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Annihilator Ideals in Almost Semilattice

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A Project on

Annihilator Ideals in Almost Semilattice

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Annihilator Ideals in Almost Semilattice

A Project Submitted to the Departhement of Mathematics in Partial Fulfilment of the Requirements for the Degree of Masters of Science in Mathematics.

By

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Advisor: Berhanu Assaye (PhD)

September 2021

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I here by certify that I have supervised, read, and evaluated this

The Project Entitled “Annihilator Ideals in Almost Semi lattice” by Azanaw Tafere Prepared under my guidance. I recommend that the project be Submitted for oral defense.

Advisor name: Berhanu Assaye (PhD)

Signature--------------------

Date-----------------------------

The Project Entitled "Annihilator Ideals in Almost Semi lattice" by Azanaw Tafere is approved for the Degree of "Masters of Sciences in Mathematics"

Board of Examiners

Name	**Signature**	**Date**
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Internal Examiner: --	---------------	-----------
External Examiner: ---------------------------------------	--------------	-----------

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Abstract

The concept of annihilator ideal is introduced in an Almost Semi lattice (ASL) L with 0. It is proved that the set of all annihilator ideals of an ASL L with 0 forms a complete Boolean algebra. The concept of annihilator preserving homomorphism is introduced in an ASL L with 0. A sufficient condition for a homomorphism to be annihilator preserving is derived. Finally, it is proved that the homomorphism image and the inverse image of an annihilator ideal are again annihilator ideals.

Chapter One

1.1 Introduction

In[9] Thomas W. Judson Stephen F. Austin State University Theory and Applications on Abstract Algebra (2011) 302 -313 introduces about the axioms of a ring give structure to the operations addition and multiplication on a set. However, we can construct algebraic structures, known as lattice and Boolean algebras that generalize other types of operations. For example, the important operations on a set are inclusions, union, and intersection. Lattices are generalizations of order relations on algebraic spaces, such as set inclusion in a set theory and inequality in the familiar number systems N, Z, Q, and R. Boolean algebras generalize the operations of intersection and union.

Let us investigate the example of the power set, P(X), of a set X more closely. The power set is a lattice that is ordered by inclusion. By the definition of the power set, the largest element in P(X) is X itself and the smallest element is $\emptyset$, the empty set. For any set A in P(X), we know that $A \cap X = A$ and $A \cup \emptyset = A$. This suggests the following definition for lattices. An element I in a poset X is a largest element if $a \leq I$ for all $a \in X$. an element o is smallest element of X if $o \leq a$ for all $a \in X$.

Let A be in P(X), Recall that the complement of A is

$A' = X \backslash A = \{x : x \in X \text{ and } x \notin A\}$.

We know that $A \cup A' = X$ and $A \cap A' = \emptyset$. We can generalize this example for lattices. A lattice L with largest element I and smallest element o is complemented if for each $a \in X$, there exist a' such that $a \vee a' = I$ and $a \wedge a' = o$.

A complemented bounded distributed lattice is Boolean algebra. For example the power set, P(X) of a set X is Boolean algebra with the operations union and intersection that is $(P(X), \cup, \cap, ', 0, 1)$, where $'$,0,1 represents a complement, zero element, and unit element respectively. $\emptyset$ is zero elements of P(X) and X itself is unit element.

There is only one reasonable way of defining what is to be meant by an ideal in a lattice. Recall that, Dedekind's definition of an ideal in a ring R is that it is a collection J of elements of R which (1) contains all multiples such as ax or ya of any of its elements a, and (2) contains the difference a-b, and hence the sum a+b, of any two of its elements a and b. By analogy, a collection J of elements of a lattice L is called an ideal if (1) it contains all multiples $a \cap x$ of any of its elements, and (2) it contains the lattice sum $a \cup b$ of any two of its elements a and b. The analogy is that the greatest lower bound, or lattice meet $a \cap b$ corresponds to product in a ring, and the least upper bound or lattice join $a \cup b$ corresponds to the sum of two elements in a ring. A Semi lattice is an algebra that satisfies the three axioms i.e. associative commutative and idempotent law. An Almost Semi lattice (ASL) was introduced by authors as an algebra (L,o) of type (2) which satisfies all most all the properties of semi lattice except possibly the commutative of o. In this paper, the concept of annihilator ideal in ASL with 0 is introduced with suitable examples and proved some basic properties of the annihilator ideal. Also proved that the set $\mathcal{A}$ (L), of all annihilator ideals of an ASL L with 0 is a complete Boolean algebra. The concept of annihilator preserving homomorphism is introduced and a sufficient condition for a homomorphism to be annihilator preserving is derived. It is proved that the image and the inverse image under a homomorphism of an annihilator ideal are again annihilator ideals. Finally, it is proved that for any ideal I of L, there exists a homomorphism f from L in to an ASL $Hom_L(I)$, of all homomorphism defined on I such that Ker(f) = I^*.

1.2 Preliminary

Definition1.2.1 [6] A Semi lattice is an algebra (S , $*$) where S is a non-empty set and $*$ is a binary operation on S satisfying:

(1) $x * (y * z) = (x * y) * z$ (associative law)

(2) $x * y = \ y * x$ (commutative law)

(3) $x * x = x$ for all $x, y, z \in S$ (idempotent law)

In other words, a semi lattice is an idempotent commutative semi group. The symbol $*$ can be replaced by any binary operation symbol, and in fact we use one of the symbols of $\wedge$,$\vee$,+ or., depending on the setting. The most natural example of a semi lattice is (p(x), ∩),or more generally any collection of subsets of X closed under intersection. A sub semi lattice of a semi lattice (S, $*$) is a sub set of S which is closed under the operation $*$. A homomorphism between two semi lattices $(S, *)$ and $(T, *)$ is a map h: S → T with the property that h(x $*$ y) =h(x) $*$ h(y) for all x ,y ∈ S. An isomorphism between two semi lattices is a homomorphism that is one to one and on to. It is worth nothing that because the operation is determined by the order and vice versa. Also, it can be easily observed that two semi lattices are isomorphic if and only if they are isomorphic as ordered sets.

Definition 1.2.2[3] An algebra (L, o) of type (2) is called almost semi lattice if it satisfies the following axioms for all x, y, z ∈L

(AS_1) x o (y o z) = (x o y) o z (associative law)

(AS_2) (x o y) o z = (y o x) o z (almost commutative law)

(AS_3) x o x = x (idempotent law)

If L has an element 0 and satisfies 0 o x = 0 along with the above properties, then L is called an ASL with 0.

Note Every semi lattice (S, o) is an ASL

Example 1.2.1 Let L = {a, b, c}. Define a binary operation o on L as below

o	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

This is an example of an ASL, but not a semi lattice, since b o c = c ≠ b = c o b.

Theorem 1.2.3[3] Let L be an ASL. Define $a \leq$ b if and only if a o b = a for all a, b ∈ L. Then ≤ is a partial ordering on L.

Definition 1.2.4[8] A partial order set (also called a poset) is a set P equipped with a binary relation ≤ which is a partial order on X; i.e ≤ satisfies the following three properties:

(1) If x ∈ P , then x ≤ x in P (reflexive property)
(2) If x, y, z ∈ P, x ≤ y in P and y ≤ x in P, then x = y (anti symmetric property)
(3) If x, y, z ∈ P, x ≤ y in P and y ≤ z in P , then x ≤ z in P (transitive property)

Lemma 1.2.5[3] Let L be an ASL with 0. Then we have the following properties.

