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LFuzzy Ideals and LFuzzy Congruence Relations in Universal Algebra

Gezahagne, Mulat

$L-$Fuzzy Ideals and $L-$Fuzzy Congruence Relations in Universal Algebra

By

Gezahagne Mulat Addis

Department of Mathematics, College of Science,
Bahir Dar University

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Advisor: Berhanu Assaye (PhD, Associate Professor)
Co-Advisor: Mihret Alamneh (PhD, Associate Professor),
Department of Mathematics, College of Science, Bahir Dar University

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Bahir Dar University

Department of Mathematics

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a Dissertation entitled "$L-$**Fuzzy Ideals and** $L-$**Fuzzy Congruence Relations in Universal Algebras**" by **Gezahagne Mulat Addis** in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Date: June 8, 2020

External Examiner-1: ______________________________

Guddati Chakradhara Rao (PhD., Professor)

External Examiner-2: ______________________________

Unsal Tekir (PhD., Professor)

Internal Examiner: ______________________________

Zelalem Teshome Walle (PhD., Assist. Professor)

Research Supervisor: ______________________________

Berhanu Assaye Alaba (PhD., Assoc. Professor)

Research Co-supervisor: ______________________________

Mihret Alamneh Taye (PhD., Assoc. Professor)

Declaration of Authorship

I here by declare that the work presented in this thesis entitled, '$L-$Fuzzy Ideals and $L-$Fuzzy Congruence Relations in Universal Algebras' is based on the original work done by me under the supervision of Dr. Berhanu Assaye Alaba, in the Department of Mathematics, College of Science, Bahir Dar University; and no part of the thesis has been presented for the award of any other degree or diploma.

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Gezahagne Mulat Addis

Abstract

In this thesis, we introduce the notion of fuzzy ideals in a more general context in universal algebras by the use of ideal terms. Fuzzy ideals generated by a fuzzy set are characterized from the fuzzy point of view as well as from the algebraic point of view. It is shown that the class of fuzzy ideals of an algebra A of a given type $\mathfrak{F}$ forms an algebraic closure fuzzy set system together with the inclusion ordering of fuzzy sets. The commutator of fuzzy ideals in universal algebras is defined as a common abstraction of the product of fuzzy normal subgroups in groups, fuzzy ideals in rings, etc. Using this commutator, we define and characterize fuzzy prime ideals, fuzzy semiprime ideals, maximal fuzzy ideals, the radical of fuzzy ideals and the space of fuzzy prime ideals in universal algebras.

On the other hand, we deal with fuzzy congruence relations and their classes so-called fuzzy congruence classes in universal algebras. We characterize fuzzy congruence relations generated by a fuzzy relation and we give a representation for fuzzy congruence relations using crisp congruences. Mainly, we make a theoretical study on fuzzy congruence classes of algebras in different varieties. Several Mal'cev type characterizations are given for a fuzzy subset of an algebra in a given variety to be a class of some fuzzy congruence. Particularly, finite characterizations are given for fuzzy congruence classes in regular and permutable varieties.

We also introduce the notion of fuzzy cosets in universal algebras by the use of coset terms. It is shown that, fuzzy ideas and more generally fuzzy congruence classes are the natural examples of fuzzy cosets. But the converse does not hold in general. We give sufficient conditions for fuzzy cosets to be a class of some fuzzy congruence relation. Moreover, the theory of fuzzy cosets is applied to characterize permutable varieties.

In some class of algebras (like groups and rings), each ideal I is the zero congruence class of a unique congruence relation denoted by I^{δ}, and the map $I \mapsto I^{\delta}$ defines a one to one correspondence between the lattice of ideals and the lattice of congruences on that algebra. A class of such algebras is called ideal determined. In this thesis, we study special fuzzy congruence classes so-called fuzzy congruence kernels and we give necessary and sufficient conditions (in a fuzzy sense) for a class of algebras to be an ideal determined. In this view, we study the structure of quotient algebras induced by fuzzy ideals in ideal determined varieties.

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Contents

List of Abbreviations

$Con(A)$	The class of all congruence relations on A
$FCon(A)$	The class of all fuzzy congruence relations on A
$I(A)$	The class of all ideals in A
$FI(A)$	The class of all fuzzy ideals in A
$C_a(A)$	The class of all cosets in A determined by the element a
$FC_a(A)$	The class of all fuzzy cosets in A determined by the element a
$L_a(A)$	The set of all congruence classes in A determined by the element a
$FL_a(A)$	The set of all fuzzy congruence classes in A determined by the element a

List of Symbols

$Z-$	The set of integers
Z_0-	The set of nonnegative integers
Z^+-	The set of positive integers
$\mathfrak{F}-$	A type (or a language) of algebras.
$\mathfrak{F}_n-$	The set of all $n-$ary fundamental operations
$\mathscr{K}-$	A class of algebras of a given type
$T(X)-$	The set of all terms over a set X
T_n-	The set of all $n-$ary terms over the given algebra A
$P_n(A)-$	The set of all $n-$ary polynomials over the algebra A
$\langle X\rangle-$	The ideal of A generated by the set X
$\langle\lambda\rangle-$	A fuzzy ideal of A generated by the fuzzy set λ
$\overline{X}^a-$	The coset of A determined by the element $a\in A$ and generated by the set X
$\overline{\lambda}^a-$	The fuzzy coset of A determined by the element $a\in A$ and generated by the fuzzy set λ
$\Theta(R)-$	The congruence relation on A generated by the relation R
$\Theta_L(\rho)-$	The fuzzy congruence relation on A generated by the fuzzy relation ρ

Dedicated to: My Wife Ephrata Fetene and to My Son Nathan Gezahagne

Publications

From this dissertation, the following four papers are published and the last four papers are communicated:

1. $L-$Fuzzy ideals in universal algebras, Annals of Fuzzy Mathematics and Informatics, 17(1), (2019), 31-39
2. Homomorphisms and $L-$Fuzzy ideals in universal algebras, International Journal of Fuzzy Mathematical Archive, 17(1) (2019),
3. $L-$Fuzzy prime ideals in universal algebras, Advances in Fuzzy Systems, Vol. 2019, Article ID 5925036, 7 pages, 2019.
4. $L-$Fuzzy semi-prime ideals in universal algebras, Korean Journal of Mathematics, 27(2), (2019), 327-340.
5. The fuzzy prime spectrum of universal algebras, (communicated).
6. $L-$fuzzy congruence relations and their classes, (communicated).
7. $L-$fuzzy cosets in universal algebras, (communicated).
8. Quotient algebras induced by $L-$fuzzy ideals, (communicated).

Introduction

In 1965, L. A. Zadeh [158] introduced the notion of a fuzzy subset of a set X as a function from X into the unit interval $[0,1]$. This idea marked a new direction and stirred the interest of researchers worldwide. It provided tools and an approach to model imprecision and uncertainty present in phenomena that do not have sharp boundaries. A lot of work on fuzzy sets has come into being with many applications to various fields such as computer science, artificial intelligence, expert systems, control systems, decision making, medical diagnosis, management science, operations research, pattern recognition, neural network and others (see [80, 135, 153, 164]).

In 1971, Rosenfeld [136] applied fuzzy sets in group theory and has formulated the concept of a fuzzy subgroup of a group. Since then, many researchers have been studying fuzzy subalgebras of several algebraic structures; fuzzy subgroups and fuzzy normal subgroups of a group (see [1, 14, 16, 32, 33, 34, 41, 57, 64, 92, 102, 129, 130]), fuzzy action of groups (see [5, 47]), fuzzy ideals of a semigroup (see [86, 106, 109, 110]), fuzzy subrings and fuzzy ideals of a ring see (see [2, 65, 66, 113, 122, 123, 132, 99]), prime and maximal fuzzy ideals of a ring (see [97, 98, 119, 120, 121, 131, 143]), fuzzy sublattices and fuzzy ideals of a lattice (see [15, 46, 26, 27]), fuzzy ideals of a pseudo-complemented semi-lattice (see [28]), fuzzy ideals of a poset (see [29, 30]), fuzzy ideals and fuzzy filters of MS-algebras (see [22, 23, 24, 25]), fuzzy ideals of BCC-algebras (see [68, 69]), fuzzy ideals of BCI/BCK algebras ([125, 87, 88]), fuzzy submodules of a module (see [3, 117, 161, 162]), fuzzy subspaces of a vector space (see [44, 58, 89]), etc.

In this thesis, we introduce and investigate the notion of fuzzy ideals in a more general context in universal algebras as a common abstraction to most of the existing theories of fuzzy ideals in different algebraic structures. We apply the general theory of algebraic fuzzy systems developed in [144, 145] to study the properties of fuzzy ideals that they have in common in different algebraic structures. In this setting, basic concepts that are connected to ideals like the generator, the commutator, primeness, semi-primeness, the prime spectrum, maximality and the radical are extended to the class of fuzzy ideals in universal algebras.

The concept of fuzzy equivalence relations was first defined by Zadeh [159] as a generalization of the concept of an equivalence relation. They have been since widely studied as a way to measure the degree of indistinguishability or similarity between the objects of a given universe of discourse, and they have shown to be useful in different contexts such as fuzzy control, approximate reasoning, fuzzy cluster analysis, etc. In literature, fuzzy equivalence relations appeared in different names such as similarity relations (see [159]), indistinguishability operators (see [152, 82, 83, 59, 60]), $\mathscr{T}-$equivalences (see [61, 62]), etc. Fuzzy congruence relations on algebraic structures are fuzzy equivalence relations which are compatible (in a fuzzy sense) with all fundamental operations of the algebra. Fuzzy congruences have been studied in different algebraic structures; in semigroups (see [56, 108, 138, 148, 157]), in groups and rings (see [91, 94, 107, 112, 128, 137, 163]), in modules and vectorspaces (see [70, 142]), in lattice structures (see [21, 46, 146]), and more generally on universal algebras (see [52, 73, 133, 134, 139, 140]). Other important concepts that we study in this thesis are fuzzy congruence relations and their classes in universal algebras. We give several Mal'cev type characterizations in a fuzzy setting. We study, the connection between fuzzy ideals and fuzzy congruence relations in a general context.

This thesis is organized in seven chapters. The first chapter contains basic concepts of universal algebras and fuzzy set theory collected from literature. The next three chapters are devoted to the development of the general theory of fuzzy ideals in universal algebras. Whereas the last three chapters are concerned with the study of fuzzy congruence relations and fuzzy congruence classes in different equational classes of algebras like regular, permutable and ideal-determined varieties.

To be more specific: in chapter two, we define fuzzy ideals of universal algebras as a normalized fuzzy set which are $\overrightarrow{y}-$closed under each ideal term $t(\overrightarrow{x},\overrightarrow{y})$ in $\overrightarrow{y}$. Examples and several characterizing theorems are given. Mainly, fuzzy ideals generated by fuzzy sets are fully characterized. Furthermore, fuzzy ideals of fuzzy subalgebras are studied in the chapter. The main concern of the third chapter is to study fuzzy prime ideals and their generalization by applying the commutator of fuzzy ideals in universal algebras. We give an internal characterization for fuzzy prime ideals of universal algebras analogous to the well-known characterization of Swamy and Swamy [143] in the case of rings. Those fuzzy ideals (not necessarily $2-$valued) in which every level subset is either the A or prime are also studied in the chapter under a name generalized fuzzy prime ideals. Moreover, we study maximal fuzzy ideals and their generalizations in the chapter.

In chapter four, the commutator of fuzzy ideals is applied to define and investigate fuzzy semi-prime ideals and the radical of fuzzy ideals in universal algebras. The radical of fuzzy ideals is described in different ways. Several characterizing theorems are given for a fuzzy ideal of an algebra A to be fuzzy semi-prime. In addition, the space of fuzzy prime ideals equipped with the hull-kernel topology is also presented in the chapter.

A. I. Mal'ceve [118] has proved that a nonempty subset H of an algebra A of some type $\mathfrak{F}$ is a class of some $\theta \in Con(A)$ if and only if for any $a,b \in H$ and any unary polynomial p over A it holds that

$$p(a) \in H \Rightarrow p(b) \in H$$

One of the main results in this thesis is that we give a Mal'cev type characterization for fuzzy subsets of an algebra A to be a class of some fuzzy congruence on A. This result is proved in Chapter five. In particular, in regular and permutable varieties, a polynomial characterization is given for fuzzy congruence classes using finite number of finitary terms. Furthermore, some equivalent conditions are also given for a variety of algebras to posses fuzzy congruence classes which are also fuzzy subuniverse.

In Chapter six, we define fuzzy cosets in universal algebras and investigate some of their properties. Fuzzy cosets generated by a fuzzy set are fully characterized. It is shown that fuzzy ideals and in general fuzzy congruence classes are the natural examples of fuzzy cosets. But the converse is not true in general. We give a sufficient condition for fuzzy cosets to be a class of some fuzzy congruence relation. Moreover, we give several characterization for a class of algebras to be permutable using fuzzy cosets.

The last chapter is about fuzzy ideals and fuzzy congruence relations. A special fuzzy congruence classes; called fuzzy congruence kernels are studied in the chapter. It is observed that fuzzy congruence classes are fuzzy ideals but the converse need not necessarily be true. We obtain a class of algebras in which fuzzy ideals are a kernel of unique fuzzy congruence. This establishes a one to one correspondence between ideals and fuzzy congruence relations. Finally, we study the structure of quotient algebras induced by fuzzy ideals in some general context. We characterize fuzzy prime ideals using their quotient structure.

Chapter 1

Preliminaries

In this chapter, we present basic notions and results that will be used throughout the thesis.

1.1 Universal Algebras

Most of the results in this section are standard and are collected from [48, 50, 74, 78]

1.1.1 Definitions and Examples

For a nonempty set A and n a nonnegative integer we define $A^0 = \{\emptyset\}$, and, for $n > 0$, A^n is the set of $n-$tuples $(a_1, a_2, ..., a_n)$ of elements from A.

Definition 1.1.1. An $n-$ary operation (or function) on a set A is any function f from A^n to A. In this case, n is the arity (or rank) of f.

An operation f on A is unary, binary, or ternary if its arity is 1,2, or 3, respectively.

Definition 1.1.2. A language (or type) of algebras is a set $\mathfrak{F}$ of function symbols such that a nonnegative integer n is assigned to each member f of $\mathfrak{F}$. This integer is called the arity (or rank) of f, and f is said to be an $n-$ary function symbol. The subset of $n-$ary function symbols in $\mathfrak{F}$ is denoted by $\mathfrak{F}_n$.

Definition 1.1.3. An algebra $\mathbf{A}$ of type $\mathfrak{F}$ is an ordered pair $(A, \mathfrak{F})$ where A is a nonempty set and $\mathfrak{F}$ is a family of finatary operations $f^{\mathbf{A}}$ on A (or a language of algebras). The set A is called the universe (or underlying set) of $\mathbf{A} = (A, F)$ and the $f^{\mathbf{A}}$'s are called the fundamental operations of $\mathbf{A}$.

In other words, an algebra is just a model for a first order language $\mathfrak{F}$ whose nonlogical symbols are finitary operation symbols f with arity $n \geq 0$.
If there is no confusion, we prefer to write just f for $f^{\mathbf{A}}$. If $\mathfrak{F}$ is finite, say $\mathfrak{F} = \{f_1, ..., f_k\}$, we often write $(A, f_1, ..., f_k)$ for (A, F), usually adopting the convention:

$$\textit{arity } f_1 \geq \textit{arity } f_2 \geq ... \geq \textit{arity } f_k$$

Whenever we deal with a class of algebras, it will be assumed that all the algebras are models for one language, i.e., that they are of the same similarity type. Thus each class $\mathscr{K}$ of algebras is attached to a language (or a type) $\mathfrak{F}$, and $\mathscr{K} \subseteq Mod(\mathfrak{F})$, where of course $Mod(\mathfrak{F})$ denotes the class of all models (or algebras) of the given type $\mathfrak{F}$. When $A \in Mod(\mathfrak{F})$ and $f \in \mathfrak{F}_k$ as above, then f^A is called the interpretation of f in A, and it is a $k-$ary operation on the set A.

Example 1.1.4. A **groupoid** is an algebra $(G, \dot{})$ of type (2), i.e., a groupoid is nonempty set G together with a binary operation $\cdot$. A **semigroup** is an associative groupoid.

Example 1.1.5. A **group** is an algebra $(G, \cdot, ^{-1}, e)$ of type $(2, 1, 0)$ in which the following identities are true:

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2. $e \cdot a = a = a \cdot e$
3. $a \cdot a^{-1} = e = a^{-1} \cdot a$

Sometimes groups are written additively in a form like $(G, +, -.0)$.

Example 1.1.6. A **ring** is an algebra $(R, +, \dot{}, -, 0, 1)$ of type $(2, 2, 1, 0, 0)$ in which the following identities are true:

1. $(R, +, -, 0)$ is an abelian group
2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

3. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Example 1.1.7. A **distributive lattice** is an algebra $(L, \vee, \wedge)$ of type $(2, 2)$ in which the following identities are true:

1. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$

2. $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

3. $a \wedge (a \vee b) = a = a \vee (a \wedge b)$

4. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ or $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

for all $a, b, c \in L$. Moreover, a bounded distributive lattice $\mathbf{L}$ is an algebra $(L, \vee, \wedge, 0, 1)$ of type $(2, 2, 0, 0)$ such that:

1. $(L, \vee, \wedge)$ is a distributive lattice.

2. $0 \wedge a = 0$ and $a \wedge 1 = a$

for all $a \in L$.

Example 1.1.8. A **Boolean algebra** is an algebra $(B, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying the following identities:

1. $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice

2. $a \wedge a' = 0$ and $a \vee a' = 1$ for all $a \in B$.

Boolean algebras were of course discovered as a result of Boole's investigations into the underlying laws of correct reasoning. Since then they have become vital to electrical engineering, computer science, axiomatic set theory, model theory, and other areas of science and mathematics.

Example 1.1.9. A **Heyting algebra** is an algebra $(H, \vee, \wedge, \rightarrow, 0, 1)$ with three binary and two nullary operations such that:

1. $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice

2. $a \to a = 1$

3. $(a \to b) \wedge b = b$ and $a \wedge (a \to b) = a \wedge b$

4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$ and $(a \vee b) \to c = (a \to c) \wedge (b \to c)$ for all $a, b, c \in H$.

Heyting algebras were introduced by Birkhoff under a different name, Brouwerian algebras, and with a different notation ($a : b$ for $a \to b$).

Example 1.1.10. An **implication algebra** is a groupoid $(A, \dot{})$ satisfying the identities

1. $(xy)x = x$;

2. $(xy)y = (yx)x$ and

3. $x(yz) = y(xz)$

Implication algebras play an important role in logic.

1.1.2 Homomorphisms and Congruence Relations

Definition 1.1.11. Let A and B be algebras of the same type $\mathfrak{F}$. A mapping $h : A \to B$ is called a homomorphism from A to B if:

$$h(f^A(a_1, a_2, ..., a_n)) = f^B(h(a_1), h(a_2), ..., h(a_n))$$

for each $n-$ary operation f in $\mathfrak{F}$ and each sequence $a_1, a_2, ..., a_n$ from A.

Definition 1.1.12. A homomorphism $h : A \to B$ is said to be:

1. a monomorphism, if it is injective (or one-one).

2. an epimorphism, if it is surjective (or onto).

3. an isomorphism, if it is bijective (or a one-to-one correspondence).

We say A is isomorphic to B, written as $A \cong B$ if there is an isomorophism from A onto B.

Note that, a monomorphism $h : A \to B$ may sometimes be called an embedding of A into B and in this case we say A can be embedded into B.

Definition 1.1.13. Let A and B be two algebras of the same type. Then B is a subalgebra of A if $B \subseteq A$ and every fundamental operation of B is the restriction of the corresponding operation of A, i.e., for each function symbol f, f^B is f^A restricted to B; we write simply $B \leq A$ to say that B is a subalgebra of A.

Definition 1.1.14. A binary relation θ on A is an equivalence relation on A if, for any elements $a, b, c \in A$, it satisfies:

1. $(a, a) \in \theta$ (reflexivity)

2. $(a, b) \in \theta \Rightarrow (b, a) \in \theta$ (symmetry)

3. $(a, b), (b, c) \in \theta \Rightarrow (a, c) \in \theta$ (transitivity)

Sometimes we may write $a\theta b$ to say that $(a, b) \in \theta$. The set of all equivalence relations on A is denoted by $Eq(A)$, and it is observed that $Eq(A)$ is a complete lattice together with the inclusion order.

Definition 1.1.15. Let A be an algebra of type $\mathfrak{F}$. A binary relation θ on A is called an admissible relation if θ satisfies the following compatibility property: for each $n-$ary $f \in \mathfrak{F}$ with $n > 0$ and $a_1, ..., a_n, b_1, ..., b_n \in A$, if $(a_i, b_i) \in \theta$ for $i = 1, ..., n$, then $(f^A(a_1, ..., a_n), f^A(b_1, ..., b_n)) \in \theta$

Definition 1.1.16. Let A be an algebra of type $\mathfrak{F}$. By a congruence relation on A we mean an admissible equivalence relation on A. The set of congruence relations on A is denoted by $Con(A)$ and it is a complete lattice together with the usual inclusion ordering $\subseteq$.

Definition 1.1.17. Let X be a set of (distinct) objects called variables. Let $\mathfrak{F}$ be a type of algebras. The set $T(X)$ of terms of type $\mathfrak{F}$ over X is the smallest set such that:

1. $X \cup \mathfrak{F}_0 \subseteq T(X)$

2. If $p_1, p_2, ..., p_n \in T(X)$ and $f \in \mathfrak{F}_n$, then the "string" $f(p_1, p_2, ..., p_n) \in T(X)$.

For $n \in Z^+$, we denote by T_n the set of all $n-$ary terms over A. For a binary function symbol "." we usually prefer to write $p_1.p_2$ to $.(p_1, p_2)$. For $pinT(X)$ we often write p as $p(x_1, ..., x_n)$ to indicate that the variables occurring in p are among $x_1, ..., x_n$. A term p is n-ary if the number of variables appearing explicitly in p is $\leq n$.

Example 1.1.18. Let $\mathfrak{F}$ consists of a single binary operation symbol $\cdot$ and let $X = \{x, y, z\}$. Then

$$x, y, z, x \cdot y, y \cdot z, x \cdot (y \cdot z), \text{ and } (x \cdot y) \cdot z$$

are some of the terms over X.

Definition 1.1.19. Given a term $p(x_1, ..., x_n)$ of type $\mathfrak{F}$ over some set X and given an algebra A of type $\mathfrak{F}$ we define a mapping $p^A : A^n \to A$ as follows:

1. If p is a variable x_i, then $p^A(a_1, .., a_n) = a_i$ for $a_1, .., a_n \in A$, i.e., p^A is the i^{th} projection map.

2. If p is of the form $f(p_1(x_1, ..., x_n), ..., p_k(x_1, ..., x_n))$ where $f \in \mathfrak{F}_k$, then

$$p^A(a_1, .., a_n) = f^A(p_1(a_1, .., a_n), ..., p_k(a_1, .., a_n))$$

In particular, if $p = f \in \mathfrak{F}$, then $p^A = f^A$. We call p^A the term function corresponding to the term p or in the older literature, derived operations.

Theorem 1.1.20. *For any algebras A and B of the same type $\mathfrak{F}$ we have the following*

1. *Let p be an $n-$ary term of type $\mathfrak{F}$, $\theta \in Con(A)$, and suppose $(a_i, b_i) \in \theta$ for $1 \leq i \leq n$. Then, $(p^A(a_1, .., a_n), p^A(b_1, .., b_n))) \in \theta$*

2. *If p is an $n-$ary term of type $\mathfrak{F}$, and $h : A \to B$ a homomorphism, then*

$$h(p^A(a_1, .., a_n)) = p^B(h(a_1), ..., h(a_n))$$

for all $a_1, .., a_n \in A$.

Definition 1.1.21. The polynomial operations of A constitute the smallest set that is closed under composition and contains the basic operations of A, the projection operations, and the constant nullary operations on A.

Definition 1.1.22. For each polynomial operation $f(x_1,...,x_n)$ of A, there exist an $(m+n)-$ary term operation $p(y_1,...,y_m,x_1,...,x_n)$ and elements $a_1,...,a_m \in A$, such that

$$f(x_1,...,x_n) = p(a_1,...,a_m,x_1,...,x_n)$$

for all $x_1,...,x_n \in A$. Specifically, if $f(x)$ is a unary polynomial over A, then there exists an $(m+1)-$ary term $p(y_1,...,y_m,x)$ and elements $a_1,...,a_m \in A$, such that

$$f(x) = p(a_1,...,a_m,x)$$

The expression algebraic function is often used in the old literature to refer to what we call a polynomial operation. For $n \in Z^+$, the set of all $n-$ary polynomials over A is denoted by $P_n(A)$.

Suppose that $\mathscr{K}$ is a class of algebras of a given type $\mathfrak{F}$ and $A \in \mathscr{K}$. Let $p = p(x_1,...,x_n)$ and $t = t(y_1,...,y_n)$ be terms of type $\mathfrak{F}$. The formula $p \approx t$ is called an equation. We write $A \models p \approx t$ to say that $p^A = t^A$, i.e., $p^A(a_1,...,a_n) = t^A(a_1,...,a_n)$ for all $a_1,...,a_n \in A$. When $A \models p \approx t$ holds, we say that $p \approx t$ is an identity (or an equation) of A. $\mathscr{K} \models p \approx t$ means that $A \models p \approx t$ for all $A \in \mathscr{K}$. If Σ is a set of equations of type $\mathfrak{F}$, then

$$Mod(\Sigma) = \{A \in Mod(\mathfrak{F}) : A \models \varepsilon \textit{ for all } \varepsilon \in \Sigma\}$$

Classes of the form $Mod(\Sigma)$, where Σ is a set of equations of $\mathfrak{F}$ are called varieties (or equational classes). By a theorem of G. Birkhoff [43], a class $\mathscr{K} \subseteq Mod(\mathfrak{F})$ is a variety if and only if $\mathscr{K}$ is closed under the formation of homomorphic images, subalgebras and products. The smallest variety containing a class $\mathscr{K} \subseteq Mod(\mathfrak{F})$ is identical with $HSP(\mathscr{K})$, where H, S and P are the operators which close classes under homomorphic images, subalgebras and direct products, respectively. We interpret these operators in such a way that the closure of $\mathscr{K}$ under any of them contains all isomorphic copies of its members.

1.1.3 Ideals in Universal Algebras

The results presented in this subsection are taken from [79, 149]. For a positive integer n, we write $\overrightarrow{a}$ to denote the $n-$tuple $\langle a_1, a_2, ..., a_n\rangle \in A^n$.

Definition 1.1.23. A term $p(\overrightarrow{x}, \overrightarrow{y})$ is said to be an ideal term in $\overrightarrow{y}$ if and only if $p(a_1, ..., a_n, 0, 0, ...0) = 0$ for all $a_1, ..., a_n \in A$.

Example 1.1.24. Let $(G, \cdot, ^{-1}, e)$ be a group. Then

1. $p(y) = y^{-1}$ is an ideal term in y
2. $p(y_1, y_2) = y_1.y_2$ is an ideal term in $\overrightarrow{y} = (y_1, y_2)$.
3. $p(x, y) = xyx^{-1}$ is an ideal term in y.

And all other ideal terms in G are the compositions of these terms only.

Example 1.1.25. Let $(R, +, \cdot, -, 0)$ be a ring. Then

1. $p(y) = -y$ is an ideal term in y
2. $p(y_1, y_2) = y_1 + y_2$ is an ideal term in $\overrightarrow{y} = (y_1, y_2)$.
3. $p(x, y) = xy$ is an ideal term in y.

And all other ideal terms in R are the compositions of these terms only.

Example 1.1.26. Let $(L, \vee, \wedge, 0, 1)$ be a distributive lattice with zero. Then

1. $p(y_1, y_2) = y_1 \vee y_2$ is an ideal term in $\overrightarrow{y} = (y_1, y_2)$.
2. $p(x, y) = x \wedge y$ is an ideal term in y.

And all other ideal terms in L are the compositions of these terms only.

Definition 1.1.27. A nonempty subset I of A is called an ideal of A if and only if for each $a_1, ..., a_n \in A, b_1, ..., b_m \in I$ and any ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$, $p(a_1, ..., a_n, b_1, ..., b_m) \in I$.

We denote the class of all ideals of A, by $I(A)$. It is easy to check that the intersection of any family of ideas of A is an ideal. So, for a subset $S \subseteq A$, always there exists a smallest ideal of A containing S which we call it the ideal of A generated by S and it is denoted by $\langle S\rangle$. Note that $x \in \langle S\rangle$ if and only if $x = p(a_1,...a_n,b_1,...,b_m)$ for some $a_1,...a_n \in A$, and $b_1,...,b_m \in S$ where $p(\overrightarrow{x},\overrightarrow{y})$ is an $(n+m)-$ary ideal term in $\overrightarrow{y}$. If $S = \{a\}$, then we write $\langle a\rangle$ instead of $\langle S\rangle$. In this case, $x \in \langle a\rangle$ if and only if $x = p(a_1,...a_n,a)$ for some $a_1,...a_n \in A$, where $p(\overrightarrow{x},\overrightarrow{y})$ is an $(n+1)-$ary ideal term in $\overrightarrow{y}$.

A nonzero element u in A is said to be a formal unit, if $A = \langle u\rangle$, i.e., A is generated by u as an ideal. A cyclic group, a ring with unity, a bounded lattice and an almost distributive lattice with maximal elements are examples of an algebra having a formal unit. A formal unit (if it exists) in an algebra is not necessarily unique (e.g., cyclic groups and almost distributive lattices may have several formal units).

Definition 1.1.28. A class $\mathscr{K}$ of algebras is called an ideal determined if every ideal I is the zero congruence class of a unique congruence relation denoted by I^{δ}. In this case the map $I \mapsto I^{\delta}$ defines an isomorphism between the lattice of ideals and congruences on A.

Definition 1.1.29. A term $t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})$ is said to be a commutator term in $\overrightarrow{y},\overrightarrow{z}$ if and only if it is an ideal term in $\overrightarrow{y}$ and an ideal term in $\overrightarrow{z}$.

Definition 1.1.30. For each $I,J \in I(A)$, their commutator $[I,J]$ is defined by:

$$[I,J] \quad = \quad \{t(\overrightarrow{a},\overrightarrow{i},\overrightarrow{j}) : \overrightarrow{a} \in A^n, \overrightarrow{i} \in I^m \textit{ and } \overrightarrow{j} \in J^k, t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z}) \textit{ is a commutator term in } \overrightarrow{y},\overrightarrow{z}\}$$

The following theorem gives a characterization for the commutator of ideals using the general commutator of congruences in ideal determined varieties.

Theorem 1.1.31. *In an ideal determined variety, the commutator $[I,J]$ of ideals I and J is the zero congruence class of the commutator congruence $[I^{\delta},J^{\delta}]$.*

For subsets H,G of A, $[H,G]$ denotes the product $[\langle H\rangle,\langle G\rangle]$. In particular, for $a,b \in A$, $[\langle a\rangle,\langle b\rangle]$ is denoted by $[a,b]$. A proper ideal P of A is called prime (respectively semi-prime) if and only if for all $I,J \in I(A)$:

$$[I,J] \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P \text{ (respectively } [I,I] \subseteq P \Rightarrow I \subseteq P)$$

It is observed that, for a proper ideal P of A to be prime (respectively semi-prime) it is necessary and sufficient that:

$$[a,b] \subseteq P \Rightarrow a \in P \text{ or } b \in P \text{ (respectively } [a,a] \subseteq P \Rightarrow a \in P)$$

for all $a,b \in A$.

Definition 1.1.32. A nonempty subset M of A is said to be an $m-$system (respectively an $n-$system) of A if for any $a,b \in M$, it holds that $M \cap [a,b] \neq \emptyset$ (respectively $M \cap [a,a] \neq \emptyset$).

1.2 $L-$Fuzzy Sets

This section is concerned to present the basic definitions and results on $L-$fuzzy sets. We begin by defining Brouwerian lattices.

1.2.1 Complete Lattices

Definition 1.2.1. A partially ordered set (poset) is a non-empty set P together with a binary relation $\leq$ which satisfies for all $x,y,z \in P$ the following conditions:

(P1) $x \leq x$ (Reflexive)

(P2) If $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetry)

(P3) If $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitivity)

An upper bound of a subset A of a poset P is an element $x \in P$ such that $a \leq x$ for all $a \in A$. The least upper bound (or suprimum) of A is an upper bound x of A for which $x \leq y$ for all upper bounds y of A. The notion of a lower bound and a greatest lower bond (or infimum) of A are defined dually. It is clear from $(P2)$ that a subset of a poset can have at most one suprimum and one infimum.

Definition 1.2.2. A lattice is a poset L in which any two elements have infimum and supremum. In this case, the infimum and supremum of $x,y \in L$ are denoted by $x \wedge y$ and $x \vee y$ respectively.

Remark. Lattices can also be defined as an algebra with two binary operations as given in Definition 1.1.7.

Definition 1.2.3. A lattice L is said to be complete if each of its subset S has both infimum and supremum in L.

Definition 1.2.4. Let L be a lattice. An element $a \in L$ is said to be compact if for a subset $A \subseteq L$ for which $\vee A$ exists, $a \leq \vee A$ implies $a \leq \vee F$ for some finite subset F of A. L is compactly generated if and only if every element in L is the supremum of a set of compact elements in L. A lattice L is called algebraic if it is complete and compactly generated.

Definition 1.2.5. By a complete Brouwerian lattice, we mean a complete lattice L satisfying the infinite meet distributive law; i.e.,

$$\alpha \wedge (\bigvee_{\beta \in M} \beta) = \bigvee_{\beta \in M} (\alpha \wedge \beta)$$

for all $\alpha \in L$ and any $M \subseteq L$.

From now onwards $\mathbf{L} = (L, \wedge, \vee, 0, 1)$ is a complete Brouwerian lattice; i.e., L is a complete lattice satisfying the infinite meet distributive law. For more detail on lattices we refer to [42, 77]

1.2.2 L-Fuzzy Sets

The results in this subsection are collected from [75, 111, 126].

Definition 1.2.6. By an $L-$ fuzzy subset of a nonempty set X, we mean a function μ from X into L, i.e, $\mu : X \to L$. In this case μ is called the membership function and the value $\mu(x)$ is thought of as the degree of membership of the element x to the $L-$fuzzy subset of X defined by the membership function μ. The set of all $L-$fuzzy subsets of X is denoted by L^X.

For the sake of convenience, we drop the prefix $L-$ and simply write fuzzy subset instead of $L-$fuzzy subset. We use lower case Greece letters $\mu, \nu, \eta ...$ to denote fuzzy subsets.

Definition 1.2.7. Let $\mu \in L^X$. Then the set $Img(\mu) = \{\mu(x) : x \in X\}$ is called the image of μ and it may sometimes be denoted by $\mu(X)$. A fuzzy subset μ of X is called normalized or unitary if $1 \in Img(\mu)$.

Definition 1.2.8. Let $\mu \in L^X$. Then μ is said to have the sup property if for every $A \subseteq X$ there exists $a \in A$ such that

$$\mu(a) = \bigvee \{\mu(x) : x \in A\}$$

The class L^X is a complete lattice with the point-wise ordering induced by the ordering of L. Using this ordering, the notions of inclusion, equality, strict inclusion, union, intersection and complement of fuzzy subsets are defined in the following way.

Definition 1.2.9. For $\mu, \nu \in L^X$:

1. $\mu \leq \nu$ (inclusion) if and only if $\mu(x) \leq \nu(x)$ for all $x \in X$.

2. $\mu = \nu$ (equality) if and only if $\mu \leq \nu$ and $\nu \leq \mu$.

3. $\mu < \nu$ (strict inclusion) if and only if $\mu \leq \nu$ and $\mu(x) \neq \nu(x)$ for at least one $x \in X$.

4. $\mu \cap \nu$ (intersection) is defined as

$$(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x) \textit{ for all } x \in X$$

where $\wedge$ is the infimum of elements in L. More generally, if $\{\mu_i\}_{i\in\Delta}$ is a family of fuzzy subsets of X, then $\bigcap_{i\in\Delta} \mu_i$ (the arbitrary intersection) is a fuzzy subset of X defined by:

$$\bigcap_{i\in\Delta} \mu_i(x) = \bigwedge_{i\in\Delta} \mu_i(x) \textit{ for all } x \in X$$

5. $\mu \cup \nu$ (union) is defined as

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) \textit{ for all } x \in X$$

where $\vee$ is the supremum of elements in L. More generally, if $\{\mu_i\}_{i\in\Delta}$ is a family of fuzzy subsets of X, then $\bigcup_{i\in\Delta}\mu_i$ (the arbitrary union) is a fuzzy subset of X defined by:

$$\bigcup_{i\in\Delta}\mu_i(x) = \bigvee_{i\in\Delta}\mu_i(x) \text{ for all } x \in X$$

Definition 1.2.10. Let $\mu \in L^X$. For $\alpha \in L$ define μ_α as follows

$$\mu_\alpha = \{x \in X : \mu(x) \geq \alpha\}$$

μ_α is called the $\alpha-$level set (the $\alpha-$cut) of μ.

Every fuzzy subset λ can be describe as follows. The theorem is taken from [93].

Theorem 1.2.11. *For any fuzzy subset λ of A and each $x \in A$, we have*

$$\lambda(x) = \bigvee\{\alpha \in L : x \in \lambda_\alpha\}$$

Definition 1.2.12. For each $H \subseteq X$, and $\alpha \in L$ we define a fuzzy subset α_H of X as follows:

$$\alpha_H(x) = \begin{cases} 1 & \text{if } x \in H \\ \alpha & \text{otherwise} \end{cases}$$

In particular 1_X (resp. 0_X) denotes the fuzzy set

$$1_X(y) = 1(\text{ resp. } 0_X(y) = 0) \text{ for all } y \in X$$

and we call it the improper (resp. the empty) fuzzy subset of X.

Definition 1.2.13. [156] For each $x \in X$ and $0 \neq \alpha \in L$, the fuzzy subset x_α of X given by:

$$x_\alpha(z) = \begin{cases} \alpha & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

is called an $L-$fuzzy point (or a fuzzy point for short) of X. In this case x is called the support of x_α and α its value. For a fuzzy subset μ of X and a fuzzy point x_α of X, we write $x_\alpha \in \mu$ whenever $\mu(x) \geq \alpha$.

We now turn our attention to defining mappings between fuzzy subsets of two sets. So let X and Y be two non-empty sets and let f be a mapping from X to Y. Then f extends to a mapping from L^X to L^Y in the following way. For each $\mu \in L^X$, $f(\mu) \in L^Y$ is defined as:

$$f(\mu)(y) = \begin{cases} 0 & \textit{if } f^{-1}(y) = \emptyset \\ \bigvee\{\mu(x) : x \in f^{-1}(y)\} & \textit{otherwise} \end{cases}$$

for all $y \in Y$, where $f^{-1}(y)$ denotes $f^{-1}(\{y\})$. $f(\mu)$ is referred to as the image of the fuzzy set μ under f. Further, for each $\nu \in L^Y$, $f^{-1}(\nu) \in L^X$ is defined as:

$$f^{-1}(\nu)(x) = \nu(f(x))$$

for all $x \in X$. $f^{-1}(\nu)$ is the pre-image (or the inverse image) of ν under f.

Definition 1.2.14. Let $f : X \to Y$ be a mapping. A fuzzy subset μ of X is said to be $f-$invariant if for each $x, y \in Y$, $f(x) = f(y)$ implies $\mu(x) = \mu(y)$.

1.2.3 Closure Systems in Fuzzy Sets

The results in this subsection are due to Swamy et.al [144] and V. Murali [133] . Let $\mathscr{C}$ be a nonempty collection of fuzzy subsets of a nonempty set X.

Definition 1.2.15. $\mathscr{C}$ is said to be a closure system in L^X if it is closed under arbitrary intersection of fuzzy sets, i.e., if for any subcollection $\mathscr{D}$ of fuzzy subsets of X in $\mathscr{C}$ it holds that

$$\bigwedge_{\mu \in \mathscr{D}} \mu \in \mathscr{C}$$

A closure system of fuzzy sets is also known as the "Moor family" of fuzzy sets.

Remark. If $\mathscr{C}$ is a closure system in L^X, then it contains the fuzzy subset 1_X of X. This is because the fuzzy set 1_X can be expressed as the infimum of an empty collection of fuzzy subsets of X.

Theorem 1.2.16. *If $\mathscr{C}$ is a closure fuzzy set system in L^X, then $(\mathscr{C}, \leq)$ forms a complete lattice, where $\leq$ is the inclusion ordering of fuzzy sets.*

Definition 1.2.17. A nonempty collection $\mathscr{C}$ of fuzzy subsets of X is called inductive if every nonempty chain in $\mathscr{C}$ has a supremum in $\mathscr{C}$.

Definition 1.2.18. An inductive closure fuzzy set system $\mathscr{C}$ in L^X is called an algebraic closure fuzzy set system.

Chapter 2

$L-$Fuzzy Ideals

Introduction

The concept of a ring ideal, a lattice ideal and a normal subgroup were extended to the notion of an ideal in a universal algebra having a constant 0 by A. Ursini [150]. However, the origin of this concept can be found in papers by R. Magari [116], E. Beutler ([38, 39, 40]), K. Fichtner [72], H. J. Hoehnke [81] and G. Matthiessen [124]. A remarkable development of the theory started with papers by A. Ursini ([149, 150, 151]), by P. Agliano ([9, 10, 11, 12, 13]), by H. P. Gumm and A. Ursini [79] and by G. Janelidze [84].

Loosely speaking, an ideal of an algebra with a constant 0 is a nonempty set which is $\overrightarrow{y}-$closed under each ideal term $t(\overrightarrow{x},\overrightarrow{y})$ in $\overrightarrow{y}$, where by an ideal term in $\overrightarrow{y}$ we mean a term $t(\overrightarrow{x},\overrightarrow{y})$ such that

$$t(a_1,...,a_n,0,0,...,0)=0$$

for all $a_1,...,a_n\in A$.

In this chapter, we introduce the notion of fuzzy ideals in universal algebras having a constant 0. This subsumes the well know structures: fuzzy normal subgroups in groups, fuzzy ideals in rings, fuzzy submodule in modules, fuzzy subspaces in vector spaces, fuzzy ideals in lattices with least element 0, etc. Mainly, we give several characterization for fuzzy ideals generated by fuzzy sets and show that the class of fuzzy ideals in universal algebras forms an algebraic closure fuzzy set system. Fuzzy ideals of fuzzy subalgebras are also defined and characterized in detail.

Throughout this thesis, unless and otherwise it is mentioned, $A \in \mathscr{K}$, where $\mathscr{K}$ is a class of algebras of a fixed type $\mathfrak{F}$ and we assume that there is an equationaly definable constant in all algebras of $\mathscr{K}$ denoted by 0.

2.1 Fuzzy Ideals: Definition and Examples

The following concept of a $\overrightarrow{y}-$closed subset of an algebra was introduced recently by R. Bělohlávek [35].

Definition 2.1.1. A subset H of A is said to be $\overrightarrow{y}$ closed under the $(n+m)-$ary term operation $p(\overrightarrow{x}, \overrightarrow{y})$ if for all $a_1,...,a_n,b_1,...,b_m \in A$,

$$b_1,...,b_m \in H \Rightarrow p(a_1,...,a_n,b_1,...,b_m) \in H$$

Analogous to this concept, we define the following in a fuzzy setting.

Definition 2.1.2. A fuzzy subset μ of A is said to be $\overrightarrow{y}$ closed under the $(n+m)-$ary term operation $p(\overrightarrow{x}, \overrightarrow{y})$ if

$$\mu(p(a_1,...,a_n,b_1,...,b_m)) \geq \mu(b_1) \wedge ... \wedge \mu(b_m)$$

for all $a_1,...,a_n,b_1,...,b_m \in A$.

We use this concept to define fuzzy ideals.

Definition 2.1.3. An $L-$fuzzy subset μ of A is said to be an $L-$fuzzy ideal of A (or shortly a fuzzy ideal of A) if and only if the following conditions are satisfied:

1. $\mu(0) = 1$ and

2. μ is $\overrightarrow{y}$ closed under each ideal term $p(\overrightarrow{x},\overrightarrow{y})$ in $\overrightarrow{y}$, i.e., for each $(n+m)-$ary ideal term $p(\overrightarrow{x},\overrightarrow{y})$ in $\overrightarrow{y}$ and each $a_1,...,a_n,b_1,...,b_m \in A$, it holds that

$$\mu(p(a_1,...,a_n,b_1,...,b_m)) \geq \mu(b_1) \wedge ... \wedge \mu(b_m)$$

We denote by $FI(A)$, the set of all fuzzy ideals of A. It follows immediately from the definition that both 1_0 and 1_A are the smallest and the largest fuzzy ideals of A respectively.

Remark. The condition (1) in the definition is just to put a restriction μ be nonempty in the fuzzy sense. In fact, one may think that, to be nonempty, it is enough to assume $\mu(0) > 1$. But for our purpose we chose the value of $\mu(0)$ to be 1, i.e., the element 0 certainly belongs to the fuzzy ideal μ.

