2021-10-04

Skew Heyting Almost Distributive Lattices

Mesere, Dessie

Bahir Dar University

College of Science

Department of Mathematics

A project report

on

Skew Heyting Almost Distributive Lattices

By

Dessie Meseret Lahush

August, 2021

Bahir Dar, Ethiopia

Bahir Dar University

College of Science

Department of Mathematics

Skew Heyting Almost Distributive Lattices

A project work submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of ‘‘Master of Science in mathematics’’.

By

Dessie Meseret Lahush

Advisor: Mihret Alemineh (PhD)

August, 2021

Bahir Dar, Ethiopia

Bahir Dar University

College of Science

Department of Mathematics

Approval of the project for defense

I hereby certify that I have supervised, read and evaluated this project entitled **Skew Heyting Almost Distributive Lattice** by **Dessie Meseret Lahush** prepared under my guidance. I recommend that the project is submitted for oral defense.

Adivisor's Name: ___________________________signature: _______date: ___/___/___

Bahir Dar University

College of Science

Department of Mathematics

“Skew Heyting Almost Distributive Lattice”

By

Dessie Meseret Lahush

A project Submitted to the Department of Mathematics, College of Science, Bahir Dar University in Partial Fulfilment of the Requirements for the Degree of **“Master of Science in Mathematics”.**

Approved by Board of Examiners

	Name	Signature	Date
External Examiner:	____________________	_______	________
Internal Examiner 1:	____________________	_______	________
Internal Examiner 2:	____________________	_______	________

Acknowledgment

First of all I would like to provide my special thanks to my advisor Dr.Mihret Alemineh (PhD) for introducing me with the new project work “Skew Heyting Almost Distributive Lattice’’ and for his valuable professional assistance, guidance and corrective feed backs throughout the years. I am very grateful to have him as an advisor. My special thanks also goes to B.Assaye, Mihret Alemineh and Yeshiwas Mebrat who write the article entitled “Skew Heyting Almost Distributive Lattice” for their work that initiates me to more about the concept of “Skew Heyting Almost Distributive Lattice” by reading their paper and other related reference suggested by them.

Finally, I would like to transfer my heartfelt thanks to my wife w/ro Mastewal Chanie for her moral support and unlimited advice during the study. My special thanks also extend to Mr.Gizachew Alemayehu for giving to help to write and technical support. In addition, I sincerely wish to thank Ministry of Education for their sponsorship to undertake my studies and Bahir Dar University College of Science department of Mathematics for providing the opportunity to do this project.

Abstract

In this project we define skew Heyting almost distributive lattice and characterize it as a skew Heyting algebra in terms of congruence relation defined on it. Moreover, we also present different conditions on which an ADL with maximal element m becomes skew HADL and a skew HADL to become skew Heyting algebra. We define an equivalence relation θ on a skew HADL and prove that θ is a congruence relation on the equivalence class$[x]_\theta$. So we generalized that each congruence class is a maximal rectangular sub-algebra. Further in order to clarify more, we give three examples that verify skew HADL. The achieved properties reveal that skew HADL generalizes skew Heyting algebra. Finally, we prove different theorems, corollaries and lemmas related to skew HADL.

Table of Contents

Content **Page**

Chapter one

Introduction and Preliminaries

1.1 Introduction

The foundation of modern theory of skew lattice was laid by Jonathan Leech in 1989. As a continuation of this studies Karin Cvetko-vah [5] develop Heyting algebra as a bounded lattice (with join and meet operation written V and Λ with least element 0 and greatest element 1) equipped with a binary operation a → b of implication such that c ∧ $a \leq$ b is equivalent to $c \leq$ a → b and it was first investigated by T. Skolem about 1920. Heyting algebra is named after a Dutch Mathematician Arend Heyting, and was introduced by G.Birkhoof and was developed by H.B.Curry around the year 1963. The notion of skew Heyting algebra was introduced by Karin Cvetko-vah [5] as a generalization of Heyting algebra with top element and need not contain bottom element.

On the other hand the concept of an almost distributive lattice (ADL) was introduced by U.M.Swamy and G.C.Rao [13] as a common abstraction of existing ring theoretic generalizations of Boolean algebra and distributive lattices. In the theory of skew lattice, it is proved that every interval in Heyting algebra is Heyting algebra. This leads to develop Heyting almost distributive lattice (HADL). For this cause G.C.Rao, Berhanu Assaye and M.V.Ratnamani introduce Heyting ADLs (HADLs) as a generalization of Heyting algebra in the class of ADLs, and they characterize an HADL in terms of the set of all of its principal ideals.

After dealing the concept of HADL, B. Assaye, M.Alemneh, and Yeshiwas Mebrat extend an ADL with maximal element m and without zero. As a result they introduce the concept of skew Heyting Almost Distributive Lattice with a number of important laws and results satisfied by skew Heyting algebra.

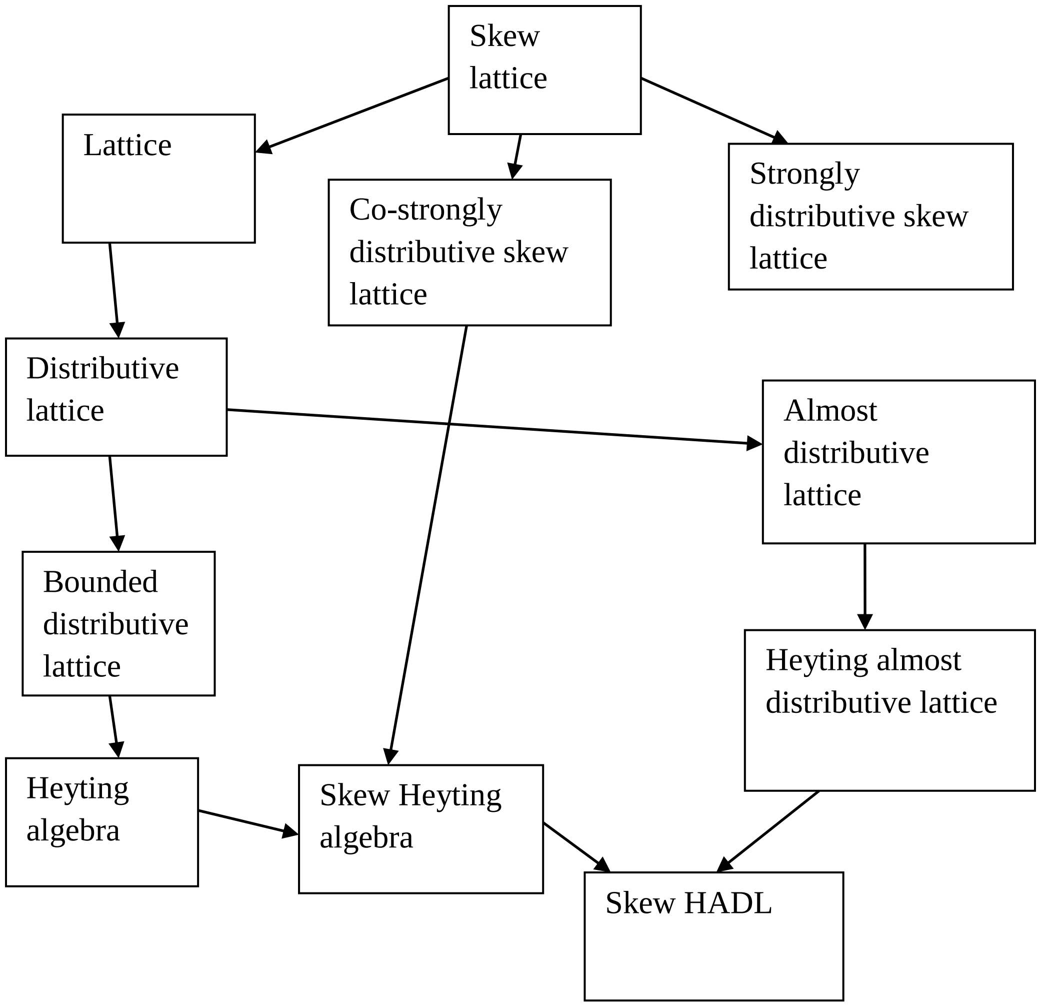
The project is mainly divided into 2 chapters. In chapter-1, the necessary definitions and results (from different sources) that used in proving lemmas, theorems and corollaries in the subsequent chapters are collected. In Chapter-2, we introduce the concept of Skew Heyting Almost Distributive Lattice (or simply Skew HADL) and derive some important properties of Skew