(1) a o (a o b) = a o b
(2) (a o b) o b = a o b
(3) b o (a o b) = a o b
(4) a o b = b o a whenever $a \leq$ b
(5) a is a minimal element of L if and only if x o a = a for all x ∈ L
(6) a o 0 = 0 for all a ∈ L
(7) a o b = 0 if and only if b o a = 0
(8) $a \leq$ b implies that a o x ≤ b o x and x o a ≤ x o b

Definition1.2.6[4] A nonempty subset I of an ASL L is said to be an ideal if x ∈ I and a ∈ L, then x o a ∈ I.

Definition 1.2.7[4] A proper ideal P of an almost semi lattice L is said to be prime ideal if for any a, b ∈ L, such that a o b ∈ P, then either a ∈ P or b ∈ P.

Theorem 1.2.8[4] Let S is a non-empty subset of L. Then (S] = $\{(\mathrm{o}^{n}_{i=1}\ s_i)\ \mathrm{o}\ x \mid x \in L,\ s_i \in S$, where $1 \leq i \leq n$ and n is a positive integer $\}$ is the smallest ideal of L containing S.

Corollary 1.2.9[4] Let L is an ASL and a ∈ L. Then (a] = {a o x | x ∈ L} is an ideal of L, and is called principal ideal generated by a.

Lemma 1.2.10[4] For any a, b in an ASL L we have the following:

(1) $a \in (b]$ if and only if $a = b \circ a$
(2) $b \in (a]$ if and only if $(b] \subseteq (a]$
(3) $(a] \subseteq (b]$ whenever $a \leq b$
(4) $(b \circ a] = (a \circ b] = (a] \cap (b]$

Corollary1.2.11 [4] Let I be an ideal of L. Then, for any a, b $\in$ L, $a \circ b \in$ I if and only if $b \circ a \in$ I.

Definition 1.2.12[3] Let L is a non-empty set. Define a binary operation o on L by x o y = y, for all x, y $\in$ L. Then (L, o) is an ASL and is called discrete ASL.

Definition1.2.13 [3] An element m $\in$ L is said to be unimaximal if m o x = x for all x $\in$ L.

Theorem1.2.14 [3] Every unimaximal element in an ASL L is a maximal element.

Definition1.2.15 [8] A lattice is an algebra (L, $\wedge$, $\vee$) of type (2, 2) satisfying for all x, y, z $\in$ L

(1) $x \wedge x = x$ and $x \vee x = x$ (idempotent law)
(2) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (commutative law)
(3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associative law)
(4) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ (absorption law)

The first three pairs of axioms say that L is both meet and join semi lattice the fourth called the absorption laws say that both operation induce the same order L. The lattice operations are sometimes denoted by. and +; for the sake of consistency we will stick such that $x \wedge y = x$ if and only if $x \vee y = y$ the binary operations $\wedge$, $\vee$ are represents meet and join respectively.

Theorem 1.2.16[8] In any lattice (L,$\vee$, $\wedge$) for any x, y, z $\in$ L the following are equivalent

(LD$\wedge$) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(RD$\wedge$) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(LD$\vee$) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(RD$\vee$) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

Definition 1.2.17[8] A lattice (L, $\vee$, $\wedge$) is called distributive lattice if it satisfies any one of the above four conditions.

Special Notations in Lattice

Here we define some special types of elements in lattices

Zero element: In a lattice L an element 0 is called the zero element of L if $0 \leq a$, for every $a \in$ L.

Unit element: An element 1 is called the unit element or the all element of L if $a \leq 1$, for every $a \in$ L

. **Bounded lattice:** A lattice L with 0, 1 is called a bounded lattice

Theorem 1.2.18[4] The set J (L), of all ideals of an ASL is a distributive lattice with respect to set inclusion, where for any I, J $\in$J (L), I $\wedge$ J = I $\cap$ J and I $\vee$ J = I $\cup$ J.

Theorem 1.2.19[8] Let (P, $\leq$) be a poset which is bounded above. If every nonempty subset of P has glb, then every nonempty subset of P has lub and hence P is a complete lattice.

Definition 1.2.20[8] A complemented distributive lattice is called a Boolean algebra.

Chapter Two

2.1Annihilators

In this section, we introduce the concept of an annihilator ideal in an almost semi lattice (ASL) L with 0 and prove some basic properties of the annihilator ideals. Also, we prove that the set $\mathcal{A}$(L), of all annihilator ideals form a complete Boolean algebra. First we begin with the following definition.

Throughout the remaining of this section, by L we mean an ASL with 0 unless otherwise specified.

Definition 2.1.1: For any non-empty subset A of an ASL L with 0 define $A^* = \{x \in L \mid x \circ a = 0$, for all $a \in A\}$. Then A^* is called the annihilator of A.

Note that, if $A = \{a\}$, then we denote $A^* = \{a\}^*$ by $[a]^*$. In the following we prove that for any non-empty subset A of L, A^* is an ideal.

Theorem 2.1.2: For any non-empty subset A of L, A^* is an ideal of L.

Proof: Since $a \circ 0 = 0$ for all $a \in A$ (by definition 1.2.5(6))

Then $0 \in A^*$ hence A^* is non-empty,

Let $x \in A^*$ and $t \in L$. Then $x \circ a = 0$ for all $a \in A$, (by definition2.1.1)

Now, let $b \in A$, then $(x \circ t) \circ b = (t \circ x) \circ b = t \circ (x \circ b) =$

$t \circ 0 = 0$ (bydefinition1.2.2 (1)and(2))

Therefore $x \circ t \in A^*$, (bydefinition2.1.1)

Thus A^* is an ideal of L. (by definition1.2.6)

Lemma2.1.3: For any subset A of L, $A \cap A^* = \{0\}$.

Proof: Suppose A is a subset of L and suppose $x \in A \cap A^*$,

Then $x \in A$ and $x \in A^*$, therefore $x \circ a = 0$, for all $a \in A$, (by definition 2.1.1)

It follows that $x = x \circ x = 0$. Therefore $A \cap A^* = \{0\}$.

Theorem2.1.4: For any ideals I, J of L, we have the following:

(1) $I^* = \bigcap_{a \in I} (a]^*$

(2) $(I \cap J)^* = (J \cap I)^*$

(3) $I \subseteq J \Rightarrow J^* \subseteq I^*$

(4) $I^* \cap J^* \subseteq (I \cap J)^*$

(5) $(I \cap J)^{**} = I^{**} \cap J^{**}$

(6) $I \subseteq I^{**}$

(7) $I^{***} = I^*$

(8) $I^* \subseteq J^* \Leftrightarrow J^{**} \subseteq I^{**}$

(9) $I \cap J = (0] \Leftrightarrow I \subseteq J^* \Leftrightarrow J \subseteq I^*$

(10) $(I \cup J)^* = I^* \cap J^*$

Proof:

(1) Let $t \in I^*$. Then $t \circ a = 0$ for all $a \in I$, hence $t \in (a]^*$ for all $a \in I$, therefore $t \in \bigcap_{a \in I} (a]^*$, thus, $I^* \subseteq \bigcap_{a \in I} (a]^*$ ------------ (1)

Conversely let $t \in \bigcap_{a \in I} (a]^*$, then $t \in (a]^*$, therefore $t \circ a = 0$, since $a \in I$, then $t \in I^*$, this implies $\bigcap_{a \in I} (a]^* \subseteq I^*$. ----------- (2)

Hence from (1) and (2) $I^* = \bigcap_{a \in I} (a]^*$.

(2) Since intersection of a set is commutative that is $I \cap J = J \cap I$, Then $(I \cap J)^* = (J \cap I)^*$.

(3) Suppose $I \subseteq J$ and $x \in J^*$ then $x \circ a = 0$ for all $a \in J$. Hence $x \circ a = 0$ for all $a \in I$, thus $x \in I^*$, and hence $J^* \subseteq I^*$.

