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L-Fuzzy Ideals and Filters of a Poset By

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L-Fuzzy Ideals and Filters of a Poset

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A Dissertation Submitted to the Department of Mathematics, College of Science, Bahir Dar University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics.

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Declaration of Authorship

I declare that the work in this dissertation entitled, "L-Fuzzy Ideals and Filters of a Poset" was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Program and that it has not been submitted for any other academic award except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Abstract

In different literatures, we have found several generalizations of ideals and filters of a lattice to an arbitrarily partially ordered set which has been studied by different scholars . In this thesis we introduce several generalizations of L-fuzzy ideals and filters of a lattice to an arbitrarily partially ordered set whose truth values are in a complete lattice satisfying the infinite meet distributive law. These are: L-fuzzy closed ideal and filter of a poset, L-fuzzy Frink ideal and filter of a poset, L-fuzzy ideal and filter of a poset in the sense of Halaš, L-fuzzy semi ideals and filters of a poset, L-fuzzy V-ideals and V-filters of a poset and m-L-fuzzy ideals and filters of a poset, where m is any cardinal number. All the definitions of L-fuzzy ideals and filters of a poset that we introduce in this thesis are generalizations of the notions of L-fuzzy ideals and filters of a lattice. We also study and establish some characterizations of them and we prove that the set of all L-fuzzy ideals of a poset forms a complete lattice with respect to point-wise ordering.

Next, by choosing the L-fuzzy ideals and filters of a poset in the sense of Halaš as an L-fuzzy ideals of a poset, we introduce the notion of L-fuzzy prime ideals, prime L-fuzzy ideals, maximal L-fuzzy ideals and L-fuzzy maximal ideals. We also study and give sufficient conditions for the existence of L-fuzzy prime ideals and prime L-fuzzy ideals in the lattice of all L-fuzzy ideals of a poset.

Lastly, we introduce the concept of L-fuzzy semi-prime ideals in a general poset. Characterizations of L-fuzzy semi-prime ideals in posets as well as characterizations of an L-fuzzy semi-prime ideal to be L-fuzzy prime ideal are obtained. Also, the relations between the L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of a poset and the L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of the lattice of all ideals of a poset are established. We also extend and prove an analogue of Stone's Theorem for finite posets, which has been studied by V. S. Kharat and K. A. Mokbel[35] using L-fuzzy semi-prime ideals.

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List of Symbols

A^u	the upper cone of set A
A^l	the lower cone of set A
$(A]_{Cl}$	the closed ideal generated by subset A of a poset Q
$(A]_{Fr}$	the Frink ideal generated by subset A of a poset Q
$(A]_{Ha}$	the ideal generated by subset A of a poset Q in the sense of Halaš
$(A]_{Se}$	the semi-ideal generated by a subset A of Q
$(A]_V$	the V-ideal generated by a subset A of Q
$\subset_m$	m-small subset, where m is any cardinal number.
L	a complete distributive lattice satisfying the infinite meet distributivity.
L^X	the set of all L-fuzzy subsets of X
$Im(\mu)$	the image of an L-fuzzy subset μ.
μ_α	the level subset of the L-fuzzy subset μ at α or the α-level subset of μ.
$(\mu]_{Cl}$	the L- fuzzy closed ideal generated by an L-fuzzy subset μ
$(\mu]_{Fr}$	the L- fuzzy Frink ideal generated by an L-fuzzy subset μ
$(\mu]_{Ha}$	the L- fuzzy ideal generated by an L-fuzzy subset μ in the sense of Halaš
$(\mu]_{Se}$	the L- fuzzy semi-ideal generated by μ
$\mathscr{I}(Q)$	the set of all ideals of a poset Q
$\mathscr{F}(Q)$	the set of all filters of a poset Q
$\mathscr{F}\mathscr{F}(Q)$	the set of all L-fuzzy filters of a poset Q
$\mathscr{F}\mathscr{I}(Q)$	the set of all L-fuzzy ideals of a poset Q
$\mathscr{F}\mathscr{C}\mathscr{I}(Q)$	the set of all L-fuzzy closed ideals of a poset Q
$\mathscr{F}\mathscr{C}\mathscr{F}(Q)$	the set of all L-fuzzy closed filters of a poset Q

$\mathscr{F}\mathscr{F}\mathscr{I}(Q)$	the set of all L-fuzzy Frink ideals of a poset Q
$\mathscr{F}\mathscr{F}\mathscr{F}(Q)$	the set of all L-fuzzy Frink filters of a poset Q
$\mathscr{F}\mathscr{S}\mathscr{I}(Q)$	the set of all L-fuzzy semi-ideals of a poset Q
$\mathscr{F}\mathscr{S}\mathscr{F}(Q)$	the set of all L-fuzzy semi-filters of a poset Q
$\mathscr{F}\mathscr{V}\mathscr{I}(Q)$	the set of all L-fuzzy V-ideals of a poset Q
$\mathscr{F}\mathscr{V}\mathscr{F}(Q)$	the set of all L-fuzzy V-filters of a poset Q
$\mathscr{P}\mathscr{F}\mathscr{I}(Q)$	the set of all prime L-fuzzy ideals of Q
$\mathscr{F}\mathscr{S}\mathscr{P}(Q)$	the set of all L-fuzzy semi-prime ideals of a poset Q

Dedicated to my lovely sons Zerabruk, Friesenaye and Abel.

Publications

From this dissertation, the following four papers are published and one paper is communicated.

1. *L*-fuzzy ideals of a poset, Ann. Fuzzy Math. Inform. 16 (3) (2018) 285-299.

2. L-Fuzzy filters of a poset, International Journal of Computing Science and Applied Mathematics 5(1) (2019), 23-29

3. *L*-fuzzy prime ideals and maximal *L*-fuzzy ideals of a poset, Ann. Fuzzy Math. Inform. 18 (1) (2019) 1-13.

4. *L*- Fuzzy semi-prime ideals of a poset, Advances in Fuzzy Systems, Vol. 2020, Article ID 1834970,, 10 pages, 2020.

5. *L*-fuzzy semi-prime ideals and separation theorems for a poset. (communicated)

Introduction

We have found several generalizations of ideals and filters of a lattice to an arbitrarily partially ordered set in different literature. Closed or normal ideals of a poset was introduced by Birkhoff [14]. Next, in 1954, the second type of ideals and filters of a poset called Frink ideals and Frink filters was introduced by O. Frink [25]. Following this P. V. Venkatanarasimhan developed the theory of semi-ideals and semi-filters in [51] and ideals and filters for a poset in [52], in 1970. These ideals (respectively, filters) are called ideals (respectively, filters) in the sense of Venkatanarasimhan or V-ideals (respectively, V-filters) for short. Next, the concept of ideals of a poset were suggested by Erné [23] in 1979. They are called m-ideals. These ideals generalize almost all ideals of a poset suggested by different authors. Latter, Halaś [28], in 1994, introduced a new type of ideal and filter of a poset which seems to be a suitable generalization of the usual concept of ideal and filter in a lattice. We will simply call an ideal (respectively, a filter) in the sense of Halaš.

On the other hand, in 1965, L.Zadeh, in his pioneering paper [54], introduced the concept of a fuzzy subset of a non-empty set X as a function from X into the unit interval $[0,1]$ to describe, study and formulate mathematically those objects which are not well defined. The theory of fuzzy sets, introduced by L.Zadeh, has evoked tremendous interest among researchers working in different area of fields. It was a new episode towards the development of science and engineering.

In 1971, A. Rosenfeld [43] applied the concept of fuzzy subset of a non-empty set to study the concept of fuzzy subgroup of a given group. The introduction of the concept of a fuzzy subgroup of a group by Rosenfeld initiated several algebraist to take up the study of fuzzy sub-algebras of several algebraic structures such as groups, rings, modules, vector-spaces, lattices,etc. More recently, in posets, Universal Algebras, Ms-Algebras, Pseudo-complemented semi-lattice, etc. His paper inspired the development of fuzzy abstract algebra. See ([17], [20], [21, 40] ,[1], [11], [16],[18], [33], [55] [10, 41, 46, 12, 53],[7, 8, 9], [2, 3, 4], [5, 6],[12].)

In 1967, as suggested by Gougen [26], the unit interval $[0,1]$ is not suffcient to take the truth values of general fuzzy statements. K. L. N. Swamy and U. M. Swamy [47] initiated that complete lattices satisfying the infinite meet distributivity are the most suitable candidates to have the truth values of general fuzzy statements.

Initiated by the above ideas and concepts, in this thesis, we introduce several generalizations of L-fuzzy ideals and filters of a lattice to an arbitrary partially ordered set whose truth values are in a complete lattice satisfying the infinite meet distributive law. In addition, by choosing one of the generalization of L-fuzzy ideals and filters of a lattice to an arbitrary partially ordered set, we introduce the notion of L-fuzzy prime ideals, prime L-fuzzy ideals, maximal L-fuzzy ideals and L-fuzzy maximal ideals. We also study and give sufficient conditions for the existence of L-fuzzy prime ideals and prime L-fuzzy ideals in the lattice of all L-fuzzy ideals of a poset. The notions of L-fuzzy semi-prime ideals in a general poset is introduced and characterized.

Throughout this work L means a non-trivial complete lattice satisfying the infinite meet distributive law: $x \wedge \sup S = \sup\{x \wedge s : s \in S\}$ for all $x \in L$ and for any subset S of L and by an L-fuzzy subset of a non-empty set X we mean a mapping from X into L.

The thesis is broadly divided into five chapters 1, 2, 3, 4 and 5. Chapter 1 is devoted to collect all the necessary preliminaries and results which will be useful in our discussions of the main text of the thesis. This chapter consists of three sections. Section 1.1 contains definitions and results related to partially ordered sets. In section 1.2, we collect definitions and some preliminary results related to type of ideals and filters studied by different scholars. Section 1.3 is devoted to study L-fuzzy subsets of an arbitrary non-empty set and we also recall the definition of L-fuzzy ideals and filters of a lattice from literature.

The main text of this thesis is in chapters 2, 3, 4 and 5. Chapter 2 is on L-fuzzy ideals of a poset. In this chapter we introduce several generalizations of L-fuzzy ideals of a lattice to an arbitrary poset whose truth values are in a complete lattice satisfying the infinite meet distributive law and give several characterizations of them. We also prove that the set of all L-fuzzy ideals of a poset forms a complete lattice with respect to point-wise ordering. This chapter consists of four sections. Section 2.1 is on L-fuzzy closed ideal of a poset which is the fuzzy version of the closed or normal ideal of a poset introduced by Birkoff[14]. Section 2.2 is devoted on L-fuzzy Frink ideal of a poset which is the fuzzy version of the Frink ideal of a poset introduced by O. Frink[25]. Section 2.3 is on L-fuzzy ideal of a poset which is the fuzzy version ideals of a poset introduced by Halaš [28] which seems to be a suitable generalization of the usual concept of L-fuzzy ideal of a lattice. Section 2.4 is devoted on L-fuzzy semi ideals and V-ideals of a poset which is the fuzzy version of semi-ideals and V-ideals of a poset introduced by Venkatanarasimhan [51, 52]. Finally, we complete this chapter by introducing m-L-fuzzy ideal which is the fuzzy version of ideals of a poset introduced by Erné [23]. This L-fuzzy ideal generalizes almost all the L-fuzzy ideals of a poset introduced in this chapter.

Chapter 3 is focused on the concept of L-fuzzy filters of a poset. It consists of four sections. In this chapter, we introduce the notion of different types of L- fuzzy filters of a poset and discuss certain properties of them analogous to those of L- fuzzy ideals of a poset introduced in Chapter 2.

Chapter 4 is on L-fuzzy prime and maximal L-fuzzy ideals of a poset Q and is subdivided into five sections. In section 4.1, we recall some definitions and crisp concepts of prime and maximal ideals of a poset from literature that will be extended to the notions of prime and maximal L-fuzzy ideals of a poset in the further sections of this chapter. In section 4.3, we introduce the notion of L-fuzzy prime ideals of a poset Q which can be characterized as the L-fuzzy subsets of a poset Q for which each α-level subset is either the whole Q or a prime ideal of Q. In section 4.3, we introduce the notion of a prime L-fuzzy ideal of a poset Q as simply a prime element in the lattice of L-fuzzy ideals of a poset Q. Prime L-fuzzy ideal of a poset Q is more stronger than L-fuzzy prime ideal of a poset Q. In section 4.4, we discuss maximal L-fuzzy ideals of a poset Q with zero which are precisely a dual atom in the lattice of L-fuzzy ideals of a poset Q. In section 4.5, we define the notion of L-fuzzy maximal ideals of a poset Q as a proper L-fuzzy ideal, for which each level subset μ_α at $\alpha \in L$ is either the whole poset Q or a maximal ideal Q.

Chapter 5 is on L-fuzzy semi-prime ideal. It consists of four sections. In section 5.1, we recall some definitions and crisp concepts of semi-prime ideals of a poset and a lattice from literature that will be extended to the notions of L-fuzzy semi-prime ideals of a poset in the further sections of this chapter. In section 5.2, we introduce the concept of an L-fuzzy semi-prime ideal in a general poset. Characterizations of L-fuzzy semi-prime ideals in posets as well as characterizations of a L-fuzzy semi-prime ideal to be L-fuzzy prime ideal are obtained. Also, L-fuzzy prime ideals in a poset are characterized. In section 5.3, we prove the set of all L-fuzzy semi-prime ideals in a poset forms a compete lattice. The relations between L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of a poset and L-fuzzy semi-prime (respectively,L-fuzzy prime) ideals of the lattice of all ideals of a poset are established. In section 5.4, we extend and prove Rav's Prime Separation Theorem for a lattice, using L-fuzzy semi-prime ideals. Lastly, we also extend and prove an analogue of Stone's Theorem for finite posets, which has been studied by V. S. Kharat and K. A. Mokbel[35] using L-fuzzy semi-prime ideals. Some counterexamples are also given.

Chapter 1

Preliminaries

In this chapter we collect the necessary preliminaries which will be useful in our discussions of the main text of the thesis. Even though these preliminaries are well known for those working in lattice theory and fuzzy set theory, it will be convenient for others to have all these elementary notions and results in the beginning of the thesis for the sake of ready reference. The proofs of most of the results presented in this chapter are either straight forward verifications or well known and hence we simply state the results without proofs.

1.1 Partially Ordered Sets

In this section we recall certain necessary concepts, terminologies and notations of partially ordered sets that will be useful in this thesis. For undefined notations and terminologies for this section, the reader is referred to Birkhoff [1961][14], Davey and Priestley [1990][19] and Grätzer[1998][27].

We begin with the definition of a partial order.

Definition 1.1.1. *Let Q be a non-empty set. A binary relation " $\leq$ " on Q is called a partial order if for all $x,y,z \in Q$, the following conditions are satisfied.*

1. *$x \leq x$ (reflexive);*

2. *$x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetric);*

3. $x \leq y$ and $y \leq x$ imply $x \leq z$ (transitive).

Definition 1.1.2. *A non-empty set Q together with the partial order " $\leq$ " on Q, denoted by $(Q, \leq)$, is called a partially ordered set or a poset.*

Let $(Q, \leq)$ be a poset and $x, y \in Q$. Then x and y are said to be comparable if $x \leq y$ or $y \leq x$. Otherwise they are called incomparable or parallel and we write $x \parallel y$. If $x \leq y$ and $x \neq y$, then we write $x < y$.

Definition 1.1.3. *A poset $(Q, \leq)$ is said to be a chain or totally ordered set if any two elements of Q are comparable. A partially ordered set $(Q, \leq)$ is called an antichain if every two distinct elements of Q are incomparable.*

Definition 1.1.4. *Let $(Q, \leq)$ be a poset. Define a binary relation " $\geq$ " on Q by:*

$$x \geq y \text{ if and only if } y \leq x \text{ for all } x, y \in Q.$$

Then " $\geq$ " is a partial order on Q and it is called the dual order of "$\leq$" and the poset $(Q, \geq)$ is the dual of the poset $(Q, \leq)$.

If Φ is a statement about posets and if, in Φ, we replace all occurrences of $\leq$ by $\geq$, we get the dual of Φ.

Now we state a principle that halves the labor of proving some results.

Duality Principle: " If a statement Φ is true in all posets, then its dual is also true in all posets."

When confusion is unlikely, we use simply the symbol Q to denote a poset $(Q, \leq)$.

Definition 1.1.5. *Let Q be a poset. Then an element $m \in Q$ is called a maximal element in Q if there is no $x \in Q$ such that $m < x$. Dually, an element $m \in Q$ is called a minimal element, if there is no $x \in Q$ such that $x < m$.*

Definition 1.1.6. *A poset Q is said to have a largest or a greatest element if there exists $x_0 \in Q$ such that $x \leq x_0$ for all $x \in Q$. Dually, a poset Q is said to have a smallest or least element if there exists $x_0 \in Q$ such that $x_0 \leq x$ for all $x \in Q$.*

If the greatest element exists in a poset Q, then it is unique and is denoted by 1. If the smallest element exists in Q, then it is unique and is denoted by 0.

Definition 1.1.7. *A poset Q is said to be bounded if it has* 0 *and* 1.

Definition 1.1.8. *A poset Q is said to satisfy the Ascending Chain Condition (ACC), if every non-empty subset of Q has a maximal element. Dually, we have the concept of Descending Chain Condition (DCC).*

The following is an important axiom of set theory, though its popular name is Zorn's lemma. It is used to prove some other equivalent axioms of set theory.

Lemma 1.1.1. *(Zorn's lemma) Let $(Q,\leq)$ be a poset in which each chain has an upper bound in Q. Then, $(Q,\leq)$ has a maximal element. Dually, if $(Q,\leq)$ is a poset in which every chain has a lower bound in Q, then Q has a minimal element.*

Definition 1.1.9. *Let Q be a poset and $A \subseteq Q$. Then an element $x \in Q$ is called an upper bound of A if $a \leq x$ for all $a \in A$. The set*

$$A^u = \{x \in Q : a \leq x \,\forall\, a \in A\}$$

of all upper bounds of A is called the upper cone of A.

Dually, an element $x \in Q$ is called a lower bound of A if $x \leq a$ for all $a \in A$. The set

$$A^l = \{x \in Q : x \leq a \,\forall\, a \in A\}$$

of all lower bounds of A is called the lower cone of A.

Definition 1.1.10. *Let Q be a poset and $A \subseteq Q$. Then an element $x_0 \in Q$ is called the least upper bound of A, denoted by* $\sup A$ *or* $\bigvee A$*, if*

$$x_0 \in A^u \text{ and } x_0 \leq x \text{ for all } x \in A^u.$$

Dually, an element $x_0 \in Q$ is called the greatest lower bound of A, denoted by $\inf A$ *or* $\bigwedge A$*, if*

$$x_0 \in A^l \text{ and } x \leq x_0 \text{ for all } x \in A^l.$$

For $x,y \in Q$, we write $x \vee y$ (read as x join y) in place of $\sup\{x,y\}$ if it exists and $x \wedge y$ (read as x meet y) in place of $\inf\{x,y\}$ if it exists.

Let Q be a poset and A,B be subsets of Q and $a,b \in Q$. Then by A^{ul} shall mean $\{A^u\}^l$ and A^{lu} shall mean $\{A^l\}^u$. The the upper cone $\{a\}^u$ is simply denoted by a^u and the upper cone $\{a,b\}^u$ is denoted by $(a,b)^u$. Further the set $\{A \cup B\}^u$ is denoted by $\{A,B\}^u$ and the set $\{A \cup \{a\}\}^u$ is denoted by $\{A,a\}^u$. Similar notations are used for lower cones.

Lemma 1.1.2. *Let Q be a poset and A,B be subsets of Q and $a \in Q$. Then*

1. $A \subseteq A^{ul}$ *and* $A \subseteq A^{lu}$*;*

2. *If* $A \subseteq B$*, then* $A^u \supseteq B^u$ *and* $A^l \supseteq B^l$*;*

3. $A^{ulu} = A^u$ *and* $A^{lul} = A^l$*;*

4. $\{a^u\}^l = a^l$ *and* $\{a^l\}^u = a^u$*.*

Lemma 1.1.3. *Let Q be a poset and A be a subset of Q. Then*

1. *if* $\sup A$ *exists, then* $A^{ul} = \{\sup A\}^l$ *and if* $\inf A$ *exists, then* $A^{lu} = \{\inf A\}^u$*;*

2. *if* $A = \emptyset$*, we have* $A^{ul} = (\emptyset^u)^l = Q^l$ *which is either empty or consists of the least element* 0 *of* Q *alone, if it exists;*

3. *if* $A = \emptyset$*, we have* $A^{lu} = (\emptyset^l)^u = Q^u$ *which is either empty or consists of the largest element* 1 *of* Q *alone, if it exists.*

4. *for any family* $\{A_i : i \in \Delta\}$ *of subsets of* Q*,*

$$\bigcup_{i\in\Delta} A_i^{ul} \subseteq (\bigcup_{i\in\Delta} A_i)^{ul}) \text{ and } \bigcup_{i\in\Delta} A_i^{lu} \subseteq (\bigcup_{i\in\Delta} A_i)^{lu}.$$

Definition 1.1.11. *A poset Q is called a join-semi-lattice, if $x \vee y$ exists for all $x,y \in Q$. Dually a poset Q is called a meet-semi-lattice, if $x \wedge y$ exists for all $x,y \in Q$.*

A poset Q is called a lattice if it is both join-semi-lattice and meet-semi-lattice.

Definition 1.1.12. *A poset Q is called a complete lattice if $\sup S$ and $\inf S$ exist for any subset S of Q. That is, if Q is closed under arbitrary supremum and infimum.*

Definition 1.1.13. *[37] A poset Q is called distributive if for all $a,b,c \in Q$,*

$$\{(a,b)^u, c\}^l = \{(a,c)^l, (b,c)^l\}^{ul}.$$

Definition 1.1.14. *An element p of a poset Q is called distributive if for all $x,y \in Q$,*

$$\{(x,y)^l, p\}^{ul} = \{(p,x)^u, (p,y)^u\}^l.$$

Dually, we have the concepts of dually distributive element of a poset Q.

Definition 1.1.15. *[19] Let Q be a poset with 0. An element p in Q is called an atom if there is no $x \in Q$ such that $0 < x < p$. That is, for any $x \in Q$, $0 \leq x \leq p$ implies either $x = 0$ or $x = p$. Dually we have the concept of a dual atom.*

Definition 1.1.16. *[19] A poset Q with 0 is called atomic poset if every non-zero element of Q contains or dominates an atom. That is, for every $0 \neq x \in Q$, there exists an atom $p \in Q$ such that $p \leq x$. Dually we have the concept of dually atomic poset.*

1.2 Ideals and Filters

In this section, we recall definitions of ideals and filters of a poset which are introduced by different scholars. The definitions of ideals and filters of a poset that we consider in this section are generalizations of the notions of ideals and filters of a lattice.

Definition 1.2.1. *(i) [14] A subset I of a poset Q is called a closed or a normal ideal of Q, if $I^{ul} \subseteq I$ (or equivalently, $I^{ul} = I$, as $I \subseteq I^{ul}$).*

Dually, a subset F of a poset Q is called a closed or normal filter of Q, if $F^{lu} \subseteq F$ (or equivalently, $F^{lu} = F$, as $F \subseteq F^{lu}$).

(ii)[25] A subset I of a poset Q is called a Frink ideal in Q if $A^{ul} \subseteq I$, whenever A is a finite subset of I.

Dually, a subset F of a poset Q is called a Frink filter in Q if $A^{lu} \subseteq F$, whenever A is a finite subset of F.

(iii) [52] A non-empty subset I of a poset Q is called a semi-ideal or an order ideal of Q, if $a \leq b$ and $b \in I$ implies $a \in I$.

Dually, a non-empty subset F of a poset Q is called a semi-filter or an order filter of Q, if $a \leq b$ and $a \in F$ implies $b \in F$.

(iv) [51] A subset I of a poset Q is called a V-ideal or an ideal in the sense of Venkatan-narasimhan, if I is a semi-ideal and for any non-empty finite subset A of I , if $\sup A$ exists, then $\sup A \in I$.

Dually, a subset F of a poset Q is called a V-filter or a filter in the sense of Venkatan-narasimhan, if F is a semi-filter and for any non-empty finite subset A of F , if $\inf A$ exists, then $\inf A \in F$.

(v) [28] A subset I of a poset Q is called an ideal in Q in the sense of Halaš, if $(a,b)^{ul} \subseteq I$ whenever $a,b \in I$.

Dually, a subset F of a poset Q is called a filter in Q in the sense of Halaš, if $(a,b)^{lu} \subseteq F$, whenever $a,b \in F$.

(vi) [27] If Q is a lattice, then a non-empty subset I of Q is an ideal in Q if I is an order-ideal and $x \vee y \in I$ whenever $x,y \in I$.

Dually If Q is a lattice, then a non-empty subset F of Q is a filter in Q if F is an order-filter and $x \wedge y \in F$ whenever $x,y \in F$.

The following definitions of ideal and filter of a poset was sugested by M. Ernè in 1979 [23].

Definition 1.2.2. *Let Q be a poset and m denote any cardinal number. Then a subset I of a poset Q is called an m-ideal in Q, if for any subset A of I of cardinality strictly less than m, written as $A \subset_m I$, we have $A^{ul} \subseteq I$.*

Dually a subset F of a poset Q is called an m-filter in Q, if for any subset A of F of cardinality strictly less than m, written as $A \subset_m F$, we have $A^{lu} \subseteq F$.

All the ideals and filters of a poset, suggested by different authors, can be deduced from m-ideals and filters of a poset as given in the following remark.

Remark 1.2.1 ([24])**.** *The following special cases are included in this general definition:*

1. *2-ideals are semi-ideals containing Q^l. Dually 2-filters are semi-filters containing Q^u.*

2. *3-ideals are ideals in the sense of Halaś containing Q^l. Dually 3-filters are filters in the sense of Halaś containing Q^u.*

3. *ω-ideals are Frink ideals containing Q^l where ω is the least infinite cardinal number. Dually ω-filters are Frink filters containing Q^u.*

   Note that the symbol ω for which $A \subset_\omega I$ means that A is a finite subset of I.

4. *Ω-ideals are closed ideals containing Q^l, where the symbol Ω for which $A \subset_\Omega I$ shall mean that A is merely a subset of I. That is, if I has cardinality κ then Ω may be interpreted as a cardinal strictly greater than κ. Dually Ω-filters are closed filters containing Q^u..*

5. *V-ideals are 2-ideals which are closed under non-empty finite supremum if it exists and containing Q^l. Dually V-filters are 2-filters which are closed under non-empty finite infimum if it exists and containing Q^u.*

Definition 1.2.3. *Let A be any subset of a poset Q. Then the smallest ideal containing A is called an ideal generated by A and is denoted by $(A]$. Dually, the smallest filter containing A is called a filter generated by A and is denoted by $[A)$. The ideal generated by a singleton set $A = \{a\}$ is called principal ideal and is denoted by $(a]$. Dually, the filter generated by a singleton set $A = \{a\}$ is called principal filter and is denoted by $[a)$.*

Note that for any subset A of Q if $\sup A$ exists then $A^{ul} = (\sup A]$ and dually, if $\inf A$ exists then $A^{lu} = [\inf A)$.

The followings are some characterizations of ideals generated by a subset A of a poset Q. We write $F \subset\subset A$ to mean F is a finite subset of A.

1. $(A]_{Cl} = \bigcup\{B^{ul} : B \subseteq A\}$ is the closed ideal or normal ideal generated by A where the union is taken overall subsets B of A. Dually, $[A)_{Cl} = \bigcup\{B^{lu} : B \subseteq A\}$ is a closed or normal filter generated by A.

2. $(A]_{Fr} = \bigcup\{F^{ul} : F \subset\subset A\}$ is the Frink ideal generated by A, where the union is taken overall finite subsets F of A. Dually, $[A)_{Fr} = \bigcup\{F^{lu} : F \subset\subset A\}$ is the Frink filter generated by A. [22]

3. Define sets $C_1 = \bigcup\{(a,b)^{ul} : a,b \in A\}$ and $C_n = \bigcup\{(a,b)^{ul} : a,b \in C_{n-1}\}$ for each positive integer $n \geq 2$, inductively. Then $(A]_{Ha} = \bigcup\{C_n : n \in \mathbb{N}\}$ is the ideal generated by A in the sense of Halaś, where $\mathbb{N}$ denotes the set of positive integers. Dually, define sets $B_1 = \bigcup\{(a,b)^{lu} : a,b \in A\}$ and $B_n = \bigcup\{(a,b)^{lu} : a,b \in B_{n-1}\}$ for each positive integer $n \geq 2$, inductively. Then $[A)_{Ha} = \bigcup\{B_n : n \in \mathbb{N}\}$ is the filter generated by A in the sense of Halaś, where $\mathbb{N}$ denotes the set of positive integers.[29]

4. $(A]_{Se} = \bigcup\{a^l : a \in A\}$ is the semi- ideal generated by A, where the union is taken overall elements a of A. Dually, $[A)_{Se} = \bigcup\{a^u : a \in A\}$ is the semi- filter generated by A.

5. Define sets $C^1 = \{x \in Q : x \leq \bigvee F,\ \emptyset \neq F \subset\subset A \text{ and } \bigvee F \text{ exists }\}$ and $C^n = \{x \in Q : x \leq \bigvee F,\ \emptyset \neq F \subset\subset C^{n-1} \text{ and } \bigvee F \text{ exists }\}$ for each positive integer $n \geq 2$, inductively. Then $(A]_V = \bigcup\{C^n : n \in \mathbb{N}\}$ is the V-ideal generated by A, where $\mathbb{N}$ denotes the set of positive integers. Dually, define sets $B^1 = \{x \in Q : \bigwedge F \leq x,\ \emptyset \neq F \subset\subset A \text{ and } \bigwedge F \text{ exists }\}$ and $B^n = \{x \in Q : \bigwedge F \leq x,\ \emptyset \neq F \subset\subset B^{n-1} \text{ and } \bigwedge F \text{ exists }\}$ for

each positive integer $n \geq 2$, inductively. Then $[A)_V = \bigcup\{B^n : n \in \mathbb{N}\}$ is the V-filter generated by A , where $\mathbb{N}$ denotes the set of positive integers. [44]

Note that if the formula defining a set involves joins (respectively, meets) , it is understood that the formula holds for all existing joins (respectively, meets), so that the running variables take all the values for which the corresponding joins (respectively, meets) exist.

6. If $a \in Q$, then $(a] = \{x \in Q : x \leq a\} = a^l$ is the principal ideal generated by a. Dually, $[a) = \{x \in Q : a \leq x\} = a^u$ is the principal filter generated by a.

7. If Q is a lattice, then $(A] = \{x \in Q : x \leq \sup F,\ F \subset\subset A\}$ is the ideal generated by A in the lattice Q. Dually, $[A) = \{x \in Q : \inf F \leq x,\ F \subset\subset A\}$ is the filter generated by A in the lattice Q. [27]

Throughout this thesis, an ideal (respectively, a filter) will mean a 3-ideal, i.e., an ideal in the sense of Halaš (respectively, a 3-filter, i.e., a filter in the sense of Halaš) unless otherwise stated.

Remark 1.2.2. *The following remarks are due to Halaš and Rachunek [30].*

1. *if Q is a lattice then a non-empty subset I of Q is an ideal as a poset if and only if it is an ideal as a lattice.*

2. *if a poset Q does not have the least element then the empty subset $\emptyset$ is an ideal in Q (since $\emptyset^{ul} = (\emptyset^u)^l = Q^l = \emptyset \subseteq \emptyset$).*

Let $\mathscr{I}(Q)$ denote the set of all ideals of Q. It is known that $(\mathscr{I}(Q), \subseteq)$ forms a complete lattice with respect to the inclusion order "$\subseteq$" with least element $\emptyset$ if Q has no 0 or $\{0\}$ if Q has 0 in which meets coincide with set intersection[30].

We call a poset Q an ideal-distributive if $(\mathscr{I}(Q), \subseteq)$ is a distributive lattice.

Lemma 1.2.1 ([29])**.** *Let $\mathscr{I}(Q)$ be the set of all ideals of a poset Q. Then $(\mathscr{I}(Q), \subseteq)$ is a lattice, where $\subseteq$ is the usual set inclusion ordering. For any I, J in $\mathscr{I}(Q)$, the the supremum $I \vee J$ of I and J in $\mathscr{I}(Q)$ is:*

$$I \vee J = (I \cup J] = \bigcup\{C_n : n \in \mathbb{N}\},$$

where $C_1 = \bigcup\{(a,b)^{ul} : a,b \in I \cup J\}$ *and* $C_n = \bigcup\{(a,b)^{ul} : a,b \in C_{n-1}\}$*, for each positive integer* $n \geq 2$ *and the infimum* $I \wedge J$ *of* I *and* J *in* $\mathscr{I}(Q)$ *is:*

$$I \wedge J = I \cap J.$$

Dually we have the following lemma.

Lemma 1.2.2. *Let* $\mathscr{F}(Q)$ *be the set of all filters of a poset* Q *and* $F, G \in \mathscr{F}(Q)$*. Then the supremum* $F \vee G$ *of* F *and* G *in* $\mathscr{F}(Q)$ *is:*

$$F \vee G = [F \cup G) = \bigcup\{B_n : n \in \mathbb{N}\},$$

where $B_1 = \bigcup\{(a,b)^{lu} : a,b \in F \cup G\}$ *and* $B_n = \bigcup\{(a,b)^{lu} : a,b \in B_{n-1}\}$*, for each positive integer* $n \geq 2$ *and the infimum* $F \wedge G$ *of* I *and* J *in* $\mathscr{F}(Q)$ *is:*

$$F \wedge G = F \cap G.$$

Definition 1.2.4 ([28])**.** *An ideal* I *of a poset* Q *is called a u-ideal, if:*

$$(a,b)^u \cap I \neq \emptyset, \text{ for all } a,b \in I.$$

Note that an easy induction shows I is a u-ideal, if $A^u \cap I \neq \emptyset$, for any finite subset A of I.

Theorem 1.2.3 ([28])**.** *Let* $\mathscr{I}(Q)$ *be the set of all ideals of* Q *and* I *and* J *be u-ideals of a poset* Q*. Then the supremum* $I \vee J$ *of* I *and* J *in* $\mathscr{I}(Q)$ *is:*

$$I \vee J = \bigcup\{(a,b)^{ul} : a \in I, b \in J\}.$$

It is known that every ideal in join-semi-lattice Q is a u- ideal. Therefore the following corollary is an easy consequence of the above theorem.

Corollary 1.2.4. *Let* I *and* J *be u- ideals of a join-semi-lattice* Q*. Then the supremum* $I \vee J$ *of* I *and* J *in* $\mathscr{I}(Q)$ *is given by:*

$$I \vee J = \{x \in Q : x \leq a \vee b \text{ for some } a \in I \text{ and } b \in J\}$$

Dually we have the following

Definition 1.2.5. *[28] A filter F of a poset Q is called an l-filter if:*

$$(x,y)^l \cap F \neq \emptyset \text{ for all } x,y \in F.$$

Note that an easy induction shows that F is an l-filter if $B^l \cap F \neq \emptyset$ for every non-empty finite subset B of F.

Theorem 1.2.5. *[28] Let $\mathscr{F}(Q)$ be the set of filters of a poset Q and F and G be l-filters of Q. Then the supremum $F \vee G$ of F and G in $\mathscr{F}(Q)$ is:*

$$F \vee G = \bigcup\{(a,b)^{lu} : a \in F, b \in G\}.$$

It is known that every filter in a meet-semi-lattice Q is an l- filter. Therefore the following corollary is an easy consequence of the above theorem.

Corollary 1.2.6. *Let F and G be l- filters of a meet-semi-lattice Q. Then the supremum $F \vee G$ of F and G in $\mathscr{F}(Q)$ is given by:*

$$F \vee G = \{x \in Q : a \wedge b \leq x \text{ for some } a \in F \text{ and } b \in G\}$$

1.3 L-Fuzzy Subsets

In this section we collect some basic concepts and properties of L-fuzzy subset from a literature. The purpose of this section is to present basic results of L-fuzzy subsets of a non-empty set that are needed in the remainder of the thesis.

L. A Zadeh, in his pioneering paper [54], introduced the notion of a fuzzy subset of a non-empty set X as a function from X into the unit interval $[0,1]$ to describe,study and formulate mathematically those objects which are not well defined. Those objects whose boundaries are not well defined are called " fuzzy objects" Fuzzy objects are often encountered in real life. For example: " the class of cleaver students " in the set of students, " the class of beautiful girls " in the set of girls, " the class of real numbers which are much greater than 1 " in the set of real numbers, and so on. All these do not constitute sets in the usual mathematical sense. Instead we call them fuzzy sets.

Fuzzy statements usually take truth values in the interval $[0,1]$ of real numbers, while the ordinary (or conventional or crisp) statements take truth values in the two element set $\{F,T\}$ or $\{0,1\}$, where F or 0 stands for 'false' and T or 1 stands for 'True'. However, [0,1] is found to be insufficient to have the truth values of certain fuzzy statements.

It was Goguen[26] who first realized that the closed unit interval $[0,1]$ of real numbers is not sufficient to have the truth values of general fuzzy statements. For example, let us consider the statement 'Bahir Dar University is a good university in Ethiopia'. This is a fuzzy statement, since 'being good' is fuzzy. The truth value of this statement may not be a real number in $[0,1]$. Being good university may have several components; good in teaching-learning process, good in research activity, good in community service, good in educational facilities, good in laboratory facilities, etc. The truth value corresponding to each component may be a real number in [0,1]. If n is the number of such components under consideration, then the truth value of the statement 'Bahir Dar University is a good university' is a n-tuple of real numbers in $[0,1]$; that is, it is an element in $[0,1]^n$. If U is the set of all universities in Ethiopia and G denotes the collection of good universities in Ethiopia, then G is not a subset of U, but it is a fuzzy subset of U, since being good is fuzzy. That is, G can be considered as a function of U into a prospective set like $[0,1]^n$, for some positive integer n.