HADL. In this chapter, we also define an equivalence relation θ on a skew HADL and shows that:

- θ is a congruence relation on each equivalence class.
- Each congruence class is a maximal rectangular sub-algebra of the equivalence class.

Moreover, we prove that every interval in a Skew HADL is Skew HADL and a binary operation → on a skew HADL H can be defined by $x \to y = ((x \vee y) \wedge m)_{(y \wedge m)} \to (y \wedge m)$.

In general the development of Skew HADL is described in the chart as follows;



1.2. Preliminaries

In this section the necessary definitions and results on; lattice, almost distributive lattices, Heyting algebras, Heyting almost distributive lattices, skew lattices and skew Heyting algebras which will be used in the next section are described.

Definition 1.2.1 ([10]) A none empty set H together with two binary operations ∨ and ∧ (read " join" and "meet" respectively) on H is called a lattice if it satisfies the following identities;

1. Commutative laws
   a. x ∨ y = y ∨ x
   b. x ∧ y = y ∧ x
2. Associative laws
   a. x ∨ (y ∨ z) = (x ∨ y) ∨ z
   b. x ∧ (y ∧ z) = (x ∧ y) ∧ z
3. Idempotent law
   a. x ∨ x = x
   b. x ∧ x = x
4. Absorption law
   a. x= x ∨ (x ∧ y)
   b. x= x ∧ (x ∨ y), for all x, y ∈H.

Example 1.2.1 ($\mathcal{P}$(A), ∨, ∧) is a lattice (under set inclusion ⊆) for any set A, where $\mathcal{P}$(A) is the power set. The join operation is the set operation of union and the meet operation is the operation intersection; that is ∨, = U and ∧ = ∩.

Remark 1.2.1 If $(H, \vee, \wedge)$ is a lattice, then an element a of H is called zero element or least element of H, if $a \wedge x = a$ for all x ∈ H. If H has a least element, then it is unique and it is denoted by 0. Similarly an element a of H is called one element or greatest element of H, if $a \wedge x = x$, for all x ∈ H. If H has a greatest element, then it is unique and it is denoted by 1.

Definition1.2.2([3]) A lattice (H, V, Λ) is called a distributive lattice if the meet and join operations distribute each other. That is; for $x, y, z \in H$ the following holds;

1. $x \, V \, (y \, \Lambda \, z) = (x \, V \, y) \, \Lambda \, (x \, V \, z)$

2. $x \Lambda (y V z) = (x \Lambda y) V (x \Lambda z)$

Definition 1.2.3 ([10]) An algebra (H, V, Λ, 0, 1) with two binary and two nullary operations is bounded distributive lattice if;

1. (H, V, Λ) is distributive lattice
2. $x \Lambda 0 = 0$ and $x V 1 = 1$, for all $x \in H$

Definition 1.2.4 ([6]) An algebra (H, ∨, ∧) of type (2, 2) is called an almost distributive lattice if it satisfies the following axioms;

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$ for all $x, y, z \in H$.

Theorem 1.2.1 ([6, 13]) Let H be an ADL with 0. Then for any w, $x, y, z \in H$, we have the following.

1. $x \vee y = x \Leftrightarrow x \wedge y = y$
2. $x \vee y = y \Leftrightarrow x \wedge y = x$
3. $x \wedge y = y \wedge x = x$ whenever $x \leq y$
4. ∧ is associative
5. $x \wedge y \wedge z = y \wedge x \wedge z$
6. $(x \vee y) \wedge z = (y \vee x) \wedge z$
7. $x \wedge y \leq y$ and $x \leq x \vee y$
8. $x \wedge (y \wedge x) = y \wedge x$ *and* $x \vee (x \vee y) = x \vee y = (x \vee y) \vee y$
9. $x \wedge x = x$ and $x \vee x = x$
10. $x \wedge 0 = 0$ and $0 \vee x = x$
11. $\{w \vee (x \vee y)\} \wedge z = \{(w \vee x) \vee y\} \wedge z$
12. *If* $x \leq z$ and $y \leq z$, then $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
13. x is less than or equal to y and written x$\leq y$ if $x \wedge y = x$

Definition 1.2.5 ([7]) If $(H, V, \wedge, 0)$ is an ADL and for any a, b ∈ H define a ≤ b if and only if a = $a \wedge b$, then ≤ is a partial order on H.

Theorem 1.2.2 ([8]) Let H be an ADL. Then the following are equivalent;

1. H is a distributive lattice.
2. ∨ is commutative in H.
3. ∧ is commutative in H.
4. ∨ is right distributive over ∧ in H.

Definition 1.2.6 ([9]) An element m in an ADL H is said to be maximal if for each a ∈ H

$m \le a$, implies that $a = m$.

Theorem 1.2.3 ([9]) Let H be an ADL. For any $m \in H$ the following are equivalent.

1. m is maximal element
2. $m \vee x = m$ for all $x \in H$
3. $m \wedge x = x$, for all $x \in H$

Definition 1.2.7 Let (H, ∨, ∧, 0) be an ADL with 0 and x, y ∈ H such that x ≤ y. Then the set

[x, y] = {a ∈ H | x ≤ a ≤ y} is called an interval in H and every interval in an ADL H is a bounded distributive lattice under the induced operations ∨ and ∧.

Definition 1.2.8 ([2, 5]) An algebra $(H, \vee, \wedge, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a Heyting algebra if it satisfies the following conditions.

1. $(H: \vee, \wedge, 0, 1)$ is a bounded distributive lattice
2. $x \rightarrow x = 1$
3. $y \le x \rightarrow y$
4. $x \wedge (x \rightarrow y) = x \wedge y$
5. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
6. $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$, for all x, $y, z \in H$.

Example 1.2.2 Let A be a total order set that has a least element 0 and greatest element 1.Then for all $x, y \in A$ and → be a binary operation on A by $x \rightarrow y = \begin{cases} 1, if\ x \le y \\ y, otherwise \end{cases}$ is Heyting algebra.

Lemma 1.2.1 ([5]) In any Heyting algebra H, for all x, y ∈ H, $x \rightarrow y = (y \vee x) \rightarrow y$

Definition 1.2.9 ([2, 6, and 12]) Let $(H, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Suppose → be a binary operation on H satisfying the following conditions;

1. $x \rightarrow x = m$

2. $(x \to y) \wedge y = y$
3. $x \wedge (x \to y) = x \wedge y \wedge m$
4. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
5. $(x \vee y) \to z = (x \to z) \wedge (y \to z)$, for all x, $y, z \in H$.