Example 2.1.4. Let $(G,\cdot,^{-1},e)$ be a group. A fuzzy subset μ of G is a fuzzy ideal of G if and only if:

1. $\mu(e) = 1$
2. $\mu(xy) \geq \mu(x) \wedge \mu(y)$
3. $\mu(y^{-1}) \geq \mu(y)$
4. $\mu(xyx^{-1}) \geq \mu(y)$

According to [137], μ is a fuzzy normal subgroup of G. Rosenfield [136], in his definition does not assume the condition $\mu(e) = 1$.

Example 2.1.5. Let $(R,+,\cdot,-,0,1)$ be a ring. A fuzzy subset μ of R is a fuzzy ideal of R if and only if:

1. $\mu(0) = 1$
2. $\mu(x+y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(-y) \geq \mu(y)$

4. $\mu(xy) \geq \mu(x) \vee \mu(y)$

These properties coincide with that of [137]. In other words μ is a normalized fuzzy ideal of R in the sense of [113].

Example 2.1.6. Let $(L, \vee, \wedge, 0)$ be a distributive lattice with least element 0. A fuzzy subset μ of L is a fuzzy ideal of L if and only if:

1. $\mu(0) = 1$
2. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$

In other words, μ is fuzzy ideal of L in the sense of [146].

2.2 Some Characterizations

Definition 2.2.1. Let $p(\overrightarrow{x})$ be an $n-$ary term operation on A. For each fuzzy subsets $\eta_1, ..., \eta_n$ of A define

$$p : L^A \times ... \times L^A \to L^A$$
$$p(\eta_1, ..., \eta_n) \mapsto \eta$$

where, $\eta(x) = \bigvee\{\eta_1(x_1) \wedge ... \wedge \eta_n(x_n) : p(x_1, ..., x_n) = x\}$. Supremum being taken over all $n-$tuples $(x_1, ..., x_n) \in A^n$ for which $p(x_1, ..., x_n) = x$.

Theorem 2.2.2. *A fuzzy subset μ of A is a fuzzy ideal of A if and only if*

1. $\mu(0) = 1$*, and*
2. *For each fuzzy subsets $\eta_1, ..., \eta_n$ of A, and each $(n+m)-$ary ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$ it holds that: $p(\eta_1, ..., \eta_n, \mu, \mu, ..., \mu) \leq \mu$*

Proof. Suppose that μ is a fuzzy ideal of A. Let $p(\overrightarrow{x},\overrightarrow{y})$ be an $(n+m)-$ary ideal term in $\overrightarrow{y}$ and $\eta_1,...,\eta_n$ be arbitrary fuzzy subsets of A. For simplicity, let us put $\eta = p(\eta_1,...,\eta_n,\mu,\mu,...,\mu)$. For $x \in A$, if there are no $(n+m)-$tuples $(x_1,...,x_n,y_1,...,y_m) \in A^{n+m}$ such that $x = p(x_1,...,x_n,y_1,...,y_m)$, then $\eta(x) = 0$ and hence $\eta(x) \leq \mu(x)$. Assume that there are such $(n+m)-$tuples. Let $(a_1,...,a_n,b_1,...,b_m) \in A^{n+m}$ be an $(n+m)-$tuple such that $x = p(a_1,...,a_n,b_1,...,b_m)$. Since $p(\overrightarrow{x},\overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$ and μ is a fuzzy ideal we get:

$$\begin{aligned} \mu(x) &\geq \mu(b_1)\wedge...\wedge\mu(b_m) \\ &\geq \eta_1(a_1)\wedge...\wedge\eta_n(a_n)\wedge\mu(b_1)\wedge...\wedge\mu(b_m) \end{aligned}$$

This implies that

$$\mu(x) \geq \bigvee\{\eta_1(x_1)\wedge...\wedge\eta_n(x_n)\wedge\mu(y_1)\wedge...\mu(y_m) : p(x_1,...,x_n,y_1,...,y_m) = x\} = \eta(x)$$

Thus, $\eta \leq \mu$. Conversely suppose that the above two conditions are satisfied. We show that μ is $\overrightarrow{y}$ closed under each $(n+m)-$ary ideal term $p(\overrightarrow{x},\overrightarrow{y})$ in $\overrightarrow{y}$. Let $p(\overrightarrow{x},\overrightarrow{y})$ be an $(n+m)-$ary ideal term in $\overrightarrow{y}$, and let $a_1,...,a_n,b_1,...,b_m \in A$. Put $x = p(a_1,...,a_n,b_1,...,b_m)$. For each $i, 1 \leq i \leq n$ define fuzzy subsets η_i of A by:

$$\eta_i(z) = \begin{cases} 1 & \textit{if } z = a_i \\ 0 & \textit{otherwise} \end{cases}$$

for all $z \in A$. Again put $\eta = p(\eta_1,...,\eta_n,\mu,\mu,...,\mu)$. By condition (2), we have $\eta(x) \leq \mu(x)$, which implies that $\mu(x) \geq \mu(b_1)\wedge...\wedge\mu(b_m)$. Therefore μ is a fuzzy ideal of A. Hence proved. □

The following theorem gives an equivalent condition for fuzzy subsets to be a fuzzy ideal in terms of their level sets.

Theorem 2.2.3. *A fuzzy subset μ of A is a fuzzy ideal of A if and only if μ_α is an ideal of A for all $\alpha \in L$.*

Proof. Suppose that μ is a fuzzy ideal of A. For any $\alpha \in L$, let $a_1,...,a_n \in A$ and $b_1,...,b_m \in \mu_\alpha$. Then, $\mu(b_i) \geq \alpha$ for all $1 \leq i \leq m$. This implies $\mu(b_1) \wedge ... \wedge \mu(b_m) \geq \alpha$. Let $p(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Since μ is given to be a fuzzy ideal of A, we have

$$\mu(p(a_1,...,a_n,b_1,...,b_m)) \geq \mu(b_1) \wedge ... \wedge \mu(b_m) \geq \alpha$$

So, we get $p(a_1,...,a_n,b_1,...,b_m) \in \mu_\alpha$ and hence each μ_α is an ideal of A. Conversely, suppose that the level subset μ_α is an ideal of A for all $\alpha \in L$. In particular μ_α is an ideal for $\alpha = 1$. So that $\mu(0) = 1$. Let $p(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$ and $a_1,...,a_n,b_1,,,.b_m \in A$. Put $\alpha = \mu(b_1) \wedge ... \wedge \mu(b_m)$. Then $b_1,...,b_m \in \mu_\alpha$. Since each μ_α is an ideal of A, we get $p(a_1,...,a_n,b_1,...,b_m) \in \mu_\alpha$. So that

$$\mu(p(a_1,...,a_n,b_1,...,b_m)) \geq \alpha = \mu(b_1) \wedge ... \wedge \mu(b_m)$$

Therefore μ is a fuzzy ideal of A. □

The previous theorem confirms that a fuzzy ideal of A is precisely a fuzzy $\mathfrak{L}$-subset of A, in the sense of [144]; where $\mathfrak{L}$ is the set of all ideals of A.

Lemma 2.2.4. *A subset I of A is an ideal of A if and only if α_I is a fuzzy ideal of A, for any $\alpha \in L - \{1\}$, where α_I is as given in Definition 1.2.12.*

Proof. Suppose that I is an ideal of A and let $\alpha \in L - \{1\}$. Then $0 \in I$ and hence $\alpha_I(0) = 1$. Let $a_1,...,a_n,b_1,...,b_m \in A$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. If $\alpha_I(b_j) = \alpha$ for some j, then

$$\alpha_I(b_1) \wedge ... \wedge \alpha_I(b_m) = \alpha \leq \alpha_I(p(a_1,...,a_n,b_1,...,b_m))$$

Assume that $\alpha_I(b_j) \neq \alpha$ for all $1 \leq j \leq m$. Then $b_1,...,b_m \in I$. By our assumption, I is an ideal of A. So, $p(a_1,...,a_n,b_1,...,b_m) \in I$ and hence

$$\alpha_I(p(a_1,...,a_n,b_1,...,b_m)) = 1 \geq \alpha_I(b_1) \wedge ... \wedge \alpha_I(b_m)$$

Thus α_I is a fuzzy ideal of A. Conversely, suppose that α_I is a fuzzy ideal of A for some $\alpha \in L - \{1\}$. Then $\alpha_I(0) = 1$ and hence $0 \in I$; that is, I is nonempty. Also, let $a_1,...,a_n \in A$, $b_1,...,b_m \in I$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Then $\alpha_I(b_j) = 1$ for all $1 \leq j \leq m$, which gives, $\alpha_I(b_1) \wedge ... \wedge \alpha_I(b_m) = 1$. Being α_I a fuzzy ideal, we obtain $\alpha_I(p(a_1,...,a_n,b_1,...,b_m)) =$

1. Equivalently, $p(a_1,...,a_n,b_1,...,b_m) \in I$ and hence I is an ideal of A. This completes the proof. □

Corollary 2.2.5. *A subset I of A is an ideal of A if and only if its characteristic mapping χ_I is a fuzzy ideal of A.*

Lemma 2.2.6. *Let μ be a fuzzy ideal of A. If A has a unit element say u, then $\mu(u) \leq \mu(x)$ for all $x \in A$.*

Lemma 2.2.7. *Let μ be a fuzzy ideal of A and $x \in A$. Then we have the following:*

1. *For each $a \in A$, if $x \in \langle a \rangle$, then $\mu(x) \geq \mu(a)$.*

2. *For any $a_1,...,a_m \in A$, if $x \in \langle \{a_1,...,a_m\} \rangle$, then $\mu(x) \geq \mu(a_1) \wedge ... \wedge \mu(a_m)$. More generally, for any nonempty subset S of A, if $x \in \langle S \rangle$, then there exist $a_1,...,a_m \in S$ such that $\mu(x) \geq \mu(a_1) \wedge ... \wedge \mu(a_m)$.*

Proof. 1. Suppose that $x \in \langle a \rangle$. Then $x = p(a_1,...,a_n,a)$ for some $a_1,...,a_n \in A$ and an $(n+1)-$ary ideal term $p(\overrightarrow{x},y)$ in y. Being μ a fuzzy ideal, we get

$$\mu(x) = \mu(p(a_1,...,a_n,a)) \geq \mu(a)$$

2. Suppose that $x \in \langle \{a_1,...,a_m\} \rangle$. Then, $x = p(b_1,...,b_n,a_1,...,a_m)$ for some $b_1,...,b_n \in A$ and some ideal term $p(\overrightarrow{x},\overrightarrow{y})$ in $\overrightarrow{y}$. So we have the following:

$$\mu(x) = \mu(p(b_1,...,b_n,a_1,...,a_m)) \geq \mu(a_1) \wedge ... \wedge \mu(a_m)$$

Hence proved.

□

Theorem 2.2.8. *A fuzzy subset μ of A is a fuzzy ideal of A if and only if for each $m \geq 0$ and each $b_1,b_2,...,b_m \in A$, if $x \in \langle \{b_1,...,b_m\} \rangle$, then $\mu(x) \geq \mu(b_1) \wedge ... \wedge \mu(b_m)$.*

Proof. One part of this theorem is proved in Lemma 6.1.6. So we proceed to the converse part. Assume the given condition is satisfied for μ. Let us put $S_m = \{b_1,...,b_m\}$. If we take $m = 0$, then $S_m = \emptyset$ and it is known that $\langle\emptyset\rangle = \{0\}$. So by our assumption, we have

$$\mu(0) \geq \bigwedge_{b\in\emptyset} \mu(b) = 1$$

Thus $\mu(0) = 1$. Let $a_1,...,a_n,b_1,...,b_m \in A$ and $p(\overrightarrow{x},\overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. If we put $S_m = \{b_1,...,b_m\}$, then one can observe that $p(a_1,...,a_n,b_1,...,b_m) \in \langle S_m\rangle$. It follows from our assumption that $\mu(p(a_1,...,a_n,b_1,...,b_m)) \geq \mu(b_1)\wedge...\wedge\mu(b_m)$. Therefore μ is a fuzzy ideal of A. Hence proved. □

In the following theorem, we give a more general setting to characterize fuzzy ideals.

Theorem 2.2.9. *A fuzzy subset μ of A is a fuzzy ideal of A if and only if for any subset S of A*

$$\mu(a) \geq \bigwedge_{x\in S} \mu(x) \textit{ for all } a \in \langle S\rangle$$

Proof. Suppose that μ is a fuzzy ideal of A. If $S = \emptyset$, then $\langle S\rangle = (0)$ and the condition holds trivially. Assume that S is nonempty and let $a \in \langle S\rangle$. Then $a = t(a_1,...,a_n,b_1,...,b_m)$ for some $b_1,...,b_m \in S$, $a_1,...,a_n \in A$ and some ideal term $t(\overrightarrow{x},\overrightarrow{y})$ in $\overrightarrow{y}$. Since μ is fuzzy ideal, it follows that

$$\mu(a) \geq \mu(b_1)\wedge...\wedge\mu(b_m) \geq \bigwedge_{b\in S} \mu(b)$$

The converse part follows from the above Theorem 6.1.7 by assuming the condition for finite sets. □

2.3 Fuzzy Ideals Generated by a Fuzzy Set

In this section, we characterize fuzzy ideals generated by fuzzy sets in different ways.

Theorem 2.3.1. *If $\{\mu_i\}_{i\in\Delta}$ is a family of fuzzy ideals of A, then $\cap_{i\in\Delta}\mu_i$ is a fuzzy ideal of A.*

Proof. Let us put $\mu = \bigcap_{i\in\Delta} \mu_i$; i.e., for each $x \in A$

$$\mu(x) = \bigwedge_{i\in\Delta} \mu_i(x)$$

for all $x \in A$. It is clear that $\mu(0) = 1$. Let $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$, and $p(\overrightarrow{x}, \overrightarrow{y})$ be an an ideal term in $\overrightarrow{y}$. Consider the following:

$$\begin{aligned}
\mu(p(a_1,\ldots,a_n,b_1,\ldots,b_m)) &= \bigwedge_{i\in\Delta} \mu_i(p(a_1,\ldots,a_n,b_1,\ldots,b_m)) \\
&\geq \bigwedge_{i\in\Delta} (\mu_i(b_1) \wedge \ldots \wedge \mu_i(b_m)) \\
&= \left(\bigwedge_{i\in\Delta} \mu_i(b_1)\right) \wedge \ldots \wedge \left(\bigwedge_{i\in\Delta} \mu_i(b_m)\right) \\
&= \mu(b_1) \wedge \ldots \wedge \mu(b_m)
\end{aligned}$$

Thus $\mu = \bigcap_{i\in\Delta} \mu_i$ is a fuzzy ideal. □

This theorem confirms that, for any fuzzy subset λ of A always there exists a smallest fuzzy ideal containing λ which we call it the fuzzy ideal of A generated by λ and is denoted by $\langle\lambda\rangle$.

Lemma 2.3.2. *Let μ and η be fuzzy subsets of A. Then*

1. $\mu \in FI(A)$ *if and only if* $\langle\mu\rangle = \mu$
2. $\mu \leq \eta \Rightarrow \langle\mu\rangle \leq \langle\eta\rangle$

It can be deduced from this lemma that the map $\mu \mapsto \langle\mu\rangle$ forms a closure operator on the lattice L^A of fuzzy subsets of A, and fuzzy ideals of A are those closed elements of L^A with respect to this closure operator.

Lemma 2.3.3. *For any subset S of A and each $\alpha \in L - \{1\}$, $\langle\alpha_S\rangle = \alpha_{\langle S\rangle}$.*

Proof. We show that $\alpha_{\langle S\rangle}$ is the smallest fuzzy ideal of A containing α_S. Since $\langle S\rangle$ is an ideal of A, it follows from Lemma 2.2.4 that $\alpha_{\langle S\rangle}$ is a fuzzy ideal of A. It is also clear that $\alpha_S \leq \alpha_{\langle S\rangle}$. Suppose λ is a fuzzy ideal of A such that $\alpha_S \leq \lambda$. Then $\lambda(s) = 1$ for all $s \in S$. More generally,

$\lambda(z) \geq \alpha$ for all $z \in A$. Let z be any element in A. If $z \notin \langle S \rangle$, then $\alpha_{\langle S \rangle}(z) = \alpha \leq \lambda(z)$. Also, if $z \in \langle S \rangle$, then $z = p(a_1, ..., a_n, s_1, ..., s_m)$ for some $a_1, ..., a_n \in A$, $s_1, ..., s_m \in S$ and some ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$. Now consider:

$$\lambda(z) = \lambda(p(a_1, ..., a_n, s_1, ..., s_m)) \geq \lambda(s_1) \wedge ... \wedge \lambda(s_m) = 1$$

So that $\alpha_{\langle S \rangle} \leq \lambda$. Therefore $\alpha_{\langle S \rangle} = \langle \alpha_S \rangle$. □

Corollary 2.3.4. *Let S be any subset of A and χ_S its characteristic function. Then $\langle \chi_S \rangle = \chi_{\langle S \rangle}$.*

In the following theorem, we characterize a fuzzy ideal generated by a fuzzy set in terms of its level sets.

Theorem 2.3.5. *For a fuzzy subset λ of A, let $\widehat{\lambda}$ be a fuzzy subset of A defined by:*

$$\widehat{\lambda}(x) = \bigvee \{\alpha \in L : x \in \langle \lambda_\alpha \rangle\}$$

for all $x \in A$. Then $\widehat{\lambda} = \langle \lambda \rangle$.

Proof. We show that $\widehat{\lambda}$ is the smallest fuzzy ideal of A containing λ. Let us first show that $\widehat{\lambda}$ is a fuzzy ideal.

1. $\widehat{\lambda}(0) = \bigvee\{\alpha \in L : 0 \in \langle \lambda_\alpha \rangle\} = 1$

2. Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Then consider:

$$\begin{aligned} (\widehat{\lambda})^m(\overrightarrow{b}) &= \bigwedge_{i=1}^{m} \widehat{\lambda}(b_i) \\ &= \bigwedge_{i=1}^{m} \{\bigvee \{\alpha_i \in L : b_i \in \langle \lambda_{\alpha_i} \rangle\}\} \\ &= \bigvee \{\bigwedge \{\alpha_i \in L : 1 \leq i \leq m\} : b_i \in \langle \lambda_{\alpha_i} \rangle\} \end{aligned}$$

If we put $\beta = \bigwedge\{\alpha_i \in L : 1 \leq i \leq m\}$, where $b_i \in \langle \lambda_{\alpha_i} \rangle$, then we get $\lambda_{\alpha_i} \subseteq \lambda_\beta$ for all $1 \leq i \leq m$, which gives that

$$\langle \lambda_{\alpha_i} \rangle \subseteq \langle \lambda_\beta \rangle$$

for all $1 \leq i \leq m$. So that $b_1,...,b_m \in \langle \lambda_\beta \rangle$. Now it follows from the above equality that;

$$\begin{aligned} (\widehat{\lambda})^m(\overrightarrow{b}) &= \bigvee \{ \bigwedge \{ \alpha_i \in L : 1 \leq i \leq m \} : b_i \in \langle \lambda_{\alpha_i} \rangle \} \\ &\leq \bigvee \{ \beta \in L : b_1,...,b_m \in \langle \lambda_\beta \rangle \} \\ &\leq \bigvee \{ \beta \in L : p(a_1...,a_n,b_1,...,b_m) \in \langle \lambda_\beta \rangle \} \\ &= \widehat{\lambda}(p(\overrightarrow{a},\overrightarrow{b})) \end{aligned}$$

Therefore $\widehat{\lambda}$ is a fuzzy ideal of A. It is also clear to see that $\lambda \leq \widehat{\lambda}$. Suppose that μ is any other fuzzy ideal of A such that $\lambda \leq \mu$. Then $\langle \lambda_\alpha \rangle \subseteq \mu_\alpha$ for all $\alpha \in L$. Now for any $x \in A$ consider:

$$\begin{aligned} \widehat{\lambda}(x) &= \bigvee \{ \alpha \in L : x \in \langle \lambda_\alpha \rangle \} \\ &\leq \bigvee \{ \alpha \in L : x \in \mu_\alpha \} \\ &= \mu(x) \quad (\text{by Theorem } 1.2.11) \end{aligned}$$

Therefore $\widehat{\lambda}$ is the smallest fuzzy ideal containing λ and hence $\widehat{\lambda} = \langle \lambda \rangle$. □

Corollary 2.3.6. *For any fuzzy subset μ of A and each $\alpha \in L$, $\langle \mu_\alpha \rangle \subseteq \langle \mu \rangle_\alpha$. Moreover, if L is a chain and μ is finite valued or equivalently if μ has sup property, then the equality holds.*

The following theorem gives a better description for $\alpha-$level cuts of the fuzzy ideal generated by a fuzzy set.

Theorem 2.3.7. *Let μ be a fuzzy subset of A and $\alpha \in L$:*

$$\langle \mu \rangle_\alpha = \bigcup \{ \bigcap_{\gamma \in M} \langle \mu_\gamma \rangle : M \subseteq L \text{ and } \alpha \leq supM \}$$

Proof. Let us put

$$H = \bigcup \{ \bigcap_{\gamma \in M} \langle \mu_\gamma \rangle : \alpha \leq supM \}$$

If $x \in H$, then $x \in \bigcap_{\gamma \in M} \langle \mu_\gamma \rangle$ for some $M \subseteq L$ with $\alpha \leq supM$; i.e., $x \in \langle \mu_\gamma \rangle$ for all $\gamma \in M$ and $\alpha \leq supM$. By Theorem 2.3.5 we have the following

$$\langle \mu \rangle(x) = \bigvee \{\beta \in L : x \in \langle \mu_\beta \rangle\}$$

So that $\langle \mu \rangle(x) \geq \gamma$ for all $\gamma \in M$. This gives $\langle \mu \rangle(x) \geq \alpha$. Thus $x \in \langle \mu \rangle_\alpha$ and hence $H \subseteq \langle \mu \rangle_\alpha$. To prove the other inequality, let us take $x \in \langle \mu \rangle_\alpha$. Then

$$\bigvee \{\beta \in L : x \in \langle \mu_\beta \rangle\} \geq \alpha$$

If we put $M = \{\beta \in L : x \in \langle \mu_\beta \rangle\}$, then $M \subseteq L$ such that $\alpha \leq supM$ and $x \in \langle \mu_\gamma \rangle$ for all $\gamma \in M$. This means that $x \in \bigcap_{\gamma \in M} \langle \mu_\gamma \rangle$ and $\alpha \leq supM$. Thus $x \in H$, and hence the proof ends. □

In the following, we give an algebraic characterization for fuzzy ideals generated by fuzzy sets.

Definition 2.3.8. For a fuzzy subset λ of A, define $\overline{\lambda}$ to be a fuzzy subset of A as follows: $\overline{\lambda}(0) = 1$ *and for* $0 \neq x \in A$

$$\overline{\lambda}(x) = \bigvee \{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, p(\overrightarrow{a}, \overrightarrow{b}) = x, \overrightarrow{a} \in A^n, p(\overrightarrow{x}, \overrightarrow{y}) \text{ is an ideal term in } \overrightarrow{y}\}$$

Theorem 2.3.9. *For any fuzzy subset λ of A, $\overline{\lambda} = \langle \lambda \rangle$.*

Proof. By Theorem 2.3.5 it is enough to show that $\overline{\lambda} = \widehat{\lambda}$. It is clear from the definition that $\overline{\lambda}(0) = \widehat{\lambda}(0)$. For each $0 \neq x \in A$, let us define two sets H_x and G_x as follows:

$$H_x = \{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, P(\overrightarrow{a}, \overrightarrow{b}) = x, \overrightarrow{a} \in A^n, P(\overrightarrow{x}, \overrightarrow{y}) \text{ is an ideal term in } \overrightarrow{y}\}$$
$$G_x = \{\alpha \in L : x \in \langle \lambda_\alpha \rangle\}$$

Clearly both H_x and G_x are nonempty subsets of L. Our claim is to show that: $\bigvee H_x = \bigvee G_x$. If $\alpha \in H_x$, then $\alpha = \lambda^m(\overrightarrow{b}) = \lambda(b_1) \wedge \ldots \wedge \lambda(b_m)$, for some $b_1, \ldots, b_m \in A$, with $p(\overrightarrow{a}, \overrightarrow{b}) = x$ for some $a_1,...,a_n \in A$, where $p(\overrightarrow{x},\overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$. That is, $\overrightarrow{b} \in (\lambda_\alpha)^m$. So that $x \in \langle \lambda_\alpha \rangle$, which gives $\alpha \in G_x$. Thus $H_x \subseteq G_x$ and hence $\bigvee H_x \leq \bigvee G_x$. To prove the other inequality it is enough to show that, for each $\alpha \in G_x$, there exists $\beta \in H_x$ such that $\alpha \leq \beta$. Let $\alpha \in G_x$. Then $x \in \langle \lambda_\alpha \rangle$; that is, $x = p(\overrightarrow{a},\overrightarrow{b})$ for some $\overrightarrow{b} \in (\lambda_\alpha)^m$, and $\overrightarrow{a} \in A^n$ where $p(\overrightarrow{x},\overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$. If we put $\beta = \lambda^m(\overrightarrow{b})$, then $\beta \in H_x$ such that $\alpha \leq \beta$. This completes the proof. □

Corollary 2.3.10. *For each $x \in A$ and $\alpha \in L - \{0\}$, the fuzzy ideal of A generated by the fuzzy point x_α is characterized as:*

$$\langle x_\alpha \rangle(z) = \begin{cases} 1 & \textit{if } z = 0 \\ \alpha & \textit{if } z \in \langle x \rangle - \{0\} \\ 0 & \textit{otherwise} \end{cases}$$

for all $z \in A$.

Notation. We write $F \subset\subset A$, to say that F is a finite subset of A.

In the following theorem we characterize fuzzy ideals generated by a fuzzy set using finitely generated crisp ideals.

Theorem 2.3.11. *For a fuzzy subset μ of A, let $\overline{\mu}$ be a fuzzy subset of A defined by:*

$$\overline{\mu}(x) = \bigvee \{ \bigwedge_{a \in F} \mu(a) : x \in \langle F \rangle, F \subset\subset A \}$$

for all $x \in A$. Then $\overline{\mu} = \langle \mu \rangle$.

Proof. It is enough if we show that $\overline{\mu} = \widehat{\mu}$. Clearly, $\overline{\mu}(0) \geq \bigwedge_{a \in F} \mu(a)$ for all finite subsets F of A (since $0 \in \langle F \rangle, \forall F \subset\subset A$). In particular,

$$\overline{\mu}(0) \geq \bigwedge_{a \in \emptyset} \mu(a) = 1$$

Thus $\overline{\mu}(0) = 1$. For each $0 \neq x \in A$, let us take the set G_x as in Theorem 2.3.9 and define a set H_x as follows:

$$H_x = \{\bigwedge_{a\in F} \mu(a) : x \in \langle F\rangle, F \subset\subset A\}$$

Our claim is to show that:

$$\bigvee\{\alpha : \alpha \in H_x\} = \bigvee\{\alpha : \alpha \in G_x\}$$

One way we show that $H_x \subseteq G_x$. If $\alpha \in H_x$, then $\alpha = \bigwedge_{a\in F} \mu(a)$ and $x \in \langle F\rangle$ for some finite subset F of A. That is, $a \in \mu_\alpha$ for all $a \in F$ and $x \in \langle F\rangle$. So that $x \in \langle \mu_\alpha\rangle$. Then $\alpha \in G_x$ and hence $H_x \subseteq G_x$. The other way, we prove that, for each $\alpha \in G_x$, there exists $\beta \in H_x$ such that $\alpha \leq \beta$. For, let $\alpha \in G_x$. Then $x \in \langle\lambda_\alpha\rangle$; that is, $x = P(\overrightarrow{a}, \overrightarrow{b})$ for some $\overrightarrow{b} \in (\lambda_\alpha)^m$, and $\overrightarrow{a} \in A^n$ where $p(\overrightarrow{x}, \overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$. Let $\overrightarrow{b} = (b_1, b_2, ..., b_m)$ and $\beta = \bigwedge_{i=1}^{m} \mu(b_i)$. Then $\beta \geq \alpha$. Moreover, if we put $F = \{b_1, b_2, ..., b_m\}$, then F is a finite subset of A such that $x \in \langle F\rangle$. Thus $\beta \in H_x$ such that $\alpha \leq \beta$. This completes the proof. □

Theorem 2.3.12. *Suppose that $\{I_\alpha\}_{\alpha\in L}$ is a family of ideals of A such that*

$$\bigcap_{\alpha\in M} I_\alpha = I_{supM}$$

for all $M \subseteq L$. Then, there is a unique fuzzy ideal μ of A for which $\mu_\alpha = I_\alpha$ for all $\alpha \in L$. Moreover, every fuzzy ideal of A is obtained in this way only.

Proof. We first show that the map $\alpha \mapsto I_\alpha$ is antitone; in the sense that, for each $\alpha, \beta \in L$ $\alpha \leq \beta \Rightarrow I_\beta \subseteq I_\alpha$. Let $\alpha, \beta \in L$ such that $\alpha \leq \beta$. Put $M = \{\alpha, \beta\}$. Then $supM = \beta$. By our hypothesis $I_\alpha \cap I_\beta = I_{supM} = I_\beta$. So that $I_\beta \subseteq I_\alpha$ and hence the map $\alpha \mapsto I_\alpha$ is antitone. Define a fuzzy subset μ of A by:

$$\mu(x) = \bigvee\{\alpha \in L : x \in I_\alpha\}$$

for all $x \in A$. Clearly μ is well defined. Our aim is to show that $\mu_\alpha = I_\alpha$ for all $\alpha \in L$. The inclusion $I_\alpha \subseteq \mu_\alpha$ follows easily from the definition of μ. To prove the other inclusion, let $x \in \mu_\alpha$. Then $\mu(x) \geq \alpha$, i.e.,

$$\bigvee\{\gamma \in L : x \in I_\gamma\} \geq \alpha$$

If we put $M = \{\gamma \in L : x \in I_\gamma\}$, then $M \subseteq L$ such that $\alpha \leq supM$ and $x \in I_\gamma$ for all $\gamma \in M$, i.e.,

$$x \in \bigcap_{\gamma \in M} I_\gamma$$

By our assumption it follows that $x \in I_{supM}$. Since $\alpha \leq supM$ and the map $\alpha \mapsto I_\alpha$ is antitone we get $x \in I_\alpha$. Thus $I_\alpha = \mu_\alpha$. This means that μ is a fuzzy subset of A for which its $\alpha-$level sets are I_α's. Each I_α being an ideal of A, it follows from Theorem 2.2.3 that μ is a fuzzy ideal. The uniqueness of μ follows from the fact $\mu_\alpha = I_\alpha$ for all $\alpha \in L$. □

2.4 The Lattice of Fuzzy Ideals

As observed in the previous section, the intersection of any family of fuzzy ideals of A is a fuzzy ideal, i.e., the subfamily $FI(A)$ of the lattice L^A is closed under arbitrary intersection of fuzzy sets. So that $(FI(A), \leq)$ forms a closure fuzzy set system and hence by Theorem 1.2.16 it is a complete lattice, where $\leq$ is a pointwise ordering of fuzzy sets. The following theorem summarizes this.

Theorem 2.4.1. *The set of all fuzzy ideals of A forms a complete lattice where the infimum and supremum of any family $\{\mu_i : i \in \Delta\}$ of fuzzy ideals of A is given by:*

$$\bigwedge \mu_i = \cap \mu_i \text{ and } \bigvee \mu_i = \langle \cup \mu_i \rangle$$

Theorem 2.4.2. *$(FI(A), \leq)$ is an algebraic closure fuzzy set system.*

Proof. By Definition 1.2.18, it is enough to show that $FI(A)$ is inductive in L^A. Let $\{\mu_i\}_{i\in\Delta}$ be a chain in $FI(A)$. Let us put

$$\eta = \bigcup_{i \in \Delta} \mu_i$$

We show that η is a fuzzy ideal of A. Clearly $\eta(0) = 1$. Let $a_1,...,a_n,b_1,...,b_m \in A$ and $p(\overrightarrow{x},\overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. First observe that, for each $m-$tuples $i_1,...,i_m \in \Delta$, there exists $k \in \{1,2,...,m\}$ such that $\mu_{i_j} \leq \mu_{i_k}$ for all $j \in \{1,2,...,m\}$. Now consider the following:

$$\begin{aligned} \eta(b_1)\wedge...\wedge\eta(b_m) &= \left(\bigvee_{i_1\in\Delta}\mu_{i_1}(b_1)\right)\wedge...\wedge\left(\bigvee_{i_m\in\Delta}\mu_{i_m}(b_m)\right) \\ &= \bigvee_{i_1,...,i_m\in\Delta}(\mu_{i_1}(b_1)\wedge...\wedge\mu_{i_m}(b_m)) \\ &\leq \bigvee_{i_k\in\Delta}(\mu_{i_k}(b_1)\wedge...\wedge\mu_{i_k}(b_m)) \\ &\leq \bigvee_{i_k\in\Delta}\mu_{i_k}(p(a_1,...,a_n,b_1,...,b_m)) \\ &= \eta(p(a_1,...,a_n,b_1,...,b_m)) \end{aligned}$$

Therefore η is a fuzzy ideal of A and this completes the proof. □

2.5 Homomorphisms and Fuzzy Ideals

Let A and B be algebras of the same type $\mathfrak{F}$. A mapping $h : A \to B$ is called a homomorphism from A to B if:

$$h(f^A(a_1,a_2,...,a_n)) = f^B(h(a_1),h(a_2),...,h(a_n))$$

for each $n-$ary operation $f \in \mathfrak{F}$ and each sequence $a_1,a_2,...,a_n$ from A. It is observed that if p is an $n-$ary term of type $\mathfrak{F}$, then

$$h(p^A(a_1,a_2,...,a_n)) = p^B(h(a_1),h(a_2),...,h(a_n))$$

for all $a_1,a_2,...,a_n \in A$.

Theorem 2.5.1. *Let $h : A \to B$ be a homomorphism. Then we have the following:*

1. *If σ is a fuzzy ideal of B, then $h^{-1}(\sigma)$ is a fuzzy ideal of A*

2. *If μ is a fuzzy ideal of A and h is surjective, then $h(\mu)$ is a fuzzy ideal of B.*

Proof. Let $h : A \to B$ be a homomorphism.

1. Suppose that σ is a fuzzy ideal of B and let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in A$. Then $h(a_1), h(a_2), ..., h(a_n), h(b_1), h(b_2), ..., h(b_m) \in B$. If $p(\overrightarrow{x}, \overrightarrow{y})$ is an $(n+m)-$ary ideal term in $\overrightarrow{y}$, then we get:

$$\sigma(p^B(h(a_1), h(a_2), ..., h(a_n), h(b_1), h(b_2), ..., h(b_m))) \geq \sigma(h(b_1)) \wedge ... \wedge \sigma(h(b_m))$$

Now consider the following:

$$\begin{aligned} h^{-1}(\sigma)(p^A(a_1, ..., a_n, b_1, ..., b_m)) &= \sigma(h(p^A(a_1, ..., a_n, b_1, ..., b_m))) \\ &= \sigma(p^B(h(a_1), ..., h(a_n), h(b_1), ..., h(b_m))) \\ &\geq \sigma(h(b_1)) \wedge ... \wedge \sigma(h(b_m)) \\ &= h^{-1}(\sigma)(b_1) \wedge ... \wedge h^{-1}(\sigma)(b_m) \end{aligned}$$

Therefore $h^{-1}(\sigma)$ is a fuzzy ideal of A.

2. Suppose that h is surjective and let μ be a fuzzy ideal of A. If $u_1, u_2, ..., u_n, v_1, v_2, ..., v_m \in B$, then there exist $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in A$ such that $h(a_i) = u_i$ and $h(b_j) = v_j$ for all i, j. If $p(\overrightarrow{x}, \overrightarrow{y})$ is an $n+m$ ideal term in $\overrightarrow{y}$, then we get:

$$\begin{aligned} h(p^A(a_1, ..., a_n, b_1, ..., b_m)) &= p^B(h(a_1), ..., h(a_n), h(b_1), ..., h(b_m)) \\ &= p^B(u_1, ..., u_n, v_1, ..., v_m) \end{aligned}$$

So that $p^A(a_1, ..., a_n, b_1, ..., b_m) \in h^{-1}(p^B(u_1, ..., u_n, v_1, ..., v_m))$. Now consider the following:

$$\begin{aligned} h(\mu)(p^B((u_1, ..., u_n, v_1, ..., v_m)) &= \bigvee\{\mu(a) : a \in h^{-1}(p^B((u_1, ..., u_n, v_1, ..., v_m))\} \\ &\geq \mu(p^A(a_1, ..., a_n, b_1, ..., b_m)) \\ &\geq \mu(b_1) \wedge ... \wedge \mu(b_m) \end{aligned}$$

Since b_j is arbitrary in $h^{-1}(v_j)$ for all $j=1,2,...,m$, it follows that

$$\begin{aligned} h(\mu)(p^B((u_1,...,u_n,v_1,...,v_m)) &\geq \left(\bigvee_{b_1\in h^{-1}(v_1)} \mu(b_1)\right) \wedge ... \wedge \left(\bigvee_{b_m\in h^{-1}(v_m)} \mu(b_m)\right) \\ &= h(\mu)(v_1)\wedge ... \wedge h(\mu)(v_m) \end{aligned}$$

Therefore $h(\mu)$ is a fuzzy ideal of B.

□

Theorem 2.5.2. *If $h: A\rightarrow B$ is an onto homomorphism, then the mapping $\mu \mapsto h(\mu)$ defines a one-to-one correspondence between the set of all h-invariant fuzzy ideals of A and the set of all fuzzy ideals of B.*

Proof. By (2) of the above theorem, $\mu \mapsto h(\mu)$ is well-defined. To show that it is onto, let σ be a fuzzy ideal of B. Put $\mu = h^{-1}(\sigma)$. Then it follows from the above theorem that μ is an $h-$invariant fuzzy ideal of A such that $h(\mu)=\sigma$. So that the map $\mu \mapsto h(\mu)$ is onto. It remains to show that it one-one. Let μ_1 and μ_2 be an h-invariant fuzzy ideals of A such that $h(\mu_1)=h(\mu_2)$. Let $x\in A$. Then $h(x)\in B$ and $h(\mu_1)(h(x))=h(\mu_2)(h(x))$. Since μ_1 is h-invariant we have $\mu_1(x)=\mu_1(a)$ for all $a\in h^{-1}(x)$. So,

$$\begin{aligned} \mu_1(x) &= \bigvee\{\mu_1(a): a\in h^{-1}(x)\} \\ &= h(\mu_1)(h(x)) \\ &= h(\mu_2)(h(x)) \\ &= \bigvee\{\mu_2(b): b\in h^{-1}(x)\} \\ &= \mu_2(x) \end{aligned}$$

Thus $\mu_1=\mu_2$ and hence the map $\mu \mapsto h(\mu)$ is a one-to-one correspondence.

□

Theorem 2.5.3. *Let $h : A \to B$ be a homomorphism, μ and ν be fuzzy ideals of A. Then*

$$h(\mu \vee \nu) = h(\mu) \vee h(\nu)$$

Proof. We show that $h(\mu \vee \nu)$ is the smallest fuzzy ideal of B containing both $h(\mu)$ and $h(\nu)$. By Theorem 2.5.1, $h(\mu \vee \nu)$ is a fuzzy ideal of B. Now let $y \in B$. If $h^{-1}(y) = \emptyset$, then $h(\mu)(y) = 0 \leq h(\mu \vee \nu)(y)$. Also if $h^{-1}(y) \neq \emptyset$, then consider the following:

$$\begin{aligned} h(\mu)(y) &= \bigvee\{\mu(x) : x \in h^{-1}(y)\} \\ &\leq \bigvee\{(\mu \vee \nu)(x) : x \in h^{-1}(y)\} \\ &= h(\mu \vee \nu)(y) \end{aligned}$$

So that $h(\mu) \leq h(\mu \vee \nu)$. Similarly, we can verify that $h(\nu) \leq h(\mu \vee \nu)$. Now for any fuzzy ideal η of B:

$$\begin{aligned} h(\mu) \leq \eta, h(\nu) \leq \eta &\Rightarrow h^{-1}(h(\mu)) \leq h^{-1}(\eta), h^{-1}(h(\nu)) \leq h^{-1}(\eta) \\ &\Rightarrow \mu \leq h^{-1}(\eta), \nu \leq h^{-1}(\eta) \\ &\Rightarrow \mu \vee \nu \leq h^{-1}(\eta) \\ &\Rightarrow h(\mu \vee \nu) \leq h(h^{-1}(\eta)) \leq \eta \end{aligned}$$

Therefore $h(\mu \vee \nu)$ is the smallest fuzzy ideal of B containing both $h(\mu)$ and $h(\nu)$. So that, $h(\mu \vee \nu) = h(\mu) \vee h(\nu)$. □

Theorem 2.5.4. *Let $h : A \to B$ be a homomorphism, and μ and ν be fuzzy ideals of A. Then*

$$h(\mu \wedge \nu) \leq h(\mu) \wedge h(\nu)$$

Moreover, if either μ or ν is $h-$invariant, then the equality holds.

Proof. Let y be any element in B. If $h^{-1}(y) = \emptyset$, then $h(\mu)(y) = 0 = h(\nu)(y) = h(\mu \wedge \nu)(y)$. Let $h^{-1}(y) \neq \emptyset$. Then consider the following:

$$\begin{aligned} h(\mu \wedge \nu)(y) &= \bigvee\{(\mu \wedge \nu)(x) : x \in h^{-1}(y)\} \\ &= \bigvee\{\mu(x) \wedge \nu(x) : x \in h^{-1}(y)\} \\ &\leq \bigvee\{\mu(a) \wedge \nu(b) : a, b \in h^{-1}(y)\} \\ &= \bigvee\{\mu(a) : a \in h^{-1}(y)\} \wedge \bigvee\{\nu(b) : b \in h^{-1}(y)\} \\ &= h(\mu)(y) \wedge h(\nu)(y) \end{aligned}$$

Therefore $h(\mu \wedge \nu) \leq h(\mu) \wedge h(\nu)$. Moreover, assume without loss of generality that μ is $h-$invariant. Then $\mu(a) = \mu(b)$, whenever $h(a) = h(b)$. Now for each $y \in B$, with $h^{-1}(y) \neq \emptyset$, consider the following:

$$\begin{aligned} h(\mu)(y) \wedge h(\nu)(y) &= \bigvee\{\mu(a) : a \in h^{-1}(y)\} \wedge \bigvee\{\nu(b) : b \in h^{-1}(y)\} \\ &= \bigvee\{\mu(a) \wedge \nu(b) : a, b \in h^{-1}(y)\} \\ &= \bigvee\{\mu(x) \wedge \nu(x) : x \in h^{-1}(y)\} \\ &= \bigvee\{(\mu \wedge \nu)(x) : x \in h^{-1}(y)\} \\ &= h(\mu \wedge \nu)(y) \end{aligned}$$

Therefore $h(\mu \wedge \nu) = h(\mu) \wedge h(\nu)$. □

Theorem 2.5.5. *Let $h : A \to B$ be a homomorphism, and σ and θ be fuzzy ideals of B. Then*

$$h^{-1}(\sigma) \vee h^{-1}(\theta) \leq h^{-1}(\sigma \vee \theta)$$

Moreover, the equality holds whenever h is surjective.

