2021-09-30

Weak LI-ideal in Lattice Implication Algebra

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A Project on

Weak LI-ideal in Lattice Implication Algebra

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August, 2021

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A project Submitted to the Department of Mathematics in Partial Fulfilment of the Requirements for the Degree of **Master of Science in Mathematics**

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Approval of the project for defence

I hereby certify that I have supervised, read and evaluated this project entitled **Weak LI-ideal in lattice implication algebra** by **Haile Asmir Gete** prepared under my guidance. I recommend that the project is submitted for oral defence.

Advisor's Name: _______________________signature: _______date: ___/___/___

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A project Submitted to the Department of Mathematics, College of Science, Bahir Dar University in Partial Fulfilment of the Requirements for the Degree of **“Master of science in Mathematics”.**

Approved by Board of Examiners

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Acknowledgments

First, I would like to say bless to GOD for all. Next I would like express my deepest graduated to my adviser Dr. Tilahun Mekonnen (PhD) for his valuable guidance, valuable comments and suggestions that helped me during the time of the project by devoting his previous time. I would like to express my gratitude to the department of mathematics at Bahir Dar University for providing the opportunity and facilitates this project work.

Finally, I would also like to thank, my friends Getalem Tesfew, Yalga Tigabu, and to all others who helped me directly or indirectly to completing this project.

Abstract

In this project work, the notions of weak LI-ideals and maximal weak LI-ideals of lattice implication algebras are introduced, respectively. The properties of WLI-ideals are investigated. Some relationships and characterizations of WLI-ideals in lattice implication algebras are studied. Furthermore, this project discussed about the extension theorems of LI-ideals in lattice implication algebras. And prove that

(1) Every ILI-ideal of a lattice implication algebra is WLI-ideal.

(2) Every lattice ideal in lattice H implication algebra is WLI-ideal.

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Chapter 1

Introduction and Preliminaries

1.1. Introduction

In the field of many-valued logical system, lattice valued plays an important role [5, 7]. Also, Goguen [1], Pavelak [3], and Novak [4] researched on this lattice-valued logic formal systems. Moreover, in order to research the many-valued logical systems whose propositional value is given in a lattice, in 1993. Xu [6] proposed the notion of lattice implication algebras and investigated many useful properties. Since then logical algebra has been extensively investigated by several researchers [1, 5, 8]. In [9] Jun, and Xu defined the concept of LI-ideal in lattice implication algebras and discussed its some properties. For the general development of lattice implication algebras, the ideal theory plays an important role [11, 12].

In this project, we understand the definitions and examples of WLI-ideals in lattice implication algebras as an extension of LI-ideals in lattice implication algebras.

1.2. Preliminaries

In this section, we preliminaries which will be useful in our discussions the main text of the project.

Definition1.2.1: [13, 15] A binary relation $\leq$ defined on a non-empty set P is called an **ordering relation** or **partial ordering relation** if it satisfies the following axioms:

1) $x \leq x$ ……………………………..…… [reflexive]
2) $x \leq y$ and $y \leq x$ implies $x = y$ ………………[anti-symmetric]
3) $x \leq y$ and $y \leq z$ implies $x \leq z$ ………………[transitive] , for any $x, y, z \in P$.

In this case, $(P, \leq)$ is called a partially ordered set or simply poset.

Definition1.2.2: [13,15] Let *A* be a subset of a poset *P*. An element $a \in$P is an **upper bound** for *A* if and only if $x \leq a$ for every $x \in A$. An element $a \in$P is the **least upper bound** of *A* **(supremum or sup A)** if *a* is an upper bound of *A* and *a* is the least from all upper bounds of *A*. Whereas, an element $a \in$P is a **lower bound** for *A* if and only if $a \leq x$ for every $x \in A$. An element a is the **greatest lower bound** of *A* **(infimum or inf A)** if *a* is a lower bound of *A* and *a* is the greatest from all lower bounds of *A*.