Then $(H, \vee, \wedge, \to, 0, m)$ is called a Heyting almost distributive lattice (HADL)

Remark 1.2.2 (1**)** Equivalent to the above definition Heyting algebra is an algebra $(H, \vee, \wedge, \to,$ $0, 1)$ such that $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice that satisfies the condition $x \wedge y \leq z$ if and only if $x \leq y \to z$.

(2) Let H be an HADL and $\to$ be a binary operation on H such that x $\to y \in H$ for any x, $y \in H$. Then $y \leq x \to y$ implies that (x $\to y) \wedge y = y$, but the converse is not always true. The converse becomes true whenever H is a lattice, and therefore an HADL becomes Heyting algebra.

(3) Every Heyting algebra is Heyting almost distributive lattice (HADL).

Theorem 1.2.4 ([2]) Let $(H, \vee, \wedge, \to, 0, m)$ be an HADL. Then the following are equivalent.

1. H is Heyting algebra.
2. For any $a, b, c \in H, a \wedge c \leq b \Leftrightarrow c \leq a \to b$
3. $b \leq a \to b$ for all $a, b \in H$.

Lemma 1.2.3 ([6]) Let m be a maximal element of a Heyting almost distributive Lattice (HADL) H. Then for any x, y, z ∈H, the following conditions hold:

a. $x \leq y \Rightarrow x \to y = m$.
b. $m \to x = x \wedge m$.
c. $x \to m = m$
d. $[\mathrm{x} \to (y \to z)] \wedge m = [(x \wedge y) \to z] \wedge m$
e. $[\mathrm{x} \to (y \to z)] \wedge m = [y \to (x \to z)] \wedge m$

Theorem 1.2.5 ([2]) Let H be an ADL with 0 and a maximal element m, then the following are equivalent;

1. H is an HADL
2. $[0, a]$ is a Heyting algebra for all $a \in H$
3. [0, m] is a Heyting algebra.

Definition 1.2.10 ([2]) Let H be a none empty set and θ be a binary relation on H (that is $\theta \subseteq H \times H$).Then θ is said to be equivalence relation on H if θ satisfies the following;

i. Reflexive:(x, x) $\in \theta, \ \forall x \in H$
ii. Symmetric:$(x,\ y) \in \theta$ implies that $(y,\ x) \in \theta, \forall x,\ y \in H$
iii. Transitive:$(x,\ y) \in \theta\ (y,\ z) \in \theta$ imply that $(x,\ z) \in \theta, \forall x,\ y \in H$, we write x$\theta y$ for $(x, y) \in \theta$

Definition 1.2.11 ([1]) An equivalence relation θ on Heyting algebra H is called a congruence relation if for all $a, b, c, d \ \in \ H$ and $(a, b) \in \theta, \ (c,\ d) \in \theta$.Then we have the following;

1. $(a \wedge c,\ b \wedge d) \in \theta$
2. $(a \vee c,\ b \vee d) \in \theta$
3. $(a \rightarrow c,\ b \rightarrow d) \in \theta$.

Lemma 1.2.4 ([2]) Let H be a Heyting algebra, then an equivalence relation θ on H is a congruence relation if and only if for any $(a, b) \in \theta, d \in H$;

1. $(a \wedge d, b \wedge d) \in \theta$
2. $(a \vee d, b \vee d) \in \theta$
3. $(a \rightarrow d, b \rightarrow d) \in \theta$
4. $(d \rightarrow a, d \rightarrow b) \in \theta$

Definition 1.2.12 ([5]) A skew lattice is an algebra $(S,\ \wedge,\ \vee)$ of type $(2, 2)$ such that $\wedge$ and $\vee$ are both idempotent and associative, and they satisfy the following absorption laws;

1. $x \wedge (x \vee y) = x = x \vee (x \wedge y)$
2. $(x \wedge y) \vee y = y = (x \vee y) \wedge y$, for all $x, y \in S$.

The natural partial order can be defined on a skew lattice S by stating that $x \leq y$ if and only if

$x \vee y = y = y \vee x$, or equivalently x $\wedge\ y = x = y \wedge x$, for $x, y \in S$.

Definition 1.2.13 ([12]) A skew lattice is called strongly distributive if for all $x, y, z \in S$ it satisfies the following identities;

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
2. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

And skew lattice is called co-strongly distributive if it satisfies the identities;

1. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

2. $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 1.2.14 ([12]) A skew lattice with top 1 is an algebra $(S, \vee, \wedge, 1)$ of type $(2, 2, 0)$ such that $(S; \wedge, \vee)$ is skew lattice and $x \vee 1 = 1 = 1 \vee x$ or equivalently $x \wedge 1 = x = 1 \wedge x \ \ \forall x \in$ S.

Definition 1.2.15 ([2, 5]) An algebra $(S, \vee, \wedge, \rightarrow, 1)$ of type $(2, 2, 2, 0)$ is said to be a skew Heyting algebra whenever the following condition satisfies;

1. $(S, \vee, \wedge, 1)$ is a co-strongly distributive skew lattice with top 1.
2. For any a $\in S$, an operation $\rightarrow_a$ can be defined on $a \uparrow = \{a \vee x \vee a \mid x \in S\}$ such that $(u \uparrow : \vee, \wedge, \rightarrow_a, 1, a)$ is a Heyting algebra with top 1 and bottom a.
3. An induced binary operation $\rightarrow$ from $\rightarrow_a$ is defined on S by

   $x \rightarrow y = (y \vee x \vee y) \rightarrow_y y$

Theorem 1.2.6 ([5]) Let $(S: \vee, \wedge, \rightarrow, 1)$ be an algebra of type (2,2,2,0) such that$(S; \vee, \wedge, 1)$ is a co-strongly distributive skew lattice with top 1.Then $(S: \vee, \wedge, \rightarrow, 1)$ is a skew Heyting algebra if and only if for all x, y, u $\in S$, we have the following axioms;

(1)$x \rightarrow y = (y \vee x \vee y) \rightarrow y$

(2)$x \rightarrow x = 1$

(3)$x \wedge (x \rightarrow y) \wedge x = x \wedge y \wedge x$

(4)$y \wedge (x \rightarrow y) = y and (x \rightarrow y) \wedge y = y$

(5)$x \rightarrow (u \vee (y \wedge z) \vee u) = (x \rightarrow (u \vee y \vee u)) \wedge (x \rightarrow (u \vee z \vee u))$

Remark 1.2.3 (1) A skew Heyting algebra becomes an ADL if it is strongly distributive skew lattice

(2) Every lattice is skew lattice

(3) If a skew Heyting algebra H contains bottom element, then it is a HADL.

Chapter two

Skew Heyting Almost Distributive Lattices

In this section, we introduce the concept of skew Heyting almost distributive lattices (skew HADLs) and, characterize it in terms of skew Heyting algebras and congruence relations defined on it. Throughout this section H stands for an ADL with a maximal element m but without 0, and

(i) For any a ∈ H, $H_a = \{x \wedge a \mid x \in H\}$

(ii) For any a ∈ H, $\rightarrow_a$ is the binary operation defined on H_a

(iii) For any b, c ∈ H, ${}_b\rightarrow$ is the binary operation defined on [b, c].

2.1 Definition and Examples of Skew Heyting Almost Distributive Lattice

As we just defined in chapter one skew Heyting algebra is a generalization of Heyting algebra. It has a top element and need not to contain bottom element.