Proof. For each $x \in A$, consider:

$$h^{-1}(\sigma)(x) = \sigma(h(x)) \leq (\sigma \vee \theta)(h(x)) = h^{-1}(\sigma \vee \theta)(x)$$

So that $h^{-1}(\sigma) \leq h^{-1}(\sigma \vee \theta)$. Similarly it can be verified that $h^{-1}(\theta) \leq h^{-1}(\sigma \vee \theta)$. Therefore $h^{-1}(\sigma) \vee h^{-1}(\theta) \leq h^{-1}(\sigma \vee \theta)$. Further, let we assume that h is surjective. To prove the equality, it is enough if we show that $h^{-1}(\sigma \vee \theta)$ is the smallest fuzzy ideal of A containing both $h^{-1}(\sigma)$ and $h^{-1}(\theta)$. From Theorem 2.5.1 we have that $h^{-1}(\sigma \vee \theta)$ is a fuzzy ideal of A. From the above inequality also we have $h^{-1}(\sigma) \leq h^{-1}(\sigma \vee \theta)$ and $h^{-1}(\theta) \leq h^{-1}(\sigma \vee \theta)$. Now let μ be any other fuzzy ideal of A such that $h^{-1}(\sigma) \leq \mu$ and $h^{-1}(\theta) \leq \mu$. Then $h(h^{-1}(\sigma)) \leq h(\mu)$ and $h(h^{-1}(\theta)) \leq h(\mu)$. Since h is surjective, it follows that $\sigma \leq h(\mu)$ and $\theta \leq h(\mu)$. So that, $\sigma \vee \theta \leq h(\mu)$, which gives $h^{-1}(\sigma \vee \theta) \leq h^{-1}(h(\mu))$. Our aim is to show that $h^{-1}(\sigma \vee \theta) \leq \mu$. Suppose not. Then there exists $a \in A$ such that $h^{-1}(\sigma \vee \theta)(a) \leq \mu(a)$. If we put $z = h(a)$, then we get $(\sigma \vee \theta)(z) \nleq h(\mu)(z)$, which is a contradiction. Therefore $h^{-1}(\sigma \vee \theta) \leq \mu$ and hence the equality holds. □

Theorem 2.5.6. *Let $h : A \to B$ be a homomorphism, and σ and θ be fuzzy ideals of B. Then*

$$h^{-1}(\sigma \wedge \theta) = h^{-1}(\sigma) \wedge h^{-1}(\theta)$$

Proof. For each $a \in A$, consider the following:

$$\begin{aligned} h^{-1}(\sigma \wedge \theta)(a) &= (\sigma \wedge \theta)(h(a)) \\ &= \sigma(h(a)) \wedge \theta(h(a)) \\ &= h^{-1}(\sigma)(a) \wedge h^{-1}(\theta)(a) \\ &= (h^{-1}(\sigma) \wedge h^{-1}(\theta))(a) \end{aligned}$$

Therefore $h^{-1}(\sigma \wedge \theta) = h^{-1}(\sigma) \wedge h^{-1}(\theta)$. □

Theorem 2.5.7. *Let $h : A \to B$ be a surjective homomorphism. For any h-invariant fuzzy subset μ of A, we have:*

$$h(\langle \mu \rangle) = \langle h(\mu) \rangle$$

Proof. For any $y \in B$, consider:

$$
\begin{aligned}
h(\langle\mu\rangle)(y) &= \bigvee\{\langle\mu\rangle(x) : x \in h^{-1}(y)\} \\
&= \bigvee\{\bigvee\{\alpha \in L : x \in \langle\mu_\alpha\rangle\} : x \in h^{-1}(y)\} \\
&= \bigvee\{\alpha \in L : x \in \langle\mu_\alpha\rangle \text{ and } h(x) = y\} \\
&= \bigvee\{\alpha \in L : y \in h(\langle\mu_\alpha\rangle)\}
\end{aligned}
$$

on the other hand

$$\langle h(\mu)\rangle(y) = \bigvee\{\alpha \in L : y \in \langle h(\mu)_\alpha\rangle\}$$

Now it is enough to show that

$$h(\langle\mu_\alpha\rangle) = \langle h(\mu)_\alpha\rangle$$

Let $z \in h(\langle\mu_\alpha\rangle)$. Then $z = h(x)$ for some $x \in \langle\mu_\alpha\rangle$. There exist $a_1, ..., a_n \in A$, $b_1, ..., b_m \in \mu_\alpha$ and an ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$ such that $x = p^A(a_1, ..., a_n, b_1, ..., b_m)$. So,

$$
\begin{aligned}
z &= h(x) \\
&= h(p^A(a_1, ..., a_n, b_1, ..., b_m)) \\
&= p^B(h(a_1), ..., h(a_n), h(b_1), ..., h(b_m))
\end{aligned}
$$

For each $j = 1, 2, ..., m$ we have

$$h(\mu)(h(b_j)) = \bigvee\{\mu(x) : x \in h^{-1}(h(b_j))\}$$

Since μ is h-invariant and each $b_j \in \mu_\alpha$, we get

$$h(\mu)(h(b_j)) = \mu(b_j) \geq \alpha$$

for all $j = 1, 2, ..., m$; that is $h(b_j) \in h(\mu)_\alpha$ for all j and $z = p^B(h(a_1), ..., h(a_n), h(b_1), ..., h(b_m))$. This means $z \in \langle h(\mu)_\alpha \rangle$. So that

$$h(\langle \mu_\alpha \rangle) \subseteq \langle h(\mu)_\alpha \rangle$$

To prove the other inclusion, let $z \in \langle h(\mu)_\alpha \rangle$. Then $z = p^B(\overrightarrow{u}, \overrightarrow{v})$ for some $u_1, ..., u_n \in B$, $v_1, ..., v_m \in h(\mu)_\alpha$ and some ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$. Since h is surjective, there exist $a_1, ..., a_n, b_1, ..., b_m \in A$ such that $h(a_i) = u_i$ and $h(b_j) = v_j$ for all $i = 1, ..., n$ and $j = 1, ..., m$. As each $v_j \in h(\mu)_\alpha$, we have $h(\mu)(h(b_j)) \geq \alpha$. Since μ is h-invariant we get $\mu(b_j) \geq \alpha$; that is, $b_j \in \mu_\alpha$ for all j. Put $x = p^A(a_1, ...a_n, b_1, ...b_m)$. Then $x \in \langle \mu_\alpha \rangle$. Moreover

$$\begin{aligned} h(x) &= h(p^A(a_1, ..., a_n, b_1, ..., b_m)) \\ &= p^B(h(a_1), ..., h(a_n), h(b_1), ..., h(b_m))) \\ &= p^B(u_1, ..., u_n, v_1, ..., v_m) \\ &= z \end{aligned}$$

That is, $z = h(x)$, where $x \in \langle \mu_\alpha \rangle$, which gives $z \in h(\langle \mu_\alpha \rangle)$. Thus $\langle h(\mu)_\alpha \rangle \subseteq h(\langle \mu_\alpha \rangle)$. Hence $h(\langle \mu_\alpha \rangle) = \langle h(\mu)_\alpha \rangle$ and this completes the proof. □

2.6 Fuzzy Ideals of Fuzzy Subalgebras

In this section, we introduce the notion of fuzzy ideals of fuzzy subalgebras in universal algebra. The following definition is due to V. Murali [133].

Definition 2.6.1. An $L-$fuzzy subset μ of A is called an $L-$fuzzy subalgebra of A if the following are satisfied:

1. $\mu(f^A) = 1$ for all nullary operation symbols f in $\mathfrak{F}$.

2. If f is an $n-$ary operation symbol with $n > 0$ and $a_1, ..., a_n \in A$, then

$$\mu(f^A(a_1, ..., a_n)) \geq \mu(a_1) \wedge ... \wedge \mu(a_n)$$

It is obvious that subalgebras of A are closed under term functions. In the sense that, if B is a subalgebra of A, $p(x_1,...,x_m)$ is an $m-$ary term and $a_1,...,a_m \in B$, then $p(a_1,...,a_m) \in B$. Similarly, one can verify that fuzzy subalgebras are also closed (in the fuzzy sense) under term functions. In the sense that, if μ is a fuzzy subalgebra of A, $p(x_1,...,x_m)$ is an $m-$ary term, then

$$\mu(p(a_1,...,a_m)) \geq \mu(a_1) \wedge ... \wedge \mu(a_m)$$

for all $a_1,...,a_m \in A$. Fundamental operations of A can be viewed as an ideal term operation on A. using this fact, one can easily verify that fuzzy ideals of A are also fuzzy subalgebras of A.

Theorem 2.6.2. *A fuzzy subset μ of A is a fuzzy subalgebra of A if and only if the level subset μ_α is a subalgebra of A for any $\alpha \in L$.*

Lemma 2.6.3. *A subset $S \subseteq A$ is a subalgebra of A if and only if α_S is a fuzzy subalgebra of A for some $\alpha \in L - \{1\}$.*

Definition 2.6.4. Let μ and η be fuzzy subalgebras of A such that $\mu \leq \eta$. Then μ is called a fuzzy ideal of η if the following holds for each $a_1,...,a_n,b_1,...,b_m \in A$ and any ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$,

$$\mu(p(a_1,...,a_n,b_1,...,b_m)) \geq \eta(a_1) \wedge ... \wedge \eta(a_n) \wedge \mu(b_1) \wedge ... \wedge \mu(b_m)$$

Lemma 2.6.5. *Every fuzzy subalgebra of A is a fuzzy ideal of itself.*

Proof. Let μ be a fuzzy subalgebra of A. Let $a_1,...,a_n,b_1,...,b_m \in A$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be an $(n+m)-$ary ideal term in $\overrightarrow{y}$. Since fuzzy subalgebras are closed under term functions, we get

$$\mu(p(a_1,...,a_n,b_1,...,b_m)) \geq \mu(a_1) \wedge ... \wedge \mu(a_n) \wedge \mu(b_1) \wedge ... \wedge \mu(b_m)$$

Therefore μ is a fuzzy ideal of itself. □

Lemma 2.6.6. *Let μ and η be fuzzy subalgebras of A. Then*

1. *$\mu \in FI(A)$ if and only if μ is a fuzzy ideal of the fuzzy set 1_A.*

2. *If $\mu \in FI(A)$ and $\mu \leq \eta$, then μ is a fuzzy ideal of η.*

3. *If $\mu \in FI(A)$, then $\mu \cap \eta$ is a fuzzy ideal of η.*

Theorem 2.6.7. *Let μ and η be fuzzy subalgebras of A. Then, μ is a fuzzy ideal of η if and only if μ_α is an ideal of η_α for all $\alpha \in L$.*

Proof. Suppose that μ is a fuzzy ideal of η. Let $\alpha \in L$. By Theorem 2.6.2, both μ_α and η_α are subalgebras of A such that $\mu_\alpha \subseteq \eta_\alpha$. Let $a_1,...,a_n \in \eta_\alpha$, $b_1,...,b_m \in \mu_\alpha$. Then

$$\eta(a_1) \wedge ... \wedge \eta(a_n) \geq \alpha \text{ and } \mu(b_1) \wedge ... \wedge \mu(b_m) \geq \alpha$$

Let $p(\overrightarrow{x}, \overrightarrow{y})$ be an $(n+m)-$ary ideal term in $\overrightarrow{y}$. Since μ is a fuzzy ideal of η, we get

$$\mu(p(\overrightarrow{a}, \overrightarrow{b})) \geq \eta(a_1) \wedge ... \wedge \eta(a_n) \wedge \mu(b_1) \wedge ... \wedge \mu(b_m) \geq \alpha$$

So that $p(\overrightarrow{a}, \overrightarrow{b}) \in \mu_\alpha$. Thus μ_α is an ideal of η_α. Conversely, suppose that μ_α is an ideal of η_α for each $\alpha \in L$. Let $a_1,...,a_n, b_1,...,b_m \in A$, and $p(\overrightarrow{x}, \overrightarrow{y})$ be an $(n+m)-$ary ideal term in $\overrightarrow{y}$. Let us put

$$\alpha = \eta(a_1) \wedge ... \wedge \eta(a_n) \wedge \mu(b_1) \wedge ... \wedge \mu(b_m)$$

Then

$$a_1,...,a_n \in \eta_\alpha \text{ and } b_1,...,b_m \in \mu_\alpha$$

Since μ_α is an ideal of η_α, we get

$$p(\overrightarrow{a}, \overrightarrow{b}) \in \mu_\alpha$$

So that

$$\mu(p(\overrightarrow{a}, \overrightarrow{b})) \geq \alpha = \eta(a_1) \wedge ... \wedge \eta(a_n) \wedge \mu(b_1) \wedge ... \wedge \mu(b_m)$$

Thus μ is a fuzzy ideal of η. Hence proved. □

Theorem 2.6.8. *Let I and B be subalgebras of A. Then, I is an ideal of B if and only if α_I is a fuzzy ideal of α_B for all $\alpha \in L-\{1\}$.*

Theorem 2.6.9. *Let $h: A \to B$ be a homomorphism. Let η and θ be fuzzy subalgebras of A and B respectively. Then we have the following:*

1. *If σ is a fuzzy ideal of θ, then $h^{-1}(\sigma)$ is a fuzzy ideal of $h^{-1}(\theta)$.*

2. *If μ is a fuzzy ideal of η and h is surjective, then $h(\mu)$ is a fuzzy ideal of $h(\eta)$.*

Theorem 2.6.10. *Let η be a fuzzy subalgebra of A. If $\{\mu_i\}_{i\in I}$ is a family of fuzzy ideals of η, then $\cap_{i\in I}\mu_i$ is a fuzzy ideal of η.*

Given a fuzzy subalgebra η of A, the above theorem shows that the family $FI(\eta)$ of all fuzzy ideals of A forms a closure fuzzy set system together with the inclusion ordering of fuzzy sets. So, by Theorem 1.2.16 it is a complete lattice. Note that its least element is $\chi_{\{0\}}$ and the largest element is η.

Theorem 2.6.11. *$(FI(\eta), \leq)$ is an algebraic closure fuzzy set system.*

Proof. By Definition 1.2.18, it is enough to show that $FI(\eta)$ is inductive in L^A. Let $\{\mu_i\}_{i\in\Delta}$ be a chain in $FI(\eta)$. Let us put

$$\mu = \bigcup_{i\in\Delta} \mu_i$$

We show that μ is a fuzzy ideal of η. Clearly $\mu(0) = 1$. Let $a_1, ..., a_n, b_1, ..., b_m \in A$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. First observe that, for each $m-$tuples $i_1, ..., i_m \in \Delta$, there exists

$k \in \{1,2,...,m\}$ such that $\mu_{i_j} \le \mu_{i_k}$ for all $j \in \{1,2,...,m\}$. Now consider the following:

$$\begin{aligned}
\eta(a_1)\wedge ... \wedge \eta(a_n) \wedge \mu(b_1) \wedge ... \wedge \mu(b_m) &= \eta(a_1)\wedge ... \wedge \eta(a_n) \wedge \left(\bigvee_{i_1\in\Delta} \mu_{i_1}(b_1)\right) \wedge ... \wedge \left(\bigvee_{i_m\in\Delta} \mu_{i_m}(b_m)\right) \\
&= \eta(a_1)\wedge ... \wedge \eta(a_n) \wedge \left(\bigvee_{i_1,...,i_m\in\Delta} (\mu_{i_1}(b_1)\wedge ... \wedge \mu_{i_m}(b_m))\right) \\
&\le \eta(a_1)\wedge ... \wedge \eta(a_n) \wedge \left(\bigvee_{i_k\in\Delta} (\mu_{i_k}(b_1)\wedge ... \wedge \mu_{i_k}(b_m))\right) \\
&= \bigvee_{i_k\in\Delta} (\eta(a_1)\wedge ... \wedge \eta(a_n) \wedge \mu_{i_k}(b_1)\wedge ... \wedge \mu_{i_k}(b_m)) \\
&\le \bigvee_{i_k\in\Delta} \mu_{i_k}(p(a_1,...,a_n,b_1,...,b_m)) \\
&= \mu(p(a_1,...,a_n,b_1,...,b_m))
\end{aligned}$$

Therefore μ is a fuzzy ideal of η and this completes the proof. □

Chapter 3

$L-$Fuzzy Prime Ideals

Introduction

In the theory of groups, the important concepts of Abelian group, solvable group, nilpotent group, the center of a group and centralizers, are all defined from the binary operation $[x,y] = x^{-1}y^{-1}xy$. Each of these notions, except centralizers of elements, may also be defined in terms of the commutator of normal subgroups. The commutator $[M,N]$ (where M and N are normal subgroups of a group) is the (normal) subgroup generated by all the commutators $[x,y]$ with $x \in M$, $y \in N$. Similarly, the commutator of ideals I and J of a ring R, written as IJ, is the ideal of R generated by all products ij and ji, with $i \in I$ and $j \in J$; i.e.,

$$IJ = \{x \in R : x = \Sigma_{i=1}^{n} y_i z_i, y_i \in I, z_i \in J\}$$

The concept of commutators has also been extended to the class of distributive lattices. This has a significant role to have a ring theoretic interpretation for the problems in order theory. For ideals I and J of a distributive lattice L, their commutator $[I,J]$ is an ideal of L generated by those elements of the form $a \wedge b$ where $a \in I$ and $b \in J$, i.e.'

$$[I,J] = \{a \wedge b : a \in I, b \in J\}$$

Swamy and Swamy [143] defined the commutator (or the product) of $L-$fuzzy ideals μ and σ of a ring R as follows:

$$[\mu,\sigma](x) = \bigvee\{\bigwedge_{i=1}^{n}(\mu(y_i)\wedge\sigma(z_i)) : x = \Sigma_{i=1}^{n} y_i z_i\}$$

for all $x \in R$. They have used this commutator to define $L-$fuzzy prime ideals of rings.

In this chapter, we define and characterize the commutator of fuzzy ideals in a more general context, in universal algebras. We use this commutator to study fuzzy prime ideals, fuzzy semi-prime ideals and the radical of fuzzy ideals in universal algebras.

3.1 The Commutator of Fuzzy Ideals

It is observed in the second chapter that, a fuzzy subset μ of A is a fuzzy ideal of A if and only if every $\alpha-$level set of μ is an ideal of A. Here we define the commutator of fuzzy ideals using their level ideals.

Definition 3.1.1. The commutator of fuzzy ideals μ and σ of A denoted by $[\mu,\sigma]$ is a fuzzy subset of A defined by:

$$[\mu,\sigma](x) = \bigvee\{\alpha\wedge\beta : \alpha,\beta \in L, x \in [\mu_\alpha,\sigma_\beta]\}$$

for all $x \in A$.

For each α,β and λ in L with $\lambda = \alpha\wedge\beta$, it can be verified that,

$$x \in [\mu_\alpha,\sigma_\beta] \Rightarrow x \in [\mu_\lambda,\sigma_\lambda]$$

So the commutator of fuzzy ideals can be equivalently redefined as follows:

$$[\mu,\sigma](x) = \bigvee\{\lambda \in L : x \in [\mu_\lambda,\sigma_\lambda]\}$$

For fuzzy subsets η and θ of A, $[\eta,\theta]$ denotes the $[\langle\eta\rangle,\langle\theta\rangle]$. In particular, for fuzzy points x_α and y_β of A, $[\langle x_\alpha\rangle,\langle y_\beta\rangle]$ is denoted by $[x_\alpha,y_\beta]$.

The following lemmas can be verified easily.

Lemma 3.1.2. *For any fuzzy ideals μ and σ of A, $[\mu,\sigma]$ is a fuzzy ideal of A such that: $[\mu,\sigma]\leq \mu\cap\sigma$.*

Lemma 3.1.3. *For any ideals I and J of A $\chi_{[I,J]}=[\chi_I,\chi_J]$.*

In the following theorem, we give an algebraic characterization for the commutator of fuzzy ideals.

Theorem 3.1.4. *For each $x\in A$, and fuzzy ideals μ and σ of A:*

$$\begin{aligned}[][\mu,\sigma](x) &= \bigvee\{\mu^m(\overrightarrow{b})\wedge\sigma^k(\overrightarrow{c}):x=t(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}),\overrightarrow{a}\in A^n,\overrightarrow{b}\in A^m,\overrightarrow{c}\in A^k,\ \textit{and}\\ &\quad t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})\ \textit{is a commutator term in}\ \overrightarrow{y},\overrightarrow{z}\}\end{aligned}$$

Proof. For each $x\in A$, let us define two sets H_x and G_x as follows:

$$\begin{aligned}H_x &= \{\mu^m(\overrightarrow{b})\wedge\sigma^k(\overrightarrow{c}):x=t(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}),\overrightarrow{a}\in A^n,\overrightarrow{b}\in A^m,\overrightarrow{c}\in A^k\\ &\quad \textit{and}\ t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})\ \textit{is a commutator term in}\ \overrightarrow{y},\overrightarrow{z}\}\\ G_x &= \{\alpha\in L:x\in[\mu_\alpha,\sigma_\alpha]\}\end{aligned}$$

Clearly both H_x and G_x are nonempty subsets of L. Our claim is to see that $\vee H_x=\vee G_x$. If $\alpha\in H_x$, then $\alpha=\mu^m(\overrightarrow{b})\wedge\sigma^k(\overrightarrow{c})$, where $x=t(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})$ for some $\overrightarrow{a}\in A^n$, $\overrightarrow{b}\in A^m$, $\overrightarrow{c}\in A^k$ and some commutator term $t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})$ in $\overrightarrow{y},\overrightarrow{z}$. That is, $\overrightarrow{b}\in(\mu_\alpha)^m$ and $\overrightarrow{c}\in(\sigma_\alpha)^k$. So that $x\in[\mu_\alpha,\sigma_\alpha]$. Then $\alpha\in G_x$ and hence $H_x\subseteq G_x$. To prove the other inequality, we show that for each $\alpha\in G_x$, there exists $\beta\in H_x$ such that $\alpha\leq\beta$. Let $\alpha\in G_x$. Then $x\in[\mu_\alpha,\sigma_\alpha]$. So that $x=t(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})$ for some $\overrightarrow{a}\in A^n$, $\overrightarrow{b}\in(\mu_\alpha)^m$, and $\overrightarrow{c}\in(\sigma_\alpha)^k$, where $t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})$ is a commutator term in $\overrightarrow{y},\overrightarrow{z}$. That is, $\mu^m(\overrightarrow{b})\geq\alpha$ and $\sigma^k(\overrightarrow{c})\geq\alpha$. If we put $\beta=\mu^m(\overrightarrow{b})\wedge\sigma^k(\overrightarrow{c})$, then $\beta\geq\alpha$ and $\beta\in H_x$. This completes the proof. □

Corollary 3.1.5. *For each $\mu, \sigma \in FI(A)$ and each $\alpha \in L$,*

$$[\mu_\alpha, \sigma_\alpha] \subseteq [\mu, \sigma]_\alpha$$

Theorem 3.1.6. *For each $\mu, \sigma \in FI(A)$ and each $\alpha \in L$,*

$$[\mu, \sigma]_\alpha = \bigcup \{ \bigcap_{\gamma \in M} [\mu_\gamma, \sigma_\gamma] : M \subseteq L \text{ and } \alpha \leq supM\}$$

Definition 3.1.7. For each $a \in A$ and $\mu \in FI(A)$, we define $[a, \mu]$ to be $[\chi_{\{a\}}, \mu]$.

The following Theorem is an easy consequence of Theorem 3.1.4.

Theorem 3.1.8. *Let $a \in A$ and $\mu \in FI(A)$. Then for each $x \in A$,*

$$[a, \mu](x) = \bigvee \{ \alpha \in L : x \in [a, \mu_\alpha]\}$$

Theorem 3.1.9. *If x_α and y_β are fuzzy points of A, then the commutator $[x_\alpha, y_\beta]$ is characterized as:*

$$[x_\alpha, y_\beta](z) = \begin{cases} 1 & \text{if } z = 0 \\ \alpha \wedge \beta & \text{if } z \in [x, y] - \{0\} \\ 0 & \text{otherwise} \end{cases}$$

The following theorem gives a finite representation for the commutator of fuzzy ideals.

Theorem 3.1.10. *For each $x \in A$, and fuzzy ideals μ and σ of A:*

$$[\mu, \sigma](x) = \bigvee \{ \bigwedge_{a \in E, b \in F} (\mu(a) \wedge \sigma(b)) : x \in [E, F], E, F \subset\subset A\}$$

Proof. For each $x \in A$, let us take the set G_x as in Theorem 3.1.4 and define a set H_x as follows:

$$H_x = \{ \bigwedge_{a \in E, b \in F} (\mu(a) \wedge \sigma(b)) : x \in [E, F], E, F \subset\subset A\}$$

Our claim is to show that $\vee H_x = \vee G_x$. If $\alpha \in H_x$, then

$$\alpha = \bigwedge_{a \in E, b \in F} (\mu(a) \wedge \sigma(b))$$

where E and F are finite subsets of A such that $x \in [E, F]$. That is, $\mu(a) \wedge \sigma(b) \geq \alpha$ for all $a \in E$ and all $b \in F$. Then $E \subseteq \mu_\alpha$ and $F \subseteq \sigma_\alpha$. So that $[E, F] \subseteq [\mu_\alpha, \sigma_\alpha]$. Thus $x \in [\mu_\alpha, \sigma_\alpha]$ and hence $\alpha \in G_x$. Therefore $H_x \subseteq G_x$. To prove the other inequality, we show that for each $\alpha \in G_x$, there exists $\beta \in H_x$ such that $\alpha \leq \beta$. Let $\alpha \in G_x$. Then, $x \in [\mu_\alpha, \sigma_\alpha]$. So $x = t(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ for some $\overrightarrow{a} \in A^n$, $\overrightarrow{b} = \langle b_1, b_2, ..., b_m \rangle \in (\mu_\alpha)^m$, and $\overrightarrow{c} = \langle c_1, c_2, ..., c_k \rangle \in (\sigma_\alpha)^k$, where $t(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ is a commutator term in $\overrightarrow{y}, \overrightarrow{z}$. That is,

$$\mu^m(\overrightarrow{b}) = \mu(b_1) \wedge .... \wedge \mu(b_m) \geq \alpha \text{ and } \sigma^k(\overrightarrow{c}) = \sigma(c_1) \wedge .... \wedge \sigma(c_m) \geq \alpha$$

If we put $E = \{b_1, b_2, ..., b_m\}$ and $F = \{c_1, c_2, ..., c_k\}$, then E and F are both finite subsets of A such that $x \in [E, F]$. Moreover, if we take

$$\beta = \bigwedge_{a \in E, b \in F} (\mu(a) \wedge \sigma(b))$$

then $\beta \in H_x$ such that $\alpha \leq \beta$. This completes the proof. □

Definition 3.1.11. For each $\mu \in FI(A)$, we define by induction:

$$\mu^{(1)} = \mu = \mu^1;$$

$$\mu^{(n+1)} = [\mu^{(n)}, \mu^{(n)}] \text{ and } \mu^{n+1} = [\mu^n, \mu]$$

Definition 3.1.12. Let $\mu, \eta \in FI(A)$.

1. μ is said to be fuzzy solvable over η, if $\mu^{(n)} \leq \eta$ for some $n \in Z_+$. μ is fuzzy solvable if it is solvable over $\chi_{(0)}$.

2. μ is said to be fuzzy nilpotent over η, if $\mu^n \leq \eta$ for some $n \in Z_+$. μ is fuzzy nilpotent if it is solvable over $\chi_{(0)}$.

Lemma 3.1.13. *A fuzzy subset μ of A is fuzzy nilpotent (resp. fuzzy solvable) if and only if μ_α is nilpotent (resp. solvable) for all $\alpha \in L$.*

Definition 3.1.14. An algebra A is called solvable (resp. nilpotent) if A is solvable (resp. nilpotent) as an ideal; i.e., if

$$A^{(n)} = (0) \quad (resp.\ A^n = (0))$$

for some $n \in Z_+$.

3.2 Fuzzy Prime Ideals

In this section we define fuzzy prime ideals and investigate some of their properties. This generalizes fuzzy prime ideals of those well known structures: semigroups [63, 141], rings [119, 131, 143], semirings [63], ternary semirings [95], Γ-rings [71], modules [3], lattices [146] and others.

Definition 3.2.1. A non-constant fuzzy ideal μ of A is called fuzzy prime if and only if:

$$[\nu, \sigma] \leq \mu \Rightarrow \text{ either } \nu \leq \mu \text{ or } \sigma \leq \mu$$

for all $\nu, \sigma \in FI(A)$.

In the following theorem we characterize fuzzy prime ideals using fuzzy points.

Theorem 3.2.2. *A non-constant fuzzy ideal μ of A is fuzzy prime if and only if for any fuzzy points x_α and y_β of A:*

$$[x_\alpha, y_\beta] \leq \mu \Rightarrow \text{ either } x_\alpha \in \mu \text{ or } y_\beta \in \mu$$

Proof. Suppose that μ satisfies the condition:

$$[x_\alpha, y_\beta] \leq \mu \Rightarrow \text{ either } x_\alpha \in \mu \text{ or } y_\beta \in \mu$$

for all fuzzy points x_α and y_β of A. Let σ and θ be fuzzy ideals of A such that $[\sigma,\theta] \leq \mu$. Suppose if possible that $\sigma \nleq \mu$ and $\theta \nleq \mu$. Then there exist $x, y \in A$ such that $\sigma(x) \nleq \mu(x)$ and $\theta(y) \nleq \mu(y)$. If we put $\alpha = \sigma(x)$ and $\beta = \theta(y)$, then x_α and y_β are fuzzy points of A such that $x_\alpha \in \sigma$ but $x_\alpha \notin \mu$ and $y_\beta \in \theta$ but $y_\beta \notin \mu$. So that $[x_\alpha, y_\beta] \leq [\sigma,\theta] \leq \mu$, but $x_\alpha \notin \mu$ and $y_\beta \notin \mu$. This contradicts to our hypothesis. Thus either $\sigma \leq \mu$ or $\theta \leq \mu$. Therefore μ is prime. The other way is clear. □

In the following theorem, we give an internal characterization for fuzzy prime ideals of universal algebras analogous to the well-known characterization of Swamy et al. [143] in the case of rings.

Theorem 3.2.3. *A non-constant fuzzy ideal μ is a fuzzy prime ideal if and only if $Img(\mu) = \{1, \alpha\}$, where α is a prime element in L and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a prime ideal of A.*

Proof. Suppose that μ is a prime fuzzy ideal. Clearly $1 \in Img(\mu)$ and since μ is non-constant there is some $a \in A$ such that $\mu(a) < 1$. We show that $\mu(a) = \mu(b)$ for all $a, b \in A - \mu_*$. Let $a, b \in A$ such that $\mu(a) < 1$ and $\mu(b) < 1$. Let us define $L-$fuzzy subsets σ and θ of A as follows:

$$\sigma(x) = \begin{cases} 1 & if\, x \in \langle a \rangle \\ 0 & otherwise \end{cases}$$

and

$$\theta(x) = \begin{cases} 1 & if\, x = 0 \\ \mu(a) & otherwise \end{cases}$$

for all $x \in A$. Then it can be verified that both σ and θ are fuzzy ideals of A. Moreover, for each $x \in A$ we have:

$$[\sigma,\theta](x) = \begin{cases} 1 & if\, x = 0 \\ \mu(a) & if\, x \in [a,a] - \{0\} \\ 0 & otherwise \end{cases}$$

Then $[\sigma,\theta] \leq \mu$. But $\sigma(a) = 1 > \mu(a)$. So $\sigma \nleq \mu$. Since μ is fuzzy prime, we get $\theta \leq \mu$, which gives $\theta(b) \leq \mu(b)$; that is, $\mu(a) \leq \mu(b)$. Similarly it can be verified that $\mu(b) \leq \mu(a)$. So that $\mu(a) = \mu(b)$ for all $a,b \in A - \mu_*$. Thus $Img(\mu) = \{1,\alpha\}$ for some $\alpha \neq 1$ in L. Next we show that the level ideal μ_* is prime. Put $P = \mu_*$ and let I and J be ideas of A such that $[I,J] \subseteq P$. Then $\chi_{[I,J]} \leq \chi_P \leq \mu$. That is, $[\chi_I,\chi_J] \leq \mu$. Since μ is fuzzy prime, either $\chi_I \leq \mu$ or $\chi_J \leq \mu$ implying that either $I \subseteq P$ or $J \subseteq P$. Therefore P is prime. It remains to show that α is a prime element in L. Let $\beta,\gamma \in L$ such that $\beta \wedge \gamma \leq \alpha$. Consider $L-$fuzzy subsets $\overline{\beta}$ and $\overline{\gamma}$ of A defined by:

$$\overline{\beta}(x) = \begin{cases} 1 & if\, x = 0 \\ \beta & otherwise \end{cases}$$

and

$$\overline{\gamma}(x) = \begin{cases} 1 & if\, x = 0 \\ \gamma & otherwise \end{cases}$$

for all $x \in A$. Then $\overline{\beta}$ and $\overline{\gamma}$ are both fuzzy ideals of A such that $[\overline{\beta},\overline{\gamma}] \leq \mu$. Since μ is $L-$fuzzy prime, either $\overline{\beta} \leq \mu$ or $\overline{\gamma} \leq \mu$. So that either $\beta \leq \alpha$ or $\gamma \leq \alpha$. Hence α is prime in L. Conversely, suppose that $Img(\mu) = \{1,\alpha\}$, where α is a prime element in L and $P = \mu_*$ is a prime ideal of A. Let σ and θ be fuzzy ideals of A such that $[\sigma,\theta] \leq \mu$. Suppose if possible that, there exist $x,y \in A$ such that $\sigma(x) \nleq \mu(x)$ and $\theta(y) \nleq \mu(y)$. Since μ is $2-$valued, $\mu(x) = \mu(y) < 1$ so that both x and y do not belong to P. Since P is prime, there exists $z \in [x,y]$ such that $z \notin P$; that is, $\mu(z) = \alpha$. Otherwise if $[x,y] \subseteq P$, then either $x \in P$ or $y \in P$. As $z \in [x,y]$, $z = t(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})$ for some $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in \langle x\rangle^m$, $\overrightarrow{c} \in \langle y\rangle^k$, where $t(\overrightarrow{u},\overrightarrow{v},\overrightarrow{w})$ is a commutator term in $\overrightarrow{v},\overrightarrow{w}$. Now consider the following:

$$\begin{aligned} \alpha &= \mu(z) \\ &\geq [\sigma,\theta](z) \\ &\geq \sigma^m(\overrightarrow{b}) \wedge \theta^k(\overrightarrow{c}) \\ &\geq \sigma(x) \wedge \theta(y) \end{aligned}$$

That is, $\sigma(x) \wedge \theta(y) \leq \alpha$. Since α is a prime element in L it follows that either $\sigma(x) \leq \alpha = \mu(x)$ or $\theta(y) \leq \alpha = \mu(y)$, which is a contradiction. Therefore μ is fuzzy prime. □

Let P be a prime ideal of A and α be a prime element in L. Consider a fuzzy subset α_P of A defined by:

$$\alpha_P(x) = \begin{cases} 1 & \text{if } x \in P \\ \alpha & \text{otherwise} \end{cases}$$

for all $x \in A$. The above theorem confirms that fuzzy prime ideals of A are only of the form α_P. This establishes a one-to-one correspondence between the class of all fuzzy prime ideals of A and the collection of all pairs (P, α) where P is a prime ideal in A and α is a prime element in L.

Corollary 3.2.4. *Let P be an ideal of A and α a prime element in L. Then P is a prime ideal if and only if α_P is a fuzzy prime ideal.*

Definition 3.2.5. A fuzzy subset λ of A is called a fuzzy $m-$system (resp. a fuzzy $n-$system) if for all $a, b \in A$, there exists $x \in [a, b]$ (resp. there exists $x \in [a, a]$) such that

$$\lambda(x) \geq \lambda(a) \wedge \lambda(b) \quad (\text{resp. } \lambda(x) \geq \lambda(a))$$

Lemma 3.2.6. *A fuzzy subset λ of A is a fuzzy $m-$system (resp. a fuzzy $n-$system) if and only if the level set λ_α is an $m-$system (resp. an $n-$system) for all $\alpha \in L$.*

Definition 3.2.7. For a normalized fuzzy subset μ of A, define a fuzzy subset μ^c of A by:

$$\mu^c(x) = \begin{cases} \bigwedge_{y \in A} \mu(y) & \text{if } \mu(x) = 1 \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in A$. We call μ^c, the generalized complement of μ.

Theorem 3.2.8. *If μ is a fuzzy prime ideal of A, then μ^c is a fuzzy $m-$system.*

Proof. Suppose that μ is prime. Then $Img(\mu) = \{1, \alpha\}$ for some prime element $\alpha \in L$ and μ_* is a prime ideal. In this case μ^c is of the form

$$\mu^c(x) = \begin{cases} \alpha & if\, x \in \mu_* \\ 1 & otherwise \end{cases}$$

for all $x \in A$. Now let $a, b \in A$. If $\mu^c(a) \wedge \mu^c(b) = \alpha$, then the result holds trivially. Let $\mu^c(a) \wedge \mu^c(b) = 1$. Then $\mu^c(a) = 1 = \mu^c(b)$, i.e., $a \notin \mu_*$ and $b \notin \mu_*$. Being μ_* a prime ideal, we get $[a, b] \nsubseteq \mu_*$. So there exists $x \in [a, b]$ such that $x \notin \mu_*$, i.e.,

$$\mu^c(x) = 1 \geq \mu^c(a) \wedge \mu^c(b)$$

Therefore μ^c is a fuzzy $m-$system. □

Theorem 3.2.9. *Suppose that μ is a fuzzy ideal of A such that $Img(\mu) = \{1, \alpha\}$ where α is a prime element in L. If μ^c is a fuzzy $m-$system, then μ is fuzzy prime.*

Proof. By Theorem 3.2.3, It is enough to show that the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a prime ideal of A. It is clear that μ_* is a proper ideal of A. Let $a, b \in A$ such that $[a, b] \subseteq \mu_*$. Then $\mu(x) = 1$ for all $x \in [a, b]$, i.e., $\mu^c(x) = \alpha$ for all $x \in [a, b]$. Since μ^c is a fuzzy $m-$system, there exists $x \in [a, b]$ such that

$$\alpha = \mu^c(x) \geq \mu^c(a) \wedge \mu^c(b)$$

So that $\mu^c(a) \wedge \mu^c(b) = \alpha$. Since α is prime either $\mu^c(a) = \alpha$ or $\mu^c(b) = \alpha$, i.e., either $\mu(a) = 1$ or $\mu(b) = 1$, which implies either $a \in \mu_*$ or $b \in \mu_*$. So that μ_* is prime and this completes the proof. □

It is natural to question our self that does every algebra in $\mathscr{K}$ has fuzzy prime ideals. Of course, probably no. In the following theorem we give a sufficient condition for an algebra A to have fuzzy prime ideals.

Theorem 3.2.10. *Let A be an algebra satisfying:*

$$x \in [x,x] \textit{ for all } x \in A$$

If $a \in A$, and μ is a fuzzy ideal of A such that $\mu(a) \leq \alpha$ where α is an irreducible element in L. Then there exists a fuzzy prime ideal θ of A such that:

$$\mu \leq \theta \text{ and } \theta(a) \leq \alpha$$

Proof. Put $\mathfrak{F} = \{\sigma \in FI(A) : \mu \leq \sigma \text{ and } \sigma(a) \leq \alpha\}$. Clearly $\mu \in \mathfrak{F}$ so that $\mathfrak{F}$ is nonempty and hence it forms a poset under the inclusion ordering of fuzzy sets. By applying Zorn's lemma we can choose a maximal element say θ in $\mathfrak{F}$. Now it is enough to show that θ is prime. Suppose not. Then there exist fuzzy ideals σ and ν of A such that $[\sigma,\nu] \leq \theta$ but $\sigma \nleq \theta$ and $\nu \nleq \theta$. Put $\theta_1 = \theta \vee \sigma$ and $\theta_2 = \theta \vee \nu$. Then θ_1 and θ_2 are fuzzy ideals of A such that $\theta \lneqq \theta_1$ and $\theta \lneqq \theta_2$. By the maximality of θ in $\mathfrak{F}$ both θ_1 and θ_2 do not belong to $\mathfrak{F}$. Thus

$$\theta_1(a) \nleq \alpha \text{ and } \theta_2(a) \nleq \alpha$$

Since α is $\wedge-$irreducible element in L, $\theta_1(a) \wedge \theta_2(a) \nleq \alpha$. Again since $a \in [a,a]$ it holds that $[\theta_1,\theta_2](a) \geq \theta_1(a) \wedge \theta_2(a)$ and hence $[\theta_1,\theta_2](a) \nleq \alpha$, implying that $\theta(a) \nleq \alpha$. This is a contradiction. Therefore θ is prime. □

If A is a non-trivial algebra such that $a \in [a,a]$ for all $a \in A$, then it can be deduced from the above theorem that fuzzy prime ideals exist in A, provided that L has irreducible elements.

Theorem 3.2.11. *Let μ be a fuzzy ideal of A and λ a fuzzy $m-$system such that $\mu \cap \lambda \leq \alpha$, where α is an irreducible element in L. Then there exists a fuzzy prime ideal θ of A such that*

$$\mu \leq \theta \text{ and } \theta \cap \lambda \leq \alpha$$

Proof. Put $\mathfrak{F} = \{\sigma \in FI(A) : \mu \leq \sigma \text{ and } \sigma \cap \lambda \leq \alpha\}$. Clearly $\mu \in \mathfrak{F}$ so that $\mathfrak{F}$ is nonempty and hence it forms a poset under the inclusion ordering of fuzzy sets. By applying Zorn's lemma we can choose a maximal element say θ in $\mathfrak{F}$. Now it is enough to show that θ is prime. Suppose not. Then there exist fuzzy ideals σ and ν of A such that $[\sigma,\nu] \leq \theta$ but $\sigma \nleq \theta$ and $\nu \nleq \theta$. Put $\theta_1 = \theta \vee \sigma$ and $\theta_2 = \theta \vee \nu$. Then θ_1 and θ_2 are fuzzy ideals of A such that $\theta \lneqq \theta_1$ and $\theta \lneqq \theta_2$. By the maximality of θ in $\mathfrak{F}$ both θ_1 and θ_2 does not belongs to $\mathfrak{F}$ so there exists $a,b \in A$ such that:

$$(\theta_1 \cap \lambda)(a) \nleq \alpha \textit{ and } (\theta_2 \cap \lambda)(b) \nleq \alpha$$

This gives $\theta_1(a) \nleq \alpha, \lambda(a) \nleq \alpha, \theta_2(b) \nleq \alpha$ and $\lambda(b) \nleq \alpha$, which implies that $\theta_1(a) \wedge \theta_2(b) \nleq \alpha$ and $\lambda(a) \wedge \lambda(b) \nleq \alpha$. Since $[\sigma,\nu] \leq \theta$, we have $[\theta_1,\theta_2] \leq \theta$. If $x \in [a,b]$, then $x = t(\overrightarrow{c},\overrightarrow{u},\overrightarrow{v})$ for some $\overrightarrow{c} \in A^n$, $\overrightarrow{u} \in \langle a \rangle^m$, $\overrightarrow{v} \in \langle b \rangle^k$ and some commutator term $t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})$ in $\overrightarrow{y},\overrightarrow{z}$. Then for each $x \in [a,b]$ it holds that:

$$\theta(x) \geq [\theta_1,\theta_2](x) \geq \theta_1(a) \wedge \theta_2(b)$$

Also we have $\theta_1(a) \wedge \theta_2(b) \nleq \alpha$, which gives that $\theta(x) \nleq \alpha$. But since $\theta \cap \lambda \leq \alpha$ and α is an irreducible element in L we get that $\lambda(x) \leq \alpha$ for all $x \in [a,b]$. This contradicts to that; λ is a fuzzy $m-$system. Therefore θ is prime. □

For a non trivial algebra A, to have a fuzzy $m-$system is a sufficient condition for A to possess fuzzy prime ideals, provided that L has irreducible elements.

3.3 Generalized Fuzzy Prime Ideals

Definition 3.3.1. A non-constant fuzzy ideal μ of A is called generalized fuzzy prime ideal if and only if each level ideal μ_α is either A or prime.

Theorem 3.3.2. *A non-constant fuzzy ideal μ of A is generalized fuzzy prime ideal if and only if*

$$\textit{either } \mu(a) \geq \bigwedge_{x\in[a,b]} \mu(x) \text{ or } \mu(b) \geq \bigwedge_{x\in[a,b]} \mu(x)$$

for all $a,b \in A$.

Proof. Suppose that μ is generalized fuzzy prime and let $a,b \in A$ such that

$$\mu(a) \ngeq \bigwedge_{x\in[a,b]} \mu(x)$$

Let us put

$$\bigwedge_{x\in[a,b]} \mu(x) = \alpha$$

Then $\mu(x) \geq \alpha$ for all $x \in [a,b]$ and $\mu(a) \ngeq \alpha$, i.e., $[a,b] \subseteq \mu_\alpha$ and $a \notin \mu_\alpha$. So that μ_α is a proper ideal of A such that $[a,b] \subseteq \mu_\alpha$. By our hypothesis μ_α is prime and hence $b \in \mu_\alpha$, i.e., $\mu(b) \geq \alpha$. Hence proved.

To prove the converse part, let $\alpha \in L$ such that μ_α is a proper ideal. We need to show that μ_α is prime. Let $a,b \in A$ such that $[a,b] \subseteq \mu_\alpha$. Then $\mu(x) \geq \alpha$ for all $x \in [a,b]$, i.e.,

$$\bigwedge_{x\in[a,b]} \mu(x) \geq \alpha$$

By our assumption, either $\mu(a) \geq \alpha$ or $\mu(b) \geq \alpha$, which implies that either $a \in \mu_\alpha$ or $b \in \mu_\alpha$. Mean that μ_α is prime and this completes the proof. □

It can be deduced from the previous theorem that, for a non constant fuzzy ideal μ of A, to be generalized fuzzy prime is equivalent to satisfy the condition:

$$\mu(a) \vee \mu(b) \geq \bigwedge_{x\in[a,b]} \mu(x)$$

for all $a,b \in A$, provided that $Img(\mu)$ is a chain.