Definition1.2.3: [12, 13] A non-empty set L together with two binary operations ***meet*** $(\wedge)$ and ***join*** $(\vee)$ on L is called **lattice** if it satisfies the following properties. For all $x, y, z \in L$

L_1: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (Commutative laws);

L_2: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *and* $x \vee (y \vee z) = (x \vee y) \vee z$ (Associative laws)

L_3: $x \wedge x = x$ and $x \vee x = x$ (Idempotent laws)

L_4: $x \vee (x \wedge y) = x \wedge (x \vee z)$ (Absorption laws)

Definition1.2.4: [12, 15] A poset $(L, \leq)$ is lattice if and only if both sup $\{x, y\}$ and inf $\{x, y\}$ exists in *L*. And we write sup $\{x, y\} = x \vee y$ inf $\{x, y\} = x \wedge y$ for $x, y \in L$

Note that: The two definition of a lattice are equivalent in the following sense:

(a) If L is a lattice by the first definition, then define $\leq$ on L by $x \leq y$ if and only if $x = x \wedge y$ and $y = x \vee y$.

(b) If L is a lattice by the second definition, then defin the operations $\vee$ and $\wedge$ by $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ respectively.

Definition1.2.5: [6, 9] A bounded lattice is an algebraic structure $L = (L, \wedge, \vee, 0, 1)$ such that $(L, \wedge, \vee)$ is a lattice and the constants $0, 1 \in L$ satisfies the following conditions.

1) for all $x \in L, x \wedge 1 = x$ and $x \vee 1 = 1$,
2) for all $x \in L, x \wedge 0 = 0$ and $x \vee 0 = x$.

The element 1 is called the upper bound or top of L and the element 0 is called the lower bound or bottom of L

Definition 1.2.6: [6, 10] Let L be a bounded lattice. A unary operation $'$ is an order-reversing involution if it is an involution, that is $(x')' = x$ for all $x, y \in L$, that inverts the ordering that is $x \leq y$ imply $y' \leq x'$.

Definition1.2.7: [6, 14] An algebra $(L, \rightarrow, ', 0, 1)$ of type $(2, 1, 0, 0)$ called implicative algebra if it satisfies the following properties

1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
2) $x \rightarrow y = y' \rightarrow x'$
3) $1 \rightarrow x = x$
4) $x \rightarrow 1 = 1$
5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$
6) $0' = 1$ for all $x, y, z \in L$

Lemma 1.2.8: [14] Let *A* be an implicative algebra, then for any element $x, y \in A$ we have:

(1) $(x')' = x$ (Involution law).

(2) $x \rightarrow x = 1$ (Identity).

(3) $x' = x \to 0$.

(4) $x \leq y$ implies $y' \leq x'$ (Order-reversing)

Definition1.2.9: [6, 7, 11] A bounded lattice $(L, \wedge, \vee, 0, 1)$ with order-reversing involution $'$ and a binary operation $\to$ is called lattice implication algebra if it satisfying the following axioms:

(L_1) $x \to (y \to z) = y \to (x \to z)$,

(L_2) $x \to x = 1$,

(L_3) $x \to y = y' \to x'$,

(L_4) $x \to y = y \to x = 1$ $imply$ $x = y$,

(L_5) $(x \to y) \to y = (y \to x) \to x$,

(L_6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(L_7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$, for all $x, y, z \in L$.

Definition1.2.10: [6, 9] A lattice implication algebra L is called a **lattice H implication** algebra if it satisfies $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in L$.

Lemma1.2.11: [6, 9] In a lattice H implication algebra L, the following holds

(1) $x \to (x \to y) = x \to y$

(2) $x \to (y \to z) = (x \to y) \to (x \to z)$

(3) $(x \to y) \to x = x$

(4) $x \vee x' = 1$

(5) $x \vee y = (x \to y) \to y$ for all $x, y, z \in L$.

Definition 1.2.12: [6, 9] In a lattice implication algebra$L$, we can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \to y = 1$.