Definition 2.1.1 Let H be an ADL with a maximal element m and without 0. Then H is said to be a skew HADL if to each a ∈ H the algebra $(H_a, \vee, \wedge, \rightarrow_a, a)$ is a skew Heyting algebra.

Example 2.1.1 Let H be an ADL with a maximal element m and b ∈ H. For any a ∈ H such that a ≤ b define a binary operation ${}_a\rightarrow$ on [a, b] by

$$x\ {}_a\rightarrow y = \begin{cases} b, if\ \ x \leq y \\ y,\ \ otherwise \end{cases}$$, Then H is a skew HADL.

Solution:

1. Since H is an ADL with maximal element m, then by **Theorem1.2.1 (12)** for all $x, y \in H$
   x ∧ y = y ∧ x and x ∨ y = y ∨ x. Hence, both ∧ and ∨ are commutative in an ADL. Then by **Theorem 1.2.2** ∨ is right distributive over ∧. Hence, H is distributive lattice.
   Therefore, H_b is co-strongly distributive skew lattice.
2. For all $x, y, z \in [a, b]$ the following holds;
   i. Since every interval in an ADL is bounded distributive lattice under the operation ∨ and ∧ implies $([a, b]: \vee,\ \wedge)$ is bounded distributive lattice.

ii. As $x \leq x$, x $_a\!\!\longrightarrow$ x = b

iii. $y \leq$ x $_a\!\!\longrightarrow$ y ....Since $y \leq b$ and y = x $_a\!\!\longrightarrow$ y

iv. x ∧ (x $_a\!\!\longrightarrow$ y) = $\begin{cases} x \wedge b = x \wedge y \wedge b = x \wedge y, if x \leq y \\ x \wedge y, otherwise \end{cases}$

v. x $_a\!\!\longrightarrow$ (y ∧ z) = $\begin{cases} b, if x \leq y \wedge z \\ y \wedge z, otherwise \end{cases}$ and

(x $_a\!\!\longrightarrow$ y) ∧ (x $_a\!\!\longrightarrow$ z) = $\begin{cases} b, if\ x \leq y\ and\ x \leq z \\ y \wedge z, otherwise \end{cases}$

In both cases x $_a\!\!\longrightarrow$ (y ∧ z) = (x $_a\!\!\longrightarrow$ y) ∧ (x $_a\!\!\longrightarrow$ z)

vi. (x ∨ y) $_a\!\!\longrightarrow$ z = $\begin{cases} b, if\ x \vee y \leq z \\ z, otherwise \end{cases}$ and (x $_a\!\!\longrightarrow$ z) ∧ (y $_a\!\!\longrightarrow$ z) = $\begin{cases} b, if x \leq z\ and\ y \leq z \\ z, otherwise \end{cases}$

Now, both cases (x ∨ y) $_a\!\!\longrightarrow$ z = (x $_a\!\!\longrightarrow$ z) ∧ (y $_a\!\!\longrightarrow$ z).

Thus by the above conditions ([a, b]: ∨, ∧, $_a\!\!\longrightarrow$) is Heyting algebra.

3. The binary operation on H_b induced from $_a\!\!\longrightarrow$ be defined by

x $\longrightarrow_b$ y = (y ∨ x) $_y\!\!\longrightarrow$ y .... [By lemma 1.2.1]

= (y ∨ x ∨ y) $_y\!\!\longrightarrow$ y........... [Since y ∨ x ∨ y= y ∨ x]

Hence, (H_b: ∨, ∧, $\longrightarrow_b$, b) is a skew Heyting algebra and therefore H is a skew HADL.■

Example 2.1.2 Let H be an ADL with a maximal element m. For any a ∈ H, the binary operation $\longrightarrow_a$ on H_a define by x $\longrightarrow_a$ y = $\begin{cases} m, if\ x \leq y \\ y, otherwise \end{cases}$ .Then H is skew HADL.

Solution;1. As verified by **Example 2.1.1** H_a is co-strongly distributive skew lattice.

2. For all x, y, z ∈ H_a the following holds;

a. x $\longrightarrow_a$ x = m

b. (x $\longrightarrow_a$ y) ∧ y = $\begin{cases} m \wedge y = y, if\ x \leq y \\ y \wedge y = y, otherwise \end{cases}$

c. x ∧ (x $\longrightarrow_a$ y) = $\begin{cases} x \wedge m = x \wedge y \wedge m, if\ x \leq y \\ x \wedge y = x \wedge y \wedge m\ , otherwise \end{cases}$

d. x $\longrightarrow_a$ (y ∧ z) = $\begin{cases} m, if\ x \leq y \wedge z \\ y \wedge z, otherwise \end{cases}$ and (x $\longrightarrow_a$ y) ∧ (x $\longrightarrow_a$ z) = $\begin{cases} m, if\ x \leq y\ and\ x \leq z \\ y \wedge z, otherwise \end{cases}$

Implies x $\longrightarrow_a$ (y ∧ z) = (x $\longrightarrow_a$ y) ∧ (x $\longrightarrow_a$ z)

e. (x ∨ y) $\longrightarrow_a$ z = $\begin{cases} m, if\ x \vee y \leq z \\ z, otherwise \end{cases}$ and

(x $\longrightarrow_a$ z) ∧ (y $\longrightarrow_a$ z) = $\begin{cases} m, if\ x \leq z\ and\ y \leq z \\ z, otherwise \end{cases}$.

Thus H_a is HADL and in both cases $y \leq x \rightarrow_a y$, then **by Theorem 1.2.4** H_a is Heyting algebra.

3. The binary operation $\rightarrow_a$ on H_a can be defined by

   $x \rightarrow_a y = (y \vee x) \rightarrow_y y$ ………………… [By lemma 1.2.1]

   $= (y \vee x \vee y) \rightarrow_y y$ ………….. [Since $(y \vee x \vee y) = (y \vee x)$]

Hence, $(H_a: \vee, \wedge, \rightarrow_a, a)$ is Skew Heyting algebra.

Therefore H is skew HADL.■

Example 2.1.3 Every Heyting algebra is skew HADL.

Solution; Suppose H is Heyting algebra.

1. Since Heyting algebra is distributive lattice and H is co-strongly distributive skew lattice.
2. Now, for all $x, y, u \in H$ and $\rightarrow$ is binary operation on H. Then the following holds.
   i. $x \rightarrow y = (y \vee x \vee y) \rightarrow y$ ……… [by Lemma 1.2.1]
   ii. $x \rightarrow x = 1$ ………….. [since H is Heyting algebra]
   iii. $x \wedge (x \rightarrow y) \wedge x = x \wedge y \wedge x$
   iv. $y \wedge (x \rightarrow y) = y$ and $(x \rightarrow y) \wedge y = y$
   v. $x \rightarrow (u \vee (y \wedge z) \vee u) = x \rightarrow ((u \vee y \vee u) \wedge (u \vee z \vee u))$…**[**Since Heyting algebra is distributive lattice]

      $\Rightarrow x \rightarrow (u \vee (y \wedge z) \vee u) = x \rightarrow (u \vee y \vee u) \wedge x \rightarrow (u \vee z \vee u)$

Then by Theorem 1.2.6 $(H: \vee, \wedge, \rightarrow, 1)$ is skew Heyting algebra.

Therefore, H is skew HADL.■

Remark 2.1.1 (1) An ADL H with maximal element m is co-strongly distributive skew lattice.