Theorem 3.3.3. *If μ is generalized fuzzy prime, then $Img(\mu)$ is a chain in L.*

Proof. Let $a,b \in A$ such that $\mu(a) \neq \mu(b)$. We show that either $\mu(a) \leq \mu(b)$ or $\mu(b) \leq \mu(a)$. Put $\alpha = \mu(a) \vee \mu(b)$. Since $\mu(a) \neq \mu(b)$ the level ideal μ_α does not contain one of $\mu(a), \mu(b)$ so that it is proper. Since μ is generalized fuzzy prime μ_α is a prime ideal in A. Now consider the following:

$$\begin{aligned} x \in [a,b] &\Rightarrow \mu(x) \geq \mu(a) \text{ and } \mu(x) \geq \mu(b) \\ &\Rightarrow \mu(x) \geq \mu(a) \vee \mu(b) = \alpha \\ &\Rightarrow x \in \mu_\alpha \end{aligned}$$

Thus $[a,b] \subseteq \mu_\alpha$. Being μ_α prime, either $a \in \mu_\alpha$ or $b \in \mu_\alpha$, i.e., either $\mu(a) \geq \alpha \geq \mu(b)$ or $\mu(b) \geq \alpha \geq \mu(a)$. Hence proved. □

Theorem 3.3.4. *Every fuzzy prime ideal of A is generalized fuzzy prime.*

Proof. Suppose that μ is fuzzy prime. By Theorem 3.2.3 μ has only two level ideals namely A and $\mu_* = \{x \in A : \mu(x) = 1\}$, which is prime. Then μ is generalized fuzzy prime. □

Theorem 3.3.5. *Suppose that μ is a fuzzy ideal of A such that $Img(\mu) = \{1, \alpha\}$, where α is a prime element in L. If μ is generalized fuzzy prime, then it is fuzzy prime.*

Proof. The proof follows from Theorem 3.2.3. □

3.4 Homomorphisms and Fuzzy Prime Ideals

Theorem 3.4.1. *Let $h : A \to B$ be a surjective homomorphism.*

1. *If μ and σ are fuzzy ideals of A, then*

$$h([\mu, \sigma]) = [h(\mu), h(\sigma)]$$

2. *If σ and θ are fuzzy ideals of B, then*

$$[h^{-1}(\sigma), h^{-1}(\theta)] \leq h^{-1}([\sigma, \theta])$$

Proof. (1) Let $y \in B$. Since h is assumed to be surjective, the set $h^{-1}(y)$ is always nonempty. By definition we have:

$$\begin{aligned} h([\mu,\eta])(y) &= \bigvee\{[\mu,\eta](x) : x \in h^{-1}(y)\} \\ &= \bigvee\{\bigvee\{\mu^m(\overrightarrow{b}) \wedge \eta^k(\overrightarrow{c}) : x = t^A(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})\} : x \in h^{-1}(y)\} \\ &= \bigvee\{\mu^m(\overrightarrow{b}) \wedge \eta^k(\overrightarrow{c}) : y = h(t^A(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}))\} \end{aligned}$$

which gives

$$h([\mu,\eta])(y) \geq \mu^m(\overrightarrow{b}) \wedge \eta^k(\overrightarrow{c}) \tag{3.4.1}$$

for any $b_1,...,b_m,c_1,...,c_k \in A$, with $y = h(t^A(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}))$ for some commutator term $t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})$ in $\overrightarrow{y},\overrightarrow{z}$ and some $a_1,...,a_n \in A$. Now let $y = t^B(\overrightarrow{u},\overrightarrow{v},\overrightarrow{w})$ be any expression of y using commutator terms, where $u_1,...,u_n,v_1,...,v_m,w_1,...,w_k \in B$. Since h is surjective there exist $a_1,...,a_n,b_1,...,b_m,c_1,...,c_k \in A$ such that $h(a_i) = u_i$, $h(b_j) = v_j$ and $h(c_r) = w_r$ for all $i = 1,2,...,n$, $j = 1,2,...,m$ and $r = 1,2,...,k$. Equivalently, each $a_i \in h^{-1}(u_i)$, $b_j \in h^{-1}(v_j)$ and $c_r \in h^{-1}(w_r)$. Now consider:

$$\begin{aligned} h(t^A(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})) &= t^B(h(a_1),...,h(a_a),h(b_1),...,h(b_m),h(c_1),...,h(c_k)) \\ &= t^B(u_1,...,u_n,v_1,...,v_m,w_1,...,w_k) \\ &= t^B(\overrightarrow{u},\overrightarrow{v},\overrightarrow{w}) \\ &= y \end{aligned}$$

So, by eq. (3.4.1) we get

$$h([\mu,\eta])(y) \geq \mu(b_1) \wedge ... \wedge \mu(b_m) \wedge \eta(c_1) \wedge ... \wedge \eta(c_k)$$

Since each b_j (respectively c_r) is arbitrary in $h^{-1}(v_j)$ (respectively in $h^{-1}(w_r)$), it follows that

$$\begin{aligned} h([\mu,\eta])(y) &\geq \left[\bigvee_{b_1\in h^{-1}(v_1)} \mu(b_1)\right] \wedge \ldots \wedge \left[\bigvee_{b_m\in h^{-1}(v_m)} \mu(b_m)\right] \wedge \left[\bigvee_{c_1\in h^{-1}(w_1)} \eta(c_1)\right] \wedge \ldots \\ &\qquad \ldots \wedge \left[\bigvee_{c_k\in h^{-1}(w_k)} \eta(c_k)\right] \\ &= h(\mu)(v_1)\wedge \ldots \wedge h(\mu)(v_m)\wedge h(\eta)(w_1)\wedge \ldots \wedge h(\eta)(w_k) \\ &= h(\mu)^m(\overrightarrow{v})\wedge h(\eta)^k(\overrightarrow{w}) \end{aligned}$$

This gives

$$\begin{aligned} h([\mu,\eta])(y) &\geq \bigvee\{h(\mu)^m(\overrightarrow{v})\wedge h(\eta)^k(\overrightarrow{w}) : y = t^B(\overrightarrow{u},\overrightarrow{v},\overrightarrow{w})\} \\ &= [h(\mu),h(\eta)](y) \end{aligned}$$

Therefore $[h(\mu),h(\eta)] \leq h([\mu,\eta])$. To prove the other inequality, consider

$$[h(\mu),h(\eta)](y) = \bigvee\{h(\mu)^m(\overrightarrow{v})\wedge h(\eta)^k(\overrightarrow{w}) : y = t^B(\overrightarrow{u},\overrightarrow{v},\overrightarrow{w})\}$$

So that

$$[h(\mu),h(\eta)](y) \geq h(\mu)^m(\overrightarrow{v})\wedge h(\eta)^k(\overrightarrow{w}) \tag{3.4.2}$$

for all $v_1,...,v_m,w_1,...,w_k \in B$, with $y = t^B(\overrightarrow{u},\overrightarrow{v},\overrightarrow{w})\}$, for some commutator term $t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})$ in $\overrightarrow{y},\overrightarrow{z}$ and some $u_1,...,u_n \in B$. Now let $y = h(t^A(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}))$ for some $a_1,...,a_n,b_1,...,b_m,c_1,...,c_k \in A$ and commutator term $t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})$ in $\overrightarrow{y},\overrightarrow{z}$. That is,

$$y = t^B(h(a_1),...,h(a_n),h(b_1),...,h(b_m)),h(c_1),...,h(c_k))$$

By eq. (3.4.2) we get

$$[h(\mu),h(\eta)](y) \geq h(\mu)(h(b_1)) \wedge \ldots \wedge h(\mu)(h(b_m)) \wedge h(\eta)(h(c_1)) \wedge \ldots \wedge h(\eta)(h(c_k))$$

using the fact that $h(\mu)(h(a)) \geq \mu(a)$ for all $a \in A$, we get the following:

$$\begin{aligned}
[h(\mu),h(\eta)](y) &\geq h(\mu)(h(b_1)) \wedge \ldots \wedge h(\mu)(h(b_m)) \wedge h(\eta)(h(c_1)) \wedge \ldots \wedge h(\eta)(h(c_k)) \\
&\geq \mu(b_1) \wedge \ldots \wedge \mu(b_m) \wedge \eta(c_1) \wedge \ldots \wedge \eta(c_k) \\
&= \mu^m(\overrightarrow{b}) \wedge \eta^k(\overrightarrow{c})
\end{aligned}$$

Since these b_i's and c_j's are arbitrary, it follows that

$$\begin{aligned}
[h(\mu),h(\eta)](y) &\geq \bigvee\{\mu^m(\overrightarrow{b}) \wedge \eta^k(\overrightarrow{c}) : y = h(t^A(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}))\} \\
&= h([\mu,\eta])(y)
\end{aligned}$$

which gives $h([\mu,\eta]) \leq [h(\mu),h(\eta)]$ and therefore the equality holds.

(2) Let $x \in A$ be any element. Then

$$\begin{aligned}
[h^{-1}(\sigma),h^{-1}(\theta)](x) &= \bigvee\{\mu^m(\overrightarrow{b}) \wedge \sigma^k(\overrightarrow{c}) : x = t(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}), \\
&\qquad \textit{where } t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z}) \textit{ is a commutator term in } \overrightarrow{y},\overrightarrow{z}\}
\end{aligned}$$

Now let $x = t^A(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})$; for some commutator term $t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})$ in $\overrightarrow{y},\overrightarrow{z}$ and $a_1,\ldots,a_n,b_1,\ldots,b_m,c_1,\ldots,c_k \in A$. Then

$$\begin{aligned}
h(x) &= h(t^A(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})) \\
&= t^B(h(a_1),\ldots,h(a_n),h(b_1),\ldots,h(b_m),h(c_1),\ldots,h(c_k))
\end{aligned}$$

Consider the following:

$$
\begin{aligned}
h^{-1}([\sigma,\theta])(x) &= [\sigma,\theta](h(x)) \\
&= \bigvee\{\sigma^m(\overrightarrow{v}) \wedge \theta^k(\overrightarrow{w}) : h(x) = t^B(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})\} \\
&\geq \sigma(h(b_1)) \wedge \ldots \wedge \sigma(h(b_m)) \wedge \theta(h(c_1)) \wedge \ldots \wedge \theta(h(c_k))) \\
&= h^{-1}(\sigma)(b_1) \wedge \ldots \wedge h^{-1}(\sigma)(b_m) \wedge h^{-1}(\theta)(c_1) \wedge \ldots \wedge h^{-1}(\theta)(c_k) \\
&= (h^{-1}(\sigma))^m(b) \wedge (h^{-1}(\theta))^k(\overrightarrow{c})
\end{aligned}
$$

Since each $a_1,\ldots,a_n,b_1,\ldots,b_m,c_1,\ldots,c_k$ are arbitrary, we get

$$
\begin{aligned}
h^{-1}([\sigma,\theta])(x) &\geq \bigvee\{(h^{-1}(\sigma))^m(\overrightarrow{b}) \wedge (h^{-1}(\theta))^k(\overrightarrow{c}) : x = t^A(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})\} \\
&= [h^{-1}(\sigma), h^{-1}(\theta)](x)
\end{aligned}
$$

Therefore $[h^{-1}(\sigma),h^{-1}(\theta)] \leq h^{-1}([\sigma,\theta])$. □

Theorem 3.4.2. *If $h : A \to B$ is an onto homomorphism and μ is an $h-$invariant fuzzy prime ideal of A, then $h(\mu)$ is a fuzzy prime ideal of B.*

Proof. Suppose that μ is an $h-$invariant fuzzy prime ideal of A. It follows from Theorem 2.5.1 that $h(\mu)$ is a fuzzy ideal of B. Let σ and θ be fuzzy ideals of B such that $[\sigma,\theta] \leq h(\mu)$. Then, $h^{-1}([\sigma,\theta]) \leq h^{-1}(h(\mu))$. Since μ is given to be an h-invariant, we have $h^{-1}(h(\mu)) = \mu$. So that, $h^{-1}([\sigma,\theta]) \leq \mu$. Also, by (2) of Theorem 3.4.1, we have $[h^{-1}(\sigma),h^{-1}(\theta)] \leq h^{-1}([\sigma,\theta])$, which gives $[h^{-1}(\sigma),h^{-1}(\theta)] \leq \mu$. Since μ is fuzzy prime, either $h^{-1}(\sigma) \leq \mu$ or $h^{-1}(\theta) \leq \mu$, which implies either $h(h^{-1}(\sigma)) \leq h(\mu)$ or $h(h^{-1}(\theta)) \leq h(\mu)$; that is, either $\sigma \leq h(\mu)$ or $\theta \leq h(\mu)$. This means $h(\mu)$ is fuzzy prime. □

Theorem 3.4.3. *If h is a homomorphism from A onto B and σ is a fuzzy prime ideal of B, then $h^{-1}(\sigma)$ is a fuzzy prime ideal of A.*

Proof. Suppose that θ is a fuzzy prime ideal of B. By Theorem 2.5.1 $h^{-1}(\theta)$ is a fuzzy ideal of A. Let μ and η be fuzzy ideals of A such that $[\mu,\eta] \leq h^{-1}(\theta)$. Then, $h([\mu,\eta]) \leq h(h^{-1}(\theta))$.

Since h is surjective, $h(h^{-1}(\theta)) = \theta$ and by (1) of Theorem 3.4.1, we have $h([\mu,\eta]) = [h(\mu),h(\eta)]$. So that $[h(\mu),h(\eta)] \leq \theta$. Since θ is fuzzy prime, either $h(\mu) \leq \theta$ or $h(\eta) \leq \theta$. This provides that either $\mu \leq h^{-1}(\theta)$ or $\eta \leq h^{-1}(\theta)$. Therefore $h^{-1}(\theta)$ is fuzzy prime. □

Theorem 3.4.4. *If $h : A \to B$ is an onto homomorphism, then the mapping $\mu \mapsto h(\mu)$ defines a one-to-one correspondence between the set of all h-invariant fuzzy prime ideals of A and the set of all fuzzy prime ideals of B.*

Proof. The above two theorems confirm that $\mu \mapsto h(\mu)$ is an onto map from the set of all h-invariant fuzzy prime ideals of A to the set of all fuzzy prime ideals of B. It remains to show that it one-one. Let μ_1 and μ_2 be an h-invariant fuzzy prime ideals of A such that $h(\mu_1) = h(\mu_2)$. Let $x \in A$. Then $h(x) \in B$ and $h(\mu_1)(h(x)) = h(\mu_2)(h(x))$. Since μ_1 is h-invariant we have $\mu_1(x) = \mu_1(a)$ for all $a \in h^{-1}(x)$. So,

$$\begin{aligned}
\mu_1(x) &= \bigvee\{\mu_1(a) : a \in h^{-1}(x)\} \\
&= h(\mu_1)(h(x)) \\
&= h(\mu_2)(h(x)) \\
&= \bigvee\{\mu_2(b) : b \in h^{-1}(x)\} \\
&= \mu_2(x)
\end{aligned}$$

Thus $\mu_1 = \mu_2$ and hence the map $\mu \mapsto h(\mu)$ is a one-to-one correspondence. □

3.5 Maximal Fuzzy Ideals

A maximal fuzzy ideal of A is a maximal element in the collection of all non-constant fuzzy ideals of A under the pointwise partial ordering of fuzzy sets.

An element $1 \neq \alpha$ in L is called a dual atom if there is no β in L such that $\alpha < \beta < 1$. In other words α is maximal in $L - \{1\}$. In the following theorem, we give an internal characterization of maximal fuzzy ideals in A.

Theorem 3.5.1. *A fuzzy ideal μ of A is maximal if and only if $Img(\mu) = \{1, \alpha\}$, where α is a dual atom in L and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a maximal ideal of A.*

Proof. Suppose that μ is maximal. Clearly $1 \in Img(\mu)$ and since μ is non-constant there is some $a \in A$ such that $\mu(a) < 1$. We first show that μ assumes exactly one value other than 1. Let $x, y \in A$ such that $\mu(x) < 1$ and $\mu(y) < 1$. Put $\alpha = \mu(x)$ and $\beta = \mu(y)$. Define fuzzy subsets $\mu_\alpha^\vee$ and $\mu_\beta^\vee$ of A as follows:

$$\mu_\alpha^\vee(z) = \mu(z) \vee \alpha \textit{ and } \mu_\beta^\vee(z) = \mu(z) \vee \beta$$

for all $z \in A$. Then it can be verified that both $\mu_\alpha^\vee$ and $\mu_\beta^\vee$ are fuzzy ideals of A such that $\mu \leq \mu_\alpha^\vee$ and $\mu \leq \mu_\beta^\vee$. By the maximality of μ we get that $\mu = \mu_\alpha^\vee$ and $\mu = \mu_\beta^\vee$. Thus $\alpha = \beta$. Therefore $Img(\mu) = \{1, \alpha\}$ for some $\alpha \in L - \{1\}$. Next we prove that α is a dual atom. Let $\beta \in L$ such that $\alpha < \beta$. Define a fuzzy subset σ of A by:

$$\sigma(z) = \begin{cases} 1 & \textit{if } \mu(z) = 1 \\ \beta & \textit{otherwise} \end{cases}$$

for all $z \in A$. Then σ is a fuzzy ideal of A such that $\mu < \sigma$. By the maximality of μ it yields that $\sigma = 1_A$; i.e., $\sigma(z) = 1$ for all $z \in A$. So $\beta = 1$. Therefore α is a dual atom. It remain to show that μ_* is a maximal ideal of A. Clearly it is a proper ideal. Let J be a proper ideal of A such that $\mu_* \subseteq J$. Define an $L-$fuzzy subset σ of A by:

$$\sigma(z) = \begin{cases} 1 & \textit{if } z \in J \\ \alpha & \textit{otherwise} \end{cases}$$

for all $z \in A$. Then σ is a non-constant fuzzy ideal of A such that $\mu \leq \sigma$. Since μ is maximal we get $\mu = \sigma$. So $\mu_* = J$. Therefore μ_* is maximal among all proper ideals of A. Conversely suppose that $Img(\mu) = \{1, \alpha\}$, where α is a dual atom in L and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a maximal ideal of A. Let σ be a non-constant fuzzy ideal of A such that $\mu \leq \sigma$. Then $\sigma(x) = 1$

for all $x \in \mu_*$ and $\alpha \leq \sigma(x)$ for all $x \in A - \mu_*$. We show that $\sigma = \mu$. Suppose not. Then there exists $x \in A$ such that $\sigma(x) > \mu(x)$. So $x \in A - \mu_*$. If $\sigma(x) = 1$, then $x \in \sigma_* = \{z \in A : \sigma(z) = 1\}$ and $\mu_* \subsetneq \sigma_* \subsetneq A$. This contradicts to the maximality of μ_*. Also if $\sigma(x) < 1$, then $\alpha \leq \sigma(x) < 1$. Again this contradicts to the hypothesis 'α is a dual atom'. Therefore $\sigma = \mu$. Hence μ is maximal. □

The above theorem shows that there is a one to one correspondence between the class of all maximal fuzzy ideals and the set of all pairs (M, α) where M is a maximal ideal in A and α is a dual atom in L.

Theorem 3.5.2. *If A is an algebra in which every maximal ideal is a prime ideal and L has a dual atom, then every maximal fuzzy ideal is a fuzzy prime ideal.*

For instance, if $[A, A] = A$ or if A has a formal unit, then every maximal fuzzy ideal of A is a fuzzy prime ideal, provided that L has a formal unit.

Theorem 3.5.3. *If $h : A \to B$ is an onto homomorphism and μ is a maximal fuzzy ideal of A, then $h(\mu)$ is a maximal fuzzy ideal of B.*

Proof. Suppose that μ is a maximal fuzzy ideal of A. Clearly, $h(\mu)$ is a fuzzy ideal of B. Let σ be a proper fuzzy ideal of B such that $h(\mu) \leq \sigma$. Then, $h^{-1}(h(\mu)) \leq h^{-1}(\sigma)$. Since μ is h-invariant, we have $\mu = h^{-1}(h(\mu))$. So that $\mu \leq h^{-1}(\sigma)$. By Theorem 2.5.1, $h^{-1}(\sigma)$ is a fuzzy ideal of A. Moreover, since σ is proper, there exists $y \in B$ such that $\sigma(y) < 1$; that is, $\sigma(y) = h(h^{-1}(\sigma))(y) < 1$, which gives $h^{-1}(\sigma)(x) < 1$ for all $x \in h^{-1}(y)$. This means, $h^{-1}(\sigma)$ is a proper fuzzy ideal of A such that $\mu \leq h^{-1}(\sigma)$. Since μ is maximal, we get that $\mu = h^{-1}(\sigma)$, which implies $h(\mu) = h(h^{-1}(\sigma)) = \sigma$. Therefore $h(\mu)$ is a maximal fuzzy ideal in A. □

Theorem 3.5.4. *If h is a homomorphism from A onto B and σ is a maximal fuzzy ideal of B, then $h^{-1}(\sigma)$ is maximal in the class of $h-$invariant fuzzy ideals of A.*

Proof. Suppose that σ is a maximal fuzzy ideal of B. By Theorem 2.5.1, $h^{-1}(\sigma)$ is a fuzzy ideal of A. Let μ be a proper h-invariant fuzzy ideal of A such that $h^{-1}(\sigma) \leq \mu$. Then $h(\mu)$ is a proper fuzzy ideal of B such that $h(h^{-1}(\sigma)) \leq h(\mu)$, which gives $\sigma \leq h(\mu)$. Being σ a maximal fuzzy ideal, it follows that $\sigma = h(\mu)$. So that $h^{-1}(\sigma) = h(h^{-1}(\mu)) = \mu$. Thus $h^{-1}(\sigma)$ is maximal in the class of $h-$invariant fuzzy ideal of A. □

Theorem 3.5.5. *If $h : A \to B$ is an onto homomorphism, then the mapping $\mu \mapsto h(\mu)$ defines aone-to-one correspondence between the set of all h-invariant maximal fuzzy ideals of A and the set of all maximal fuzzy ideals of B.*

3.6 Generalized Maximal Fuzzy Ideals

In this section, A is assumed to have a formal unit say u.

Theorem 3.6.1. *Let M be a maximal ideal of A and μ be a non-constant fuzzy ideal. Then $\chi_M \leq \mu$ if and only if $\mu = \alpha_M$ for some $\alpha \in L - \{1\}$.*

Proof. Suppose that $\chi_M \leq \mu$. Then $(\chi_M)_\alpha \subseteq \mu_\alpha$ for all $\alpha \in L$. In particular, $M \subseteq \mu_*$, where $\mu_* = \{x \in A : \mu(x) = 1\}$. Since μ is non-constant, μ_* is a proper ideal. M being maximal, it holds that $M = \mu_*$. We show that μ attains exactly one value other than 1. Let $a, b \in A$ such that $\mu(a) < 1$ and $\mu(b) < 1$. If we put $\alpha = \mu(a)$ and $\beta = \mu(b)$, then M is properly contained in the level ideal μ_α. Since M is maximal, it follows that $\mu_\alpha = A$. So that $b \in \mu_\alpha$, i.e., $\mu(b) \geq \alpha = \mu(a)$. Similarly, by interchanging a and b we can show that $\mu(a) \geq \mu(b)$ and hence the $\mu(a) = \mu(b)$. This confirms that $\mu = \alpha_M$ for some $\alpha \in L - \{0\}$. The converse part is straight forward. □

Definition 3.6.2. A non-constant fuzzy ideal μ of A is called generalized maximal fuzzy ideal if and only if each level ideal μ_α is either A or a maximal ideal in A.

Theorem 3.6.3. *Let M be a proper ideal of A and $\alpha \in L - \{1\}$. Then M is maximal if and only if α_M is generalized maximal fuzzy ideal.*

Corollary 3.6.4. *A proper ideal M of A is maximal if and only if χ_M is generalized maximal fuzzy ideal.*

Theorem 3.6.5. *A non-constant fuzzy ideal μ of A is generalized maximal fuzzy ideal if and only if $\mu = \alpha_M$ for some maximal ideal M of A and $\alpha \in L - \{1\}$.*

Proof. If $\mu = \alpha_M$ for some $\alpha \in L - \{0\}$ and some maximal ideal M, then it is clear from the definition that α_M is generalized fuzzy maximal. Conversely, assume that μ is generalized fuzzy maximal. Then every level ideal of A is either A or maximal. If we put $M = \{x \in A : \mu(x) = 1\}$, then M is a maximal ideal of A such that $\chi_M \leq \mu$. By Theorem 3.6.1, there exists some $\alpha \in L - \{1\}$ such that $\mu = \alpha_M$. Hence proved. □

Chapter 4

$L-$Fuzzy Semi-Prime Ideals

Introduction

Several attempts have been made to fuzzify the concept of semiprime ideals in rings (see [66, 98, 96, 104, 109, 160]). Dixit et al. [66] and Zahedi [160] have defined semiprime fuzzy ideal of a ring R as a fuzzy ideal μ, which satisfies the condition, that if $\sigma^n \leq \mu$ ($\sigma^2 \leq \mu$), then $\sigma \leq \mu$ for all fuzzy ideals σ of R. These definitions, however, make no reference to the grade of membership of an element of R. With this view, later Kumbhojkar et al. [104] defined fuzzy semiprime ideals of a ring R as a fuzzy ideal μ, satisfying the condition: $\mu(x^2) = \mu(x)$ for all $x \in R$. This helps to see the effect of semi-primeness on the elements of R.

In this chapter, we define fuzzy semi-prime ideals of universal algebras in the sense of [66] by applying the commutator of fuzzy ideals given in the previous chapter. We show that this definition is equivalent to that of [104]. In addition, we define the radical of fuzzy ideals in universal algebras and give several characterizing theorems describing the properties of fuzzy semi-prime ideals. In the last section, we study the space of fuzzy prime ideals equipped with the hull kernel topology.

4.1 Fuzzy Semi-Prime Ideals

Definition 4.1.1. A fuzzy ideal μ of A is called fuzzy semi-prime if:

$$[\theta,\theta] \leq \mu \Rightarrow \theta \leq \mu$$

for all $\theta \in FI(A)$.

It is clear that every fuzzy prime ideal is fuzzy semi-prime.

Theorem 4.1.2. *A fuzzy ideal μ of A is fuzzy semi-prime if and only if μ_α is semi-prime for all $\alpha \in L$.*

Proof. Suppose that μ is fuzzy semi-prime and let $\alpha \in L$. Let I be an ideal of A such that $[I,I] \subseteq \mu_\alpha$. We show that $I \subseteq \mu_\alpha$. Define a fuzzy subset σ of A as follows:

$$\sigma(x) = \begin{cases} 1 & if\, x = 0 \\ \alpha & if\, x \in I - \{0\} \\ 0 & otherwise \end{cases}$$

for all $x \in A$. Then it is easy to check that σ is a fuzzy ideal of A. Moreover, for each $x \in A$ we have:

$$[\sigma,\sigma](x) = \begin{cases} 1 & if\, x = 0 \\ \alpha & if\, x \in [I,I] - \{0\} \\ 0 & otherwise \end{cases}$$

It follows from our hypothesis; $[I,I] \subseteq \mu_\alpha$ that $[\sigma,\sigma] \leq \mu$. Since μ is fuzzy semi-prime, $\sigma \leq \mu$. Thus the level ideal σ_α which is precisely I will be included in μ_α and hence μ_α is semi-prime. Conversely, suppose that μ_α is semi-prime for all $\alpha \in L$. Let $\sigma \in FI(A)$ such that $[\sigma,\sigma] \leq \mu$. Then $[\sigma,\sigma]_\alpha \leq \mu_\alpha$ for all $\alpha \in L$. By Corollary 3.1.5, we have

$$[\sigma_\alpha,\sigma_\alpha] \subseteq [\sigma,\sigma]_\alpha \leq \mu_\alpha$$

Since each μ_α is semi-prime, it follows that $\sigma_\alpha \subseteq \mu_\alpha$ for all $\alpha \in L$. Thus $\sigma \leq \mu$ and hence μ is fuzzy semi-prime. □

Theorem 4.1.3. *An ideal I of A is semi-prime if and only if its characteristic function χ_I is fuzzy semi-prime.*

Proof. Suppose that I is semi-prime. Let μ be a fuzzy ideal of A such that $[\mu,\mu] \leq \chi_I$. We show that $\mu \leq \chi_I$. Suppose not. There exists $x \in A - I$ such that $\mu(x) > 0$. Since I is semi-prime, $[x,x] \nsubseteq I$. Choose an element a in $[x,x]$ and $a \notin I$. We can verify that $[\mu,\mu](a) \geq \mu(x) > 0$, which is a contradiction. Therefore χ_I is fuzzy semi-prime. The converse part is straight forward. □

Theorem 4.1.4. *A non-constant fuzzy ideal μ of A is fuzzy semi-prime if and only if for any fuzzy point x_α of A:*

$$[x_\alpha, x_\alpha] \leq \mu \Rightarrow x_\alpha \in \mu$$

Proof. Suppose that μ satisfies the condition:

$$[x_\alpha, x_\alpha] \leq \mu \Rightarrow x_\alpha \in \mu$$

for each fuzzy point x_α of A. We show that μ is fuzzy semi-prime. Let σ be a fuzzy ideal of A such that $[\sigma,\sigma] \leq \mu$. Suppose on contrary that $\sigma \nleq \mu$. Then there exists $x \in A$ such that $\sigma(x) \nleq \mu(x)$. If we put $\alpha = \sigma(x)$, then x_α is a fuzzy point of A such that $x_\alpha \in \sigma$ but $x_\alpha \notin \mu$. So $[x_\alpha, x_\alpha] \leq [\sigma,\sigma] \leq \mu$, but $x_\alpha \notin \mu$. This contradicts to our hypothesis. Thus $\sigma \leq \mu$ and therefore μ is fuzzy semi-prime. The converse part is clear. □

Theorem 4.1.5. *A fuzzy ideal μ of A is fuzzy semi-prime if and only if:*

$$\mu(a) \geq \bigwedge\{\mu(x) : x \in [a,a]\} \tag{4.1.1}$$

for all $a \in A$.

Proof. Suppose that μ is fuzzy semi-prime. We use contradiction. Assume that there exists $a \in A$ such that

$$\mu(a) \ngeq \bigwedge\{\mu(x) : x \in [a,a]\}$$

Put $\alpha = \wedge\{\mu(x) : x \in [a,a]\}$ and define a fuzzy subset θ of A by:

$$\theta(x) = \begin{cases} 1 & if\, x = 0 \\ \alpha & if\, x \in \langle a \rangle - \{0\} \\ 0 & otherwise \end{cases}$$

for all $x \in A$. Then θ is a fuzzy ideal of A such that for each $x \in A$ we have:

$$[\theta,\theta](x) = \begin{cases} 1 & if\, x = 0 \\ \alpha & if\, x \in [a,a] - \{0\} \\ 0 & otherwise \end{cases}$$

So that $[\theta,\theta] \leq \mu$. Since μ is fuzzy semi-prime it yields that $\theta \leq \mu$. This is a contradiction, because $\theta(a) \nleq \mu(a)$. Therefore the inequality 4.1.1 holds for all $a \in A$. Conversely, suppose that the inequality 4.1.1 holds for all $a \in A$. Let θ be any fuzzy ideal of A such that $[\theta,\theta] \leq \mu$. We show that $\theta \leq \mu$. Suppose not. Then there exists $a \in A$ such that $\theta(a) \nleq \mu(a)$. For each $x \in [a,a]$, we can verify that $[\theta,\theta](x) \geq \theta(a)$. Since $[\theta,\theta] \leq \mu$, it yields that $\mu(x) \geq \theta(a)$ for all $x \in [a,a]$. So that

$$\mu(a) \geq \bigwedge\{\mu(x) : x \in [a,a]\} \geq \theta(a)$$

This is a contradiction. Therefore μ is fuzzy semi-prime. □

Theorem 4.1.6. *If $h : A \to B$ is an onto homomorphism and μ is an $h-$invariant fuzzy semi-prime ideal of A, then $h(\mu)$ is a fuzzy semi-prime ideal of B.*

Proof. Suppose that μ is an $h-$invariant fuzzy semi-prime ideal of A. It follows from Theorem 2.5.1 that $h(\mu)$ is a fuzzy ideal of B. Let θ be fuzzy a ideal of B such that $[\theta,\theta] \leq h(\mu)$. Then,

$h^{-1}([\theta,\theta]) \leq h^{-1}(h(\mu))$. Since μ is given to be h-invariant, we have $h^{-1}(h(\mu)) = \mu$. So that, $h^{-1}([\theta,\theta]) \leq \mu$. Also, by (2) of Theorem 3.4.1, we have $[h^{-1}(\theta),h^{-1}(\theta)] \leq h^{-1}([\theta,\theta])$, which gives $[h^{-1}(\theta),h^{-1}(\theta)] \leq \mu$. Since μ is fuzzy semi-prime, we get $h^{-1}(\theta) \leq \mu$, which implies that $h(h^{-1}(\theta)) \leq h(\mu)$; that is, $\theta \leq h(\mu)$. This means $h(\mu)$ is fuzzy semi-prime. □

Theorem 4.1.7. *If h is a homomorphism from A to B and σ is a fuzzy semi-prime ideal of B, then $h^{-1}(\sigma)$ is a fuzzy semi-prime ideal of A.*

Proof. Suppose that θ is a fuzzy semi-prime ideal of B. By Theorem 2.5.1 $h^{-1}(\theta)$ is a fuzzy ideal of A. Let μ be a fuzzy ideal of A such that $[\mu,\mu] \leq h^{-1}(\theta)$. Then, $h([\mu,\mu]) \leq h(h^{-1}(\theta))$. Since h is surjective, $h(h^{-1}(\theta)) = \theta$ and by (1) of Theorem 3.4.1, we have $h([\mu,\mu]) = [h(\mu),h(\mu)]$. So that $[h(\mu),h(\mu)] \leq \theta$. Since θ is fuzzy semi-prime, we get $h(\mu) \leq \theta$. This provides that $\mu \leq h^{-1}(\theta)$. Therefore $h^{-1}(\theta)$ is fuzzy semi-prime. □

Theorem 4.1.8. *If $h : A \to B$ is an onto homomorphism, then the mapping $\mu \mapsto h(\mu)$ defines a one-to-one correspondence between the set of all h-invariant fuzzy semi-prime ideals of A and the set of all fuzzy semi-prime ideals of B.*

Proof. The proof is similar to that of Theorem 3.4.4. □

4.2 The Radical of Fuzzy Ideals

According to [149], the prime radical of an ideal I of A, denoted by $\sqrt{I}$ is the intersection of all prime ideals of A containing I. Here we define the prime radical of fuzzy ideals using their level ideals.

Definition 4.2.1. For a fuzzy ideal μ of A, its prime radical of μ denoted by $\sqrt{\mu}$ is defined as a fuzzy subset of A such that, for each $x \in A$:

$$\sqrt{\mu}(x) = \alpha \text{ if and only if } x \in \sqrt{\mu_\alpha} \text{ and } x \notin \sqrt{\mu_\beta} \text{ for all } \beta \nleq \alpha \text{ in } L.$$

If the algebra we are assuming is a ring, then this definition coincides with that of Kumar [97].

Lemma 4.2.2. *Let μ be a fuzzy ideal of A and $x \in A$. Then*

$$\sqrt{\mu}(x) = \bigvee\{\alpha \in L : x \in \sqrt{\mu_\alpha}\}$$

Lemma 4.2.3. *The following holds for all $\mu, \nu \in FI(A)$:*

1. $\sqrt{(\mu_\alpha)} = (\sqrt{\mu})_\alpha$ *for all* $\alpha \in L$
2. $\mu \leq \sqrt{\mu}$
3. $\mu \leq \nu \Rightarrow \sqrt{\mu} \leq \sqrt{\nu}$

Lemma 4.2.4. *For any $\mu \in FI(A)$, $\sqrt{\mu}$ is a fuzzy ideal of A.*

Proof. It is clear that $\sqrt{\mu}(0) = 1$. Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$ and $P(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Then consider:

$$\begin{aligned}
(\sqrt{\mu})^m(\overrightarrow{b}) &= \bigwedge\{\sqrt{\mu}(b_i) : 1 \leq i \leq m\} \\
&= \bigwedge\{\bigvee\{\alpha_i \in L : b_i \in \sqrt{\mu_{\alpha_i}}\} : 1 \leq i \leq m\} \\
&= \bigvee\{\bigwedge\{\alpha_i \in L : 1 \leq i \leq m\} : b_i \in \sqrt{\mu_{\alpha_i}}\}
\end{aligned}$$

If we put $\beta = \wedge\{\alpha_i \in L : 1 \leq i \leq m\}$, then we get $\mu_{\alpha_i} \subseteq \mu_\beta$ for all $1 \leq i \leq m$. This implies that $\sqrt{\mu_{\alpha_i}} \subseteq \sqrt{\mu_\beta}$ for all $1 \leq i \leq m$. Then we have the following:

$$\begin{aligned}
(\sqrt{\mu})^m(\overrightarrow{b}) &= \bigvee\{\bigwedge\{\alpha_i \in L : 1 \leq i \leq m\} : b_i \in \sqrt{\mu_{\alpha_i}}\} \\
&\leq \bigvee\{\beta \in L : b_i \in \sqrt{\mu_\beta},\ \textit{for all } 1 \leq i \leq m\} \\
&= \bigvee\{\beta \in L : \overrightarrow{b} \in (\sqrt{\mu_\beta})^m\} \\
&\leq \bigvee\{\beta \in L : P(\overrightarrow{a}, \overrightarrow{b}) \in \sqrt{\mu_\beta}\} \\
&= \sqrt{\mu}(P(\overrightarrow{a}, \overrightarrow{b}))
\end{aligned}$$

Therefore $\sqrt{\mu}$ is a fuzzy ideal of A. □

Lemma 4.2.5. *For any $\mu \in FI(A)$, $\sqrt{\mu}$ is fuzzy semi-prime.*

Proof. The proof follows from (1) of Lemma 4.2.3 and Theorem 4.1.2. □

Lemma 4.2.6. *For any $\theta \in FI(A)$, if θ is fuzzy semi-prime, then $\sqrt{\theta} = \theta$.*

Proof. Suppose that θ is fuzzy semi-prime. By Theorem 4.1.2, every level ideal θ_α is semi-prime. By the equivalence in (3.5) of [149], we get $\sqrt{\theta_\alpha} = \theta_\alpha$ for all $\alpha \in L$. This confirms that $\sqrt{\theta} = \theta$. □

Corollary 4.2.7. *For any $\mu \in FI(A)$, $\sqrt{\sqrt{\mu}} = \sqrt{\mu}$.*

Lemma 4.2.8. *For any $\mu \in FI(A)$, if θ is a fuzzy semi-prime ideal of A such that $\mu \leq \theta$, then $\sqrt{\mu} \leq \theta$.*

Proof. The proof is straight forward. □

Corollary 4.2.9. *For any $\mu \in FI(A)$,*

$$\sqrt{\mu} = \cap\{\theta : \theta \text{ is a fuzzy semi-prime ideal of } A, \mu \leq \theta\}$$

Lemma 4.2.10. *For any $\mu, \nu \in FI(A)$,*

$$\sqrt{[\mu,\nu]} = \sqrt{\mu \cap \nu} = \sqrt{\mu} \cap \sqrt{\nu}$$

Proof. For any $x \in A$, it is clear to see that:

$$\sqrt{[\mu,\nu]}(x) \leq \sqrt{\mu \cap \nu}(x) \leq \sqrt{\mu}(x) \wedge \sqrt{\nu}(x)$$

It is enough to show that $\sqrt{\mu}(x) \wedge \sqrt{\nu}(x) \leq \sqrt{[\mu,\nu]}(x)$. Let $\alpha \in L$ such that $\sqrt{\mu}(x) \wedge \sqrt{\nu}(x) = \alpha$. Then $x \in (\sqrt{\mu})_\alpha = \sqrt{\mu_\alpha}$ and $x \in (\sqrt{\nu})_\alpha = \sqrt{\nu_\alpha}$. So that $x \in P$ for all prime ideals P containing μ_α (respectively ν_α). Let Q be any prime ideal of A such that $[\mu,\nu]_\alpha \subseteq Q$. Since $[\mu,\nu]_\alpha = [\mu_\alpha, \nu_\alpha]$, we get that either $\mu_\alpha \subseteq Q$ or $\nu_\alpha \subseteq Q$. So that $x \in Q$. Thus $x \in \sqrt{[\mu,\nu]_\alpha}$ and hence $\sqrt{[\mu,\nu]}(x) \geq \alpha = \sqrt{\mu}(x) \wedge \sqrt{\nu}(x)$. □

Theorem 4.2.11. *If L is a chain and μ is a fuzzy ideal of A satisfying the sup property, then*

$$\sqrt{\mu} = \cap\{\theta : \theta \text{ is a fuzzy prime ideal of } A, \mu \leq \theta\}$$

Proof. Let $x \in A$ and $\alpha \in L$. Suppose that $\sqrt{\mu}(x) = \alpha$. Then $x \in \sqrt{\mu_\alpha}$ and $x \notin \sqrt{\mu_\beta}$ for all $\beta \nleq \alpha$. So that $x \in P$ for all prime ideals P of A with $\mu_\alpha \subseteq P$. Let θ be any fuzzy prime ideal of A such that $\mu \leq \theta$. By Theorem 3.2.3, $Img(\theta) = \{1, \beta\}$, where $\beta \in L - \{1\}$ and the set $\theta_* = \{x \in A : \theta(x) = 1\}$ is a prime ideal of A.

Case(1) If $\beta \geq \alpha$, then it is clear that $\theta(x) \geq \alpha$.

Case(2) If $\beta < \alpha$, then we can verify that $\theta_\alpha = \theta_*$ (which is a prime ideal of A) and $\mu_\alpha \subseteq \theta_\alpha = \theta_*$. That is, θ_* is a prime ideal of A containing μ_α. So that $x \in \theta_*$ and hence $\theta(x) \geq \alpha$.

Therefore

$$\bigwedge\{\theta(x) : \theta \text{ is a fuzzy prime ideal of } A, \mu \leq \theta\} \geq \alpha$$

To prove the other side of the inequality, Let

$$\alpha = \bigwedge\{\theta(x) : \theta \text{ is a fuzzy prime ideal of } A, \mu \leq \theta\}$$

Then $\theta(x) \geq \alpha$ for all fuzzy prime ideals θ of A with $\mu \leq \theta$. Let P be any prime ideal of A such that $\mu_\alpha \subseteq P$. We show that $x \in P$. If $x \in \mu_\alpha$, then it is clear. Assume that $x \notin \mu_\alpha$. Then $\mu(x) < \alpha$. Put $\beta = \vee\{\mu(y) : y \notin P\}$. Since μ has the sup-property, $\beta < \alpha$. Let us define a fuzzy subset θ_P of A as follows:

$$\theta_P(z) = \begin{cases} 1 & \text{if } z \in P \\ \beta & \text{otherwise} \end{cases}$$

for all $z \in A$. Then θ_P is a fuzzy prime ideal of A such that $\mu \leq \theta_P$. Thus $\theta_P(x) \geq \alpha > \beta$ and hence $\theta_P(x) = 1$. So that $x \in P$, which implies that $x \in \sqrt{\mu_\alpha}$. This confirms that the equality holds. □

Theorem 4.2.12. *If the commutator* [,] *of ideals in* A *is finitary and* μ *has the sup-property, then*

$$\sqrt{\mu}(a) = \bigvee\{\bigwedge_{x\in(a)^{(n)}} \mu(x) : n \in Z_+\}$$

for all $a \in A$.

Proof. Let $\alpha \in L$ such that $\bigvee\{\bigwedge_{x\in(a)^{(n)}} \mu(x) : n \in Z_+\} = \alpha$. Then there exists $n \in Z_+$ such that $(a)^{(n)} \subseteq \mu_\alpha$. If P is any prime ideal containing μ_α, then $(a)^{(n)} \subseteq P$. So that $a \in P$. Thus $a \in \sqrt{\mu_\alpha}$ and hence $\sqrt{\mu}(a) \geq \alpha$. To prove the other side of the inequality, let $\beta = \sqrt{\mu}(a)$. Then it follows from Corollary 4.2.9 that, $\theta(a) \geq \beta$ for all fuzzy semi-prime ideals θ of A with $\mu \leq \theta$. We need to show that

$$\bigvee\{\bigwedge_{x\in(a)^{(n)}} \mu(x) : n \in Z_+\} \geq \beta$$

Suppose not. Then

$$\beta \nleq \bigwedge_{x\in(a)^{(n)}} \mu(x) \text{ for all } n \in Z_+$$

That is; for each $n \in Z_+$, $(a)^{(n)} \nsubseteq \mu_\beta$. Then the set

$$\mathfrak{F} = \{I \in \mathscr{I}(A) : \mu_\beta \subseteq I, (a)^{(n)} \nsubseteq I \text{ for all } n \in Z_+\}$$

is nonempty. Moreover, $\mathfrak{F}$ together with the usual inclusion order forms a poset satisfying the hypothesis of Zorn's Lemma (here we use the condition; [,] is finitary). So that $\mathfrak{F}$ has a maximal element, say M. Our aim is to show that M is semi-prime. Take $b \notin M$. Then $M \vee \langle b\rangle \notin \mathfrak{F}$. By the property of $\mathfrak{F}$, $(a)^{(n)} \subseteq M \vee \langle b\rangle$ for some $n \in Z_+$. Then

$$\begin{aligned}
(a)^{(n+1)} &= [(a)^{(n)}, (a)^{(n)}] \\
&\subseteq [M \vee \langle b\rangle, M \vee \langle b\rangle] \\
&= [M, M] \vee [M, \langle b\rangle] \vee [b, b] \\
&\subseteq M \vee [b, b]
\end{aligned}$$

So that $M \vee [b,b] \notin \mathfrak{F}$ and hence $[b,b] \nsubseteq M$. Therefore M is a semi-prime ideal of A such that $\mu_\beta \subseteq M$ such that $a \notin M$. Put

$$\lambda = \vee\{\mu(y) : y \in A - M\}$$

Since μ has the sup-property, $\beta \nleq \lambda$. Now define a fuzzy subset θ_M of A as follows:

$$\theta_M(z) = \begin{cases} 1 & if\, z \in M \\ \lambda & otherwise \end{cases}$$

for all $z \in A$. Then θ_M is fuzzy semi-prime ideal of A such that $\mu \leq \theta_M$. But $\theta_M(a) = \lambda \ngeq \beta$, which is a contradiction. Therefore the equality holds. □

Theorem 4.2.13. *Let $\mu \in FI(A)$. If the commutator [,] of ideals in A is associative and finitary, then for each $x \in A$:*

$$\sqrt{\mu}(x) = \bigvee\{\alpha \in L : \exists n \in Z_+ \text{ such that } (x_\alpha)^{(n)} \leq \mu\}$$

where x_α is a fuzzy point of A.