Lemma1.2.13: [6, 9] In a lattice implication algebra L, the following holds

(1) $0 \to x = 1$

(2) $1 \to x = x$

(3) $x \to 1 = 1$

(4) $x \to 0 = x \to 0$

(5) $(x \to y) \leq (y \to z) \to (x \to z)$

(6) $x \vee y = (x \to y) \to y$ and $x \wedge y = ((x' \to y') \to y)'$

(7) $x \leq y$ implies $(y \to z) \leq (x \to z)$ and $(z \to x) \leq (z \to y)$

(8) $x \leq (x \to y) \to y$

(9) $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$ (De Morgan's law) for all $x, y, z \in L$

Definition1.2.14: [7, 9, 11] Let L be lattice implication algebra. An LI-ideal A is non-empty subset of L such that for any $x, y \in L$,

$(LI1).\ 0 \in A,$

$(LI2).\ (x \to y)' \in A$ and $y \in A$ imply $x \in A$.

Theorem 1.2.15: [9] Let A be LI-ideal of a lattice implication algebra L and $x \in A$. If $y \leq x$ then $y \in A$. for all $y \in L$.

Definition1.2.16: [9, 15] Let L be a non-empty set and $(L, \vee, \wedge)$ be a lattice. A non-empty subset I of L is called a lattice ideal if:

(I_1) $x \in I, y \in L$ and $y \leq x$ imply that $y \in I$,

(I_2) $x, y \in I$ implies $x \vee y \in I$.

Theorem1.2.17: [9] Let L be a lattice implication algebra. Every LI-ideal A of L is a lattice ideal.

Proof Suppose that A is an LI-ideal of L.

Claim A is lattice ideal.

Let $y \leq x$ for $x \in A$, $y \in L$, then $(y \to x)' = 1' = 0 \in A$ imply $y \in A$ (since A is LI-ideal)

Suppose $y \in A$, and $(x \to y)' \in A$ then $x \in A$ (since A is LI-ideal). Moreover $((x \vee y) \to y)' \in A$ and $y \in A$ imply $(x \vee y) \in A$. Therefore A is a lattice ideal■

Theorem1.2.18: [9] In lattice H implication algebra L, every lattice ideal is an LI-ideal.

Proof: Let A be lattice ideal of L. Assume that $(x \to y)' \in A$ and $y \in A$.

Note that

$$y \vee (x \to y)' = (y \to (x \to y)') \to (x \to y)' \text{ (by Lemma1.2.11(5))}$$

$$= ((x \to y) \to y') \to (x \to y)' \text{ (by Definition 1.2.7 (2))}$$

$$= (x \to y) \to (y')' \text{ [by lemma 1.2.11]}$$

$$= (x \to y) \to y \text{ (by involution)}$$

$$= x \vee y \text{ (by Lemma1.2.11(5))}$$

It follows from Definition 1.2.16(I_2) $x \vee y = y \vee (x \to y)'$

Since $x \leq x \vee y$ by Definition 1.2.16(I_1) we have $x \in A$. and $0 \in A$. Hence, A is an LI-ideal of L.

Definition1.2.19: [7] A non-empty subset A of a lattice implication algebra L is said to be **Implicative LI-ideal** (ILI-ideal) if it satisfies

$(ILI1)$ $0 \in A$, and

$(ILI2)$ $(((x \to y)' \to y)' \to z)' \in A$ and $z \in A$ imply

$(x \to y)' \in A$,for $x, y, z \in L$.

Theorem1.2.20: [11] Let A be a non-empty subset of a lattice implication algebra L. Then A is an ILI-ideal of L if and only if it satisfies for all $z \in A$ and $y, (x \to y)' \in L, ((x \to y)' \to y)' \leq z$ implies $(x \to y)' \in A$.

Proof. Suppose A is an ILI-ideal of L.