(2) Let H be a skew HADL. If we include 0 to H, then for each a ∈ H, $H_a = [0, a]$ is a skew Heyting algebra. Particularly if a = m, then [0, m] is a skew Heyting algebra. Thus for each b ∈ [0, m], $([b, m]: \vee, \wedge, {}_b\rightarrow, b, m)$ is a Heyting algebra. Now, taking b = 0 makes $([0, m]: \vee, \wedge, {}_b\rightarrow, b, m)$ is a Heyting algebra and Hence by **Theorem 1.2.5** H is an HADL.

(3) From **Theorem 1.2.1(12)** and **Theorem 1.2.2** we can deduce that in a Skew HADL H with maximal element m;

i. Both ∨ and ∧ are commutative in H.
ii. ∨ and ∧ are distributive to each other in H.

2.2 Properties of Skew Heyting Almost Distributive Lattice and Characterize it by Skew Heyting Algebra

In the above examples we have seen, there are an algebra which satisfies definition of skew HADL and every Heyting algebra is skew HADL. In this section we give necessarily and sufficient condition for an ADL with maximal element m to become skew HADL and investigate some of its algebraic properties.

On the rest of this section, H stands for a skew HADL (H, ∨, ∧, →, m) unless otherwise specified.

Theorem 2.2.1 Let (H: ∨, ∧, m) be an ADL with a maximal element m. Then (H, ∨, ∧, →, m) is a skew HADL if and only if the following conditions hold;

1. For any b ∈ H, ([b, m], ∨, ∧, $_b$→, b, m) is a HADL
2. A binary operation → on H can be defined by
   $x \to y = ((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\to (y \wedge m)$.

Proof, (1) Let H is a skew HADL. Since H has maximal element(s), for each b ∈H there exist a maximal element m such that b≤ m and we have $H_m = \{b \wedge m \mid b \in H\} = \{b \mid b \in H\}$, hence, b ∈ H_m. From the definition of skew HADL for any a ∈ H, H_a is a skew Heyting algebra. In particular for a = m, H_m is a skew Heyting algebra. Consequently, from the definition of skew Heyting algebra for any b ∈ H_m, ([b, m], ∨, ∧, $_b$→, b, m) is a Heyting algebra so that ([b, m], ∨, ∧, $_b$→, b, m) is a HADL.

(2) Since H_m is a skew Heyting algebra, the induced operation $\to_m$ on H_m from $_b$→ on

[b, m] is given by $x \to_m y = (y \vee x \vee y)\ {}_y\!\!\to y$.

Thus it is possible to define a binary operation → on H by $x \to y = (x \wedge m) \to_m (y \wedge m)$. But,

$(x \wedge m) \to_m (y \wedge m) = ((y \wedge m) \vee (x \wedge m) \vee (y \wedge m))\ {}_{(y \wedge m)}\!\!\to (y \wedge m)$ …[As H_m is Skew Heyting algebra]

$= ((y \vee x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\to (y \wedge m)$………… [Right distributive of ∧ over ∨]

$= ((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\to (y \wedge m)$……. [Since in an ADL $y \vee x \vee y = x \vee y$]

And hence $x \to y = ((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\to (y \wedge m)$

Conversely, suppose conditions (1) and (2) hold and let a ∈ H. Then H_a is a co-strongly distributive skew lattice. By (1) for any b ∈ H_a, [b, m] is a HADL. Since H has maximal element and for some maximal element m in H, a ≤ m so that [b, a] ⊆ [b, m]. **Theorem 1.2.5** asserts that ([b, a], ∨, ∧, $_b$→, b, a) is a Heyting algebra. The maximal element in H_a is a, thus using (2) it is possible to define $\rightarrow_a$ on H_a by

$$x \rightarrow_a y = ((x \vee y) \wedge a)\ {}_{(y \wedge a)}\!\!\rightarrow (y \wedge a).$$

But $((x \vee y) \wedge a)\ {}_{(y \wedge a)}\!\!\rightarrow (y \wedge a) = ((y \vee x \vee y) \wedge a)\ {}_{(y \wedge a)}\!\!\rightarrow (y \wedge a)$ … [Since $y \vee x \vee y = x \vee y$]

$$= (y \vee x \vee y)\ {}_y\!\!\rightarrow y \text{.................... [By 2]}$$

And hence $x \rightarrow_a y = (y \vee x \vee y)\ {}_y\!\!\rightarrow y$.

Therefore (H_a, ∨, ∧, $\rightarrow_a$, a) is a skew Heyting algebra so that H is a skew HADL.■

Corollary 2.2.1 For any a ∈ H, [a, m] is a Heyting algebra.

Proof, Since H is skew HADL with maximal element m. For each a ∈ H, a ≤ m. Hence, a ∈ H_m. Now, from the definition of skew HADL, H_m is skew Heyting algebra. Then by definition of skew Heyting algebra for any a ∈ H_m, ([a, m], ∨, ∧ $_a$→ a, m) is Heyting algebra.■

Lemma 2.2.1 For any b ∈ H, [b, m] is a skew HADL.

Proof, From **Corollary 2.2.1** for any b ∈ H, [b, m] is Heyting algebra. Following this for any c ∈ [b, m], we have [b, c] is a Heyting algebra. Since for all $x, y \in [b, m]$ the induced binary operation $_b$→ on [b, m] from $_b$→ on [b, c] is defined as $x\ {}_b\!\!\rightarrow y = (y \vee x \vee y)\ {}_y\!\!\rightarrow y$ so that [b, c] is a skew Heyting algebra with top element c. Therefore, [b, m] is a skew HADL.■

Lemma 2.2.2 Let H be skew HADL. Then for any $a \in H, H_a$ is skew HADL.

Proof, Suppose that H is skew HADL. Then for any $a \in H, H_a$ is skew Heyting algebra. Take any $b \in H_a$, as $b \in H, H_b$ is also skew Heyting algebra. Therefore H_a is skew HADL.■

Corollary 2.2.2 Let H be a skew HADL. If x, y ∈ H such that x ≤ y and a, b ∈ [y, m], then

$a\ {}_x\!\!\rightarrow b = a\ {}_y\!\!\rightarrow b$.

Proof, Let x, y ∈ H such that x ≤ y, Then [y, m] ⊆ [x, m], if a, b ∈ [y, m], then by closurity of binary operation $a\ {}_y\!\!\rightarrow b \in [y, m]$ and hence $a\ {}_y\!\!\rightarrow b \in [x, m]$. By **Corollary 2.2.1** [x, m] and [y, m] are Heyting algebras. Since a, b ∈ [x, m], a $_x\!\!\longrightarrow$ b also belongs to [x, m]. The maximal element characterization of a $_x\!\!\longrightarrow$ b and a $_y\!\!\longrightarrow$ b on the Heyting algebra [x, m] forces the two elements are equal.■

Lemma 2.2.3 Let H be a skew HADL. Then the following conditions hold;

1. $x \rightarrow y = (x\ _{(y \wedge m)}\!\!\longrightarrow y) \wedge m$
2. $x \wedge (x \rightarrow y) = x \wedge y \wedge m$
3. $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$
4. $x \rightarrow y = (x \rightarrow y) \wedge m$, for all, x, y, z ∈ H.