Proof. For each $\alpha > 0$ and $n \in Z_+$, we first show that $(x)^{(n)} \subseteq \mu_\alpha$ if and only if $(x_\alpha)^{(n)} \leq \mu$. It is clear that $(x)^{(n)} \subseteq \mu_\alpha$ if and only if $\mu(z) \geq \alpha$ for all $z \in (x)^{(n)}$. On the other hand we can verify that:

$$(x_\alpha)^{(n)}(z) = \begin{cases} 1 & if\, z = 0 \\ \alpha & if\, z \in (x)^{(n)} - \{0\} \\ 0 & otherwise \end{cases}$$

for all $z \in A$. Therefore $(x)^{(n)} \subseteq \mu_\alpha$ if and only if $(x_\alpha)^{(n)} \leq \mu$. Now consider the following:

$$\begin{aligned} \sqrt{\mu}(x) &= \bigvee\{\alpha \in L : x \in \sqrt{\mu_\alpha}\} \\ &= \bigvee\{\alpha \in L : \exists n \in Z_+, (x)^{(n)} \subseteq \mu_\alpha\} \\ &= \bigvee\{\alpha \in L : \exists n \in Z_+, (x_\alpha)^{(n)} \leq \mu\} \end{aligned}$$

□

Theorem 4.2.14. *If the commutator* [,] *of ideals in A is finitary, then*

$$\sqrt{\mu} = \cup\{\eta \in L^A : \exists n \in Z_+ \text{ such that } \eta^{(n)} \leq \mu\}$$

Proof. For each $x \in A$, let us define two sets H_x and G_x as follows:

$$\begin{aligned} H_x &= \{\alpha \in L : x \in \sqrt{\mu_\alpha}\} \\ G_x &= \{\eta(x) : \eta \in L^A \text{ such that } \eta^{(n)} \leq \mu \text{ for some } n \in Z_+\} \end{aligned}$$

Clearly both H_x and G_x are subsets of L. Our aim is to show that $H_x = G_x$ for all $x \in A$. Let $\alpha \in H_x$ (without loss of generality we can assume that $\alpha > 0$). Then $x \in \sqrt{\mu_\alpha}$. Since the commutator [,] of ideals in A is finitary, there exists $n \in Z_+$ such that $(x)^{(n)} \subseteq \mu_\alpha$. Thus $(x_\alpha)^{(n)} \leq \mu$. If we take η to be the fuzzy point x_α, then $\eta \in L^A$, with $\eta(x) = \alpha$ such that $\eta^{(n)} \leq \mu$ for some $n \in Z_+$. Therefor $\alpha \in G_x$. So that $H_x \subseteq G_x$. Also let $\alpha \in G_x$. Then there exists $\eta \in L^A$ such that $\alpha = \eta(x)$ and $\eta^{(n)} \leq \mu$ for some $n \in Z_+$. Consider the fuzzy point x_α. Since $\eta(x) = \alpha$, $x_\alpha \in \eta$. So that $(x_\alpha)^{(n)} \leq \eta^{(n)} \leq \mu$. Then $(x_\alpha)^{(n)} \leq \mu$, which implies that $(x)^{(n)} \subseteq \mu_\alpha$. That is, $x \in \sqrt{\mu_\alpha}$. So that $\alpha \in H_x$. Therefore $H_x = G_x$. □

Theorem 4.2.15. *If the commutator* [,] *of ideals in A is finitary, then*

$$\sqrt{\mu} = \cup\{\eta \in FI(A) : \exists n \in Z_+ \text{ such that } \eta^{(n)} \leq \mu\}$$

Proof. For each $\eta \in L^A$ and $n \in Z_+$, it yields that $\eta^{(n)} = \langle\eta\rangle^{(n)}$. So that the proof of this theorem follows from Theorem 4.2.14. □

4.3 The Fuzzy Prime Spectrum

The space of prime ideals of universal algebras equipped with the hull-kernel topology was first studied by Agliano in [7]. He was considering algebras in ideal determined varieties. More generally, in the paper [6], Agliano defined and studied the space of prime congruences so called prime spectra in modular varieties.

The space of fuzzy prime ideals called fuzzy prime spectrum of rings were studied by Kumar [100] and Kumbhojkar [103]. The spectrum of prime $L-$submodules was studied by R. Ameri and R. Mahjoob in [31]. In this section, we study the space fuzzy prime ideals of universal algebras equipped with the hull-kernel topology. For the standard topological concepts we refer to [90].

We begin by giving the following notations.

1. $Y = \{P : P \text{ is a prime ideal of } A\}$

2. $X = \{\mu : \mu \text{ is a fuzzy prime ideal of } A\}$

3. For any subset $S \subseteq A$:

$$V(S) = \{P \in Y : S \subseteq P\} \text{ and } D(S) = \{P \in Y : S \nsubseteq P\}$$

4. For any fuzzy subset θ of A:

$$V(\theta) = \{\mu \in X : \theta \leq \mu\} \text{ and } D(\theta) = \{\mu \in X : \theta \nleq \mu\}$$

It is immediate from the definition that $D(1_A) = X = V(0_A)$ and $D(0_A) = \emptyset = V(1_A)$.

Lemma 4.3.1. *For any fuzzy subset θ of A,*

$$D(\theta) = D(\langle\theta\rangle) \text{ and } V(\theta) = V(\langle\theta\rangle)$$

Proof. Let $\mu \in D(\theta)$. Then μ is a fuzzy prime ideal of A such that $\theta \nleq \mu$. Since $\theta \leq \langle\theta\rangle$, it follows that $\langle\theta\rangle \nleq \mu$ and hence $\mu \in D(\langle\theta\rangle)$. Thus $D(\theta) \subseteq D(\langle\theta\rangle)$. Also, if $\mu \in D(\langle\theta\rangle)$, then $\langle\theta\rangle \nleq \mu$. If we are assuming that $\theta \leq \mu$, then $\langle\theta\rangle \leq \mu$ which is impossible. Thus $\mu \in D(\theta)$ and hence the equality holds. Similarly, we can verify that $V(\theta) = V(\langle\theta\rangle)$. □

Lemma 4.3.2. *For any two fuzzy subsets θ and σ of A:*

1. $\theta \leq \sigma \Rightarrow V(\sigma) \subseteq V(\theta)$

2. $\theta \leq \sigma \Rightarrow D(\theta) \subseteq D(\sigma)$

Lemma 4.3.3. *If L is a chain and θ and σ are fuzzy ideals of A having the sup property, then the following are equivalent::*

1. $D(\theta) = D(\sigma)$

2. $V(\theta) = V(\sigma)$

3. $V(\theta_\alpha) = V(\sigma_\alpha)$ *for all* $\alpha \in L$.

4. $\sqrt{\theta_\alpha} = \sqrt{\sigma_\alpha}$ *for all* $\alpha \in L$.

5. $\sqrt{\theta} = \sqrt{\sigma}$

Proof. $(1) \Rightarrow (2)$ is trivial. We proceed to show $(2) \Rightarrow (3)$. Suppose that $V(\theta) = V(\sigma)$. Let $\alpha \in L$. Let P be a prime ideal of A such that $P \in V(\theta_\alpha)$. Then $\theta_\alpha \subseteq P$. Put $\beta = \vee\{\theta(x) : x \notin P\}$. Since θ has sup property, we get $\beta < \alpha$. Let us define a fuzzy subset θ_P of A by:

$$\theta_P(x) = \begin{cases} 1 & \textit{if } x \in P \\ \beta & \textit{otherwise} \end{cases}$$

for all $x \in A$. Since L is a chain, β is a prime element in L and hence θ_P is a fuzzy prime ideal of A. Moreover, $\theta \leq \theta_P$. So that $\theta_P \in V(\theta) = V(\sigma)$. Thus $\sigma \leq \theta_P$. Now consider the following:

$$\begin{aligned} x \in \sigma_\alpha &\Rightarrow \sigma(x) \geq \alpha \\ &\Rightarrow \theta_P(x) \geq \alpha > \beta \\ &\Rightarrow \theta_P(x) = 1 \\ &\Rightarrow x \in P \end{aligned}$$

Thus $\sigma_\alpha \subseteq P$ and hence $P \in V(\sigma_\alpha)$. So that $V(\theta_\alpha) \subseteq V(\sigma_\alpha)$. By symmetry, we can also verify that $V(\sigma_\alpha) \subseteq V(\theta_\alpha)$ and therefore the equality holds.

(3) ⇒ (4). Suppose that $V(\theta_\alpha) = V(\sigma_\alpha)$ for all $\alpha \in L$. For each $\alpha \in L$, consider the following:

$$\begin{aligned} \sqrt{(\theta_\alpha)} &= \cap\{P : P \textit{ is a prime ideal of } A \text{ such that } \theta_\alpha \subseteq P\} \\ &= \cap\{P : P \in V(\theta_\alpha)\} \\ &= \cap\{P : P \in V(\sigma_\alpha)\} \\ &= \cap\{P : P \textit{ is a prime ideal of such that } \sigma_\alpha \subseteq P\} \\ &= \sqrt{(\sigma_\alpha)} \end{aligned}$$

(4) ⇒ (5). Suppose that $\sqrt{(\theta_\alpha)} = \sqrt{(\sigma_\alpha)}$ for all $\alpha \in L$. For each $x \in A$, we have

$$\begin{aligned} \sqrt{\theta}(x) &= \bigvee\{\alpha \in L : x \in \sqrt{(\theta_\alpha)}\} \\ &= \bigvee\{\alpha \in L : x \in \sqrt{(\sigma_\alpha)}\} \\ &= \sqrt{\sigma}(x) \end{aligned}$$

Therefore $\sqrt{\theta} = \sqrt{\sigma}$. (5) ⇒ (1) is clear and the proof ends. □

Lemma 4.3.4. *For any $\mu, \nu \in FI(A)$:*

1. $D(\theta \vee \sigma) = D(\theta) \cup D(\sigma)$

2. $D([\theta, \sigma]) = D(\theta \wedge \sigma) = D(\theta) \cap V(\sigma)$

3. $V(\mu \vee \nu) = V(\mu) \cap V(\nu)$

4. $V(\mu \wedge \nu) = V(\mu) \cup V(\nu)$

Proof. The proof of (3) and (4) is dual to the proof of (1) and (2) respectively. So that we prove only (1) and (2).

1. Suppose that $\mu \in D(\theta \vee \sigma)$. Then μ is a fuzzy prime ideal of A such that

$$\theta \vee \sigma \nleq \mu$$

If $\theta \leq \mu$ and $\sigma \leq \mu$, then we get $\theta \vee \sigma \leq \mu$, which is impossible. So that either $\theta \nleq \mu$ or $\sigma \nleq \mu$; that is, either $\mu \in D(\theta)$ or $\mu \in D(\sigma)$ and hence $D(\theta \vee \sigma) \subseteq D(\theta) \cup D(\sigma)$. To prove the other inclusion, let μ be a fuzzy prime ideal of A such that either $\mu \in D(\theta)$ or $\mu \in D(\sigma)$. Then either $\theta \nleq \mu$ or $\sigma \nleq \mu$, which implies that $\theta \vee \sigma \nleq \mu$. So that $\mu \in D(\theta \vee \sigma)$. Thus $D(\theta) \cup D(\sigma) \subseteq D(\theta \vee \sigma)$ and hence the equality holds.

2. For any fuzzy prime ideal μ of A and each $\theta, \sigma \in FI(A)$ it holds that

$$[\theta, \sigma] \leq \mu \text{ if and only if } \theta \wedge \sigma \leq \mu$$

This provides that the equality $D([\theta, \sigma]) = D(\theta \wedge \sigma)$. So it is enough to show $D([\theta, \sigma]) = D(\theta) \cap D(\sigma)$. Let $\mu \in D([\theta, \sigma])$. Then μ is a fuzzy prime ideal of A such that $[\theta, \sigma] \nleq \mu$. Since $[\theta, \sigma] \leq \theta$ and $[\theta, \sigma] \leq \sigma$, we get $\theta \nleq \mu$ and $\sigma \nleq \mu$; that is, $\mu \in D(\theta) \cap D(\sigma)$, and hence $D([\theta, \sigma]) \subseteq D(\theta) \cap D(\sigma)$. To prove the other inclusion let μ be a fuzzy prime ideal of A such that $\mu \in D(\theta)$ and $\mu \in D(\sigma)$. Then $\theta \nleq \mu$ and $\sigma \nleq \mu$. Being μ a fuzzy prime ideal we get $[\theta, \sigma] \nleq \mu$. So $\mu \in D([\theta, \sigma])$ and hence $D(\theta) \cap D(\sigma) \subseteq D([\theta, \sigma])$. Therefore the the equality holds.

□

Lemma 4.3.5. *For any subset S of A and its characteristic mapping χ_S*

$$D(\chi_S) = \{\alpha_P : \text{ where } \alpha \text{ is a prime element in } L \text{ and } P \in D(S)\}$$

Proof. Let us put

$$\mathscr{S} = \{\alpha_P : \text{ where } \alpha \text{ is a prime element in } L \text{ and } P \in D(S)\}$$

Let α be a prime element in L and $P \in D(S)$. Then P is a prime ideal of A such that $S \nsubseteq P$. There exists $a \in S$ and $a \notin P$. So that

$$\alpha_P(a) = \alpha < 1 = \chi_S(a)$$

This implies that $\chi_S \nleq \alpha_P$. Since α_P is a fuzzy prime ideal, it belongs to $D(\chi_S)$. So $\mathscr{S} \subseteq D(\chi_S)$. To prove the other side inclusion, let $\mu \in D(\chi_S)$. Then μ is a fuzzy prime ideal of A such that $\chi_S \nleq \mu$. By Theorem 3.2.3, there exists a prime element $\alpha \in L$ and a prime ideal P of A such that $\mu = \alpha_P$. Now from $\chi_S \nleq \mu = \alpha_P$, there exists $a \in A$ such that $\chi_S(a) \nleq \alpha_P(a)$, which implies that $\alpha_P(a) < 1$ and hence $a \notin P$. If $a \notin S$, then

$$0 = \chi_S(a) \leq \alpha_P(a)$$

which is impossible and hence $a \in S$. Thus $S \nsubseteq P$ and so $P \in D(S)$. This gives $\mu = \alpha_P \in \mathscr{S}$. Therefore $D(\chi_S) \subseteq \mathscr{S}$ and hence the equality holds. □

Theorem 4.3.6. *The collection*

$$\mathscr{T} = \{D(\theta) : \theta \text{ is a fuzzy ideal of } A\}$$

is a topology on X.

Proof. As $D(1_0) = \emptyset$ and $D(1_A) = X$, then $\mathscr{T}$ contains both $\emptyset$ and X. Also for any fuzzy ideals θ_1 and θ_2 of A, it is shown in Lemma 4.3.4 that $D(\theta_1) \cap D(\theta_2) = D(\theta_1 \wedge \theta_2)$. This shows that $\mathscr{T}$ is closed under finite intersections. Further, let $\{\theta_i : i \in I\}$ be any family of fuzzy ideals of A. We verify that $\cup_{i\in I} D(\theta_i) = D(\vee_{i\in I}\theta_i)$. Suppose that $\mu \in D(\vee_{i\in I}\theta_i)$. Then $\vee_{i\in I}\theta_i \nleq \mu$ which implies that $\theta_i \nleq \mu$ for some $i \in I$. Otherwise if $\theta_i \leq \mu$ for each $i \in I$, then it would be true that $\vee_{i\in I}\theta_i \leq \mu$, which is impossible. Thus $\mu \in \cup_{i\in I} D(\theta_i)$ and hence $D(\vee_{i\in I}\theta_i) \subseteq \cup_{i\in I} D(\theta_i)$. To prove the other inclusion, let $\mu \in \cup_{i\in I} D(\theta_i)$. Then $\mu \in D(\theta_i)$ for some $i \in I$; that is, $\theta_i \nleq \mu$ for some $i \in I$. Since $\theta_i \leq \vee_{i\in I}\theta_i$, we get $\vee_{i\in I}\theta_i \nleq \mu$. So that

$\mu \in D(\vee_{i\in I}\theta_i)$. Whence $\cup_{i\in I}D(\theta_i) \subseteq D(\vee_{i\in I}\theta_i)$ and hence the equality holds. Therefore $\mathscr{T}$ is closed under arbitrary union and hence it is a topology on X. □

Definition 4.3.7. The topological space $(X,\mathscr{T})$ is called the fuzzy prime spectrum of A and it is denoted by $F-spec(A)$.

Lemma 4.3.8. *For any fuzzy points x_α, y_β of A.*

$$D(x_\alpha)\cap D(y_\beta) = D([x_\alpha, y_\beta])$$

Lemma 4.3.9. *Let L be a chain. For any $\alpha > 0$ in L, $D(x_\alpha) = \emptyset$ if and only if $x \in \sqrt{0}$.*

Proof. Let $0 < \alpha \in L$. Suppose that $D(x_\alpha) = \emptyset$ and let P be a prime ideal in A. Since L is a chain, 0 is a prime element in L. Consider the fuzzy subset 0_P of A defined by:

$$0_P(z) = \begin{cases} 1 & \text{if } z \in P \\ 0 & \text{otherwise} \end{cases}$$

for all $z \in A$. Then it is clear that 0_P is a fuzzy prime ideal of A. By our assumption, $0_P \notin D(x_\alpha)$ and heence $x_\alpha \leq 0_P$. So that

$$0 < \alpha = x_\alpha(x) \leq 0_P(x)$$

This gives $x \in P$. Since P is arbitrary, we can conclude that

$$x \in \cap\{P : P \text{ is a prime ideal of } A\} = \sqrt{0}$$

Conversely suppose that $x \in \sqrt{0}$. Then $x \in P$ for each prime ideal P of A. Let μ be a fuzzy prime ideal of A. Our aim is to show that $x_\alpha \in \mu$. Since the set

$$\mu_* = \{a \in A : \mu(a) = 1\}$$

is a prime ideal, we get $x \in \mu_*$ and hence $\mu(x) = 1 \geq \alpha$. So that $x_\alpha \in \mu$. Thus $\mu \notin D(x_\alpha)$, which gives $D(x_\alpha) = \emptyset$. □

Lemma 4.3.10. *Suppose that A has a unit element say u and L is a chain. For any $\alpha > 0$ in L, $D(x_\alpha) = X$ if and only if x is a unit.*

Proof. Suppose that $D(x_\alpha) = X$. Then $x_\alpha \notin \mu$ for all $\mu \in X$. So $\mu(x) < 1$ for all $\mu \in X$. This implies that, for each prime ideal P of A, $x \notin P$. If we are assuming that x is not a unit, then by applying Zorn's Lemma we can find a prime ideal P of A containing x which is a contradiction. Thus x is a unit element in A. Conversely, suppose that x is a unit. Then $x \notin P$ for each prime ideal P of A. Now let μ be any fuzzy prime ideal of A. Then the set

$$\mu_* = \{a \in A : \mu(a) = 1\}$$

is a prime ideal in A. So $x \notin \mu_*$, that is, $\mu(x) \neq 1$. If we take $\alpha = 1$, then the fuzzy point $x_\alpha \notin \mu$ and hence $\mu \in D(x_\alpha)$. Thus $D(x_\alpha) = X$. □

Theorem 4.3.11. *Let $A, B \in \mathscr{K}$ and let $h : A \to B$ be a surjective homomorphism.*

1. *If $\mu \in F-Spec(B)$, then $h^{-1}(\mu) \in F-Spec(A)$. Hence h induces a map h_* from $F-Spec(B)$ to $F-Spec(A)$, defined by $h_*(\mu) = h^{-1}(\mu)$.*
2. *The map h_* is a homeomorphism from $F-Spec(B)$ to the class of h−-invariant fuzzy prime ideals of A.*

Lemma 4.3.12. *The subfamily $\mathscr{B} = \{D(x_\alpha) : x \in A, \alpha \in L - \{0\}\}$ of $\mathscr{T}$ is a base for $\mathscr{T}$.*

Proof. Let θ be any fuzzy ideal of A and $\mu \in D(\theta)$. Then μ is a fuzzy prime ideal of A such that $\theta \nleq \mu$. There exists $x \in A$ such that $\theta(x) \nleq \mu(x)$. If we put $\beta = \theta(x)$, then $\beta > 0$, and the fuzzy point $x_\beta \in \theta$ and $x_\beta \notin \mu$. So that $\mu \in D(x_\beta) \subseteq D(\theta)$. Thus $\mathscr{B}$ forms a base for $\mathscr{T}$. □

Lemma 4.3.13. *If A has a unit element u, then for each prime element $\alpha \in L$, the set*

$$A_\alpha = \{\mu \in X : Im(\mu) = \{1, \alpha\}\}$$

is a compact subspace of X.

Proof. Remember that A_α can be made a subspace of X by the relativized topology $\mathscr{T}_\alpha$ where

$$\mathscr{T}_\alpha = \{D(\theta) \cap A_\alpha : \theta \in FI(A)\}$$

If we put $L^\alpha = \{\gamma \in L : \gamma \nleq \alpha\}$, then L^α is a nonempty subset of L. Moreover, it is clear to show that the family

$$\mathscr{B}_\alpha = \{D(x_\gamma) \cap A_\alpha : x \in A \text{ and } \gamma \in L^\alpha\}$$

constitutes a base for $\mathscr{T}_\alpha$. Suppose that the family

$$\mathfrak{C} = \{D((x_i)_t) \cap A_\alpha : i \in \Delta \text{ and } \gamma \in K \subseteq L^\alpha\}$$

is a basic open cover for A_α. If we take $r = \vee\{t : t \in K\}$, then $r \nleq \alpha$ and the family $\{D((x_i)_r) \cap A_\alpha : i \in \Delta\}$ covers A_α. Now consider the following:

$$\begin{aligned} A_\alpha &= \bigcup_{i\in\Delta}[D((x_i)_r) \cap A_\alpha] \\ &= A_\alpha \cap \bigcup_{i\in\Delta}[D((x_i)_r)] \\ &= A_\alpha \cap D\left(\cup_{i\in\Delta}(x_i)_r\right) \\ &= A_\alpha \cap (X - V[\cup_{i\in\Delta}(x_i)_r]) \end{aligned}$$

which implies that $A_\alpha \cap V[\cup_{i\in\Delta}(x_i)_r] = \emptyset$. For any prime ideal P of A, consider the fuzzy ideal α_P of A as given in Definition1.2.12. It is shown in Corollary 3.2.4 that α_P is fuzzy prime and hence $\alpha_P \in A_\alpha$. Since $A_\alpha \cap V[\cup_{i\in\Delta}(x_i)_r] = \emptyset$, it yields that $\alpha_P \notin V[\cup_{i\in\Delta}(x_i)_r]$. So that, $\cup_{i\in\Delta}(x_i)_r \nleq \alpha_P$. If $(x_i)_r \in \alpha_P$ for all $i \in \Delta$, then $\cup_{i\in\Delta}(x_i)_r \leq \alpha_P$ which is impossible. So there exists $j \in \Delta$ such that $(x_j)_r \notin \alpha_P$, implying that $r \nleq \alpha_P(x_j)$. Then $\alpha_P(x_j) \neq 1$ and hence $\alpha_P(x_j) = \alpha$; that is, $x_j \notin P$. Mean that, for each prime ideal P of A, there exists $j \in \Delta$ such that $x_j \notin P$. Equivalently saying that, every prime ideal P does not contains the ideal $\langle\{x_i : i \in \Delta\}\rangle$.

So $\langle\{x_i : i \in \Delta\}\rangle = A$, and hence $u \in \langle\{x_i : i \in \Delta\}\rangle$. Then

$$u = p(a_1,...,a_n,x_{i_1},...,x_{i_m})$$

for some $a_1,...,a_n \in A$, $i_1,i_2,...,i_m \in \Delta$. We show that

$$V[\bigcup_{j=1}^{m}(x_{i_j})_r] \cap A_\alpha = \emptyset$$

Suppose on contrary that there is some

$$\mu \in V[\bigcup_{j=1}^{m}(x_{i_j})_r] \cap A_\alpha$$

which implies that $\mu(x_{i_j}) \geq r$ for all $1 \leq j \leq m$. Since $r \nleq \alpha$, we get $\mu(x_{i_j}) = 1$ for all $1 \leq j \leq m$. Now consider:

$$\begin{aligned} \mu(u) &= \mu\left(p(a_1,...,a_n,x_{i_1},...,x_{i_m})\right) \\ &\geq \bigwedge\{\mu(x_{i_j}) : 1 \leq j \leq m\} \\ &= 1 \end{aligned}$$

Since $\mu(u) \leq \mu(x)$ for all $x \in A$, it follows that μ is constant which is a contradiction. Therefore

$$V\left(\bigcup_{j=1}^{m}(x_{i_j})_r\right) \cap A_\alpha = \emptyset$$

Hence the subfamily $\{D((x_{i_j})_r) : 1 \leq j \leq m\}$ finitely covers A_α and therefore A_α is compact. □

Theorem 4.3.14. *The space X is a T_0 space.*

Proof. Let μ and θ be fuzzy prime ideals of A such that $\mu \neq \theta$. Then either $\mu \nleq \theta$ or $\theta \nleq \mu$. Without loss of generality we can assume that $\mu \nleq \theta$. Then there exists $x \in A$ such that $\mu(x) \nleq \theta(x)$. Let us put $\alpha = \mu(x)$. Then x_α is a fuzzy point of A such that $x_\alpha \in \mu$ and $x_\alpha \notin \theta$; that is,

$\mu \notin D(x_\alpha)$ and $\theta \in D(x_\alpha)$. This means $D(x_\alpha)$ is an open set in X containing θ but not contain μ. Therefore X is a T_0 space. □

Theorem 4.3.15. *For any $\mu \neq \nu \in X$, $\nu \in \overline{\{\mu\}}$ if and only if $\mu \leq \nu$.*

Proof. Let $\mu \neq \nu \in X$. Suppose that $\nu \in \overline{\{\mu\}}$. Then $\mu \in U$ for each neighborhood U of ν in X. Since neighborhoods of ν in X are of the form $D(\theta)$ for some fuzzy ideal θ of A with $\theta \nleq \nu$, it is equivalent to say that $\mu \in D(\theta)$, and hence $\theta \nleq \mu$, for all fuzzy ideals θ of A with $\theta \nleq \nu$. In other words, for any fuzzy ideal θ of A the following holds:

$$\theta \leq \mu \Rightarrow \theta \leq \nu$$

which gives that $\mu \leq \nu$. Conversely, suppose that $\mu \leq \nu$ and let U be a neighborhood of ν in X. Then $U = D(\theta)$ for some fuzzy subset θ of A with $\theta \nleq \nu$. Since $\mu \leq \nu$, we get $\theta \nleq \mu$, which gives that $\mu \in D(\theta) = U$. So that, $\{\mu\} \cap U \neq \emptyset$. Therefore $\nu \in \overline{\{\mu\}}$. □

Corollary 4.3.16. *For each $\mu \in X$,*

$$V(\mu) = \overline{\{\mu\}}$$

Chapter 5

$L-$Fuzzy Congruence Relations

Introduction

Given an algebra A of a given type $\mathfrak{F}$, the set $Con(A)$ of all its congruence relations forms an algebraic lattice for which its compact elements are those finitely generated congruences. It is called the congruence lattice (or the structural lattice) of A. This gives a lattice theoretic interpretation for A. For some class of algebras, the congruence lattice may be modular or distributive. One can understand the algebraic nature of the algebra A by studying the properties of its congruence lattice.

To determine congruences on a given algebra we need to know what are the congruence classes. Hence also properties of these classes (alias blocks) can be used to indicate the structure of the corresponding quotient algebra. Moreover, one can find out that in some special, but often considered algebras it may happen that one (either arbitrary or fixed) congruence class determines the whole congruence. Among these algebras there are e. g. groups, rings, Boolean algebras, implication algebras, relatively complemented lattices etc. Having an algebra, one may be interested in the question which subsets or subalgebras can be classes of suitable congruences. The most general solution was proposed by A. I. Mal'cev in his pioneering paper [118]. Further investigations can be found in [35, 36, 37, 54, 50].

In this chapter, we study fuzzy congruence relations and their classes; so called fuzzy congruence classes in universal algebras. Fuzzy congruence relations generated by a fuzzy relation are fully characterized in different ways. The main result in this chapter is that, we give several

Mal'cev type characterization for a fuzzy subset of an algebra A in a given variety to be a class of some fuzzy congruence on A. Some equivalent conditions are also given for a variety of algebras to posses fuzzy congruence classes which are also fuzzy subuniverse. Special fuzzy congruence classes; called fuzzy congruence kernels are characterized in a more general context in universal algebras.

5.1 Fuzzy Congruences

In semigroups and vector-spaces, fuzzy congruence relations generated by a fuzzy relation were studied in [138] and [142] respectively. The main purpose of this section is to characterize fuzzy congruence relations generated by a fuzzy relation in general universal algebras. By a fuzzy relation on A, we mean a fuzzy subset of $A \times A$. The following definition is due to [144].

Definition 5.1.1. A fuzzy relation Θ on A is said to be:

1. reflexive if: $\Theta(x,x) = 1$ for all $x \in A$
2. symmetric if: $\Theta(x,y) = \Theta(y,x)$ for all $x,y \in A$
3. transitive if for each $x,z \in A$: $\Theta(x,z) \geq \Theta(x,y) \wedge \Theta(y,z)$ for all $y \in A$.

A reflexive, symmetric and transitive fuzzy relation on A is called a fuzzy equivalence relation on A.

Definition 5.1.2. A fuzzy relation Θ on A is said to be compatible, if

$$\Theta(f^A(x_1,x_2,\dots,x_n), f^A(y_1,y_2,\dots,y_n)) \geq \Theta(x_1,y_1) \wedge \dots \wedge \Theta(x_n,y_n)$$

for every $n \in Z^+$, $f \in \mathfrak{F}_n$ and all $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n \in A$.

Note that, compatible fuzzy relations on A are those fuzzy subuniverses of $A \times A$, where $A \times A$ is equipped with the product algebra. This fact is often expressed in the way that Θ is said to have the substitution property with respect to each fundamental operations $f \in \mathfrak{F}$. According to Werner [154], compatible fuzzy relations may also be refereed as admissible fuzzy relations.

Definition 5.1.3. A fuzzy congruence on A is a fuzzy equivalence relation on A which is compatible with all fundamental operations of A.

We denote by $FCon(A)$ the class of all fuzzy congruence relations on A.

Lemma 5.1.4. *If $t(x_1,...,x_m)$ is an $m-$ary term operation on A and $\Theta \in FCon(A)$, then it holds that*

$$\Theta(t(a_1,...,a_m),t(b_1,...,b_m)) \geq \Theta(a_1,b_1) \wedge ... \wedge \Theta(a_m,b_m)$$

for all $a_1,...,a_m,b_1,...,b_m \in A$.

Proof. For the term t, let $l(t)$ be the number of occurences of n-ary operation symbols in t for. We use induction on $l(t)$. If $l(t)=0$, then either $t=x_j$ for some j, whence

$$\Theta(t^A(a_1,...,a_n),t^A(b_1,...,b_n)) = \Theta(a_j,b_j) \geq \bigwedge_{i=1}^{n} \Theta(a_i,b_i)$$

or $t=f^A$ for some $f \in \mathscr{F}_0$, whence

$$\Theta(t^A(a_1,...,a_n),t^A(b_1,...,b_n)) = \Theta(f^A,f^A) = 1 \geq \bigwedge_{i=1}^{n} \Theta(a_i,b_i)$$

Now let $l(t)>0$ and assume the result to be true for every term q with $l(q)<l(t)$. Then we know t is of the form

$$f^A(t_1(x_1,...,x_n),...,t_k(x_1,..,x_n)),$$

as $l(t_i)<l(t)$ for each $j=1,2,...,k$, it follows from the induction hypothesis that for $1 \leq j \leq k$,

$$\Theta(t_j(a_1,...,a_n),t_j(b_1,...,b_n)) \geq \bigwedge_{i=1}^{n} \Theta(a_i,b_i)$$

By the compatibility of Θ, we get

$$\begin{aligned}\Theta(t(a_1,...,a_n),t(b_1,...,b_n)) &= \Theta(f^A(t_1(a_1,...,a_n),...,t_k(a_1,..,a_n)),f^A(t_1(b_1,...,b_n),...,t_k(b_1,..,b_n)))\\ &\geq \bigwedge_{j=1}^{k}\Theta(t^A(a_1,...,a_n),t^A(b_1,...,b_n))\\ &\geq \bigwedge_{i=1}^{n}\Theta(a_i,b_i)\end{aligned}$$

□

For a fuzzy relation Θ on A and each $\alpha \in L$, remember that

$$\Theta_\alpha = \{(x,y) \in A\times A : \Theta(x,y) \geq \alpha\}$$

is the $\alpha-$level relation of Θ on A. The following lemma is simple but a useful characterization of fuzzy congruences using their level relations.

Lemma 5.1.5. *Let Θ be a fuzzy relation on A. Then the following hold:*

1. *Θ is a fuzzy equivalence relation on A if and only if Θ_α is an equivalence relation on A for all $\alpha \in L$.*

2. *Θ is a fuzzy congruence relation on A if and only if Θ_α is a congruence relation on A for all $\alpha \in L$.*

From the above theorem, one can conclude that fuzzy congruence relations are fuzzy $\mathfrak{L}-$subsets of $A\times A$ (in the sense of [144]), where $\mathfrak{L}-$ is a the class of congruence relations on A.

Theorem 5.1.6. *The intersection of any family of fuzzy congruence relations on A is a fuzzy congruence on A.*

Given a fuzzy relation ρ on A, it follows from the above theorem that, always there exists a smallest fuzzy congruence relation on A containing ρ, which we call it the fuzzy congruence on A generated by ρ. It is denoted by $\Theta_L(\rho)$. Particularly, for a fuzzy subset λ of A, we write $\Theta_L(\lambda)$ to denote the fuzzy congruence $\Theta_L(\lambda\times\lambda)$.

Definition 5.1.7. [133, 134] For any two fuzzy relations Θ and Φ on A, their composition $\Theta \circ \Phi$ is a fuzzy relation on A given by:

$$\Theta \circ \Phi(x,y) = \bigvee \{\Theta(x,z) \wedge \Phi(y,z) : z \in A\}$$

for all $x, y \in A$. For a positive integer n, by Θ^n, we mean $\Theta \circ \Theta \circ \ldots \circ \Theta$ (n copies).

V. Murali [133, 134] has characterized the supremum of two fuzzy congruence relations as follows.

Theorem 5.1.8. *For any $\Theta, \Phi \in FCon(A)$, we have*

$$\Theta \vee \Phi = \bigcup_{n=1}^{\infty} (\Theta \circ \Phi \circ \Theta)^n$$

Theorem 5.1.9. *The class $FCon(A)$ of all fuzzy congruence relations on A forms an algebraic closure fuzzy set system under the inclusion ordering of fuzzy sets.*

We turn our attention to characterize the fuzzy congruence $\Theta_L(\rho)$ generated a fuzzy relation ρ. Note that we use the notation $\Theta(R)$ to denote the crisp congruence on A generated by the crisp relation R. The following theorem gives a natural characterization for $\Theta_L(\rho)$ using level relations.

Theorem 5.1.10. *Let ρ be a normalized fuzzy relation on A. Then $\Theta_L(\rho)$ can be characterized as:*

$$\Theta_L(\rho)(x,y) = \bigvee \{\alpha \in L : (x,y) \in \Theta(\rho_\alpha)\}$$

for all $x, y \in A$.

Proof. For $x, y \in A$ let us define a fuzzy relation Γ of A by:

$$\Gamma(x,y) = \bigvee \{\alpha \in L : (x,y) \in \Theta(\rho_\alpha)\}$$

Our aim is to show that Γ is the smallest fuzzy congruence on A containing ρ. We first show that Γ is a fuzzy congruence on A. Clearly it is reflexive and symmetric. To prove transitivity, let $x,y,z \in A$.

$$\begin{aligned}\Gamma(x,y)\wedge\Gamma(y,z) &= \bigvee\{\alpha\in L:(x,y)\in\Theta(\rho_\alpha)\}\wedge\bigvee\{\beta\in L:(y,z)\in\Theta(\rho_\beta)\}\\ &= \bigvee\{\alpha\wedge\beta:(x,y)\in\Theta(\rho_\alpha)\text{ and }(y,z)\in\Theta(\rho_\beta)\}\end{aligned}$$

Let $a,b \in L$ such that $(x,y) \in \Theta(\rho_\alpha)$ and $(y,z) \in \Theta(\rho_\beta)$. If we put $\lambda = \alpha \wedge \beta$, then $\lambda \in L$ such that $\rho_\alpha \subseteq \rho_\lambda$ and $\rho_\beta \subseteq \rho_\lambda$, which gives $\Theta(\rho_\alpha) \subseteq \Theta(\rho_\lambda)$ and $\Theta(\rho_\beta) \subseteq \Theta(\rho_\lambda)$, i.e., $\Theta(\rho_\lambda)$ contains both (x,y) and (y,z). By the transitive property of $\Theta(\rho_\lambda)$, $(x,z) \in \Theta(\rho_\lambda)$. Now we have the following:

$$\begin{aligned}\Gamma(x,y)\wedge\Gamma(y,z) &= \bigvee\{\alpha\wedge\beta:(x,y)\in\Theta(\rho_\alpha)\text{ and }(y,z)\in\Theta(\rho_\beta)\}\\ &\leq \bigvee\{\lambda\in L:(x,z)\in\Theta(\rho_\lambda)\}\\ &= \Gamma(x,z)\end{aligned}$$

Therefore Γ is transitive and hence a fuzzy equivalence relation on A. No it remains to show that Γ is compatible. Let $n > 0$, $f \in \mathfrak{F}_n$ and $a_1,...,a_n,b_1,...,b_n \in A$.

$$\begin{aligned}\bigwedge_{i=1}^{n}\Gamma(a_i,b_i) &= \bigwedge_{i=1}^{n}\bigvee\{\alpha_i\in L:(a_i,b_i)\in\Theta(\rho_{\alpha_i})\}\\ &= \bigvee\{\bigwedge_{i=1}^{n}\alpha_i:(a_i,b_i)\in\Theta(\rho_{\alpha_i})\}\end{aligned}$$

Let $\alpha_1,...,\alpha_n \in L$ be such that $(a_i,b_i) \in \Theta(\rho_{\alpha_i})$. If we put

$$\lambda = \bigwedge_{i=1}^{n}\alpha_i$$

then $\lambda \in L$ and $\Theta(\rho_{\alpha_i}) \subseteq \Theta(\rho_\lambda)$ for all $i = 1,2,...,n$, which gives $(a_i,b_i) \in \Theta(\rho_\lambda)$ for $i = 1,2,...,n$. Using the compatible property of $\Theta(\rho_\lambda)$ we get $(f^A(a_1,...,a_n), f^A(b_1,...,b_n)) \in$

$\Theta(\rho_\lambda)$. Now consider the following:

$$
\begin{aligned}
\bigwedge_{i=1}^{n} \Gamma(a_i,b_i) &= \bigvee\{\bigwedge_{i=1}^{n} \alpha_i : (a_i,b_i) \in \Theta(\rho_{\alpha_i})\} \\
&\leq \bigvee\{\lambda \in L : (a_i,b_i) \in \Theta(\rho_\lambda) \text{ for } i = 1,\ldots,n\} \\
&\leq \bigvee\{\lambda \in L : (f^A(a_1,\ldots,a_n), f^A(b_1,\ldots,b_n)) \in \Theta(\rho_\lambda)\} \\
&= \Gamma(f^A(a_1,\ldots,a_n), f^A(b_1,\ldots,b_n))
\end{aligned}
$$

Thus Γ is compatible and hence it as a fuzzy congruence on A. Remember that, any fuzzy subset μ of A can be expressed as follows: for each $x \in A$

$$\mu(x) = \bigvee\{\alpha \in L : x \in \mu_\alpha\}$$

Similarly, for each $x, y \in A$ we have:

$$
\begin{aligned}
\rho(x,y) &= \bigvee\{\alpha \in L : (x,y) \in \rho_\alpha\} \\
&\leq \bigvee\{\alpha \in L : (x,y) \in \Theta(\rho_\alpha)\} \\
&= \Gamma(x,y)
\end{aligned}
$$

Mean that $\rho \leq \Gamma$. Further, let Ψ be any fuzzy congruence on A such that $\rho \leq \Psi$. Then, for each $\alpha \in L$, the level relation Ψ_α is a congruence on A such that $\rho_\alpha \subseteq \Psi_\alpha$, which gives that $\Theta(\rho_\alpha) \subseteq \Theta(\Psi_\alpha) = \Psi_\alpha$. Now for each $x, y \in A$:

$$
\begin{aligned}
\Gamma(x,y) &= \bigvee\{\alpha \in L : (x,y) \in \Theta(\rho_\alpha)\} \\
&\leq \bigvee\{\alpha \in L : (x,y) \in \Psi_\alpha\} \\
&= \Psi(x,y)
\end{aligned}
$$

Thus Γ is the smallest fuzzy congruence on A containing ρ. This completes the proof. $\square$

Using the fact that $FCon(A)$ together with the point-wise ordering of fuzzy sets is an algebraic closure fuzzy set system we have another characterization for $\Theta_L(\rho)$ as follows.

Theorem 5.1.11. *Let ρ be a normalized fuzzy relation on A. Then*

$$\Theta_L(\rho)(x,y) = \bigvee\{\bigwedge_{(a,b)\in F} \rho(a,b) : F \subset\subset A\times A, (x,y)\in\Theta(F)\}$$

for all $x,y\in A$, where $F \subset\subset A\times A$ is to say that F is a finite subset of $A\times A$.

Proof. For each $x,y\in A$, let us define two sets $G_{x,y}$ and $H_{x,y}$ as follows:

$$\begin{aligned} G_{x,y} &= \{\bigwedge_{(a,b)\in F} \rho(a,b) : F \subset\subset A\times A, (x,y)\in\Theta(F)\} \\ H_{x,y} &= \{\alpha\in L : (x,y)\in\Theta(\rho_\alpha)\} \end{aligned}$$

Clearly, both $G_{x,y}$ and $H_{x,y}$ are nonempty subsets of L. By Theorem 5.1.10, it is enough to show that $\vee G_{x,y} = \vee H_{x,y}$. Let $\alpha\in G_{x,y}$. Then there exists a finite subset F of $A\times A$ such that

$$\alpha = \bigwedge_{(a,b)\in F} \rho(a,b) \text{ and } (x,y)\in\Theta(F)$$

i.e., $\rho(a,b)\geq\alpha$ for all $(a,b)\in F$, which gives $F\subseteq\rho_\alpha$. This implies $\Theta(F)\subseteq\Theta(\rho_\alpha)$. Whence $(x,y)\in\Theta(\rho_\alpha)$. So that $\alpha\in H_{x,y}$. Thus $G_{x,y}\subseteq H_{x,y}$ and hence $\bigvee G_{x,y}\leq\bigvee H_{x,y}$. To prove the other inequality, let $\alpha\in H_{x,y}$. Then $(x,y)\in\Theta(\rho_\alpha)$. By Corollary 1.0.3 of [50] we can find a finite subset F of ρ_α such that $(x,y)\in\Theta(F)$. F being contained in ρ_α, we get

$$\bigwedge_{(a,b)\in F} \rho(a,b)\geq\alpha$$

If we put $\beta = \bigwedge_{(a,b)\in F}\rho(a,b)$, then $\alpha\leq\beta$ and $\beta\in G_{x,y}$, i.e., for each $\alpha\in H_{x,y}$, we can find a $\beta\in G_{x,y}$ with $\alpha\leq\beta$. This confirms that $\vee H_{x,y}\leq\vee G_{x,y}$ and this completes the proof. □

Definition 5.1.12. [50] A mapping $p : A \to A$ is called a translation of A if there exist an $n \in Z^+$, an $n-$ary operation $f \in \mathfrak{F}$, an $i \in \{1,2,...,n\}$ and $a_1,...,a_{i-1},a_{i+1},...,a_n$ in A such that

$$f^A(a_1,,,.a_{i-1},x,a_{i+1},...,a_n) = p(x)$$

for all $x \in A$.

The following theorem gives a necessary and sufficient condition for fuzzy equivalence relations to be a fuzzy congruence by the use of translations.

Theorem 5.1.13. *A fuzzy equivalence relation Θ on A is a fuzzy congruence on A if and only if it is compatible with all translations of A, i.e.,*

$$\Theta(p(x),p(y)) \geq \Theta(x,y)$$

for all $x,y \in A$ and all translations p of A.

Proof. If Θ is a fuzzy congruence on A, then it is clear that it is compatible with all translations of A. Conversely, suppose that Θ is compatible with all translations of A. Let $n \in Z^+, f \in \mathfrak{F}_n$ and $a_1,..,a_n,b_1,...,b_n \in A$. For each $i \in \{1,...,n\}$ define $p_i : A \to A$ by

$$p_i(x) = f^A(b_1,...,b_{i-1},x,a_{i+1},...,a_n)$$

for all $x \in A$. It can be easily verified that each p_i is a translation of A. Moreover, we have the following:

$$p_1(a_1) = f^A(a_1,a_2,...,a_n)$$

$$p_n(b_n) = f^A(b_1,b_2,...,b_n) \text{ and}$$

$$p_i(b_i) = p_{i+1}(a_{i+1}) \text{ for } i = 1,...,n-1$$

Now using the compatibility and transitive property of Θ we get the following:

$$\begin{aligned}\bigwedge_{i=1}^{n}\Theta(a_i,b_i) &\leq \left(\bigwedge_{i=1}^{n-1}\Theta(p_i(a_i),p_{i+1}(a_{i+1}))\right)\wedge\Theta(p_n(a_n),p_n(b_n))\\ &\leq \Theta(p_1(a_1),p_n(b_n))\\ &= \Theta(f^A(a_1,\ldots,a_n),f^A(b_1,\ldots,b_n))\end{aligned}$$

That is, Θ is compatible with all fundamental operations of A and hence it is a fuzzy congruence on A. □

Unary polynomials over A can be identified as a translation of A. The following corollary immediately follows from this fact.