Let $z \in A$ and $(x \to y)' \in L$. $((x \to y)' \to y)' \leq z$

$\Leftrightarrow ((x \to y)' \to y)' \to z = 1$

$\Leftrightarrow (((x \to y)' \to y)' \to z)' = 0 \in A$.

Then we have $(((x \to y)' \to y)' \to z)' \in A$ implies $(x \to y)' \in A$ holds.

Moreover A is an ILI-ideal of L.■

Definition1.2.21: [7, 8, 10] A subset F of a lattice implication algebra L is called a *filter* of L if it satisfies:

(F$_1$) $1 \in F$,

(F$_2$) $x \in F$ and $x \to y \in F$ then $y \in F$ for all $x, y \in L$.

Theorem1.2.22: [11] Let A be a non-empty subset of a lattice implication algebra L. Then A is a filter of L if and only if A' is an LI-ideal of L.

Proof: Assume that A is a filter of L, then $1 \in A$, and so $0' = 1 \in A'$.

Let $(x \to y) \in A'$ and $y \in A'$ for all $x, y \in L$,

let $(x \to y) = u'$ and $y = v'$ for some $u, v \in A$.

Thus $v \to x = x \to v = (x \to y)')' = (u')' = u \in A$.

Since A is a filter, we have $x' \in A$, and so $x = (x')' \in A'$.

This prove that A' is an LI-ideal of L.

Conversely, suppose that A' is an LI-ideal of L. Then $1 \in A$ since $1' = 0 \in A'$.

Let $x, y \in L$ be such that $x \in A$ and $x \to y \in A$.

Then $x' \in A'$and $(y' \to x')' = (x \to y)' \in A'$,

as A´ is an LI-ideal, it follows from definition of LI-ideal (LI2) that y´ ∈ A or y ∈A. Hence A is a filter of L.

Chapter 2

Weak LI-ideal in lattice implication algebra

In this chapter, we discuss the definition of weak LI-ideal of lattice implication algebra which is analogous to LI-ideal of lattice implication algebra and some basic result, give examples of weak LI-ideals in lattice implication algebra.

2.1 Definitions and examples of WLI-ideal in lattice implication algebra

In this section we discuss the notion of weak LI-ideal in lattice implication algebra and examples of weak LI-ideal in lattice implication algebra.

Definition2.1.1: A non-empty subset A of a lattice implication algebra L is called a weak LI-ideals of L if it satisfy the condition

$((x \to y)' \to y)' \in A$ if $(x \to y)' \in A$, for all x, y ∈ L.

The following example shows that there exists a WLI-ideal in lattice implication algebra.

Example 2.1.1: Let $A = \{1, 0\}$ be a set where 0 is the smallest and 1 is the largest element in L.

Now take $x = 0, y = 1$, then $(0 \to 1)' = 0 \in A$ implies $((0 \to 1)' \to 1)' = 0 \in A$;

and also take $x = 1, y = 0$, then $(1 \to 0)' = 1 \in A$ implies

$((1 \to 0)' \to 0)' = 1 \in A$.

Hence A is WLI-ideal.

Example 2.1.2. Let $B = \{0\}$ be a set.

Let $x = 0, y = 0$, then $(0 \to 0)' = 1' = 0 \in B$

implies $((0 \to 0)' \to 0)' = (1' \to 0)' = (0 \to 0)' = 1' = 0 \in B$. Therefore B is WLI-ideal.

Definition 2.1.2: [11] Let L be a lattice implication algebra, a non-empty subset A of L is called a weak filter of L if it satisfy the condition:

If $(x \to y) \in A$, then $x \to (x \to y) \in A$, for all $x, y \in L$.

Theorem 2.1.3: Let L be a lattice implication algebra, $A \subseteq L$ is an LI-ideal of L. Then A is a WLI-ideal of L.

Proof: Suppose that A is an LI-ideal of L.

Claim: A is a WLI-ideal of L.