Proof, Let H is a skew HADL and $x, y, z \in H$. Since by **corollary 2.2.1** for any $y \in H$ $[y, m]$ is Heyting algebra and $y \wedge m \leq y$ implies $[y, m] \subseteq [y \wedge m, m]$, Thus by **theorem 1.2.5**

$[y \wedge m, m]$ is HADL. Now, we have;

1. $(x\ _{(y \wedge m)}\!\!\longrightarrow y) \wedge m = (x\ _{(y \wedge m)}\!\!\longrightarrow y) \wedge m \wedge m$ …………….. [Since m ∧ m = m]

$= (x\ _{(y \wedge m}\!\!\longrightarrow y) \wedge (x\ _{(y \wedge m)}\!\!\longrightarrow m) \wedge m$ …………… [By Lemma 1.2.3(3)]

$= ((x\ _{(y \wedge m)}\!\!\longrightarrow (y \wedge m)) \wedge m$ ………………….. [Since [y ∧ m, m] is HADL]

$= (x\ _{(y \wedge m)}\!\!\longrightarrow (m_{(y \wedge m)}\!\!\longrightarrow y)) \wedge m$ …………….. [By lemma 1.2.3(2)]

$= (m\ _{(y \wedge m)}\!\!\longrightarrow (x_{(y \wedge m)}\!\!\longrightarrow y)) \wedge m$ ………………. [By Lemma 1.2.3(5)]

$= ((x \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow y) \wedge m \wedge m$ ………………… [By Lemma 1.2.3(4)]

$= ((x \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow y) \wedge ((x \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow m) \wedge m$ … [Since $[(x \wedge m)\ _{(y \wedge m)} \longrightarrow m = m$]

$= ((x \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow (y \wedge m)) \wedge m$ ……………… [By Definition 1.2.9 (4)]

$= ((x \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow (y \wedge m)) \wedge ((y \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow (y \wedge m))$…[Since $(y \wedge m)_{(y \wedge m)}\!\!\longrightarrow (y \wedge m) = m$]

$= ((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow (y \wedge m)$ ………………………… [By definition 1.2.9(5)]

$= x \rightarrow y$ ……………………………… [By Theorem 2.2.1(2)]

(2) $x \wedge (x \rightarrow y) = x \wedge \{((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)\}$ ……. [By Theorem 2.2.1(2)]

$= m \wedge x \wedge \{((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow (y \wedge m)\}$… [Since m is maximal element]

$= x \wedge m \wedge \{((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\longrightarrow (y \wedge m)\}$………….. [∧ is commutative in ADL]

$= x \wedge (x \vee y) \wedge m \wedge \{((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)\}$……. [Absorption law in ADL]

$= x \wedge \{((x \vee y) \wedge m) \wedge \{((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)\}\}$… [∧ is associative in ADL]

$= x \wedge \{((x \vee y) \wedge m) \wedge y\}$…………… [By Definition of Heyting algebra]

$= x \wedge \{(x \vee y) \wedge (y \wedge m)\}$……….. [Associative and commutative laws of ∧ in ADL]

$= x \wedge \{((x \vee y) \wedge y) \wedge m)\}$………………………….. [∧ is associative in ADL]

$= x \wedge (y \wedge m)$……………………………………. [Absorption law in ADL]

$= x \wedge y \wedge m.$

(3) $(x \vee y) \rightarrow z = \{((x \vee y) \vee z) \wedge m\}\ {}_{(z \wedge m)}\!\!\rightarrow (z \wedge m)$……… [By Theorem 2.2.1(2)]

$= \{((x \vee y) \vee z \vee z) \wedge m\}\ {}_{(z \wedge m)}\!\!\rightarrow (z \wedge m)$ ……………….... [Idempotent law in ADL]

$= \{((z \vee x) \vee (y \vee z)) \wedge m\}\ {}_{(z \wedge m)}\!\!\rightarrow (z \wedge m)$ …… [Commutative & associative of ∨ from Theorem 1.2.2(11)]

$= \{((x \vee z) \wedge m)\ {}_{(z \wedge m)}\!\!\rightarrow (z \wedge m)\} \wedge \{((y \vee z) \wedge m)\ {}_{(z \wedge m)}\!\!\rightarrow (z \wedge m)\}$…[Since $[z \wedge m, m]$ is HADL]

$= (x \rightarrow z) \wedge (y \rightarrow z)$…………………………… [By Theorem 2.2.1(2)]

(4) $(x \rightarrow y) \wedge m = ((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)) \wedge m$ … [Since H is skew HADL]

$= \{((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)\} \wedge ((y \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m))$… [By Definition of HADL]

$= \{((x \vee y) \vee y) \wedge m\}\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)$………………… [By Definition 1.2.8 (5)]

$= ((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)$……………… [Since $(x \vee y) \vee y = x \vee y$]

$= x \rightarrow y$……………………………………… [As H is skew HADL]. ■

Corollary 2.2.3 Let x, y ∈ H, then $(x\ {}_{y}\!\!\rightarrow y) \wedge m = (x \rightarrow y) \wedge m$.

Proof, Let H is a skew HADL and $x, y \in H$. Then

$(x \rightarrow y) \wedge m = ((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)) \wedge m$

$= \{((x \vee y) \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)\} \wedge ((y \wedge m)\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m))$ … [By Definition 1.2.9(1)]

$= \{((x \vee y) \vee y) \wedge m\}\ {}_{(y \wedge m)}\!\!\rightarrow (y \wedge m)$ ……………… [Since$[y \wedge m, m]$ is HADL]

$= ((x \lor y) \land m)\ _{(y \land m)}\!\!\rightarrow (y \land m)$ .... [Since $(x \lor y) \lor y = x \lor y$]

$= ((x \land m)\ _{(y \land m)}\!\!\rightarrow (y \land m)) \land ((y \land m)\ _{(y \land m)}\!\!\rightarrow (y \land m))$ ...... [Since $[y \land m, m]$ is HADL]

$= ((x \land m)\ _{(y \land m)}\!\!\rightarrow y) \land ((x \land m)\ _{(y \land m)}\!\!\rightarrow m) \land ((y \land m)\ _{(y \land m)}\!\!\rightarrow (y \land m))$

$= ((x \land m)\ _{(y \land m)}\!\!\rightarrow y) \land m \land m$

$= (m\ _{(y \land m)}\!\!\rightarrow (x\ _{(y \land m)}\!\!\rightarrow y)) \land m$ .... [By Lemma 1.2.3 (5)]

$= (x\ _{(y \land m)}\!\!\rightarrow (m\ _{(y \land m)}\!\!\rightarrow y)) \land m$ .... [By Lemma 1.2.3 (4)]

$= (x\ _{(y \land m)}\!\!\rightarrow (y \land m)) \land m$ .... [By Lemma 1.2.3(2)]

$= (x\ _{(y \land m)}\!\!\rightarrow y) \land (x\ _{(y \land m)}\!\!\rightarrow m) \land m$ .... [Since $[y \land m, m]$ is HADL]

$= (x\ _{(y \land m)}\!\!\rightarrow y) \land m \land m$ .... [By Lemma 1.2.3(3)]

$= (x\ _{y}\!\!\rightarrow y) \land m$ .... [By Corollary 2.2.2] ■

Lemma 2.2.4 Let $x, y, z \in H$ *such that* $x \land m = y \land m$. Then the following statements hold;

1. $x \rightarrow y = m$
2. $x \rightarrow z = y \rightarrow z$ and $z \rightarrow x = z \rightarrow y$

Proof, Since H is skew HADL with maximal element m and for $x, y \in H, x \land m = y \land m$.

Then, (1) $x \rightarrow y = ((x \lor y) \land m)\ _{(y \land m)}\!\!\rightarrow (y \land m)$

$= ((x \land m) \lor (y \land m))\ _{(y \land m)}\!\!\rightarrow (y \land m)$

$= ((y \land m) \lor (y \land m))\ _{(y \land m)}\!\!\rightarrow (y \land m)$.......... [Since $x \land m = y \land m$]

$= (y \land m))\ _{(y \land m)}\!\!\rightarrow (y \land m)$.............. [Idempotent law]

$= m$

Therefore, $x \land m = y \land m$ implies $x \rightarrow y = m$.