Corollary 5.1.14. *A fuzzy equivalence relation Θ on A is a fuzzy congruence on A if and only if it is compatible with all unary polynomials on A.*

Definition 5.1.15. [55] A transitive closure of a fuzzy relation ρ on A is the smallest transitive fuzzy relation on A containing ρ.

Note that, if ρ is a reflexive (symmetric or compatible) relation on A, then its transitive closure is so respectively.

Lemma 5.1.16. *[55] The transitive closure of a reflexive fuzzy relation ρ on A is given by the formula*

$$\bigcup_{n=1}^{\infty}\rho^n$$

where the power of ρ is formed with respect to the relational products.

In the following theorem, we give an algebraic characterization for $\Theta_L(\rho)$.

Theorem 5.1.17. *If ρ is a normalized fuzzy relation on A, then $\Theta_L(\rho)$ is the transitive closure of the fuzzy relation λ on A defined as follows: for each $x,y\in A$, $\lambda(x,x)=1$ and for $x\neq y$,*

$$\lambda(x,y)=\bigvee\{\rho(a,b):\{x,y\}=\{p(a),p(b)\},p\in P_1(A)\}$$

Proof. Clearly, λ is reflexive and symmetric fuzzy relation on A with $\rho \leq \lambda$. We first show that λ is compatible with all unary polynomials over A. Let $x, y \in A$ and $q \in P_1(A)$. If $x = y$, then $q(x) = q(y)$ and

$$\lambda(x,y) = 1 = \lambda(q(x), q(y))$$

Let $x \neq y$. Then

$$\lambda(x,y) = \bigvee \{\rho(a,b) : \{x,y\} = \{p(a), p(b)\}, p \in P_1(A)\}$$

and

$$\lambda(q(x), q(y)) = \bigvee \{\rho(c,d) : \{q(x), q(y)\} = \{p(c), p(d)\}, p \in P_1(A)\}$$

For any $p \in P_1(A)$ and every $a, b \in A$, if $x = p(a)$ and $y = p(b)$, then $q(x) = q(p(a))$ and $q(y) = q(p(b))$. Since the composition of unary polynomials is a unary polynomial, we can find a $t \in P_1(A)$ such that $q(x) = t(a)$ and $q(y) = t(b)$. This implies that

$$\lambda(x,y) \leq \lambda(q(x), q(y))$$

Mean that, λ is compatible with all unary polynomials over A. If we define Φ to be the transitive closure of λ, then it can be verified that Φ is the smallest fuzzy equivalence relation on A which is compatible with all unary polynomials over A with $\lambda \leq \Phi$. It follows from Corollary 5.1.14 that $\Phi = \Theta_L(\lambda)$. Our aim is to show that $\Phi = \Theta_L(\rho)$. Clearly $\rho \leq \Phi$. Now let Ψ be any fuzzy congruence on A with $\rho \leq \Psi$. It suffices to show that $\lambda \leq \Psi$. For $x \neq y$ consider the following:

$$\begin{aligned}
\lambda(x,y) &= \bigvee \{\rho(a,b) : \{x,y\} = \{p(a), p(b)\}, p \in P_1(A)\} \\
&\leq \bigvee \{\Psi(a,b) : \{x,y\} = \{p(a), p(b)\}, p \in P_1(A)\} \\
&\leq \Psi(x,y)
\end{aligned}$$

That is, $\lambda \leq \Psi$ and this completes the proof. $\square$

Corollary 5.1.18. *Let ρ be a normalized fuzzy relation on A. Then*

$$\begin{aligned}\Theta_L(\rho)(x,y) \;=\; & \bigvee\{\bigwedge_{i=1}^{n}\rho(b_i,c_i) : \exists a_0,a_1,...,a_n \in A, p_1,...,p_n \in P_1(A), n>0 \\ & \textit{such that } a_0 = x, a_n = y \textit{ and } \{a_{i-1},a_i\} = \{p_i(b_i),p_i(c_i)\}\}\end{aligned}$$

for all $x,y \in A$.

5.2 Fuzzy Congruence Classes

Remember that for a binary relation R on A and $S \subseteq A$, the set $R[S]$ is a subset of A given by:

$$R[S] = \{y \in A : (x,y) \in R \textit{ for some } x \in S\}$$

In particular, if $S = \{a\}$, then we write $R[a]$ instead of $R[S]$. If θ is a congruence on A (in the usual sense) and $a \in A$, then the set

$$\theta[a] = \{b \in A : (a,b) \in \theta\}$$

is called the congruence class of θ determined by a. Some authors denote this set as $[a]\theta$ (or a/θ). Analogous to this classical concept, we define the following.

Definition 5.2.1. For a fuzzy subset μ of A and a fuzzy relation Θ on A we define μ/Θ to be a fuzzy subset of A as follows:

$$\mu/\Theta(x) = \bigvee\{\mu(y) \wedge \Theta(x,y) : y \in A\}$$

for all $x \in A$. Also for each $a \in A$, we define a/Θ to be $\chi_{\{a\}}/\Theta$.

It is observed that

$$a/\Theta(x) = \Theta(a,x) \text{ for all } x \in A$$

If Θ is a fuzzy congruence on A and $a \in A$, then we call a/Θ the fuzzy congruence class of A determined by Θ and a. Sometimes we may write Θ_a to denote a/Θ.

It was proved by A. I. Mal'cev [118] that a nonempty subset H of A is a class of some $\theta \in Con(A)$ if and only if for each $a, b \in H$ and any unary polynomial p on A it holds that:

$$p(a) \in H \Rightarrow p(b) \in H$$

Analogous to this well known characterization, we give the following lemma in the fuzzy sense.

Lemma 5.2.2. *A normalized fuzzy subset μ of A is a class of some fuzzy congruence on A if and only if*

$$\mu(x) \wedge \mu(y) \wedge \mu(p(x)) = \mu(p(y)) \wedge \mu(x) \wedge \mu(y)$$

for each $x, y \in A$ and each $p \in P_1(A)$.

Proof. Suppose that $\mu = a/\Theta$ for some $\Theta \in FCon(A)$ and $a \in A$. Let $x, y \in A$ and $p \in P_1(A)$. It follows from Corollary 5.1.14 that

$$\Theta(p(x), p(y)) \geq \Theta(x, y)$$

Also, by the transitive property of Θ, we have

$$\Theta(x, y) \geq \Theta(x, a) \wedge \Theta(a, y) = \mu(x) \wedge \mu(y)$$

Consider the following:

$$\begin{aligned}\mu(p(y)) &= \Theta(a,p(y))\\ &\geq \Theta(a,p(x))\wedge\Theta(p(x),p(y))\\ &\geq \Theta(a,p(x))\wedge\Theta(x,y)\\ &\geq \Theta(a,p(x))\wedge\mu(x)\wedge\mu(y)\\ &= \mu(p(x))\wedge\mu(x)\wedge\mu(y)\end{aligned}$$

which gives

$$\mu(x)\wedge\mu(y)\wedge\mu(p(y))\geq\mu(p(x))\wedge\mu(x)\wedge\mu(y)$$

By symmetry, the equality holds. Conversely, suppose that the condition of the theorem holds. Let us define a fuzzy relation ρ on A as follows: for each $x,y\in A$, $\rho(x,x)=1$ and for $x\neq y$

$$\rho(x,y)=\bigvee\{\mu(b)\wedge\mu(c):x=p(b),y=p(c),p\in P_1(A)\}$$

It is clear that ρ is a reflexive and symmetric fuzzy relation on A such that $\mu\times\mu\leq\rho$. Moreover, one can easily verify that ρ is compatible with all unary polynomials over A. Since μ is normalized, we can choose $x\in A$ with $\mu(x)=1$. Our aim is to show that $x/\Theta_L(\mu)=\mu$. Let $y\in A$. If $y=x$, then it is clear that $x/\Theta_L(\mu)(y)\leq\mu(y)$. Let $y\neq x$. Then $x/\Theta_L(\mu)(y)=\Theta_L(\mu)(x,y)$. By Theorem 5.1.17 $\Theta_L(\mu)$ is the transitive closure of ρ. So it follows from Lemma 5.1.16 that

$$\Theta_L(\mu)(x,y)=\bigvee_{n=1}^{\infty}\rho^n(x,y)$$

We show that $\rho^n(x,y)\leq\mu(y)$ for all $n\in Z^+$. Let $n\in Z^+$. It is clear that

$$\rho^n(x,y)=\bigvee\{\bigwedge_{i=1}^{n}\rho(x_{i-1},x_i):x_0,x_1,\dots,x_n\in A,x_0=x\text{ and }x_n=y\}$$

Let $x_0, x_1, ..., x_n \in A$ such that $x_0 = x$ and $x_n = y$. We show that

$$\bigwedge_{i=1}^{n} \rho(x_{i-1}, x_i) \leq \mu(y)$$

We use induction on n. If $n = 1$, then $x_0 = x$ and $x_1 = y$. So that

$$\bigwedge_{i=1}^{n} \rho(x_{i-1}, x_i) = \rho(x, y) = \bigvee\{\mu(b) \wedge \mu(c) : x = p(b), y = p(c), p \in P_1(A)\}$$

For any $a, b \in A$, and $p \in P_1(A)$; if $p(a) = x$ and $p(b) = y$, then it follows from our assumption (the condition of the theorem) that

$$\mu(a) \wedge \mu(b) \wedge \mu(x) = \mu(y) \wedge \mu(a) \wedge \mu(b)$$

Since $\mu(x) = 1$, it holds that

$$\mu(a) \wedge \mu(b) = \mu(y) \wedge \mu(a) \wedge \mu(b)$$

which gives

$$\mu(a) \wedge \mu(b) \leq \mu(y)$$

Since a, b and p are arbitrary, it follows that

$$\rho(x, y) = \bigvee\{\mu(b) \wedge \mu(c) : x = p(b), y = p(c), p \in P_1(A)\} \leq \mu(y)$$

Let $n > 0$ and assume the result to be true for $n - 1$; i.e.,

$$\bigwedge_{i=1}^{n-1} \rho(x_{i-1}, x_i) \leq \mu(x_{n-1})$$

Now consider the following:

$$\begin{aligned}
\bigwedge_{i=1}^{n} \rho(x_{i-1},x_i) &= \left(\bigwedge_{i=1}^{n-1} \rho(x_{i-1},x_i)\right) \wedge \rho(x_{n-1},x_n) \\
&\leq \mu(x_{n-1}) \wedge \rho(x_{n-1},x_n) \\
&= \mu(x_{n-1}) \wedge \rho(x_{n-1},y) \\
&= \mu(x_{n-1}) \wedge \bigvee\{\mu(b) \wedge \mu(c) : x_{n-1} = p(b), y = p(c), p \in P_1(A)\} \\
&= \bigvee\{\mu(x_{n-1}) \wedge \mu(b) \wedge \mu(c) : x_{n-1} = p(b), y = p(c), p \in P_1(A)\}
\end{aligned}$$

If $b, c \in A$ such that $x_{n-1} = p(b), y = p(c)$ for some $p \in P_1(A)$, then by condition of the theorem we get

$$\mu(x_{n-1}) \wedge \mu(b) \wedge \mu(c) = \mu(y) \wedge \mu(b) \wedge \mu(c) \leq \mu(y)$$

which implies that

$$\bigvee\{\mu(x_{n-1}) \wedge \mu(b) \wedge \mu(c) : x_{n-1} = p(b), y = p(c), p \in P_1(A)\} \leq \mu(y)$$

which gives

$$\bigwedge_{i=1}^{n} \rho(x_{i-1},x_i) \leq \mu(y)$$

Since $x_0, x_1, ..., x_n$ are arbitrary in A with $x_0 = x$ and $x_n = y$, it follows that

$$\rho^n(x,y) \leq \mu(y)$$

This is true for all $n \in Z^+$, which implies that

$$\bigvee_{n=1}^{\infty} \rho^n(x,y) \leq \mu(y)$$

i.e.,

$$\Theta_L(\mu)(x,y) \leq \mu(y)$$

Therefore $x/\Theta_L(\mu) \leq \mu$. The inequality $\mu \leq x/\Theta_L(\mu)$ is straightforward and hence the equality holds. This completes the proof. □

Notation. For each $a \in A$, let us define two sets $L_a(A)$ and $FL_a(A)$ respectively as follows:

$$\begin{aligned} L_a(A) &= \{\theta[a] : \theta \in Con(A)\} \\ FL_a(A) &= \{a/\Theta : \Theta \in FCon(A)\} \end{aligned}$$

Lemma 5.2.3. *Let μ be a fuzzy subset of A and $a \in A$. Then $\mu \in FL_a(A)$ if and only if the level subset $\mu_\alpha \in L_a(A)$ for all $\alpha \in L$.*

Proof. Suppose that $\mu \in FL_a(A)$ and let $\alpha \in L$. Then, by the above theorem, $\mu(a) = 1$ and μ satisfies the equality

$$\mu(x) \wedge \mu(y) \wedge \mu(p(x)) = \mu(p(y)) \wedge \mu(x) \wedge \mu(y)$$

for each $x, y \in A$ and all $p \in P_1(A)$. Let $x, y \in \mu_\alpha$. For any $p \in P_1(A)$, if $p(x) \in \mu_\alpha$, then

$$\mu(x) \wedge \mu(y) \wedge \mu(p(x)) \geq \alpha$$

by the above equality we get $\mu(p(y)) \geq \alpha$. So that $p(y) \in \mu_\alpha$, i.e., μ_α satisfies the Mal'cev condition and hence it is the class of some $\theta \in Con(A)$ and this completes the proof. □

In the following we give an alternative proof independent of the Mal'cev theorem.

Proof. **(An alternative proof for Lemma 5.2.3)**
Suppose that $\mu \in FL_a(A)$ and let $\alpha \in L$. Then $\mu = a/\Theta$ for some $\Theta \in FCon(A)$; i.e., for each $x \in A$

$$\mu(x) = \Theta(a, x)$$

It is an easy task to observe that $\mu_\alpha = \Theta_\alpha[a]$, and hence $\mu_\alpha \in L_a(A)$ for all $\alpha \in L$. Conversely, suppose that $\mu_\alpha \in L_a(A)$ for all $\alpha \in L$. Then for each $\alpha \in L$ there exists a congruence θ on A

such that $\mu_\alpha = \theta[a]$. Let us put

$$\phi_\alpha = \cap\{\theta \in Con(A) : \mu_\alpha = \theta[a]\}$$

Then ϕ_α is a congruence on A such that $\mu_\alpha = \phi_\alpha[a]$ and $\phi_\alpha \subseteq \phi_\beta$ whenever $\beta \leq \alpha$. Now define a fuzzy relation Θ_μ on A by:

$$\Theta_\mu(x,y) = \bigvee\{\alpha \in L : (x,y) \in \phi_\alpha\}$$

for all $x,y \in A$. Since each ϕ_α is a congruence relation on A and the map $\alpha \mapsto \phi_\alpha$ is an antitone, one can easily verify that Θ_μ is a fuzzy congruence on A. Moreover, for each $x \in A$ consider:

$$\begin{aligned} a/\Theta_\mu(x) &= \Theta_\mu(a,x) \\ &= \bigvee\{\alpha \in L : (a,x) \in \phi_\alpha\} \\ &= \bigvee\{\alpha \in L : x \in \phi_\alpha[a]\} \\ &= \bigvee\{\alpha \in L : x \in \mu_\alpha\} \\ &= \mu(x) \end{aligned}$$

Therefore μ coincide with the fuzzy congruence class of Θ_μ determined by a. So that $\mu \in FL_a(A)$. □

5.3 Fuzzy Congruence Classes in Regular and Permutable Varieties

In this section, a finite characterization is given for fuzzy congruence classes in regular and permutable varieties.

Definition 5.3.1. An algebra A is called regular if each of its congruences is determined by every single class, i.e., if for every $\theta, \phi \in Con(A)$ and every $a \in A$,

$$\theta[a] = \phi[a] \Rightarrow \theta = \phi$$

A class of algebras is called regular if each of its members has this property.

Theorem 5.3.2. *An algebra A is regular if and only if each of its fuzzy congruences is determined by every single fuzzy class, i.e., if for every $\Theta, \Phi \in FCon(A)$ and every $a \in A$,*

$$a/\Theta = a/\Phi \Rightarrow \Theta = \Phi$$

Definition 5.3.3. An algebra is called congruence permutable (or simply permutable) if any two of its congruences permute, i.e., if $\theta \circ \phi = \phi \circ \theta$ for all $\theta, \phi \in Con(A)$. A class of algebras is called permutable if each of its members has this property.

The following fundamental theorem gives a simple description for permutable varieties, and it is due to A. I. Mal'cev (see [118]).

Theorem 5.3.4. *A variety is permutable if and only if there exists a ternary term p with*

$$p(x,x,z) = z \text{ and } p(x,z,z) = x$$

Definition 5.3.5. The ternary term p described in Theorem 5.3.4 is called a Mal'cev term for regular and permutable varieties.

A Mal'cev condition on admissible relations was applied by Werner in [154]. In the following theorem we adopt his theorem in a fuzzy setting so that it could be used in the latter sections.

Theorem 5.3.6. *Let A be an algebra in a permutable variety, $\mathscr{K}$. Then the following conditions hold:*

1. *Each admissible reflexive fuzzy relation on A is symmetric.*
2. *Each admissible reflexive fuzzy relation on A is transitive.*
3. *Each admissible reflexive fuzzy relation on A is a fuzzy congruence on A.*

Proof. Given that $\mathscr{K}$ is permutable. So, it has the Mal'cev term $p(x,y,z)$.

1. Let Θ be an admissible reflexive fuzzy relation on A, and $x,y \in A$. Then we have

$$x = p(x,y,y) \text{ and } y = p(x,x,y)$$

Now consider:

$$\begin{aligned}\Theta(x,y) &= \Theta(p(x,y,y),p(x,x,y)) \\ &\geq \Theta(x,x) \wedge \Theta(y,x) \wedge \Theta(y,y) \\ &= \Theta(y,x)\end{aligned}$$

Similarly we can verify that $\Theta(y,x) \geq \Theta(x,y)$. Thus $\Theta(x,y) = \Theta(y,x)$ and hence Θ is symmetric.

2. Let Θ be an admissible reflexive fuzzy relation on A, and $x,y,z \in A$. Then we have

$$x = p(x,y,y) \text{ and } y = p(z,z,y)$$

Then consider the following:

$$\begin{aligned}\Theta(x,y) &= \Theta(p(x,y,y),p(z,z,y)) \\ &\geq \Theta(x,z) \wedge \Theta(y,z) \wedge \Theta(y,y) \\ &= \Theta(x,z) \wedge \Theta(y,z) \\ &= \Theta(x,z) \wedge \Theta(z,y)\end{aligned}$$

Since z is arbitrary in A we get

$$\Theta(x,y) \geq \bigvee\{\Theta(x,z) \wedge \Theta(z,y) : z \in A\}$$

Therefore Θ is transitive. Hence proved.

3. The proof follows from (1) and (2).

□

In fact, a simple logical arrangement shows that each of the above three conditions are equivalent to each other. Moreover, one (and hence all) of these conditions is necessary and sufficient for a variety $\mathscr{K}$ to be congruence permutable.

The following Mal'cev type characterization was derived independently in [51, 67].

Theorem 5.3.7. *A variety $\mathscr{K}$ is regular and permutable if and only if there exist $n \geq 1$ ternary terms $t_1,...,t_n$ and a $(3+n)-$ary term t such that*

(*) $t_i(x,x,z) = z$ *for all* $i = 1,2,...,n$.

(**) $x = t(x,y,z,t_1(x,y,z),...,t_n(x,y,z))$.

(***) $y = t(x,y,z,z,z,...,z)$

By applying these terms, in regular and permutable varities, a finite characterization was given by Bělohlávek and Chagda for a subset C of A to be a congruence class.

Theorem 5.3.8. *[36, 37] Let $\mathscr{K}$ be a regular and permutable variety, $A \in \mathscr{K}$ and $\emptyset \neq C \subseteq A$. Then C is a class of some $\theta \in Con(A)$ if and only if the following conditions hold:*

1. *if $t_i(a_j,b_j,c) \in C$ for $c \in C$, $i = 1,2,...,n$, $j = 1,2,...,m$ and f is an $m-$ary fundamental operation, then*

$$t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c) \in C$$

2. *if $c,d \in C, a \in A$ and $t_i(a,d,c) \in C$ for $i = 1,2,...,n$, then $a \in C$*

3. *if $c,d \in C$, then $t_i(d,c,c) \in C$ for $i = 1,2,...,n$,*

Parallel to this theorem, we state and prove the following theorem in a fuzzy setting:

Theorem 5.3.9. *Let $\mathscr{K}$ be a regular and permutable variety and $A \in \mathscr{K}$. A normalized fuzzy subset μ of A is a fuzzy congruence class of some $\Theta \in FCon(A)$ if and only if the following conditions hold:*

1. *For each $m-$ary fundamental operation f,*

$$\mu\left(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c\right) \geq \left(\bigwedge_{i,j=1}^{n,m} \mu(t_i(a_j,b_j,c))\right) \wedge \mu(c)$$

for all $i = 1,2,...,n$

2. *For any $a,b,c \in A$,*

$$\mu(a) \geq \mu(b) \wedge \mu(c) \wedge \left(\bigwedge_{i=1}^{n} \mu(t_i(a,b,c))\right)$$

3. *For any $b,c \in A$,*

$$\mu(t_i(b,c,c)) \geq \mu(b) \wedge \mu(c)$$

for all $i = 1,2,...,n$.

Proof. Suppose that μ satisfies the conditions $(1),(2)$ and (3). Since μ is given to be normalized we can choose and fix an element $c \in A$ with $\mu(c) = 1$. Now define a fuzzy relation Θ on A by:

$$\Theta(a,b) = \mu(t_1(a,b,c)) \wedge ... \wedge \mu(t_n(a,b,c))$$

for all $a,b \in A$. Let us first prove that Θ is a fuzzy congruence on A. For any $a \in A$:

$$\begin{aligned}\Theta(a,a) &= \mu(t_1(a,a,c)) \wedge ... \wedge \mu(t_n(a,a,c)) \\ &= \mu(c) \wedge ... \wedge \mu(c) \\ &= 1\end{aligned}$$

So Θ is reflexive. Also let $a_1,...,a_m,b_1,...,b_m \in A$ and let f be an $m-$ary operation on A. Then consider the following:

$$\begin{aligned}
\Theta(f^A(a_1,...a_m),f^A(b_1,...,b_m)) &= \bigwedge_{i=i}^{n}\mu(t_i(f^A(a_1,...a_m),f^A(b_1,...,b_m),c)) \\
&\geq \left(\bigwedge_{i,j=1}^{n,m}\mu(t_i(a_j,b_j,c))\right)\wedge\mu(c) \\
&= \bigwedge_{i,j=1}^{n,m}\mu(t_i(a_j,b_j,c)) \\
&= \bigwedge_{j=1}^{m}\Theta(a_j,b_j)
\end{aligned}$$

Therefore Θ is an admissible reflexive fuzzy relation on A. Since $\mathscr{K}$ is congruence permutable, it follows from Theorem 5.3.6 that Θ is a fuzzy congruence on A. Our aim is to show that $\mu = c/\Theta$. For any $x \in A$, consider:

$$\begin{aligned}
c/\Theta(x) &= \Theta(x,c) \\
&= \mu(t_1(x,c,c))\wedge...\wedge\mu(t_n(x,c,c)) \\
&\geq \mu(x)\wedge\mu(c)(\textit{ by condition }(3)) \\
&= \mu(x)
\end{aligned}$$

On the other hand, by condition (2) we have

$$\begin{aligned}
\mu(x) &\geq \mu(c)\wedge\left(\bigwedge_{i=1}^{n}\mu(t_i(x,c,c))\right) \\
&= \bigwedge_{i=1}^{n}\mu(t_i(x,c,c)) \\
&= \Theta(x,c) \\
&= c/\Theta(x)
\end{aligned}$$

Therefore $\mu = c/\Theta$ (the fuzzy congruence class of Θ determined by c). Conversely, suppose that $\mu = x/\Theta$ for some $\Theta \in FCon(A)$ and $x \in A$. Then $\mu_\alpha = \Theta_\alpha[x]$ for all $\alpha \in L$.

1. Let $a_1,...,a_m,b_1,...,b_m \in A$ and f be an $m-$ary operation on A. Let us put

$$\alpha = \left(\bigwedge_{i,j=1}^{n,m} \mu(t_i(a_j,b_j,c))\right) \wedge \mu(c)$$

Then $c \in \mu_\alpha$ and $t_i(a_j,b_j,c) \in \mu_\alpha$ for all $i = 1,...,n$ and $j = 1,...,m$. This gives

$$\Theta(t_i(a_j,b_j,c),c) \geq \alpha$$

for all $i = 1,...,n$ and $j = 1,...,m$. By $(**)$ and $(***)$ of Theorem 5.3.7, for each $j = 1,2,...,m$ we obtain

$$\begin{aligned} a_j &= t(a_j,b_j,c,t_1(a_j,b_j,c),...,t_n(a_j,b_j,c)) \\ b_j &= t(a_j,b_j,c,c,...,c) \end{aligned}$$

Using the compatible property of Θ, we get the following for each $j = 1,2,...,m$:

$$\begin{aligned} \Theta(a_j,b_j) &= \Theta(q(a_j,b_j,c,t_1(a_j,b_j,c),...,t_n(a_j,b_j,c)),t(a_j,b_j,c,c,...,c)) \\ &\geq \Theta(t_1(a_j,b_j,c),c) \wedge ... \wedge \Theta(t_n(a_j,b_j,c),c) \\ &\geq \alpha \end{aligned}$$

So that

$$\bigwedge_{j=1}^{m} \Theta(a_j,b_j) \geq \alpha$$

Again using the compatible property of Θ, we get,

$$\Theta(f^A(a_1,...,a_m),f^A(b_1,...,b_m)) \geq \alpha$$

By $(*)$ of Theorem 5.3.7, $t_i(f^A(b_1,...,b_m), f^A(b_1,...,b_m), c) = c$ for all $i = 1,2,...,n$. Now consider the following:

$$\begin{aligned}\Theta(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c), c) &= \Theta(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c), t_i(f^A(b_1,...,b_m),\\ &\qquad f^A(b_1,...,b_m), c))\\ &\geq \Theta(f^A(a_1,...,a_m), f^A(b_1,...,b_m))\\ &\geq \alpha\end{aligned}$$

So that $(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c), c) \in \Theta_\alpha$, which gives

$$t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c) \in \Theta_\alpha[c] = \mu_\alpha$$

Thus $\mu(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c) \geq \alpha$ and hence the result holds.

2. Let $a,b,c \in A$. Put
$$\alpha = \mu(b) \wedge \mu(c) \wedge \left(\bigwedge_{i=1}^{n} \mu(t_i(a,b,c))\right)$$
Then $b,c \in \mu_\alpha$ and $t_i(a,b,c) \in \mu_\alpha$ for all $i = 1,2,...,n$. Since μ_α is a congruence class of Θ_α we get $(t_i(a,b,c),c) \in \Theta_\alpha$ and $(b,c) \in \Theta_\alpha$; i.e., $\Theta(b,c) \wedge \Theta((t_i(a,b,c),c)) \geq \alpha$ for all $i = 1,2,...,n$. By $(**)$ and $(***)$ of Theorem 5.3.7 we obtain

$$\begin{aligned}a &= t(a,b,c,t_1(a,b,c),...,t_n(a,b,c))\\ b &= t(a,b,c,c,...,c)\end{aligned}$$

So that
$$\Theta(a,b) \geq \Theta((t_1(a,b,c),c) \wedge ... \wedge \Theta((t_n(a,b,c),c) \geq \alpha$$

Then $(a,b) \in \Theta_\alpha$, which gives $a \in \Theta_\alpha[b] = \Theta_\alpha[c] = \mu_\alpha$. Thus $a \in \mu_\alpha$ and hence the result holds.

3. Let $b,c \in A$. If we put $\alpha = \mu(b) \wedge \mu(c)$, then $b,c \in \mu_\alpha$ which gives that $\Theta(b,c) \geq \alpha$.

Again by $(*)$ of Theorem 5.3.7, we have $c = t_i(c,c,c)$ for all $i = 1,2,...,n$. For each $i = 1,2,...,n$, consider the following:

$$\Theta(t_i(b,c,c),c) = \Theta(t_i(b,c,c),t_i(c,c,c)) \geq \Theta(b,c) \geq \alpha$$

This completes the proof. □

The following theorem gives another description for fuzzy congruence classes in regular and permutable varieties.

Theorem 5.3.10. *Let $\mathscr{K}$ be a regular and permutable variety and $A \in \mathscr{K}$. A normalized fuzzy subset μ of A is a fuzzy congruence class of some $\Theta \in FCon(A)$ if and only if μ is $\overrightarrow{y}-$closed under the following terms:*

1. *for each $m \in Z^+$, each $f \in \mathfrak{F}_m$ and every $i = 1,2,...,n$,*

$$q_i(x_1,...,x_m,x'_1,...,x'_m,y,y_{11},...,y_{1n},...,y_{m1},...,y_{mn}) = t_i(f(t(x_1,x'_1,y,y_{11},...,y_{1n}),...,$$
$$...,t(x_m,x'_m,y,y_{m1},...,y_{mn})),f(x'_1,...,x'_m),y)$$

2. *$q(x,y,y',y_1,...,y_n) = t(x,y,y',y_1,...,y_n)$*

3. *$d_1(x,y_1,y_2) = t_1(y_1,y_2,y_2)$,...,$d_n(x,y_1,y_2) = t_n(y_1,y_2,y_2)$*

where t and t_i's are those terms obtained in Theorem 5.3.7.

Proof. If μ is a class of some $\Theta \in FCon(A)$, then the $\overrightarrow{y}-$closedness under the terms listed in $(i)-(iii)$ follows immediately from the substitution property of Θ. Conversely, suppose that μ is $\overrightarrow{y}-$closed under the terms given in $(i)-(iii)$. It suffices to show μ satisfies the conditions $(i)-(iii)$ of Theorem 5.3.9.

(i) Let $m \in Z^+, f \in \mathfrak{F}_m$ and $a_1, ..., a_m, b_1, ..., b_m, c \in A$. From Theorem 5.3.7 we have the following:

$$\begin{aligned} a_1 &= t(a_1, b_1, c, t_1(a_1, b_1, c), ..., t_n(a_1, b_1, c)) \\ a_2 &= t(a_2, b_2, c, t_1(a_2, b_2, c), ..., t_n(a_2, b_2, c)) \\ &\vdots \\ a_m &= t(a_m, b_m, c, t_1(a_m, b_m, c), ..., t_n(a_m, b_m, c)) \end{aligned}$$

Then for each $i = 1, 2, ..., n$

$$\begin{aligned} t_i(f^A(a_1, ..., a_m), f^A(b_1, ..., b_m), c) &= t_i(f(t(a_1, b_1, c, t_1(a_1, b_1, c), ..., t_n(a_1, b_1, c)), ... \\ &..., \ t(a_m, b_m, c, t_1(a_m, b_m, c), ..., t_n(a_m, b_m, c))), f(b_1, ..., b_m), c) \\ &= q_i(a_1, ..., a_m, b_1, ..., b_m, y, y_{11}, ..., y_{1n}, ..., y_{m1}, ..., y_{mn}) \end{aligned}$$

where $y = c$ and each $y_{ij} = t_i(a_j, b_j, c)$ for $i = 1, ..., n$ and $j = 1, ..., m$.

μ being $\overrightarrow{y}-$closed under each term q_i, it follows that

$$\mu(q_i(a_1, ..., a_m, b_1, ..., b_m, y, y_{11}, ..., y_{1n}, ..., y_{m1}, ..., y_{mn})) \geq \left(\bigwedge_{i,j=1}^{n,m} \mu(y_{ij}) \right) \wedge \mu(y)$$

This is equivalent to

$$\mu\left(t_i(f^A(a_1, ..., a_m), f^A(b_1, ..., b_m), c\right) \geq \left(\bigwedge_{i,j=1}^{n,m} \mu(t_i(a_j, b_j, c)) \right) \wedge \mu(c)$$

proving the condition (i) of Theorem 5.3.9.

(ii) Let $a, b, c \in A$. By Theorem 5.3.7

$$\begin{aligned} a &= t(a, b, c, t_1(a, b, c), ..., t_n(a, b, c)) \\ &= q(a, y, y', y_1, ..., y_n) \end{aligned}$$

where $y = b, y' = c$ and $y_i = t_i(a,b,c)$ for each $i = 1,2,\ldots,n$. Since μ is $\overrightarrow{y}-$closed under the term q it holds that

$$\mu(q(a,y,y',y_1,\ldots,y_n)) \geq \mu(y) \wedge \mu(y') \wedge \bigwedge_{i=1}^{n} \mu(y_i)$$

which gives

$$\mu(a) \geq \mu(b) \wedge \mu(c) \wedge \left(\bigwedge_{i=1}^{n} \mu(t_i(a,b,c)) \right)$$

Hence proving (ii) of Theorem 5.3.9.

(iii) μ being $\overrightarrow{y}-$closed under the terms $d_1,\ldots,d_n$ directly implies, μ satisfies the condition (iii) of Theorem 5.3.9. This completes the proof. □

5.4 Fuzzy Congruence Classes in Regular Varieties

Now assumption of permutablility is omitted. Three Mal'cev type conditions characterizing regular varieties were published in 1970 independently by B. Cs*á*k*á*ny [53], G. Gr*ä*atzer [76] and R. Wille [155]. We modified that of B. Cs*á*k*á*ny as follows.

Theorem 5.4.1. *A variety $\mathscr{K}$ is regular if and only if there exist an $n \in Z^+$, $t_1,\ldots,t_n \in T_3$ and $q_1,\ldots,q_n \in T_5$ satisfying the following identities:*

$$t_1(x,x,z) = \ldots = t_n(x,x,z) = z,$$

$$q_1(t_1(x,y,z),z,x,y,z) = x$$

$$q_i(z,t_i(x,y,z),x,y,z) = q_{i+1}(t_{i+1}(x,y,z),z,x,y,z) \text{ for } i = 1,\ldots,n-1$$

$$\text{and } q_n(z,t_n(x,y,z),x,y,z) = y$$

The following theorem is an independent characterization of fuzzy congruence classes in regular variety using the terms obtained in Theorem 5.4.1. In fact, it is the fuzzy version of the theorem of [37].

Theorem 5.4.2. *Let $\mathscr{K}$ be a regular variety and $A \in \mathscr{K}$. A normalized fuzzy subset μ of A is a class of some $\Theta \in FCon(A)$ if and only if the following conditions hold:*

1. *If $m \in Z^+, f \in \mathfrak{F}_n$ and $a_1, ..., a_m, b_1, ..., b_m, c \in A$, then*

$$\mu(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c)) \geq \mu(c) \wedge \left(\bigwedge_{i,j=1}^{n,m} \mu(t_i(a_j, b_j, c)) \right)$$

2. *For any $a, b, c, d \in A$,*

$$\mu(t_i(a,d,c)) \geq \mu(c) \wedge \left(\bigwedge_{i=1}^{n} (\mu(t_i(a,b,c)) \wedge \mu(t_i(b,d,c))) \right)$$

3. *For and $c, d \in A$, and each $i = 1, ..., n$,*

$$\mu(t_i(c,d,d)) \geq \mu(c) \wedge \mu(d)$$

4. *For each $a, c \in A$,*

$$\mu(a) \geq \mu(c) \wedge \left(\bigwedge_{i=1}^{n} \mu(t_i(a,c,c)) \right)$$

Proof. First assume μ to be a class of some fuzzy congruence on A, i.e., $\mu = x/\Theta$ for some $x \in A$ and some $\Theta \in FCon(A)$, which gives $\mu_\alpha = \Theta_\alpha[x]$ for all $\alpha \in L$.

1. Let $m \in Z^+, f \in \mathfrak{F}^m$ and $a_1, ..., a_m, b_1, ..., b_m, c \in A$. If we put

$$\alpha = \mu(c) \wedge \left(\bigwedge_{i,j=1}^{n,m} \mu(t_i(a_j, b_j, c)) \right)$$

then $c \in \mu_\alpha$ and $t_i(a_j, b_j, c) \in \mu_\alpha$ for all $i = 1, ...n$ and all $j = 1, ..., m$. μ_α being a class of the congruence Θ_α we get

$$\Theta(c, t_i(a_j, b_j, c)) \geq \alpha \text{ for all } i = 1, ..., n \text{ and } j = 1, ..., m$$

We show that $\Theta(a_i,b_j) \geq \alpha$ for all $j = 1,...,m$. By Theorem 5.4.1, we have

$$\begin{aligned} a_j &= q_1(t_1(a_j,b_j,c),c,a_j,b_j,c) \\ b_j &= q_n(c,t_n(a_j,b_j,c),a_j,b_j,c) \end{aligned}$$

Now consider the following:

$$\begin{aligned} \Theta(a_j,b_j) &= \Theta(q_1(t_1(a_j,b_j,c),c,a_j,b_j,c),b_j) \\ &\geq \alpha \wedge \Theta(q_1(c,t_1(a_j,b_j,c),a_j,b_j,c),b_j) \\ &= \alpha \wedge \Theta(q_2(t_2(a_j,b_j,c),c,a_j,b_j,c),b_j) \\ &\geq \alpha \wedge \alpha \wedge \Theta(q_3(c,t_3(a_j,b_j,c),a_j,b_j,c),b_j) \\ &= \alpha \wedge \Theta(q_3(t_3(a_j,b_j,c),c,a_j,b_j,c),b_j) \\ &\vdots \\ &= \alpha \wedge \Theta(q_n(t_n(a_j,b_j,c),c,a_j,b_j,c),b_j) \\ &\geq \alpha \wedge \Theta(q_n(c,t_n(a_j,b_j,c),a_j,b_j,c),b_j) \\ &= \alpha \wedge \Theta(b_j,b_j) \\ &= \alpha \end{aligned}$$

So that

$$\bigwedge_{j=1}^{m} \Theta(a_j,b_j) \geq \alpha$$

Again using the substitution property of Θ, we get,

$$\Theta(f^A(a_1,...,a_m),f^A(b_1,...,b_m)) \geq \alpha$$

By Theorem 5.4.1, $t_i(f^A(b_1,...,b_m), f^A(b_1,...,b_m), c) = c$ for all $i = 1,2,...,n$. Now consider the following:

$$\begin{aligned}\Theta(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c), c) &= \Theta(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c),\\ &\qquad t_i(f^A(b_1,...,b_m), f^A(b_1,...,b_m), c))\\ &\geq \Theta(f^A(a_1,...,a_m), f^A(b_1,...,b_m))\\ &\geq \alpha\end{aligned}$$

So that $(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c), c) \in \Theta_\alpha$, which gives

$$t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c) \in \Theta_\alpha[c] = \mu_\alpha$$

Thus $\mu(t_i(f^A(a_1,...,a_m), f^A(b_1,...,b_m), c) \geq \alpha$ and hence the result holds.

2. For any $a,b,c,d \in A$, let us put

$$\alpha = \mu(c) \wedge \left(\bigwedge_{i,j=1}^{n,m} \mu(t_i(a_j,b_j,c))\right)$$

Then $c \in \mu_\alpha$ and $t_i(a,b,c), t_i(b,d,c) \in \mu_\alpha$ for all $i = 1,...,n$. Since μ_α is a class of the congruence Θ_α, we get

$$\Theta(c, t_i(a,b,c)) \geq \alpha \text{ and } \Theta(c, t_i(b,d,c)) \geq \alpha$$

We show that $\Theta(a,b) \geq \alpha$ and $\Theta(b,d) \geq \alpha$. From Theorem 5.4.1 we can write a and b as follows:

$$\begin{aligned}a &= q_1(t_1(a,b,c), c, a, b, c)\\ b &= q_n(c, t_n(a,b,c), a, b, c)\end{aligned}$$

Now consider the following:

$$\begin{aligned}
\Theta(a,b) &= \Theta(q_1(t_1(a,b,c),c,a,b,c),b) \\
&\geq \alpha \wedge \Theta(q_1(c,t_1(a,b,c),a,b,c),b) \\
&= \alpha \wedge \Theta(q_2(t_2(a,b,c),c,a,b,c),b) \\
&\vdots \\
&= \alpha \wedge \Theta(q_n(t_n(a,b,c),c,a,b,c),b) \\
&\geq \alpha \wedge \Theta(q_n(c,t_n(a,b,c),a,b,c),b) \\
&= \alpha \wedge \Theta(b,b) \\
&= \alpha
\end{aligned}$$

Similarly we can show that $\Theta(b,d) \geq \alpha$. By the transitive property of Θ it follows that

$$\Theta(a,d) \geq \Theta(a,b) \wedge \Theta(b,d) \geq \alpha$$

Again by Theorem 5.4.1, we can write c as $c = t_i(a,a,c)$. So for each $i = 1,...,n$ we got the following:

$$\begin{aligned}
\Theta(c,t_i(a,d,c)) &= \Theta(t_i(a,a,c),t_i(a,d,c)) \\
&\geq \Theta(a,d) \\
&\geq \alpha
\end{aligned}$$

So that $(c,t_i(a,d,c)) \in \Theta_\alpha$, which gives $t_i(a,d,c) \in \Theta_\alpha[c] = \mu_\alpha$. Thus $\mu(t_i(a,d,c)) \geq \alpha$ and hence the result holds.

3. For any $c,d \in A$, let us put $\mu(c) \wedge \mu(d) = \alpha$. Then $c,d \in \mu_\alpha$. This is equivalent to

$$\Theta(c,d) \geq \alpha$$

By Theorem 5.4.1, we can be write c as $c = t_i(c,c,c)$. Now for each $i = 1,2,...,n$ consider the following:

$$\begin{aligned}\Theta(t_i(c,d,d),c) &= \Theta(t_i(c,d,d),t_i(c,c,c)) \\ &\geq \Theta(c,d) \\ &\geq \alpha\end{aligned}$$

So that $(t_i(c,d,d),c) \in \Theta_\alpha$, which gives $t_i(c,d,d) \in \Theta_\alpha[c] = \mu_\alpha$. Thus $\mu(t_i(c,d,d)) \geq \alpha$. Hence proved.

4. Let $a,c \in A$. If we put

$$\alpha = \mu(a) \geq \mu(c) \wedge \left(\bigwedge_{i=1}^{n} \mu(t_i(a,c,c)) \right)$$

then $c \in \mu_\alpha$ and $t_i(a,c,c) \in \mu_\alpha$ for all $i = 1,2,...,n$. By following the same procedure as we have done in (2), we can show that $\Theta(a,c) \geq \alpha$. So that $(a,b) \in \Theta_\alpha$ which implies that $a \in \Theta_\alpha[c] = \mu_\alpha$. Thus $\mu(a) \geq \alpha$ and hence the proved.

Conversely, assume the conditions $(1)-(4)$ hold. Since μ is given to be a normalized fuzzy set, we can choose and fix an element $c \in A$ with $\mu(c) = 1$. Let us define a fuzzy relation Φ on A by:

$$\Phi(x,y) = \bigwedge_{i=1}^{n} \mu(t_i(x,y,c))$$

for all $x,y \in A$. Clearly Φ is reflexive. Compatibility and transitivity of Φ follow from (1) and (2), respectively. So, one can easily observe that each of the the level relations of Φ is reflexive, transitive and compatible. It was proved in [53] that regular varieties are $(n+1)-$permutable. Again for a variety $\mathscr{K}$ to be $(n+1)-$permutable, it is necessary and sufficient that for each $A \in \mathscr{K}$, every reflexive and transitive compatible binary relation on A is a congruence on A (see [49]). This implies that Φ_α is a congruence on A for all $\alpha \in L$. It follows from Lemma 5.1.5 that Φ is a fuzzy congruence on A. Using (3) and (4) one can easily verify that $\mu = c/\Phi$. Hence proved.

□

Theorem 5.4.3. *Let μ be a normalized fuzzy subset of an algebra A belonging to a regular variety and assume that for any $a,b,c,d \in A$ and each $i = 1,2,...,n$*

$$\mu(t_i(a,d,c)) \geq \mu(t_i(a,b,c)) \wedge \mu(t_i(b,d,c))$$

Then μ is a class of some $\Theta \in FCon(A)$ if and only if the following conditions hold:

1. *For each $m \in Z^+, f \in \mathfrak{F}_m$ and each $i,j,k = 1,2,...,n$, μ is $\overrightarrow{y}$−closed under the term*

$$d_{ijk}(x_1,...,x_m,x'_1,...,x'_m,y,y',y'',y_1,...,y_m,y'_1,...,y'_m) = t_i(t_j(f^A(q_k(y_1,y,x_1,x'_1,y),...,$$
$$...,q_k(y_m,y,x_m,x'_m,y)),f^A(x'_1,...,x'_m),y),t_j(f^A(q_k(y'_1,y',x_1,x'_1,y'),...,q_k(y'_m,y',x_m,x'_m,y')),$$
$$f^A(x'_1,...,x'_m),y'),y'')$$

2. *μ is $\overrightarrow{y}$−closed under the terms*

$$d_i(x_1,y_1,y_2) = t_i(y_1,y_2,y_2)$$

   For all $i = 1,2,..,n$.