(4) Since $I \cap J \subseteq I, J$ by (3) we get $I^*, J^* \subseteq (I \cap J)^*$, therefore $I^* \cap J^* \subseteq (I \cap J)^*$.

(5) Let $I, J \in J(L)$. Then we have $I \cap J \subseteq I, J$. Hence by (3) we get $I^*, J^* \subseteq (I \cap J)^*$. It follows that $(I \cap J)^{**} \subseteq I^{**}, J^{**}$. Thus $(I \cap J)^{**} \subseteq I^{**} \cap J^{**}$.-------- (1)

Conversely, let $x \in I^{**} \cap J^{**}$ and $y \in (I \cap J)^*$, then for any $i \in I$ and $j \in J$, we have $i \circ j \in I \cap J$. Hence $(y \circ i) \circ j = y \circ (i \circ j) = 0$, therefore $y \circ i \in J^*$, again, since x

$\in J^{**}$ and $y \circ i \in J^{*}$ we get $(x \circ y) \circ i = x \circ (y \circ i) = 0$, hence $x \circ y \in I^{*}$. Since $x \in I^{**}$, we get $x \circ y \in I^{**}$. Thus $x \circ y \in I^{*} \cap I^{**} = \{0\}$, hence $x \circ y = 0$, therefore $x \in (I \cap J)^{**}$ thus $I^{**} \cap J^{**} \subseteq (I \cap J)^{**}$------- (2)

Hence from (1) and (2) $(I \cap J)^{**} = I^{**} \cap J^{**}$.

(6) Suppose $x \in I$ and $y \in I^{*}$. Then $y \circ a = 0$ for all $a \in I$, in particular, $y \circ x = 0$, hence $x \in I^{**}$,thus $I \subseteq I^{**}$.

(7) Suppose $x \in I^{*}$ and $a \in I^{**}$, then $x \circ a \in I^{*} \cap I^{**} = \{0\}$, hence $x \circ a = 0$, therefore $x \in I^{***}$.

This implies $I^{*} \subseteq I^{***}$----------- (1)

Conversely $I \subseteq I^{**}$ by (6) and $I^{***} \subseteq I^{*}$ by (3) ---------------- (2)

Therefore from (1) and (2) $I^{*} = I^{***}$.

(8)Suppose $I^{*} \subseteq J^{*}$, now by (3) $J^{**} \subseteq I^{**}$.

Again by (3) $I^{***} \subseteq J^{***}$ and by (7) $I^{*} = I^{***}$, thus $I^{*} \subseteq J^{*}$.

(9)Suppose $I \cap J = (0]$, let $x \in I$ and $a \in J$, then we get $x \circ a \in I$ and $x \circ a \in J$, hence $x \circ a \in I \cap J = (0]$. Therefore $x \circ a = 0$, *it follows that* $x \in J^{*}$, *thus* $I \subseteq J^{*}$.

Conversely, suppose $I \subseteq J^{*}$, let $x \in I \cap J$, then $x \in I$ and $x \in J$, since $I \subseteq J^{*}$, $x \in J^{*}$, it follows that $x \circ x = 0$, therefore $x = 0$,thus $I \cap J = (0]$.

And suppose $I \cap J = (0]$, let $x \in J$ and $a \in I$, then we get $x \circ a \in J$ and $x \circ a \in I$, hence $x \circ a \in I \cap J = (0]$, therefore $x \circ a = 0$, it follows that $x \in I^{*}$, thus $J \subseteq I^{*}$.

Conversely, suppose $J \subseteq I^{*}$, let $x \in I \cap J$, then $x \in I$ and $x \in J$, since $J \subseteq I^{*}$, $x \in I^{*}$, it follows that $x \circ x = 0$, therefore $x = 0$, thus $I \cap J = (0]$.

(10)We have $I, J \subseteq I \cup J$, therefore by (3), we get $(I \cup J)^{*} \subseteq I^{*}, J^{*}$, hence $(I \cup J)^{*} \subseteq I^{*} \cap J^{*}$------ (1)

Conversely, let $x \in I^* \cap J^*$, then $x \in I^*$ and $x \in J^*$, hence $x \circ a = 0$, for all $a \in I$ and $x \circ b = 0$, for all $b \in J$, therefore $x \circ t = 0$, for all $t \in I \cup J$ and hence $x \in (I \cup J)^*$

Thus $I^* \cap J^* \subseteq (I \cup J)^*$------------(2)

Therefore from (1) and (2) $I^* \cap J^* = (I \cup J)^*$.

Corollary 2.1.5 If $\{I_i \mid i \in \Delta\}$ is a family of ideals of L, then $(\bigcap_{i \in \Delta} I_i)^{**} = \bigcap_{i \in \Delta} (I_i)^{**}$

Proof: by theorem 2.1.4(5) $(I \cap J)^{**} = I^{**} \cap J^{**}$, this implies that $(I_1 \cap I_2 \cap I_3 \ldots)^{**} = I_1^{**} \cap I_2^{**} \cap I_3^{**} \ldots$, therefore $(\bigcap_{i \in \Delta} I_i)^{**} = \bigcap_{i \in \Delta} (I_i)^{**}$

Theorem2.1.6: For any $x, y \in L$, we have the following:

1) $x \leq y \Rightarrow [y]^* \subseteq [x]^*$
2) $[x]^* \subseteq [y]^* \Rightarrow [y]^{**} \subseteq [x]^{**}$
3) $x \in [x]^{**}$
4) $(x]^* = [x]^*$
5) $(x] \cap [x]^* = \{0\}$
6) $[x \circ y]^* = [y \circ x]^*$
7) $[x]^* \cap [y]^* \subseteq [x \circ y]^*$
8) $[x \circ y]^{**} = [x]^{**} \cap [y]^{**}$
9) $[x]^{***} = [x]^*$
10) $[x]^* \subseteq [y]^*$ if and only if $[y]^{**} \subseteq [x]^{**}$

Proof:

1) Suppose $x, y \in L$ such that $x \leq y$ and $t \in [y]^*$, then $t \circ y = 0$.

   Since $x \leq y$, we get $t \circ x \leq t \circ y = 0$. Lemma 1.2.5(8)

   This implies $t \circ x = 0$ and hence $t \in [x]^*$, thus $[y]^* \subseteq [x]^*$.

2) Suppose $[x]^* \subseteq [y]^*$, let $s \in [y]^{**}$, then $s \circ a = 0$ for all $a \in [y]^*$, hence $s \circ a = 0$ for all $a \in [x]^*$, therefore $s \in [x]^{**}$, thus $[y]^{**} \subseteq [x]^{**}$.

3) Let $t \in [x]^*$, then $t \circ x = 0$ and since $t \circ x = x \circ t = 0$ (by lemma 1.2.5 (7), and since $[x]^{**}$ is annihilator ideals of $[x]^*$, then $x \in [x]^{**}$.
4) Let $t \in (x]^*$, then $t \circ s = 0$ for all $s \in (x]$, in particular $t \circ x = 0$, since $x \in (x]$, hence $t \in [x]^*$, therefore $(x]^* \subseteq [x]^*$-----------(1)

Conversely, suppose $t \in [x]^*$, then $t \circ x = 0$, let $s \in (x]$, then $x \circ s = s$, now, $t \circ s = t \circ (x \circ s) = (t \circ x) \circ s = 0 \circ s = 0$, hence $t \circ s = 0$ for all $s \in (x]$, therefore $t \in (x]^*$,

Hence $[x]^* \subseteq (x]^*$-------- (2)

Therefore from (1) and (2) $[x]^* = (x]^*$.