(2) $x \rightarrow z = ((x \lor z) \land m)\ _{(z \land m)}\!\!\rightarrow (z \land m)$ .... [By Theorem 2.2.1 (2)]

$= ((x \land m) \lor (z \land m))\ _{(z \land m)}\!\!\rightarrow (z \land m)$

$= ((y \land m) \lor (z \land m))\ _{(z \land m)}\!\!\rightarrow (z \land m)$ ....[Since $x \land m = y \land m$]

$= ((y \vee z) \wedge m)_{(z \wedge m)} \rightarrow (z \wedge m)$

$= y \rightarrow z$................................ [By Theorem 2.2.1 (2)]

And similarly, $z \rightarrow x = ((z \vee x) \wedge m)_{(x \wedge m)} \rightarrow (x \wedge m)$ .... [By Theorem 2.2.1 (2)]

$= ((z \wedge m) \vee (x \wedge m))_{(x \wedge m)} \rightarrow (x \wedge m)$

$= ((z \wedge m) \vee (y \wedge m))_{(y \wedge m)} \rightarrow (y \wedge m)$ .... [Since $x \wedge m = y \wedge m$]

$= ((z \vee y) \wedge m)_{(y \wedge m)} \rightarrow (y \wedge m)$

$= z \rightarrow y$ .... [By Theorem 2.2.1(2)] ■

Definition 2.2.1([11]) For any congruence relation θ on an ADL H and a $\in H$, Now define

$[a]_\theta$= {b$\in H$ ı$(a,b) \in \theta$} and it is called congruence class containing a.

Definition 2.2.2 ([5]) Let D is congruence on a skew lattice S, each congruence class is rectangular sub algebra if x $\wedge y = y \vee x$ for all x, y$\in D$.

Theorem 2.2.2 Let H be an ADL with a maximal element m. If H is a skew HADL and θ defined by $\theta = \{(x,y) \in H \times H \mid x \wedge y = y$ and $y \wedge x = x\}$ is a relation on H, then the following conditions hold;

1. θ *is a congruence relation on* H
2. The congruence classes are the maximal rectangular sub algebras of H.

Proof, Suppose H is a skew HADL.

(1) Let $x, y, z, a \in H$, since $x \wedge x = x$ so that θ is reflexive and let $x \wedge y = y$ as H is ADL $x \wedge y = y = y \wedge x$. Hence θ is symmetric.

Assume that $x\,\theta\,y$ and $y\,\theta\,z$.Then $x \wedge y = y$, $y \wedge x = x$,$y \wedge z = z$.
And z ∧ y = y.
As $z \wedge x = z \wedge y \wedge x = y \wedge x = x$ and
x ∧ z = y ∧ x ∧ z = x ∧ y ∧ z = y ∧ z = z, it follows that $x\,\theta\,z$. Consequently θ is transitive and hence it is an equivalence relation. To show that θ is a congruence relation it suffices to show θ satisfies **Lemma (1.2.4).** Given that $x\,\theta\,y$ and $a \in H$, Then
$(y \wedge a) \wedge (x \wedge a) = y \wedge x \wedge a$ .... [Since H is an ADL]
$= x \wedge a$

And $(x \wedge a) \wedge (y \wedge a) = x \wedge y \wedge a = y \wedge a$, and hence$(x \wedge a)\, \theta\, (y \wedge a)$.

Also $(x \vee a) \wedge (y \vee a) = ((x \vee a) \wedge y) \vee ((x \vee a) \wedge a)$ … [Left distributive of ∧ over ∨]

$= ((x \wedge y) \vee (a \wedge y)) \vee a$

$= y \vee a$

And $(y \vee a) \wedge (x \vee a) = ((y \vee a) \wedge x) \vee ((y \vee a) \wedge a)$

$= ((y \wedge x) \vee (a \wedge x)) \vee a$

$= x \vee a$

Hence, $(x \vee a)\, \theta\, (y \vee a)$

Finally, it remains to show that $(x \rightarrow a)\, \theta\, (y \rightarrow a)$ *and* $(a \rightarrow x)\, \theta\, (a \rightarrow y)$ hold.

From the property of ADLs given by **Theorem (1.2.1)** and the given conditions,

$x \wedge y = y$ *and* $y \wedge x = x$, One can observe that

$x \wedge m = y \wedge x \wedge m$

$= x \wedge y \wedge m$

$= y \wedge m$

Thus, by (2) of **Lemma (2.2.4)** $a \rightarrow x = a \rightarrow y$ and $x \rightarrow a = y \rightarrow a$.

Then, $x \rightarrow a \wedge y \rightarrow a = y \rightarrow a \wedge y \rightarrow a$

$= (y \vee y) \rightarrow a$ ……………….. [By Lemma 2.2.3(3)]

$= y \rightarrow a$

And $y \rightarrow a \wedge x \rightarrow a = x \rightarrow a \wedge x \rightarrow a$

$= (x \vee x) \rightarrow a$

$= x \rightarrow a$.

Similarly, $a \rightarrow x \wedge a \rightarrow y = a \rightarrow y \wedge a \rightarrow y$

$= (a \vee a) \rightarrow y$

$= a \to y.$

And $a \to y \wedge a \to x = a \to x \wedge a \to x$

$= (a \vee a) \to x$

$= a \to x.$

This implies $(x \to a)\, \theta\, (y \to a)$ and $(a \to x)\, \theta\, (a \to y)$.

Hence θ is a congruence relation.

(2) Suppose $x, y \in [z]_\theta$, Then $x\, \theta\, y$. Hence $y \vee x = y \vee (y \wedge x) = y = x \wedge y$.

Therefore, each congruence class is rectangular sub algebra of L.

Let R be the set of all rectangular sub algebras of L. Now take an arbitrary congruence class $[x]_\theta$ for some $x \in L$. Let T be a rectangular sub algebra of L such that $[x]_\theta \subseteq T$ and let $r \in T$. Since $x \in T$ and T is rectangular sub algebra of L. Now $r \vee x = x \wedge r$ and $x \vee r = r \wedge x$.

Since in an ADL $(r \wedge x) \wedge r = x \wedge r$ and thus, $x \wedge r = (r \wedge x) \wedge r = (x \vee r) \wedge r = r$ which implies that

$r \vee x = x \wedge r = r$ and

$r \wedge x = x \vee r = x \vee (x \wedge r) = x$. Hence, $r \in [x]_\theta$

Therefore $T \subseteq [x]_\theta$ and now we conclude that $[x]_\theta = T$. Hence $[x]_\theta$ is a maximal element of R, i.e. each congruence class is a maximal rectangular sub algebra of L.■

Theorem 2.2.3 Let H be an ADL with a maximal element m and $0 \notin H$. Then H is a skew HADL if and only if for any $a, z \in H$ such that $a \in H_z$ and $w, x, y \in [a, z]$, the following conditions hold;

1. $x \leq w \,{}_a\!\!\to x$
2. $(x \,{}_a\!\!\to y) \wedge w = w$ if and only if $y \wedge w \wedge x = w \wedge x$.

Proof, Let H is a skew HADL.