It is well known that the interval $[0,1]$ of real numbers is a chain under the usual ordering of real numbers; while $[0,1]^n$, when $n > 1$, is not a chain under the co-ordinate wise ordering. However, $[0,1]^n$ satisfies certain rich lattice theoretic properties; it is a complete lattice satisfying the infinite meet distributive law. For this reason, K. L. N. Swamy and U. M. Swamy [47] initiated that complete lattices satisfying the infinite meet distributivity are the most suitable candidates to take the truth values of general fuzzy statements. In this section we introduce the notion of an L-fuzzy subset of a set X , where

L is a given complete lattice satisfying the infinite meet distributive law.

Definition 1.3.1. *A complete lattice $(L, \leq)$ is said to be satisfy the infinite meet distributive law if for any $a \in L$ and $S \subseteq L$,*

$$a \wedge \sup S = \sup\{a \wedge x : x \in S\}.$$

Definition 1.3.2 ([26])**.** *Let X be a non-empty set and let L be a complete lattice satisfying the infinite meet distributive law. An L-fuzzy subset μ of X is a mapping from X into L. The set of all L-fuzzy subsets of X is is denoted by L^X.*

Note that if L is a unit interval of real numbers, then μ is the usual fuzzy subset of X originally introduced by L. Zadeh [54].

For any α in L, the constant L-fuzzy subset of Q which maps all elements of Q onto α is denoted by $\overline{\alpha}$.

Definition 1.3.3. *Let μ be an L-fuzzy subset of a non-empty set X. Then the set $\{\mu(x) : x \in X\}$ is called the image of μ, and is denoted by $Im(\mu)$.*

Definition 1.3.4. *An L- fuzzy subset μ of a non-empty set X is said to have the sup-property if for every non-empty subset A of X, the supremum of $\{\mu(x) : x \in A\}$ is attained at a point of A. That is, there exists an $x_o \in A$ such that $\mu(x_o) = \sup\{\mu(x) : x \in A\}$.*

Definition 1.3.5 ([39])**.** *Let μ and σ be L-fuzzy subsets of a non-empty set X. The union of L-fuzzy subsets μ and σ of X, denoted by $\mu \cup \sigma$, is an L-fuzzy subset of X defined by: for all $x \in X$:*

$$(\mu \cup \sigma)(x) = \mu(x) \vee \sigma(x)$$

and the intersection of L-fuzzy subsets μ and σ of X, denoted by $\mu \cap \sigma$, is a fuzzy subset of X defined by: for all $x \in X$,

$$(\mu \cap \sigma)(x) = \mu(x) \wedge \sigma(x).$$

More generally, the union and intersection of any non-empty family $\{\mu_i\}_{i \in \Delta}$ of L-fuzzy subsets of X, denoted by $\bigcup_{i \in \Delta} \mu_i$ and $\bigcap_{i \in \Delta} \mu_i$ respectively, are defined by:

$$(\bigcup_{i\in\Delta}\mu_i)(x) = \sup_{i\in\Delta}\mu_i(x) \text{ and } (\bigcap_{i\in\Delta}\mu_i)(x) = \inf_{i\in\Delta}\mu_i(x),$$

for all $x \in X$, respectively.

Definition 1.3.6 ([26]). *For any L-fuzzy subsets μ and σ in L^x, define a binary relation " $\subseteq$ " on L^X by:*

$$\mu \subseteq \sigma \text{ if and only if } \mu(x) \leq \sigma(x), \text{ for all } x \in X$$

It can be easily verified that $\subseteq$ is a partial order on the set L^X of L- fuzzy subsets of X. The partial ordering " $\subseteq$ " is called the point wise ordering or inclusion ordering of fuzzy subsets.

Theorem 1.3.1. *[39] Let X be a non-empty set and L be a complete lattice satisfying the infinite meet distributivity. Then $(L^X, \subseteq)$ forms a complete lattice, in which, for any $\{\mu_i\}_{i\in\Delta} \subseteq L^X$,*

$$\sup_{i\in\Delta}\mu_i = \bigcup_{i\in\Delta}\mu_i \text{ and } \inf_{i\in\Delta}\mu_i = \bigcap_{i\in\Delta}\mu_i$$

where " $\subseteq$ " is the point wise ordering given in Definition 1.3.6 above.

Definition 1.3.7 ([46]). *Let μ be an L-fuzzy subset of X. Then for each $\alpha \in L$, the set $\mu_\alpha = \{x : \mu(x) \geq \alpha\}$ is called the level subset of μ at α or the α-level subset of μ.*

Lemma 1.3.2. *Let X be a non-empty set and L be a complete lattice satisfying the infinite meet distributivity. Let $\mu.\sigma \in L^X$. Then*

1. *$\alpha, \beta \in L$ and $\alpha \leq \beta$ imply $\mu_\alpha \supseteq \mu_\beta$;*
2. *$\mu \subseteq \sigma \Leftrightarrow \mu_\alpha \subseteq \sigma_\alpha$ for all $\alpha \in L$;*
3. *$\mu = \sigma \Leftrightarrow \mu_\alpha = \sigma_\alpha$ for all $\alpha \in L$.*

The following theorems show some basic properties of level subsets of L- fuzzy subsets of a nonempty set X.

Theorem 1.3.3. *Suppose that $\{\mu_i\}_{i\in\Delta} \subseteq L^X$. Then for any $\alpha \in L$,*

1. $(\bigcup_{i\in\Delta}\mu_i)_\alpha \supseteq \bigcup_{i\in\Delta}(\mu_i)_\alpha$;

2. $(\bigcap_{i\in\Delta}\mu_i)_\alpha = \bigcap_{i\in\Delta}(\mu_i)_\alpha$

Moreover, when L is a finite chain, we have equality in (1).

Theorem 1.3.4. *Let $\mu \in L^X$ and $\{\alpha_i\}_{i\in\Delta}$ be a non-empty subset of L such that $\beta = \bigwedge_{i\in\Delta}\alpha_i$ and $\alpha = \bigvee_{i\in\Delta}\alpha_i$. Then*

1. $\mu_\beta \supseteq \bigcup_{i\in\Delta}\mu_{\alpha_i}$;

2. $\mu_\alpha = \bigcap_{i\in\Delta}\mu_{\alpha_i}$.

Lemma 1.3.5 ([36])**.** *Let μ be an L- fuzzy subset of a nonempty set X. Then $\mu(x) = \sup\{\alpha \in L : x \in \mu_\alpha\}$, for all $x \in X$.*

Definition 1.3.8 ([50])**.** *For each x in X and $0 \neq \alpha$ in L, we define $x_\alpha \in L^X$ as follow:*

$$x_\alpha(y) = \begin{cases} \alpha & if\ y = x \\ 0 & otherwise \end{cases}$$

for each $y \in X$ and we call it an L-fuzzy point of X.

An L- fuzzy point x_α of X is said to be belongs to an L- fuzzy subset μ of X, written as $x_\alpha \in \mu$, if $\alpha \leq \mu(x)$. Evidently, every L-fuzzy subset μ can be expressed as the union of all the L-fuzzy points of a non-empty set X which belongs to μ

For any non-empty set X and for any element $\alpha \in L$ we write $\overline{\alpha}$ to denote the constant map of X into L which maps every element of X onto α. In particular $\overline{0}$ is called the zero map of X into L and observe that $\overline{0} \subseteq \mu$ for any $\mu \in L^X$.

Definition 1.3.9. *[46] An L-fuzzy subset μ of a lattice X with 0 is said to be an L -fuzzy ideal of X, if $\mu(0) = 1$ and $\mu(a \vee b) = \mu(a) \wedge \mu(b)$ for all $a, b \in X$.*

Dually, an L-fuzzy subset μ of a lattice X with 1 is said to be an L -fuzzy filter of X, if $\mu(1) = 1$ and $\mu(a \wedge b) = \mu(a) \wedge \mu(b)$ for all $a, b \in X$.

Chapter 2

L-Fuzzy Ideals

In this chapter we introduce several generalizations of L-fuzzy ideals of a lattice to an arbitrary poset whose truth values are in a complete lattice satisfying the infinite meet distributive law and give several characterizations of them. The different types of L-fuzzy ideals of a poset that we introduce in this chapter are generalizations of the notions of L-fuzzy ideals of a lattice. We also prove that the set of all each types of L-fuzzy ideals of a poset forms a complete lattice with respect to point-wise ordering. Throughout this work L stands for a non-trivial complete lattice satisfying the infinite meet distributive law and throughout this chapter Q stands for a poset $(Q, \leq)$ with 0 unless otherwise stated.

2.1 L-Fuzzy Closed Ideals

In this section, we introduce the fuzzy version of the closed or normal ideal of a poset introduced by Birkoff[14]. We also prove and characterize certain properties of L-fuzzy closed ideals of a poset. In particular, we prove that the set of all L-fuzzy closed ideals of a poset form a complete lattice. We shall begin with its definition.

Definition 2.1.1. *An L- fuzzy subset μ of a poset Q is called an L- fuzzy closed ideal of Q, if it satisfies the following conditions:*

(i) $\mu(0) = 1$,

(ii) for any subset A of Q, $\mu(x) \geq \inf\{\mu(a) : a \in A\}\ \forall x \in A^{ul}$.

First of all, note that any L-fuzzy closed ideal μ of a poset Q is not the constant map $\overline{0}$. We prove the following lemma, which facilitates to identify any (crisp) closed ideal of Q with an L- fuzzy closed ideal of Q.

Lemma 2.1.1. *A subset I of Q is a closed ideal if and only if its characteristic map χ_I is an L-fuzzy closed ideal of Q.*

Proof. Suppose I is a closed ideal of Q. Recall that the characteristic map χ_I of I from Q into L is an L-fuzzy subset of Q given by:

$$\chi_I(x) = \begin{cases} 1 & if\ x \in I \\ 0 & otherwise \end{cases}$$

for each $x \in Q$. Note that $\chi_I \neq \overline{0}$ if and only if $I \neq \emptyset$.

Since $\emptyset \subseteq I$, we have $\{0\} = \emptyset^{ul} = Q^l \subseteq I$ and so $\chi_I(0) = 1$. Again let $A \subseteq Q$ and $x \in A^{ul}$. Then if $A \subseteq I$, we have $\inf\{\chi_I(a) : a \in A\} = 1$ and $x \in A^{ul} \subseteq I^{ul} = I$. Thus

$$\chi_I(x) = 1 = \inf\{\chi_I(a) : a \in A\}.$$

Again if $A \nsubseteq I$, then it is clear that $\inf\{\chi_I(a) : a \in A\} = 0$ and so

$$\chi_I(x) \geq 0 = \inf\{\chi_I(a) : a \in A\}.$$

Hence for any $A \subseteq Q$, we have

$$\chi_I(x) \geq \inf\{\chi_I(a) : a \in A\}, \text{ for all } x \in A^{ul}.$$

Therefore χ_I is an L-fuzzy closed ideal of Q.

Conversely, suppose that χ_I is an L- fuzzy closed ideal. Since $\chi_I(0) = 1$, we have $0 \in I$, i.e., $Q^l = \{0\} \subseteq I$. Let $x \in I^{ul}$. Then, by hypothesis, we have

$$\chi_I(x) \geq \inf\{\chi_I(a) : a \in I\} = 1 \text{ and so } \chi_I(x) = 1$$

and this implies that $x \in I$. Therefore $I^{ul} \subseteq I$ and so I is a closed ideal of Q. □

Note that, for any $I \subseteq Q$ and $\alpha \in L$, the α–level subset of the characteristic map $\chi_I : Q \longrightarrow L$ of I is Q if $\alpha = 0$ and I if $\alpha \neq 0$. Since Q is always a closed ideal of Q, it follows from the above theorem that χ_I is an L- fuzzy closed ideal of Q if and only if the α–level subset of χ_I a closed ideal of Q for each $\alpha \in L$.

The following lemma is a generalization of the above Lemma 2.1.1, which characterize any L- fuzzy closed ideal of Q in terms of its level subsets.

Lemma 2.1.2. *An L- fuzzy subset μ of a poset Q is an L- fuzzy closed ideal of Q if and only if the α–level subset μ_α of μ is a closed ideal of Q, for all $\alpha \in L$.*

Proof. Let μ be an L- fuzzy closed ideal of a poset Q and $\alpha \in L$. Then $\mu(0) = 1 \geq \alpha$. Thus $0 \in \mu_\alpha$, i.e., $Q^l = \{0\} \subseteq \mu_\alpha$. Again let $x \in (\mu_\alpha)^{ul}$. Then

$$\mu(x) \geq \inf\{\mu(a) : a \in \mu_\alpha\} \geq \alpha.$$

This implies that $x \in \mu_\alpha$. Therefore $(\mu_\alpha)^{ul} \subseteq \mu_\alpha$. So μ_α is a closed ideal of Q.

Conversely, suppose that μ_α is a closed ideal of Q for all $\alpha \in L$. In particular, μ_1 is a closed ideal. Since $\{0\} = Q^l \subseteq \mu_1$, we have $0 \in \mu_1$. and hence $\mu(0) = 1$. Again let A be any subset of Q and put $\alpha = \inf\{\mu(a) : a \in A\}$. Then $\mu(a) \geq \alpha$, $\forall a \in A$. Thus $A \subseteq \mu_\alpha$ and so we have $A^{ul} \subseteq \mu_\alpha^{ul} = \mu_\alpha$. Therefore

$$\mu(x) \geq \alpha = \inf\{\mu(a) : a \in A\} \text{ for all } x \in A^{ul}.$$

Hence μ is an L-fuzzy closed ideal of Q. □

Corollary 2.1.3. *Let μ be an L-fuzzy closed ideal of a poset Q. Then μ is anti-tone in the sense that $\mu(x) \geq \mu(y)$, whenever $x \leq y$.*

Proof. Let μ be an L-fuzzy closed ideal of a poset Q. Let $x, y \in Q$ such that $x \leq y$. Put $\mu(y) = \alpha$. Then $y \in \mu_\alpha$. Since μ_α is a closed ideal of Q and $y \in \mu_\alpha$, we have $y^l = \{y\}^{ul} \subseteq (\mu_\alpha)^{ul} = \mu_\alpha$. Thus $x \leq y \Rightarrow x \in y^l \Rightarrow x \in \mu_\alpha$. So $\mu(x) \geq \alpha = \mu(y)$. Therefore μ is anti-tone. □

Example 2.1.4. *Consider the poset* $([0,1],\leq)$ *of closed interval* $[0,1]$ *in the real number system with the usual order "$\leq$". Let* $L=\{0,\alpha,1\}$ *where* $1>\alpha>0$. *Let* $\mu\in L^{[0,1]}$ *defined by:*

$$\mu(x)=\begin{cases}1 & if\ x\in[0,\frac{1}{3}]\\ \alpha & if\ x\in[0,\frac{1}{2}]-[0,\frac{1}{3}]\\ 0 & if\ otherwise\end{cases}$$

for all $x\in[0,1]$.

Then, since $\mu_0=[0,1]$, $\mu_\alpha=[0,\frac{1}{2}]$, *and* $\mu_1=[0,\frac{1}{3}]$ *are all closed ideals of the poset* $([0,1],\leq)$, *by Lemma 2.1.2, we have* μ *is an L-fuzzy closed ideal of the poset* $([0,1],\leq)$.

Theorem 2.1.5. *Let* $x\in Q$ *and* $\alpha\in L$. *Define an L- fuzzy subset* α_x *of* Q *by:*

$$\alpha_x(y)=\begin{cases}1 & if\ y\in(x]\\ \alpha & if\ y\notin(x],\end{cases}$$

for all $y\in Q$. *Then* α_x *is an L-fuzzy closed ideal of* Q.

Proof. Since $0\in(x]$, we clearly have $\alpha_x(0)=1$. Let $A\subseteq Q$ and $y\in A^{ul}$. If $A\subseteq(x]$, then $y\in A^{ul}\subseteq(x]^{ul}=x^{lul}=x^l=(x]$. So $\alpha_x(y)=1$ and $\alpha_x(a)=1$ for all $a\in A$. Therefore $\alpha_x(y)=1=\inf\{\alpha_x(a):a\in A\}$. If $A\nsubseteq(x]$, then there exists $a_0\in A$ such that $a_0\notin(x]$. This implies that $\inf\{\mu(a):a\in A\}=\alpha$. Thus $\alpha_x(y)\geq\alpha=\inf\{\mu(a):a\in A\}$. So in either cases, we have

$$\alpha_x(y)\geq\inf\{\mu(a):a\in A\},\ \text{for all } y\in A^{ul}.$$

Hence α_x is an L-fuzzy closed ideal of Q. □

Definition 2.1.2. *The L-fuzzy closed ideal* α_x *of* Q *defined above is called the* α*-level principal L-fuzzy closed ideal corresponding to* x.

Lemma 2.1.6. *The intersection of any family of L- fuzzy closed ideals of a poset* Q *is an L-fuzzy closed ideal of* Q.

Proof. Let $\{\mu_i : i \in \Delta\}$ be any L-fuzzy closed ideal of Q. Now we claim that $\bigcap_{i\in\Delta}\mu_i$ is an L-fuzzy closed ideal of Q. If $\Delta = \emptyset$, then it is clear that, $\bigcap_{i\in\Delta}\mu_i = \overline{1}$, which is an L-fuzzy closed ideal of Q. Assume that $\Delta \neq \emptyset$. Since $\mu_i(0) = 1$, for all $i \in \Delta$, we have $\bigcap_{i\in\Delta}\mu_i(0) = \inf\{\mu_i(0) : i \in \Delta\} = 1$. Again let $A \subseteq Q$ and $x \in A^{ul}$. Then

$$\begin{aligned}(\bigcap_{i\in\Delta}\mu_i)(x) &= \inf\{\mu_i(x) : i \in \Delta\}\\ &\geq \inf\{\inf\{\mu_i(a) : a \in A\} : i \in \Delta\}\\ &= \inf\{\inf\{\mu_i(a) : i \in \Delta\} : a \in A\}\\ &= \inf\{(\bigcap_{i\in\Delta}\mu_i)(a) : a \in A\}\end{aligned}$$

Therefore $\bigcap_{i\in\Delta}\mu_i$ is an L-fuzzy closed ideal of Q. □

Definition 2.1.3. *Let μ be an L- fuzzy subset of a poset Q. Then the smallest L- fuzzy closed ideal of Q containing μ is called an L- fuzzy closed ideal generated by μ and is denoted by $(\mu]_{Cl}$.*

Theorem 2.1.7. *Let $\mathscr{FCI}(Q)$ be the set of all L-fuzzy closed ideals of a poset Q and μ be an L fuzzy subset of Q. Then $(\mu]_{Cl} = \bigcap\{\theta \in \mathscr{FCI}(Q) : \mu \subseteq \theta\}$.*

Proof. Put $X = \{\theta \in \mathscr{FCI}(Q) : \mu \subseteq \theta\}$. Then, by Lemma 2.1.6, $\bigcap_{\theta\in X}\theta$ is an L-fuzzy closed ideal of Q and it is clear that $\mu \subseteq \bigcap_{\theta\in X}\theta$. Let σ be any L-fuzzy closed ideal of Q such that $\mu \subseteq \sigma$. This implies that $\sigma \in X$ and hence $\bigcap_{\theta\in X}\theta \subseteq \sigma$.

This shows that $\bigcap_{\theta\in X}\theta$ is the smallest L-fuzzy closed ideal of Q containing μ. Therefore $(\mu]_{Cl} = \bigcap\{\theta \in \mathscr{FCI}(Q) : \mu \subseteq \theta\}$. □

Theorem 2.1.8. *Let $(A]_{Cl}$ be a closed ideal generated by subset A of Q and χ_A be its characteristics function. Then $\chi_{(A]_{Cl}} = (\chi_A]_{Cl}$, that is, the L- fuzzy closed ideal generated by χ_A is the characteristic map of the closed ideal generated by A.*

Proof. Now we prove $\chi_{(A]_{Cl}}$ that it is the smallest L-fuzzy closed ideal of Q containing χ_A. Since $(A]_{Cl}$ is a closed ideal of Q, by Lemma 2.1.1, we have $\chi_{(A]_{Cl}}$ is an L-fuzzy closed ideal. Again since $A \subseteq (A]_{Cl}$, we have $\chi_A \subseteq \chi_{(A]_{Cl}}$.

Let μ be any L-fuzzy closed ideal of Q such that $\chi_A \subseteq \mu$. Then $\mu(a) = 1$, for all $a \in A$. Now we claim that $\chi_{(A]_{Cl}} \subseteq \mu$. Let $x \in Q$. If $x \notin (A]_{Cl}$, then

$$\chi_{(A]_{Cl}}(x) = 0 \leq \mu(x).$$

If $x \in (A]_{Cl}$, then $x \in B^{ul}$, for some subset B of A. and hence

$$\chi_{(A]_{Cl}}(x) = 1 = \inf\{\chi_A(b) : b \in B\} \leq \inf\{\mu(b) : b \in B\} \leq \mu(x).$$

So that $\chi_{(A]_{Cl}}(x) \leq \mu(x)$, for all $x \in Q$. Therefore $\chi_{(A]_{Cl}} = (\chi_A]_{Cl}$. □

The following theorem is also another characterization of L-fuzzy closed ideal of a poset Q .

Theorem 2.1.9. *An L-fuzzy subset μ of a poset Q is an L-fuzzy closed ideal if and only if for any subset A of Q,*

$$\mu(x) \geq \inf\{\mu(a) : a \in A\}, \textit{ for all } x \in (A]_{Cl}.$$

Proof. Suppose that μ is an L-fuzzy closed ideal of Q. Let $A \subseteq Q$ and $x \in (A]_{Cl}$. Then $x \in B^{ul}$ for some $B \subseteq A$. Then, since μ is an L-fuzzy closed ideal of Q, we clearly have

$$\mu(x) \geq \inf\{\mu(b) : b \in B\}$$

and since $B \subseteq A$, we have

$$\inf\{\mu(b) : b \in B\} \geq \inf\{\mu(a) : a \in A\}.$$

This implies that, for any subset A of Q, we have

$$\mu(x) \geq \inf\{\mu(a) : a \in A\} \text{ for all } x \in (A]_{Cl}.$$

Conversely suppose that μ satisfies the given condition. Let $A \subseteq Q$ and $x \in A^{ul}$. Then, since $A^{ul} \subseteq (A]_{Cl}$, by hypothesis, we have

$$\mu(x) \geq \inf\{\mu(a) : a \in A\} \text{ for all } x \in A^{ul}.$$

In particular, if $A = \emptyset$, then ,by hypothesis, we have

$$\mu(x) \geq \inf\{\mu(a) : a \in \emptyset\}, \text{ for all } x \in (\emptyset]_{Cl}.$$

But since $inf\{\mu(a) : a \in \emptyset\} = 1$ and $(\emptyset]_{Cl} = \{0\}$, we clearly have $\mu(0) = 1$.

Therefore μ is an L-fuzzy closed ideal of Q. □

The following theorem characterizes any L-fuzzy closed ideal of Q generated by an L-fuzzy subset of Q in terms of closed ideals generated by its level subsets.

Theorem 2.1.10. *Let μ be an L-fuzzy subset of Q. Then the L-fuzzy subset $\hat{\mu}$ of Q defined by:*

$$\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_{Cl}\}, \textit{for all } x \in Q$$

is an L- fuzzy closed ideal of Q generated by μ.

Proof. We show that $\hat{\mu}$ is the smallest L- fuzzy closed ideal containing μ.

Let $x \in Q$. Then since $\mu_\alpha \subseteq (\mu_\alpha]_{Cl}$, we have

$$\mu(x) = \sup\{\alpha \in L : x \in \mu_\alpha\} \leq \sup\{\alpha \in L : x \in (\mu_\alpha]_{Cl}\} = \hat{\mu}(x).$$

Therefore $\mu \subseteq \hat{\mu}$.

Again since $0 \in Q^l \subseteq (\mu_\alpha]_{Cl}$, for all $\alpha \in L$ and in particular $0 \in (\mu_1]_{Cl}$, we have

$$\hat{\mu}(0) = \sup\{\alpha \in L : 0 \in (\mu_\alpha]_{Cl}\} \geq 1,$$

and so $\hat{\mu}(0) = 1$. Again let A be any subset of Q and $x \in A^{ul}$. Now, if $A = \emptyset$, then it is clear that $A^{ul} = \{0\}$ and $\inf\{\hat{\mu}(a) : a \in A\} = 1$. Thus we have

$$\inf\{\hat{\mu}(a) : a \in A\} = 1 = \hat{\mu}(0) = \hat{\mu}(x).$$

Again let $A \neq \emptyset$ Then we have

$$\begin{aligned} \inf\{\hat{\mu}(a) : a \in A\} &= \inf\{\sup\{\alpha_a : a \in (\mu_{\alpha_a}]_{Cl}\} : a \in A\} \\ &= \sup\{\inf\{\alpha_a : a \in A\} : a \in (\mu_{\alpha_a}]_{Cl}\} \end{aligned}$$

Put $\lambda = \inf\{\alpha_a : a \in A\}$. Then $\lambda \leq \alpha_a$ for all $a \in A$. This implies that $(\mu_{\alpha_a}]_{Cl} \subseteq (\mu_\lambda]_{Cl}$ for all $a \in A$. Therefore $A \subseteq (\mu_\lambda]_{Cl}$ and so $x \in A^{ul} \subseteq ((\mu_\lambda]_{Cl})^{ul} = (\mu_{\lambda_a}]_{Cl}$.

Hence

$$\begin{aligned}\inf\{\hat{\mu}(a) : a \in A\} &= \sup\{\inf\{\alpha_a : a \in A\} : a \in (\mu_{\alpha_a}]_{Cl}\} \\ &\leq \sup\{\lambda \in L : x \in (\mu_\lambda]_{Cl}\} \\ &= \hat{\mu}(x).\end{aligned}$$

Therefore $\hat{\mu}$ is an L-fuzzy closed ideal of Q.

Again let θ be any L-fuzzy closed ideal of Q such that $\mu \subseteq \theta$. Then $\mu_\alpha \subseteq \theta_\alpha$. for any $\alpha \in L$. This implies that

$$(\mu_\alpha]_{Cl} \subseteq (\theta_\alpha]_{Cl} = \theta_\alpha \text{ for any } \alpha \in L.$$

So for any $x \in Q$, we have

$$\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_{Cl}\} \leq \sup\{\alpha \in L : x \in \theta_\alpha\} = \theta(x).$$

Hence $\hat{\mu} \subseteq \theta$. Therefore $\hat{\mu} = (\mu]_{Cl}$. □

In the following we give an algebraic characterization of L-fuzzy closed ideal generated by an L-fuzzy subset of Q.

Theorem 2.1.11. *Let μ be an L- fuzzy subset of Q. Then the fuzzy subset $\overline{\mu}_{Cl}$ defined by*

$$\overline{\mu}_{Cl}(x) = \begin{cases} 1 & if x = 0 \\ \sup\{\inf_{a\in A}\mu(a) : A \subseteq Q \; and \;\; x \in A^{ul}\} & if x \neq 0 \end{cases}$$

is an L-fuzzy closed ideal of Q generated by μ.

Proof. It is enough to show that $\overline{\mu}_{Cl} = \hat{\mu}$, where $\hat{\mu}$ is the L- fuzzy closed ideal given in the Theorem 2.1.10 above. Let $x \in Q$. If $x = 0$, then $\bar{\mu}(x) = 1 = \hat{\mu}(x)$.

Let $x \neq 0$. Put

$$A_x = \{\inf_{a\in A}\mu(a) : A \subseteq Q \; and \;\; x \in A^{ul}\} \text{ and } B_x = \{\alpha : x \in (\mu_\alpha]_{Cl}\}.$$

Now we claim that $\sup A_x = \sup B_x$. Let $\alpha \in A_x$. Then $\alpha = \inf_{a\in A}\mu(a)$, for some subset A of Q such that $x \in A^{ul}$. This implies that $\alpha \leq \mu(a)$, for all $a \in A$. Thus $A \subseteq \mu_\alpha \subseteq (\mu_\alpha]_{Cl}$. Since $(\mu_\alpha]_{Cl}$ is a closed ideal, we have $x \in A^{ul} \subseteq ((\mu_\alpha]_{Cl})^{ul} \subseteq (\mu_\alpha]_{Cl}$. So $x \in (\mu_\alpha]_{Cl}$, i.e., $\alpha \in B_x$. Hence $A_x \subseteq B_x$. Therefore $\sup A_x \leq \sup B_x$.

Again let $\alpha \in B_x$. Then $x \in (\mu_\alpha]_C$. Since $(\mu_\alpha]_{Cl} = \bigcup\{A^{ul} : A \subseteq \mu_\alpha\}$, we have $x \in A^{ul}$, for some subset A of μ_α. This implies that $\mu(a) \geq \alpha$, for all $a \in A$. Thus $\inf\{\mu(a) : a \in A\} \geq \alpha$. Put $\beta = \inf\{\mu(a) : a \in A\}$. Then $\beta \in A_x$. Thus for each $\alpha \in B_x$, we get $\beta \in A_x$ such that $\alpha \leq \beta$. So $\sup A_x \geq \sup B_x$. Hence $\sup A_x = \sup B_x$.

Therefore $\overline{\mu}_{Cl} = \hat{\mu} = (\mu]_{Cl}$. □

Theorem 2.1.11 yields the following.

Theorem 2.1.12. *The set $\mathscr{FCI}(Q)$ of all L-fuzzy closed ideals of Q forms a complete lattice, in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\mu_i$ of any family $\{\mu_i : i \in \Delta\}$ of L-fuzzy closed ideals of Q respectively are given by:*

$sup_{i\in\Delta}\mu_i = \overline{(\bigcup_{i\in\Delta}\mu_i)}_{Cl}$ where $\overline{(\bigcup_{i\in\Delta}\mu_i)}_{Cl}$ is given by:

$$\overline{(\bigcup_{i\in\Delta}\mu_i)}_{Cl}(x) = \begin{cases} 1 & if\ x = 0 \\ \sup\{\inf_{a\in A}(\bigcup_{i\in\Delta}\mu_i)(a) : A \subseteq Q\ and\ \ x \in A^{ul}\} & if\ x \neq 0 \end{cases}$$

for all $x \in Q$ and $\inf_{i\in\Delta}\mu_i = \bigcap_{i\in\Delta}\mu_i$, where $\bigcap_{i\in\Delta}\mu_i$ is given by:

$$(\bigcap_{i\in\Delta}\mu_i)(x) = \inf_{i\in\Delta}\mu_i(x) \text{ for all } x \in Q.$$

.

Corollary 2.1.13. *For any μ and θ in $\mathscr{FCI}(Q)$, the supremum $\mu \vee \theta$ and the infimum $\mu \wedge \theta$ of μ and θ ,respectively are: $\mu \vee \theta = \overline{(\mu \cup \theta)}_{Cl}$ where $\overline{(\mu \cup \theta)}_{Cl}$ is given by:*

$$\overline{(\mu \cup \theta)}_{Cl}(x) = \begin{cases} 1 & if\ x = 0 \\ \sup\{\inf_{a\in A}(\mu \cup \theta)(a) : A \subseteq Q\ and\ \ x \in A^{ul}\} & if\ x \neq 0 \end{cases},$$

for all $x \in Q$*. and* $\mu \wedge \theta = \mu \cap \theta$*, where* $\mu \cap \theta$ *is given by:*

$$(\mu \cap \theta)(x) = \mu(x) \wedge \theta(x) \text{ for all} x \in Q.$$

2.2 L-Fuzzy Frink Ideals

In this section we introduce the fuzzy version of the ideals of a poset introduced by O. Frink [25].

Definition 2.2.1. *An L- fuzzy subset* μ *of* Q *is called an L- fuzzy Firink ideal, if it satisfies the following conditions:*

(i) $\mu(0) = 1$,

(ii) for any finite subset F *of* Q*,* $\mu(x) \geq \inf\{\mu(a) : a \in F\}\ \forall x \in F^{ul}$.

The following lemma characterizes any L- fuzzy Frink ideal of Q in terms of its level subset whose proof is similar to the proof of Lemma 2.1.2.

Lemma 2.2.1. *An L- fuzzy subset* μ *of* Q *is an L- fuzzy Frink ideal of* Q *if and only if* μ_α *is a Frink ideal of* Q*, for all* $\alpha \in L$.

As the consequence of the above lemma we have the following corollary.

Corollary 2.2.2. *A subset* I *of* Q *is a Frink ideal of* Q *if and only if its characteristic map* χ_I *is an L-fuzzy Frink ideal of* Q*.*

Theorem 2.2.3. *Let* $x \in Q$ *and* $\alpha \in L$*. Define an L- fuzzy subset* α_x *of* Q *by*

$$\alpha_x(y) = \begin{cases} 1 & if\ y \in (x] \\ \alpha & if\ y \notin (x], \end{cases}$$

for all $y \in Q$*. Then* α_x *is an L-fuzzy frink ideal of* Q*.*

Definition 2.2.2. *The L-fuzzy Frink ideal* α_x *defined in Theorem 2.2.3 above is called the* α*-level principal L-fuzzy Frink ideal corresponding to* x*.*

The following theorem shows that an L- fuzzy Frink ideal of a poset is a natural generalization of an L- fuzzy ideal of a lattice.

Theorem 2.2.4. *Let $(Q,\leq)$ be a lattice. An L-fuzzy subset μ of Q is an L- fuzzy Frink ideal in the poset Q if and only it an L-fuzzy ideal in the lattice Q.*

Proof. Let μ be an L-fuzzy Frink ideal in the poset Q and $a,b\in Q$. Then, by definition, $\mu(0)=1$ and since $F=\{a,b\}\subset\subset Q$ and $a\vee b\in F^{ul}$, we have

$$\mu(a\vee b)\geq\inf\{\mu(x):x\in F\}=\mu(a)\wedge\mu(b).$$

Again since μ is anti-tone, we have

$$\mu(a\vee b)\leq\mu(a)\text{ and }\mu(a\vee b)\leq\mu(b).$$

So we have $\mu(a)\wedge\mu(b)\leq\mu(a\vee b)$. Therefore $\mu(a\vee b)=\mu(a)\wedge\mu(b)$ and hence μ is an L-fuzzy ideal in the lattice Q.

Conversely suppose that μ is an L-fuzzy ideal in the lattice Q. Then clearly $\mu(0)=1$. Again et $F\subset\subset Q$ and $x\in F^{ul}$. Then x is a lower bound of F^u. Since $\sup F\in F^u$, we have $x\leq\sup F$ and hence we have

$$\mu(x)\geq\mu(\sup F)=\inf\{\mu(a):a\in F\}.$$

Therefore μ is an L-fuzzy Frink ideal in the poset Q. □

Lemma 2.2.5. *The intersection of any family of L-fuzzy Frink-ideals of a poset Q is an L-fuzzy Frink ideal.*

Definition 2.2.3. *Let μ be an L- fuzzy subset of a poset Q. Then the smallest L- fuzzy Frink ideal of Q containing μ is called an L- fuzzy Frink ideal generated by μ and is denoted by $(\mu]_{Fr}$.*

Theorem 2.2.6. *Let $\mathscr{F}\mathscr{F}\mathscr{I}(Q)$ be the set of all L-fuzzy Frink ideals of a poset Q and μ be an L fuzzy subset of Q. Then $(\mu]_{Fr}=\bigcap\{\theta\in\mathscr{F}\mathscr{F}\mathscr{I}(Q):\mu\subseteq\theta\}$.*

Theorem 2.2.7. *Let $(A]_{Fr}$ be a Frink-ideal generated subset A of Q and χ_A be its characteristics functions. Then $(\chi_A]_{Fr} = \chi_{(A]_{Fr}}$.*

Theorem 2.2.8. *An L-fuzzy subset μ of Q is an L-fuzzy Frink ideal if and only if for any finite subset F of Q,*

$$\mu(x) \geq \inf\{\mu(a) : a \in F\} \textit{ for all } x \in (F]_{Fr}.$$

Proof. Suppose that μ is an L-fuzzy ideal of Q. Let $F \subset\subset Q$ and $x \in (F]_{Fr}$. Then $x \in B^{ul}$ for some $B \subseteq F$. Then, since $B \subset\subset Q$ and μ is an L-fuzzy Frink ideal of Q, we have

$$\mu(x) \geq \inf\{\mu(b) : b \in B\} \geq \inf\{\mu(a) : a \in F\}.$$

Thus , for any finite subset F of Q, we have

$$\mu(x) \geq \inf\{\mu(a) : a \in F\} \text{ for all } x \in (F]_{Fr}.$$

Conversely suppose that μ satisfies the given condition. Now we show that μ is an L-fuzzy Frink ideal. Let $F \subset\subset Q$ and $x \in F^{ul}$. Since $x \in F^{ul} \subseteq (F]_{Fr}$, by hypothesis, we have

$$\mu(x) \geq \inf\{\mu(a) : a \in F\}.$$

Again since $\emptyset$ is finite subset of Q, by hypothesis, we have

$$\mu(x) \geq \inf\{\mu(a) : a \in \emptyset\}, \text{ for all } x \in (\emptyset]_{Fr}.$$

But since $inf\{\mu(a) : a \in \emptyset\} = 1$ and $(\emptyset]_{Fr} = \{0\}$, we clearly have $\mu(0) = 1$. Therefore μ is an L-fuzzy Frink ideal of Q. □

Now we give a characterization of any L-fuzzy Frink ideal generated by an L-fuzzy subset of Q in terms of Frink ideals generated by its level subset.

Theorem 2.2.9. *For any L-fuzzy subset μ of Q, define an L-fuzzy subset $\hat{\mu}$ of Q by:*

$$\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_{Fr}\}, \textit{for all } x \in Q.$$

Then $\hat{\mu}$ is an L-fuzzy Frink ideal of Q generated by μ.

Proof. We show that $\hat{\mu}$ is the smallest L- fuzzy Frink ideal containing μ. Let $x \in Q$. Then as $\mu_\alpha \subseteq (\mu_\alpha]_{Fr}$, we have

$$\mu(x) = \sup\{\alpha \in L : x \in \mu_\alpha\} \leq \sup\{\alpha \in L : x \in (\mu_\alpha]_{Fr}\} = \hat{\mu}(x).$$

Hence $\mu \subseteq \hat{\mu}$.