3. *For each $a,b,c,d \in A$,*

$$\mu(q_1(d,c,a,c,c)) \geq \mu(c) \wedge \mu(d) \wedge \mu(t_1(a,c,c)) \wedge ... \wedge \mu(t_n(a,c,c))$$

Proof. If μ is a class of some $\Theta \in FCon(A)$, then the $\overrightarrow{y}$−closedness under the terms listed in (i) and (iii) follows immediately from the substitution property of Θ. We proceed to prove (iii). By our hypothesis $\mu = x/\Theta$ for some $x \in A$ and $\Theta \in FCon(A)$, which gives $\mu_\alpha = \Theta_\alpha[x]$. Let us put

$$\alpha = \mu(c) \wedge \mu(d) \wedge \mu(t_1(a,c,c)) \wedge ... \wedge \mu(t_n(a,c,c))$$

Then $c, d \in \mu_\alpha$ and $t_i(a, c, c) \in \mu_\alpha$ for all $i = 1, 2, ..., n$. This is equivalent to

$$\Theta(c, d) \wedge \Theta(c, t_i(a, c, c)) \wedge \Theta(d, t_i(a, c, c)) \geq \alpha$$

Also observe that

$$\begin{aligned}
\Theta(c, q_1(d, c, a, c, c)) &= \Theta(c, q_1(c, t_1(a, c, c), a, c, c)) \wedge \Theta(q_1(c, t_1(a, c, c), a, c, c), q_1(d, c, a, c, c)) \\
&\geq \Theta(c, q_1(c, t_1(a, c, c), a, c, c)) \wedge \alpha \\
&= \Theta(c, q_2(t_2(a, c, c), c, a, c, c)) \wedge \alpha \\
&\geq \Theta(c, q_2(c, t_2(a, c, c), a, c, c)) \wedge \alpha \\
&= \Theta(c, q_3(t_3(a, c, c), c, a, c, c)) \wedge \alpha \\
&\vdots \\
&\geq \Theta(c, q_n(c, t_n(a, c, c), a, c, c)) \wedge \alpha \\
&= \Theta(c, c) \wedge \alpha \\
&= \alpha
\end{aligned}$$

Thus $(c, q_1(d, c, a, c, c)) \in \Theta_\alpha$, which implies $q_1(d, c, a, c, c) \in \Theta_\alpha[c] = \mu_\alpha$. So that $\mu(q_1(d, c, a, c, c)) \geq \alpha$. Hence proved. Conversely, assume conditions $(i) - (iii)$ are satisfied. Since μ is normalized we can choose and fix an element $c \in A$ with $\mu(c) = 1$. Define a fuzzy relation Θ on A by: for each $x, y \in A$,

$$\Theta(x, y) = \bigwedge_{i=1}^{n} \mu(t_i(x, y, c))$$

Clearly Θ is reflexive. Transitivity of Θ follows from the assumption of the theorem and compatibility follows from the conditions (i) and (ii), i.e., Θ is reflexive, transitive and compatible fuzzy relation on A. By using the same argument as in the previous theorem we can show that Θ is a fuzzy congruence on A such that $\mu = c/\Theta$. Hence proved. □

5.5 Fuzzy Congruence Classes Which are Fuzzy Subuniverses

Remember that a subset B of A is called a subuniverse of A if for all $n > 0$, ever $n-$ary operation f and any $b_1,...,b_n \in B$ it holds that $f^A(b_1,...,b_n) \in B$. Analogous to this, we have the following definition.

Definition 5.5.1. A fuzzy subset μ of A is called a fuzzy subuniverse of A, if for all $n > 0$, every $n-$ary operation f and any $b_1,...,b_n \in A$ it holds that

$$\mu(f^A(b_1,...,b_n)) \geq \mu(b_1) \wedge ... \wedge \mu(b_n)$$

Theorem 5.5.2. *For a variety $\mathscr{K}$, the following conditions are equivalent:*

1. *At least one class of each congruence of every nonempty $A \in \mathscr{K}$ is a subuniverse of A.*
2. *There exists a unary term t with $p(t(x),t(x),...t(x)) = t(x)$ for every term p.*
3. *At least one fuzzy class of each fuzzy congruence of every nonempty $A \in \mathscr{K}$ is a fuzzy subuniverse of A.*

Proof. $(1) \Leftrightarrow (2)$ is proved by B. Cs*á*k*á*ny in [54]. So we proceed to prove $(2) \Rightarrow (3)$. Let $A \in \mathscr{K}$ and $\Theta \in FCon(A)$. For each $a \in A$ we show that the fuzzy class $t(a)/\Theta$ of Θ is a fuzzy subuniverse of A, where t is the unary term satisfying (2). Let $f \in \mathfrak{F}$ be an $n-$ary operation, $n > 0$ and $x_1,...,x_n \in A$. Then consider the following:

$$\begin{aligned} t(a)/\Theta(x_1) \wedge ... \wedge t(a)/\Theta(x_n) &= \Theta(t(a),x_1) \wedge ... \wedge \Theta(t(a),x_n) \\ &\leq \Theta(f^A(t(a),...,t(a)), f^A(x_1,...,x_n)) \\ &= \Theta(t(a), f^A(x_1,...,x_n)) \\ &= t(a)/\Theta(f^A(x_1,...,x_n)) \end{aligned}$$

Thus $t(a)/\Theta$ is a fuzzy subuniverse of A.

The proof of $(3) \Rightarrow (1)$ follows from the fact that every congruence relation on A can be identified as a fuzzy congruence relation by its characteristic function. □

Theorem 5.5.3. *For a variety $\mathscr{K}$, the following conditions are equivalent:*

1. *At most one class of each congruence of $A \in \mathscr{K}$ is a subuniverse of A.*

2. *There exists $n \in Z^{+}$, $p_1,...,p_n \in T_6$ and $u_1,...,u_n \in T_1$ satisfying the following identities:*

$$p_1(u_1(x),u_1(y),x,y,x,y) = x$$

$$p_i(x,y,u_i(x),u_i(y),x,y) = p_{i+1}(u_{i+1}(x),u_{i+1}(y),x,y,x,y) \text{ for } i = 1,2,...,n-1$$

$$p_n(x,y,u_n(x),u_n(y),x,y) = y$$

3. *At most one fuzzy class of each fuzzy congruence on $A \in \mathscr{K}$ is a fuzzy subuniverse of A.*

Proof. The proof of (1) $\Leftrightarrow$ (2) is given in [54]. So we proceed to prove (2) $\Rightarrow$ (3). Let $A \in \mathscr{K}$ and $\Theta \in FCon(A)$. Suppose that μ_1 and μ_2 are two fuzzy classes of Θ which are fuzzy subuniverses of A. Let $a,b \in A$ such that $\mu_1 = a/\Theta$ and $\mu_2 = b/\Theta$. Then $\mu_1(a) = 1$ and $\mu_2(b) = 1$. We first show that $\Theta(a,b) = 1$. Since μ_1 and μ_2 are fuzzy subuniverses of A, we get

$$\mu_1(u_i(a)) \geq \mu_1(a) = 1 \text{ and } \mu_2(u_i(b)) \geq \mu_2(b) = 1$$

for each i. So that

$$\Theta(a,u_i(a)) = 1 = \Theta(b,u_i(b))$$

for all i. Now consider the following:

$$\begin{aligned}
\Theta(a,b) &= \Theta(p_1(u_1(a),u_1(b),a,b,a,b),b) \\
&\geq \Theta(p_1(a,b,u_1(a),u_1(b),a,b),b) \\
&= \Theta(p_2(u_2(a),u_2(b),a,b,a,b),b) \\
&\geq \Theta(p_2(a,b,u_2(a),u_2(b),a,b),b) \\
&\vdots \\
&\geq \Theta(p_n(a,b,u_n(a),u_n(b),a,b),b) \\
&= \Theta(b,b) \\
&= 1
\end{aligned}$$

Now for any $x \in A$,

$$\begin{aligned}
\mu_1(x) &= a/\Theta(x) \\
&= \Theta(a,x) \\
&\geq \Theta(a,b) \wedge \Theta(b,x) \\
&= \Theta(b,x) \\
&= b/\Theta(x) \\
&= \mu_2(x)
\end{aligned}$$

By symmetry, we can also show that $\mu_2(x) \geq \mu_1(x)$, i.e., $\mu_1(x) = \mu_2(x)$ for all $x \in A$ and hence $\mu_1 = \mu_2$.

The proof of $(3) \Rightarrow (1)$ follows from the fact that every congruence relation on A can ve identified as a fuzzy congruence relation by its characteristic function. □

Definition 5.5.4. An algebra A is called idempotent if for each of its fundamental operations f it holds $f(x,x,...,x) = x$ in A. A class of algebras is called idempotent if each of its members has this property.

Theorem 5.5.5. *For an algebra A, the following conditions are equivalent:*

1. *Each congruence class of A is a subuniverse.*
2. *A is idempotent.*
3. *Each fuzzy congruence class of A is a fuzzy subuniverse.*

Proof. The equivalence of (1) and (2) is proved in [54]. We prove $(2) \Rightarrow (3)$. Suppose that A is idempotent. Let $a \in A$, $\Theta \in FCon(A)$, $n > 0, f \in \mathfrak{F}_n$ and $x_1, \ldots, x_n \in A$. Then consider the following:

$$\begin{aligned} a/\Theta(x_1) \wedge \ldots \wedge a/\Theta(x_n) &= \Theta(a, x_1) \wedge \ldots \wedge \Theta(a, x_n) \\ &\leq \Theta(f^A(a, a, \ldots, a), f^A(x_1, \ldots, x_n)) \\ &= \Theta(a, f^A(x_1, \ldots, x_n)) \\ &= a/\Theta(f^A(x_1, \ldots, x_n)) \end{aligned}$$

Therefore a/Θ is a fuzzy subuniverse of A. The proof of $(3) \Rightarrow (1)$ follows from the fact that every congruence relation on A can ve identified as a fuzzy congruence relation by its characteristic function. $\square$

Chapter 6

$L-$Fuzzy Cosets

Cosets in universal algebra were first introduced by P. Agliano in [8] and later studied by R. B$\breve{e}$lohl$\acute{a}$vek [35] under a name 'convex sets'. Agliano, in his paper [8], has defined cosets using coset terms. He gives a natural structure to the set of congruence classes containing a given element of the algebra, and relates the properties of this structure to general features of the variety generated by the algebra. It comes out that such kind of results are better understood if we consider the set of congruence classes containing a given element as a subset of a generally richer family of subset of the algebra; called 'cosets' of the algebra-which in fact is endowed with the very natural structure of an algebraic lattice. It is observed that in many classical cases cosets are very well-known structures: in the case of groups left-cosets (or right -cosets) determined by normal subgroups, in rings cosets determined by ideals.

In this chapter, we define $L-$fuzzy cosets in universal algebras and investigate some of their properties. We give necessary and sufficient conditions for a class of algebras to be congruence permutable.

6.1 Fuzzy Cosets

Recall from [8] that, a term $t(\overrightarrow{x},\overrightarrow{y})$ is said to be a cost term in $\overrightarrow{y}$ if $t(a_1,...,a_n,b,b...,b)=b$ for all $a_1,...,a_n,b\in A$, and cosets of A are those nonempty subsets of A which are $\overrightarrow{y}-$closed under each coset term $t(\overrightarrow{x},\overrightarrow{y})$ in $\overrightarrow{y}$. In the following, we define fuzzy cosets.

Definition 6.1.1. An $L-$fuzzy subset μ of A is said to be an $L-$fuzzy coset of A (or shortly a fuzzy coset of A) if and only if the following conditions are satisfied:

1. $\mu(a) = 1$ for some $a \in A$.

2. If $t(\overrightarrow{x}, \overrightarrow{y})$ is a coset term in $\overrightarrow{y}$ and $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in A$, then

$$\mu(t(a_1, a_2, ..., a_n, b_1, b_2, ..., b_m)) \geq \mu(b_1) \wedge \mu(b_2) \wedge ... \wedge \mu(b_m)$$

In this case, we say that μ is a fuzzy coset of A determined by a. For each $a \in A$, we denote by $FC_a(A)$ the set of all fuzzy cosets of A determined by a.

The following theorem gives an equivalent condition for fuzzy subsets to be a fuzzy coset in terms of their level sets.

Theorem 6.1.2. *Let $a \in A$. A fuzzy subset μ of A is a fuzzy coset of A determined by a if and only if μ_α is a coset of A containing a for all $\alpha \in L$.*

Proof. Suppose that μ is a fuzzy coset of A determined by a. Then $\mu(a) = 1$. So $a \in \mu_\alpha$ for all $\alpha \in L$. Also, for any $\alpha \in L$, let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in (\mu_\alpha)^m$ and $t(\overrightarrow{x}, \overrightarrow{y})$ be a coset term in $\overrightarrow{y}$. Since $\mu(t(\overrightarrow{a}, \overrightarrow{b})) \geq \mu^m(\overrightarrow{b}) \geq \alpha$, we get $t(\overrightarrow{a}, \overrightarrow{b}) \in \mu_\alpha$ and hence each μ_α is a coset of A containing a. Conversely suppose that the level subset μ_α is a coset of A containing a for all $\alpha \in L$. In particular μ_α is a coset of A containing a for $\alpha = 1$. So that $\mu(a) = 1$. Let $t(\overrightarrow{x}, \overrightarrow{y})$ be a coset term in $\overrightarrow{y}$ and $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$. Put $\mu^m(\overrightarrow{b}) = \alpha$. Then $\overrightarrow{b} \in (\mu^m)_\alpha = (\mu_\alpha)^m$. Since each μ_α is a coset we get, $t(\overrightarrow{a}, \overrightarrow{b}) \in \mu_\alpha$. So that $\mu(t(\overrightarrow{a}, \overrightarrow{b})) \geq \alpha = \mu^m(\overrightarrow{b})$. Therefore μ is a fuzzy coset of A determined by a. □

This theorem confirms that a fuzzy coset of A determined by a is precisely a fuzzy $\mathfrak{L}$-subset of A (in the sense of [144]), where $\mathfrak{L}$ is the set of all cosets of A containing a.

Lemma 6.1.3. *Let $a \in A$ and $H \subseteq A$. For $\alpha \in L - \{1\}$, let α_H be as given in Definition 1.2.12. Then, H is a coset of A containing a if and only if α_H is a fuzzy coset of A determined by a for some $\alpha \in L - \{1\}$.*

Corollary 6.1.4. *Let $a \in A$. A subset H of A is a coset of A containing a if and only if its characteristic function χ_H is a fuzzy coset of A determined by a.*

Lemma 6.1.5. *Every fuzzy ideal is a fuzzy coset determined by* 0.

Proof. It is enough to show that every coset term is an ideal term. □

Lemma 6.1.6. *Let $a \in A$ and $\mu \in FC_a(A)$. Then, for any $a_1, ..., a_m \in A$, if $x \in \overline{\{a_1, ..., a_m\}}^a$, then $\mu(x) \geq \mu(a_1) \wedge ... \wedge \mu(a_m)$. More generally, for any nonempty subset S of A, if $x \in \overline{S}^a$, then there exist $a_1, ..., a_m \in S$ such that $\mu(x) \geq \mu(a_1) \wedge ... \wedge \mu(a_m)$.*

Proof. Suppose that $x \in \overline{\{a_1, ..., a_m\}}^a$. Then, $x = p(b_1, ..., b_n, a_1, ..., a_m)$ for some $b_1, ..., b_n \in A$ and some coset term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$. So we have the following:

$$\mu(x) = \mu(p(b_1, ..., b_n, a_1, ..., a_m)) \geq \mu(a_1) \wedge ... \wedge \mu(a_m)$$

Hence proved. □

Theorem 6.1.7. *Let $a \in A$. Then, $\mu \in FC_a(A)$ if and only if for each $m \geq 0$ and each $b_1, b_2, ..., b_m \in A$, if $x \in \overline{\{a_1, ..., a_m\}}^a$, then $\mu(x) \geq \mu(b_1) \wedge ... \wedge \mu(b_m)$.*

Proof. One part of this theorem is proved in the above Lemma. So we proceed to the converse part. Assume the given condition is satisfied for μ. Let us put $S_m = \{b_1, ..., b_m\}$. If we take $m = 0$, then $S_m = \emptyset$ and it is known that $\overline{\emptyset}^a = \{a\}$. So by our assumption, we have

$$\mu(a) \geq \bigwedge_{b \in \emptyset} \mu(b) = 1$$

Thus $\mu(a) = 1$. Let $a_1, ..., a_n, b_1, ..., b_m \in A$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be a coset term in $\overrightarrow{y}$. If we consider the set $S_m = \{b_1, ..., b_m\}$, then one can observe that

$$p(a_1, ..., a_n, b_1, ..., b_m) \in \overline{\{b_1, ..., b_m\}}^a$$

It follows from our assumption that$\mu(p(a_1, ..., a_n, b_1, ..., b_m)) \geq \mu(b_1) \wedge ... \wedge \mu(b_m)$. Therefore $\mu \in FC_a(A)$. Hence proved. □

In the following theorem, we give a more general setting to characterize fuzzy cosets.

Theorem 6.1.8. *Let $a \in A$. Then, $\mu \in FC_a(A)$ if and only if for any subset S of A*

$$\mu(b) \geq \bigwedge_{x \in S} \mu(x) \text{ for all } b \in \overline{S}^a$$

Proof. Suppose that $\mu \in FC_a(A)$. If $S = \emptyset$, then $\overline{S}^a = \{a\}$ and the condition holds trivially. Assume that S is nonempty and let $b \in \overline{S}^a$. Then $b = t(a_1, \ldots, a_n, b_1, \ldots, b_m)$ for some $b_1, \ldots, b_m \in S$, $a_1, \ldots, a_n \in A$ and some coset term $t(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$. Since μ is fuzzy a fuzzy coset, it follows that

$$\mu(b) \geq \mu(b_1) \wedge \ldots \wedge \mu(b_m) \geq \bigwedge_{x \in S} \mu(x)$$

The converse part follows from the above theorem by assuming the condition for finite sets. $\square$

6.2 Fuzzy Cosets Generated by a Fuzzy Set

This section is devoted to characterize fuzzy cosets generated by a fuzzy set.

Theorem 6.2.1. *Let $a \in A$. If $\{\mu_i\}_{i \in \Delta}$ is a family of fuzzy cosets of A determined by a, then $\cap_{i \in \Delta} \mu_i$ is a fuzzy coset of A determined by a.*

This theorem confirms that, for each $a \in A$ and any fuzzy subset λ of A with $\lambda(a) = 1$, always there exists a smallest fuzzy coset determined by a containing λ which we call it the fuzzy coset of A determined by a generated by λ and is denoted by $\overline{\lambda}^a$. Note also that, for a subset X of A and $a \in X$, we denote by $\overline{X}^a$ the coset of generated by X.

Lemma 6.2.2. *Let $a \in A$, $S \subseteq A$ and $a \in S$. Then $\overline{\chi_S}^a = \chi_{\overline{S}^a}$.*

Proof. We show that $\chi_{\overline{S}^a}$ is the smallest fuzzy coset of A determined by a such that $\chi_S \leq \chi_{\overline{S}^a}$. Since $\overline{S}^a$ is a coset of A containing a, it follows from Corollary 6.1.4 that $\chi_{\overline{S}^a}$ is a fuzzy coset of A determined by a. It is also clear that $\chi_S \leq \chi_{\overline{S}^a}$. Let λ be any fuzzy coset of A determined by a such that $\chi_S \leq \lambda$. Then $\lambda(s) = 1$ for all $s \in S$ and hence $\lambda(a) = 1$. Let z be any element in

A. If $z \notin \overline{S}^a$, then $\chi_{\overline{S}^a}(z) = 0 \leq \lambda(z)$. I $z \in \overline{S}^a$, then $z = t(\overrightarrow{a}, \overrightarrow{s})$ for some $\overrightarrow{a} \in A^n$, $\overrightarrow{s} \in S^m$ and some coset term $t(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$. Now consider:

$$\lambda(z) \quad = \quad \lambda(t(a_1, ..., a_n, s_1, ..., s_m)) \geq \lambda(s_1) \wedge \lambda(s_2) \wedge ... \wedge \lambda(s_m) = 1$$

So that $\chi_{\overline{S}^a} \leq \lambda$. Therefore $\chi_{\overline{S}^a} = \overline{\chi_S}^a$. □

For any fuzzy subset λ of A, recall from Theorem 1.2.11 that:

$$\lambda(x) = \bigvee\{\alpha \in L : x \in \lambda_\alpha\}$$

for all $x \in A$. In the following theorem we characterize fuzzy cosets generated by a fuzzy set in terms of their level sets.

Theorem 6.2.3. *Let $a \in A$. For a fuzzy subset λ of A with $\lambda(a) = 1$, let Λ_1^a be a fuzzy subset of A defined by:*

$$\Lambda_1^a(x) = \bigvee\{\alpha \in L : x \in \overline{(\lambda_\alpha)}^a\} \text{ for all } x \in A$$

Then $\Lambda_1^a = \overline{\lambda}^a$.

Proof. We show that Λ_1^a is the smallest fuzzy coset of A determined by a containing λ. Let us first show that Λ_1^a is a fuzzy coset. Since $\lambda(a) = 1$, it is clear that $\Lambda_1^a(a) = 1$. Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$ and $t(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Then consider:

$$\begin{aligned}
(\Lambda_1^a)^m(\overrightarrow{b}) &= \bigwedge\{\Lambda_1^a(b_i) : 1 \leq i \leq m\} \\
&= \bigwedge\{\bigvee\{\alpha_i \in L : b_i \in \overline{(\lambda_\alpha)}^a\} : 1 \leq i \leq m\} \\
&= \bigvee\{\bigwedge\{\alpha_i \in L : 1 \leq i \leq m\} : b_i \in \overline{(\lambda_\alpha)}^a\}
\end{aligned}$$

If we put $\beta = \wedge\{\alpha_i \in L : 1 \leq i \leq m\}$, then we get $\lambda_{\alpha_i} \subseteq \lambda_\beta$ for all $1 \leq i \leq m$. So that

$$\begin{aligned}(\Lambda_1^a)^m(\overrightarrow{b}) &= \bigvee\{\bigwedge\{\alpha_i \in L : 1 \leq i \leq m\} : b_i \in \overline{(\lambda_\alpha)}^a\} \\ &\leq \bigvee\{\beta \in L : \overrightarrow{b} \in b_1, \ldots, b_m \in \overline{(\lambda_\alpha)}^a\} \\ &\leq \bigvee\{\beta \in L : t(\overrightarrow{a}, \overrightarrow{b}) \in \overline{(\lambda_\alpha)}^a\} \\ &= \Lambda_1^a(t(\overrightarrow{a}, \overrightarrow{b}))\end{aligned}$$

Therefore Λ_1^a is a fuzzy coset of A determined by a. It is also clear to see that $\lambda \leq \Lambda_1^a$. Suppose that μ is any other fuzzy coset of A determined by a such that $\lambda \leq \mu$. Then $\overline{(\lambda_\alpha)}^a \subseteq \mu_\alpha$ for all $\alpha \in L$. Now for any $x \in A$ consider:

$$\begin{aligned}\Lambda_1^a(x) &= \bigvee\{\alpha \in L : x \in \overline{(\lambda_\alpha)}^a\} \\ &\leq \bigvee\{\alpha \in L : x \in \mu_\alpha\} \\ &= \mu(x)\end{aligned}$$

Therefore Λ_1^a is the smallest fuzzy coset of A determined by a containing λ. Thus $\Lambda_1^a = \overline{\lambda}^a$. □

Corollary 6.2.4. *Let $a \in A$ and μ a fuzzy subset of A such that $\mu(a) = 1$. Then*

$$\overline{(\mu_\alpha)}^a \subseteq (\overline{\mu}^a)_\alpha \quad \text{for all } \alpha \in L$$

Moreover, if L is a chain and μ is finite valued or equivalently if μ has sup property, then the equality holds.

Theorem 6.2.5. *Let μ be a fuzzy subset of A and $\alpha \in L$:*

$$(\overline{\mu}^a)_\alpha = \bigcup\{\bigcap_{\gamma \in M} \overline{(\mu_\gamma)}^a : M \subseteq L \text{ and } \alpha \leq supM\}$$

Proof. The proof is similar to that of Theorem 2.3.7. □

In the following we give an algebraic characterization of fuzzy cosets generated by fuzzy sets.

Definition 6.2.6. Let $a \in A$. For a fuzzy subset λ of A with $\lambda(a) = 1$, let us define a fuzzy subset Λ_2^a of A as follows:

$\Lambda_2^a(a) = 1$ *and for* $a \neq x \in A$

$$\begin{aligned} \Lambda_2^a(x) &= \bigvee\{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, t(\overrightarrow{a}, \overrightarrow{b}) = x \text{ where} \\ &\quad \overrightarrow{a} \in A^n, t(\overrightarrow{x}, \overrightarrow{y}) \text{ is a coset term in } \overrightarrow{y}\} \end{aligned}$$

Theorem 6.2.7. *Let $a \in A$. For a fuzzy subset λ of A with $\lambda(a) = 1$, we have $\Lambda_2^a = \overline{\lambda}^a$.*

Proof. By Theorem 6.2.3, it is enough to show that $\Lambda_2^a = \Lambda_1^a$. For each $x \neq a$ in A, let us define two sets H_x and G_x as follows:

$$\begin{aligned} H_x &= \{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, t(\overrightarrow{a}, \overrightarrow{b}) = x \\ &\quad \text{where } \overrightarrow{a} \in A^n, t(\overrightarrow{x}, \overrightarrow{y}) \text{ is coset term in } \overrightarrow{y}\} \\ G_x &= \{\alpha \in L : x \in \overline{\lambda_\alpha}^a\} \end{aligned}$$

Clearly both H_x and G_x are subsets of L. Our claim is to see that:

$$\bigvee\{\alpha : \alpha \in H_x\} = \bigvee\{\alpha : \alpha \in G_x\}$$

We first show that $H_x \subseteq G_x$. If $\alpha \in H_x$, then $\alpha = \lambda^m(\overrightarrow{b})$, for some $\overrightarrow{b} \in A^m$, such that $t(\overrightarrow{a}, \overrightarrow{b}) = x$ for some $\overrightarrow{a} \in A^n$ where $t(\overrightarrow{x}, \overrightarrow{y})$ is a coset term in $\overrightarrow{y}$. That is, $\overrightarrow{b} \in (\lambda_\alpha)^m$ and so that $x \in \overline{\lambda_\alpha}^a$. Then $\alpha \in G_x$ and hence $H_x \subseteq G_x$. Thus

$$\bigvee\{\alpha : \alpha \in H_x\} \leq \bigvee\{\alpha : \alpha \in G_x\}$$

To prove the inequality, we show that for each $\alpha \in G_x$, there exists $\beta \in H_x$ such that $\alpha \leq \beta$. Let $\alpha \in G_x$. Then $x \in \overline{\lambda_\alpha}^a$; that is, $x = t(\overrightarrow{a}, \overrightarrow{b})$ for some $\overrightarrow{b} \in (\lambda_\alpha)^m$, and $\overrightarrow{a} \in A^n$ where $t(\overrightarrow{x}, \overrightarrow{y})$

is an ideal term in $\overrightarrow{y}$. If we put $\beta = \lambda^m(\overrightarrow{b})$, then we get $\beta \in H_x$ and $\alpha \leq \beta$. This completes the proof. □

In the following theorem, we give a finite characterization for fuzzy cosets.

Theorem 6.2.8. *Let $a \in A$. For a fuzzy subset λ of A with $\lambda(a) = 1$, let us define a fuzzy subset Λ_3^a of A by:*

$\Lambda_3^a(a) = 1$ and for each $a \neq x \in A$:

$$\Lambda_3^a(x) = \bigvee\{\bigwedge_{y \in F} \lambda(y) : x \in \overline{F}^a, F \subset\subset A\}$$

Then $\Lambda_3^a = \overline{\lambda}^a$.

Proof. It is enough if we show that $\Lambda_3^a = \Lambda_1^a$. For each $a \neq x \in A$, let us take the set G_x as in Theorem 6.2.7 and define a set H_x as follows:

$$H_x \quad = \quad \{\bigwedge_{a \in F} \lambda(a) : x \in x \in \overline{F}^a, F \subset\subset A\}$$

Our claim is to show that:

$$\bigvee\{\alpha : \alpha \in H_x\} = \bigvee\{\alpha : \alpha \in G_x\}$$

We first show that $H_x \subseteq G_x$. $\alpha \in H_x$, implies that $\alpha = \bigwedge_{a \in F} \lambda(a)$ and $x \in \overline{F}^a$, for some finite subset F of A. That is, $a \in \lambda_\alpha$ for all $a \in F$ and $x \in \overline{F}^a$. So that $x \in \overline{\lambda_\alpha}^a$. Then $\alpha \in G_x$ and hence $H_x \subseteq G_x$. Next we show that, for each $\alpha \in G_x$, there exists $\beta \in H_x$ such that $\alpha \leq \beta$. Let $\alpha \in G_x$. Then $x \in \overline{\lambda_\alpha}^a$; that is, $x = t(\overrightarrow{a}, \overrightarrow{b})$ for some $\overrightarrow{b} \in (\lambda_\alpha)^m$, and $\overrightarrow{a} \in A^n$ where $t(\overrightarrow{x}, \overrightarrow{y})$ is a coset term in $\overrightarrow{y}$. Let $\overrightarrow{b} = \langle b_1, b_2, ..., b_m\rangle$ and $\beta = \bigwedge_{i=1}^m \mu(b_i)$. Then $\beta \geq \alpha$. Moreover, if we put $F = \{b_1, b_2, ..., b_m\}$, then F is a finite subset of A such that $x \in \overline{F}^a$. Thus $\beta \in H_x$ such that $\alpha \leq \beta$. This completes the proof. □

Theorem 6.2.9. *Let $a \in A$. Suppose that $\{H_\alpha\}_{\alpha \in L}$ is a subfamily of $C_a(A)$ such that*

$$\bigcap_{\alpha \in M} H_\alpha = H_{supM}$$

for all $M \subseteq L$. Then, there is a unique fuzzy coset μ of A determined by μ for which $\mu_\alpha = H_\alpha$ for all $\alpha \in L$.

Proof. The proof is similar to that of Theorem 2.3.12 □

6.3 The Lattice of Fuzzy Cosets

As observed in Theorem 6.2.1, for each $a \in A$ the class $FC_a(A)$ is closed under arbitrary intersection of fuzzy sets. So that $(FC_a(A), \leq)$ forms a closure fuzzy set system and hence by Theorem 1.2.16 it is a complete lattice, where $\leq$ is a pointwise ordering of fuzzy sets. The following theorem summarizes this.

Theorem 6.3.1. *Let $a \in A$. Then the set $FC_a(A)$ of all fuzzy cosets of A determined by a forms a complete lattice where the infimum and supremum of any sub-family $\{\mu_i : i \in \Delta\}$ of $FC_a(A)$ is given by:*

$$\bigwedge \mu_i = \cap \mu_i \text{ and } \bigvee \mu_i = \overline{(\cup \mu_i)}^a$$

The least and the largest elements in $FC_a(A)$ are $\chi_{\{a\}}$ and 1_A respectively.

Theorem 6.3.2. *$(FC_a(A), \leq)$ is an algebraic closure fuzzy set system.*

Proof. By Definition 1.2.18, it is enough to show that $FC_a(A)$ is inductive in L^A. Let $\{\mu_i\}_{i \in \Delta}$ be a chain in $FC_a(A)$. Let us put

$$\eta = \bigcup_{i \in \Delta} \mu_i$$

We show that η is a fuzzy coset of A determined by a. Clearly $\eta(a) = 1$. Let $a_1, ..., a_n, b_1, ..., b_m \in A$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be a coset term in $\overrightarrow{y}$. First observe that, for each $m-$tuples $i_1, ..., i_m \in \Delta$, there exists $k \in \{1,2,...,m\}$ such that $\mu_{i_j} \leq \mu_{i_k}$ for all $j \in \{1,2,...,m\}$. Now consider the following:

$$\begin{aligned}
\eta(b_1) \wedge ... \wedge \eta(b_m) &= \left(\bigvee_{i_1 \in \Delta} \mu_{i_1}(b_1)\right) \wedge ... \wedge \left(\bigvee_{i_m \in \Delta} \mu_{i_m}(b_m)\right) \\
&= \bigvee_{i_1,...,i_m \in \Delta} (\mu_{i_1}(b_1) \wedge ... \wedge \mu_{i_m}(b_m)) \\
&\leq \bigvee_{i_k \in \Delta} (\mu_{i_k}(b_1) \wedge ... \wedge \mu_{i_k}(b_m)) \\
&\leq \bigvee_{i_k \in \Delta} \mu_{i_k}(p(a_1,...,a_n,b_1,...,b_m)) \\
&= \eta(p(a_1,...,a_n,b_1,...,b_m))
\end{aligned}$$

Therefore η is a fuzzy coset of A determined by a and this completes the proof. □

6.4 Fuzzy Cosets and Fuzzy Congruences

Remember that, for each $a \in A$, $FL_a(A)$ denotes the set

$$FL_a(A) = \{a/\Theta : \Theta \in FCon(A)\}$$

The following lemma shows that fuzzy congruence classes are fuzzy cosets.

Lemma 6.4.1. *For each $a \in A$,*

$$FL_a(A) \subseteq FC_a(A)$$

Proof. Let $\Theta \in FCon(A)$. We show that $a/\Theta \in FC_a(A)$. Clearly $a/\Theta(a) = 1$. Let $a_1,...,a_n,b_1,...,b_m \in A$ and $t(\overrightarrow{x},\overrightarrow{y})$ be a coset term in $\overrightarrow{y}$. Consider:

$$\begin{aligned}
\Theta_a(b_1) \wedge ... \wedge \Theta_a(b_m) &= \Theta(a,b_1) \wedge ... \wedge \Theta(a,b_m) \\
&\leq \Theta(t(a_1,...,a_n,a,a,...,a),t(a_1,...,a_n,b_1,...,b_m)) \\
&= \Theta(a,t(a_1,...,a_n,b_1,...,b_m)) \\
&= a/\Theta(t(a_1,...,a_n,b_1,...,b_m))
\end{aligned}$$

Therefore $a/\Theta \in FC_a(A)$. □

Theorem 6.4.2. *Let $a \in A$. If $\Theta \in FCon(A)$ and $\mu \in FC_a(A)$, then $\mu/\Theta \in FC_a(A)$.*

Proof. Since $\mu/\Theta \in FC_a(A)$, we have $\mu(a) = 1$. Now consider

$$\begin{aligned}\mu/\Theta(a) &= \bigvee\{\mu(b) \wedge \Theta(a,b) : b \in A\}\\ &\geq \mu(a) \wedge \Theta(a,a)\\ &= 1\end{aligned}$$

Thus $\mu/\Theta(a) = 1$. Let $t(\overrightarrow{x}, \overrightarrow{y})$ be a coset term in $\overrightarrow{y}$ and $a_1, ..., a_n, b_1, ..., b_m \in A$. Then, the proof is similar to that of Theorem 7.2.1 to show that

$$\mu/\Theta(t(a_1, ..., a_n, b_1, ..., b_m)) \geq \mu/\Theta(b_1) \wedge ... \wedge \mu/\Theta(b_m)$$

□

Lemma 6.4.3. *Let $a \in A$. For each $\Theta, \Phi \in FCon(A)$ we have:*

$$(a/\Theta)/\Phi = a/(\Theta \circ \Phi)$$

Proof. For any $x \in A$, consider the following:

$$\begin{aligned}(a/\Theta)/\Phi(x) &= \bigvee\{(a/\Theta)(y) \wedge \Phi(x,y) : y \in A\}\\ &= \bigvee\{\Theta(a,y) \wedge \Phi(x,y) : y \in A\}\\ &= \Theta \circ \Phi(a,x)\\ &= a/(\Theta \circ \Phi)(x)\end{aligned}$$

Thus, $(a/\Theta)/\Phi = a/(\Theta \circ \Phi)$. □

Lemma 6.4.4. *Let $a \in A$. For each $\Theta, \Phi \in FCon(A)$ we have:*

$$a/\Theta \bigvee_a a/\Phi \leq (a/\Theta)/\Phi \leq a/(\Theta \vee \Phi)$$

where $\bigvee_a$ denotes the supremum of fuzzy cosets in $FC_a(A)$.

Proof. To prove the first inequality let us put $\lambda = a/\Theta \cup a/\Phi$. Then λ is a fuzzy subset of A (not necessarily a fuzzy coset) such that

$$a/\Theta \bigvee_a a/\Phi = \overline{\lambda}^a$$

Let $x \in A$. Then by Theorem 6.2.7

$$\overline{\lambda}^a(x) = \bigvee\{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, t(\overrightarrow{a}, \overrightarrow{b}) = x \text{ where } \overrightarrow{a} \in A^n, t(\overrightarrow{x}, \overrightarrow{y}) \text{ is a coset term in } \overrightarrow{y}\}$$

Again from Lemma 6.4.3, we have

$$(a/\Theta)/\Phi(x) = \bigvee\{\Theta(a,y) \wedge \Phi(x,y) : y \in A\}$$

Let us define two sets H_x and G_x as follows:

$$H_x = \{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, t(\overrightarrow{a}, \overrightarrow{b}) = x \text{ where } \overrightarrow{a} \in A^n, t(\overrightarrow{x}, \overrightarrow{y}) \text{ is a coset term in } \overrightarrow{y}\}$$
$$G_x = \{\Theta(a,y) \wedge \Phi(x,y) : y \in A\}$$

Then both H_x and G_x are nonempty subsets of L. Our aim is to show that $\bigvee H_x \leq \bigvee G_x$ for all $x \in A$. Let $\alpha \in H_x$. Then $\alpha = \lambda^m(\overrightarrow{b})$, for some $\overrightarrow{b} \in A^m$, such that $t(\overrightarrow{a}, \overrightarrow{b}) = x$ for some $\overrightarrow{a} \in A^n$ where $t(\overrightarrow{x}, \overrightarrow{y})$ is a coset term in $\overrightarrow{y}$. That is, $b_1, \ldots, b_m \in \lambda_\alpha = (a/\Theta)_\alpha \cup (a/\Phi)_\alpha$. Without loss of generality, we can assume that $b_1, \ldots, b_k \in (a/\Theta)_\alpha$ and $b_{k+1}, \ldots, b_m \in (a/\Phi)_\alpha$. So that

$$\Theta(a,b_1) \wedge \ldots \wedge \Theta(a,b_k) \geq \alpha \text{ and } \Phi(a,b_{k+1}) \wedge \ldots \wedge \Phi(a,b_m) \geq \alpha$$

We have $x = t(\overrightarrow{a}, b_1, ..., b_k, b_{k+1}, ..., b_m)$ and since $t(\overrightarrow{x}, \overrightarrow{y})$ is a coset term in $\overrightarrow{y}$, we get $a = t(\overrightarrow{a}, a, ..., a, a, ..., a)$. Put $y = t(\overrightarrow{a}, b_1, ..., b_k, a, a, ..., a)$ and consider the following:

$$\begin{aligned} \Theta(a,y) &= \Theta(t(\overrightarrow{a}, a, ..., a, a, ..., a), t(\overrightarrow{a}, b_1, ..., b_k, a, a, ..., a)) \\ &\geq \Theta(a, b_1) \wedge ... \wedge \Theta(a, b_k) \\ &\geq \alpha \end{aligned}$$

and

$$\begin{aligned} \Phi(x,y) &= \Phi(t(\overrightarrow{a}, b_1, ..., b_k, b_{k+1}, ..., b_m), t(\overrightarrow{a}, b_1, ..., b_k, a, a, ..., a)) \\ &\geq \Phi(b_{k+1}, a) \wedge ... \wedge \Phi(b_m, a) \\ &\geq \alpha \end{aligned}$$

If we put $\beta = \Theta(a,y) \wedge \Phi(x,y)$, then $\beta \in G_x$ such that $\alpha \leq \beta$, which gives that $\bigvee H_x \leq \bigvee G_x$. Therefore

$$a/\Theta \bigvee_a a/\Phi \leq (a/\Theta)/\Phi$$

The last inequality follows from the fact that $\Theta \circ \Phi \leq \Theta \vee \Phi$ and this completes the proof. □

6.5 Characterizing Congruence Permutable Varieties

Definition 6.5.1. A class $\mathscr{K}$ of algebras is called congruence permutable if the following holds for each $A \in \mathscr{K}$ and each $\theta, \phi \in Con(A)$:

$$\theta \circ \phi = \phi \circ \theta$$

In the following theorem we give an equivalent condition for a variety $\mathscr{K}$ of algebras to be congruence permutable.

Theorem 6.5.2. *A class $\mathscr{K}$ of algebras is congruence permutable if and only if*

$$\Theta \circ \Phi = \Phi \circ \Theta$$

for each $A \in \mathscr{K}$ and all $\Theta, \Phi \in FCon(A)$.

Proof. Suppose that $\mathscr{K}$ is congruence permutable. Let $\Theta, \Phi \in FCon(A)$. For any $x, y \in A$, let us define two sets $H_{x,y}$ and $G_{x,y}$ as follows:

$$\begin{aligned} H_{x,y} &= \{\Theta(x,z) \wedge \Phi(y,z) : z \in A\} \\ G_{x,y} &= \{\Theta(y,z) \wedge \Phi(x,z) : z \in A\} \end{aligned}$$

Then both $H_{x,y}$ and $G_{x,y}$ are nonempty subsets of L. Our claim is to show that

$$\bigvee H_{x,y} = \bigvee G_{x,y} \text{ for all } x, y \in A$$

Let $\alpha \in H_{x,y}$. Then $\alpha = \Theta(x,z) \wedge \Phi(y,z)$ for some $z \in A$. So that $\Theta(x,z) \geq \alpha$ and $\Phi(y,z) \geq \alpha$; i.e., $(x,z) \in \Theta_\alpha$, $(y,z) \in \Phi_\alpha$ and both Θ_α and Φ_α are congruence relations on A, which gives that $(x,y) \in \Phi_\alpha \circ \Theta_\alpha$. Since $\mathscr{K}$ is congruence permutable, we get $(x,y) \in \Theta_\alpha \circ \phi_\alpha$. So there exists some $u \in A$ such that $(x,u) \in \phi_\alpha$ and $(y,u) \in \Theta_\alpha$; that is,

$$\Theta(y,u) \wedge \Phi(x,u) \geq \alpha$$

Thus $\bigvee G_{x,y} \geq \alpha$. Since α is arbitrary in $H_{x,y}$, we get $\bigvee G_{x,y} \geq \alpha$ for all $\alpha \in H_{x,y}$ and hence

$$\bigvee G_{x,y} \geq \bigvee H_{x,y}$$

Similarly, by interchanging Θ and Φ we can show that

$$\bigvee G_{x,y} \leq \bigvee H_{x,y}$$

Therefore the equality holds. The converse part of this theorem follows from the fact that every congruence relation on A can be identified by as a fuzzy congruence on A by its characteristic mapping. □

Corollary 6.5.3. *A class $\mathscr{K}$ of algebras is congruence permutable if and only if*

$$\Theta \circ \Phi = \Theta \vee \Phi$$

for each $A \in \mathscr{K}$ and all $\Theta, \Phi \in FCon(A)$.

Theorem 6.5.4. *A class $\mathscr{K}$ of algebras is congruence permutable if and only if for each $A \in \mathscr{K}$ and each $a \in A$, the map $f_a : FCon(A) \to FC_a(A)$ defined by:*

$$f_a(\Theta) = a/\Theta$$

is a lattice homomorphism.