Let $(x \to y)' \in A$ for all $x, y \in L$. Then,

$(((x \to y)' \to y)' \to (x \to y)')' = (((x \to y)')' \to ((x \to y)' \to y)')'$

(by Definition 1.2.9 of (L_3)).

$= ((x \to y) \to ((x \to y)' \to y)))'$ (Involution law)

$= ((x \to y) \to (y' \to (x \to y)))'$ (by Definition 1.2.9 of(L_3)).

$= (y' \to ((x \to y) \to (x \to y)))'$ (by Definition 1.2.7 of(1)).

$=(y' \to 1)' = 1' = 0 \in A$,

i.e., $(((x \to y)' \to y)' \to (x \to y)')' \in A$. Thus, $((x \to y)' \to y)' \in A$ as A is an LI-ideal and $(x \to y)' \in A$.

Therefore A is a WLI-ideal of L.

2.2 Some relationships and characterizations of WLI-ideals in lattice implication algebras

Theorem 2.2.1: Let L be lattice implication algebra. Every ILI-ideal A of L is WLI-ideal.

Proof: Suppose that A is an ILI-ideal of a lattice implication algebra L, and $(x \to y)' \in A$ for all $x, y \in L$.

Claim: A is a WLI-ideal of L

Since $(((((x \to y)' \to y)' \to 0)' \to 0)' \to (x \to y)')'$

$= ((((x \to y)' \to y)' \to 0)' \to (x \to y)')'$ (by Lemma1.2.8(3))

$= (((x \to y)' \to y)' \to (x \to y)')'$

$= ((x \to y) \to ((x \to y)' \to y))'$ (by Definition 1.2.9 of(L_3)).

$$= ((x \to y) \to (y' \to (x \to y))'$$

$$= (y' \to (x \to y) \to (x \to y))'$$

$$= (y' \to 1)' = 0 \in A.$$

Hence we obtain $((x \to y)' \to y)' \in A$ by A is an ILI-ideal of L. Therefore A is a WLI-ideal of L.

Theorem 2.2.2: Let A is a non-empty subset of a lattice implication algebra L and $A' = \{x' : x \in A\}$, then A' is a WLI-ideal of L if and only if A is a weak filter of L.

Proof: Assume that for any $x, y \in L, A$ is a weak filter of L.

Since $(x \to y) \in A$, then $(x \to y)' \in A'$ implies $(x \to (x \to y))' \in A'$

i.e., if $(y' \to x')' \in A'$ then $((y' \to x')' \to x')' \in A'$.

Thus,A' is a WLI-ideal of L.

Conversely, Let A' is a WLI-ideal of L and $(x \to y)' \in A'$ implies $((x \to y)' \to y)' \in A'$ for all $x, y \in L$. Since $x \to y = y' \to x' \in A$;

$$(((x \to y)' \to y)')' = (x \to y)' \to y = (y' \to (y' \to x')) \in A.$$

Moreover, we get $y' \to x' \in A$ implies $(y' \to (y' \to x')) \in A$ holds.

Hence A is a weak filter of L.

Theorem 2.2.3: Every lattice ideal in lattice H implication algebra L is a WLI-ideal of L

Proof: Let L be a lattice H implication algebra, A is lattice ideal and $(x \to y)' \in A, y \in A$ for all $x, y \in L$.

For $y \vee (x \to y)' = y \vee (x' \vee y)' = x \vee y$.

Hence $x \vee y \in A$.

It follows

$$y \vee ((x \to y)' \to y)' = y \vee (((x' \vee y)')' \vee y)'$$

$$= y \vee ((x' \vee y) \vee y)'$$

$= y \vee (x \wedge y')$

$= x \vee y$. So that $y \vee ((x \to y)' \to y)' \in A$ since

$((x \to y)' \to y)' \leq y \vee ((x \to y)' \to y)'$.

Therefore $((x \to y)' \to y)' \in A$ as A is lattice ideal of L.