Again since $0 \in Q^l \subseteq (\mu_\alpha]_{Fr}$, for all $\alpha \in L$ and in particular $0 \in (\mu_1]_{Fr}$, we have

$$\hat{\mu}(0) = \sup\{\alpha \in L : 0 \in (\mu_\alpha]_{Fr}\} \geq 1,$$

and so $\hat{\mu}(0) = 1$. Again let $F \subset\subset Q$ and $x \in F^{ul}$. Then we have

$$\begin{aligned} \inf\{\hat{\mu}(a) : a \in F\} &= \inf\{\sup\{\alpha_a : a \in (\mu_{\alpha_a}]_{Fr}\} : a \in F\} \\ &= \sup\{\inf\{\alpha_a : a \in F\} : a \in (\mu_{\alpha_a}]_{Fr}\} \end{aligned}$$

Put $\lambda = \inf\{\alpha_a : a \in F$. This implies that $\lambda \leq \alpha_a$, for all $a \in F$. Therefore $(\mu_\alpha]_{Fr} \subseteq (\mu_\lambda]_{Fr}$ for all $a \in F$. Thus $F \subseteq (\mu_{\lambda_a}]_{Fr}$ and so $x \in F^{ul} \subseteq (\mu_{\lambda_a}]_{Fr}$. Therefore

$$\begin{aligned} \inf\{\hat{\mu}(a) : a \in F\} &= \sup\{\inf\{\alpha_a : a \in F\} : a \in (\mu_{\alpha_a}]_{Fr}\} \\ &\leq \sup\{\lambda \in L : x \in (\mu_\lambda]_{Fr}\} \\ &= \hat{\mu}(x). \end{aligned}$$

Therefore $\hat{\mu}$ is an L-fuzzy Frink ideal of Q.

Again let θ be any L-fuzzy Frink ideal of Q such that $\mu \subseteq \theta$. Then $\mu_\alpha \subseteq \theta_\alpha$. for any $\alpha \in L$. This implies that $(\mu_\alpha]_{Fr} \subseteq (\theta_\alpha]_{Fr} = \theta_\alpha$ for any $\alpha \in L$. So for any $x \in Q$,

$$\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_{Fr}\} \leq \sup\{\alpha \in L : x \in \theta_\alpha\} = \theta(x).$$

Hence $\hat{\mu} \subseteq \theta$. Therefore $\hat{\mu} = (\mu]_{Fr}$. □

In the following we give an algebraic characterization of L-fuzzy Frink ideals generated by L-fuzzy subsets. We write $F \subset\subset Q$ to mean that F a finite subset of Q.

Theorem 2.2.10. *Let μ be an L-fuzzy subset of Q. Then the L- fuzzy subset $\overline{\mu}_{Fr}$ defined by:*

$$\overline{\mu}_{Fr}(x) = \begin{cases} 1 & if\ x = 0 \\ \sup\{\inf_{a\in F}\mu(a) : F \subset\subset Q \ and \ x \in F^{ul}\} & if\ x \neq 0 \end{cases}$$

is an L- fuzzy Frink ideal of Q generated by μ.

Proof. It is enough to show that $\overline{\mu}_{Fr} = \hat{\mu}$, where $\hat{\mu}$ is the L- fuzzy Frink ideal given in the theorem 2.2.9 above. Let $x \in Q$. If $x = 0$, then $\bar{\mu}(x) = 1 = \hat{\mu}(x)$.
Let $x \neq 0$. Put

$$A_x = \{\inf_{a\in F}\mu(a) : F \subset\subset Q \ \text{ and } \ x \in F^{ul}\} \ \text{ and } \ B_x = \{\alpha : x \in (\mu_\alpha]_{Fr}\}.$$

Now we show $\sup A_x = \sup B_x$. Let $\alpha \in A_x$. Then $\alpha = \inf_{a\in F}\mu(a)$, for some finite subset F of Q such that $x \in F^{ul}$. This implies that $\alpha \leq \mu(a)$, for all $a \in F$. Thus $F \subseteq \mu_\alpha \subseteq (\mu_\alpha]_{Fr}$. Since $(\mu_\alpha]_{Fr}$ is a Frink ideal, we have $x \in F^{ul} \subseteq (\mu_\alpha]_{Fr}$. So $x \in (\mu_\alpha]_{Fr}$, i.e., $\alpha \in B_x$. Hence $A_x \subseteq B_x$. Therefore $\sup A_x \leq \sup B_x$.

Again let $\beta \in B_x$. Then $x \in (\mu_\beta]_{Fr}$. Since $(\mu_\beta]_{Fr} = \bigcup\{F^{ul} : F \subseteq \ \mu_\beta\}$, we have $x \in F^{ul}$, for some finite subset F of μ_β. This implies that $\inf\{\mu(a) : a \in A\} \geq \beta$. Put $\alpha = \inf\{\mu(a) : a \in F\}$. Then since $x \in F^{ul}, F \subset\subset Q$, we have $\alpha \in A_x$. Thus for each $\beta \in B_x$, we get $\alpha \in A_x$ such that $\alpha \geq \beta$. So $\sup A_x \geq \sup B_x$. Hence $\sup A_x = \sup B_x$. Therefore $\overline{\mu}_{Fr} = \hat{\mu} = (\mu]_{Fr}$. ☐

Theorem 2.2.10 yields the following.

Theorem 2.2.11. *The set $\mathscr{F}\mathscr{F}\mathscr{I}(Q)$ of all L-fuzzy Frink ideal of Q forms a complete lattice, in which the supremum $\sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\mu_i$ of any family $\{\mu_i : i \in \Delta\}$ of L-fuzzy Frink ideals of Q are given by:*

$$\sup_{i\in\Delta}\mu_i = \overline{(\bigcup_{i\in\Delta}\mu_i)}_{Fr} \quad \text{and} \quad \inf_{i\in\Delta}\mu_i = \bigcap_{i\in\Delta}\mu_i$$

.

Corollary 2.2.12. *For any μ and θ in $\mathscr{F}\mathscr{F}\mathscr{I}(Q)$, the supremum $\mu \vee \theta$ and the infimum $\mu \wedge \theta$ of μ and θ, respectively are:*

$$\mu \vee \theta = (\overline{\mu \cup \theta})_{Fr} \quad \text{and} \quad \mu \wedge \theta = \mu \cap \theta.$$

2.3 L-Fuzzy Ideals in the Sense of Halaś

Now we introduce the fuzzy version ideals of a poset introduced by Halaš [28] which seems to be a suitable generalization of the usual concept of L-fuzzy ideal of a lattice.

Definition 2.3.1. *An L- fuzzy subset μ of Q is called an L- fuzzy ideal in the sense of Halaś, if it satisfies the following conditions:*

(i) $\mu(0) = 1$,

(ii) for any $a, b \in Q$, $\mu(x) \geq \mu(a) \wedge \mu(b)$, for all $x \in (a,b)^{ul}$.

The following lemma characterizes any L- fuzzy ideal of a poset in the sense of Halaś in terms of its level-subsets.

Lemma 2.3.1. *An L- fuzzy subset μ of Q is an L- fuzzy ideal of Q in the sense of Halaś if and only if μ_α is an ideal of Q, for all $\alpha \in L$.*

Proof. Let μ be an L- fuzzy ideal of Q in the sense of Halaś and $\alpha \in L$. Then since $\mu(0) = 1 \geq \alpha$, we have $0 \in \mu_\alpha$, i.e., $Q^l = \{0\} \subseteq \mu_\alpha$. Let $a, b \in \mu_\alpha$. Then clearly, $\mu(a) \wedge \mu(b) \geq \alpha$. Let $x \in (a,b)^{ul}$. Then $\mu(x) \geq \mu(a) \wedge \mu(b) \geq \alpha$. This implies that $x \in \mu_\alpha$. Therefore $(a,b)^{ul} \subseteq \mu_\alpha$. So μ_α is an ideal of Q.

Conversely, suppose that μ_α is an ideal of Q, for all $\alpha \in L$. In particular, μ_1 is an ideal of Q. Since $\{0\} = Q^l \subseteq \mu_1$, we have $0 \in \mu_1$ and hence $\mu(0) = 1$. Again let $a, b \in Q$. Put $\alpha = \mu(a) \wedge \mu(b)$. This implies that $a, b \in \mu_\alpha$. Since, by hypothesis, μ_α is an ideal of Q we have $(a,b)^{ul} \subseteq \mu_\alpha$. This implies that $\mu(x) \geq \alpha = \mu(a) \wedge \mu(b)$ for all $x \in (a,b)^{ul}$. Therefore μ is an L-fuzzy ideal of Q in the sense of Halaś. $\square$

Corollary 2.3.2. *A subset I of Q is an ideal of Q if and only if its characteristic map χ_I is an L-fuzzy ideal of Q in the sense of Halaś.*

Lemma 2.3.3. *If μ is an L- fuzzy ideal of Q in the sense of Halaś, then the following assertions hold:*

1. *for any $x,y \in Q$, $\mu(x) \geq \mu(y)$, whenever $x \leq y$, i.e., μ is anti-tone.*

2. *for any $x,y \in Q$, $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$, whenever $x \vee y$ exists.*

The following theorem shows that an L- fuzzy ideal in the sense of Halaś of the poset is a natural generalization of an L- fuzzy ideal of a lattice.

Theorem 2.3.4. *Let $(Q, \leq)$ be a lattice. Then an L-fuzzy subset μ of Q is an L- fuzzy ideal in the sense of Halaś in the poset Q if and only if it is an L-fuzzy ideal in the lattice Q.*

Proof. Let μ be an L-fuzzy ideal in the sense of Halaś in the poset Q and $a,b \in Q$. Then $\mu(0) = 1$. Since $a \vee b \in (a,b)^{ul}$, we have $\mu(a \vee b) \geq \mu(a) \wedge \mu(b)$. Since μ is anti-tone, we have $\mu(a) \geq \mu(a \vee b)$ and $\mu(b) \geq \mu(a \vee b)$. Thus $\mu(a) \wedge \mu(b) \geq \mu(a \vee b)$. So $\mu(a \vee b) = \mu(a) \wedge \mu(b)$. Hence μ is an L-fuzzy ideal in the lattice Q.

Conversely, suppose that μ is an L-fuzzy ideal in the lattice Q. Let $a,b \in Q$ and $x \in (a,b)^{ul}$. Then $x \leq y$, for all $y \in (a,b)^u$. Since $a \vee b \in (a,b)^u$, we have $x \leq a \vee b$. Thus $\mu(x) \geq \mu(a \vee b) = \mu(a) \wedge \mu(b)$. So μ is an L-fuzzy ideal in the sense of Halaś in the poset Q. □

Definition 2.3.2. *Let μ be an L-fuzzy subset of Q. The smallest L- fuzzy ideal in the sense of Halaś of Q containing μ is called an L- fuzzy ideal in the sense of Halaś generated by μ and is denoted by $(\mu]_{Ha}$.*

Lemma 2.3.5. *The intersection of any family of L-fuzzy ideals in the sense of Halaś of a poset Q is an L- fuzzy deal in the sense of Halaś.*

Theorem 2.3.6. *Let $\mathscr{F}\mathscr{I}(Q)$ be the set of all L-fuzzy ideals in the sense of Halaś of a poset Q and μ be an L fuzzy subset of Q. Then $(\mu]_{Ha} = \bigcap\{\theta \in \mathscr{F}\mathscr{I}(Q) : \mu \subseteq \theta\}$.*

Theorem 2.3.7. *Let $(A]_{Ha}$ be an ideal generated by a subset A of Q in the sense of Halaś and χ_A be characteristics functions of A. Then $(\chi_A] = \chi_{(A]_{Ha}}$.*

Proof. Since $(A]_{Ha}$ is an ideal of Q, by Corollary 2.3.2, we have $\chi_{(A]_{Ha}}$ is an L-fuzzy ideal in the sense of Halaś. Again since $A \subseteq (A]_{Ha}$, we have $\chi_A \subseteq \chi_{(A]_{Ha}}$. Let μ be any L-fuzzy ideal of Q in the sense of Halaś such that $\chi_A \subseteq \mu$. Now we claim $\chi_{(A]_{Ha}} \subseteq \mu$. Let $x \in Q$. If $x \notin (A]_{Ha}$, then $\chi_{(A]_{Ha}}(x) = 0 \leq \mu(x)$. Let $x \in (A]_{Ha}$. Since $\chi_A \subseteq \mu$, we have $A \subseteq \mu_1$. This implies that $(A]_{Ha} \subseteq (\mu_1]_{Ha} = \mu_1$. Thus $x \in \mu_1$ and hence $\mu(x) = 1$. Therefore $\chi_{(A]_{Ha}}(x) = 1 = \mu(x)$. So that $\chi_{(A]_{Ha}}(x) \leq \mu(x)$, for all $x \in Q$. Hence the claim holds. □

The following result is another characterization of an L-fuzzy ideal in the sense of Halaś.

Theorem 2.3.8. *An L-fuzzy subset μ of Q is an L-fuzzy ideal in the sense of Halaś if and only if for any $F \subset\subset Q$,*

$$\mu(x) \geq \bigwedge_{a\in F} \mu(a) \text{ for all } x \in (F]_{Ha}.$$

Proof. Suppose that μ is an L-fuzzy ideal in the sense of Halaś of Q. Let $F \subset\subset Q$ and put $\alpha = \inf\{\mu(a) : a \in F\}$. Then $\mu(a) \geq \alpha$ for all $a \in F$ and hence $F \subseteq \mu_\alpha$. Clearly, by Lemma 2.3.1, μ_α is an ideal. Therefore $(F]_{Ha} \subseteq \mu_\alpha$ and hence $\mu(x) \geq \alpha = \bigwedge_{a\in F} \mu(a)$ for all $x \in (F]_{Ha}$.

Conversely suppose that μ satisfies the given condition. Now since $\emptyset$ is finite and $(\emptyset]_{Ha} = \{0\}$ we have

$$\mu(0) \geq \inf\{\mu(a) : a \in \emptyset\} = 1 \text{ and hence } \mu(0) = 1.$$

Let $a, b \in Q$ such that $x \in (a,b)^{ul}$. Put $F = \{a,b\}$. Then it is clear that $x \in (a,b)^{ul} \subseteq C_1 \subseteq (F]_{Ha}$. Thus, by hypothesis, we have

$$\mu(x) \geq \inf\{\mu(y) : y \in F\} = \mu(a) \wedge \mu(b).$$

Therefore μ is an L-fuzzy ideal in the sense of Halaś of Q. □

In the rest of this dissertation, by an L- fuzzy ideal of a poset will mean an L-fuzzy ideal in the sense of Halaś.

Now we give a characterization of any L-fuzzy ideal generated by an L-fuzzy subset of Q in terms of ideals generated by its level subset.

Theorem 2.3.9. *For any L-fuzzy subset μ of Q, define an L-fuzzy subset $\hat{\mu}$ of Q by:*

$$\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_{Ha}\}, \text{for all } x \in Q.$$

Then $\hat{\mu}$ is an L-fuzzy ideal of Q generated by μ.

Definition 2.3.3. *Let μ be an L-fuzzy subset of Q and $\mathbb{N}$ be a set of positive integers. Define L-fuzzy subsets $C_1^\mu, C_2^\mu, \cdots, C_n^\mu \cdots$, of Q, inductively, as follow: for each $x \in Q$,*

$$C_1^\mu(x) = \sup\{\mu(a) \wedge \mu(b) : x \in (a,b)^{ul}\}$$

and for each $n \in \mathbb{N} - \{1\}$,

$$C_n^\mu(x) = \sup\{C_{n-1}^\mu(a) \wedge C_{n-1}^\mu(b) : x \in (a,b)^{ul}\}.$$

Lemma 2.3.10. *The set $\{C_n^\mu : n \in \mathbb{N}\}$ forms a chain and C_n^μ is antitone for each $n \in \mathbb{N}$.*

Proof. Let $x \in Q$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} C_{n+1}^\mu(x) &= \sup\{C_n^\mu(a) \wedge C_n^\mu(b) : x \in (a,b)^{ul}\} \\ &\geq C_n^\mu(x) \wedge C_n^\mu(x) \ (\textit{since } x \in x^l = (x,x)^{ul}) \\ &= C_n^\mu(x), \ \forall x \in Q. \end{aligned}$$

Thus $C_n^\mu \subseteq C_{n+1}^\mu$, for each $n \in \mathbb{N}$. So $\{C_n^\mu : n \in \mathbb{N}\}$ is a chain.

Let $x, y \in Q$ such that $x \leq y$.. Then

$$\begin{aligned} C_n^\mu(y) &= \sup\{C_{n-1}^\mu(a) \wedge C_{n-1}^\mu(b) : y \in (a,b)^{ul}\} \\ &\leq \sup\{C_{n-1}^\mu(a) \wedge C_{n-1}^\mu(b) : x \in (a,b)^{ul}\} = C_n^\mu(x) \end{aligned}$$

Thus C_n^μ is antitone for each $n \in \mathbb{N}$., □

Now we give a characterization of an L-fuzzy ideal generated by an L- fuzzy subset of a poset Q.

Theorem 2.3.11. *The L- fuzzy subset $\overline{\mu}_{Ha}$ defined by: for all $x \in Q$,*

$$\overline{\mu}_{Ha}(x) = \begin{cases} 1 & if\ x = 0 \\ \sup\{C_n^{\mu}(x) : n \in \mathbb{N}\} & if\ x \neq 0 \end{cases}$$

is an L-fuzzy ideal generated by μ.

Proof. Now we claim that $\overline{\mu}_{Ha}$ is the smallest L-fuzzy ideal containing μ. Now, for any $x \in Q$, we have

$$\begin{aligned} \overline{\mu}_{Ha}(x) &\geq C_1^{\mu}(x) \\ &= \sup\{\mu(a) \wedge \mu(b) : x \in (a,b)^{ul}\} \\ &\geq \mu(x) \wedge \mu(x) \ \ (since\ x \in (x,x)^{ul}) \\ &= \mu(x) \end{aligned}$$

Thus $\mu \subseteq \overline{\mu}_{Ha}$. By definition of $\overline{\mu}_{Ha}$, $\overline{\mu}_{Ha}(0) = 1$. Let $a, b \in Q$ and $x \in (a,b)^{ul}$. If $a = 0$ or $b = 0$. then it is clear that $x \leq a$ or $x \leq b$ and since C_n^{μ} is antitone for each $n \in \mathbb{N}$., we have $\overline{\mu}_{Ha}(a) \wedge \overline{\mu}_{Ha}(b) \leq \overline{\mu}_{Ha}(x)$. Let $a \neq 0$ and $b \neq 0$. Then

$$\begin{aligned} \overline{\mu}_{Ha}(a) \wedge \overline{\mu}_{Ha}(b) &= \sup\{C_n^{\mu}(a) : n \in \mathbb{N}\} \wedge \sup\{C_m^{\mu}(b) : m \in \mathbb{N}\} \\ &= \sup\{C_n^{\mu}(a) \wedge C_m^{\mu}(b) : n, m \in \mathbb{N}\} \\ &\leq \sup\{C_k^{\mu}(a) \wedge C_k^{\mu}(b) : k \in \mathbb{N}\} \text{ where } k = \max\{m,n\} \\ &\leq C_{k+1}^{\mu}(x) \ \ (\text{Since } x \in (a,b)^{ul}) \\ &\leq \sup\{C_n^{\mu}(x) : n \in \mathbb{N}\} \\ &= \overline{\mu}_{Ha}(x) \end{aligned}$$

So $\overline{\mu}_{Ha}$ is an L-fuzzy ideal of Q.

Again let θ be any L-fuzzy ideal of Q such that $\mu \subseteq \theta$. Now we show that the statement " $C_n^{\mu} \subseteq \theta$ for all $n \in \mathbb{N}$ " is true. Now for any $x \in Q$, we have

$$\begin{aligned} C_1^{\mu}(x) &= \sup\{\mu(a) \wedge \mu(b) : x \in (a,b)^{ul}\} \\ &\leq \sup\{\theta(a) \wedge \theta(b) : x \in (a,b)^{ul}\} \leq \theta(x). \end{aligned}$$

. This implies that $C_1^{\mu} \subseteq \theta$. Hence the statement is true for $n = 1$. Assume $C_n^{\mu} \subseteq \theta$ for some $n > 1$. Now for any $x \in Q$, we have

$$\begin{aligned} C_{n+1}^{\mu}(x) &= \sup\{C_n^{\mu}(a) \wedge C_n^{\mu}(b) : x \in (a,b)^{ul}\} \\ &\leq \sup\{\theta(a) \wedge \theta(b) : x \in (a,b)^{ul}\} \leq \theta(x). \end{aligned}$$

Thus $C_{n+1}^{\mu} \subseteq \theta$. Thus, by mathematical induction, we have $C_n^{\mu} \subseteq \theta$ for all $n \in \mathbb{N}$. Let $x \in Q$. If $x = 0$, then we have $\overline{\mu}_{Ha}(x) = 1 = \theta(x)$. Let $x \neq 0$. Then

$$\overline{\mu}_{Ha}(x) = sup\{C_n^{\mu}(x) : n \in \mathbb{N}\} \leq \theta(x)$$

Hence $\overline{\mu}_{Ha} \subseteq \theta$. Therefore $\overline{\mu}_{Ha} = (\mu]_{Ha}$. □

Theorem 2.3.11 yields the following.

Theorem 2.3.12. *The set $\mathscr{F}\mathscr{I}(Q)$ of all L-fuzzy ideal of Q forms a complete lattice, in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\mu_i$ of any family $\{\mu_i : i \in \Delta\}$ in $\mathscr{F}\mathscr{I}(Q)$ respectively are:*

$$sup_{i\in\Delta}\mu_i = \overline{(\bigcup_{i\in\Delta}\mu_i)}_{Ha} \text{ and } \inf_{i\in\Delta}\mu_i = \bigcap_{i\in\Delta}\mu_i.$$

Corollary 2.3.13. *For any μ and $\theta \in \mathscr{F}\mathscr{I}(Q)$ the supremum $\mu \vee \theta$ and the infimum $\mu \wedge \theta$ of μ and θ respectively are:*

$$\mu \vee \theta = \overline{(\mu \cup \theta)}_{Ha} \quad \text{and} \quad \mu \wedge \theta = \mu \cap \theta.$$

Theorem 2.3.14. *Let $x \in Q$ and $\alpha \in L$. Define an L- fuzzy subset α_x of Q by*

$$\alpha_x(y) = \begin{cases} 1 & if\ y \in (x] \\ \alpha & if\ y \notin (x], \end{cases}$$

for all $y \in Q$. Then α_x is an L-fuzzy ideal of Q.

Definition 2.3.4. *The L-fuzzy ideal α_x defined above is called the α-level principal L-fuzzy ideal corresponding to x.*

Definition 2.3.5. *An L-fuzzy ideal μ of a poset Q is called a u-L-fuzzy ideal, if for any $a,b \in Q$, there exists $x \in (a,b)^u$ such that $\mu(x) = \mu(a) \wedge \mu(b)$.*

Note that this property is immediately extends from $\{a,b\}$ to any finite subset of Q. That is, if μ is a u-L-fuzzy ideal then there exists $x \in F^u$ such that $\mu(x) = \inf\{\mu(a) : a \in F\}$.

Lemma 2.3.15. *An L- fuzzy ideal μ of Q is a u-L-fuzzy ideal of Q if and only if μ_α is a u-ideal of Q, for all $\alpha \in L$.*

Proof. Suppose that μ is a u-L-fuzzy ideal and $\alpha \in L$. Since μ is an L- fuzzy ideal, μ_α is an ideal. Let $a,b \in \mu_\alpha$. Then $\mu(a) \wedge \mu(b) \geq \alpha$. Since μ is a u- L- fuzzy ideal, there exists $x \in (a,b)^u$ such that $\mu(x) = \mu(a) \wedge \mu(b)$. So $\mu(x) \geq \alpha$. Hence $x \in \mu_\alpha \cap (a,b)^u$ and thus $\mu_\alpha \cap (a,b)^u \neq \emptyset$. Therefore μ_α is a u- ideal of Q.

Conversely, suppose that μ_α is a u- ideal of a poset Q, for all $\alpha \in L$. Then μ is an L-fuzzy ideal of Q. Let $a,b \in Q$ and put $\alpha = \mu(a) \wedge \mu(b)$. Then $\mu_\alpha \cap (a,b)^u \neq \emptyset$. Let $x \in \mu_\alpha \cap (a,b)^u$. Then $x \in \mu_\alpha$ and $x \in (a,b)^u$. This implies that

$$\mu(x) \geq \alpha = \mu(a) \wedge \mu(b) \text{ and } a \leq x,\ b \leq x.$$

Since μ is anti-tone, we have $\mu(a) \geq \mu(x)$ and $\mu(b) \geq \mu(x)$. Thus $\mu(a) \wedge \mu(b) \geq \mu(x)$. So there exists $x \in (a,b)^u$ such that $\mu(x) = \mu(a) \wedge \mu(b)$.

Hence μ is a u-L-fuzzy ideal of Q. □

Corollary 2.3.16. *Let $(Q,\leq)$ be a poset with* 1 *and let $x \in Q$ and $\alpha \in L$. Then the α-level principal L-fuzzy ideal corresponding to x is a u-L-fuzzy ideal of Q.*

Remark 2.3.1. *Not every L-fuzzy ideal is a u-L-fuzzy ideal. For example consider the poset $(Q \leq)$ depicted in the Fig. 2.1 below. Define a fuzzy subset $\mu : Q \longrightarrow [0,1]$ of Q by:*

$$\mu(0) = 1,\ \mu(a) = \mu(b) = 0.9,\ \mu(c) = \mu(d) = \mu(1) = 0.7.$$

Then μ is an L-fuzzy ideal but not a u- L-fuzzy ideal. This is because $a,b \in Q$ and there is no x in $(a,b)^u = \{1,,c,d\}$ such that $\mu(x) = \mu(a) \wedge \mu(b)$.

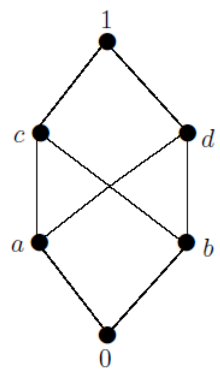

Fig. 2.1

However, if Q is a join semi-lattice, then we have

Theorem 2.3.17. *Let $(Q,\leq)$ be a join-semi-lattice. Then every L-fuzzy ideal of Q is a u-L-fuzzy ideal of Q.*

Proof. Let μ be an L-fuzzy ideal of a join-semi-lattice Q. Let $a,b \in Q$. Since Q is a join semi-lattice, $a \vee b$ exists and it is clear that $a \vee b \in (a,b)^u$ and $\mu(a \vee b) = \mu(a) \wedge \mu(b)$. Therefore μ is a u-L-fuzzy ideal of Q. □

Theorem 2.3.18. *Let μ and θ be u- L-fuzzy ideals of Q. Then the supremum $\mu \vee \theta$ of μ and θ in $\mathscr{F}\mathscr{I}(Q)$ is given by: for all $x \in Q$.*

$$(\mu \vee \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^{ul}\}.$$

Proof. Let σ be an L-fuzzy subset of Q defined by: for each $x \in Q$,

$$\sigma(x) = \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^{ul}\}.$$

We claim that σ is the smallest L-fuzzy ideal of Q containing $\mu \cup \theta$.

Let $x \in Q$. Then

$$\begin{aligned}
\sigma(x) &= \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^{ul}\} \\
&\geq \mu(x) \wedge \theta(0), \ (since\ x \in (x,0)^{ul}) \\
&= \mu(x) \wedge 1 = \mu(x).
\end{aligned}$$

Thus $\sigma \supseteq \mu$. Similarly, we can show $\sigma \supseteq \theta$. So $\sigma \supseteq \mu \cup \theta$.

Let $a,b \in Q$ and $x \in (a,b)^{ul}$. Then

$$\begin{aligned}
\sigma(a) \wedge \sigma(b) &= \sup\{\mu(c) \wedge \theta(d) : a \in (c,d)^{ul}\} \wedge \sup\{\mu(e) \wedge \theta(f) : b \in (e,f)^{ul}\} \\
&= \sup\{\mu(c) \wedge \theta(d) \wedge \mu(e) \wedge \theta(f) : a \in (c,d)^{ul}, b \in (e,f)^{ul}\} \\
&\leq \sup\{\mu(c) \wedge \theta(d) \wedge \mu(e) \wedge \theta(f) : a,b \in (c,d,e,f)^{ul}\} \\
&= \sup\{\mu(c) \wedge \mu(e) \wedge \theta(d) \wedge \theta(f) : a,b \in (c,d,e,f)^{ul}\}.
\end{aligned}$$

Since μ and θ are u-L-fuzzy ideals, for each c,e and d,f, there are $r \in (c,e)^u$ and $s \in (d,f)^u$ such that $\mu(r) = \mu(c) \wedge \mu(e)$ and $\theta(s) = \theta(d) \wedge \theta(f)$. Since $r \in (c,e)^u$ and $s \in (d,f)^u$, $\{c,d,e,f\}^{ul} \subseteq \{s,r\}^{ul}$. Thus $a,b \in \{s,r\}^{ul}$. So $(a,b)^{ul} \subseteq \{s,r\}^{ul}$ and thus $x \in \{s,r\}^{ul}$. Hence for all $x \in (a,b)^{ul}$, we have

$$\begin{aligned}
\sigma(a) \wedge \sigma(b) &\leq \sup\{\mu(c) \wedge \mu(e) \wedge \theta(d) \wedge \theta(f) : a,b \in (c,d,e,f)^{ul}\} \\
&\leq \sup\{\mu(r) \wedge \theta(s) : x \in (r,s)^{ul}\} \\
&\leq \sup\{\sigma(r) \wedge \sigma(s) : x \in (r,s)^{ul}\} \\
&\leq \sigma(x)
\end{aligned}$$

Therefore σ is an L-fuzzy ideal of Q.

Let ϕ be any L-fuzzy ideal of Q such that $\mu \cup \theta \subseteq \phi$.

Then for any $x \in Q$, we have

$$\begin{aligned} \sigma(x) &= \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^{ul}\} \\ &\leq \sup\{\phi(a) \wedge \phi(b) : x \in (a,b)^{ul}\} \\ &\leq \phi(x). \end{aligned}$$

Thus $\sigma \subseteq \phi$. So $\sigma = (\mu \cup \theta]_{Ha} = \mu \vee \theta$. □

It is known that every L-fuzzy ideal in join-semi-lattice Q is a u-L-fuzzy ideal. Therefore the following corollary is an easy consequence of the above theorem.

Corollary 2.3.19. *Let μ and θ be u- L-fuzzy ideals of a join-semi-lattice Q. Then the supremum $\mu \vee \theta$ of μ and θ in $\mathscr{F}\mathscr{I}(Q)$ is given by:*

$$(\mu \vee \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : x \leq a \vee b\}, \text{for all } x \in Q.$$

2.4 L-Fuzzy Semi Ideals and V-Ideals

Now we introduce the fuzzy version of semi-ideals and V-ideals of a poset introduced by Venkatanarasimhan in [51, 52].

Definition 2.4.1. *An L- fuzzy subset μ of Q is called an L- fuzzy semi-ideal or L-fuzzy order ideal, if it satisfies the following conditions:*

1. *$\mu(0) = 1$;*

2. *for any $a \in Q$, $\mu(x) \geq \mu(a)$, for all $x \in a^l$.*

Definition 2.4.2. *An L- fuzzy semi-ideal μ of Q is called an L- fuzzy V-ideal, if for any non-empty finite subset F of Q, if $\sup F$ exists, then*

$$\mu(\sup F) \geq \inf\{\mu(a) : a \in F\}.$$

Lemma 2.4.1. *An L- fuzzy subset μ of Q is an L- fuzzy semi ideal (respectively, V-ideal) of Q if and only if μ_α is a semi ideal (respectively, V-ideal) of Q, for all $\alpha \in L$.*

Corollary 2.4.2. *A subset I of Q is a semi ideal (respectively, V-ideal) of Q if and only if its characteristic map χ_I is an L-fuzzy semi ideal (respectively, V-ideal) of Q.*

Lemma 2.4.3. *The intersection of any family of L-fuzzy semi-ideals (respectively, V-ideals) is an L- fuzzy semi-ideal (respectively,V-ideal).*

Definition 2.4.3. *Let μ be an L- fuzzy subset of a poset Q. The L- fuzzy semi-ideal generated by μ, denoted by $(\mu]_{Se}$, is the smallest L- fuzzy semi-ideal of Q containing μ.*

Definition 2.4.4. *Let μ be an L- fuzzy subset of a poset Q. The L- fuzzy V-ideal generated by μ, denoted by $(\mu]_V$, is the smallest L- fuzzy V-ideal of Q containing μ.*

Theorem 2.4.4. *Let $\mathscr{FSI}(Q)$ be the set of all L-fuzzy semi-ideals of a poset Q and μ be an L fuzzy subset of Q. Then $(\mu]_{Se} = \bigcap\{\theta \in \mathscr{FSI}(Q) : \mu \subseteq \theta\}$.*

Theorem 2.4.5. *Let $\mathscr{FVI}(Q)$ be the set of all L-fuzzy V-ideals of a poset Q and μ be an L fuzzy subset of Q. Then $(\mu]_V = \bigcap\{\theta \in \mathscr{FVI}(Q) : \mu \subseteq \theta\}$.*

Theorem 2.4.6. *Let $(A]_{Se}$ be a semi-ideal generated by a subset A of Q and χ_A be the characteristics functions of A. Then $(\chi_A]_{Se} = \chi_{(A]_{Se}}$.*

Theorem 2.4.7. *Let $(A]_V$ be a semi-ideal generated by a subset A of Q and χ_A be the characteristics functions of A. Then $(\chi_A]_V = \chi_{(A]_V}$.*

In the following two theorems we give a characterization of any *L*-fuzzy semi-ideal and *L*-fuzzy V-ideal generated by an *L*-fuzzy subset of *Q* in terms of its level subset.

Theorem 2.4.8. *For any L-fuzzy subset μ of Q, define an L-fuzzy subset $\hat{\mu}$ of Q by:*

$$\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_{Se}\}, \textit{for all } x \in Q.$$

Then $\hat{\mu}$ is an L-fuzzy semi-ideal of Q generated by μ.

Proof. We show $\hat{\mu}$ is the smallest L-fuzzy semi-ideal containing μ. It is clear that $\mu \subseteq \hat{\mu}$ and $\hat{\mu}(0) = 1$. Let $a \in Q$ and $x \in a^l$. Now, let $\alpha \in L$ such that $a \in (\mu_\alpha]_{Se}$. Then since $x \in a^l$, we have $x \in (\mu_\alpha]_{Se}$ and hence $\{\alpha : x \in a^l, a \in (\mu_\alpha]_{Se}\} \subseteq \{\alpha : x \in (\mu_\alpha]_{Se}\}$. Therefore

$$\hat{\mu}(a) = \sup\{\alpha : a \in (\mu_\alpha]_{Se}\} \leq \sup\{\alpha : x \in (\mu_\alpha]_{Se} = \hat{\mu}(x).$$

Therefore $\hat{\mu}$ is an L-fuzzy semi-ideal.

Again let θ be any L-fuzzy semi ideal of Q such that $\mu \subseteq \theta$. Then $\mu_\alpha \subseteq \theta_\alpha$, for any $\alpha \in L$ and hence $(\mu_\alpha]_{Se} \subseteq (\theta_\alpha]_{Se} = \theta_\alpha$. So for any $x \in Q$,

$$\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_{Se}\} \leq \sup\{\alpha \in L : x \in \theta_\alpha\} = \theta(x).$$

Hence $\hat{\mu} \subseteq \theta$. This proves that $\hat{\mu}$ is the smallest L-fuzzy semi ideal containing μ. Hence $\hat{\mu} = (\mu]_{se}$. □

Theorem 2.4.9. *For any L-fuzzy subset μ of Q, define an L-fuzzy subset $\hat{\mu}$ of Q by* $\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_V\}$, *for all $x \in Q$. Then $\hat{\mu}$ is an L-fuzzy V-ideal of Q generated by μ.*

Proof. Clearly, by Theorem 2.4.8 given above, $\hat{\mu}$ is an L-fuzzy semi ideal containing μ.. Let $\emptyset \neq F \subseteq Q$ and $\sup F$ exists in Q. Then we have

$$\begin{aligned}\inf\{\hat{\mu}(a) : a \in F\} &= \inf\{\sup\{\alpha_a : a \in (\mu_{\alpha_a}]_V\} : a \in F\} \\ &= \sup\{\inf\{\alpha_a : a \in F\} : a \in (\mu_{\alpha_a}]_V\}\end{aligned}$$

Put $\lambda = \inf\{\alpha_a : a \in F$. This implies that $\lambda \leq \alpha_a$, for all $a \in F$. Therefore $(\mu_\alpha]_V \subseteq (\mu_\lambda]_V$ for all $a \in F$. Thus $F \subseteq (\mu_{\lambda_a}]_V$ and so $\sup F \in (\mu_{\lambda_a}]_V$. Therefore

$$\begin{aligned}\inf\{\hat{\mu}(a) : a \in F\} &= \sup\{\inf\{\alpha_a : a \in F\} : a \in (\mu_{\alpha_a}]_V\} \\ &\leq \sup\{\lambda \in L : \sup F \in (\mu_\lambda]_V\} \\ &= \hat{\mu}(\sup F).\end{aligned}$$

Therefore $\hat{\mu}$ is an L-fuzzy V-ideal of Q.

Again let θ be any L-fuzzy V-ideal of Q such that $\mu \subseteq \theta$. Then it is easy to show that $\hat{\mu} \subseteq \theta$. Therefore $\hat{\mu} = (\mu]_V$. □

In the following we give an algebraic characterization of an L-fuzzy semi-ideal generated by an L-fuzzy subset.

Theorem 2.4.10. *Let μ be an L-fuzzy subset of Q. Then the L-fuzzy subset $\overline{\mu}_{Se}$ defined by:*

$$\overline{\mu}_{Se}(x) = \begin{cases} 1 & if\ x = 0 \\ \sup\{\mu(a) : a \in Q, x \in a^l\} & if\ x \neq 0 \end{cases}$$

, for all $x \in Q$ is an L- fuzzy semi-ideal of Q generated by μ.