5) Let $t \in (x] \cap [x]^*$, then $t \in (x]$ and $t \in [x]^*$, hence $x \circ t = t$ and $t \circ x = 0$, hence $t = 0$, thus $(x] \cap [x]^* = \{0\}$.
6) Let $t \in [x \circ y]^*$, then $t \circ (x \circ y) = 0$ and since $t \circ (x \circ y) = t \circ (y \circ x)$ by definition 1.2.2(2) $t \circ (y \circ x) = 0$, then $t \in [y \circ x]^*$, this implies $[x \circ y]^* \subseteq [y \circ x]^*$----------(1)

Conversely let $t \in [y \circ x]^*$, then $t \circ (y \circ x) = 0$ and since $t \circ (y \circ x) = t \circ (x \circ y)$ by definition 1.2.2(2) $t \circ (x \circ y) = 0$, then $t \in [x \circ y]^*$, this implies $[y \circ x]^* \subseteq [x \circ y]^*$-----(2)

Therefore from (1) and (2) $[x \circ y]^* = [y \circ x]^*$.

7) Suppose $t \in [x]^* \cap [y]^*$, then $t \in [x]^*$ and $t \in [y]^*$, therefore $t \circ x = 0$ and $t \circ y = 0$ and hence $t \circ (x \circ y) = 0$, it follows that $t \in [x \circ y]^*$, thus $[x]^* \cap [y]^* \subseteq [x \circ y]^*$.
8) Let $x, y \in L$, then we have $x \circ y \leq y$ and $y \circ x \leq x$, therefore by (1), we get $[y]^* \subseteq [x \circ y]^*$ and $[x]^* \subseteq [y \circ x]^* = [x \circ y]^*$, hence $[x]^*, [y]^* \subseteq [x \circ y]^*$, therefore by (2), $[x \circ y]^{**} \subseteq [x]^{**}, [y]^{**}$, and hence $[x \circ y]^{**} \subseteq [x]^{**} \cap [y]^{**}$----------------(1)

Conversely suppose $t \in [x]^{**} \cap [y]^{**}$ and $s \in [x \circ y]^*$, then $t \in [x]^{**}$, $t \in [y]^{**}$ and $s \circ (x \circ y) = 0$, it follows that $s \circ x \in [y]^*$, since $t \in [y]^{**}$, we get $t \circ (s \circ x) = 0$, it follows that $t \circ s \in [x]^*$, now, $t \circ s = (t \circ t) \circ s = t \circ (t \circ s) = 0$, since $t \in [x]^{**}$, therefore $t \in [x \circ y]^{**}$, hence $[x]^{**} \cap [y]^{**} \subseteq [x \circ y]^{**}$---------------------(2)

Therefore from (1) and (2) $[x \circ y]^{**} = [x]^{**} \cap [y]^{**}$.

9) Let $t \in [x]^*$ and $s \in [x]^{**}$, then $t \circ s = [x]^* \cap [x]^{**} = \{0\}$, hence $t \in [x]^{***}$, therefore $[x]^* \subseteq [x]^{***}$--------------------(1)
conversely let $s \in [x]^{***}$, then $s \circ x = 0$ since $x \in [x]^{**}$ by (3), thus $s \in [x]^*$ since $s \circ x = 0$, therefore $[x]^{***} \subseteq [x]^*$---------------------(2)

This implies from (1) and (2) $[x]^{***} = [x]^*$.

10) Suppose $[x]^* \subseteq [y]^*$, let $s \in [y]^{**}$, then $s \circ a = 0$ for all $a \in [y]^*$, hence $s \circ a = 0$ for all $a \in [x]^*$, therefore $s \in [x]^{**}$, thus $[y]^{**} \subseteq [x]^{**}$.

Conversely suppose $[y]^{**} \subseteq [x]^{**}$, let $t \in [x]^{***}$, then $t \circ a = 0$ for all $a \in [x]^{**}$, hence $t \circ a = 0$ for all $a \in [y]^{**}$, thus $t \in [y]^{***}$, therefore $[x]^{***} \subseteq [y]^{***}$ and since $[x]^{***} = [x]^*$ and $[y]^{***} = [y]^*$ by(9),then $[x]^* \subseteq [y]^*$

Recall that M_o is the least element in the distributive lattice J (L) of all ideals of L which contains precisely all minimal elements in L. In the following, we define annihilator of a non-empty set in another form.

Definition2.1.7 For any non-empty subset S of L, define $[S]^* = \{x \in L \mid x \circ s \in M_o \text{ for all } s \in S\}$.

Theorem 2.1.8 Let L is an ASL with a minimal element, then, for any non-empty subset S of L, $[S]^*$ is an ideal of L.

Proof: Suppose L has a minimal element, since S is non-empty subset of L and L has minimal element by hypothesis, then $[S]^*$ is non-empty, let $x \in [S]^*$ and $t \in L$, then $x \circ s \in M_o$ for all $s \in S$, let $s \in S$, now, consider $(x \circ t) \circ s = (t \circ x) \circ s = t \circ (x \circ s) = x \circ s$, since $x \circ s \in M_o$ which is minimal.

Hence $(x \circ t) \circ s \in M_o$, therefore $x \circ t \in [S]^*$, thus $[S]^*$ is an ideal of L.

Corollary 2.1.9 For any non-empty set S of L, $[S]^* = [(S]]^*$, where (S] is an ideal generated by S.

Proof: Let $x \in [S]^*$, $x \circ s \in M_o$ for all $s \in S$, then $x \in [(S]]^*$ since $s \in S$, then $[S]^* \subseteq [(S]]^*$ -------------- (1)

Conversely let $x \in [(S]]^*$, then $x \circ s \in M_o$ for all $s \in S$, then $x \in [S]^*$, because $s \in S$, Then $[(S]]^* \subseteq [S]^*$ ----------------- (2)

Therefore from (1) and (2) $[(S]]^* = [S]^*$.

Lemma2.1.10 Let L be an ASL with 0, then for any a, b $\in$ L, we have the following

(1) $x \in [a]^* \Leftrightarrow (x] \in \{(a]\}^*$
(2) $[a]^* = [b]^* \Leftrightarrow \{(a]\}^* = \{(b]\}^*$

Proof: (1) Suppose $x \in [a]^* \Leftrightarrow x \circ a = 0 \Leftrightarrow (x] \cap (a] = (0] \Leftrightarrow (x] \in \{(a]\}^*$.

Conversely let $(x] \in \{(a]\}^* \Leftrightarrow (x] \cap (a] = (0] \Leftrightarrow x \circ a = 0 \Leftrightarrow x \in [a]^*$

2) Suppose $[a]^* = [b]^*$, then $(x] \in \{(a]\}^* \Leftrightarrow (x] \cap (a] = (0]$

$\Leftrightarrow x \circ a = 0$

$\Leftrightarrow x \in [a]^*$

$\Leftrightarrow x \in [b]^*$

$\Leftrightarrow x \circ b = 0$

$\Leftrightarrow (x \circ b] = (0]$

$\Leftrightarrow (x] \cap (b] = (0]$

$\Leftrightarrow (x] \in \{(b]\}^*$

Therefore $\{(a]\}^* = \{(b]\}^*$.

Conversely, suppose $\{(a]\}^* = \{(b]\}^*$, then

$x \in [a]^* \Leftrightarrow (x] \in \{(a]\}^*$

$\Leftrightarrow (x] \in \{(b]\}^*$

$\Leftrightarrow (x] \cap (b] = (0]$

$\Leftrightarrow (x \circ b] = (0]$

$\Leftrightarrow x \circ b = 0$

$\Leftrightarrow x \in [b]^*$

Therefore $[a]^* = [b]^*$.

Corollary2.1.11 Let L is an ASL with a minimal element, then for any non-empty subset of L, $(S] \cap [S]^* = M_o$.

