condition.

REFERENCES

1. Abdisa, Lencha Tamiru. (2021). One Dimensional Heat Equation and its Solution by the Methods of Separation of Variables, Fourier Series and Fourier Transform.
2. Boyce, William E, DiPrima, Richard C, & Meade, Douglas B. (2017). Elementary differential equations: John Wiley & Sons.
3. Cochran, William T, Cooley, James W, Favin, David L, Helms, Howard D, Kaenel, Reginald A, Lang, William W, . . . Welch, Peter D. (1967). What is the fast Fourier transform? Proceedings of the IEEE, 55(10), 1664-1674.
4. Duffy, Dean G. (2015). Green's functions with applications: Chapman and Hall/CRC.
5. Hahn, David W, & Özisik, M Necati. (2012). Heat conduction: John Wiley & Sons.
6. Hildebrand, Tor, & Rüegsegger, Peter. (1997). A new method for the model-independent assessment of thickness in three-dimensional images. Journal of microscopy, 185(1), 67-75.
7. Lange, Kenneth. (1999). The Fourier Transform. Numerical Analysis for Statisticians, 221-234.
8. Narasimhan, Thiruppudaimarudhur N. (1999). Fourier's heat conduction equation: History, influence, and connections. Reviews of Geophysics, 37(1), 151-172.
9. Njogu, SAMMY KIHARA. (2009). Heat Equations And Their Applications (One And Two Dimension Heat Equations). University of NAIROBI.
10. Pfeffer, WF. (1986). The divergence theorem. Transactions of the American Mathematical Society, 295(2), 665-685.
11. Qin, Qing-Hua. (2010). Green's function and boundary elements of multifield materials: Elsevier.
12. Stakgold, Ivar, & Holst, Michael J. (2011). Green's functions and boundary value problems (Vol. 99): John Wiley & Sons.
13. Subani, Norazlina, Jamaluddin, Faizzuddin, Mohamed, Muhammad Arif Hannan, & Badrolhisam, Ahmad Danial Hidayatullah. (2020). Analytical solution of homogeneous one-dimensional heat equation with Neumann boundary conditions. Paper presented at the Journal of Physics: Conference Series.
14. Watanabe, Kazumi, & Watanabe, K. (2014). Integral transform techniques for green's function: Springer.
15. Wolff, Sebastian, & Bucher, Christian. (2013). Distance fields on unstructured grids: Stable interpolation, assumed gradients, collision detection and gap function. Computer methods in applied mechanics and engineering, 259, 77-92.