Proof. Now we claim $\overline{\mu}_{Se}$ is the smallest L- fuzzy semi-ideal of Q containing μ. Let $x \in Q$. Then since $x \in x^l$, we have

$$\mu(x) \leq \sup\{\mu(a) : x \in a^l\} \leq \overline{\mu}_{Se}(x).$$

Therefore $\mu \subseteq \overline{\mu}_{Se}$. By definition, $\overline{\mu}_{Se}(0) = 1$. Let $a \in Q$ and $x \in a^l$. If $a = 0$, then $x = 0$ and hence $\overline{\mu}_{Se}(a) = 1 = \overline{\mu}_{Se}(x)$. Let $a \neq 0$. Then

$$\begin{aligned} \overline{\mu}_{Se}(a) &= \sup\{\mu(y) : a \in y^l\} \\ &\leq \sup\{\mu(y) : x \in y^l\} \leq \overline{\mu}_{Se}(x). \end{aligned}$$

Thus $\overline{\mu}_{Se}$ is an L- fuzzy semi-ideal of Q. Let θ be any L- fuzzy semi-ideal of Q such that $\mu \subseteq \theta$. Let $x \in Q$. If $x = 0$, then $\overline{\mu}_{Se}(0) = 1 = \theta(0)$. Let $x \neq 0$. Then

$$\begin{aligned} \overline{\mu}_{Se}(x) &= \sup\{\mu(a) : x \in a^l\} \\ &\leq \sup\{\theta(a) : x \in a^l\} \leq \theta(x) \end{aligned}$$

Hence the claim is true. Therefore $\overline{\mu}_{Se} = (\mu]_{Se}$ □

Theorem 2.4.10 yields the following.

Theorem 2.4.11. *The set $\mathscr{F}\mathscr{S}\mathscr{I}(Q)$ of all L-fuzzy semi-ideal of Q forms a complete lattice, in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\mu_i$ of any family $\{\mu_i : i \in \Delta\}$ of L-fuzzy semi-ideals of Q are given by:*

$$\sup_{i\in\Delta}\mu_i = \overline{(\bigcup_{i\in\Delta}\mu_i)_{Se}} \quad \text{and} \quad \inf_{i\in\Delta}\mu_i = \bigcap_{i\in\Delta}\mu_i.$$

Corollary 2.4.12. *For any μ and θ in $\mathscr{F}\mathscr{S}\mathscr{I}(Q)$, the supremum $\mu\vee\theta$ and the infimum $\mu\wedge\theta$ of μ and θ, respectively are:*

$$\mu\vee\theta = \overline{(\mu\cup\theta)_{Se}} \quad \text{and} \quad \mu\wedge\theta = \mu\cap\theta.$$

Definition 2.4.5. *Let μ be a fuzzy subset of Q and $\mathbb{N}$ be a set of positive integers. Define L-fuzzy subsets $C^1_\mu, C^2_\mu, \cdots, C^n_\mu \cdots$, of Q, inductively, as follow: for each $x \in Q$*

$$C^1_\mu(x) = \sup\{\textstyle\bigwedge_{a\in F}\mu(a) : x \leq \bigvee F, \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\}$$

and for each $n \in \mathbb{N} - \{1\}$

$$C^n_\mu(x) = \sup\{\textstyle\bigwedge_{a\in F} C^{n-1}_\mu(a) : x \leq \bigvee F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\}.$$

Lemma 2.4.13. *The set $\{C^n_\mu : n \in \mathbb{N}\}$ forms a chain and for each $n \in \mathbb{N}$, $C^n_\mu(x) \geq C^n_\mu(y)$ whenever $x \leq y$.*

Proof. Let $x \in Q$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} C^{n+1}_\mu(x) &= \sup\{\bigwedge_{a\in F} C^n_\mu(a) : x \leq \bigvee F, \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\ &\geq C^n_\mu(x) \quad (\text{Since x} \leq x = \bigvee\{x\} \text{ and } \emptyset \neq \{x\} \subset\subset Q) \end{aligned}$$

Thus $C^{n+1}_\mu(x) \geq C^n_\mu(x)$ for all $x \in Q$ and hence $C^n_\mu \subseteq C^{n+1}_\mu$, for each $n \in \mathbb{N}$. So the set $\{C^n_\mu : n \in \mathbb{N}\}$ is a chain.

Let $x \leq y$. Then

$$\begin{aligned} C_\mu^n(y) &= \sup\{\bigwedge_{a\in F} C_\mu^{n-1}(a) : y \leq \bigvee F, \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\ &\leq \sup\{\bigwedge_{a\in F} C_\mu^{n-1}(a) : x \leq \bigvee F, \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\ &= C_\mu^n(x) \end{aligned}$$

Therefore $C_\mu^n(x) \geq C_\mu^n(y)$ whenever $x \leq y$. That is C_μ^n is anti-tone for all $n \in \mathbb{N}$. □

Now we give a characterization of an L-fuzzy V-ideal generated by an L- fuzzy subset of a poset Q.

Theorem 2.4.14. *The L- fuzzy subset $\overline{\mu}_V$ defined by: for all $x \in Q$,*

$$\overline{\mu}_V(x) = \begin{cases} 1 & if\ x = 0 \\ \sup\{C_\mu^n(x) : n \in \mathbb{N}\} & if x \neq 0 \end{cases}$$

is an L-fuzzy V-ideal generated by μ.

Proof. Now we claim that $\overline{\mu}_V$ is the smallest L-fuzzy ideal containing μ. Let $x \in Q$. Then since $x \leq x = \bigvee\{x\}$ and $\emptyset \neq \{x\} \subset\subset Q$, we have

$$\begin{aligned} \mu(x) &\leq \sup\{\bigwedge_{a\in F} \mu(a) : x \leq \bigvee F, \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\ &= C_\mu^1(x) \\ &\leq \sup\{C_\mu^n(x) : n \in \mathbb{N}\} \\ &\leq \overline{\mu}_V(x) \end{aligned}$$

Therefore $\mu \subseteq \overline{\mu}_V$. By definition of $\overline{\mu}_V$, we have $\overline{\mu}_V(0) = 1$. Let $a \in Q$ and $x \in a^l$. Then if $a = 0$ then $x = 0$ and hence $\overline{\mu}_V(a) = 1 = \overline{\mu}_V(x)$. Let $a \neq 0$. Then

$$\overline{\mu}_V(a) = \sup\{C_\mu^n(a) : n \in \mathbb{N}\} \leq \sup\{C_\mu^n(x) : n \in \mathbb{N}\} = \overline{\mu}_V(x)$$

So $\overline{\mu}_V$ is an L-fuzzy semi-ideal of Q.

Again let $\emptyset \neq F \subset\subset Q$ such that $\sup F$ exists.

$$\begin{aligned}
\bigwedge_{a\in F} \overline{\mu}_V(a) &= \bigwedge_{a\in F} \sup\{C_\mu^n(a) :: n \in \mathbb{N}\} \\
&= \sup\{\bigwedge_{a\in F} C_\mu^n(a) :: n \in \mathbb{N}\} \\
&\leq \sup\{C_\mu^{n+1}(\sup F) : n \in \mathbb{N}\} \;\; (\text{since } \sup F \leq \sup F) \\
&= \overline{\mu}_V(\sup F)
\end{aligned}$$

So $\overline{\mu}_V$ is an L-fuzzy V-ideal of Q.

Again let θ be any L-fuzzy V-ideal of Q such that $\mu \subseteq \theta$. Now we show that $C_\mu^n \subseteq \theta$ for all $n \in \mathbb{N}$. Here we apply mathematical induction. Let $x \in Q$.

$$\begin{aligned}
C_\mu^1(x) &= \sup\{\bigwedge_{a\in F} \mu(a) : x \leq \sup F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\
&\leq \sup\{\bigwedge_{a\in F} \theta(a) : x \leq \sup F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\
&\leq \sup\{\theta(\sup F) : x \leq \sup F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\
&\leq \theta(x)
\end{aligned}$$

Hence $C_\mu^1 \subseteq \theta$. Therefore the statement is true for $n = 1$. Assume $C_\mu^n \subseteq \theta$ for some $n > 1$.

$$\begin{aligned}
C_\mu^{n+1}(x) &= \sup\{\bigwedge_{a\in F} C_\mu^n(a) : x \leq \sup F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\
&\leq \sup\{\bigwedge_{a\in F} \theta(a) : x \leq \sup F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\
&\leq \sup\{\theta(\sup F) : x \leq \sup F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\} \\
&\leq \theta(x)
\end{aligned}$$

Thus by mathematical induction, we have $\theta(x) \geq C_\mu^n(x)\ \forall n \in \mathbb{N}.$. So for any $x \in Q$, we have $\theta(x) \geq sup\{C_\mu^n(x) : n \in \mathbb{N}\} = \overline{\mu}_V(x)$. Therefore $\overline{\mu}_V \subseteq \theta$. □

Theorem 2.4.14 yields the following.

Theorem 2.4.15. *The set $\mathscr{FVI}(Q)$ of all L-fuzzy V-ideal of Q forms a complete lattice under point-wise ordering " $\subseteq$ "; in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\mu_i$ of any family $\{\mu_i : i \in \Delta\}$ in $\mathscr{FVI}(Q)$ respectively are:*

$$sup_{i\in\Delta}\mu_i = \overline{(\bigcup_{i\in\Delta}\mu_i)}_V \text{ and } \inf_{i\in\Delta}\mu_i = \bigcap_{i\in\Delta}\mu_i.$$

Corollary 2.4.16. *For any μ and $\theta \in \mathscr{FVI}(Q)$ the supremum $\mu \vee \theta$ and the infimum $\mu \wedge \theta$ of μ and θ respectively are:*

$$\mu \vee \theta = \overline{(\mu \cup \theta)}_V \text{ and } \mu \wedge \theta = \mu \cap \theta.$$

Now we study the relationships among the types of L-fuzzy ideals introduced in this chapter.

Theorem 2.4.17. *The following implications hold.*

1. *L- fuzzy closed ideal $\Longrightarrow$ L-fuzzy Frink ideal $\Longrightarrow$ L-fuzzy V-ideal $\Longrightarrow$ L-fuzzy semi-ideal,*

2. *L- fuzzy closed ideal $\Longrightarrow$ L-fuzzy Frink ideal $\Longrightarrow$ L-fuzzy ideal $\Longrightarrow$ L- fuzzy semi-ideal.*

Proof. (L- fuzzy closed ideal) $\Longrightarrow$ (L-fuzzy Frink ideal): It is clear.

(L-fuzzy Frink ideal) $\Longrightarrow$ (L-fuzzy V-ideal) : Let μ be an L-fuzzy Frink ideal. Then $\mu(0) = 1$. Let $a \in Q$ and $x \in a^l$. This implies that $x \leq a$, and since μ is anti-tone, we have $\mu(x) \geq \mu(a)$. Then μ is an L-fuzzy semi-ideal. Let F be a non-empty subset of Q such that $\sup F$ exists. Since $\sup F \in F^{ul}$ and μ is an L-fuzzy Frink ideal, we have $\mu(\sup F) \geq \inf\{\mu(a) : a \in F\}$. This proves that μ is an L-fuzzy V-ideal of Q.

(L-fuzzy Frink ideal) $\Longrightarrow$ (L-fuzzy ideal): Let μ be an L-fuzzy Frink ideal. Then $\mu(0) = 1$. Let $a,b \in Q$ and $x \in (a,b)^{ul}$. Put $F = \{a,b\}$. Since $F = \{a,b\} \subset\subset Q$ and μ is an L Frink ideal, we have $\mu(x) \geq \inf\{\mu(y) : y \in F\} = \mu(a) \wedge \mu(b)$. This proves that μ is an L-fuzzy ideal of Q.

(L-fuzzy V-ideal) $\Longrightarrow$ (L-fuzzy semi-ideal) and (L-fuzzy ideal) $\Longrightarrow$ (L-fuzzy semi-ideal are clear. □

The following examples show that the converse of the above implications do not hold in general.

Example 2.4.18. *Consider the Poset* $([0,1],\leq)$ *with the usual ordering. Define a fuzzy subset* $\mu : [0,1] \longrightarrow [0,1]$ *by:*

$$\mu(x) = \begin{cases} 1 & if\ x \in [0,\frac{1}{2}) \\ 0 & if\ x \in [\frac{1}{2},1]. \end{cases}$$

Then μ *is L-fuzzy Frink ideal but not L- fuzzy closed ideal.*

Example 2.4.19. *Consider the poset* $(Q,\leq)$ *depicted in the Fig. 2.2 given below. Define a fuzzy subset* $\mu : Q \longrightarrow [0,1]$ *by:* $\mu(0) = 1$, $\mu(a) = \mu(b) = 0.8$ *and* $\mu(c) = 0.6$.

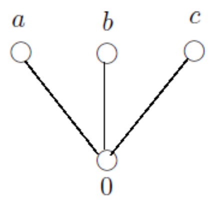

Fig. 2.2

Then μ *is L-fuzzy V-ideal but not L-fuzzy Frink-ideal. Because,* $F = \{0,a,b\} \subset\subset Q$ *and* $c \in F^{ul} = \{0,a,b,c\}$ *but* $\mu(c) = 0.6 \not\geq 0.8 = \inf\{\mu(x) : x \in F\}$.

Example 2.4.20. *Consider the poset* $(Q,\leq)$ *depicted in the Fig. 2.3 given below. Define a fuzzy subset* $\mu : Q \longrightarrow [0,1]$ *by:* $\mu(0) = \mu(a) = 1$, $\mu(a') = \mu(b') = \mu(c') = \mu(d') = \mu(1) = 0.2$, $\mu(b) = 0.6$, $\mu(c) = 0.5$ *and* $\mu(d) = 0.7$.

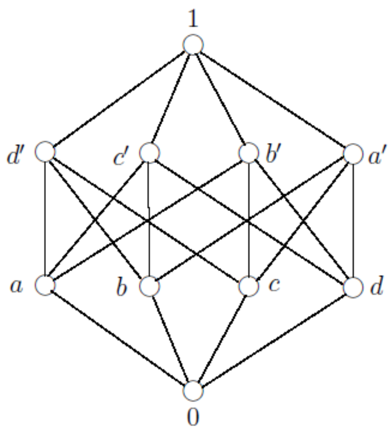

Fig. 2.3

Then μ is L-fuzzy ideal but not L-fuzzy Frink-ideal. Because, $F = \{0,a,b,c\} \subseteq Q$ and $d' \in F^{ul} = \{0,a,b,c,d'\}$ but $\mu(d') = 0.2 \ngeq 0.5 = \inf\{\mu(x) : x \in F\}$.

Example 2.4.21. *Consider the poset $(Q,\leq)$ depicted in the Fig. 2.4 given below. Define a fuzzy subset $\mu : Q \longrightarrow [0,1]$ by: $\mu(0) = 1$, $\mu(a) = 0.8$, $\mu(b) = 0.9$, $\mu(c) = 0.2$ and $\mu(1) = 0$.*

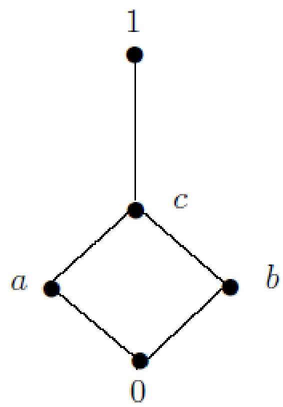

Fig. 2.4

Then μ is L-fuzzy semi-ideal but μ is neither L-fuzzy V-ideal nor L-fuzzy ideal. Since $\emptyset \neq F = \{0,a,b\} \subset\subset Q$ and $\sup F = c \in Q$, $\mu(c) = 0.2 \ngeq 0.8 = \inf\{\mu(x) : x \in F\}$, and since $a,b \in Q$ and $c \in (a,b)^{ul}$, but $\mu(c) = 0.2 \ngeq 0.8 = \mu(a) \wedge \mu(b)$.

Theorem 2.4.22. *Every u- L-fuzzy ideal is an L- fuzzy Frink ideal.*

Proof. suppose μ is a u- L-fuzzy ideal. Then $\mu(0) = 1$. Let F be a finite subset of Q. If $F = \emptyset$, then it is clear that $F^{ul} = \{0\}$ and $\mu(0) = 1 = \inf\{\mu(x) : x \in F\}$. Let $\emptyset \neq F$. Since μ is a u- L-fuzzy ideal, then there is $y \in F^u$ such that $\mu(y) = inf\{\mu(a) : a \in F\}$. Let $x \in F^{ul}$. Then $x \leq s$, $\forall s \in F^u$. Since $y \in F^u$, $x \leq y$. Thus $\mu(x) \geq \mu(y) = inf\{\mu(a) : a \in F\}$. So

$$\mu(x) \geq inf\{\mu(a) : a \in F\}, \text{ for all } x \in F^{ul}.$$

Hence μ is an L-fuzzy Frink ideal of Q. □

Now we complete this chapter by introducing the following definition of L-fuzzy ideal of a poset which generalizes all the L-fuzzy ideals of a poset introduced in this chapter.

Definition 2.4.6. *Let m be any cardinal number. An L- fuzzy subset μ of Q is an m- L-fuzzy ideal, if it satisfies the following conditions:*

(i) $\mu(0) = 1$,

(ii) for any subset A of Q of cardinality strictly less than m, written as, $A \subset_m Q$,

$\mu(x) \geq \inf\{\mu(a) : a \in A\}$, $\forall x \in A^{ul}$.

Remark 2.4.1. *The following special cases are included in this general definition:*

1. *Ω-L-fuzzy ideals are L-fuzzy closed ideals, where Ω is a cardinal number greater than the cardinal number of Q.*

2. *ω- L-fuzzy ideals-ideals are L-fuzzy Frink ideals, where ω is the smallest infinite cardinal number.*

3. *3- L-fuzzy ideals are L-fuzzy ideals in the sense of Halaś.*

4. 2-*L-fuzzy ideals are L-fuzzy semi-ideals.*

5. *L-fuzzy V-ideals are 2-L-fuzzy ideals and if for any non-empty finite subset F of Q, if* $\sup F$ *exists, then* $\mu(\sup F) \geq \inf\{\mu(a) : a \in F\}$.

Chapter 3

L-Fuzzy Filters

In this chapter, we introduce the notions of different types of L- fuzzy filters of a poset and discuss certain properties of them analogous to those of L- fuzzy ideals of a poset. The different types of L-fuzzy filters of a poset that we introduce in this chapter are generalizations of the notion of L-fuzzy filters of a lattice. We also prove that the set of each type of L-fuzzy filters of a poset forms a complete lattice with respect to point-wise ordering. Throughout this chapter, Q stands for a poset $(Q, \leq)$ with 1 unless otherwise stated.

3.1 *L*-Fuzzy Closed Filters

In this section, we introduce the notion of L-fuzzy closed filters of a poset which is the fuzzy version of the dual closed or normal ideal of a poset introduced by Birkoff[14]. We also prove and characterize certain elementary properties of L-fuzzy closed filters. In particular, we prove that the set of all L-fuzzy closed filters of a poset forms a complete lattice.

Wee start our discussion with the definition of L-fuzzy closed filters of a poset.

Definition 3.1.1. *An L-fuzzy subset η of Q is called an L- fuzzy closed filter if it fulfills the following conditions:*

1. $\eta(1) = 1$ *and*

2. *for any subset S of Q,* $\eta(x) \geq \inf\{\eta(a) : a \in S\}\ \forall x \in S^{lu}$.

In the following we prove that any crisp closed filter of Q can be identified with an L-fuzzy closed filter of Q.

Lemma 3.1.1. *A subset F of Q is a closed filter if and only if its characteristic map χ_F is an L-fuzzy closed filter of Q.*

Proof. Suppose that F is a closed filter of Q. Since $\{1\} = Q^u \subseteq F$, we have $\chi_F(1) = 1$. Again let S be any subset of Q and $x \in S^{lu}$. Then if $S \subseteq F$, we have $x \in S^{lu} \subseteq F^{lu} = F$ and $\chi_F(a) = 1$ for all $a \in S$. Therefore

$$\chi_F(x) = 1 = \inf\{\chi_F(a) : a \in S\}.$$

Again if $S \nsubseteq F$, then there is $c \in S$ such that $c \notin F$ and hence $\inf\{\chi_F(a) : a \in S\} = 0$. Thus

$$\chi_F(x) \geq 0 = \inf\{\chi_F(a) : a \in S\}.$$

Thus in either cases, we have,

$$\chi_F(x) \geq \inf\{\chi_F(a) : a \in S\} \text{ for all } x \in S^{lu}.$$

Therefore χ_F is an L-fuzzy closed filter of Q.

Conversely suppose that χ_F is an L-fuzzy closed filter. Since $\chi_F(1) = 1$ we have $1 \in F$, that is $Q^u = \{1\} \subseteq F$. Let $x \in F^{lu}$. Then, by hypothesis, we have

$$\chi_F(x) \geq \inf\{\chi_F(a) : a \in F\} = 1.$$

This implies that $\chi_F(x) = 1$ and hence $x \in F$. Therefore $F^{lu} \subseteq F$ and hence F is a closed filter of Q. □

The following result characterizes any L-fuzzy closed filter of Q in terms of its level subsets.

Lemma 3.1.2. *Let η be in L^Q. Then η is an L- fuzzy closed filter of Q if and only if η_α is a closed filter of Q, for all $\alpha \in L$.*

Proof. Let η be an L- fuzzy closed filter of Q and $\alpha \in L$. Then $\eta(1) = 1 \geq \alpha$ and hence $1 \in \eta_\alpha$, i.e., $Q^u = \{1\} \subseteq \eta_\alpha$. Again let $x \in (\eta_\alpha)^{lu}$. Then

$$\eta(x) \geq \inf\{\eta(a) : a \in \eta_\alpha\} \geq \alpha$$

so that $x \in \eta_\alpha$. Therefore $(\eta_\alpha)^{lu} \subseteq \eta_\alpha$ and hence η_α is a closed filter of Q.

Conversely suppose that η_α is a closed filter of Q for all $\alpha \in L$. In particular η_1 is a closed filter of Q. Since $\{1\} = Q^u \subseteq \eta_1$, we have $\eta(1) = 1$. Again let S be any subset of Q and $x \in S^{lu}$. Put $\alpha = \inf\{\eta(a) : a \in S\}$. Then $\eta(a) \geq \alpha$ for all $a \in S$. This implies that $S \subseteq \eta_\alpha$ and hence $x \in S^{lu} \subseteq \eta_\alpha^{lu} = \eta_\alpha$. Therefore

$$\eta(x) \geq \alpha = \inf\{\eta(a) : a \in S\}.$$

Thus η is an L-fuzzy closed filter of Q. □

Note that any L- fuzzy closed filter of a poset Q must be necessarily an iso-tone as shown in the following Lemma.

Lemma 3.1.3. *Let η be an L-fuzzy closed filter of a poset Q. Then η is iso-tone, in the sense that $\eta(x) \leq \eta(y)$ whenever $x \leq y$.*

Proof. Let $x, y \in Q$ such that $x \leq y$. Put $\eta(x) = \alpha$. Since η an L- fuzzy closed filter we have η_α is a closed filter of Q and hence $(\eta_\alpha)^{lu} = \eta_\alpha$.
Now $\eta(x) = \alpha \Rightarrow x \in \eta_\alpha \Rightarrow x^u = \{x\}^{lu} \subseteq (\eta_\alpha)^{lu} = \eta_\alpha$.
Thus $x \leq y \Rightarrow y \in x^u \Rightarrow y \in \eta_\alpha$ and hence $\eta(x) = \alpha \leq \eta(y)$. □

Theorem 3.1.4. *Let $x \in Q$ and $\alpha \in L$. Define an L- fuzzy subset α^x of Q by*

$$\alpha^x(y) = \begin{cases} 1 & if\ y \in [x) \\ \alpha & if\ y \notin [x), \end{cases}$$

for all $y \in Q$. Then α^x is an L-fuzzy closed filter of Q.

Proof. Since $1 \in [x)$, $\alpha^x(1) = 1$. Let $S \subseteq Q$ and $y \in S^{lu}$. If $S \subseteq [x)$, then

$y \in S^{lu} \subseteq [x)^{lu} = x^{ulu} = x^u = [x)$ and $\alpha^x(s) = 1$ for all $s \in S$.

This implies that $\alpha^x(y) = 1 = \inf\{\mu(s) : s \in S\}$.

If $S \nsubseteq [x)$, then there exists $s_0 \in S$ such that $s_0 \notin [x)$. This implies that $\inf\{\mu(s) : s \in S\} = \alpha$. Thus $\alpha^x(y) \geq \alpha = \inf\{\mu(s) : s \in S\}$. So in either cases, we have

$$\alpha^x(y) \geq \inf\{\mu(s) : s \in S\}, \text{ for all } y \in S^{lu}.$$

Hence α^x is an L-fuzzy closed filter of Q. □

Definition 3.1.2. *The L-fuzzy closed filter α^x defined above is called α-level principal L-fuzzy closed filter corresponding to x.*

Lemma 3.1.5. *The intersection of any family of L-fuzzy closed filters of a poset Q is an L-fuzzy closed filter of Q.*

Definition 3.1.3. *Let η be an L- fuzzy subset of a poset Q. Then the smallest L- fuzzy closed filter of Q containing η, denoted by $[\eta)_{Cl}$, is called the L- fuzzy closed filter generated by η.*

Theorem 3.1.6. *Let $\mathscr{FCF}(Q)$ be the set of all L-fuzzy closed filters of a poset Q and η be an L fuzzy subset of Q. Then $[\eta)_{Cl} = \bigcap\{\theta \in \mathscr{FCF}(Q) : \eta \subseteq \theta\}$.*

Theorem 3.1.7. *Let $[S)_{Cl}$ be a closed filter generated by a subset S of Q and χ_S be its characteristics function. Then $\chi_{[S)_{Cl}} = [\chi_S)_{Cl}$.*

Proof. Now we claim that $\chi_{[S)_{Cl}}$ is the smallest L- fuzzy closed filter containing χ_S. Since $[S)_{Cl}$ is a closed filter of Q, by lemma 3.1.1, we have $\chi_{[S)_{Cl}}$ is an L- fuzzy closed filter. Again since $S \subseteq [S)_{Cl}$, we clearly have $\chi_S \subseteq \chi_{[S)_{Cl}}$. Let η be any L-fuzzy closed filter such that $\chi_S \subseteq \eta$. Then $\eta(a) = 1$ for all $a \in S$. Now we claim that $\chi_{[S)_{Cl}} \subseteq \eta$. Let $x \in Q$. If $x \notin [S)_{Cl}$, then $\chi_{[S)}(x) = 0 \leq \eta(x)$. If $x \in [S)_{Cl}$, then $x \in T^{lu}$ for some subset T of S. Therefore

$$\eta(x) \geq \inf\{\eta(b) : b \in T\} = 1 = \chi_{[S)_{Cl}}(x).$$

Hence in either cases, $\chi_{[S)_{Cl}}(x) \leq \eta(x)$ for all $x \in Q$ and hence $\chi_{[S)_{Cl}} \subseteq \eta$. Therefore $\chi_{[S)_{Cl}} = [\chi_S)_{Cl}$. □

Theorem 3.1.8. *An L-fuzzy subset η of Q is an L-fuzzy closed filter if and only if for any subset S of Q,*

$$\eta(x) \geq \inf\{\eta(s) : s \in S\} \textit{ for all } x \in [S)_{Cl}.$$

Proof. Suppose that η is an L-fuzzy closed filter of Q. Let $S \subseteq Q$ and $x \in [S)_{Cl}$. . Then $x \in B^{lu}$ for some $B \subseteq S$. Then, since η is an L-fuzzy closed filter of Q, we clearly have

$$\eta(x) \geq \inf\{\eta(b) : b \in B\} \geq \inf\{\eta(s) : s \in S\}.$$

Thus, for any subset S of Q, we have

$$\eta(x) \geq \inf\{\eta(s) : s \in S\} \text{ for all } x \in [S)_{Cl}.$$

Conversely suppose that η is an L-fuzzy subset of Q satisfying the given condition. Let $S \subseteq Q$ and $x \in S^{lu}$. Then, since $S^{lu} \subseteq [S)_{Cl}$, by hypothesis, we have

$$\eta(x) \geq \inf\{\eta(s) : s \in S\}$$

In particular, if $S = \emptyset$, by hypothesis, we have

$$\eta(x) \geq \inf\{\eta(s) : s \in \emptyset\}, \text{ for all } x \in [\emptyset)_{Cl}.$$

But since $inf\{\eta(s) : s \in \emptyset\} = 1$ and $[\emptyset)_{Cl} = \{1\}$, we clearly have $\eta(1) = 1$.
Therefore η is an L-fuzzy closed filter of Q. □

In the following theorem we characterize an L- fuzzy closed filter generated by an L-fuzzy subset of Q in terms of its level closed filters.

Theorem 3.1.9. *Let $\eta \in L^Q$. Then the L-fuzzy subset $\hat{\eta}$ of Q defined by:*

$$\hat{\eta}(x) = \sup\{\alpha \in L : x \in [\eta_\alpha)_{Cl}\}, \textit{for all } x \in Q.$$

is an L-fuzzy closed filter of Q generated by η, where $[\eta_\alpha)_{Cl}$ is a closed filter generated by the set η_α.

Proof. Now we show that $\hat{\eta}$ is the smallest L-fuzzy closed filter containing η.
Now for any $x \in Q$, we have

$$\eta(x) = \sup\{\alpha : x \in \eta_\alpha\} \leq \sup\{\alpha : x \in [\eta_\alpha)_{Cl}\} = \hat{\eta}(x).$$

Thus $\eta \subseteq \hat{\eta}$. It is clear that $\hat{\eta}(1) = 1$ Let S be any subset of Q and $x \in S^{lu}$. Then we have

$$\begin{aligned} \inf\{\hat{\eta}(s) : s \in S\} &= \inf\{\sup\{\alpha_s : s \in [\eta_{\alpha_s})_{Cl}\} : s \in S\} \\ &= \sup\{\inf\{\alpha_s : s \in S\} : s \in [\eta_{\alpha_s})_{Cl}\} \end{aligned}$$

Put $\lambda = \inf\{\alpha_s : s \in S\}$. Then $\lambda \leq \alpha_s$ for all $s \in S$. This implies that $[\eta_{\alpha_s})_{Cl} \subseteq [\eta_\lambda)_{Cl}$ for all $s \in S$. Therefore $S \subseteq [\eta_\lambda)_{Cl}$ and so $x \in S^{lu} \subseteq ([\eta_\lambda)_{Cl})^{lu} = [\mu_{\lambda_a})_{Cl}$. Hence

$$\begin{aligned} \inf\{\hat{\eta}(s) : s \in S\} &= \sup\{\inf\{\alpha_s : s \in S\} : s \in [\eta_{\alpha_s})_{Cl}\} \\ &\leq \sup\{\lambda \in L : x \in [\eta_\lambda)_{Cl}\} \\ &= \hat{\eta}(x). \end{aligned}$$

Therefore $\hat{\eta}$ is an L-fuzzy closed filter of Q. Again let θ be any L-fuzzy closed filter of Q such that $\eta \subseteq \theta$. Then $\eta_\alpha \subseteq \theta_\alpha$ for any $\alpha \in L$ and hence $[\eta_\alpha)_{Cl} \subseteq [\theta_\alpha)_{Cl} = \theta_\alpha$.

Thus for any $x \in Q$, we have

$$\hat{\eta}(x) = \sup\{\alpha \in L : x \in [\eta_\alpha)\} \leq \sup\{\alpha \in L : x \in \theta_\alpha\} = \theta(x).$$

Therefore $\hat{\eta} \subseteq \theta$. This proves that $\hat{\eta} = [\eta)_{Cl}$. □

In the following we give an algebraic characterization of an L-fuzzy Closed filter of Q generated by an L-fuzzy subset of Q.

Theorem 3.1.10. *Let $\eta \in L^Q$. Then the L-fuzzy subset $\overline{\eta}_{Cl}$ defined by:*

$$\overline{\eta}_{Cl}(x) = \begin{cases} 1 & if\ x = 1 \\ \sup\{\inf_{s\in S} \eta(s) : x \in S^{lu}, S \subseteq Q\} & if\ x \neq 1 \end{cases}$$

is an L-fuzzy closed filter of Q generated by η.

Proof. It is enough to show that $\overline{\eta}_{Cl} = \hat{\eta}$ where $\hat{\eta}$ is an L-fuzzy subset defined in the Theorem 3.1.9. Let $x \in Q$. If $x = 1$, then $\overline{\eta}(x) = 1 = \hat{\eta}(x)$. Let $x \neq 0$. Put

$$A_x = \{\inf_{s\in S} \eta(s) : S \subseteq Q \text{ and } x \in S^{lu}\} \text{ and } B_x = \{\alpha : x \in [\eta_\alpha)_{Cl}\}.$$

Now we claim that $\sup A_x = \sup B_x$. Let $\alpha \in A_x$. Then $\alpha = \inf_{s\in S} \eta(s)$ for some subset S of Q such that $x \in S^{lu}$. This implies that $\alpha \leq \eta(s)$ for all $s \in S$ and hence $S \subseteq \eta_\alpha \subseteq [\eta_\alpha)_{Cl}$. Thus $x \in S^{lu} \subseteq ([\eta_\alpha)_{Cl})^{lu} = [\eta_\alpha)_{Cl}$. This implies that $\alpha \in B_x$ and hence $A_x \subseteq B_x$. Therefore $\sup A_x \leq \sup B_x$.

Again let $\alpha \in B_x$.Then $x \in [\eta_\alpha)_{Cl}$. Since $[\eta_\alpha)_{Cl} = \bigcup\{S^{lu} : S \subseteq \eta_\alpha\}$, we have $x \in S^{lu}$ for some subset S of η_α. This implies that $\eta(s) \geq \alpha$ for all $s \in S$ and hence

$$\inf\{\eta(s) : s \in S\} \geq \alpha$$

Thus $\beta = \inf\{\eta(s) : s \in S\} \in A_x$. Thus for each $\alpha \in B_x$ we get $\beta \in A_x$ such that $\alpha \leq \beta$ and hence $\sup A_x \geq \sup B_x$. Therefore $\sup A_x = \sup B_x$ and hence $\overline{\eta}_{Cl} = \hat{\eta}$. □

The above theorem yields the following.

Theorem 3.1.11. *Let $\mathscr{FCF}(Q)$ be the set of all L-fuzzy closed filters of a poset Q. Then the pair $(\mathscr{FCF}(Q), \subseteq)$ forms a complete lattice with respect to the point wise ordering "$\subseteq$", in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\eta_i$ of any family $\{\eta_i : i \in \Delta\}$ in $\mathscr{FCF}(Q)$ are given by:*

$$sup_{i\in\Delta}\eta_i = \overline{(\bigcup_{i\in\Delta}\{\eta_i\})}_{Cl} \quad \textit{and} \quad \inf_{i\in\Delta}\eta_i = \bigcap_{i\in\Delta}\eta_i.$$

Corollary 3.1.12. *For any L-fuzzy closed filters η and ν of Q, the supremum $\eta \vee \nu$ and the infimum $\eta \wedge \nu$ of η and ν in $\mathscr{FCF}(Q)$ respectively are:*

$$\eta \vee \nu = \overline{(\eta \cup \nu)}_{Cl} \quad \textit{and} \quad \eta \wedge \nu = \eta \cap \nu.$$

3.2 L-Fuzzy Frink Filters

Now we introduce the fuzzy version of a filter (dual ideal) of a poset introduced by O. Frink [25].

Definition 3.2.1. *An L- fuzzy subset η of Q is an L- fuzzy Firink filter if it satisfies the following conditions:*

1. $\eta(1) = 1$ *and*

2. *for any finite subset F of Q, $\eta(x) \geq \inf\{\eta(a) : a \in F\}$ $\forall x \in F^{lu}$*

Lemma 3.2.1. *Let $\eta \in L^Q$. Then η is an $L-$ fuzzy Frink filter of Q if and only if η_α is a Frink filter of Q for all $\alpha \in L$.*

Lemma 3.2.2. *Let η be fuzzy Frink filter of a poset Q. Then η is iso-tone, in the sense that $\eta(x) \leq \eta(y)$ whenever $x \leq y$.*

Corollary 3.2.3. *A subset S of Q is a Frink filter of Q if and only if its characteristic map χ_S is an L-fuzzy Frink filter of Q.*

The following theorem shows that an L-fuzzy Frink filter of a poset is a natural generalizations of an L-fuzzy filter of a lattice.

Theorem 3.2.4. *Let $(Q, \leq)$ be a lattice and $\eta \in L^Q$. Then η is an L- fuzzy Frink filter in the poset Q if and only if it is an L-fuzzy filter in the lattice Q.*

Proof. Let η be an L-fuzzy Frink filter in the poset Q and $a, b \in Q$. Then $\eta(1) = 1$ and since $F = \{a, b\} \subset\subset Q$ and $a \wedge b \in F^{lu}$, we have

$$\eta(a \wedge b) \geq \inf\{\eta(x) : x \in F\} = \eta(a) \wedge \eta(b).$$

Again since η is iso-tone, we have

$$\eta(a \wedge b) \leq \eta(a) \text{ and } \eta(a \wedge b) \leq \eta(b).$$

This implies that $\eta(a \wedge b) \leq \eta(a) \wedge \eta(b)$ and hence $\eta(a \wedge b) = \eta(a) \wedge \eta(b)$.
Therefore η is an L-fuzzy filter in the lattice Q.

Conversely suppose that η ba an L-fuzzy filter in the lattice Q. Then $\eta(1) = 1$. Let $F \subset\subset Q$ and $x \in F^{lu}$. Then x is an upper bound of F^l. Since $\inf F \in F^l$, we have $x \geq \inf F$ and hence

$$\eta(x) \geq \eta(\inf F) = \inf\{\eta(a) : a \in F\}.$$

Therefore η is an L-fuzzy Frink filter in the poset Q. □

Theorem 3.2.5. *Let $x \in Q$ and $\alpha \in L$. Define an L- fuzzy subset α^x of Q by*

$$\alpha^x(y) = \begin{cases} 1 & if\ y \in [x) \\ \alpha & if\ y \notin [x), \end{cases}$$

for all $y \in Q$. Then α^x is an L-fuzzy Frink filter of Q.