Theorem 2.2.4: Let L be a lattice H implication algebra, if

$\boldsymbol{A}\ (\boldsymbol{t}) = \{x \in L: (x \to t)' = 0\}$ for all elements of L, then $A(t)$ is a WLI-ideal of L.

Proof: Suppose that $(x \to y)' \in A(t)$ for all $x, y \in L$, then

$((x \to y)' \to t)' = 0 \Leftrightarrow ((x \to y) \vee t)' = 0$,

i.e, $(x \to y)' \wedge t' = 0$. Since $(((x \to y)' \to y)' \to t)' = (((x \to y) \vee y)' \to t)'$

$= (((x \to y) \vee y) \vee t)'$

$= ((x \to y)' \wedge y') \wedge t'$

$= ((x \to y)' \wedge t) \wedge y' = 0$

$= 0 \wedge y' = 0$, we have $((x \to y)' \to y)' \in A(t)$.

Therefore $A(t)$ is a WLI-ideal of L by Definition 2.1.1

Theorem 2.2.5: Let L be a lattice H implication algebra, if A is an LI-ideal of L then $A_t = \{x \in L: (x \to t)' \in A\}$ is a WLI-ideal for any $t \in L$.

Proof: Assume that $(x \to y)' \in A_t$ for all $x, y \in L$, then $((x \to y)' \to t)' \in A$. Since

$((((x \to y)' \to y)' \to t)' \to ((x \to y)' \to t)')'$

$= (((x \to y)' \to t) \to (((x \to y)' \to y)' \to t))'$

$= (((x \to y)' \to y)' \to (((x \to y)' \to y)' \to t) \to t))' = (((x \to y)' \to y)' \to ((x \to y)' \vee t))'$

$= ((((x \to y)' \to y)' \to (x \to y)') \vee (((x \to y)' \to y)' \to t))'$

$= (((x \to y) \to ((x \to y)' \to y)) \vee (t' \to ((x \to y)' \to y))'$

$= (((x \to y) \to ((x \to y)' \to y)) \vee (t' \to ((x \to y)' \to y))'$

$= ((y' \to I) \vee (y' \to (t' \to (x \to y))))'$

$= (I \vee (y' \rightarrow (t' \rightarrow (x \rightarrow y))))'$

$= 0 \wedge (((x \rightarrow y)' \rightarrow t)' \rightarrow y)'$

$= 0 \wedge ((x \rightarrow y) \vee t)' \wedge y' = 0 \in A.$

Note that if A is an LI-ideal of L, then $(((x \rightarrow y)' \rightarrow y)' \rightarrow t)' \in A$.

Hence $((x \rightarrow y)' \rightarrow y)' \in A_t$.

Theorem 2.2.6: Let L be a lattice implication algebra, $\{A_i : i \in I\}$ is the set of WLI-ideal of L for I is an index set, then $\bigcup_{i \in I} A_i$ and $\bigcap_{i \cup I} A_i$ are WLI-ideals.

Proof: let $(x \rightarrow y)' \in \bigcup_{i \in I} A_i$ for all $x, y \in L$, then there exists $i \in I$ such that $(x \rightarrow y)' \in A_i$.

Since A_i is WLI-ideal, which imply that $((x \rightarrow y)' \rightarrow y)' \in A_i$ for some $i \in I$.

Hence, we get $((x \rightarrow y)' \rightarrow y)' \in \bigcup_{i \in I} A_i$.

By Definition 2.1.1 $\bigcup_{i \in I} A_i$ is a WLI-ideal of L.

Suppose that $(x \rightarrow y)' \in \bigcap_{i \cup I} A_i$ for any $x, y \in L$, then $(x \rightarrow y)' \in A_i$ for any $i \in I$.

Since A_i is a WLI-ideal, we have $((x \rightarrow y)' \rightarrow y)' \in A_i$ for any $i \in I$.

Thus we have $((x \rightarrow y)' \rightarrow y)' \in \bigcap_{i \cup I} A_i$.