(1) For any $z \in H$, H_z is a skew Heyting algebra. Let $a \in H_z$ and $w, x, y \in [a, z]$. Since [a, z] is Heyting algebra. Then, $x \leq w \,{}_a\!\!\to x$ ………………… [By Definition 1.2.8(3)]

(2) Assume that $(x \,{}_a\!\!\to y) \wedge w = w$. Then,

$w \wedge x = ((x \,_a\!\!\rightarrow y) \wedge w) \wedge x$

$= x \wedge ((x \,_a\!\!\rightarrow y) \wedge w)$………… [Commutative of ∧ on ADL]

$= (x \wedge (x \,_a\!\!\rightarrow y)) \wedge w)$………… [Associative of ∧ on ADL]

$= (x \wedge y) \wedge w$…………….. [Since [a, z] is Heyting algebra]

$= y \wedge w \wedge x.$

On the other hand given that $y \wedge w \wedge x = w \wedge x$, Then

$x \,_a\!\!\rightarrow (w \wedge x) = x \,_a\!\!\rightarrow (y \wedge w \wedge x)$

$= (x \,_a\!\!\rightarrow y) \wedge (x \,_a\!\!\rightarrow w) \wedge (x \,_a\!\!\rightarrow x)$

$= (x \,_a\!\!\rightarrow y) \wedge (x \,_a\!\!\rightarrow w) \wedge z$ ……………… (∗)

But, $x \,_a\!\!\rightarrow (w \wedge x) = (x \,_a\!\!\rightarrow w) \wedge (x \,_a\!\!\rightarrow x)$

$= (x \,_a\!\!\rightarrow w) \wedge z$ ……………….. (∗∗)

Hence, from (∗) and (∗∗) we have $(x \,_a\!\!\rightarrow w) \wedge z = (x \,_a\!\!\rightarrow y) \wedge (x \,_a\!\!\rightarrow w) \wedge z$

Therefore, $(x \,_a\!\!\rightarrow y) \wedge w = (x \,_a\!\!\rightarrow y) \wedge (x \,_a\!\!\rightarrow w) \wedge w$ … [Since $w \leq x \,_a\!\!\rightarrow w$ & $(x \,_a\!\!\rightarrow w) \wedge w = w$]

$= (x \,_a\!\!\rightarrow y) \wedge (x \,_a\!\!\rightarrow w) \wedge z \wedge w$ …. [Since z is maximum element in [a, z]]

$= (x \,_a\!\!\rightarrow w) \wedge z \wedge w$

$= z \wedge (x \,_a\!\!\rightarrow w) \wedge w$

$= z \wedge w$

$= w$

Conversely, let $z \in H$ and assume that (1) and (2) hold.

Now for any $a \in H_z$ take $c, d, e \in [a, z]$. By (1), $c \leq e \,_a\!\!\rightarrow c$ such that $e \,_a\!\!\rightarrow c \in [a, z]$ and then

$d \leq e \,_a\!\!\rightarrow c \Leftrightarrow (e \,_a\!\!\rightarrow c) \wedge d = d$ ……. [By Theorem 1.2.1 (3 and 13)]

$\Leftrightarrow c \wedge d \wedge e = d \wedge e$ ………………………[By (2)]

$\Leftrightarrow d \wedge e \wedge c = d \wedge e$………………………..[Commutative of ∧ on ADL]

$\Leftrightarrow d \wedge e \leq c$ …………………………….. [By Theorem 1.2.1(13 and 3)]

Hence [a, z] is a Heyting algebra.

One can define an induced binary operation $\rightarrow_z$ on H_z by $x \rightarrow_z y = (y \vee x \vee y)\ _y\!\!\rightarrow y$

Hence H_z is a skew Heyting algebra and therefore H is a skew HADL.■

Corollary 2.2.4 For any x, y ∈ H, $((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m) = m$ if and only if $y \wedge x = x$.

Proof, Assume $((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m) = m$. Then

$x = m \wedge x$

$= \{((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)\} \wedge x$

$= x \wedge \{((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)\} \wedge x$ ... [Idempotent law and commutative of ∧ on ADL]

$= x \wedge ((x \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)) \wedge m \wedge x$ ……………. [By Lemma 2.2.3 (3)]

$= x \wedge m \wedge ((x \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)) \wedge x$ ………….. [Commutative of ∧ in ADL]

$= x \wedge m \wedge y \wedge m \wedge x$ …………………….. [By Lemma 2.2.3(2)]

$= m \wedge y \wedge x$ ……………………………. [Since H is an ADL]

$= y \wedge x$

Hence $y \wedge x = x$.

Conversely, suppose $y \wedge x = x$. Then,

$((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m) = \{((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)\} \wedge m$ ….. [By Lemma 2.2.3(4)]

$= \{((x \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)\} \wedge ((y \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m))$

$= (((x \vee y) \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)$

$= ((((y \wedge x) \vee y) \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)$…… [Since $y \wedge x = x$]

$= ((y \vee y) \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)$ ………… [As $(y \wedge x) \vee y = y$]

$= (y \wedge m)\ _{(y \wedge m)}\!\!\rightarrow (y \wedge m)$

$= m$. ■

Conclusion

Skew HADL is defined by using skew Heyting algebra and express in terms of almost distributive lattice with different conditions. In this project work; ADL, HADL, Heyting algebra and skew Heyting algebra are used:

- To define skew HADL and related concepts
- To prove heorems, corollaries and lemmas which are expressed in a skew HADL

Finally, we conclude that an equivalence relation θ on a skew HADL is a congruence relation on each equivalence class and each congruence class is a maximal rectangular sub-algebra of the equivalence class.

In this project work, from the development of skew Heyting almost distributive lattice by B.Assaye, M.Alemineh Yeshiwas Mebrat I have learnt how Mathematical journals are published?

References

[1] B. Assaye, M.Alemneh, and D. Nigussie congruence relations on Heyting algebra international journal of mathematics virtual institute Vol. 9 (2019), 245-253

[2] B. Assaye, M.Alemneh, and Yeshiwas Mebrat skew semi- Heyting Almost Distributive Lattice international journal of mathematics and its applications volume 5,(2017), 359-369

[3] Christopher Policastro, Lecture note on algebraic combinatrics March 1, 2011

[4] K. Cvetko-Vah Non-commutative frames of Heyting algebras arXive:1702.04949v2 [math. RA]20Dec2017.

[5] K. Cvetko-Vah. On skew Heyting algebras. Ars.Mathematica contemporanea,12 (1) (2017), 37–50

[6] G. C. Rao. Almost Distributive Lattices, PhD thesis, Department of Mathematics, Andhra University, 1980.

[7] G. C Rao and B. Assaye closed Elements in of Heyting in Almost Distributive Lattice. International Jour. of computational cognition.

[8] G.C Rao and Naveen Kakuman pseudo-supplemented Almost Distributive Lattice Department of Mathematics Andhra University Viskhapatnam-530003 Andhra Pradesh India.

[9] G.C Rao and M.Sambassive Rao Annulets in ADL European Journal of pure and applied mathematics

[10] G.Gratzer Lattice theory first concept & Distributive Lattice, University of Mantobia.

[11] Gezahagn Mulat Addis, on congruence relations an almost distributive lattice-II Asian-European Journal of mathematics vol 13,No-1(2021)

[12] M.Sambassive Rao and G.C Rao The Space of Prime Ordered Filters of Heyting Almost Distributive Lattices South east Asian Bulletin of mathematics,(2015) 39:865-875.

[13] U. M. Swamy and G. C. Rao. Almost distributive lattices, J. Austral. Math. Soc. 31(1) (1981), 77–91.