Lemma 3.2.6. *The intersection of any family of L-fuzzy Frink-filters of a poset Q is an L-fuzzy Frink filter of Q.*

Definition 3.2.2. *Let $\eta \in L^Q$. The smallest L- fuzzy Frink filter of Q containing η is called an L- fuzzy Frink filter generated by η and is denoted by $[\eta)_{Fr}$.*

Theorem 3.2.7. *Let $\mathscr{F}\mathscr{F}\mathscr{F}(Q)$ be the set of all L-fuzzy Frink filters of a poset Q and η be an L fuzzy subset of Q. Then $[\eta)_{Fr} = \bigcap\{\theta \in \mathscr{F}\mathscr{F}\mathscr{F}(Q) : \eta \subseteq \theta\}$.*

Theorem 3.2.8. *Let $[S)_{Fr}$ be a Frink-filter generated by a subset S of Q and χ_S be its characteristic function. Then $[\chi_S)_{Fr} = \chi_{[S)_{Fr}}$.*

Theorem 3.2.9. *An L-fuzzy subset η of Q is an L-fuzzy Frink filter if and only if for any finite subset F of Q,*

$$\eta(x) \geq \inf\{\eta(a) : a \in F\}\ for\ all\ x \in [F)_{Fr}.$$

Proof. Suppose that η is an L-fuzzy Frink filter of Q. Let $F \subset\subset Q$ and $x \in [F)_{Fr}$. Then $x \in B^{lu}$ for some $B \subseteq F$. Then, since η is an L-fuzzy Frink filter of Q, we have

$$\eta(x) \geq \inf\{\eta(a) : a \in B\} \geq \inf\{\eta(a) : a \in F\}.$$

Conversely suppose that η be an L-fuzzy subset satisfying the given condition. Since $\emptyset \subset\subset Q$ and $1 \in [\emptyset)_{Fr}$, by hypothesis, we have

$$\eta(1) \geq \inf\{\eta(a) : a \in \emptyset\} = 1.$$

Therefore $\eta(1) = 1$. Let $F \subset\subset Q$ and $x \in F^{lu}$. Then since $F^{lu} \subseteq [F)_{Fr}$, by hypothesis, we have $\eta(x) \geq \inf\{\eta(a) : a \in F\}$. Therefore η is an L-fuzzy Frink filter of Q. □

The following theorem gives a characterization of L-Fuzzy Frink filter generated by an L- fuzzy subset of Q in terms of its level subset.

Theorem 3.2.10. *Let $\eta \in L^Q$. Define an L- fuzzy subset $\hat{\eta}$ of Q by:*

$$\hat{\eta}(x) = \sup\{\alpha \in L : x \in [\eta_\alpha)_{Fr}\} \text{ for all } x \in Q,$$

where $[\eta_\alpha)_{Fr}$ is a Frink filter generated by η_α. Then $\hat{\eta}$ is an L-fuzzy Frink filter of Q generated by η.

In the following we give an algebraic characterization of L-fuzzy Frink filterss generated by L-fuzzy subsets.

Theorem 3.2.11. *Let η be an L-fuzzy subset of Q. Then the L-fuzzy subset $\overline{\eta}_{Fr}$ defined by*

$$\overline{\eta}_{Fr}(x) = \begin{cases} 1 & if\ x = 1 \\ \sup\{\inf_{a \in F} \eta(a) : F \subset\subset Q,\ x \in F^{lu}\} & if\ x \neq 1 \end{cases}$$

is an L- fuzzy Frink filter of Q generated by η.

Theorem 3.2.12. *Let $\mathscr{F}\mathscr{F}\mathscr{F}(Q)$ be the set of all L-fuzzy Frink filters of Q. Then the pair $(\mathscr{F}\mathscr{F}\mathscr{F}(Q), \subseteq)$ forms a complete lattice with respect to point wise ordering " $\subseteq$ ", in which the supremum and the inifimum of any family $\{\eta_i : i \in \Delta\}$ in $\mathscr{F}\mathscr{F}\mathscr{F}(Q)$, respectively are:*

$$sup_{i \in \Delta}\eta_i = \overline{(\bigcup_{i \in \Delta}\{\eta_i\}}_{Fr} \quad and \quad \inf_{i \in \Delta} \eta_i = \bigcap_{i \in \Delta} \eta_i.$$

Corollary 3.2.13. *For any L-fuzzy Frink filters η and ν of Q, the supremum $\eta \vee \nu$ and the infimum $\eta \wedge \nu$ of η and ν in $\mathscr{F}\mathscr{F}\mathscr{F}(Q)$, respectively are:*

$$\eta \vee \nu = \overline{(\eta \cup \nu)}_{Fr} \ and \ \eta \wedge \nu = \eta \cap \nu.$$

3.3 L-Fuzzy Filters in the Sense of Halaś

In this section we introduce the fuzzy version filters of a poset introduced by Halaš [28] which seems to be a suitable generalization of the usual concept of L-fuzzy filter of a lattice.

Definition 3.3.1. *$\eta \in L^Q$ is called an L- fuzzy filter of Q in the sense of Halaś if it fulfills the following:*

1. $\eta(1) = 1$ *and*
2. *for any* $a, b \in Q$ *,* $\eta(x) \geq \eta(a) \wedge \eta(b)$ *for all* $x \in (a,b)^{lu}$

Lemma 3.3.1. *$\eta \in L^Q$ is an L- fuzzy filter in the sense of Halaś if and only if η_α is a filter of Q in the sense of Halaś for all $\alpha \in L$.*

Corollary 3.3.2. *A subset S of Q is a filter of Q in the sense of Halaś if and only if its characteristic map χ_S is an L- fuzzy filter of Q in the sense of Halaś.*

Lemma 3.3.3. *If η is an L- fuzzy filter of Q in the sense of Halaś, then the following assertions hold:*

1. *for any* $x, y \in Q$*,* $\eta(x) \leq \eta(y)$ *whenever* $x \leq y$*; i.e.,* η *is iso-tone;*
2. *for any* $x, y \in Q$*,* $\eta(x \wedge y) \geq \eta(x) \wedge \eta(y)$ *whenever* $x \wedge y$ *exists.*

The following theorem shows that an L-fuzzy filter of a poset is a natural generalizations of an L-fuzzy filter of a lattice.

Theorem 3.3.4. *Let $(Q, \leq)$ be a lattice. Then an L-fuzzy subset η of Q is an L- fuzzy filter in the sense of Halaś in the poset Q if and only if it is an L-fuzzy filter in the lattice Q.*

Proof. Let $(Q, \leq)$ be a lattice. Let η be an L-fuzzy filter in the sense of Halaś in the poset Q and $a, b \in Q$. Then $\eta(1) = 1$. Since $a \wedge b \in [a \wedge b) = (a,b)^{lu}$, we have

$$\eta(a \wedge b) \geq \eta(a) \wedge \eta(b).$$

Also since η is iso-tone, we have

$$\eta(a) \geq \eta(a \wedge b) \text{ and } \eta(b) \geq \eta(a \wedge b).$$

This implies that

$$\eta(a) \wedge \eta(b) \geq \eta(a \wedge b) \text{ and so } \eta(a \wedge b) = \eta(a) \wedge \eta(b).$$

Hence η is an L-fuzzy filter in the lattice Q.

Conversely, suppose that η is an L-fuzzy filter in the lattice Q. Then $\eta(1) = 1$ and $\eta(a \wedge b) = \eta(a) \wedge \eta(b)$ for any $a, b \in Q$. Now let $a, b \in Q$ and $x \in (a,b)^{lu}$. Then $x \geq y$, for all $y \in (a,b)^l$. Since $a \wedge b \in (a,b)^l$, we have $x \geq a \wedge b$. Thus

$$\eta(x) \geq \eta(a \wedge b) = \eta(a) \wedge \eta(b).$$

So η is an L-fuzzy filter in the sense of Halaś in the poset Q. This completes the proof. □

Definition 3.3.2. *Let η be an L-fuzzy subset of Q. Then the L- fuzzy filter generated by η in the sense of Halaś, denoted by $[\eta)_{Ha}$, is the smallest L- fuzzy filter in the sense of Halaś containing η.*

Lemma 3.3.5. *The intersection of any family of L-fuzzy filters of Q in the sense of Halaś is an L- fuzzy filter of Q in the sense of Halaś.*

Theorem 3.3.6. *Let $\mathscr{F}\mathscr{F}(Q)$ be the set of all L-fuzzy filters in the sense of Halaś of a poset Q and μ be an L fuzzy subset of Q. Then $[\eta)_{Ha} = \bigcap\{\theta \in \mathscr{F}\mathscr{F}(Q) : \mu \subseteq \theta\}$.*

Theorem 3.3.7. *Let $[S)_{Ha}$ be a filter generated by subset S of Q in the sense of Halaś and χ_S be its characteristic functions. Then $[\chi_S)_{Ha} = \chi_{[S)_{Ha}}$.*

Theorem 3.3.8. *An L-fuzzy subset η of Q is an L-fuzzy filter in the sense of Halaś if and only if for any $F \subset\subset Q$,*

$$\eta(x) \geq \bigwedge_{a \in F} \mu(a) \text{ for all } x \in [F)_{Ha}.$$

In the rest of this dissertation by an L- fuzzy filter of a poset will mean an L- fuzzy filter in the sense of Halaś.

Now we give a characterization of an L- fuzzy filter generated by an L-fuzzy subset of a poset Q.

Theorem 3.3.9. *For any L-fuzzy subset η of Q, define an L-fuzzy subset $\hat{\eta}$ of Q by:*

$$\hat{\eta}(x) = \sup\{\alpha \in L : x \in [\eta_\alpha)_{Ha}\}, \text{for all } x \in Q.$$

Then $\hat{\eta}$ is an L-fuzzy filter of Q generated by η.

Definition 3.3.3. *Let η be an L- fuzzy subset of Q and $\mathbb{N}$ be a set of positive integers. Define L-fuzzy subsets of Q inductively as follow:*

$$B_1^\eta(x) = \sup\{\eta(a) \wedge \eta(b) : x \in (a,b)^{lu}\}$$

and for each $n \geq 2$ and $a, b \in Q$

$$B_n^\eta(x) = \sup\{B_{n-1}^\eta(a) \wedge B_{n-1}^\eta(b) : x \in (a,b)^{lu}\}$$

Lemma 3.3.10. *The set $\{B_n^\eta : n \in \mathbb{N}\}$ forms a chain and each B_n^η is isotone.*

Proof. Let $x \in Q$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} B_{n+1}^\eta(x) &= \sup\{B_n^\eta(a) \wedge B_n^\eta(b) : x \in (a,b)^{lu}\} \\ &\geq B_n^\eta(x) \wedge B_n^\eta(x) \ (since \ \ x \in x^u = (x,x)^{lu}) \\ &= B_n^\eta(x), \ \forall\, x \in Q. \end{aligned}$$

Thus $B_n^\eta \subseteq B_{n+1}^\eta$, for each $n \in \mathbb{N}$. So $\{B_n^\eta : n \in \mathbb{N}\}$ is a chain. Again let $x, y \in Q$ such that $x \leq y$. Now

$$\begin{aligned} B_n^\eta(x) &= \sup\{B_{n-1}^\eta(a) \wedge B_{n-1}^\eta(b) : x \in (a,b)^{lu}\} \\ &\leq \sup\{B_{n-1}^\eta(a) \wedge B_{n-1}^\eta(b) : y \in (a,b)^{lu}\} = B_n^\eta(y) \end{aligned}$$

Hence B_n^η is isotone for each $n \in \mathbb{N}$. □

Now we give a characterization of an L-fuzzy filterl generated by an L- fuzzy subset of a poset Q.

Theorem 3.3.11. *The L- fuzzy subset $\overline{\eta}_{Ha}$ defined by: for all $x \in Q$,*

$$\overline{\eta}_{Ha}(x) = \begin{cases} 1 & if\ x = 1 \\ \sup\{B_n^{\eta}(x) : n \in \mathbb{N}\} & if\ x \neq 1 \end{cases}$$

is an L- fuzzy filter generated by η.

Proof. Now we show that $\hat{\eta}$ is the smallest L-fuzzy filter containing η. Let $x \in Q$. Then

$$\begin{aligned} \overline{\eta}_{Ha}(x) &= \sup\{B_n^{\eta}(x) : n \in \mathbb{N}\} \\ &\geq B_1^{\eta}(x) \\ &= \sup\{\eta(a) \wedge \eta(b) : x \in (a,b)^{lu}\} \\ &\geq \eta(x) \wedge \eta(x) \ \ (since\ x \in (x,x)^{lu}) \\ &= \eta(x) \ \ \forall\, x \in Q. \end{aligned}$$

Therefore $\eta \subseteq \overline{\eta}_{Ha}$.

Let $a, b \in Q$ and $x \in (a,b)^{lu}$. If $a = 1$ or $b = 1$. then $b \leq x$ or $a \leq x$. Since B_n^{η} is isotone for each $n \in \mathbb{N}$, we clearly have $\overline{\eta}_{Ha}(a) \wedge \overline{\eta}_{Ha}(b) \leq \overline{\eta}_{Ha}(x)$. Let $a \neq 1$ and $b \neq 1$. Then

$$\begin{aligned} \overline{\eta}_{Ha}(a) \wedge \overline{\eta}_{Ha}(b) &= \sup\{B_n^{\eta}(a) : n \in \mathbb{N}\} \wedge \sup\{B_m^{\eta}(b) : m \in \mathbb{N}\} \\ &= \sup\{B_n^{\eta}(a) \wedge B_m^{\eta}(b) : n, m \in \mathbb{N}\} \\ &\leq \sup\{B_k^{\eta}(a) \wedge B_k^{\eta}(b) : k \in \mathbb{N}\} \text{ where } k = \max\{m,n\} \\ &\leq B_{k+1}^{\eta}(x) \ \ (\text{Since } x \in (a,b)^{ul}) \\ &\leq \sup\{B_k^{\eta}(x) : k \in \mathbb{N}\} \\ &= \overline{\eta}_{Ha}(x) \end{aligned}$$

So $\overline{\eta}_{Ha}$ is an L-fuzzy filter of Q.

Again let θ be any L-fuzzy filter of Q such that $\eta \subseteq \theta$. Now we show that the statement "$B_n^\eta \subseteq \theta$ for all $n \in \mathbb{N}$" is true. Now for any $x \in Q$, we have

$$\begin{aligned} B_1^\eta(x) &= \sup\{\eta(a) \wedge \eta(b) : x \in (a,b)^{lu}\} \\ &\leq \sup\{\theta(a) \wedge \theta(b) : x \in (a,b)^{lu}\} \leq \theta(x) \end{aligned}$$

This implies that $B_1^\eta \subseteq \theta$. Hence the statement is true for $n = 1$. Assume $B_n^\eta \subseteq \theta$ for some $n > 1$. Now for any $x \in Q$, we have

$$\begin{aligned} B_{n+1}^\eta(x) &= \sup\{B_n^\eta(a) \wedge B_n^\eta(b) : x \in (a,b)^{lu}\} \\ &\leq \sup\{\theta(a) \wedge \theta(b) : x \in (a,b)^{lu}\} \\ &\leq \theta(x). \end{aligned}$$

Thus $B_{n+1}^\eta \subseteq \theta$. Thus, by mathematical induction, we have $B_n^\mu \subseteq \theta$ for all $n \in \mathbb{N}$. Let $x \in Q$. If $x = 1$, then we have $\overline{\eta}_{Ha}(x) = 1 = \theta(x)$. Let $x \neq 1$. Then

$$\overline{\eta}_{Ha}(x) = sup\{B_n^\eta(x) : n \in \mathbb{N}\} \leq \theta(x)$$

Hence $\overline{\eta}_{Ha} \subseteq \theta$. Therefore $\overline{\eta}_{Ha} = [\eta)_{Ha}$. □

The above theorem yields the following.

Theorem 3.3.12. *Let $\mathscr{F}\mathscr{F}(Q)$ be the set of all L-fuzzy filter of Q. Then $(\mathscr{F}\mathscr{F}(Q), \subseteq)$ forms a complete lattice with respect to the point wise ordering " $\subseteq$ ", in which the supremum and the inifimum of any family $\{\eta_i : i \in \Delta\}$ in $\mathscr{F}\mathscr{F}(Q)$ respectively are :*

$$sup_{i\in\Delta}\eta_i = \overline{(\bigcup_{i\in\Delta}\eta_i)}_{Ha} \;\; \textit{and} \;\; \inf_{i\in\Delta}\eta_i = \bigcap_{i\in\Delta}\eta_i.$$

Corollary 3.3.13. *For any L-fuzzy filter η and ν of Q, the supremum $\eta \vee \nu$ and the inifimum $\eta \wedge \nu$ of η and ν in $\mathscr{F}\mathscr{F}(Q)$ respectively are: for any $x \in Q$,*

$$\eta \vee \nu = \overline{(\eta \cup \nu)}_{Ha} \;\; \textit{and} \;\; \eta \wedge \nu = \eta \cap \nu.$$

Theorem 3.3.14. *Let $x \in Q$ and $\alpha \in L$. Define an L- fuzzy subset α^x of Q by:*

$$\alpha^x(y) = \begin{cases} 1 & if\ y \in [x) \\ \alpha & if\ y \notin [x) \end{cases}$$

for all $y \in Q$. Then α^x is an L-fuzzy filter of Q.

Definition 3.3.4. *The L-fuzzy filter α^x defined above is called α-level principal fuzzy filter corresponding to x.*

Definition 3.3.5. *An L-fuzzy filter μ of a poset Q is called an l-L-fuzzy filter if for any $a,b \in Q$, there exists $x \in (a,b)^l$ such that $\mu(x) = \mu(a) \wedge \mu(b)$.*

Lemma 3.3.15. *An $L-$ fuzzy filter μ of Q is an l-L- fuzzy filter of Q if and only if μ_α is an l-filter of Q for all $\alpha \in L$.*

Proof. Suppose that μ is an l-L-fuzzy filter of Q and $\alpha \in L$. Since μ is an L-fuzzy filter, μ_α is a filter of Q in the sense of Halaś. Let $a,b \in \mu_\alpha$. Then

$$\mu(a) \geq \alpha \text{ and } \mu(b) \geq \alpha \text{ and hence } \mu(a) \wedge \mu(b) \geq \alpha.$$

Also since μ is an l- L- fuzzy filter there exists $x \in (a,b)^l$ such that

$$\mu(x) = \mu(a) \wedge \mu(b) \text{ and hence } \mu(x) \geq \alpha.$$

Therefore $x \in \mu_\alpha \cap (a,b)^l$ and hence $\mu_\alpha \cap (a,b)^l \neq \emptyset$. Therefore μ_α is an l-filter of a poset Q.

Conversely suppose μ_α is an l- filter of a poset Q for all $\alpha \in L$. Then μ is an L- fuzzy filter. Let $a,b \in Q$ and put $\alpha = \mu(a) \wedge \mu(b)$. Then $\mu_\alpha \cap (a,b)^l \neq \emptyset$. Let $x \in \mu_\alpha \cap (a,b)^l$. Then $x \in \mu_\alpha$ and $x \in (a,b)^l$. This implies that

$$\mu(x) \geq \alpha = \mu(a) \wedge \mu(b) \text{ and } x \leq a,\ x \leq b.$$

Since μ is iso-tone we have $\mu(x) \leq \mu(a)$ and $\mu(x) \leq \mu(b)$ and hence $\mu(x) \leq \mu(a) \wedge \mu(b)$. Therefore there exists $x \in (a,b)^l$ such that $\mu(x) = \mu(a) \wedge \mu(b)$. Therefore μ is an l-L-fuzzy filter. □

Corollary 3.3.16. *Let $(Q, \leq)$ be a poset with 0 and let $x \in Q$ and $\alpha \in L$. Then the α-level principal fuzzy filter corresponding to x is an l-L-fuzzy filter.*

Remark 3.3.1. *Not every L-fuzzy filter is an l-L-fuzzy filter. For example consider the poset $(Q \leq)$ depicted in the Fig. 3.1 below and define a fuzzy subset $\eta : Q \longrightarrow [0,1]$ by:*

$$\eta(1) = 1,\ \eta(c) = \eta(d) = 0.9,\ \eta(a) = \eta(b) = \eta(0) = 0.2.$$

Then η is an L-fuzzy filter which is not an l- L-fuzzy filter. This is because $c, d \in Q$ and there is no x in $(c,d)^l = \{0,,a,b\}$ such that $\eta(x) = \eta(c) \wedge \eta(d)$.

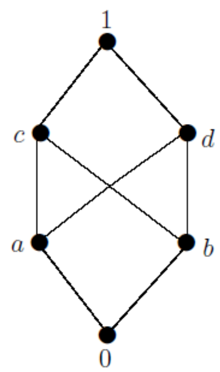

Fig. 3.1

However, if Q is a meet semi-lattice, then we have

Theorem 3.3.17. *Let $(Q, \leq)$ be a meet-semi-lattice. Then every L-fuzzy filter of Q is an l-L-fuzzy filter of Q.*

Proof. Let η be an L-fuzzy filter of a meet-semi-lattice Q. Let $a, b \in Q$. Since Q is a meet-semi-lattice, $a \wedge b$ exists and it is clear that $a \wedge b \in (a,b)^l$ and $\eta(a \wedge b) = \eta(a) \wedge \eta(b)$. Therefore η is an l-L-fuzzy filter of Q. □

Theorem 3.3.18. *Let η and ν be l- L-fuzzy filters of Q. Then the supremum $\eta \vee \nu$ of η and ν in $\mathscr{F}\mathscr{F}(Q)$ is given by:*

$$(\eta \vee \nu)(x) = \sup\{\eta(a) \wedge \nu(b) : x \in (a,b)^{lu}\} \text{ for all } x \in Q.$$

Proof. Let σ be an L-fuzzy subset of Q defined by $\sigma(x) = \sup\{\eta(a)\wedge\nu(b) : x \in (a,b)^{lu}\}\ \forall x \in Q$. Now we claim σ is the smallest L-fuzzy filter of Q containing $\eta\cup\nu$. Let $x\in Q$. Then

$$\begin{aligned}\sigma(x) &= \sup\{\eta(a)\wedge\nu(b) : x\in(a,b)^{lu}\}\\ &\geq \eta(x)\wedge\nu(1),\ (since\ x\in(x,1)^{lu})\\ &= \eta(x)\wedge 1 = \eta(x)\end{aligned}$$

and hence $\sigma\supseteq\eta$. Similarly we can show that $\sigma\supseteq\nu$. Therefore $\sigma\supseteq\eta\cup\nu$.
Let $a,b\in Q$ and $x\in(a,b)^{lu}$. Then

$$\begin{aligned}\sigma(a)\wedge\sigma(b) &= \sup\{\eta(c)\wedge\nu(d) : a\in(c,d)^{lu}\}\wedge\sup\{\eta(e)\wedge\nu(f) : b\in(e,f)^{lu}\}\\ &= \sup\{\eta(c)\wedge\nu(d)\wedge\eta(e)\wedge\nu(f) : a\in(c,d)^{lu}, b\in(e,f)^{lu}\}\\ &\leq \sup\{\eta(c)\wedge\eta(e)\wedge\nu(d)\wedge\nu(f) : a,b\in(c,d,e,f)^{lu}\}\end{aligned}$$

Again since η and ν are l-L-fuzzy filters, for each c,e and d,f there are $r\in(c,e)^{l}$ and $s\in(d,f)^{l}$ such that $\eta(r) = \eta(c)\wedge\eta(e)$ and $\nu(s) = \nu(d)\wedge\nu(f)$. Now

$$\begin{aligned}r\in(c,e)^{l} and\ s\in(d,f)^{l} &\Rightarrow \{c,d,e,f\}^{lu}\subseteq\{s,r\}^{lu}\\ &\Rightarrow a,b\in\{s,r\}^{lu}\\ &\Rightarrow (a,b)^{lu}\subseteq\{s,r\}^{lu}\\ &\Rightarrow x\in\{s,r\}^{lu}\\ \text{Thus}\ \ \sigma(a)\wedge\sigma(b) &\leq \sup\{\eta(c)\wedge\eta(e)\wedge\nu(d)\wedge\nu(f) : a,b\in(c,d,e,f)^{lu}\}\\ &\leq \sup\{\eta(r)\wedge\nu(s) : x\in(r,s)^{lu}\}\\ &\leq \sup\{\sigma(r)\wedge\sigma(s) : x\in(r,s)^{lu}\}\\ &\leq \sigma(x)\end{aligned}$$

Therefore σ is an L-fuzzy filter of Q.

Let ϕ be any L-fuzzy filter of Q such that $\eta \cup \nu \subseteq \phi$. Now for any $x \in Q$, we have

$$\begin{aligned}\sigma(x) &= \sup\{\eta(a) \wedge \nu(b) : x \in (a,b)^{lu}\} \\ &\leq \sup\{\phi(a) \wedge \phi(b) : x \in (a,b)^{lu}\} \\ &\leq \phi(x)\end{aligned}$$

and hence $\sigma \subseteq \phi$. Therefore $\sigma = [\eta \cup \nu)_{Ha} = \eta \vee \nu$, that is σ is the supremum of η and ν in $\mathscr{F}\mathscr{F}(Q)$. □

It is known that every L-fuzzy filter in meet-semi-lattice Q is an l-L-fuzzy filter. Therefore the following corollary is an easy consequence of the above theorem.

Corollary 3.3.19. *Let η and ν be l- L-fuzzy ideals of a meet-semi-lattice Q. Then the supremum $\eta \vee \nu$ of η and ν in $\mathscr{F}\mathscr{F}(Q)$ is given by:*

$$(\eta \vee \nu)(x) = \sup\{\eta(a) \wedge \nu(b) : a \wedge b \leq x\}, \text{for all } x \in Q.$$

3.4 *L*-Fuzzy Semi Filterss and V-Filters

In this section we introduce the fuzzy version of semi-filters and V-filters of a poset introduced by P.V. Venkatanarasimhan in [51] and in [52].

Definition 3.4.1. *η in L^Q is said to be an L-fuzzy semi-filter or L-fuzzy order filter if it satisfies the following conditions:*

1. *$\eta(1) = 1$;*
2. *for any $a \in Q$, $\eta(x) \geq \eta(a)$, for all $x \in a^u$.*

Definition 3.4.2. *An L- fuzzy semi-filter η of Q is called an L- fuzzy V-filter, if for any non-empty finite subset F of Q, if $\inf F$ exists, then*

$$\eta(\inf F) \geq \inf\{\eta(a) : a \in F\}.$$

Lemma 3.4.1. *An L- fuzzy subset η of Q is an L- fuzzy semi filter (respectively, L-fuzzy V-filter) of Q if and only if η_α is a semi-filter (respectively, V-filter) of Q, for all $\alpha \in L$.*

Corollary 3.4.2. *A subset S of Q is a semi filter (V-filter) of Q if and only if its characteristic map χ_S is an L-fuzzy semi filter (respectively, L- fuzzy V-filter) of Q.*

Lemma 3.4.3. *The intersection of any family of L-fuzzy semi-filters (respectively, L-fuzzy V-filters) of Q is an L- fuzzy semi-filter (respectively, L-fuzzy V-filter).*

Definition 3.4.3. *Let η be an L- fuzzy subset of a poset Q. The smallest L- fuzzy semi-filter of Q containing η is called an L- fuzzy semi-filter generated by η and is denoted by $[\mu)_{Se}$.*

Definition 3.4.4. *Let η be an L- fuzzy subset of a poset Q. The smallest L- fuzzy V-filter of Q containing η is called an L- fuzzy V-filter generated by η and is denoted by $[\mu)_V$.*

Theorem 3.4.4. *Let $\mathscr{F}\mathscr{S}\mathscr{F}(Q)$ be the set of all L-fuzzy semi-filters of a poset Q and η be an L fuzzy subset of Q. Then $[\mu)_{Se} = \bigcap\{\theta \in \mathscr{F}\mathscr{S}\mathscr{F}(Q) : \eta \subseteq \theta\}$.*

Theorem 3.4.5. *Let $\mathscr{F}\mathscr{V}\mathscr{F}(Q)$ be the set of all L-fuzzy V-filters of a poset Q and η be an L fuzzy subset of Q. Then $[\eta)_V = \bigcap\{\theta \in \mathscr{F}\mathscr{V}\mathscr{F}(Q) : \eta \subseteq \theta\}$.*

Theorem 3.4.6. *Let $[A)_{Se}$ be a semi-filter generated by a subset A of Q and χ_A be the characteristics functions of A. Then $\chi_{[A)_{Se}} = [\chi_A)_{Se}$.*

Theorem 3.4.7. *Let $[A)_V$ be a semi-filter generated by a subset A of Q and χ_A be the characteristics functions of A. Then $[\chi_A)_V = \chi_{[A)_V}$.*

In the following two theorems we give a characterization of any L-fuzzy semi-filter and L-fuzzy V-filter generated by an L-fuzzy subset of Q in terms of their level subsets.

Theorem 3.4.8. *For any L-fuzzy subset η of Q, define an L-fuzzy subset $\hat{\eta}$ of Q by:*

$$\hat{\eta}(x) = \sup\{\alpha \in L : x \in [\mu_\alpha)_{Se}\}, \textit{for all } x \in Q.$$

Then $\hat{\eta}$ is an L-fuzzy semi-filter of Q generated by η.

Proof. We show that $\hat{\eta}$ is the smallest L-fuzzy semi-filter containing μ. It is clear that $\eta \subseteq \hat{\eta}$ and $\hat{\eta}(1) = 1$.

Now let $a \in Q$, $x \in a^u$. Again let $\alpha \in L$ such that $a \in [\eta_\alpha)_{Se}$. Then since $x \in a^u$ and $a \in [\eta_\alpha)_{Se}$ imply that $x \in [\eta_\alpha)_{Se}$. So $\{\alpha : a \in [\eta_\alpha)_{Se}\} \subseteq \{\alpha : x \in [\eta_\alpha)_{Se}$ and hence

$$\hat{\eta}(a) = \sup\{\alpha : a \in [\eta_\alpha)_{Se}\} \leq \sup\{\alpha : x \in [\eta_\alpha)_{Se} = \hat{\eta}(x).$$

Therefore $\hat{\eta}$ is an L-fuzzy semi-filter.

Again let θ be any L-fuzzy semi filter of Q such that $\eta \subseteq \theta$. Then $\eta_\alpha \subseteq \theta_\alpha$, for any $\alpha \in L$ and hence $[\eta_\alpha)_{Se} \subseteq [\theta_\alpha)_{Se} = \theta_\alpha$. So for any $x \in Q$,

$$\hat{\eta}(x) = \sup\{\alpha \in L : x \in [\eta_\alpha)_{Se}\} \leq \sup\{\alpha \in L : x \in \theta_\alpha\} = \theta(x).$$

Hence $\hat{\eta} \subseteq \theta$. Therefore $\hat{\eta} = [\eta)_{se}$. □

Theorem 3.4.9. *For any L-fuzzy subset η of Q, define an L-fuzzy subset $\hat{\eta}$ of Q by $\hat{\eta}(x) = \sup\{\alpha \in L : x \in [\eta_\alpha)_V\}$, for all $x \in Q$. Then $\hat{\eta}$ is an L-fuzzy V-ideal of Q generated by η.*

In the following we give an algebraic characterization of an L-fuzzy semi-filter generated by an L-fuzzy subsets.

Theorem 3.4.10. *Let η be an L-fuzzy subset of Q. Then the L-fuzzy subset $\overline{\eta}_{Se}$ defined by:*

$$\overline{\eta}_{Se}(x) = \begin{cases} 1 & if\ x = 1 \\ \sup\{\eta(a) : a \in Q, x \in a^u\} & if\ x \neq 1 \end{cases}$$

, for all $x \in Q$ is an L- fuzzy semi-filter of Q generated by η.

Proof. Now we claim $\overline{\eta}_{Se}$ is the smallest L- fuzzy semi-filter of Q containing η. Let $x \in Q$. If $x = 1$, then $\eta(x) \leq 1 = \overline{\eta}_{Se}(x)$. Let $x \neq 1$. Then since $x \in x^u$, we have $\eta(x) \leq \sup\{\eta(a) : a \in Q, x \in a^u\} = \overline{\eta}_{Se}(x)$. Therefore $\eta \subseteq \overline{\eta}_{Se}$.

Again let $a \in Q$ such that $x \in a^u$. Now if $x = 1$, then, by definition of $\overline{\eta}_{Se}$, we have $\overline{\eta}_{Se}(a) \leq 1 = \overline{\eta}_{Se}(x)$. Let $x \neq 1$. Since $a \leq x$, it is clear that $a \neq 1$ and hence

$$\begin{aligned}\overline{\eta}_{Se}(a) &= \sup\{\eta(b) : b \in Q, a \in b^u\}\\ &\leq \sup\{\eta(b) : b \in Q, x \in b^u\} \text{ (...since } x \in a^u)\\ &= \overline{\eta}_{Se}(x).\end{aligned}$$

Thus $\overline{\eta}_{Se}$ is an L- fuzzy semi-filter of Q. Let θ be any L- fuzzy semi-filter of Q such that $\eta \subseteq \theta$. Let $x \in Q$. If $x = 1$, then $\overline{\eta}_{Se}(1) = 1 = \theta(1)$. Let $x \neq 1$. Then

$$\begin{aligned}\overline{\eta}_{Se}(x) &= \sup\{\eta(a) : a \in Q, x \in a^u\}\\ &\leq \sup\{\theta(a) : a \in Q, x \in a^u\}\\ &\leq \theta(x), \text{ as } \theta \text{ is an } L\text{- fuzzy semi-filter of } Q.\end{aligned}$$

Hence the claim is true. Therefore $\overline{\eta}_{Se} = (\eta]_{Se}$. □

The above theorem yields the following.

Theorem 3.4.11. *The set $\mathscr{F}\mathscr{S}\mathscr{F}(Q)$ of all L-fuzzy semi-filters of Q forms a complete lattice, in which the supremum $sup_{i\in\Delta}\eta_i$ and the inifimum $\inf_{i\in\Delta}\eta_i$ of any family $\{\eta_i : i \in \Delta\}$ of L-fuzzy semi-filters of Q are given by:*

$$\sup_{i\in\Delta}\eta_i = (\overline{\bigcup_{i\in\Delta}\eta_i})_{Se} \quad \text{and} \quad \inf_{i\in\Delta}\eta_i = \bigcap_{i\in\Delta}\eta_i.$$

Corollary 3.4.12. *For any η and ν in $\mathscr{F}\mathscr{S}\mathscr{F}(Q)$, the supremum $\eta \vee \nu$ and the infimum $\eta \wedge \nu$ of η and ν, respectively are:*

$$\eta \vee \nu = (\overline{\eta \cup \nu})_{Se} \quad \text{and} \quad \eta \wedge \nu = \eta \cap \nu.$$

Definition 3.4.5. *Let η be an L- fuzzy subset of Q and $\mathbb{N}$ be a set of positive integers. Define L-fuzzy subsets $B^1_\eta, B^2_\eta, \cdots, B^n_\eta \cdots$, of Q, inductively, as follow: for each $x \in Q$*

$$B^1_\eta(x) = \sup\{\textstyle\bigwedge_{a\in F}\eta(a) : \bigwedge F \leq x, \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\}$$

and for each $n \in \mathbb{N} - \{1\}$,

$$B_\eta^n(x) = \sup\{\bigwedge_{a\in F} B_\eta^{n-1}(a) : \bigwedge F \leq x,\ \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\}$$

Lemma 3.4.13. *The set* $\{B_\eta^n : n \in \mathbb{N}\}$ *forms a chain and for each* $n \in \mathbb{N}$, $B_\eta^n(x) \leq B_\eta^n(y)$ *whenever* $x \leq y$.

Proof. Let $x \in Q$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} B_\eta^{n+1}(x) &= \sup\{\bigwedge_{a\in F} B_\eta^n(a) : \bigwedge F \leq x, \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\} \\ &\geq B_\eta^n(x) \quad (Since\ \ \bigwedge\{x\} = x \leq x\ and\ \{x\} \subset\subset Q \\ &= B_\eta^n(x),\ \forall\, x \in Q. \end{aligned}$$

Therefore $B_\eta^n \subseteq B_\eta^{n+1}$, for each $n \in \mathbb{N}$. So $\{B_\eta^n : n \in \mathbb{N}\}$ is a chain.

Let $x \leq y$. Then

$$\begin{aligned} B_\eta^n(x) &= \sup\{\bigwedge_{a\in F} B_\eta^{n-1}(a) : \bigwedge F \leq x, \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\} \\ &\leq \sup\{\bigwedge_{a\in F} B_\eta^{n-1}(a) : \bigwedge F \leq y, \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\} \\ &= B_\eta^n(y) \end{aligned}$$

Therefore $B_\eta^n(x) \leq B_\eta^n(y)$ whenever $x \leq y$. That is C_μ^n is isotone for all $n \in \mathbb{N}$. □

Now we give a characterization of an L-fuzzy V-filter generated by an L- fuzzy subset of a poset Q.

Theorem 3.4.14. *The L- fuzzy subset* $\overline{\eta}_V$ *defined by: for all* $x \in Q$,

$$\overline{\eta}_V(x) = \begin{cases} 1 & if\ x = 1 \\ \sup\{B_\eta^n(x) : n \in \mathbb{N}\} & if\ x \neq 1 \end{cases}$$

is an L-fuzzy V-filter generated by η.

Proof. Now we claim that $\overline{\eta}_V$ is the smallest L-fuzzy ideal containing η. Let $x \in Q$. Then since $\bigwedge\{x\} = x \leq x$ and $\emptyset \neq \{x\} \subset\subset Q$, we have

$$\begin{aligned}\eta(x) &\leq \sup\{\bigwedge_{a\in F}\mu(a) : \bigwedge F \leq x, \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\}\\ &= B^1_\eta(x)\\ &\leq \sup\{B^n_\eta(x) : n \in \mathbb{N}\}\\ &\leq \overline{\eta}_V(x)\end{aligned}$$

Therefore $\eta \subseteq \overline{\eta}_V$. By definition of $\overline{\eta}_V$, we have $\overline{\eta}_V(1) = 1$. Let $a \in Q$ and $x \in a^u$. Then if $a = 1$ then $x = 1$ and hence $\overline{\eta}_V(a) = 1 = \overline{\eta}_V(x)$. Let $a \neq 1$. Then

$$\overline{\eta}_V(a) = \sup\{B^n_\eta(a) : n \in \mathbb{N}\} \leq \sup\{B^n_\eta(x) : n \in \mathbb{N}\} = \overline{\eta}_V(x)$$

. So $\overline{\eta}_V$ is an L-fuzzy semi-filter of Q.