Proof: we have M_o is the least element in J (L), therefore $M_o \subseteq (S] \cap [S]^*$------ (1)

Conversely, let $t \in (S] \cap [S]^*$, then $t \in (S]$ and $t \in [S]^* = [(S]]^*$, thus $t \circ s \in M_o$ for all $s \in (S]$, in particular $t = t \circ t \in M_o$, since $t \in (S]$, therefore $t \in M_o$ and hence $(S] \cap [S]^* \subseteq M_o$------(2)

Thus from (1) and (2) $M_o = (S] \cap [S]^*$.

Now, we define the concept of annihilator ideal in an ASL L with 0.

Definition 2.1.12 Let L be an ASL with 0, an ideal I of L is called an annihilator ideal if $I = S^*$ for some non-empty subset S of L.

It can be easily seen that if I is annihilator ideal, then $I = I^{**}$. Note that, the set of all annihilator ideals of L is denoted by $\mathcal{A}$ (L). In the following, we give some examples of annihilator ideals.

Example2.1.1 Let X be a discrete ASL with 0 and with at least two elements, other than 0. Then $(X^n, \circ, 0')$ is an ASL with zero $0' = (0, 0, . . ., 0)$, where o defined coordinate- wise. Put, $I = \{(0 \ . \ . \ ._i,\ldots, 0) \mid a_i \in X\}$. Then I is an ideal of X^n. Also, $I^* = \{(a_1, a_2,\ldots,a_{i-1}, 0, a_{i+1},\ldots,a_n) \mid a_i \in X\}$ and $I^{**} = \{(0,\ldots,a_i,\ldots,0) \mid a_i \in X\} = I$. Hence I is an annihilator ideal of L.

Example2.1.2 Let (R, +, . , 0) be a commutative ring with unity. For any $a \in L$, let a^0 be the unique idempotent element in L such that $aR = a^0R$. For any $x, y \in R$, define $x \circ y = x^0y$.

Then (R, +, ., 0) is ASL with 0.

Now, consider I = $(x^0]$ and J = $(1- x^0]$. Since x^0 o $(1- x^0) = 0$, we get that $(x^0] \subseteq (1- x^0]^*$ and $(1- x^0] \subseteq (x^0]^*$. Now, $a \in (x^0]^*$ implies a o $x^0 = 0$. So $a^0x^0 = 0$, Now, $(1- x^0) = a - ax^0 = a - 0 = a$, hence $a \in (1 - x^0]$, thus $(x^0]^* \subseteq (1 - x^0]$ = J. and $J^* = (1 - x^0]^* = (x^0]$ = I. Hence I and J are the annihilator ideals in L.

Example2.1.3 Let L = {0, a, b, c} and defined o on L as follows:

o	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then (L, o, 0) is an ASL with 0. Consider the set I = $\{0, a\} \subseteq$ L, then I is an ideal in L, now, $I^* = \{0, c\}$ and also $I^{**} = \{0, a\}$ = I. Thus I is an annihilator ideal in L. Similarly consider the set J = $\{0, c\} \subseteq$ L, then J is an ideal of L, now $J^* = \{0, a\}$ and also $J^{**} = \{0, c\}$ = J, thus J is another annihilator ideal in L.

Example2.1.4 Let L = {0, a, b, c} and defined o on L as follows:

o	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then (L,o, 0) is an ASL with 0. Consider the ideal I = {0, c}, then $I^{**} = (0]^*$ = L, therefore I is not an annihilator ideal in L.

In the following, we prove some properties of annihilator ideals.

Theorem2.1.13 For I, J $\in \mathcal{A}(L)$, we have $I \cap J = (I^* \cup J^*)^*$.

Proof: Since $I^*, J^* \subseteq I^* \cup J^*$, we get $(I^* \cup J^*)^* \subseteq I^{**}, J^{**}$ by theorem 2.1.4(3)

And since $I = I^{**}$ and $J = J^{**}$, this implies $(I^* \cup J^*)^* \subseteq$ I, J.

Therefore $(I^* \cup J^*)^* \subseteq I \cap J$---------------------- (1)

Conversely, suppose $x \in I \cap J$ and $y \in I^* \cup J^*$, then $y \in I^*$ or $y \in J^*$.

Since $x \in I \cap J$ and $y \in I^* \cup J^*$, $x \circ y \in I \cap J^*$ or $x \circ y \in J \cap J^*$.

It follows that $x \circ y = 0$.

Therefore $x \in (I^* \cup J^*)^*$ and hence $I \cap J \subseteq (I^* \cup J^*)^*$------------- (2)

Thus from (1) and (2) $I \cap J = (I^* \cup J^*)^*$.

Theorem 2.1.14 Let L be an ASL, then for any ideal of L, $I^e = \{(a] \mid a \in I\}$ is an ideal of PJ (L), moreover, I is prime if and only if I^e is prime

Proof: Suppose I is an ideal of L, then $I^e = \{(a] \mid a \in I\}$, now, we shall prove that I^e is an ideal of PJ (L), since I is nonempty, it follows that I^e is nonempty, let $(a] \in I^e$ and $(t] \in$ PJ (L), then $a \in I$ and $t \in L$, therefore $a \circ t \in I$, hence $(a] \circ (t] = (a \circ t] \in I^e$, thus I^e is an ideal of PJ (L).

Suppose I is a prime ideal of L, we shall prove that I^e is a prime ideal of PJ (L), let $(a], (b] \in$PJ (L) such that $(a] \circ (b] \in I^e$, then $(a \circ b] \in I^e$, therefore $(a \circ b] = (t]$ for some $t \in I$, since $a \circ b \in (a \circ b] = (t]$, $a \circ b = t \circ (a \circ b)$, therefore $a \circ b \in I$, since I is prime, either $a \in I$ or $b \in I$, it follows that $(a] \in I^e$ or $(b] \in I^e$, thus I^e is a prime ideal of PJ (L).

Conversely, suppose I^e is a prime ideal of PJ (L), let $a, b \in L$ such that $a \circ b \in I$, then $(a] \circ (b] = (a \circ b] \in I^e$, therefore $(a] \in I^e$ or $(b] \in I^e$, hence $(a] = (s]$ or $(b] = (t]$ for some $s, t \in I$, therefore $a = s \circ a \in I$ and hence I is prime.

Recall that, for any ideal I in L $I^e = \{(a] \mid a \in I\}$ is an ideal of an ASL PJ(L) of all principal ideal in L. Now, we prove the following theorem which expresses the relation between ideals of L and ideals of PJ (L).

Theorem 2.1.15 Let L is an ASL with 0, then I is an annihilator ideal in L if and only if I^e is an annihilator ideal in PJ(L).

Proof: Suppose I is an annihilator ideal in L, since I^e is an ideal, Then $I^e \subseteq I^{e**}$, by theorem 2.1.4(6)

Let $(a] \in I^{e**}$ and $b \in I^*$, then for any $c \in I$, $(b] \cap (c] = (b \circ c] = (0]$

Hence $(b] \in I^{e*}$, since $(a] \in I^{e**}$, we get $(a] \cap (b] = (0]$.

Therefore $(a \circ b] = (0]$, which implies that $a \circ b = 0$.

Hence $a \in I^{**} = I$, it follows that $(a] \in I^e$, therefore $I^{e**} \subseteq I^e$.

Hence $I^e = I^{e**}$.

Thus I^e is an annihilator ideal of PJ(L).

Conversely, suppose I^e is an annihilator ideal in PJ(L).

We have always $I \subseteq I^{**}$, by theorem 2.1.4(6)

Let $a \in I^{**}$ and $(b] \in I^{e*}$, now, for any $c \in I$, $(c] \in I^e$, hence $(b] \cap (c] = (0]$.