Proof. Suppose that $\mathscr{K}$ is congruence permutable. For any $\Theta, \Phi \in FCon(A)$, it is clear that

$$a/(\Theta \wedge \Phi) = a/\Theta \wedge a/\Phi$$

So it is enough to show that $a/(\Theta \vee \Phi) = (a/\Theta) \bigvee_a (a/\Phi)$. One inequality is given in Lemma 6.4.4. To prove the other inequality, let $x \in A$. Since $\mathscr{K}$ is congruence permutable, it follows from Corollary 6.5.3 that $\Theta \circ \Phi = \Theta \vee \Phi$. Then we have

$$\begin{aligned} a/(\Theta \vee \Phi)(x) &= \Theta \vee \Phi(a,x) \\ &= \Theta \circ \Phi(a,x) \\ &= \bigvee \{\Theta(a,y) \wedge \Phi(x,y) : y \in A\} \end{aligned}$$

For an arbitrary y in A, let us put

$$\alpha = \Theta(a,y) \wedge \Phi(x,y)$$

So we have $a/\Theta(y) \geq \alpha$ and $\Phi(x,y) \geq \alpha$. Since, by our assumption, $\mathscr{K}$ is congruence permutable, it has a Mal'cev term $p(u,v,w)$ such that the following equations are valid in $\mathscr{K}$

$$p(u,u,w) \approx w \approx p(w,u,u)$$

Let us put $b_1 = a, b_2 = y$ and $b_3 = p(a,y,x)$. Since we can write a as $a = p(a,x,x)$ we have the following:

$$\begin{aligned}\Phi(a,b_3) &= \Phi(p(a,x,x),p(a,y,x)) \\ &\geq \Phi(x,y) \\ &\geq \alpha\end{aligned}$$

That is $a/\Phi(b_3) \geq \alpha$. If we put, for simplicity, $\mu = (a/\Theta)\bigvee_a(a/\Phi)$, then μ is a fuzzy coset of A determined by a such that

$$\mu(b_1) \wedge \mu(b_2) \wedge \mu(b_3) \geq \alpha$$

Let us define a $(2+3)-$ary term $t(\overrightarrow{x},\overrightarrow{y})$ by:

$$t(x_1,x_2,y_1,y_2,y_3) = p(p(x_1,y_1,x_2),p(x_1,y_2,x_2),y_3)$$

Then $t(x_1,x_2,y_1,y_2,y_3)$ is a coset term in y_1,y_2,y_3. To verify this consider

$$t(x_1,x_2,y,y,y) = p(p(x_1,y,x_2),p(x_1,y,x_2),y) = y$$

Moreover, consider the following

$$\begin{aligned}t(a,x,b_1,b_2,b_3) &= t(a,x,a,y,p(a,y,x)) \\ &= p(p(a,a,x),p(a,y,x),p(a,y,x)) \\ &= p(a,a,x) \\ &= x\end{aligned}$$

Since μ is a fuzzy coset, we get

$$\mu(x) \geq \mu(b_1) \wedge \mu(b_2) \wedge \mu(b_3) \geq \alpha$$

Since $\alpha = \Theta(a,y) \wedge \Phi(x,y)$ and y is arbitrary in A, we can conclude that

$$\mu(x) \geq \bigvee \{\Theta(a,y) \wedge \Phi(x,y) : y \in A\}$$

Therefore $a/(\Theta \vee \Phi) \leq (a/\Theta) \bigvee_a (a/\Phi)$ and hence the equality holds. Conversely suppose that the map $f_a : FCon(A) \rightarrow FC_a(A)$ defined by:

$$f_a(\Theta) = a/\Theta$$

is a lattice homomorphism. Then it follows from Lemma 6.4.3 and 6.4.4 that

$$a/(\Theta \vee \Phi) = a/(\Theta \circ \Phi) = a/\Theta \bigvee_a a/\Phi$$

for all $a \in A$, $\Theta, \Phi \in FCon(A)$. For any $a, b \in A$, consider

$$\begin{aligned} \Theta \vee \Phi(a,b) &= a/(\Theta \vee \Phi)(b) \\ &= a/(\Theta \circ \Phi)(b) \\ &= \Theta \circ \Phi(a,b) \end{aligned}$$

Therefore by Corollary 6.5.3, $\mathscr{K}$ is congruence permutable. □

Theorem 6.5.5. *The following are equivalent:*

1. $FL_a(A) = FC_a(A)$ *for all* $a \in A$.

2. $FL_a(A)$ *is a dual ideal of* $FC_a(A)$.

Proof. It is known that every lattice is a dual ideal of it self. This makes trivial the proof of $(1) \Rightarrow (2)$. To prove $(2) \Rightarrow (1)$, assume that $FL_a(A)$ is a dual ideal of $FC_a(A)$. The inclusion

$FL_a(A) \subseteq FC_a(A)$ is proved in Lemma 6.4.1. To prove the other inclusion let $\mu \in FC_a(A)$. Consider the the zero fuzzy congruence 0_A on A defined by:

$$0_A(x,y) = \begin{cases} 1 & if\, x = y \\ 0 & otherwise \end{cases}$$

for all $x, y \in A$. Since $\mu(a) = 1$, we have $a/0_A \leq \mu$; that is, $a/0_A \in FL_a(A)$ and $\mu \in FC_a(A)$ such that $a/0_A \leq \mu$. Since $FL_a(A)$ is a dual ideal of $FC_a(A)$, we get $\mu \in FL_a(A)$. So $FC_a(A) \subseteq FL_a(A)$ and hence the equality holds. □

Theorem 6.5.6. *If $\mathscr{K}$ is congruence permutable, then*

$$FL_a(A) = FC_a(A)$$

Proof. It is enough to show that $FC_a(A) \subseteq FL_a(A)$. Let $\mu \in FC_a(A)$.

Claim. For each unary algebraic polynomial $f(x)$ on A, we show that

$$\mu(x) \wedge \mu(y) \wedge \mu(f(x)) = \mu(x) \wedge \mu(y) \wedge \mu(f(y))$$

for all $x, y \in A$. Put

$$\alpha = \mu(x) \wedge \mu(y) \wedge \mu(f(x))$$

Then $\alpha \leq \mu(x) \wedge \mu(y)$ and $\alpha \leq \mu(f(x))$. Let $a_1, a_2, ..., a_n \in A$ and $t(x_1, x_2, .., x_n, y)$ be a term on A such that $f(z) = t(a_1, a_2, .., a_n, z)$ for all $z \in A$. Since $\mathscr{K}$ is congruence permutable, it has a Mal'cev term p as given in Theorem 6.5.4. Now let us define a term $r(x_1, x_2, .., x_n, y_1, y_2, y_3)$ by:

$$r(x_1, x_2, .., x_n, y_1, y_2, y_3) = p(y_1, t(x_1, x_2, .., x_n, y_2), t(x_1, x_2, .., x_n, y_3))$$

Then it can be verified that $r(\overrightarrow{x}, y_1, y_2, y_3)$ is a coset term in y_1, y_2, y_3. Moreover,

$$\begin{aligned} r(a_1, a_2, ..., a_n, f(x), x, y) &= p(f(x), p(t(a_1, a_2, ..., a_n, x)), p(t(a_1, a_2, ..., a_n, y))) \\ &= p(f(x), f(x), f(y)) \\ &= f(y) \end{aligned}$$

Being μ a fuzzy coset and $r(\overrightarrow{x}, y_1, y_2, y_3)$ a coset term in y_1, y_2, y_3, we have the following:

$$\begin{aligned} \mu(f(y)) &= \mu(r(a_1, a_2, ..., a_n, f(x), x, y)) \\ &\geq \mu(f(x)) \wedge \mu(x) \wedge \mu(y) \\ &= \alpha \end{aligned}$$

So that $\mu(f(y)) \geq \alpha$. Since $\mu(x) \wedge \mu(y) \geq \alpha$, we get $\mu(f(y)) \wedge \mu(x) \wedge \mu(y) \geq \alpha$; that is,

$$\mu(f(y)) \wedge \mu(x) \wedge \mu(y) \geq \mu(f(x)) \wedge \mu(x) \wedge \mu(y)$$

The other inequality can be proved by interchanging x and y, and hence the equality holds. By Lemma 5.2.2, $\mu \in FL_a(A)$ and this completes the proof. □

Theorem 6.5.7. *If for each $\Theta, \Phi \in FCon(A)$ there exists $\Psi \in FCon(A)$ such that*

$$a/\Theta \bigvee_a a/\Phi = a/\Psi$$

for all $a \in A$, then $\mathscr{K}$ is congruence permutable.

Proof. Let $\Theta, \Phi \in FCon(A)$. Then there exists $\Psi \in FCon(A)$ such that

$$a/\Theta \bigvee_a a/\Phi = a/\Psi$$

for all $a \in A$. Our aim is to show that $\Psi = \Theta \circ \Phi$. For any $a, b \in A$ consider:

$$\begin{aligned}
\Theta \circ \Phi(x,y) &= \bigvee\{\Theta(x,a) \wedge \Phi(y,a) : a \in A\} \\
&= \bigvee\{a/\Theta(x) \wedge a/\Phi(y) : a \in A\} \\
&\leq \bigvee\{a/\Psi(x) \wedge a/\Psi(y) : a \in A\} \\
&= \bigvee\{\Psi(x,a) \wedge \Psi(y,a) : a \in A\} \\
&\leq \Psi(x,y)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Psi(x,y) &= y/\Psi(x) \\
&= (y/\Theta \bigvee_y y/\Phi)(x) \\
&\leq y/(\Theta \circ \Phi)(x) \quad \textit{(By Lemma 6.4.3 and 6.4.4)} \\
&= \Theta \circ \Phi(x,y)
\end{aligned}$$

Therefore $\Psi = \Theta \circ \Phi$. Since $a/\Theta \bigvee_a a/\Phi = a/\Phi \bigvee_a a/\Theta$, we get $\Theta \circ \Phi = \Phi \circ \Theta$. Thus, by Theorem 6.5.2, $\mathscr{K}$ is congruence permutable. □

Chapter 7

$L-$Fuzzy Ideals and $L-$Fuzzy Congruences

Introduction

Congruences turn out to be useful in order to construct a new, so-called quotient algebra, from a given one. This construction is an algebraic counterpart to a situation from real life known in science by the term "abstraction". In this process we neglect those properties of a given object which cannot be distinguished by a congruence and we form a new and rough structure having only those properties which have their origin in the structure of congruence classes. Fuzzy congruence relations and specifically fuzzy ideals are also important to construct quotient algebras analogous to crisp congruences. The main purpose of this chapter is to study quotient algebras induced by fuzzy ideals in ideal determined varieties.

In the first section, we deal with fuzzy congruence kernels which are fuzzy congruence classes determined by 0. It is observed that fuzzy congruence kernels are fuzzy ideals. But the converse does not holds in general. Section 2 is devoted to the study of those class of algebras in which every fuzzy ideal is a class of a unique fuzzy congruence relation. Finally, in section 3, we study the structure of quotient algebras induced by fuzzy ideals in ideal determined varieties.

7.1 Fuzzy Congruence Kernels

In some special cases it is not possible to characterize every class of every fuzzy congruence on A but sometimes it is possible to characterize the kernel (the so-called fuzzy congruence kernel).

Definition 7.1.1. An algebra with 0 is an algebra with a constant unary term 0. A variety with 0 is a variety with a constant unary term (equationally definable constant) 0.

This constant is usually denoted by 0. Sometimes it may also be denoted by 1 or another symbol.

Definition 7.1.2. Let A be an algebra with 0 and Θ a fuzzy equivelence relation on A. Then $0/\Theta$ is called the kernel of Θ. A fuzzy subset μ of A is called a fuzzy congruence kernel if it is the kernel of some fuzzy congruence Θ on A.

Lemma 7.1.3. *Every fuzzy congruence kernel is fuzzy ideal.*

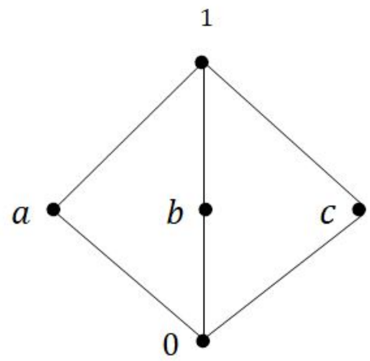
Proof. Let $\Theta \in FCon(A)$. Clearly $0/\Theta(0) = 1$. Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Then consider:

$$\begin{aligned}
0/\Theta(p(\overrightarrow{a}, \overrightarrow{b})) &= \Theta(0, p(\overrightarrow{a}, \overrightarrow{b})) \\
&= \Theta(p(\overrightarrow{a}, \overrightarrow{0}), p(\overrightarrow{a}, \overrightarrow{b})) \\
&= \Theta(p(a_1, a_2, ..., a_n, 0, 0, ..., 0), p(a_1, a_2, ..., a_n, b_1, b_2, ..., b_m) \\
&\geq \Theta(a_1, a_1) \wedge ... \wedge \Theta(a_n, a_n) \wedge \Theta(0, b_1) \wedge ... \wedge \Theta(0, b_m) \\
&= \Theta(0, b_1) \wedge ... \wedge \Theta(0, b_m) \\
&= 0/\Theta(b_1) \wedge ... \wedge 0/\Theta(b_m)
\end{aligned}$$

Therefore $0/\Theta$ is a fuzzy ideal of A. □

Remark. But every fuzzy ideal is not in general the kernel of some fuzzy congruence relation. This is verified in the following example.

Example 7.1.4. Let $A = \{0, a, b, c, 1\}$ be the lattice given in the following diagram:



Let L be the real interval $[0,1]$ and μ be an $L-$fuzzy subset of A defined by:

$$\mu(0) = 1, \mu(a) = 0.7 \text{ and } \mu(b) = \mu(c) = \mu(1) = 0.4$$

Then μ is a fuzzy ideal of A. Suppose if possible that μ is a kernel of some fuzzy congruence Θ on A, i.e., $\mu = 0/\Theta$. Now consider the following:

$$\begin{aligned} \mu(c) &= \Theta(0,c) \\ &\geq \Theta(1,b) \\ &\geq \Theta(0,a) \\ &= \mu(a) \end{aligned}$$

This is a contradiction. Thus μ is not a kernel of any fuzzy congruence on A.

Definition 7.1.5. An algebra with 0 is called permutable at 0 if $(\theta \circ \phi)[0] = (\phi \circ \theta)[0]$ for each of its congruences θ, ϕ. A class of algebras with 0 is called permutable at 0 if each of its members has this property.

It can be easily verified that A is permutable at 0 if and only if $0/(\Theta \circ \Phi) = 0/(\Phi \circ \Theta)$ for each of its fuzzy congruences Θ, Φ.

The following characterization was developed by H.-P. Gumm and A. Ursini [79] :

Theorem 7.1.6. *A variety with 0 is permutable at* 0 *if and only if there exists a binary term t with*

$$t(x,x) = 0 \text{ and } t(x,0) = x$$

Definition 7.1.7. The term t occurring in Theorem 7.1.6 is called subtractive term (or the difference term).

For any nonnegative integer n and every $s \in T_{n+1}$ let w_s denote the $(n+3)-$ary term defined by

$$w_s(x_1,\ldots,x_{n+3}) = t(s(x_{n+1},x_1,\ldots,x_n),t(s(x_{n+2},x_1,\ldots,x_n),x_{n+3}))$$

Theorem 7.1.8. *Let $\mathscr{K}$ be a permutable at* 0 *variety, $A \in \mathscr{K}$. A fuzzy subset μ of A is a fuzzy congruence kernel if and only if the following conditions hold:*

1. $\mu(0) = 1$ *and*
2. *for every $s \in T_{n+1}$, μ is $\overrightarrow{y}-$closed under the term $w_s(x_1,\ldots,x_n,y_1,y_2,y_3)$, i.e.,*

$$\mu(w_s(x_1,\ldots,x_n,y_1,y_2,y_3)) \geq \mu(y_1) \wedge \mu(y_2) \wedge \mu(y_3)$$

Proof. Suppose that $\mu = 0/\Theta$ for some $\Theta \in FCon(A)$. Clearly, $\mu(0) = 1$ (proving (1)). First observe that

$$\begin{aligned} w_s(x_1,\ldots,x_n,0,0,0) &= t(s(0,x_1,\ldots,x_n),t(s(0,x_1,\ldots,x_n),0)) \\ &= t(s(0,x_1,\ldots,x_n),s(0,x_1,\ldots,x_n)) \\ &= 0 \end{aligned}$$

Now consider the following:

$$\begin{aligned}
\mu((w_s(x_1,...,x_n,y_1,y_2,y_3))) &= \Theta(0,w_s(x_1,...,x_n,y_1,y_2,y_3))) \\
&= \Theta(w_s(x_1,...,x_n,0,0,0),w_s(x_1,...,x_n,y_1,y_2,y_3)) \\
&\geq \Theta(x_1,x_1)\wedge ...\wedge \Theta(x_n,x_n)\wedge \Theta(0,y_1)\wedge \Theta(0,y_2)\wedge \Theta(0,y_3) \\
&= \mu(y_1)\wedge \mu(y_2)\wedge \mu(y_3)
\end{aligned}$$

proving (2). Conversely, suppose that μ satisfies the conditions (1) and (2). We show that μ satisfies the condition of Lemma 5.2.2. Let $a,b\in A$ and $p\in P_1(A)$. Then there exist an $n\in Z^+, s\in T_{n+1}$ and $a_1,...,a_n\in A$ such that

$$p(x) = s(x,a_1,...,a_n) \text{ for all } x\in A$$

By the property of the difference term t, we have the following:

$$\begin{aligned}
p(b) &= t(p(b),0) \\
&= t(p(b),t(p(a),p(a))) \\
&= t(s(b,a_1,...,a_n),t(s(a,a_1,...,a_n),p(a))) \\
&= w_s(a_1,...,a_n,b,a,p(a))
\end{aligned}$$

By (2), it follows that

$$\mu(p(b))\geq \mu(a)\wedge \mu(b)\wedge \mu(p(a))$$

which implies that

$$\mu(a)\wedge \mu(b)\wedge \mu(p(b))\geq \mu(a)\wedge \mu(b)\wedge \mu(p(a))$$

Similarly, by interchanging a and b, we get

$$\mu(a) \wedge \mu(b) \wedge \mu(p(a)) \geq \mu(a) \wedge \mu(b) \wedge \mu(p(b))$$

Hence the equality holds. By Lemma 5.2.2 , μ is a class of some fuzzy congruence Θ on A and since $\mu(0) = 1$ it holds that

$$\mu = 0/\Theta$$

Hence proved. □

Lemma 7.1.9. *In a permutable at 0 variety, every fuzzy ideal is a kernel of some fuzzy congruence relation.*

Theorem 7.1.10. *Let $\mathscr{K}$ be a permutable variety with 0 and μ a fuzzy subset of A with $\mu(0) = 1$. Define a fuzzy relation Θ_μ of A by:*

$$\Theta_\mu(x,y) = \mu(p(x,y,0))$$

for all $x, y \in A$, where p is the Mal'cev term.

1. *$\Theta_\mu \in FCon(A)$ if and only if for every $m \in Z^+, f \in \mathfrak{F}_m$ and all $a_1, ..., a_m, b_1, ..., b_m \in A$, it holds that:*

$$\mu(f^A(a_1,...,a_m), f^A(b_1,...,b_m)) \geq \mu(p(a_1,b_1,0)) \wedge ... \wedge \mu(p(a_m,b_m,0))$$

2. *If $\Theta_\mu \in FCon(A)$, then it is the largest fuzzy congruence on A with kernel μ.*

Proof. 1. If $\Theta_\mu \in FCon(A)$, then by the substitution property of Θ_μ the condition holds. If, conversely, μ sitisfies the condition of the theorem, the Θ_μ compatible. It is also clear that Θ_μ is reflexive, i.e., Θ_μ is a reflexive and compatible fuzzy relation on A. Since $\mathscr{K}$ is permutable, it follows from Theorem 5.3.6 that $\Theta_\mu \in FCon(A)$.

2. Suppose that $\Theta_\mu \in FCon(A)$. First for each $x \in A$ observe that,

$$\begin{aligned}\mu(x) &= \mu(p(x,0,0)) \\ &= \Theta_\mu(x,0) \\ &= 0/\Theta_\mu(x)\end{aligned}$$

So that μ is the kernel of Θ_μ. Now let Φ be any other fuzzy congruence on A with kernel μ, i.e., $\mu = 0/\Phi$. For any $a,b \in A$

$$\begin{aligned}\Theta_\mu(a,b) &= \mu(p(a,b,0)) \\ &= 0/\Phi(p(a,b,0)) \\ &= \Phi(0,p(a,b,0)) \\ &= \Phi(p(a,a,0),p(a,b,0)) \\ &\geq \Phi(a,b)\end{aligned}$$

Thus Θ_μ is the largest fuzzy congruence on A with kernel μ.

□

Example 7.1.11. In the variety of pseudo-complemented semi lattices, fuzzy congruence kernels are characterized in [28].

7.2 Ideal Determined Varieties

In groups (resp. rings) it is well known that congruence relations are in one-to-one correspondence with normal subgroups (resp. ideals). Where as, this correspondence does not holds in lattices. it is proved by M. Samhan in [137] that there is a one-to-one correspondence between fuzzy normal subgroups (resp. fuzzy ideals) and fuzzy congruences of a group (resp. a ring). In this section, we study those class of algebras for which fuzzy ideals and fuzzy congruence relations are in one-to-one correspondence. Such a class of algebras is called ideal determined variety.

Theorem 7.2.1. *If μ is a fuzzy ideal of A and Θ a fuzzy congruence on A, then μ/Θ is a fuzzy ideal of A, where μ/Θ is as given in Definition 5.2.1.*

Proof. Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$ and $p(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Consider

$$\begin{aligned}
(\mu/\Theta)^m(\overrightarrow{b}) &= \bigwedge_{i=1}^{m} \mu/\Theta(b_i) \\
&= \bigwedge_{i=1}^{m} \bigvee\{\mu(x_i) \wedge \Theta(x_i, b_i) : x_i \in A\} \\
&= \bigvee\{\left(\bigwedge_{i=1}^{m} \mu(x_i)\right) \wedge \left(\bigwedge_{i=1}^{m} \Theta(x_i, b_i)\right) : x_1, \ldots, x_m \in A\} \\
&= \bigvee\{(\mu^m(\overrightarrow{x})) \wedge \left(\bigwedge_{i=1}^{m} \Theta(x_i, b_i)\right) : x_1, \ldots, x_m \in A\} \\
&\leq \bigvee\{\mu(p(\overrightarrow{a}, \overrightarrow{x})) \wedge \Theta(p(\overrightarrow{a}, \overrightarrow{x}), p(\overrightarrow{a}, \overrightarrow{b})) : \overrightarrow{x} \in A^m\} \\
&\leq \bigvee\{\mu(y) \wedge \Theta(y, p(\overrightarrow{a}, \overrightarrow{b})) : y \in A\} \\
&= \mu/\Theta(p(\overrightarrow{a}, \overrightarrow{b}))
\end{aligned}$$

Therefore μ/Θ is a fuzzy ideal of A. □

Definition 7.2.2. [79] A class $\mathscr{K}$ of algebras is called an ideal determined if every ideal I is the zero congruence class of a unique congruence relation denoted by I^δ. In this case the map $I \mapsto I^\delta$ defines an isomorphism between the lattice of ideals and congruences on A.

Theorem 7.2.3. *A class $\mathscr{K}$ of algebras is an ideal determined if and only if every fuzzy ideal μ is the zero fuzzy congruence class of a unique fuzzy congruence relation denoted by Θ^μ.*

Proof. Suppose that $\mathscr{K}$ is an ideal determined variety. Let μ be any fuzzy ideal of A. Then μ_α is an ideal of A for all $\alpha \in L$; that is, for each $\alpha \in L$, there is a unique congruence relation on A denoted by $(\mu_\alpha)^\delta$ for which μ_α is its zero congruence class. Now define a fuzzy relation Θ^μ on A as follows:

$$\Theta^\mu(x, y) = \bigvee\{\alpha \in L : (x, y) \in (\mu_\alpha)^\delta\}$$

for all $x,y \in A$. We first show that Θ^{μ} is a fuzzy congruence relation on A. Clearly it is reflexive and symmetric. To show that Θ^{μ} is transitive consider:

$$\begin{aligned}\Theta^{\mu}(x,y)\wedge\Theta^{\mu}(y,z) &= \bigvee\{\alpha\in L:(x,y)\in(\mu_{\alpha})^{\delta}\}\wedge\bigvee\{\beta\in L:(y,z)\in(\mu_{\beta})^{\delta}\}\\ &= \bigvee\{\alpha\wedge\beta:(x,y)\in(\mu_{\alpha})^{\delta},(y,z)\in(\mu_{\beta})^{\delta}\}\end{aligned}$$

If we put $\gamma=\alpha\wedge\beta$, then we get $\mu_{\alpha}\subseteq\mu_{\gamma}$ and $\mu_{\beta}\subseteq\mu_{\gamma}$. It follows from the fact $I\subseteq J\Rightarrow I^{\delta}\subseteq J^{\delta}$ that $(\mu_{\alpha})^{\delta}\subseteq(\mu_{\alpha})^{\delta}$. Thus,

$$\begin{aligned}\Theta^{\mu}(x,y)\wedge\Theta^{\mu}(y,z) &= \bigvee\{\alpha\wedge\beta:(x,y)\in(\mu_{\alpha})^{\delta},(y,z)\in(\mu_{\beta})^{\delta}\}\\ &\leq \bigvee\{\gamma:(x,y),(y,z)\in(\mu_{\gamma})^{\delta}\}\\ &\leq \bigvee\{\gamma:(x,z)\in(\mu_{\gamma})^{\delta}\}\\ &= \Theta^{\mu}(x,z)\end{aligned}$$

Therefore it is transitive and hence it is a fuzzy equivalence relation. Let $x_1,\ldots,x_n,y_1,\ldots,y_n\in A$ and f be an $n-$ary operation. Then

$$\begin{aligned}\bigwedge\{\Theta^{\mu}(x_i,y_i):1\leq i\leq n\} &= \bigwedge\{\bigvee\{\alpha_i\in L:(x_i,y_i)\in(\mu_{\alpha_i})^{\delta}\}:1\leq i\leq n\}\\ &= \bigvee\{\bigwedge\{\alpha_i\in L:1\leq i\leq n\}:(x_i,y_i)\in(\mu_{\alpha_i})^{\delta}\}\end{aligned}$$

If we put $\gamma=Inf\{\alpha_i\in L:1\leq i\leq n\}$, then we get $\mu_{\alpha_i}\subseteq\mu_{\gamma}$ for all $i=1,2,\ldots,n$ which implies that $(\mu_{\alpha_i})^{\delta}\subseteq(\mu_{\alpha})^{\delta}$for all $i=1,2,\ldots,n$. Thus,

$$\begin{aligned}\bigwedge\{\Theta^{\mu}(x_i,y_i):1\leq i\leq n\} &= \bigvee\{\bigwedge\{\alpha_i\in L:1\leq i\leq n\}:(x_i,y_i)\in(\mu_{\alpha_i})^{\delta}\}\\ &\leq \bigvee\{\gamma\in L:(x_i,y_i)\in(\mu_{\gamma})^{\delta},\forall\, i=1,2,\ldots,n\}\\ &\leq \bigvee\{\gamma\in L:(f(x_1,\ldots,x_n),f(y_1,\ldots,y_n))\in(\mu_{\gamma})^{\delta}\}\\ &= \Theta^{\mu}(f(x_1,\ldots,x_n),f(y_1,\ldots,y_n))\end{aligned}$$

Therefore Θ^{μ} is a fuzzy congruence relation on A. Now, we show that the kernel of Θ^{μ} is precisely μ; for,

$$\begin{aligned} 0/\Theta^{\mu}(x) &= \Theta^{\mu}(x,0) \\ &= \bigvee\{\alpha \in L : (x,0) \in (\mu_{\alpha})^{\delta}\} \\ &= \bigvee\{\alpha \in L : (\mu_{\alpha})^{\delta}[x] = (\mu_{\alpha})^{\delta}[0]\} \\ &= \bigvee\{\alpha \in L : (\mu_{\alpha})^{\delta}[x] = \mu_{\alpha}\} \\ &= \bigvee\{\alpha \in L : x \in \mu_{\alpha}\} \\ &= \mu(x) \end{aligned}$$

To prove the uniqueness of such a fuzzy congruence, let us take any fuzzy congruence Φ on A for which $0/\Phi = \mu$. Then $\Phi_{\alpha}[0] = \mu_{\alpha}$ for all $\alpha \in L$; that is, μ_{α} is kernel of the congruence relation Φ_{α}. By the uniqueness of the congruence $(\mu_{\alpha})^{\delta}$ we get $\Phi_{\alpha} = (\mu_{\alpha})^{\delta}$ for all $\alpha \in L$ and hence $\Phi = \Theta^{\mu}$. Therefore Θ^{μ} is the unique fuzzy congruence on A for which $0/\Theta^{\mu} = \mu$. In this case, the map $\mu \mapsto \Theta^{\mu}$ defines an order isomorphism between the lattice of fuzzy ideals and the lattice of fuzzy congruence relations on A. We see from Corollary 2.3.4 that every ideal of A can be identified as a fuzzy ideal by its characteristic mapping. This proofs the converse part. □

Lemma 7.2.4. *If $\mathscr{K}$ is ideal determined and $A \in \mathscr{K}$, then*

$$\mu/\Theta = \mu \vee 0/\Theta$$

Proof. Let us first see that μ/Θ contains both μ and $0/\Theta$. For each $x \in A$, consider

$$\begin{aligned} \mu/\Theta(x) &= \bigvee\{\mu(y) \wedge \Theta(y,x) : y \in A\} \\ &\geq \mu(y) \wedge \Theta(y,x) \quad \textit{for all } y \in A \end{aligned}$$

In particular for $y = x$; that is, $\mu/\Theta(x) \geq \mu(x)$. So that $\mu \leq \mu/\Theta$. Also, if we take $y = 0$ we get

$\mu/\Theta(x) \geq \Theta(0,x) = 0/\Theta(x)$ and hence $0/\Theta \leq \mu/\Theta$. Thus $\mu \vee 0/\Theta \leq \mu/\Theta$. To prove the other inequality, let us put $\lambda = \mu \cup 0/\Theta$. Then λ is a fuzzy subset of A such that $\langle\lambda\rangle = \mu \vee 0/\Theta$. For each $x \in A$, it follows from Theorem 2.3.8 that

$$\begin{aligned}\langle\lambda\rangle(x) &= \bigvee\{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, t(\overrightarrow{a}, \overrightarrow{b}) = x \textit{ where} \\ &\quad \overrightarrow{a} \in A^n, t(\overrightarrow{x}, \overrightarrow{y}) \textit{ is an ideal term in } \overrightarrow{y}\}\end{aligned}$$

Since $\mu, \mu/\Theta$ and Θ_0 are all fuzzy ideals we have

$$\mu/\Theta(0) = 1 = (\mu \vee 0/\Theta)(0)$$

For each $0 \neq x \in A$, let us define two sets H_x and G_x as follows:

$$\begin{aligned}H_x &= \{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, P(\overrightarrow{a}, \overrightarrow{b}) = x \\ &\quad \textit{where } \overrightarrow{a} \in A^n, P(\overrightarrow{x}, \overrightarrow{y}) \textit{ is an ideal term in } \overrightarrow{y}\} \\ G_x &= \{\mu(y) \wedge \Theta(y,x) : y \in A\}\end{aligned}$$

Clearly both H_x and G_x are subsets of L. Our claim is to see that: $\vee H_x \leq \vee G_x$. Let $\alpha \in H_x$. Then $\alpha = \lambda^m(\overrightarrow{b})$, for some $\overrightarrow{b} \in A^m$, such that $t(\overrightarrow{a}, \overrightarrow{b}) = x$ for some $\overrightarrow{a} \in A^n$ where $t(\overrightarrow{x}, \overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$. That is, $b_1, \ldots b_m \in \lambda_\alpha = \mu_\alpha \cup (\Theta_0)_\alpha$. Without loss of generality we can assume that $b_1, \ldots b_k \in \mu_\alpha$ and $b_{k+1}, \ldots b_m \in (0/\Theta)_\alpha$. So we have

$$\mu(b_1) \wedge \ldots \wedge \mu(b_k) \geq \alpha \textit{ and } \Theta(b_{k+1}, 0) \wedge \ldots \wedge \Theta(b_m, 0) \geq \alpha$$

Let us put $y = t(\overrightarrow{a}, b_1, \ldots b_k, 0, 0, \ldots, 0)$. Then

$$\mu(y) \geq \mu(b_1) \wedge \ldots \wedge \mu(b_k) \geq \alpha \text{ and } \Theta(x,y) \geq \Theta(b_{k+1}, 0) \wedge \ldots \wedge \Theta(b_m, 0) \geq \alpha$$

If we put $\beta = \mu(y) \wedge \Theta(x,y)$, then $\beta \in G_x$ such that $\alpha \leq \beta$. This confirms that $\vee H_x \leq \vee G_x$ for all $x \in A$. Therefore $\mu/\Theta = \mu \vee \Theta_0$. □

If $\mathscr{K}$ is an ideal determined class of algebras, then the supremum of two fuzzy ideals is easy to describe. This could be done in the following way. If μ and ν are fuzzy ideals of $A \in \mathscr{K}$ and Θ^{ν} is the unique fuzzy congruence on A for which $\nu = (\Theta^{\nu})_0$, then Lemma 7.2.4 confirms that $\Theta^{\nu}[\mu]$ is the supremum of μ and ν.

It is proved by the use of Mal'cev condition in [79] that a class $\mathscr{K}$ of algebras is an ideal determined if and only if for some positive integer m, there are binary terms $d_1, d_2, \ldots, d_m, d_{m+1}$ such that:

$$d_1(y,z) = d_2(y,z) =, \ldots, = d_m(y,z) = 0 \Rightarrow y = z \quad \textit{and}$$

$$d_{m+1}(y,y) = 0, \quad d_{m+1}(0,y) = y$$

In this case for an ideal I the congruence I^{δ} is characterized as follows:

$$I^{\delta} = \{(a,b) \in A \times A : d_i(a,b) \in I, \quad \textit{for all } 1 \leq i \leq m\}$$

Similarly for a fuzzy ideal μ of A we characterize the unique fuzzy congruence Θ^{μ} of A as follows:

$$\Theta^{\mu}(a,b) = \bigwedge_{i=1}^{m} \mu(d_i(a,b))$$

7.3 Quotient Algebra Induced by Fuzzy Ideals

The various constructions of quotient groups and quotient rings by fuzzy subgroups and fuzzy ideals respectively was done by different scholars (see [2, 105, 101, 114, 115, 127, 163]). More generally, quotient algebras of a given type induced by fuzzy congruences were studied in [139] and [134]. In this section, we study quotient algebras induced by fuzzy ideals in ideal determined varieties. We begin by defining quotient algebras induced by fuzzy congruence relations. Given a fuzzy congruence relation Θ on A and $x \in A$, consider the fuzzy congruence class Θ_x of A determined by Θ and x.

Lemma 7.3.1. *Let Θ be a fuzzy congruence on A. For any $x, y \in A$, the following hold:*

1. *$x/\Theta = y/\Theta$ if and only if $\Theta(x,y) = 1$.*
2. *either $x/\Theta = y/\Theta$ or there exists $\alpha \in L - \{1\}$ such that*

$$x/\Theta \cap y/\Theta \leq \alpha$$

Definition 7.3.2. Let us define a set A/Θ by:

$$A/\Theta = \{x/\Theta : x \in A\}$$

Then A/Θ can be made into an algebra of the same type as A in the following way: If $f \in \mathfrak{F}$ is nullary, then

$$f^{A/\Theta} = f^A/\Theta$$

If $f \in \mathfrak{F}$ is $n-$ary, $n > 0$ and $a_1, ..., a_n \in A$, then

$$f^{A/\Theta}(a_1/\Theta, ..., a_n/\Theta) = f^A(a_1, ..., a_n)/\Theta$$

Theorem 7.3.3. *Let Θ be a fuzzy congruence on A. If Θ_* denotes the level relation:*

$$\Theta_* = \{(x,y) \in A \times A : \Theta(x,y) = 1\}$$

Then it is clear that Θ_* is a congruence relation (crisp) on A. Moreover,

$$A/\Theta \cong A/\Theta_*$$

Proof. Define $h : A/\Theta \to A/\Theta_*$ by

$$h(x/\Theta) = \Theta_*[x]$$

for all $x \in A$. We first show that h is well defined. Let $x, y \in A$ such that $x/\Theta = y/\Theta$. Then by (1) of Lemma 7.3.1, $\Theta(x, y) = 1$, which gives $(x, y) \in \Theta_*$; i.e., $\Theta_*[x] = \Theta_*[y]$ and hence h is well defined. To show that h is a homomorphism, let $f \in \mathfrak{F}$ be $n-$ary, $n > 0$ and $a_1, ..., a_n \in A$. Then consider:

$$\begin{aligned} h(f^{A/\Theta}(a_1/\Theta, ..., a_n/\Theta)) &= h(f^A(a_1, ..., a_n)/\Theta) \\ &= \Theta_*[f^A(a_1, ..., a_n)] \\ &= f^{A/\Theta_*}(\Theta_*[a_1], ..., \Theta_*[a_n]) \\ &= f^{A/\Theta_*}(h(a_1/\Theta), ..., h(a_n/\Theta)) \end{aligned}$$

Thus h is a homomorphism. It is also clear that h is surjective. It remains to show that h is injective. Let $x, y \in A$ such that $\Theta_*[x] = \Theta_*[y]$. Then $(x, y) \in \Theta_*$ which means $\Theta(x, y) = 1$. By Lemma 7.3.1, we get that $x/\Theta = y/\Theta$, i.e., h is injective and hence it is an isomorphism. □

Lemma 7.3.4. *Let Θ be a fuzzy congruence on A and A/Θ its quotient. For any $x, y \in A$, the following hold in A/Θ:*

1. $\langle x/\Theta \rangle = \{z/\Theta : z \in \langle x \rangle\}$
2. $[x/\Theta, y/\Theta] = \{z/\Theta : z \in [x, y]\}$

Proof. 1. Let us define two sets G and H as follows

$$\begin{aligned} G &= \langle x/\Theta \rangle \\ H &= \{z/\Theta : z \in \langle x \rangle\} \end{aligned}$$

Our aim is to show that $G = H$. Let $a/\Theta \in G$. Then there exist $a_1, ..., a_n \in A$ and an $(n+1)-$ary ideal term $t(\overrightarrow{x}, y)$ in y such that

$$\begin{aligned} a/\Theta &= t^{A/\Theta}(a_1/\Theta, ..., a_n/\Theta, x/\Theta) \\ &= t^A(a_1, ..., a_1, x)/\Theta \end{aligned}$$

If we put $b = t^A(a_1, ..., a_n, x)$, then $b \in \langle x \rangle$ such that $a/\Theta = b/\Theta$. Thus $\Theta_a \in H$ and hence $G \subseteq H$. Conversely, let $a/\Theta \in H$. Then $a/\Theta = b/\Theta$ for some $b \in \langle x \rangle$. There exist $b_1, ..., b_n \in A$ and an $(n+1)-$ary ideal term $t(\overrightarrow{x}, y)$ in y such that $b = t^A(b_1, ..., b_n, x)$. Now consider the following:

$$\begin{aligned} a/\Theta &= b/\Theta \\ &= t^A(b_1, ..., b_n, x)/\Theta \\ &= t^{A/\Theta}(b_1/\Theta, ..., b_n/\Theta, x/\Theta) \end{aligned}$$

Mean that $a/\Theta \in \langle x/\Theta \rangle$; i.e., $H \subseteq G$ and hence the equality holds.

2. Let us define two sets G and H as follows

$$\begin{aligned} G &= [x/\Theta, y/\Theta] \\ H &= \{z/\Theta : z \in [x, y]\} \end{aligned}$$

Our aim is to show that $G = H$. Let $a/\Theta \in G$. Then there exist $a_1, ..., a_n \in A$ and an $(n+1+1)-$ary commutator term $t(\overrightarrow{x}, y, z)$ in y, z such that

$$\begin{aligned} a/\Theta &= t^{A/\Theta}(a_1/\Theta, ..., a_n/\Theta, x/\Theta, y/\Theta) \\ &= \Theta_{t^A(a_1, ..., a_n, x, y)} \end{aligned}$$

If we put $b = t^A(a_1, ..., a_n, x, y)$, then $b \in [x, y]$ such that $a/\Theta = b/\Theta$. Thus $a/\Theta \in H$ and hence $G \subseteq H$. Conversely, let $a/\Theta \in H$. Then $a/\Theta = b/\Theta$ for some $b \in [x,y]$. There exist $b_1,...,b_n \in A$ and an $(n+1+1)-$ary commutator term $t(\overrightarrow{x},y,z)$ in y,z such that $b = t^A(b_1,...,b_n,x,y)$. Now consider the following:

$$\begin{aligned} a/\Theta &= b/\Theta \\ &= t^A(b_1,...,b_n,x,y)/\Theta \\ &= t^{A/\Theta}(b_1/\Theta,...,b_n/\Theta,x/\Theta,y/\Theta) \end{aligned}$$

Mean that $a/\Theta \in [x/\Theta, y/\Theta]$; i.e., $H \subseteq G$ and hence the equality holds.

□

Let $\mathscr{K}$ be an ideal determined variety and $A \in \mathscr{K}$. As observed in the previous section, each fuzzy ideal μ of A is the zero fuzzy congruence class of the unique fuzzy congruence relation on A denoted by Θ^μ.

Definition 7.3.5. Let $\mathscr{K}$ be an ideal determined variety and $A \in \mathscr{K}$. For a fuzzy ideal μ of A, A/μ denotes the quotient algebra of A induced by the fuzzy congruence Θ^μ and call it the quotient algebra of A induced by μ.

For a fuzzy ideal μ of A and each $x \in A$, we define μ_x to be the fuzzy congruence class of Θ^μ determined by x. So that we have the following.

Lemma 7.3.6. *Let μ be a fuzzy ideal of $A \in \mathscr{K}$ and $x,y \in A$. If $\mathscr{K}$ is an ideal determined, then the following hold:*

1. *$\mu_x = \mu_y$ if and only if $\Theta^\mu(x,y) = 1$ if and only if $\mu(d_i(x,y)) = 1$ for all $i = 1,2,...,m+1$, where d_i's are those binary terms given in the previous section.*

2. *either $\mu_x = \mu_y$ or there exists $\alpha \in L - \{1\}$ such that*

$$\mu_x \cap \mu_y \leq \alpha$$

Analogous to Theorem 7.3.3 we have the following theorem.

Theorem 7.3.7. *Let μ be a fuzzy ideal of $A \in \mathscr{K}$. If μ_* denotes the level set:*

$$\mu_* = \{x \in A : \mu_x = \mu_0\}$$

Then it is clear that μ_* is an ideal of A. Moreover,

$$A/\mu \cong A/\mu_*$$

Proof. The proof follows from Theorem 7.3.3. □

Our attention now turns to characterize fuzzy prime ideals using their quotient structure. Let us first define an important concept.

Definition 7.3.8. [149] For any $B \in \mathscr{K}$, define

$$D(B) = \bigcup_{0 \neq y \in A} ((y) : (0))$$

where for each $y \in A$ and $I \in I(A)$,

$$(\langle y \rangle : I) = \{x \in L : [x, y] \subseteq I\}$$

Theorem 7.3.9. *If μ is a fuzzy prime ideal of A, then*

$$D(A/\mu) = (0)$$

Proof. Let $\mu_x \in D(A/\mu)$. Then $\mu_x \in (\langle \mu_y \rangle : \langle \mu_0 \rangle)$ for some $y \in A$ with $\mu_y \neq \mu_0$, i.e., $[\mu_x, \mu_y] = \langle \mu_0 \rangle$. By (2) of Lemma 7.3.4

$$\{\mu_z : z \in [x, y]\} = \{\mu_0\}$$

which gives that $\mu_z = \mu_0$ for all $z \in [x, y]$, i.e.,

$$\Theta^{\mu}(z, 0) = 1 \text{ for all } z \in [x, y]$$

which is equivalent to

$$\bigwedge_{i=1}^{m} \mu(d_i(z,0)) = 1 \text{ for all } z \in [x,y]$$

Since $d_m(z,0) = z$, we get $\mu(z) = 1$ for all $z \in [x,y]$, i.e. $[x,y] \subseteq \mu_*$. Since μ_* is prime either $x \in \mu_*$ or $y \in \mu_*$. So that either $\mu(x) = 1$ or $\mu(y) = 1$, which gives either $\mu_x = \mu_0$ or $\mu_y = \mu_0$. Since $\mu_y = \mu_0$ is impossible, we get $\mu_x = \mu_0$ (the zero element in $D(A/\mu)$). Therefore $D(A/\mu) = (0)$. □

Theorem 7.3.10. *Suppose that μ is a fuzzy ideal of A such that $Img(\mu) = \{1,\alpha\}$ where α is a prime element in L. If $D(A/\mu) = (0)$, then μ is fuzzy prime.*

Proof. By Theorem 3.2.3, it is enough to show that $\mu_* = \{x \in A : \mu(x) = 1\}$ is a prime ideal of A. Clearly it is is a proper ideal. Let $a,b \in A$ such that $[a,b] \subseteq \mu_*$. Then $\mu(x) = 1$ for all $x \in [a,b]$. Then $\mu(x) = 1$ for all $x \in [a,b]$, i.e., $\mu_x = \mu_0$ for all $x \in [a,b]$. By (2) of Lemma 7.3.4

$$[\mu_a, \mu_b] = \{\mu_x : x \in [a,b]\} \subseteq \langle \mu_0 \rangle$$

Then by our assumption either $\mu_a = \mu_0$ or $\mu_b = \mu_0$, which is equivalent to that either $a \in \mu_*$ or $b \in \mu_*$. Thus μ_* is prime and hence proved. □

Conclusion and further recomendations

The notion of fuzzy ideals of universal algebras is introduced as a common abstraction to most of the existing theories of fuzzy ideals in different algebraic structures by applying the general theory of algebraic fuzzy systems. In this setting, basic concepts that are connected to ideals like the generator, the commutator, primeness, semi-primeness, the prime spectrum, maximality and the radical are extended to the class of fuzzy ideals in universal algebras. On the other hand, fuzzy congruence relations and their classes in universal algebras are studied in the dissertation. Several Mal'cev type characterizations are given for fuzzy congruence classes in general algebraic structures. Special fuzzy congruence classes called fuzzy congruence kernels are also studied in different algebraic structures. Furthermore, the structure of quotient algebras induced by fuzzy ideals is studied in ideal determined varieties. In addition, the notion fuzzy cosets in universal algebras is introduced as a generalization of fuzzy ideals ang fuzzy congruence classes. This notion is applied to characterize those congruence permutable varieties.

It is under investigation by the author to extend the notion of relative annihilators, annihilator ideals, $\alpha-$ideals and deductive systems in universal algebras to the fuzzy setting.

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