Again let $\emptyset \neq F \subset\subset Q$ such that $\inf F$ exists.

$$\begin{aligned}\bigwedge_{a\in F}\overline{\eta}_V(a) &= \bigwedge_{a\in F}\sup\{B^n_\eta(a) :: n \in \mathbb{N}\}\\ &= \sup\{\bigwedge_{a\in F}B^n_\eta(a) :: n \in \mathbb{N}\}\\ &\leq B^{n+1}_\eta(\inf F) \ \ (\text{since } \inf F \leq \inf F)\\ &\leq \sup\{B^n_\eta(\inf F) : n \in \mathbb{N}\}\\ &= \overline{\eta}_V(\inf F)\end{aligned}$$

So $\overline{\eta}_V$ is an L-fuzzy V-filter of Q.

Again let θ be any L-fuzzy V-filter of Q such that $\eta \subseteq \theta$. Now we show that $B^n_\eta \subseteq \theta$ for all $n \in \mathbb{N}$. Let $x \in Q$. Then

$$B^1_\eta(x) = \sup\{\bigwedge_{a\in F}\eta(a) : \inf F \leq x,\ \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\}$$

$$\begin{aligned}
&\leq \sup\{\bigwedge_{a\in F}\theta(a) : \inf F \leq x,\ \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\}\\
&\leq \sup\{\bigwedge_{a\in F}\theta(\inf F) : \inf F \leq x,\ \emptyset \neq F \subset\subset Q \text{ and } \bigwedge F \text{ exists}\}\\
&\leq \theta(x)
\end{aligned}$$

Therefore $B^1_\eta \subseteq \theta$ Hence the statement is true for $n = 1$.

Assume $B^n_\eta \subseteq \theta$ for some $n > 1$.

$$\begin{aligned}
B^{n+1}_\eta(x) &= \sup\{\bigwedge_{a\in F} C^n_\eta(a) : x \leq \sup F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\}\\
&\leq \sup\{\bigwedge_{a\in F}\theta(a) : \inf F \leq x,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\}\\
&\leq \sup\{\theta(\sup F) : x \leq \sup F,\ \emptyset \neq F \subset\subset Q \text{ and } \bigvee F \text{ exists}\}\\
&\leq \theta(x)
\end{aligned}$$

Thus, by mathematical induction, we have $\theta(x) \geq B^n_\eta(x)\ \forall n \in \mathbb{N}$. . So for any $x \in Q$, we have

$\theta(x) \geq sup\{B^n_\eta(x) : n \in \mathbb{N}\} = \overline{\eta}_V(x)$. Therefore $\overline{\eta}_V \subseteq \theta$. □

Theorem 3.4.14 yields the following.

Theorem 3.4.15. *The set $\mathscr{F}\mathscr{V}\mathscr{F}(Q)$ of all L-fuzzy V-filter of Q forms a complete lattice under point-wise ordering " $\subseteq$ "; in which the supremum $sup_{i\in\Delta}\eta_i$ and the inifimum $\inf_{i\in\Delta}\eta_i$ of any family $\{\eta_i : i \in \Delta\}$ in $\mathscr{F}\mathscr{V}\mathscr{F}(Q)$ respectively are:*

$$sup_{i\in\Delta}\eta_i = (\overline{\bigcup_{i\in\Delta}\eta_i})_V \quad \text{and} \quad \inf_{i\in\Delta}\eta_i = \bigcap_{i\in\Delta}\eta_i.$$

Corollary 3.4.16. *For any η and $\nu \in \mathscr{F}\mathscr{V}\mathscr{F}(Q)$ the supremum $\eta \vee \nu$ and the infimum $\eta \wedge \nu$ of η and ν respectively are:*

$$\eta \vee \nu = (\overline{\eta \cup \nu})_V \quad \text{and} \quad \eta \wedge \nu = \eta \cap \nu.$$

Now we study the relationships among types of L-fuzzy filters introduced in this chapter.

Theorem 3.4.17. *The following implications hold.*

1. *L-fuzzy closed filter* $\Longrightarrow$*L- fuzzy Frink filter* $\Longrightarrow$ *L-fuzzy* V*-filter* $\Longrightarrow$ *L-fuzzy semi-filter.*

2. *L- fuzzy closed filter* $\Longrightarrow$ *L-fuzzy Frink filter* $\Longrightarrow$ *L-fuzzy filter* $\Longrightarrow$ *L- fuzzy semi-filter.*

The following examples show that the converse of the above implications do not hold in general.

Example 3.4.18. *Consider the poset* $([0,1],\leq)$ *with the usual ordering. Define a fuzzy subset* $\eta : [0,1] \longrightarrow [0,1]$ *by*

$$\eta(x) = \begin{cases} 1 & if\ x \in (\frac{1}{2}, 1] \\ 0 & if\ x \in [0, \frac{1}{2}] \end{cases}$$

Then η *is an L- fuzzy Frink filter but not an L- fuzzy closed filter.*

Example 3.4.19. *Consider the poset* $(Q,\leq)$ *depicted in the Fig. 3.2 below. Define a fuzzy subset* $\nu : Q \longrightarrow [0,1]$ *by:*

$\nu(1) = \nu(a') = 1$*,* $\nu(a) = \nu(b) = \nu(c) = \nu(d) = \nu(0) = 0.2$*,* $\nu(b') = 0.6$*,* $\nu(c') = 0.5$ *and* $\nu(d') = 0.7$.

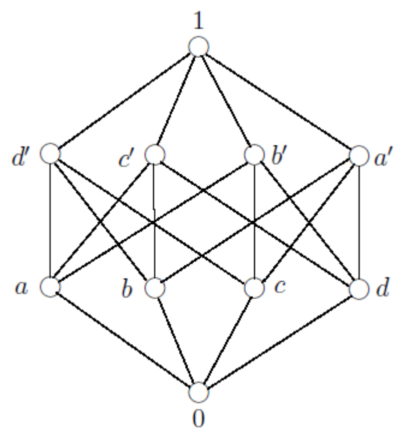

Fig. 3.2

Then ν is an L- fuzzy filter but not an L- fuzzy Frink-filter.

This is because $F = \{1, a', b', c'\} \subset\subset Q$ and $d \in F^{lu} = \{1, a', b', c', d\}$ but $\nu(d) = 0.2 \not\geq 0.5 = \inf\{\nu(x) : x \in F\}$

Example 3.4.20. *Consider the poset $(Q, \leq)$ depicted in the Fig. 3.3 below. Define a fuzzy subset $\theta : Q \longrightarrow [0,1]$ by $\theta(U) = 1$, $\theta(L) = \theta(M) = 0.8$ and $\theta(N) = 0.6$.*

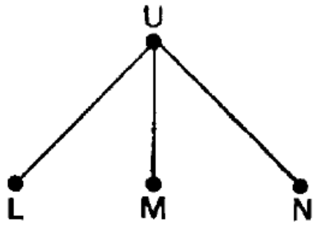

Figure 3.3

Then θ is an L- fuzzy V-filter but not an L- fuzzy Frink-filter.

This is because $F = \{U, L, M\} \subset\subset Q$ and $N \in F^{lu} = \{U, L, M, N\}$ but $\theta(N) = 0.6 \not\geq 0.8 = \inf\{\theta(x) : x \in F\}$.

Example 3.4.21. *Consider the poset $(Q, \leq)$ depicted in the Fig. 3.4 below. Define a fuzzy subset $\sigma : Q \longrightarrow [0,1]$ by $\sigma(1) = 1$, $\sigma(a) = 0.8$, $\sigma(b) = 0.9$ and $\sigma(0) = 0.2$.*

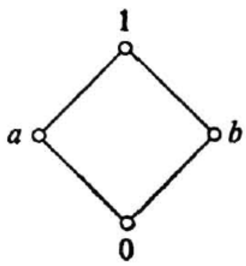

Fig. 3.4

Then σ is an L-fuzzy semi-filter but not an L-fuzzy filter. This is because $a, b \in Q$ and $0 \in (a, b)^{lu}$ but $\sigma(0) = 0.2 \not\geq 0.8 = \sigma(a) \wedge \sigma(b)$.

Theorem 3.4.22. *Every l-L-fuzzy filter of a poset Q is an L- fuzzy Frink filter of Q.*

Proof. suppose that η is an l-L-fuzzy filter. Let F be a finite subset of Q.

Then there exists $y \in F^l$ such that $\eta(y) = inf\{\eta(a) : a \in F\}$.

$$\begin{aligned} Again\ x \in F^{lu} &\Rightarrow s \leq x\, \forall s \in F^l \\ &\Rightarrow y \leq x\ \ (since\, y \in F^l) \\ &\Rightarrow \eta(x) \geq \eta(y) = inf\{\eta(a) : a \in F\} \end{aligned}$$

Therefore η is an L-fuzzy Frink filter of Q. □

Now we complete this chapter by introducing the following definition which generalize all the L-fuzzy filters of a poset introduced in this chapter.

Definition 3.4.6. *An L- fuzzy subset η of Q is an m- L- fuzzy filter, if it satisfies the following conditions:*

(i) $\eta(0) = 1$,

(ii) for any subset A of Q of cardinality strictly less than m, written as, $A \subset_m Q$,

$$\eta(x) \geq \inf\{\eta(a) : a \in A\},\ \forall x \in A^{lu}, \text{ where } m \text{ is any cardinal.}$$

Remark 3.4.1. *The following special cases are included in this general definition:*

1. *Ω-L-fuzzy filters are L-fuzzy closed filters, where Ω is a cardinal number greater than the cardinal number of Q.*

2. *ω- L-fuzzy filters-ideals are L-fuzzy Frink filters, where ω the smallest infinite cardinal number.*

3. *3- L-fuzzy filters are L-fuzzy filters in the sense of Halaś.*

4. *2-L-fuzzy filters are L-fuzzy semi-filters.*

5. *L-fuzzy V-filters are 2-L-fuzzy filters and if for any non-empty finite subset F of Q, if* $\inf F$ *exists, then* $\eta(\inf F) \geq \inf\{\eta(a) : a \in F\}$.

Chapter 4

L-Fuzzy Prime and Maximal L-Fuzzy Ideals

A prime ideal in a poset was introduced by Halaś and Rachůnek [30] in 1995. Next in 2006, Erné [22] did a systematic investigation and comparison of various prime and maximal ideal theorems in partially ordered sets. He proved that the following conditions are equivalent on poset Q.

1. For each ideal I and each down-directed subset D of Q disjoint from I, there is a prime ideal containing I and disjoint from D.

2. Each ideal of Q is an intersection of prime ideals.

3. For $a, b \in Q$ with $a \nleq b$, there is a prime ideal P with $a \notin P$ but $b \in P$.

4. Q is ideal distributive.

Also, the theory of prime ideals in a poset has been further developed by V. S. Kharat and K. A. Mokbel [34] in 2009, Joshi and Mundlik [32] in 2013 and Erné and Joshi [24] in 2015.

On the other hand, U. M. Swamy and K. L. N. Swamy [47] introduced the concept of L-fuzzy prime ideals in rings and U. M. Swamy and D. V. Raju [46] in lattices with truth values in a complete lattice satisfying the infinite meet distributive law and latter Koguep et al. [36] discussed certain properties of prime fuzzy ideals of lattices when the truth values are taken from the interval $[0,1]$ of real numbers.

We have already introduced several generalizations of L-fuzzy ideals and filters of a lattice to an arbitrary poset in chapter 2 and 3, respectively. In this chapter, by choosing, the L-fuzzy ideals and filters of a poset in the sense of Halaš as L-fuzzy ideals and filters of a poset, we introduce and present certain comprehensive results on the notion of L-fuzzy prime ideals, prime L-fuzzy ideals, maximal L-fuzzy ideals and L-fuzzy maximal ideals by applying the general theory of algebraic fuzzy systems introduced in [48] and [49]. We also study the existence of L-fuzzy prime ideals and prime L-fuzzy ideals in the lattice $(\mathscr{F}\mathscr{I}(Q), \subseteq)$ of L-fuzzy ideals of a poset.

4.1 Prime and Maximal Ideals

In this section, we recall some definitions and crisp concepts of prime and maximal ideals of a poset from a literature that will be extended to the notions of L-fuzzy prime ideals, prime and maximal L-fuzzy ideals of a poset in the further sections of this chapter.

An ideal I (respectively, a filter F) of a poset Q is called proper if $I \neq Q$ (respectively, $F \neq Q$). Now, we consider the concept of a prime ideal introduced by Halaš[28] and Halaš and Rachůnek [30] as given in the following

Definition 4.1.1 ([30]). *A proper ideal P of a poset Q is called prime, if for all $a, b \in Q$, $(a,b)^l \subseteq P$ implies $a \in P$ or $b \in P$.*

The set of all prime ideals of a poset Q is denoted by $\mathscr{P}(Q)$. A proper ideal I of a poset Q is prime if it is a prime element in the lattice $\mathscr{I}(Q)$ of ideals of a poset Q, as we show in the next result.

Theorem 4.1.1. *A proper ideal P of a poset Q is prime if and only if for any two ideals I, J of Q with $I \cap J \subseteq P$, we have $I \subseteq P$ or $J \subseteq P$.*

Proof. Suppose that P is a prime ideal and I and J be ideals of Q such that $I \cap J \subseteq P$. Now we claim either $I \subseteq P$ or $J \subseteq P$. Suppose on the contrary that $I \cap J \subseteq P$ and $I \nsubseteq P$ and $J \nsubseteq P$. Then three exist elements $a \in I$ and $b \in J$ such that $a, b \notin P$. Then, by the hypothesis, $(a,b)^l \nsubseteq P$. Since $(a,b)^l \subseteq a^l \subseteq I$ and $(a,b)^l \subseteq b^l \subseteq J$, we have $(a,b)^l \subseteq I \cap J \subseteq P$, which is a contradiction. Hence the claim is true.

Conversely, suppose that let $a, b \in Q$ such that $(a,b)^l \subseteq P$. Then $(a] \cap (b] = (a,b)^l \subseteq P$. Then, by the hypothesis, we have either $(a] \subseteq P$ or $(b] \subseteq P$ and so, $a \in P$ or $b \in P$. Therefore P is a prime ideal of Q. □

Definition 4.1.2. *[30] A proper filter P of a poset Q is called prime if for all $a, b \in Q$, $(a,b)^u \subseteq I$ implies $a \in P$ or $b \in P$.*

Dually, a proper filter F of a poset Q is prime if it is a prime element in the lattice $\mathscr{F}(Q)$ of filters of a poset Q, as we show in the next result.

Theorem 4.1.2. *A proper filter P of a poset Q is prime if and only if for any two filters F, G of Q with $F \cap G \subseteq P$, we have $F \subseteq P$ or $G \subseteq P$.*

Definition 4.1.3. *For an ideal I of a poset Q and $a \in Q$, we define the set*

$$I : a = \{x \in Q : (a,x)^l \subseteq I\}$$

Lemma 4.1.3. *Let I be an ideal of a poset Q. Then $I : a$ is a semi-ideal of Q for any $a \in Q$.*

Proof. Let I be an ideal of a poset Q. Since $(a,0)^l = \{0\} \subseteq I$, $0 \in I : a$. Again let $x, y \in Q$ such that $y \leq x$ with $x \in I : a$. Then $(a,x)^l \subseteq I$ which implies $(a,y)^l \subseteq (a,x)^l \subseteq I$. Hence $y \in I : a$. □

Remark 4.1.1. *For an ideal I of Q, $I : a$ need not be an ideal of Q for all $a \in Q$. For example consider the poset depicted in the Fig. 4.1 given below.*

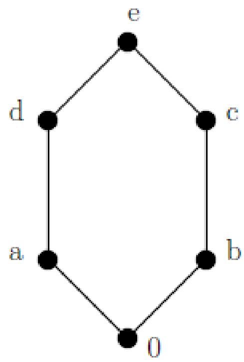

Fig. 4.1

Consider the ideal $I = \{0, a\}$ *of* Q*, but* $I : d = \{0, a, b, c\}$ *is not an ideal as* $(a, b)^{ul} = e^l = Q \nsubseteq I : d$

Lemma 4.1.4. *[34] Let* I *be an ideal of a poset* Q *and* $a, b, x \in Q$*. Then* $(a, b)^l \subseteq I : x$ *if and only if* $(x, a, b)^l \subseteq I$*.*

Proof. Suppose that $(a, b)^l \subseteq I : x$. Let $z \in (x, a, b)^l$. Then, as $(x, a, b)^l \subseteq (a, b)^l$, we have $z \in I : x$. This implies that $z \in (x, z)^l \subseteq I$. Therefore $(x, a, b)^l \subseteq I$.

Conversely suppose that $(x, a, b)^l \subseteq I$. Let $z \in (a, b)^l$. Then it is clear that $(z, x)^l \subseteq (x, a, b)^l \subseteq I$. This implies that $z \in I : x$. Therefore $(a, b)^l \subseteq I : x$. □

Lemma 4.1.5. *[32] Let* I *be an ideal of an atomic poset* Q *and let* p *be a dually distributive atom of* Q *such that* $p \notin I$*. Then* $I : p$ *is a u-ideal.*

The following theorem is on the existence of prime ideals in atomic posets whose proof is given from V. Joshi, N. Mundlik work in[32].

Theorem 4.1.6. ***[The existence of prime ideals in atomic posets]***

Let Q *be an atomic poset and let* p *be an atom of* Q*. If* p *is a dually distributive element of* Q *such that* $p \notin I$ *for an ideal* I*, then there exists a prime u-ideal* P *such that* $I \subseteq P$ *and* $p \notin P$*. Conversely, for any ideal* I *such that* $p \notin I$*, if there exists a prime ideal* P *such that* $I \subseteq P$ *and* $p \notin P$ *then* p *is a dually distributive element.*

Definition 4.1.4. *A proper ideal M of a poset Q is called a maximal ideal if the only ideal properly containing M is Q. That is, a proper ideal M of Q is called maximal ideal of if, for any ideal I of Q, $M \subseteq I$ implies that either $M = I$ or $I = Q$.*

The set of all maximal ideals of a poset Q is denoted by $\mathscr{M}ax(Q)$.

Note that the existence of maximal ideals in a poset Q is not guaranteed. Consider the following example.

Example 4.1.7. *Let Q be a chain without largest element. If M is a proper ideal of Q, then we can choose $a \in Q - M$ and another element $b \in Q$ such that $a < b$. Then, it can be easily checked that $(a]$ is a proper ideal of Q containing M properly; that is, $M \subsetneqq (a] \subsetneqq Q$ and hence M is not a maximal ideal of Q. Therefore Q has no maximal ideals.*

However, if a poset Q possesses 1 then, by Zorn's Lemma, there exists a maximal ideal in Q.

A maximal ideal of a poset need not be a prime ideal. For, consider the following.

Example 4.1.8. *Consider the poset depicted in the Fig. 3.2 given in chapter 3. $M = \{0, a, b, c, d\}$ is a maximal ideal because for each $x \in Q - M$ there exists $y \in M$ such that $(x, y)^{ul} = Q$. But M is not a prime ideal, since $(a', b')^l = \{0, c, d\} \subseteq M$ and $a' \notin M$ and $b' \notin M$.*

How ever, if the posrt Q is an ideal distributive poset, then every maximal ideal of X is a prime ideal.

Theorem 4.1.9. *Let Q be an ideal distributive poset and M be a maximal ideal of Q. Then M is a prime ideal.*

Proof. Let M be a maximal ideal of Q. Then, by definition, $M \neq Q$. Let $a, b \in Q$ such that $(a, b)^l \subseteq M$. Now we claim that either $a \in M$ or $b \in M$. Suppose on the contrary that $a \notin M$ and $b \notin M$. Then, by maximality of M, we have $M \vee (a] = Q$ and $M \vee (b] = Q$. Now $(a, b)^l \subseteq M \Rightarrow (a] \cap (b] \subseteq M$. Since Q is an ideal distributive poset, $(\mathscr{I}(Q), \subseteq)$ is a distributive lattice and hence

$$M = M \vee ((a] \cap (b]) = (M \vee (a]) \cap (M \vee (b]) = Q$$

which is a contradiction. Hence the claim is true. Therefore M is a prime ideal of Q. □

4.2 L-Fuzzy Prime Ideals

In this section we introduce the notion of L-fuzzy prime ideals of a poset which is the fuzzy version of prime ideals which was introduced by Halaś and Rachůnek [30].

Let us recall from [8] that an L-fuzzy subset μ of a poset Q with 0 is called an L-fuzzy ideal of Q if $\mu(0) = 1$ and for any $a, b \in Q$, $\mu(x) \geq \mu(a) \wedge \mu(b)$, for all $x \in (a,b)^{ul}$.

Note that for any α in L, the constant L-fuzzy subset of Q which maps all elements of Q onto α is denoted by $\overline{\alpha}$.

Definition 4.2.1. *An L-fuzzy ideal μ of a poset Q is called proper, if μ is not the constant map $\overline{1}$. That is, if there exists an element $a \in Q$ such that $\mu(a) \neq 1$.*

Recall that a proper L-fuzzy ideal μ of a lattice X is called an L-fuzzy prime ideal, if $\mu(a \wedge b) = \mu(a)$ or $\mu(b)$ for any $a, b \in X$ (See [36]).

Now we introduce the notion of L-fuzzy prime ideal of a poset Q.

Definition 4.2.2. *A proper L-fuzzy ideal μ of a poset Q is called an L-fuzzy prime, if for any $a, b \in Q$,*

$$inf\{\mu(x) : x \in (a,b)^l\} = \mu(a) \text{ or } \mu(b).$$

The following result characterizes any L-fuzzy prime ideal of a poset in terms of its level-subset.

Theorem 4.2.1. *An L-fuzzy ideal μ of a poset Q is an L-fuzzy prime if and only if for any $\alpha \in L$, either $\mu_\alpha = Q$ or μ_α a prime ideal of Q.*

Proof. Suppose that μ is an L-fuzzy prime ideal of Q and $\alpha \in L$. Since μ is an L-fuzzy ideal, clearly μ_α is an ideal of Q. Suppose that $\mu_\alpha \neq Q$. Now for any $a, b \in Q$,

$$\begin{aligned}(a,b)^l \subseteq \mu_\alpha &\Rightarrow \mu(x) \geq \alpha \ \forall x \in (a,b)^l \\ &\Rightarrow \inf\{\mu(x) : x \in (a,b)^l\} \geq \alpha \\ &\Rightarrow \mu(a) \geq \alpha \text{ or } \mu(b) \geq \alpha \\ &\Rightarrow a \in \mu_\alpha \text{ or } b \in \mu_\alpha.\end{aligned}$$

Conversely, suppose that $\mu_\alpha = Q$ or μ_α is a prime ideal of Q, for each $\alpha \in L$. Let $a, b \in Q$ and put $\alpha = inf\{\mu(x) : x \in (a,b)^l\}$. Then clearly, $x \in \mu_\alpha \ \forall x \in (a,b)^l$, that is, $(a,b)^l \subseteq \mu_\alpha$. Thus, by hypothesis, we have either $a \in \mu_\alpha$ or $b \in \mu_\alpha$. This implies that

$$\mu(a) \geq \alpha = inf\{\mu(x) : x \in (a,b)^l\}) \text{ or } \mu(b) \geq \alpha = inf\{\mu(x) : x \in (a,b)^l\}.$$

Also since μ is anti-tone, we clearly have

$$\mu(a) = inf\{\mu(x) : x \in (a,b)^l\} \text{ or } \mu(b) = inf\{\mu(x) : x \in (a,b)^l\}.$$

So μ is an L-fuzzy prime ideal of Q. □

The following result also characterizes an L-fuzzy prime ideals of a poset Q.

Corollary 4.2.2. *Let μ be a proper L-fuzzy ideal of a poset Q. Then μ is an L-fuzzy prime ideal of Q if and only if $Im(\mu)$ is a chain in L and for any $a, b \in Q$*

$$\inf\{\mu(x) : x \in (a,b)^l\} = \mu(a) \vee \mu(b).$$

Proof. Let μ be an L-fuzzy prime ideal of Q and $a, b \in Q$. Then $\mu(a), \mu(b) \in Im(\mu)$. Put $\alpha = \mu(a) \vee \mu(b)$. Now we show that $(a,b)^l \subseteq \mu_\alpha$. Now

$x \in (a,b)^l \Rightarrow x \leq a$ and $x \leq b \Rightarrow \mu(x) \geq \mu(a)$ and $\mu(x) \geq \mu(b) \Rightarrow$
$\mu(x) \geq \mu(a) \vee \mu(b) = \alpha \Rightarrow x \in \mu_\alpha$.

Thus $(a,b)^l \subseteq \mu_\alpha$. Again since $\mu_\alpha = Q$ or a prime ideal of Q, we have either $a \in \mu_\alpha$ or $b \in \mu_\alpha$. This implies that

$$\mu(a) \geq \alpha = \mu(a) \vee \mu(b) \geq \mu(b) \text{ or } \mu(b) \geq \alpha = \mu(a) \vee \mu(b) \geq \mu(a).$$

So $Im(\mu)$ is a chain in L and we clearly have

$$\inf\{\mu(x) : x \in (a,b)^l\} = \mu(a) \vee \mu(b).$$

The converse is straight forward. □

Lemma 4.2.3. *Let μ be an L- fuzzy ideal of Q. Then for any $a,b \in Q$,*

$$inf\{\mu(x) : x \in (a,b)^l\} = \mu(a \wedge b),$$

whenever $a \wedge b$ exists in Q.

Proof. Put $X = \{\mu(x) : x \in (a,b)^l\}$.

$$\begin{aligned} Now\, x \in (a,b)^l &\Rightarrow x \leq a \text{ and } x \leq b \\ &\Rightarrow x \leq a \wedge b \\ &\Rightarrow \mu(x) \geq \mu(a \wedge b) \end{aligned}$$

Then $\mu(x) \geq \mu(a \wedge b)$ for all $x \in (a,b)^l$. Thus $\mu(a \wedge b)$ is a lower bound of X. Let α be any lower bound of X. Then $\alpha \leq \mu(x)$, for all $x \in (a,b)^l$. Since $a \wedge b \in (a,b)^l$, we have $\alpha \leq \mu(a \wedge b)$. Thus $inf\{\mu(x) : x \in (a,b)^l\} = \mu(a \wedge b)$. □

Corollary 4.2.4. *Let $(Q, \leq)$ be a lattice. Then an L-fuzzy ideal μ of Q is an L- fuzzy prime ideal in the poset Q if and only if it is an L-fuzzy prime ideal in the lattice Q.*

Now, given an L-fuzzy ideal of a poset Q and any element in Q, we define the following L-fuzzy subset of Q as follow:.

Definition 4.2.3. *Let μ be an L-fuzzy ideal of Q and $x \in Q$. Define an L-fuzzy subset $\mu : x$ of Q by:*

$$(\mu : x)(y) = \inf\{\mu(z) : z \in (x,y)^l\} \text{ for all } y \in Q.$$

By the Definition 4.2.3, observe that an L-fuzzy ideal μ of Q is an L-fuzzy prime ideal if for any $a,b \in Q$,

$$(\mu : a)(b) = \mu(a) \text{ or } \mu(b).$$

Now we have the following lemma.

Lemma 4.2.5. *Let μ be an L-fuzzy ideal of Q and $x \in Q$. Then $\mu : x$ is an L-fuzzy semi ideal containing μ.*

Proof. Now $(\mu : x)(0) = \inf\{\mu(z) : z \in (x,0)^l\} = \inf\{\mu(z) : z = 0\} = \mu(0) = 1$ Therefore $(\mu : x)(0) = 1$.
Again let $a \in Q$ and $y \in a^l$. Now

$$\begin{aligned} (\mu : x)(y) &= \inf\{\mu(w) : w \in (x,y)^l\} \\ &\geq \inf\{\mu(w) : w \in (x,a)^l\} \quad (\text{ Since } (x,y)^l \subseteq (x,a)^l) \\ &= (\mu : x)(a) \end{aligned}$$

Therefore $\mu : x$ is an L-fuzzy semi ideal. Again let $y \in Q$. Since $\mu(z) \geq \mu(y)\ \forall z \in (x,y)^l$, we have

$$(\mu : x)(y) = \inf\{\mu(z) : z \in (x,y)^l\} \geq \mu(y)$$

Hence $\mu \subseteq \mu : x$. This proves the lemma. □

Note that for any $x, y \in Q$, observe that

$$(\mu : x)(y) = (\mu : y)(x)$$

Remark 4.2.1. *For an L-fuzzy ideal μ of a poset Q $\mu : x$ need not be an L-fuzzy ideal of Q for all $x \in Q$. For example consider the poset $(Q, \leq)$ depicted in the Fig. 4.2 given below. Define a fuzzy subset $\mu : Q \longrightarrow [0.1]$ by:*

$$\mu(0) = 1, \mu(a) = 0.8, \mu(b) = \mu(c) = \mu(d) = \mu(e) = 0.2.$$

Then μ is an L-fuzzy ideal of Q and$\mu : d$ is a fuzzy subset of Q given by:

$$(\mu : d)(0) = (\mu : d)(b) = (\mu : d)(c) = 1, \ (\mu : d)(a) = 0.8 \quad and$$
$$(\mu : d)(d) = (\mu : d)(e) = 0.2.$$

Observe that $e \in (a,b)^{ul}$ but $(\mu : d)(e) = 0.2 \not\geq 0.8 = (\mu : d)(a) \wedge (\mu : d)(b)$. This implies that $\mu : d$ is not an L-fuzzy ideal of Q.

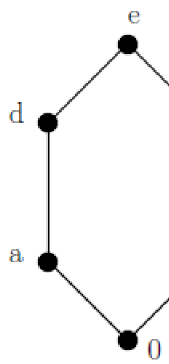

Fig. 4.2

Lemma 4.2.6. *Let μ be an L-fuzzy ideal of a poset Q and $x \in Q$. Then the following hold:*

1. $(\mu : x)_\alpha = \mu_\alpha : x$*, for any* $\alpha \in L$.
2. $\inf\{(\mu : x)(y) : y \in (a,b)^l\} = \inf\{\mu(y) : y \in (x,a,b)^l\}$
3. $\mu : x = \overline{1}$ *if and only if* $\mu(x) = 1$.

Proof. (1) Now we have

$$\begin{aligned} y \in (\mu : x)_\alpha &\Leftrightarrow (\mu : x)(y) \geq \alpha \\ &\Leftrightarrow \inf\{\mu(w) : w \in (x,y)^l\} \geq \alpha \\ &\Leftrightarrow \mu(w) \geq \alpha \ for \ all \ w \in (x,y)^l \\ &\Leftrightarrow w \in \mu_\alpha \ for \ all \ w \in (x,y)^l \\ &\Leftrightarrow (x,y)^l \subseteq \mu_\alpha \\ &\Leftrightarrow y \in \mu_\alpha : x. \end{aligned}$$

Therefore $(\mu : x)_\alpha = \mu_\alpha : x$

(2) Let $A = \{\mu(y) : y \in (x,a,b)^l\}$ and $B = \{(\mu : x)(y) : y \in (a,b)^l\}$. Now we claim that $\inf A = \inf B$. Put $\alpha = \inf A$. Then

$$\begin{aligned}
\alpha \leq \mu(z)\ \ \forall z \in (x,a,b)^l \ &\Rightarrow\ (x,a,b)^l \subseteq \mu_\alpha \\
&\Rightarrow\ (a,b)^l \subseteq \mu_\alpha : x = (\mu : x)_\alpha \\
&\Rightarrow\ (\mu : x)(y) \geq \alpha\ \ \forall y \in (a,b)^l \\
&\Rightarrow\ \inf\{(\mu : x)(y) : y \in (a,b)^l\} \geq \alpha. \\
&\Rightarrow\ \inf B \geq \inf A.
\end{aligned}$$

To prove the other side of the inequality, let $\beta = \inf B$. Then

$$\begin{aligned}
\beta \leq (\mu : x)(y)\ \ \forall y \in (a,b)^l \ &\Rightarrow\ (a,b)^l \subseteq (\mu : x)_\beta = \mu_\beta : x \\
&\Rightarrow\ (x,a,b)^l \subseteq \mu_\beta \\
&\Rightarrow\ \mu(y) \geq \beta\ \ \forall y \in (x,a,b)^l \\
&\Rightarrow\ \inf\{\mu(y) : y \in (x,a,b)^l\} \geq \beta. \\
&\Rightarrow\ \inf A \geq \inf B.
\end{aligned}$$

Hence the claim is true.

(3) Let $\mu : x = \overline{1}$. Then $(\mu : x)(y) = 1$, for all$y \in Q$. Thus, in particular, $(\mu : x)(x) = 1$.

$$\begin{aligned}
(\mu : x)(x) = 1 \ &\Rightarrow\ \inf\{\mu(y) : y \in (x,x)^l\} = 1 \\
&\Rightarrow\ \mu(y) = 1\ \forall y \in (x,x)^l \\
&\Rightarrow\ \mu(x) = 1. \quad (\text{since} x \in (x,x)^l)
\end{aligned}$$

Conversely suppose that $\mu(x) = 1$. Now since, for any $y \in Q$, $(\mu : x)(y) = \inf\{\mu(z) : z \in (x,y)^l\} \geq \mu(x) = 1$, we have $(\mu : x)(y) = 1$ for all $y \in Q$. Therefore $\mu : x = \overline{1}$. □

The next result is also a characterization of an L-fuzzy ideal to be an L-fuzzy prime ideal in a poset Q.

Theorem 4.2.7. *Let μ be a proper L-fuzzy ideal of a poset Q. Then μ is an L-fuzzy prime ideal of Q if and only if $\mu : a = \mu$ for all $a \in Q$ such that $\mu(a) \neq 1$.*

Proof. Suppose that μ is an L-fuzzy prime ideal of Q and let $a \in Q$ such that $\mu(a) \neq 1$. Now we claim that $\mu : a = \mu$. Now for any $x \in Q$, we have $(\mu : a)(x) = \mu(x)$ or $\mu(a)$. Suppose that $(\mu : a)(x) = \mu(a)$. This implies that $\mu : a$ is a constant map $\overline{\mu(a)}$ Then $(\mu : a)(x) = \mu(a) \neq 1$. which is a contradiction to the fact that $(\mu : a)(0) = 1$.

Therefore $(\mu : a)(x) = \mu(x)$ for all $x \in Q$ and hence $\mu : a = \mu$.

Conversely suppose that the given condition holds. Let $a,,b \in Q$. Now we claim that

$$(\mu : a)(b) = \mu(a) \text{ or } \mu(b).$$

Suppose that $(\mu : a)(b) \neq \mu(a)$. Then $\inf\{\mu(x) : x \in (a,b)^l\} \nleq \mu(a)$. This implies that $\mu(a) \neq 1$. Thus, by hypothesis we have $\mu : a = \mu$ and hence

$$(\mu : a)(b) = \mu(b).$$

Therefore μ is an L-fuzzy prime ideal of Q. □

Lemma 4.2.8. *Let μ be an L-fuzzy ideal of an atomic poset Q and let p be a dually distributive atom of Q such that $\mu(p) \neq 1$. Then $\mu : p$ is a u-L-fuzzy ideal of Q.*

Proof. Since p is an atom in Q, for any $x \in Q$, we have $(p,x)^l = \{0,p\}$ if $p \leq x$ and $\{0\}$ otherwise. Thus, for any $x \in Q$,

$$(\mu : p)(x) = \begin{cases} 1 & if\ p \nleq x \\ \mu(p) & if\ p \leq x \end{cases}$$

First let us show that $\mu : p$ is an L-fuzzy ideal of Q. It is clear that $(\mu : p)(0) = 1$. Let $a,b \in Q$ and $x \in (a,b)^{ul}$. If $(\mu : p)(a) = \mu(p)$ or $(\mu : p)(b) = \mu(p)$, we have

$$(\mu : p)(a) \wedge (\mu : p)(b) = \mu(p) \leq (\mu : x)(p) = (\mu : p)(x).$$

Let $(\mu : p)(a) = 1 = (\mu : p)(b)$. Then $p \nleq a$ and $p \nleq b$. This implies that $(p,a)^l = \{0\} = (p,b)^l$. Now we claim that $p \nleq x$. Suppose that $p \leq x$. Then $p \in (a,b)^{ul}$ and since p is dually distributive, we have

$$\{0, p\} = p^l = \{p, (a,b)^u\}^l = \{(p,a)^l, (p,b)^l\}^{ul} = \{0\},$$

which is a contradiction. Thus $p \nleq x$ and hence $(\mu : p)(x) = 1$. This implies that

$$(\mu : p)(a) \wedge (\mu : p)(b) = 1 = (\mu : p)(x).$$

Thus, for any $a, b \in Q$ and $x \in (a,b)^{ul}$, we have,

$$(\mu : p)(a) \wedge (\mu : p)(b) \leq (\mu : p)(x)$$

and hence $\mu : p$ is an L-fuzzy ideal of Q.