Therefore $(b \circ c] = (0]$, which implies that $b \circ c = 0$, therefore $b \in I^*$.

Now, $a \in I^{**}$ and $b \in I^*$ and hence $a \circ b = 0$, therefore $(a] \cap (b] = (a \circ b] = (0]$.

It follows that $(a] \in I^{e**} = I^e$, thus $a \in I$, hence $I^{**} \subseteq I$, we get that $I = I^{**}$.

Therefore I is an annihilator ideal in L.

Theorem2.1.16 Let L is an ASL with 0, then the set $\mathcal{A}$(L) of all annihilator ideals of L forms a complete Boolean Algebra, on its own.

Proof: Let $I, J \in \mathcal{A}(L)$, define $I \wedge J = I \cap J$ and $I \vee J = (I^* \cap J^*)^*$.

Since $I, J \in \mathcal{A}(L)$, $I^{**} = I$ and $J^{**} = J$, hence $(I \cap J)^{**} = I^{**} \cap J^{**} = I \cap J$.

This implies $I \cap J \in \mathcal{A}(L)$.

Also, $(I \vee J)^{**} = ((I^* \cap J^*)^*)^{**} = (I^* \cap J^*)^{***} = (I^* \cap J^*)^* = I \vee J$.

This implies $I \vee J \in \mathcal{A}(L)$

It can be easily seen that with respect to set inclusion, for any $I, J, K \in \mathcal{A}(L)$.

1) $I \subseteq I$ (reflexive)
2) $I \subseteq J$ and $J \subseteq I$,this implies $I = J$ (anti symmetric)
3) $I \subseteq J$ and $J \subseteq K$, this implies $I \subseteq K$ (transitivity)

Therefore ($\mathcal{A}$(L), $\subseteq$) is a poset.

I $\cap$ J is the glb of I, J, now, we have I, J $\subseteq$ I $\cup$ J, by theorem 2.1. 4(10) we get $I^* \cap J^* = (I \cup J)^* \subseteq I^*, J^*$ and hence $I^{**}, J^{**} \subseteq (I^* \cap J^*)^*$, it follows that I, J $\subseteq$ I $\vee$ J. Therefore I $\vee$ J is an upper bound of I, J.

Suppose H $\in\mathcal{A}$(L) is an upper bound of I, J, then I, J $\subseteq$ H, by theorem 2.1.4(3) ,$H^* \subseteq I^*, J^*$, therefore $H^* \subseteq I^* \cap J^*$ and hence $(I^* \cap J^*)^* \subseteq H^{**}$. Thus I $\vee$ J $\subseteq$ H and hence I $\vee$ J is a lub of I, J. This implies that ($\mathcal{A}$ (L), $\wedge$, $\vee$) is a lattice.

Since $(0]^* = L$ and $L^* = (0]$, it follows that (0], L $\in \mathcal{A}$(L). (0] and L are the least and greatest elements of $\mathcal{A}$(L) , therefore ($\mathcal{A}$(L), $\wedge$, $\vee$) is a bounded lattice.

Let I $\in\mathcal{A}$ (L), then $I^* \in \mathcal{A}$(L) since $I^* = I^{***}$, also, $I \cap I^* = (0]$ and $I \vee I^* = (I^* \cap I^{**})^* = (I^* \cap I)^* = (0]^* = L$. Thus I^* is a complement of I, therefore ($\mathcal{A}$ (L), $\wedge$, $\vee$, *, (0], L) is a complemented lattice.

Let I, J, K $\in\mathcal{A}$(L) , we shall prove that I $\vee$ (J $\wedge$ K) = (I $\vee$J) $\wedge$ (I $\vee$ K), I $\vee$ (J $\wedge$ K) $\subseteq$ (I $\vee$ J) $\wedge$ (I $\vee$ K), now, we prove that (I $\vee$ J) $\wedge$ (I $\vee$ K) $\subseteq$ I $\vee$ (J $\wedge$ K).

We first prove that (I $\vee$ J) $\wedge$ K $\subseteq$ I $\vee$ (J $\wedge$ K), we have $I \cap K \cap [I^* \cap (J \cap K)^*] = (0]$, it follows by theorem 2.1.4(9) $K \cap I^* \cap (J \cap K)^* \subseteq I^*$.

And also $K \cap I^* \cap (J \cap K)^* \subseteq J^*$, hence $K \cap I^* \cap (J \cap K)^* \subseteq I^* \cap J^*$, therefore $[K \cap I^* \cap (J \cap K)^*] \cap (I^* \cap J^*)^* = (0]$, hence $I^* \cap (J \cap K)^* \cap [K \cap (I^* \cap J^*)^*] = (0]$, thus $K \cap (I^* \cap J^*)^* \subseteq [I^* \cap (J \cap K)^*]^*$, hence, we get (I $\vee$ J) $\wedge$ K $\subseteq$ I $\vee$ (J $\cap$ K), now, (I $\vee$ J) $\cap$ (I $\vee$ K) $\subseteq$ I $\vee$ [J $\cap$ (I $\vee$ K)] = I $\vee$ [(I $\vee$ K) $\cap$ J] $\subseteq$ I $\vee$ [I $\vee$ (K $\cap$ J)], therefore $\mathcal{A}$(L)is distributive lattice

Thus ($\mathcal{A}$ (L), $\wedge$, $\vee$, *, (0], L) is a Boolean Algebra

Since $(\bigcap_{i\in\Delta} A_i)^{**} = \bigcap_{i\in\Delta}(A_i)^{**} = \bigcap_{i\in\Delta} A_i$ for each $A_i \in \mathcal{A}$ (L), this implies

($\mathcal{A}$(L) ,$\wedge$, $\vee$, *, (0], L) is a complete Boolean Algebra

2.2Annihilator Preserving Homomorphism

In this section, we introduce the concept of annihilator preserving homomorphism and derive a sufficient condition for a homomorphism to be annihilator preserving. We prove that the image and inverse image of annihilator ideal are again annihilator ideals. Finally, we prove that for any ideal I of L there exists a homomorphism f from L in to $Hom_L(I)$ such that Ker(f) = I^*.

Definition 2.2.1 Let L and L′ are two ASLs with zero elements 0 and 0′ respectively, then a mapping$f$: L → L′ is called a homomorphism if it satisfies the following:

(1) $f(a \circ b) = f(a) \circ f(b)$ for all a, b ∈ L
(2) $f(0) = 0'$.

The kernel of the homomorphism$f$: L → L′ (both L and L′ are ASLs with 0 and 0′ respectively) is defined by Ker (f) ={x ∈ L| $f(x) = 0'$}, and Ker (f) is an ideal of L.

Lemma 2.2.2 Let L and L′ are two ASLs with zero elements 0 and 0′ respectively and$f$: L → L′ is a homomorphism, then we have the following:

(1) If f is onto, then for any ideal I of L, f(I) is an ideal of L′.
(2) For any ideal J of L′, $f^{-1}(J)$ is an ideal of L containing Ker (f).

Proof: (1) Suppose f is onto and I is an ideal of L, then $f(I) = \{f(x) \mid x \in I\}$ is non-empty.

Let $f(x) \in f(I)$ and b ∈ L′, since f is onto, there exists a ∈ L such that b = $f(a)$

now, $f(x) \circ b = f(x) \circ f(a) = f(x \circ a) \in f(I)$ since $x \circ a \in I$, thus f(I) is an ideal of L′.

(2) Suppose J is an ideal of L′, we have $f^{-1}(J) = \{x \in L \mid f(x) \in J\}$.

Since J is an ideal of L′, $0' = f(0) \in J$. Hence $f^{-1}(J)$ is non-empty.