Therefore $\bigcap_{i \cup I} A_i$ is a WLI-ideal.

Remark 2.2.7: Let L be a lattice implication algebra, the intersection of a WLI-ideal of L is also a WLI-ideal by theorem 2.2.6

Suppose $A \subseteq L$, the smallest WLI-ideal containing A is called the WLI-ideal generated by A and denoted by $\langle A \rangle$

Definition 2.2.8: Let L be a lattice implication algebra, a WLI-ideal A of L is called a maximal WLI-ideal if it is not equal to L, and it is a maximal element of the set of all WLI-ideals with respect to set inclusion (or the relationship of one set being a subset of another set)

In what follows, for any $a \in L$,

$$L_a^1 = \{((x_1 \to y_1)' \to y_1)' : x_1, y_1 \in L, (x_1 \to y_1)' = a\};$$
$$L_a^2 = \{((x_2 \to y_2)' \to y_2)' : x_2,\ y_2 \in L, (x_2 \to y_2)' \in L_a^1\};$$
$$L_a^3 = \{((x_3 \to y_3)' \to y_3)' : x_3,\ y_3 \in L, (x_3 \to y_3)' \in L_a^2\};$$
$$L_a^4 = \{((x_4 \to y_4)' \to y_4)' : x_4,\ y_4 \in L, (x_4 \to y_4)' \in L_a^3\};$$
$$\vdots \qquad\qquad \vdots$$
$$L_a^n = \{((x_n \to y_n)' \to y_n)' : x_n,\ y_n \in L, (x_n \to y_n)' \in L_a^{n-1}\}.$$

It is easy to check.

$$((x_i \to y_i)' \to y_i)' = (((x_i \to y_i)' \to y_i)' \to 0)';$$

$$(((x_i \to y_i)' \to y_i)' \to y_i)' \le ((x_i \to y_i)' \to y_i)'.$$

Hence $L_a^n \subseteq L_a^{n-1} \cdots L_a^4 \subseteq L_a^3 \subseteq L_a^2 \subseteq L_a^1$ and denoted by $T_a = \bigcap_{i=1}^{\infty} L_a^i$

Theorem 2.2.9: Let L be a lattice implication algebra, then T_a is a WLI-ideal for any $a \in L$.

Proof. Suppose that $(x_i \to y_i)' \in T_a$ for any $x_i, y_i \in L$, then there exists $i \geq 1$ and it is the element of the set of $\{0,1,2,3 \ldots, \ldots\}$ such that $(x_i \to y_i)' \in L_a^i$.

Hence $((x_i \to y_i)' \to y_i)' \in L_a^{i+1}$, i.e., $((x_i \to y_i)' \to y_i)' \in T_a$. Therefore T_a is a WLI-ideal of L by Definition 2.1.1.

Theorem 2.2.10: Let L be a lattice implication algebra, $x, a \in L$, then $x \in T_a$ if and only if there exist $k \in N^+, x_k, x_{k-1}, x_2, x_1 \in L$, and $y_k, y_{k-1}, y_2, y_1 \in L$ if it satisfies the follows conditions:

(1) $(x_1 \to y_1)' = a$;

(2) $(x_i \to y_i)' \in L_a^{i-1}$, and $(x_i \to y_i)' = ((x_{i-1} \to y_{i-1})' \to y_{i-1})'$;

(3) $((x_k \to y_k)' \to y_k)' = x$.

Proof: Assume that the conditions 1 to 3 hold, then $x = ((x_k \to y_k)' \to y_k)' \in L_a^{k-1}$ for every k = 2, 3, 4...

$\Rightarrow x \in \bigcap_{i=2}^{\infty} L_a^{k-1} = Ta$

Therefore $x \in Ta$.

Conversely suppose x $\in T_a$, then there exist $k \in N^+$ such that $x \in L_a^k$ by $T_a = \bigcap_{i=1}^{\infty} L_a^i$, i.e. $\exists x_k, y_k \in L$ such that $x = ((x_k \to y_k)' \to y_k)'$.