Next we show that $\mu : p$ is a u-L-fuzzy ideal of Q. Let $a, b \in Q$. Now we claim that there exists $x \in (a,b)^u$ such that

$$(\mu : p)(x) = (\mu : p)(a) \wedge (\mu : p)(b).$$

Let $(\mu : p)(a) = \mu(p)$ or$(\mu : p)(b) = \mu(p)$. Then

$$(\mu : p)(a) \wedge (\mu : p)(b) = \mu(p) \text{ and } p \leq a \text{ or } p \leq b.$$

Now we claim that $p \leq x$ for some $x \in (a,b)^u$. Suppose on the contrary that $p \nleq x$ for all $x \in (a,b)^u$, i.e., $p \notin \{p, (a,b)^u\}^l$. As $(p,a)^l \cup (p,b)^l = \{0, p\}$ and p is dually distributive we have

$$p \notin \{p, (a,b)^u\}^l = \{(p,a)^l, (p,b)^l\}^{ul} = \{0, p\},$$

which is a contradiction. Hence the calim is true Therefore, in this case, there exists $x \in (a,b)^u$ such that $(\mu : p)(x) = \mu(p)$ and hence

$$(\mu : p)(x) = (\mu : p)(a) \wedge (\mu : p)(b).$$

Let $(\mu : p)(a) = 1 = (\mu : p)(b)$. Then

$$(\mu : p)(a) \wedge (\mu : p)(b) = 1 \text{ and } p \nleq a,\ p \nleq b.$$

Now we claim that $p \nleq x$ for some $x \in (a,b)^u$. Suppose on the contrary that $p \leq x$ for all $x \in (a,b)^u$. This implies that $p \in (a,b)^{ul}$ and $(\mu : p)(x) = \mu(p)$. Thus in this case we have

$$\mu(p) = (\mu : p)(p) \geq (\mu : p)(a) \wedge (\mu : p)(b) = 1$$

which contradicts to the fact that $\mu(p) \neq 1$. Hence there exists $x \in (a,b)^u$ such that $p \nleq x$. Thus, in this case, we have $(\mu : p)(x) = 1 = (\mu : p)(a) \wedge (\mu : p)(b)$. Hence in either cases there exists $x \in (a,b)^u$ such that

$$(\mu : p)(x) = (\mu : p)(a) \wedge (\mu : p)(b)$$

Therefore $\mu : p$ is a u-L-fuzzy ideal of Q. □

Now, we prove the existence of L-fuzzy prime ideals in atomic posets.

Theorem 4.2.9. *Let Q be an atomic poset p is an atom in Q. Then, if p is a dually distributive such that $\mu(p) \neq 1$ for an L-fuzzy ideal of μ of Q, then there exists a u-L-fuzzy prime ideal θ of Q such that $\mu \subseteq \theta$ and $\theta(p) \neq 1$. Conversely, for any L-fuzzy ideal μ of Q such that $\mu(p) \neq 1$, if there exists an L-fuzzy prime ideal θ of Q such that $\mu \subseteq \theta$ and $\theta(p) \neq 1$, then p is a dually distributive element.*

Proof. Let $\mathscr{S} = \{\sigma \in \mathscr{F}\mathscr{I}(Q) : \mu \subseteq \sigma \text{ and } \sigma(p) \neq 1\}$. Since $\mu \in \mathscr{S}$, $\mathscr{S}$ is non empty set and hence it forms a poset under the point wise ordering " $\subseteq$ ". By applying Zorn's Lemma we can choose a maximal element say θ in $\mathscr{S}$. Thus, since $\theta(p) \neq 1$, by Lemma 4.2.8, $\theta : p$ is a u-L-fuzzy ideal of Q. It is easy to observe that $\mu \subseteq \theta : p$ and $(\theta : p)(p) \neq 1$. Thus, by maximality of θ, we have $\theta : p = \theta$. Now we show that θ is an L-fuzzy prime ideal of Q. Let $a \in Q$ such that $\theta(a) \neq 1$. Now for any $x \in Q$, we have

$$\begin{aligned} (\theta : a)(x) &= \inf\{\theta(z) : z \in (a,x)^l\} \\ &= \inf\{(\theta : p)(z) : z \in (a,x)^l\} \text{ (...since } \theta = \theta : p) \end{aligned}$$

$$
\begin{aligned}
&= \inf\{\theta(z) : z \in (p,a,x)^l\} \\
&= \inf\{(\theta : x)(z) : z \in (p,a)^l\} \\
&= (\theta : x)(p) \text{ or } 1 \text{ (according as } p \leq a \text{ or } p \nleq a) \\
&= (\theta : p)(x) \text{ or } 1 \\
&= \theta(x) \text{ or } 1 \text{ (...since } \theta = \theta : p)
\end{aligned}
$$

This implies that either $\theta : a = \theta$ or $\theta : a = \overline{1}$. Suppose that $\theta : a = \overline{1}$. This implies that $\theta(a) = 1$, which is a contradiction to the fact that $\theta(a) \neq 1$. Hence $\theta : a = \theta$ for all $a \in Q$ such that $\theta(a) \neq 1$. Hence, by Theorem 4.2.7, θ is an L-fuzzy prime ideal of Q.

Conversely suppose that for any L-fuzzy ideal μ of Q with $\mu(p) \neq 1$, there exists an L-fuzzy prime ideal θ of Q such that $\mu \subseteq \theta$ and $\theta(p) \neq 1$. To show p is dually distributive, it is enough to show that

$$\{p,(a,b)^u\}^l \subseteq \{(p,a)^l,(p,b)^l\}^{ul} \text{ for any } a,b \in Q.$$

If $p \notin (a,b)^{ul}$ then $\{p,(a,b)^u\}^l = \{0\}$ and hence the inclusion follows immediately. Assume that $p \in (a,b)^{ul}$. Then $\{p,(a,b)^u\}^l = \{p,0\}$. Now we claim that $p \in \{(p,a)^l,(p,b)^l\}^{ul}$. On the contrary, suppose that $p \notin \{(p,a)^l,(p,b)^l\}^{ul}$. Then there exists $x \in \{(p,a)^l,(p,b)^l\}^u$ such that $p \nleq x$. This implies that $\chi_{(x]}(p) \neq 1$. By the hypothesis, there exists a fuzzy prime ideal θ such that $\chi_{(x]} \subseteq \theta$ and $\theta(p) \neq 1$. Since $(p,a)^l,(p,b)^l \subseteq \{(p,a)^l,(p,b)^l\}^{ul} \subseteq x^l = (x]$ and $p \notin (x]$, $(p,a)^l = \{0\} = (p,b)^l$ and hence

$$\inf\{\theta(y) : y \in (p,a)^l\} = 1 \text{ and } \inf\{\theta(y) : y \in (p,b)^l\} = 1.$$

Now, since θ is an L-fuzzy prime ideal with $\theta(p) \neq 1$, we have $\theta : p = \theta$ and hence

$$\theta(a) = (\theta : p)(a) = 1 \text{ and } \theta(b) = (\theta : p)(b) = 1.$$

Again as $p \in (a,b)^{ul}$, we have $\theta(p) \geq \theta(a) \wedge \theta(b) = 1$ which is a contradiction to the fact that $\theta(p) \neq 1$. Hence the claim is true. □

4.3 Prime L-Fuzzy Ideals

In this section, we introduce a prime L- fuzzy ideal of a poset Q which is a prime element in the lattice $\mathscr{F}\mathscr{I}(Q)$ of L-fuzzy ideals of Q.

Recall that an element $\alpha \neq 1$ in a lattice L is said to be prime if for any $t, s \in L$, $t \wedge s \leq \alpha$ implies either $s \leq \alpha$ or $t \leq \alpha$.

Definition 4.3.1. *Let μ and θ be L-fuzzy subsets of a poset Q. The product of μ and θ, denoted by $\mu *_l \theta$, is defined as:*

$$(\mu *_l \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^l\} \text{ for any } x \in Q.$$

For any L-fuzzy points x_α and y_β of a poset Q, it is clear that, for any $z \in Q$:

$$(x_\alpha *_l y_\beta)(z) = \begin{cases} \alpha \wedge \beta & if\ z \in (x,y)^l \\ 0 & otherwise \end{cases}$$

.

Lemma 4.3.1. *Let $x \in Q$ and $\alpha \in L$. Define an L- fuzzy subset $(\alpha,0)_{(x]}$ of Q by*

$$(\alpha,0)_{(x]}(y) = \begin{cases} 1 & if\ y = 0 \\ \alpha & if\ y \in (x] - \{0\} \\ 0 & if\ y \notin (x], \end{cases}$$

for all $y \in Q$. Then $(\alpha,0)_{(x]} = (x_\alpha]$.

Proof. We show that $(\alpha,0)_{(x]}$ is the smallest L-fuzzy ideal containing the fuzzy point x_α. Since for any $\beta \in L$, $(\alpha,0)_{(x]})_\beta = Q$ or $(x]$ or $\{0\}$, which is an ideal of Q, we have $(\alpha,0)_{(x]}$ is an L-fuzzy ideal of Q.

Again since $x \in (x]$, we have $\alpha \leq (\alpha,0)_{(x]})(x)$ and hence $x_\alpha \in (\alpha,0)_{(x]}$. Again let μ be any L-fuzzy ideal of Q such that $x_\alpha \in \mu$. Then $\alpha \leq \mu(x)$. Now we show that

$(\alpha,0)_{(x]}) \subseteq \mu$. Let $y \in Q$. Now if $y \notin (x]$, $(\alpha,0)_{(x]})(y) = 0 \leq \mu(y)$. Let $y \in (x]$. Then $y \leq x$. If $y = 0$, then $(\alpha,0)_{(x]})(y) = 1 = \mu(y)$. Again if $y \neq 0$, then $x \neq 0$ and hence $(\alpha,0)_{(x]})(y) = \alpha \leq \mu(x) \leq \mu(y)$. Thus in all cases, we have

$$(\alpha,0)_{(x]})(y) \leq \mu(y), \text{ for all } y \in Q.$$

So $(\alpha,0)_{(x]}) \subseteq \mu$. Therefore $(\alpha,0)_{(x]} = (x_\alpha]$. □

Now we have the following lemma.

Lemma 4.3.2. *Let x_α and y_β be any L-fuzzy points of a poset Q. Then*

$$(x_\alpha *_l y_\beta] = (\alpha \wedge \beta, 0)_{(x,y)^l} = (x_\alpha] \cap (y_\beta]$$

Now we give the definition a prime L-fuzzy ideal of a poset.

Definition 4.3.2. *A proper L-fuzzy ideal μ of a poset Q is called a prime L-fuzzy ideal, if for any L-fuzzy ideals σ and θ of Q,*

$$\sigma \cap \theta \subseteq \mu \text{ implies } \sigma \subseteq \mu \text{ or } \theta \subseteq \mu.$$

In the following theorem we characterize prime L-fuzzy ideals using L-fuzzy points of a poset Q.

Theorem 4.3.3. *A proper L-fuzzy ideal μ of a poset Q is prime L- fuzzy ideal if and only if for any L-fuzzy points x_α and y_β of Q:*

$$x_\alpha *_l y_\beta \subseteq \mu \Rightarrow \text{either } x_\alpha \in \mu \text{ or } y_\beta \in \mu.$$

Proof. Suppose that μ is a prime L- fuzzy ideal of Q. Let x_α and y_β be L-fuzzy points in Q such that $x_\alpha *_l y_\beta \subseteq \mu$. Then

$$\begin{aligned} x_\alpha *_l y_\beta \subseteq \mu &\Rightarrow (x_\alpha *_l y_\beta] \subseteq \mu \\ &\Rightarrow (x_\alpha] \cap (y_\beta] \subseteq \mu \\ &\Rightarrow (x_\alpha] \subseteq \mu \text{ or } (y_\beta] \subseteq \mu \\ &\Rightarrow \alpha \leq \mu(x) \text{ or } \beta \leq \mu(y) \\ &\Rightarrow x_\alpha \in \mu \text{ or } y_\beta \in \mu. \end{aligned}$$

Conversely, suppose that the given condition holds. Suppose that μ is not a prime L-fuzzy ideal of Q.. Then there exist L-fuzzy ideals σ and θ of Q such that $\sigma \cap \theta \subseteq \mu$ and $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$.

Then there exist $x, y \in Q$ such that $\sigma(x) \nleq \mu(x)$ and $\theta(y) \nleq \mu(y)$. If we put $\alpha = \sigma(x)$ and $\beta = \theta(y)$, then we clearly have $x_\alpha \in \sigma$ and $y_\beta \in \theta$ and $x_\alpha \notin \mu$ and $y_\beta \notin \mu$. Then by hypothesis, we have $x_\alpha *_l y_\beta \nsubseteq \mu$.

But, it is clear that $x_\alpha *_l y_\beta \subseteq \sigma \cap \theta \subseteq \mu$. But this contradicts the fact that $x_\alpha *_l y_\beta \nsubseteq \mu$. So μ is a prime L- fuzzy ideal of Q. □

In the following we characterize prime L-fuzzy ideal of a poset Q in terms of prime ideals of Q and prime elements of L.

Lemma 4.3.4. *Let I be an ideal of a poset Q and $1 \neq \alpha \in L$. Then the L-fuzzy subset α_I of a poset Q defined by:*

$$\alpha_I(x) = \begin{cases} 1 & if\ x \in I \\ \alpha & if\ x \notin I \end{cases},$$

for all $x \in Q$ is an L-fuzzy ideal of Q.

The L-fuzzy ideal α_I defined in Lemma 4.3.4 above is called the α-level L-fuzzy ideal of Q corresponding to the ideal I.

Corollary 4.3.5. *Let I and J are ideals in Q and $1 \neq \alpha \in L$ and $1 \neq \beta \in L$. Then $\alpha_I \subseteq \beta_J$ if and only if $I \subseteq J$ and $\alpha \leq \beta$.*

Theorem 4.3.6. *Let P be an ideal of a poset Q and $1 \neq \alpha \in L$. Then α_P is a prime L-fuzzy ideal of Q if and only if P is a prime ideal of Q and α is a prime element in L.*

Proof. Suppose that α_P is a prime L-fuzzy ideal of Q. Now we show that P is a prime ideal of Q and α is a prime element in L. Since α_P is proper, we have $P \neq Q$ and $\alpha \neq 1$. Let $a, b \in Q$ such that $(a, b)^l \subseteq P$. Now

$$
\begin{aligned}
(a,b)^l \subseteq P &\Rightarrow (a] \cap (b] \subseteq P \\
&\Rightarrow \alpha_{(a]\cap(b]} \subseteq \alpha_P \\
&\Rightarrow \alpha_{(a]} \cap \alpha_{(b]} \subseteq \alpha_P \\
&\Rightarrow \alpha_{(a]} \subseteq \alpha_P \text{ or } \alpha_{(b]} \subseteq \alpha_P \\
&\Rightarrow (a] \subseteq P \text{ or } (b] \subseteq P \\
&\Rightarrow a \in P \text{ or } b \in P.
\end{aligned}
$$

Thus P is a prime ideal of Q. Again let $\beta, \gamma \in L$ such that $\beta \wedge \gamma \leq \alpha$.

$$
\begin{aligned}
\text{Now} \quad \beta \wedge \gamma \leq \alpha &\Rightarrow (\beta \wedge \gamma)_P \subseteq \alpha_P \\
&\Rightarrow \beta_P \cap \gamma_P \subseteq \alpha_P \\
&\Rightarrow \beta_P \subseteq \alpha_P \text{ or } \gamma_P \subseteq \alpha_P \\
&\Rightarrow \beta \leq \alpha \text{ or } \gamma \leq \alpha.
\end{aligned}
$$

Thus α is a prime element in L.

Conversely, suppose that P is a prime ideal of Q and α is a prime element in L. Clearly, α_P is a proper L-fuzzy ideal of Q. Suppose that α_P is not a prime L-fuzzy prime ideal of Q. Then there exist L-fuzzy ideals μ and σ of Q such that

$$\mu \cap \sigma \subseteq \alpha_P \text{ and } \mu \nsubseteq \alpha_P \text{ and } \sigma \nsubseteq \alpha_P.$$

Then there exist $a, b \in Q$ such that

$$\mu(a) \nleq \alpha_P(a) \text{ and } \sigma(b) \nleq \alpha_P(b).$$

This implies that $\mu(a) \nleq \alpha$ and $\sigma(b) \nleq \alpha$ and $a \notin P$ and $b \notin P$. Since α is prime element in L and P is a prime ideal of Q, we have $\mu(a) \wedge \sigma(b) \nleq \alpha$ and $(a,b)^l \nsubseteq P$. Thus there exists $y \in (a,b)^l$ such that $y \notin P$. Then it is clear that

$$(\mu \cap \sigma)(y) \nleq \alpha = \alpha_P(y).$$

So $\mu \cap \sigma \nsubseteq \alpha_P$ which is a contradiction. Thus α_P is a prime L-fuzzy ideal of Q. □

Theorem 4.3.7. *Let μ be an L-fuzzy ideal of Q. Then μ is a prime L-fuzzy ideal of Q if and only if there exist prime ideal of P of Q and prime element α in L such that $\mu = \alpha_P$.*

Proof. Suppose that μ is a prime L-fuzzy ideal of Q. Since μ is proper it assumes at least two values. Since $\mu(0) = 1$, 1 is necessarily in $Im(\mu)$. Suppose that $\alpha, \beta \in Im(\mu)$ other than 1. Then there exist $a, b \in Q$ such that $\mu(a) = \alpha$ and $\mu(b) = \beta$. Now we claim that $\alpha = \beta$. Now put $P = \mu_1 = \{x \in Q : \mu(x) = 1\}$. Consider the L-fuzzy ideals $\chi_{(a]}$ and α_P Now we show $\chi_{(a]} \cap \alpha_P \subseteq \mu$. Let $x \in Q$. if $x \notin (a]$, then we have

$$(\chi_{(a]} \cap \alpha_P)(x) = \chi_{(a]}(x) \wedge \alpha_P(x) = 0 \wedge \alpha_P(x) = 0 \leq \mu(x).$$

Let $x \in (a]$. Now in this case, if $x \in P$, we have

$$(\chi_{(a]} \cap \alpha_P)(x) = \chi_{(a]}(x) \wedge \alpha_P(x) = 1 \wedge 1 = 1 = \mu(x).$$

Again if $x \notin P$, then we have

$$(\chi_{(a]} \cap \alpha_P)(x) = \chi_{(a]}(x) \wedge \alpha_P(x) = 1 \wedge \alpha = \alpha = \mu(a) \leq \mu(x).$$

Therefore in either cases, we have $(\chi_{(a]} \cap \alpha_P)(x) \leq \mu(x)$, for all $x \in Q$ and so

$$\chi_{(a]} \cap \alpha_P \subseteq \mu.$$

But as μ is a prime L-fuzzy ideal of Q, we have

$$\chi_{(a]} \subseteq \mu \text{ or } \alpha_P \subseteq \mu$$

But as $\chi_{(a]}(a) = 1 \nleq \alpha = \mu(a)$, we have $\chi_{(a]} \nsubseteq \mu$. Therefore $\alpha_P \subseteq \mu$. In particular, since $b \notin P$, we get that $\alpha = \alpha_P(b) \leq \mu(b) = \beta$.

In similar fashion, we can show that $\beta \leq \alpha$ and hence $\alpha = \beta$. So μ assumes exactly one value say α other than 1 and hence $\mu = \alpha_P$.

Now we remain show that α is a prime element in L and P a prime ideal of Q.

Let $\beta, \gamma \in L$ such that $\beta \wedge \gamma \leq \alpha$. This implies that

$$\beta_P \cap \gamma_P = (\beta \wedge \gamma)_P \subseteq \alpha_P = \mu.$$

Since μ is prime, we have either $\beta_P \subseteq \mu = \alpha_P$ or $\gamma_P \subseteq \mu = \alpha_P$. This implies that $\beta \leq \alpha$ or $\gamma \leq \alpha$. Thus α a is a prime element in in L.

Again to show P is a prime ideal, let $a, b \in Q$ such that $(a,b)^l \subseteq P$.

$$\begin{aligned} (a,b)^l \subseteq P \quad &\Rightarrow \quad (a] \cap (b] \subseteq P \\ &\Rightarrow \quad \alpha_{(a] \cap (b]} \subseteq \alpha_P = \mu \\ &\Rightarrow \quad \alpha_{(a]} \cap \alpha_{(b]} \subseteq \alpha_P = \mu \end{aligned}$$

Since μ is a prime L-fuzzy ideal, we have either

$$\alpha_{(a]} \subseteq \mu = \alpha_P \text{ or } \alpha_{(b]} \subseteq \mu = \alpha_P$$

This implies that $(a] \subseteq P$ or $(b] \subseteq P$. and hence either $a \in P$ or $b \in P$. Therefore P is a prime ideal of Q.

The converse part of this theorem follows from Theorem 4.3.6. □

Corollary 4.3.8. *Let $L = [0,1]$. Then a proper ideal P of Q is prime if and only if its characteristic map χ_P is a prime L-fuzzy ideal of Q.*

Note that we write α_P for the prime L-fuzzy ideal of Q corresponding to the pair (P,α) and $\mathscr{PFI}(Q)$ for the set of all prime L-fuzzy ideal of Q. Now the following result follows from the above theorem.

Corollary 4.3.9. *There is a one-to-one correspondence between the class $\mathscr{PFI}(Q)$ of all prime L-fuzzy ideals of Q and the collection of all pairs (P,α), where P is a prime ideal of Q and α is a prime element in L.*

Example 4.3.10. *Consider the poset $(Q,\leq)$ depicted in the Fig. 4.3 below. Define a fuzzy subset $\mu : Q \longrightarrow [0,1]$ by:*

$$\mu(0) = \mu(e) = \mu(a) = 1 \text{ and } \mu(b) = \mu(c) = \mu(d) = 0.5.$$

Then μ is a prime L-fuzzy ideal of Q.

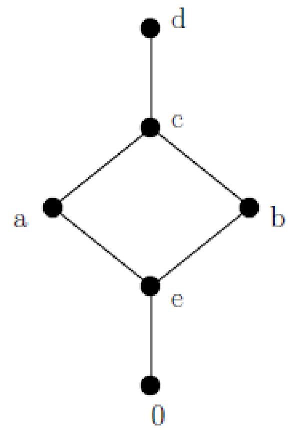

Fig. 4.3

Definition 4.3.3. *An L-fuzzy subset η of Q is said to be an L fuzzy down directed, if for any $a,b \in Q$, there exists $x \in (a,b)^l$ such that $\eta(x) \geq \eta(a) \wedge \eta(b)$.*

Now we prove the following theorem which is analogous to Stone's Prime ideal Theorem in distributive lattices[45].

Theorem 4.3.11. *Let the lattice $(\mathscr{F}\mathscr{I}(Q), \subseteq)$ of all L-fuzzy ideals of Q is distributive, $\mu \in \mathscr{F}\mathscr{I}(Q)$ and α is a prime element in L. If η is an L-fuzzy down directed subset of Q such that such that $\mu \cap \eta \subseteq \overline{\alpha}$, then there exists a prime L-fuzzy ideal θ of Q such that $\mu \subseteq \theta$ and $\theta \cap \eta \subseteq \overline{\alpha}$.*

Proof. Let $\mathscr{S} = \{\sigma \in \mathscr{F}\mathscr{I}(Q) : \mu \subseteq \sigma \text{ and } \sigma \cap \eta \subseteq \overline{\alpha}\}$. Since $\mu \in \mathscr{S}$, $\mathscr{S}$ is non empty and hence it forms a poset under the point wise ordering " $\subseteq$ ". By applying Zorn's Lemma, we can choose a maximal element say θ in $\mathscr{S}$. Now we show that θ is a prime L-fuzzy ideal of Q. Suppose not. Then there exist L-fuzzy ideals ν_1 and ν_2 of Q such that $\nu_1 \cap \nu_2 \subseteq \theta$ but $\nu_1 \nsubseteq \theta$ and $\nu_2 \nsubseteq \theta$. Put

$$\theta_1 = \theta \vee \nu_1 \text{ and } \theta_2 = \theta \vee \nu_2.$$

Then clearly, θ_1 and θ_2 are L-fuzzy ideals containing θ properly. Thus by maximality of θ, both θ_1 and θ_2 do not belong to $\mathscr{S}$. So there exist $a,b \in Q$ such that

$$(\theta_1 \cap \eta)(a) \nleq \alpha \text{ and } (\theta_2 \cap \eta)(b) \nleq \alpha.$$

Let $z \in (a,b)^l$. Then it is clear that

$$((\theta_1 \cap \eta)(z) \nleq \alpha \ \text{ and } \ (\theta_2 \cap \eta)(z) \nleq \alpha.$$

Since α a prime element in L, we have $(\theta_1 \cap \theta_2) \cap \eta)(z) \nleq \alpha$. Now

$$\begin{aligned} ((\theta_1 \cap \theta_2) \cap \eta)(z) \nleq \alpha \ &\Rightarrow \ ((\theta \vee \nu_1) \cap (\theta \vee \nu_2) \cap \eta)(z) \nleq \alpha \\ &\Rightarrow \ ((\theta \vee (\nu_1 \cap \nu_2)) \cap \eta)(z) \nleq \alpha \\ &\Rightarrow \ (\theta \cap \eta)(z) \nleq \alpha \quad (\text{ since } \nu_1 \cap \nu_2 \subseteq \theta) \end{aligned}$$

which is a contradiction to the fact that $\theta \cap \eta \subseteq \overline{\alpha}$. So $\nu_1 \cap \nu_2 \subseteq \theta$ implies $\nu_1 \subseteq \theta$ or $\nu_2 \subseteq \theta$. Hence, θ is a prime L-fuzzy ideal of Q. □

Corollary 4.3.12. *Let μ be in the distributive lattice $(\mathscr{F}\mathscr{I}(Q), \subseteq)$ of all L-fuzzy ideals of Q and $a \in Q$. If $\mu(a) \leq \alpha$, where α is a prime element in L, then there exists a prime L-fuzzy ideal θ of Q such that $\mu \subseteq \theta$ and $\theta(a) \leq \alpha$.*

Proof. Apply Theorem 4.3.11 to μ and $\eta = \chi_{[a)}$ □

Theorem 4.3.13. *Every prime L-fuzzy ideal of a poset Q is an L-fuzzy prime ideal of Q.*

Proof. Let μ be a prime L-fuzzy ideal of a poset Q. Then there exists a prime ideal P of Q and a prime element α of L such that $\mu = \alpha_P$. Thus clearly μ is proper and $Im(\mu) = \{\alpha, 1\}$ is a chain.

Let $a, b \in Q$. Now if $(a,b)^l \subseteq P$, then $\mu(x) = 1$, for all $x \in (a,b)^l$ and hence $\inf\{\mu(x) : x \in (a,b)^l\} = 1$. Again since P is a prime ideal, $(a,b)^l \subseteq P$ implies either $a \in P$ or $b \in P$ and hence, either $\mu(a) = 1$ or $\mu(b) = 1$. Therefore

$$\mu(a) \vee \mu(b) = 1 = \inf\{\mu(x) : x \in (a,b)^l\}.$$

Again if $(a,b)^l \nsubseteq P$, then there exists $y \in (a,b)^l$ such that $y \notin P$ and thus

$$\mu(y) = \alpha = \inf\{\mu(x) : x \in (a,b)^l\}.$$

Again $(a,b)^l \nsubseteq P$ implies $a \notin P$ and $b \notin P$. Thus $\mu(a) = \mu(b) = \alpha$. So

$$\mu(a) \vee \mu(b) = \alpha = \inf\{\mu(x) : x \in (a,b)^l\}.$$

. Therefore in either cases, we have $\mu(a) \vee \mu(b) = \inf\{\mu(x) : x \in (a,b)^l\}$.

Thus, by Corollary 4.2.2 , μ is an L-fuzzy prime ideal of Q. □

Remark 4.3.1. *The converse of the above theorem is not true. For example consider the poset* $(Q \leq)$ *depicted in the Fig.4.4 below and define a fuzzy subset* $\mu : Q \longrightarrow [0,1]$ *by:*

$$\mu(0) = 1,\ \mu(a) = \mu(b) = 0.8,\ \mu(c) = \mu(d) = \mu(e) = \mu(1) = 0.$$

Then μ *is an L-fuzzy prime ideal of Q but not a prime L-fuzzy ideal of Q as* μ *assumes two values other than* 1.

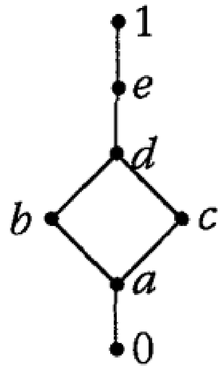

Fig. 4.4

4.4 Maximal L-Fuzzy Ideal

In this section, we introduce the notion of maximal L-fuzzy ideal of a poset which is a dual atom in the lattice of L-fuzzy ideals of a poset Q, that is, a maximal element in the set of all proper L-fuzzy ideals of a poset Q.

Recall that an element α in a bounded lattice L is said to be a dual atom if there is no $\beta \in L$ such that $\alpha < \beta < 1$.

Definition 4.4.1. *A proper L-fuzzy ideal μ of a poset Q is said to be a maximal L-fuzzy ideal of Q, if μ is a dual atom in the lattice $\mathscr{F}\mathscr{I}(Q)$ of all L-fuzzy ideals of Q under point wise ordering "$\subseteq$". That is, if there is no proper L-fuzzy ideal θ of Q such that $\mu \subsetneqq \theta$.*

Lemma 4.4.1. *Let μ be an L-fuzzy ideal of a poset Q and $\alpha \in L$. Then $\mu \cup \overline{\alpha}$ is an L-fuzzy ideal of Q containing μ.*

Proof. Clearly $\mu \subseteq \mu \cup \overline{\alpha}$ and $(\mu \cup \overline{\alpha})(0) = 1$. Again let $a, b \in Q$ and $x \in (a,b)^{ul}$. Then

$$\begin{aligned}(\mu \cup \overline{\alpha})(x) &= \mu(x) \vee \alpha \\ &\geq (\mu(a) \wedge \mu(b)) \vee \alpha \\ &= (\mu(a) \vee \alpha) \wedge (\mu(b)) \vee \alpha) \\ &= (\mu \cup \overline{\alpha})(a) \wedge (\mu \cup \overline{\alpha})(b).\end{aligned}$$

Thus $\mu \cup \overline{\alpha}$ is an L-fuzzy ideal of Q containing μ. □

Lemma 4.4.2. *Let μ be a maximal L-fuzzy ideal of a poset Q. Then $Im(\mu)$ is a chain.*

Proof. Let $\alpha, \beta \in Im(\mu)$. Then there exist $a, b \in Q$ such that $\mu(a) = \alpha$ and $\mu(b) = \beta$. Then, by Lemma 4.4.1, $\mu \cup \overline{\alpha}$ is an L-fuzzy ideal of Q. Since $\mu \subseteq \mu \cup \overline{\alpha}$ and μ is a maximal L-fuzzy ideal of Q, we have either, $\mu = \mu \cup \overline{\alpha}$ or $\mu \cup \overline{\alpha} = \overline{1}$.

If $\mu = \mu \cup \overline{\alpha}$, then we have

$$\beta = \mu(b) = (\mu \cup \overline{\alpha})(b) = \mu(b) \vee \alpha = \beta \vee \alpha.$$

Thus $\alpha \leq \beta$. If $\mu \vee \overline{\alpha} = \overline{1}$, then we have

$$(\mu \cup \overline{\alpha})(a) = 1 = (\mu \cup \overline{\alpha})(b).$$

This implies that $\mu(a) \vee \alpha = \mu(b) \vee \alpha$, that is, $\alpha = \beta \vee \alpha$ and hence $\beta \leq \alpha$.

So $Im(\mu)$ is a chain. □

Lemma 4.4.3. *Let μ be a maximal L-fuzzy ideal of Q. Then μ attains exactly one value other than* 1.

Proof. Since μ is an L-fuzzy ideal of Q, we have $\mu(0) = 1$. Thus $1 \in Im\mu$. Let $\alpha, \beta \in Im\mu$ other than 1. Then there exist $a, b \in Q$ such that $\alpha = \mu(a)$ and $\beta = \mu(b)$. Then $\mu \cup \overline{\alpha}$ and $\mu \cup \overline{\beta}$ are L-fuzzy ideals of Q containing μ. Again since

$$(\mu \cup \overline{\alpha})(a) = \alpha \neq 1 = \overline{1}(a) \text{ and } (\mu \cup \overline{\beta})(b) = \beta \neq 1 = \overline{1}(b)$$

we have $\mu \cup \overline{\alpha} \neq \overline{1}$ and $\mu \cup \overline{\beta} \neq \overline{1}$. Thus, by maximality of μ, we have $\mu = \mu \cup \overline{\alpha} = \mu \cup \overline{\beta}$. Thus, in particular, we have

$$\beta = \mu(b) = (\mu \cup \overline{\alpha})(b) = \mu(b) \vee \alpha = \beta \vee \alpha \,.$$

and

$$\alpha = \mu(a) = (\mu \cup \overline{\beta})(a) = \mu(a) \vee \beta = \alpha \vee \beta$$

Therefore $\alpha = \alpha \vee \beta = \beta$. So μ assumes exactly one value other than 1. □

The following theorem gives a characterization of a maximal L-fuzzy ideal of a poset.

Theorem 4.4.4. *An L-fuzzy subset μ of Q is a maximal L-fuzzy ideal of Q if and only if there exist a maximal ideal M of Q and a dual atom α in L such that $\mu = \alpha_M$.*

Proof. Suppose that μ is a maximal L-fuzzy ideal of Q. Put $M = \{x \in Q : \mu(x) = 1\}$. Then, by the Lemma 4.4.3, μ assumes exactly one value, say α other than 1. Therefore $\mu = \alpha_M$.

Now we show that M is a maximal ideal of Q and α is a dual element in L. Since μ is proper, it is clear that M is proper. Let I be a proper ideal of Q such that $M \subseteq I$. Then

$$\mu = \alpha_M \subseteq \alpha_I \subset \overline{1}.$$

By maximality of μ, we have that $\alpha_M = \alpha_I$. Thus $M = I$. So M is a maximal ideal of Q. Again let $\beta \in L$ such that $\alpha \leq \beta < 1$. Then

$$\mu = \alpha_M \subseteq \beta_M \subset \overline{1}.$$

Thus, by the maximality of μ, we have $\alpha_M = \beta_M$. So $\alpha = \beta$.

Hence α is a dual atom in L.

Conversely suppose that $\mu = \alpha_M$, where M is a maximal ideal in Q and α is a dual atom in L. Since M is proper, there exists $a \in Q$ such that $a \notin M$ and hence $\mu(a) = \alpha_M(a) = \alpha \neq 1$. Therefore μ is proper. Let θ be any proper L-fuzzy ideal of Q such that $\mu \subseteq \theta \subset \overline{1}$. This implies that

$$M = \mu_1 \subseteq \theta_1 \subset Q.$$

Thus, by the maximality of M, we have $M = \theta_1 = \{x \in Q : \theta(x) = 1\}$.

Let $x \in Q$. If $x \in M$, then $\mu(x) = 1 = \theta(x)$. If $x \notin M$, then we have

$$\mu(x) = \alpha \leq \theta(x) < 1.$$

Since α is a dual atom in L, we have $\mu(x) = \alpha = \theta(x)$. Thus $\mu = \alpha_M = \theta$.

So μ is a maximal L-fuzzy ideal of Q. □

Now we have the following corollary; which is an immediate consequence of Theorem 4.4.4.

Corollary 4.4.5. *There is a one-to-one correspondence between the class of all maximal L-fuzzy ideals of Q and the collection of all pairs* (M, α)*, where M is a maximal ideal of Q and* α *is a dual atom in L.*

Remark 4.4.1. *Since the interval* $[0,1]$ *of real numbers has no dual atom, so is* $[0,1]^n$ *for any positive integer n, there is no maximal L-fuzzy ideal of Q if* $L = [0,1]^n$.

Example 4.4.6. *Consider the poset* $(Q, \leq)$ *depicted in the Fig. 4.4 above and the distributive lattice L in the Fig. 4.5 below. Define an L- fuzzy subset* $\mu : Q \longrightarrow L$ *by:* $\mu(0) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = \mu(e) = 1$ *and* $\mu(1) = a$. *Then* μ *is a maximal L-fuzzy ideal as* $\mu = \alpha_M$*, where* $\alpha = a$ *is a dual atom in L and* $M = \{0, a, b, c, d, e\}$ *is a maximal ideal of Q.*

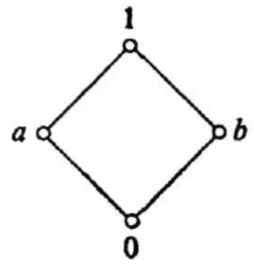

Fig. 4.5

Since L is a distributive lattice, every dual atom in L is prime and hence we have the following.

Corollary 4.4.7. *If Q is a poset in which every maximal ideal is a prime ideal then every maximal L-fuzzy ideal of Q is a prime L-fuzzy ideal of Q.*

Remark 4.4.2. *The converse of the above corollary is not true. Example 4.3.10 is a prime L-fuzzy ideal which is not a maximal L-fuzzy ideal of the given poset as there is no dual atom in $L = [0,1]$.*

4.5 L-Fuzzy Maximal Ideals

In this section, we define the notion of L-fuzzy maximal ideals of a poset Q as a proper L-fuzzy ideal, for which each level subset μ_α at $\alpha \in L$ is either the whole poset Q or a maximal ideal Q.

First, we prove the following result which gives a motivation in defining the concept of L-fuzzy maximal ideals of a poset Q.

Lemma 4.5.1. *Let M be a maximal ideal of a poset Q and χ_M be its characteristic map. Let μ be a proper L-fuzzy ideal of Q such that $\chi_M \subseteq \mu$. Then μ assumes exactly two values.*

Proof. Since $\mu(0) = 1, 1 \in Im\mu$. Since μ is proper, there exists $a \in Q$ such that $\mu(a) \neq 1$. Thus $\mu_1 = \{x \in Q : \mu(x) = 1\}$ is a proper ideal of Q. Again $\chi_M \subseteq \mu$ implies that $M \subseteq \mu_1 \subset Q$. The, by maximality of M, we have $M = \mu_1$.