Let x $\in f^{-1}$(J) and $a \in$ L, then f(x) $\in$ J and $f(a) \in f$(L) $\subseteq$ L′, therefore f(x o a) = f(x) o $f(a) \in$ J.

Hence x o $a \in f^{-1}$(J), thus f^{-1}(J) is an ideal of L, let x $\in$Ker (f), then f(x) = 0 $\in$ J.

Hence x $\in f^{-1}$(J), therefore f^{-1}(J) is an ideal of L containing Ker (f).

Lemma 2.2.3 Let L and L′are two ASLs with 0 and 0′ respectively and let$f$: L$\rightarrow$ L′ is a homomorphism, then $f((a]) \subseteq (f(a)]$ ($a \in$ L), moreover, if f is onto, then $f((a]) = (f(a)]$.

Proof: Let f(x) $\in f((a])$, then x $\in (a]$ and hence x = a o x, it follows that f(x) = $f(a$ o x) = $f(a)$ o f(x), therefore f(x) $\in (f(a)]$, thus $f((a]) \subseteq (f(a)]$ ----------- (1)

Now, suppose f is onto, let t $\in (f(a)]$, since f is onto, there exists x $\in$ L such that f(x) = t, it follows that f(x) = $f(a)$ of(x) = $f(a$ o x) $\in f((a])$, therefore t = f(x) $\in f((a])$, hence $(f(a)] \subseteq f((a])$ ------------ (2)

Thus from (1) and (2)$f((a]) = (f(a)]$.

Theorem 2.2.4 Let L and L′ are two ASLs with 0 and 0′ respectively and$f$: L$\rightarrow$ L′ be a homomorphism. Then for any non-empty subset A of L, we have $f(A^*) \subseteq (f(A))^*$.

Proof: Let $a \in f(A^*)$ and y $\in f$(A), then there exists b $\in A^*$ and x $\in$ A such that a = f(b) and y = f(x), now, a o y = f(b) of(x) = f(b o x) =f(0) = 0′, therefore a o y = 0′, hence $a \in (f(A))^*$, thus $f(A^*) \subseteq (f(A))^*$.

Definition 2.2.5 Let L, L′ be ASLs with zero elements 0 and 0′ respectively and let f : L $\rightarrow$ L′ be a homomorphism, then f is called annihilator preserving if $f(A^*) = (f(A))^*$, for any $\{0\} \subset A \subset L$.

Example 2.2.1 Let A = {0, a} and B = {0, b_1, b_2} be two discrete ASLs. Write L = A×B = {(0, 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)}. Then (L, o, 0) is an ASL with 0 under point-wise operations, where the zero elements in L is 0 = (0, 0). Let L′ = {0, d, e, f} be another ASL in which the operation o is defined as follows:

o	0	d	e	f
0	0	0	0	0
d	0	d	0	d
e	0	0	e	e
f	0	d	e	f

Now, define the mapping f : L ⟶ L′ by $f((0, 0)) = 0$, $f((a, 0)) = d$, $f((0, b_1)) = f((0, b_2)) = e$, $f((a, b_1)) = f((a, b_2)) = f$. Then f is a homomorphism from L on to L′.

Definition2.2.6 An element a in an ASL with 0 is said to be dense element if $[a]^* = \{0\}$.

It can be easily observed that every unimaximal element is dense. But, dense element need not be unimaximal. Consider the following example.

Example2.2.1 Let A = {0, a} and B = {a, b_1, b_2} are two discrete ASLs. Let L = A × B = {(0 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)}. Define a binary operation o on L under point-wise:

o	(0, 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(0, b_1)	(0, 0)	(0, b_1)	(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)
(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)
(a, 0)	(0, 0)	(0, 0)	(0, 0)	(a, 0)	(a, 0)	(a, 0)
(a, b_1)	(0, 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)
(a, b_2)	(0 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)

Now, let L′ = {(0, 0), (0, b_1), (0, b_2), (a, b_1), (a, b_2)}, then L′ is a sub ASL of (L, o, 0′), in L′, (a, b_1), (a, b_2) are only unimaximal elements, now, $((0, b_1)]^* = \{(0, 0)\}$, so that $(0, b_1)$ is a dense element, but not a unimaximal element in L′, because $(0, b_1) \circ (a, b_1) = (0, b_1) \neq (a, b_1)$. Similarly $(0, b_2)$ is also a dense element but not unimaximal element.

Definition 2.2.7 An ASL L with 0 is said to be dense if $[a]^* = \{0\}$ for all $a\ (\neq 0) \in$ L.

Theorem2.2.8 Let L and L′ are two ASLs with zero elements 0 and 0′ respectively and let $f : L \longrightarrow L'$ be a homomorphism, if Ker $(f) = \{0\}$ and f is onto, then both f and f^{-1} are annihilator preserving.

Proof: Let A be a subset of L such that $(0] \subset A \subset L$, then $f(A^*) \subseteq (f(A))^*$ by theorem 2.2.4---------- (1)

Now, let $x \in (f(A))^*$, since f is onto, there exists $y \in L$ such that $f(y) = x \in (f(A))^*$, hence $f(y) \circ f(a) = 0'$ for all $a \in A$, this implies that $f(y \circ a) = 0'$ and hence $y \circ a \in$ Ker$(f) = \{0\}$, it follows that $y \circ a = 0$ for all $a \in A$, therefore $y \in A^*$, hence $x = f(y) \in f(A^*)$

This implies $(f(A))^* \subseteq f(A^*)$ ------------------- (2)

From (1) and (2) $(f(A))^* = f(A^*)$

Therefore f is annihilator preserving

Again, let $(0] \subset A \subset L'$, it is enough to prove that $f^{-1}(A^*) = (f^{-1}(A))^*$, let $x \in (f^{-1}(A))^*$, then $x \circ a = 0$ for all $a \in f^{-1}(A)$, hence $x \circ a = 0$ for all $f(a) \in A$, it follows that $f(x) \circ f(a) = f(x \circ a) = f(0) = 0'$ for all $f(a) \in A$, therefore $f(x) \in A^*$ and hence $x \in f^{-1}(A^*)$.

Thus $(f^{-1}(A))^* \subseteq f^{-1}(A^*)$ ------------------------ (1)

Conversely, suppose $x \in f^{-1}(A^*)$ and $a \in f^{-1}(A)$, then $f(x) \in A^*$ and $f(a) \in A$, hence $f(x \circ a) = f(x) \circ f(a) = 0'$, thus $x \circ a \in$ Ker $(f) = \{0\}$, therefore $x \circ a = 0$, for all $a \in f^{-1}(A)$, hence $x \in (f^{-1}(A))^*$, therefore $f^{-1}(A^*) \subseteq (f^{-1}(A))^*$ ----------------- (2)

From (1) and (2) we get $f^{-1}(A^*) = (f^{-1}(A))^*$.

Therefore f^{-1} is annihilator preserving

Theorem2.2.9 Let L and L' are two ASLs with zero elements 0 and $0'$ respectively, let $f : L \longrightarrow L'$ be annihilator preserving homomorphism such that Ker (f) = {0}, then $A^* = B^*$ if and only if $(f(A))^* = (f(B))^*$ for any non-empty subsets A and B of L.

Proof: Suppose $A^* = B^*$, then $f(A^*) = f(B^*)$.
Since f is annihilator preserving, $(f(A))^* = (f(B))^*$.
Conversely, assume that $(f(A))^* = (f(B))^*$, let $t \in A^*$, then $t \circ a = 0$ for all $a \in A$, hence $f(t \circ a) = f(0) = 0'$, therefore $f(t) \circ f(a) = 0'$ for all $a \in A$, it follows that $f(t) \in (f(A))^*$ and hence $f(t) \in (f(B))^*$, therefore $f(t) \circ f(b) = 0'$ for all $b \in B$, hence $f(t \circ b) = 0'$, therefore $t \circ b \in Ker(f) = \{0\}$ for all $b \in B$, we get $t \circ b = 0$ for all $b \in B$, therefore $t \in B^*$.