Thus, we have $(x \to y)' \in L_a^{k-1}$. Since there exist $x_{k-1}, y_{k-1} \in L$ such that $(x_k \to y_k)' = ((x_{k-1} \to y_{k-1})' \to y_{k-1})'$ for $x_k \to y_k \in L_a^{k-1}$, and so we get $x_{k-1} \to y_{k-1} \in L_a^{k-2}$.

It follows that we can be obtaining sequences $x_k, x_{k-1}, x_2, x_1 \in L$ and $y_k, y_{k-1}, y_2, y_1 \in L$ such that three conditions hold.■

Theorem 2.2.11: Let L be a lattice implication algebra, then $T_a = \langle a \rangle$ for any $a \in L$.

Proof: Suppose that $a \in T_a$then $\langle a \rangle \subseteq T_a$ by Theorem2.2.9

On the other hand, let $a \in T_a$then there exist $k \in N^+$ such that $x_k, x_{k-1}, x_2, x_1 \in L$ and $y_k, y_{k-1}, y_2, y_1 \in L$ satisfy the following conditions.

(1) $(x \to y)' = a;$

(2) $(x_i \to y_i)' \in L_a^{i-1}$, and $(x_i \to y_i)' = ((x_{i-1} \to y_{i-1})' \to y_{i-1})' (i = 2,3 \dots);$

(3) $((x_k \to y_k)' \to y_k)' = x.$

Moreover, we have $(x_i \to y_i)' \in \langle a \rangle \ (i = 2,3 \dots k),$

i.e. $T_a = \langle a \rangle$. Consequently, the result is valid.

Theorem 2.2.12: Let L be a lattice implication algebra, $A \subseteq L$. Then $\langle A \rangle = \bigcap_{a \in A} \langle a \rangle$.

Proof: Since $a \in \langle a \rangle$ for all $a \in A$, we have $A \subseteq \bigcap_{a \in A} \langle a \rangle$.

Thus $\langle A \rangle \subseteq \bigcap_{a \in A} \langle a \rangle$.

On the other hand, if $\forall a \in A$ then $\langle a \rangle \subseteq \langle A \rangle$.

Hence we obtain $\bigcap_{a \in A} \langle a \rangle \subseteq \langle A \rangle$.

Thus we have $\langle A \rangle = \bigcap_{a \in A} \langle a \rangle$.

Corollary 2.2.13: Let L be a lattice implication algebra, $A \subseteq L, B \subseteq L$ and $A \subseteq B$. Then $\langle A \rangle \supseteq \langle B \rangle$.

Proof: Suppose L be a lattice implication algebra, $A \subseteq B$ and $x \in B$. Then $x \in \langle B \rangle$ (since $\langle B \rangle$ is the smallest LI-ideals of L containing $x \in \bigcap_{b \in B} \langle b \rangle$.

By Theorem 2.2.12. $\langle B \rangle = \bigcap_{b \in B} \langle b \rangle$

$\Rightarrow x \in \langle a \rangle$ for all $b \in B$

$\Rightarrow x \in \langle a \rangle$ for all $a \in A$ (since $A \subseteq B$)

$\Rightarrow x \in \bigcap_{a \in A} \langle a \rangle = \langle A \rangle$ by Theorem 2.2.12.

Therefore $\langle B \rangle \subseteq \langle A \rangle$■

Conclusion

In this paper, we proposed the notion of WLI-ideals and maximal WLI-ideals in lattice implication algebras, discussed some of their properties with examples, also the extension theorem of WLI-ideals in lattice implication algebras are discussed. As a result, every ILI-ideal of a lattice implication algebra is WLI-ideal and every lattice ideal in lattice H implication algebra is WLI-ideal. The above work will used to for future study about the structure of WLI-ideals in lattice implication algebras and develop corresponding many-valued logic system.

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