This implies $A^* \subseteq B^*$------------------------ (1)

Conversely let $t \in B^*$, then $t \circ a = 0$ for all $a \in B$, hence $f(t \circ a) = f(0) = 0'$, therefore $f(t) \circ f(a) = 0'$ for all $a \in B$, it follows that $f(t) \in (f(B))^*$ and hence $f(t) \in (f(A))^*$, therefore $f(t) \circ f(b) = 0'$ for all $b \in A$, hence $f(t \circ b) = 0'$, therefore $t \circ b \in Ker(f) = \{0\}$ for all $b \in A$, we get $t \circ b = 0$ for all $b \in A$, therefore $t \in A^*$, hence $B^* \subseteq A^*$--------------------(2)

Therefore from (1) and (2) $A^* = B^*$.

Theorem 2.2.10 Let L and L' are two ASLs with zero elements 0 and $0'$ respectively and let $f : L \longrightarrow L'$ is homomorphism, and then we have the following

(a) If f is annihilator preserving and onto, then $f(I)$ is annihilator ideal of L' for every annihilator ideal I of L.
(b) If f^{-1} preserves annihilators, then $f^{-1}(J)$ is an annihilator ideal of L for every annihilator ideal J of L'.

Proof: (a) Suppose f is annihilator preserving homomorphism which is onto and suppose I is an annihilator ideal of L, then by lemma 2.2.2(1)$f(I)$ is an ideal of L', since f is annihilator preserving, $(f(I))^{**} = ((f(I)^*)^* = (f(I^*))^* = f(I^{**}) = f(I)$, therefore $f(I)$ is an annihilator ideal of L'.

(b) Suppose f^{-1} preserves annihilators, let J be an annihilator ideal of L', then by lemma 2.2.2(2), $f^{-1}(J)$ is an ideal of L. Since f^{-1} preserves annihilators, we get $(f^{-1}(J))^{**} = ((f^{-1}(J))^*)^* = (f^{-1}(J^*))^* = f^{-1}(J^{**}) = f^{-1}(J)$. Therefore $f^{-1}(J)$ is an annihilator ideal of L.

Corollary 2.2.11 Let L and L' be two ASLs with zero elements 0 and $0'$ respectively and let $f : L \longrightarrow L'$ be homomorphism such that f^{-1} preserves annihilators, then Ker (f) is an annihilator ideal of L.

Proof: Since Ker $(f) = f^{-1}((0'])$ and $(0']$ is annihilator ideal of L', by the above theorem 2.2.10, Ker (f) is an annihilator ideal of L.

Recall that if I is an ideal of an ASL L with 0, then I is a sub ASL with 0. Finally we prove that if I is an ideal of L, then there exists a homomorphism whose kernel is the annihilator of I. First we need the following lemma.

Lemma 2.2.12 Let L is an ASL with 0 and I be an ideal of L, then the set $Hom_L(I)$, of all endomorphism on I is an ASL under the operation o defined on $Hom_L(I)$ by $(f \text{ o } g)(x) = f(x) \text{ o } g(x)$ for all $x \in I$.

Proof: Since I is non-empty Hom_L (I) is a non-empty set and the identity map on I belonging to Hom_L (I). Also, $Hom_L(I)$ is an ASL under the binary operation o, now, define $f_o: I \longrightarrow I$ by $f_o(x) = 0$ for all $x \in I$, then $f_o \in Hom_L$ (I). Also, for any $f \in Hom_L(I)$ and $x \in I$, consider, $(f_o \text{ o} f)(x) = f_o(x) \text{ o} f(x) = 0 \text{ o } f(x) = 0 = f_o(x)$, therefore $f_o \text{ o } f = f_o$, hence f_o is the zero element in Hom_L (I). Thus Hom_L (I) is an ASL with zero element f_o.

Theorem 2.2.13 Let L is an ASL with 0, then for any ideal I of L there exists a homomorphism f from L to Hom_L (I) such that Ker $(f) = I^*$.

Proof: Let I be an ideal of L, now, fix $r \in L$ and define $\theta_r: I \longrightarrow I$ by $\theta_r(x) = x \text{ o } r$ for all $x \in I$. We shall prove that $\theta_r \in Hom_L(I)$, since I is an ideal of L, we get $\theta_r(x) = x \text{ o } r \in I$, and let $x, y \in I$, then $\theta_r(x \text{ o } y) = (x \text{ o } y) \text{ o } r = (x \text{ o } y) \text{ o } (r \text{ o } r) = x \text{ o } (y \text{ o } (r \text{ o } r)) = x \text{ o } ((y \text{ o } r) \text{ o } r) = x \text{ o } ((r \text{ o } y) \text{ o } r) = (x \text{ o } (r \text{ o } y)) \text{ o } r = ((x \text{ o } r) \text{ o } y) \text{ o } r = (x \text{ o } r) \text{ o } (y \text{ o } r) = \theta_r(x) \text{ o } \theta_r(y)$. Also, $\theta_r(0) = 0 \text{ o } r = 0$. Thus θ_r is a homomorphism, hence $\theta_r \in Hom_L(I)$. Now, define $f : L \longrightarrow Hom_L(I)$ by $f(r) = \theta_r$ for all $r \in L$, and, now, let $r, s \in L$, then $f(r \text{ o } s) = \theta_{r \text{ o } s}$, now, for any $x \in I$, $\theta_{r \text{ o } s}(x) = x \text{ o } (r \text{ o } s) = (x \text{ o } x) \text{ o } (r \text{ o } s) = x \text{ o } (x \text{ o } (r \text{ o } s)) = x \text{ o } ((x \text{ o } r) \text{ o } s) = x \text{ o } ((r \text{ o } x) \text{ o } s) = x \text{ o } (r \text{ o } (x \text{ o } s)) = (x \text{ o } r) \text{ o } (x \text{ o } s) = \theta_r(x) \text{ o } \theta_s(x) = (\theta_r \text{ o } \theta_s)(x)$, therefore $\theta_{r \text{ o } s} = \theta_r \text{ o } \theta_s$, thus $f(r \text{ o } s) = f(r) \text{ o } f(s)$. Also, $f(0) = \theta_0$. Now, $\theta_0(x) = x \text{ o } 0 = 0 = f_o(x)$ for all $x \in I$. Therefore $\theta_0 = f_o$. Thus f is a homomorphism. Hence Ker (f) is an ideal of L. We now prove that Ker$f = I^*$. Consider,

$r \in Ker(f) \Leftrightarrow f(r) = \theta_0$, which is the zero element of $Hom_L(I)$.

$\Leftrightarrow \theta_r = \theta_0$

$\Leftrightarrow \theta_r(x) = \theta_0(x)$ for all $x \in I$

$\Leftrightarrow x \text{ o } r = \theta_0(x) = 0$ for all $x \in I$

$\Leftrightarrow r \in I^*$

Therefore Ker $(f) = I^*$.

3. Conclusion

In this project we discussed the concept of semi lattice, almost semi lattice and lattice and gave the definition of these mathematical structures, and also discussed about ideals and annihilator ideals in almost semi lattice with 0. And proved some properties of annihilator ideals and gave some examples of annihilator ideals. Also we introduced the concept of Boolean algebra and the concept of homomorphism of two almost semi lattice are discussed and some annihilator preserving homeomorphisms are derived.

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