Let $\alpha, \beta \in Im\mu$ other than 1. Then there exists $a, b \in Q$ such that $\alpha = \mu(a)$ and $\beta = \mu(b)$. Now we claim that $\alpha = \beta$. Now it is clear to see that $M = \mu_1 \subset \mu_\alpha \subseteq Q$ and $M = \mu_1 \subset \mu_\beta \subseteq Q$. Thus, by maximality of M, we have $\mu_\alpha = Q = \mu_\beta$. Thus $a \in \mu_\beta$ and $b \in \mu_\alpha$. This implies that

$$\alpha = \mu(a) \geq \beta \text{ and } \beta = \mu(b) \geq \alpha \text{ and so } \alpha = \beta.$$

Therefore μ assumes exactly one value other than 1. □

Definition 4.5.1. *A proper L-fuzzy ideal μ of Q is called an L-fuzzy maximal ideal of Q if, for each $\alpha \in L$, either $\mu_\alpha = Q$ or a maximal ideal of Q.*

Lemma 4.5.2. *Let μ be an L-fuzzy maximal ideal of Q. Then $\mu_1 = \{x \in Q : \mu(x) = 1\}$ is a maximal ideal of Q.*

Proof. Since μ is proper, $\mu(a) \neq 1$ for some $a \in Q$ and hence $\mu_1 = \{x \in Q : \mu(x) = 1\}$ is a proper ideal of Q. Thus, by definition, μ_1 is a maximal ideal of Q. □

Theorem 4.5.3. *Every L-fuzzy maximal ideal of Q assumes exactly two values.*

Proof. Let μ be an L-fuzzy maximal ideal of Q. Then, by Lemma 4.5.2, $\mu_1 = \{x \in Q : \mu(x) = 1\}$ is a maximal ideal of Q. Put $M = \mu_1$. Then $\chi_M \subseteq \mu$ and hence by Lemma 4.5.1, $\mu = \alpha_M$ for some $1 \neq \alpha \in L$. Thus μ assumes exactly two values. □

Theorem 4.5.4. *A proper L-fuzzy ideal μ of a poset Q is an L-fuzzy maximal ideal of Q if and only if $\mu = \alpha_M$ for some maximal ideal M of Q and $1 \neq \alpha \in L$.*

Proof. Suppose that μ is an L-fuzzy maximal ideal of Q. Put $M = \mu_1 = \{x \in Q : \mu(x) = 1\}$. Then, by Lemma 4.5.2, M is a maximal ideal of Q. Also, by Theorem 4.5.3, μ assumes exactly two values. Clearly 1 is the value of μ. Let α be the only values of μ other than 1. Then for any $x \in Q$,

$$\mu(x) = \begin{cases} 1 & if\ x \in M \\ \alpha & if\ x \notin M \end{cases},$$

and hence $\mu = \alpha_M$.

Conversely suppose that $\mu = \alpha_M$ for some maximal ideal M of Q and $1 \neq \alpha \in L$. Now, for any $\beta \in L$, clearly we have $\mu_\beta = Q$ or M. Thus, by definition, μ is an L-fuzzy maximal ideal of Q. □

Corollary 4.5.5. *A proper ideal M of Q is a maximal ideal of Q if and only if the characteristic map χ_M of M is an L-fuzzy maximal ideal of Q.*

Since any dual atom $\alpha \neq 1$, we have the following corollary.

Corollary 4.5.6. *Every maximal L-fuzzy ideal of Q is an L-fuzzy maximal ideal of Q.*

Remark 4.5.1. *The converse of Corollary 4.5.6 is not true. For example consider $L = [0,1]$ and M be any maximal ideal of any poset Q with 0. Then for any $1 \neq \alpha \in L$, α_M is an L-fuzzy maximal ideal of Q, but it is not a maximal L-fuzzy ideal of Q.*

Corollary 4.5.7. *If Q is an ideal distributive poset then every L-fuzzy maximal ideal of Q is an L-fuzzy prime ideal of Q.*

Proof. Let μ be an L-fuzzy maximal ideal of Q. Then $\mu = \alpha_M$ for some maximal ideal M of Q and $1 \neq \alpha \in L$. Since Q is an ideal distributive poset, by Theorem 4.1.9, M is a prime ideal and hence for any $\beta \in L$, μ_β is either Q or M. Then, by Theorem 4.2.1, μ is an L-fuzzy prime ideal of Q. □

Note that the converse of Corollary 4.5.7, is note true. For example consider the ideal distributive posets $(Q \leq)$ depicted in the Fig. 4.4. Observe that $P = (b]$ is a prime ideal of Q_1 but not a maximal ideal of Q_1. Define a fuzzy subset μ of Q by:

$$\mu(x) = \begin{cases} 1 & if\ x \in P \\ 0.5 & if\ x \notin P \end{cases},$$

for all $x \in Q$. Then μ is a fuzzy prime ideal of Q which is not a fuzzy maximal.

Chapter 5

L-Fuzzy Semi-prime Ideals

In 1989, Y. Rav [42] introduced and studied semiprime ideals in lattices. Also, he studied the lattice of semi-prime ideals and he proved the analogue of the prime separation theorem for semi-prime ideals. Also he proved the connections between primness and semi-primeness in lattices. In 2009, V. S. Khart and K. A. Mokbel, [34] introduced the concept of a semi-prime ideal in general poset and they obtained characterizations of semi-prime ideals in posets as well as condition of semi-prime to be prime. They also prove an analogue of Stone's Theorem for finite posets using semi-prime ideals in [35]. In 2013, K. A. Mokbel and V. S. Khart [38] obtained several characterization of 0- distributive posets by using the prime ideals as well as semi-prime ideals of a poset.

In this chapter we introduce the concept of L-fuzzy semi-prime ideal in a general poset. Characterizations of L-fuzzy semi-prime ideals in posets as well as characterizations of an L-fuzzy semi-prime ideal to be L-fuzzy prime ideal are obtained. Also, the relations between the L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of a poset and the L-fuzzy semi-prime (respectively, L-fuzzy prime) of the lattice of all ideals of a poset are established. We extend and prove Rav's Prime Separation Theorem for a lattice, using L-fuzzy semi-prime ideals. Lastly, we also extend and prove an analogue of Stone's Theorem for finite posets using L-fuzzy semi-prime ideals.

5.1 Semi-prime Ideals

In this section, we recall some definitions and crisp concepts of semi-prime ideals of a poset and a lattice that will be extended to the notions of L-fuzzy semi-prime ideals of a poset in the further sections of this chapter.

Throughout this chapter, an ideal will mean a 3-ideal (i.e., an ideal in the sense of Halaš) unless otherwise stated.

Now, we consider the concept of a semi-prime ideal introduced by V. S. Khart and K. A. Mokbel, [34] in a poset and by Y. Rav [42] in a lattice, as given in the following.

Definition 5.1.1. *[34] An ideal I of a poset Q is called a semi-prime ideal of Q if for all $x, y, z \in Q$,*

$$(x,y)^l \subseteq I \text{ and } (x,z)^l \subseteq I \text{ imply } \{x,(y,z)^u\}^l \subseteq I.$$

Dually we have the concept semi-prime filter of a poset Q.

Definition 5.1.2. *[42] An ideal I of a lattice X is called a semi-prime ideal of X if for all $x, y, z \in X$,*

$$x \wedge y \in I \text{ and } x \wedge z \in I \text{ together imply } x \wedge (y \vee z) \in I.$$

Dually we have the concept semi-prime filter of a lattice X.

For an ideal I and an element a in a poset Q, define a set $I : a$ by:

$$I : a = \{x \in Q : (a,x)^l \subseteq I\}.$$

The following are some properties of ideals in a poset using the set defined above.

Lemma 5.1.1. *[34] Let I be an ideal of a poset Q. Then the following statements hold:*

1. *$\{x,(a,b)^u\}^l \subseteq I$ if and only if $(a,b)^{ul} \subseteq I : x$.*

2. *$I : x = Q$ if and only if $x \in I$.*

Definition 5.1.3. *[35] Let I be an ideal of a poset Q. An element $i \in Q$ is called an I-atom if the following conditions hold:*

1. $i \notin I$
2. *for* $x \in Q$ *with* $x < i$ *implies* $x \in I$.

Dually, we have the concept of F-dual atom for a given filter F of Q.

We state the following concepts that are essentially introduced for lattices by Rav [42].

Let Q be a given poset and $(\mathscr{I}(Q), \subseteq)$ be the lattice of all ideals of a poset. Define an extension of an ideal I of Q, denoted by I^e, as

$$I^e = \{J \in \mathscr{I}(Q) : J \subseteq I\}$$

and for an ideal λ of the lattice $(\mathscr{I}(Q), \subseteq)$ of all ideals of a poset Q, define the contraction of λ, denoted by λ^c, as

$$\lambda^c = \bigcup\{J : J \in \lambda\}.$$

M. H. Stone [45], in his famous paper, proved the Separation Theorem for prime ideals in the case of distributive lattices as follows.

Theorem 5.1.2. *[45] Let X be a distributive lattice. Let I be an ideal of X and D be a dual ideal of X such that $I \cap D = \emptyset$. Then there exists a prime ideal P of X such that $I \subseteq P$ and $P \cap D = \emptyset$.*

The following is Y. Rav's Separation Theorem for semi-prime ideals in Lattice as stated below.

Theorem 5.1.3. *[42] Let X be a lattice containing an ideal I and a filter F such that $I \cap F = \emptyset$. If I is semi-prime, then there exists a semi-prime filter G such that $F \subseteq G$ and $I \cap G = \emptyset$. A dual result holds if F is semiprime.*

V. S. Kharat and K. A. Mokbel[35] proved an analogue of the Stone Theorem for a finite poset stated as follow:

Theorem 5.1.4. *Let Q be a finite poset, I be a semi-prime ideal of Q such that $I \cap K = \emptyset$ for an l-filter K in Q. Then there exists a semi-prime filter F such that $K \subseteq F$ and $I \cap F = \emptyset$.*

5.2 *L*-Fuzzy Semi-prime Ideals

In this section we introduce the concept of an L-fuzzy semi-prime ideal in a general poset. Characterizations of L-fuzzy semi-prime ideals in posets as well as characterizations of an L-fuzzy semi-prime ideal to be L-fuzzy prime ideal are obtained.

Definition 5.2.1. *An L-fuzzy ideal μ of a poset Q is called an L-fuzzy semi-prime ideal if for all $a,b,c \in Q$,*

$$\mu(z) \geq \inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \ \forall z \in \{a,(b,c)^u\}^l.$$

Dually we have the concept of L-fuzzy semi-prime filter of Q.

Definition 5.2.2. *An L-fuzzy ideal μ of a lattice Q is called an L-fuzzy semi-prime ideal of Q, if for all $a,b,c \in Q$,*

$$\mu(a \wedge (b \vee c)) = \mu(a \wedge b) \wedge \mu(a \wedge c).$$

Dually we have the concept of L-fuzzy semi-prime filter of a lattice Q.

Lemma 5.2.1. *An L- fuzzy ideal μ of Q is an L- fuzzy semi-prime ideal of Q if and only if μ_α is a semi-prime ideal of Q for all $\alpha \in L$.*

Proof. Suppose that μ is an L-fuzzy semi-prime ideal and $\alpha \in L$. Then clearly μ_α is an ideal of Q. Let $a,b,c \in Q$ such that $(a,b)^l \subseteq \mu_\alpha$ and $(a,c)^l \subseteq \mu_\alpha$ and $z \in \{a,(b,c)^u\}^l$. Then $\mu(x) \geq \alpha \ \forall x \in (a,b)^l$ and $\mu(y) \geq \alpha \ \forall y \in (a,b)^l$. This implies that

$$\inf\{\mu(x) : x \in (a,b)^l\} \geq \alpha \text{ and } \inf\{\mu(y) : y \in (a,c)^l\} \geq \alpha.$$

Therefore $\inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \geq \alpha$. Since μ is an L- fuzzy semi-prime ideal and $z \in \{a,(b,c)^u\}^l$ we have

$$\mu(z) \geq \inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \geq \alpha.$$

This implies that $z \in \mu_\alpha$ for all $z \in \{a,(b,c)^u\}^l$ and hence $\{a,(b,c)^u\}^l \subseteq \mu_\alpha$. Therefore μ_α is a semi-prime ideal of a poset Q.

Conversely suppose that μ_α is a semi-prime ideal of Q for all $\alpha \in L$. Then, clearly, μ is an L- fuzzy ideal of Q. Let $a,b,c \in Q$ and put

$$\alpha = inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\}.$$

Then

$$inf\{\mu(x) : x \in (a,b)^l\} \geq \alpha \text{ and } inf\{\mu(y) : y \in (a,c)^l\} \geq \alpha.$$

That is, $\mu(x) \geq \alpha\ \forall x \in (a,b)^l$ and $\mu(y) \geq \alpha\ \forall y \in (a,c)^l$. This implies that $(a,b)^l \subseteq \mu_\alpha$ and $(a,c)^l \subseteq \mu_\alpha$. Thus, since μ_α is a semi-prime ideal of Q, we have $\{a,(b,)^u\}^l \subseteq \mu_\alpha$. Therefore

$$\mu(z) \geq \alpha = inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \text{ for all } z \in \{a,(b,c)^u\}^l$$

and hence μ is an L-fuzzy semi-prime ideal of Q. □

Corollary 5.2.2. *A subset I of a poset Q is a semi-prime ideal of Q if and only if its characteristic map χ_I is an L-fuzzy semi-prime ideal of Q.*

Recall that if μ is an L- fuzzy ideal of Q and $a \wedge b$ exists for $a,b \in Q$, then

$$inf\{\mu(x) : x \in (a,b)^l\} = \mu(a \wedge b),$$

Theorem 5.2.3. *Let $(Q,\leq)$ be a lattice. Then an L-fuzzy ideal of Q is an L- fuzzy semi-prime ideal in the poset Q if and only if it is an L-fuzzy semi-prime ideal in the lattice Q.*

Proof. Let μ be an L-fuzzy semi-prime ideal in the poset Q and $a,b,c \in Q$. Then, since $a \wedge (b \vee c) \in \{a,(b,c)^u\}^l$, we have

$$\begin{aligned}\mu(a\wedge(b\vee c)) &\geq \inf\{\mu(x)\wedge\mu(y) : x\in(a,b)^l, y\in(a,c)^l\}\\ &= inf\{\mu(x) : x\in(a,b)^l\}\wedge inf\{\mu(y) : y\in(a,c)^l\}\\ &= \mu(a\wedge b)\wedge\mu(a\wedge c).\end{aligned}$$

Again since $a\wedge b\leq a\wedge(b\vee c)$, $a\wedge c\leq a\wedge(b\vee c)$ and μ is anti-tone we clearly have

$$\mu(a\wedge(b\vee c))\leq\mu(a\wedge b)\wedge\mu(a\wedge c)$$

Therefore μ is an L-fuzzy semi-prime ideal in the lattice Q.

Conversely, suppose that μ is an L-fuzzy semi-prime ideal in the lattice Q. Let $a,b,c\in Q$ and $z\in\{a,(b,c)^u\}^l$. Then $z\leq a$ and $z\leq t$, for all $t\in(a,b)^u$. Since $a\vee b\in(a,b)^u$, we have $z\leq a\vee b$. This implies that $z\leq a\wedge(b\vee c)$ and hence

$$\begin{aligned}\mu(z) &\geq \mu(a\wedge(b\vee c))\\ &= \mu(a\wedge b)\wedge\mu(a\wedge c)\\ &= inf\{\mu(x) : x\in(a,b)^l\}\wedge inf\{\mu(y) : y\in(a,c)^l\}\\ &= inf\{\mu(x)\wedge\mu(y) : x\in(a,b)^l, y\in(a,c)^l\}\end{aligned}$$

So μ is an L-fuzzy semi-prime ideal in the poset Q. □

Recall that for any L-fuzzy ideal μ of a poset Q and $x\in Q$, the L-fuzzy subset $\mu : x$ of Q is given by:

$$(\mu : x)(y)=\inf\{\mu(z) : z\in(x,y)^l\} \text{ for all } y\in Q.$$

and $\mu : x$ is an L-fuzzy semi-ideal of Q.

Lemma 5.2.4. *Let μ be an L-fuzzy ideal of a poset Q and $x\in Q$. Then*

$$\inf\{(\mu : x)(y) : y\in(a,b)^{ul}\}=\inf\{\mu(y) : y\in\{x,(a,b)^u\}^l\}$$

Proof. . Let $A = \{\mu(y) : y \in \{x,(a,b)^u\}^l\}$ and $B = \{(\mu : x)(y) : y \in (a,b)^{ul}\}$. Now we claim that $\inf A = \inf B$. Put $\alpha = \inf A$. Then

$$\begin{aligned}
\alpha \leq \mu(y)\ \ \forall y \in \{x,(a,b)^u\}^l &\Rightarrow \{x,(a,b)^u\}^l \subseteq \mu_\alpha \\
&\Rightarrow (a,b)^{ul} \subseteq \mu_\alpha : x = (\mu : x)_\alpha \\
&\Rightarrow (\mu : x)(y) \geq \alpha\ \ \forall y \in (a,b)^{ul} \\
&\Rightarrow \inf\{(\mu : x)(y) : y \in (a,b)^{ul}\} \geq \alpha. \\
&\Rightarrow \inf B \geq \inf A.
\end{aligned}$$

To prove the other side of the inequality, let $\beta = \inf B$. Then

$$\begin{aligned}
\beta \leq (\mu : x)(y)\ \ \forall y \in (a,b)^{ul} &\Rightarrow (a,b)^{ul} \subseteq (\mu : x)_\beta = \mu_\beta : x \\
&\Rightarrow \{x,(a,b)^u\}^l \subseteq \mu_\beta \\
&\Rightarrow \mu(y) \geq \beta\ \ \forall y \in \{x,(a,b)^u\}^l \\
&\Rightarrow \inf\{\mu(y) : y \in \{x,(a,b)^u\}^l\} \geq \beta. \\
&\Rightarrow \inf A \geq \inf B.
\end{aligned}$$

Hence the claim is true.

□

Now we present a characterization of an L-fuzzy semi-prime ideal of a poset Q in terms of $\mu : x$ where μ is an L-fuzzy ideal of Q and $x \in Q$.

Theorem 5.2.5. *An L-fuzzy ideal μ of a poset Q is an L-fuzzy semi-prime ideal if and only if $\mu : x$ is an L-fuzzy ideal for all $x \in Q$, in fact, an L-fuzzy semi-prime ideal for all $x \in Q$.*

Proof. Let μ be an L-fuzzy semi-prime ideal of Q and $x \in Q$. First let us show that $\mu : x$ is an L-fuzzy ideal of Q. Now $(\mu : x)(0) = \inf\{\mu(y) : y \in (x,0)^l\} = \mu(0) = 1$. Again let

$a,b \in Q$ and $z \in (a,b)^{ul}$. Then

$$\begin{aligned}(\mu : x)(a) \wedge (\mu : x)(b) &= \inf\{\mu(w) : w \in (x,a)^l\} \wedge \inf\{\mu(u) : u \in (x,b)^l\} \\ &= \inf\{\mu(w) \wedge \mu(u) : w \in (x,a)^l, u \in (x,b)^l\} \\ &\leq \mu(v) \quad \text{for all } v \in \{x,(a,b)^u\}^l\end{aligned}$$

This implies that

$$\begin{aligned}(\mu : x)(a) \wedge (\mu : x)(b) &\leq \inf\{\mu(v) : v \in \{x,(a,b)^u\}^l\} \\ &= \inf\{(\mu : x)(v) : v \in (a,b)^{ul}\} \\ &\leq (\mu : x)(z). \text{ (Since } z \in (a,b)^{ul}\text{)}\end{aligned}$$

Therefore $\mu : x$ is an L-fuzzy ideal of Q for all $x \in Q$.

Now we show that $\mu : x$ is an L-fuzzy semi-prime ideal of Q. Let $a,b,c \in Q$ and $z \in \{a,(b,c)^u\}^l$.

$$\begin{aligned}\text{Now} \quad & \inf\{(\mu : x)(u) \wedge (\mu : x)(w) : u \in (a,b)^l, w \in (a,c)^l\} \\ &= \inf\{(\mu : x)(u) : u \in (a,b)^l\} \wedge \inf\{(\mu : x)(w) : w \in (a,c)^l\} \\ &= \inf\{\mu(u) : u \in (x,a,b)^l\} \wedge \inf\{\mu(w) : w \in (x,a,c)^l\} \\ &= \inf\{(\mu : b)(u) : u \in (x,a)^l\} \wedge \inf\{(\mu : c)(w) : w \in (x,a)^l\} \\ &= \inf\{(\mu : b)(s) \wedge (\mu : c)(s) : s \in (x,a)^l\} \\ &\leq \inf\{(\mu : b)(s) \wedge (\mu : c)(s) : s \in \{x,a,(b,c)^u\}^l\} \\ &= \inf\{(\mu : s)(b) \wedge (\mu : s)(c) : s \in \{x,a,(b,c)^u\}^l = (x,a)^l \cap (b,c)^{ul}\} \\ &\leq \inf\{(\mu : s)(s) : s \in \{x,a,(b,c)^u\}^l\} \\ &= \inf\{\mu(s) : s \in \{x,a,(b,c)^u\}^l\}\end{aligned}$$

Now, since $z \in \{a,(b,c)^u\}^l$, we have $(x,z)^l \subseteq \{x,a,(b,c)^u\}^l$ and hence

$$\begin{aligned}
\inf\{(\mu:x)(u)\wedge(\mu:x)(w) : u\in(a,b)^l, w\in(a,c)^l\} &\leq \inf\{\mu(s) : s\in\{x,a,(b,c)^u\}^l\} \\
&\leq \inf\{\mu(s) : s\in(x,z)^l\} \\
&= (\mu:x)(z)
\end{aligned}$$

Therefore $\mu : x$ is an L-fuzzy semi-prime ideal of Q.

Conversely suppose that $\mu : x$ is an L-fuzzy ideal of Q for all $x \in Q$. Now we show that μ is an L-fuzzy semi-prime ideal of Q. Let $a,b,c \in Q$ and $z \in \{a,(b,c)^u\}^l$. Then

$$\begin{aligned}
& inf\{\mu(x)\wedge\mu(y) : x\in(a,b)^l, y\in(a,c)^l\} \\
= \;& inf\{\mu(x) : x\in(a,b)^l\}\wedge\inf\{\mu(y) : y\in(a,c)^l\} \\
= \;& (\mu:a)(b)\wedge(\mu:a)(c) \\
\leq \;& (\mu:a)(t)\ for\ all\ t\in(b,c)^{ul}
\end{aligned}$$

This implies that

$$\begin{aligned}
inf\{\mu(x)\wedge\mu(y) : x\in(a,b)^l, y\in(a,c)^l\} &\leq \inf\{(\mu:a)(t) : t\in(b,c)^{ul}\} \\
&= \inf\{\mu(t) : t\in\{a,(b,c)^u\}^l\} \\
&\leq \mu(z).
\end{aligned}$$

Thus

$$inf\{\mu(x)\wedge\mu(y) : x\in(a,b)^l, y\in(a,c)^l\} \leq \mu(z) \text{ for all } z\in\{a,(b,c)^u\}^l$$

and hence μ is an L-fuzzy semi-prime ideal of Q. □

Recall that a proper L-fuzzy ideal μ of a poset Q is called an L-fuzzy prime, if for any $a,b \in Q$,

$$inf\{\mu(x) : x\in(a,b)^l\} = \mu(a) \text{ or } \mu(b).$$

The following result establishes a connection between L-fuzzy prime ideals and L-fuzzy semi-prime ideals of a poset Q.

Lemma 5.2.6. *Every L-fuzzy prime ideal of a poset Q is an L-fuzzy semi-prime ideal.*

Proof. Let μ be an L-fuzzy prime ideal of Q. Let $a,b,c \in Q$. Then since μ is an L-fuzzy prime ideal of Q, we clearly have

$$\inf\{\mu(x) : x \in (a,b)^l\} = \mu(a) \text{ or } \mu(b) \text{ and } \inf\{\mu(y) : y \in (a,c)^l\} = \mu(a) \text{ or } \mu(c).$$

Let $z \in \{a,(b,c)^u\}^l = a^l \cap (b,c)^{ul}$. Then $z \leq a$ and $z \in (b,c)^{ul}$. Now if

$\inf\{\mu(x) : x \in (a,b)^l\} = \mu(a)$ or $\inf\{\mu(y) : y \in (a,c)^l\} = \mu(a)$, then we have

$$\begin{aligned} \mu(z) \geq \mu(a) &\geq inf\{\mu(x) : x \in (a,b)^l\} \wedge \inf\{\mu(y) : y \in (a,c)^l\} \\ &= inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \end{aligned}$$

Again if $\inf\{\mu(x) : x \in (a,b)^l\} \neq \mu(a)$ and $\inf\{\mu(y) : y \in (a,c)^l\} \neq \mu(a)$ then we have

$$\inf\{\mu(x) : x \in (a,b)^l\} = \mu(b) \text{ and } \inf\{\mu(y) : y \in (a,c)^l\} = \mu(c).$$

Now since $z \in (b,c)^{ul}$ and μ is an L-fuzzy ideal, we have

$$\begin{aligned} \mu(z) \geq \mu(b) \wedge \mu(c) &= inf\{\mu(x) : x \in (a,b)^l\} \wedge \{\mu(y) : y \in (a,c)^l\} \\ &= inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\}. \end{aligned}$$

Hence, in either cases, we have

$$\mu(z) \geq inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \text{ for all } z \in \{a,(b,c)^u\}^l.$$

Hence μ is an L-fuzzy semi-prime ideal of Q. □

Remark 5.2.1. *The converse of the above lemma is not is not true. For example consider the poset $(Q \leq)$ depicted in the Fig. 5.1 below. Define a fuzzy subset $\mu : Q \longrightarrow [0,1]$ by:*

$$\mu(0) = 1 \textit{ and } \mu(a) = \mu(b) = \mu(1) = 0.5.$$

Then μ is a fuzzy semi-prime ideal but not a fuzzy prime ideal of Q. This is because $a,b \in Q$ and $\inf\{\mu(x) : x \in (a,b)^l\} = \mu(0) = 1 \neq 0.5 = \mu(a) = \mu(b)$.

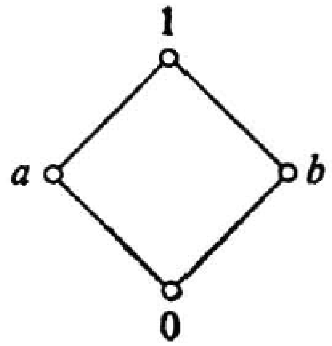

Fig. 5.1

Now before we prove some other characterizations of L-fuzzy semi-primeness and L-fuzzy primeness in the case of a poset satisfying DCC, we introduce the notion of a μ-atom of an L-fuzzy ideal μ of a poset.

Definition 5.2.3. *Let μ be an L-fuzzy ideal of a poset Q and $\alpha \in L$. An element i in Q is called a μ-atom with respect to α, if it satisfies the following conditions:*

1. $\alpha \nleq \mu(i)$ *and*
2. $\alpha \leq \mu(x)$ *whenever* $x < i$.

Example 5.2.7. *Consider the poset depicted in the Fig 5.3 given below on page 131. Define a fuzzy subset* $\mu : Q \longrightarrow [0,1]$ *by:*

$$\mu(0) = \mu(a) = 1,\ \mu(b) = 0.7,\ \mu(c) = 0.6 \text{ and } \mu(d) = 0.8,$$
$$\mu(a') = \mu(b') = \mu(c') = \mu(d') = \mu(1) = 0.2.$$

. Then it is easy to see that μ is an L-fuzzy ideal of Q and a' is a μ-atom with respect to $\alpha = 0.6$ *in* $[0,1]$.

Lemma 5.2.8. *There always exists a μ-atom for every proper L-fuzzy ideal μ in a poset Q satisfying DCC with respect to some α in L.*

Proof. Let Q be a poset satisfying DCC and μ be a proper L-fuzzy ideal of Q. Then there exists $a \in Q$ such that $\mu(a) \neq 1$. This implies that there exists $\alpha \in L$ such that $\alpha \nleq \mu(a)$. Put $I = \{x \in Q : \mu(x) \geq \alpha\}$. Then $Q - I$ is a non-empty subset of Q and as Q is satisfying DCC, $Q - I$ has a minimal element, say i, such that $i \leq a$. Now we claim that i is a μ-atom with respect to α. Since $i \in Q - I$ we have $\alpha \nleq \mu(i)$. Let $x < i$. Then, by the minimality of i, $x \notin Q - I$ and hence $\mu(x) \geq \alpha$. Hence the claim is true. □

Remark 5.2.2. *Lemma 5.2.8 gives a guarantee that if μ be an L-fuzzy ideal of a poset Q satisfying DCC and $\alpha \nleq \mu(a)$ for some $a \in Q$ and $\alpha \in L$, then there exists a μ-atom i in Q with respect to α such that $i \leq a$.*

Lemma 5.2.9. *Any two distinct μ-atoms of an L-fuzzy ideal μ of a poset Q with respect to any $\alpha \in L$ are incomparable.*

Proof. Let μ be an L-fuzzy ideal of Q and i and j be any two distinct μ-atoms with respect to $\alpha \in L$. Then, by definition, we have $\alpha \nleq \mu(i)$ and $\mu(x) \geq \alpha$ whenever $x < i$ and $\alpha \nleq \mu(j)$ and $\mu(y) \geq \alpha$ whenever $y < j$. Now we show that i and j are incomparable. Suppose not. Then $i < j$ or $j < i$, i.e., $\mu(i) \geq \alpha$ or $\mu(j) \geq \alpha$ which is a contradiction to the fact that $\alpha \nleq \mu(j)$ and $\alpha \nleq \mu(j)$. Hence i and j are incomparable. □

Remark 5.2.3. *From Lemma 5.2.9 above, we can deduce that if i and j are μ-atoms in a poset Q with respect to some α in L such that $i \leq j$ then $i = j$.*

Lemma 5.2.10. *Let μ be an L-fuzzy semi-prime ideal of a poset Q satisfying DCC. Then $\mu : i$ is a u- L-fuzzy ideal for every μ-atom i in Q with respect to 1 in L.*

Proof. Let i be a μ-atom in Q with respect to 1 in L. Since μ is an L-fuzzy semi-prime ideal, by Theorem 5.2.5, $\mu : i$ is an L-fuzzy ideal of Q. Now we show that $\mu : i$ is a u-L-fuzzy ideal. Suppose on the contrary that $\mu : i$ is not a u-L-fuzzy ideal. Then there exist $a, b \in Q$ such that

$$(\mu : i)(a) \wedge (\mu : i)(b) \nleq (\mu : i)(x) \text{ for all } x \in (a,b)^u.$$

This implies that there exists $y \in (i,x)^l$ such that

$$(\mu : i)(a) \wedge (\mu : i)(b) \nleq \mu(y) \text{ for all } x \in (a,b)^u.$$

Thus, by Remark 5.2.2, there exists a μ-atom, say j, with respect to $\alpha = (\mu : i)(a) \wedge (\mu : i)(b)$ such that $j \leq y$. Since $j \leq y$ and $y \in (i,x)^l$, we have $j \leq i$ and hence $\mu(j) \geq \mu(i)$. This implies that $\alpha \nleq \mu(i)$. Again let $z < i$. Then $\mu(z) = 1 \geq \alpha$. Therefore i is also a μ-atom with respect to α. Also since $j \leq y \leq i$, by Remark 5.2.3, we have $j = y = i$. This implies that $i \in (i,x)^l$ and hence $i \leq x$ for all $x \in (a,b)^u$ and so $i \in (a,b)^{ul}$. Since $\mu : i$ is an ideal we have

$$\alpha = (\mu : i)(a) \wedge (\mu : i)(b) \leq (\mu : i)(i) = \mu(i),$$

which is a contradiction to the fact that $\alpha \nleq \mu(i)$. Therefore $\mu : i$ a u-L-fuzzy ideal. □

Theorem 5.2.11. *Let μ be an L-fuzzy semi-prime ideal of a poset Q satisfying DCC. Then $\mu : i$ is an L-fuzzy prime ideal of Q for every μ-atom $i \in Q$ with respect to* 1 *in L.*

Proof. Let μ be an L-fuzzy semi-prime ideal of a poset Q satisfying DCC and i is a μ-atom in Q with respect to 1 in L. Then, by Lemma 5.2.10, $\mu : i$ is a u-L-fuzzy ideal. Now we remain to show that $\mu : i$ is an L-fuzzy prime ideal. Since $\mu(i) \neq 1$, by Lemma 5.2.4, $\mu : i \neq \overline{1}$. Hence $\mu : i$ is proper. Let $a, b \in Q$ and suppose that

$$\inf\{(\mu : i)(x) : x \in (a,b)^l\} \neq (\mu : i)(a).$$

Put $\alpha = \inf\{(\mu : i)(x) : x \in (a,b)^l\}$. Since $(\mu : i)(a) = \inf\{\mu(y) : y \in (a,i)^l\}$ there exists y_1 in $(i,a)^l$ such that $\alpha \nleq \mu(y_1)$. Then, by Remark 5.2.2, there exists a μ-atom, say j in Q with respect to α such that $j \leq y_1$. It is also clear that i is also a μ-atom with respect to α. Since $j \leq y_1 \leq i$, by Remark 5.2.3, we must have $j = y_1 = i$, and therefore $i \leq a$, i.e., $(i,a)^l = i^l$. Thus we have

$$\begin{aligned}
\inf\{(\mu : i)(x) : x \in (a,b)^l\} &= \inf\{\mu(y) : y \in (i,a,b)^l\} \\
&= \inf\{\mu(y) : y \in (i,,b)^l\} \\
&= (\mu : i)(b).
\end{aligned}$$

This proves that $\mu : i$ is an L-fuzzy prime ideal for every μ-atom $i \in Q$ with respect to 1. □

Theorem 5.2.12. *Let μ be an L-fuzzy ideal of a poset Q satisfying DCC and let $\mu : i$ is an L-fuzzy ideal for every μ-atom $i \in Q$. Then μ is an L-fuzzy semi-prime ideal Q.*

Proof. Let $\mu : i$ is an L-fuzzy ideal for any μ-atom i in Q. Let $a,b,c \in Q$. Now we claim $\inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \leq \mu(z)$ for all $z \in \{a,(b,c)^u\}^l$. Suppose not. Then there exists $z_1 \in \{a,(b,c)^u\}^l = a^l \cap (b,c)^{ul}$ such that

$$\inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \nleq \mu(z_1).$$

Hence, by Remark 5.2.2, there exists a μ-atom j in Q with respect to $\alpha = \inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\}$ in L such that $j \leq z_1$. Then, by hypothesis, $\mu : j$ is an L-fuzzy ideal. Again, since $(j,b)^l \subseteq (a,b)^l$ and $(j,c)^l \subseteq (a,c)^l$, we have

$$\begin{aligned}
\alpha &= \inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \\
&= \inf\{\mu(x) : x \in (a,b)^l\} \wedge \inf\{\mu(y) : y \in (a,c)^l\} \\
&\leq \inf\{\mu(x) : x \in (j,b)^l\} \wedge \inf\{\mu(y) : y \in (j,c)^l\} \\
&= (\mu : j)(b) \wedge (\mu : j)(c) \\
&\leq (\mu : j)(j) \quad (\textit{since } j \in (b,c)^{ul}) \\
&= \mu(j)
\end{aligned}$$

which is a contradiction to the fact that j is a μ-atom wit respect to α. Therefore μ is an L-fuzzy semi-prime ideal of Q. □

Definition 5.2.4. *[9] A proper L-fuzzy ideal μ of a poset Q is said to be maximal L-fuzzy ideal if μ is a maximal element in the set of all proper L-fuzzy ideals of Q. That is, if there is no proper L-fuzzy ideal θ of Q such that $\mu \subsetneqq \theta$.*

The following result gives another characterization for L-fuzzy semiprime ideals to be L-fuzzy prime.

Theorem 5.2.13. *Every maximal L-fuzzy semi-prime ideal of a poset Q is an L-fuzzy prime ideal.*

Proof. Let μ be a maximal L-fuzzy semi-prime ideal of a poset Q, that is, maximal among all proper L-fuzzy semi-prime ideals of a poset Q . Let $a, b \in Q$. Then, by Theorem 5.2.5, $\mu : b$ is an L-fuzzy semi-prime ideal. Since $\mu \subseteq \mu : b$, by maximality of μ, we have either $\mu = \mu : b$ or $\mu : b = \overline{1}$. If $\mu : b = \overline{1}$ then, by Lemma 5.2.4, $\mu(b) = 1$. Thus

$$\inf\{\mu(x) : x \in (a,b)^l\} = (\mu : b)(a) = \overline{1}(a) = 1 = \mu(b).$$

Again if $\mu = \mu : b$, then we have

$$\inf\{\mu(x) : x \in (a,b)^l\} = (\mu : b)(a) = \mu(a).$$

Thus in either cases we have

$$\inf\{\mu(x) : x \in (a,b)^l\} = \mu(a) \text{ or } \mu(b) \text{ for all } a, b \in Q.$$

Hence μ is an L-fuzzy prime ideal of Q. □

As a consequence we have the following corollary.

Corollary 5.2.14. *Let μ be a maximal L-fuzzy ideal of Q. Then μ is an L-fuzzy semi-prime ideal Q if and only if μ is an L-fuzzy prime ideal.*

The following is a characterization of an L-fuzzy ideal to be L-fuzzy prime ideal in terms of a μ-atom in a poset Q satisfying DCC.

Theorem 5.2.15. *Let μ be an L-fuzzy ideal of a poset Q satisfying DCC. Then μ is an L-fuzzy prime ideal Q if and only if Q has exactly one μ-atom with respect to each α in L.*

Proof. Let μ be an L-fuzzy prime ideal of a poset Q satisfying DCC. Since μ is proper, by Lemma 5.2.8, there exists a μ-atom in Q with respect to some α in L. Now we claim that Q has exactly one μ-atom with respect to α in L. Suppose not. Let $i, j \in Q$ be any distinct μ-atoms in Q with respect to α in L. Then, by Lemma 5.2.9, i, j are incomparable and $\mu(x) \geq \alpha$ for all $x < i$ and $\mu(y) \geq \alpha$ for all $y < j$. This implies that

$\inf\{\mu(x) : x \in (i,j)^l\} \geq \alpha$. Since $\inf\{\mu(x) : x \in (i,j)^l\} = \mu(i)$ or $\mu(j)$, we have $\mu(i) \geq \alpha$ or $\mu(j) \geq \alpha$, which is a contradiction. Therefore Q has exactly one μ-atom with respect to α in L.

Conversely suppose that Q has exactly one μ-atom, say i, with respect to some α in L. Now we show that μ is an L-fuzzy prime ideal. Since $\alpha \nleq \mu(i)$, we have $\mu(i) \neq 1$ and hence μ is proper. Now we show that for any $a,b \in Q$

$$\inf\{\mu(x) : x \in (a,b)^l\} = \mu(a) \text{ or } \mu(b).$$

Suppose not. Thus there exist $a,b \in Q$ such that

$$\inf\{\mu(x) : x \in (a,b)^l\} \nleq \mu(a) \text{ and } \inf\{\mu(x) : x \in (a,b)^l\} \nleq \mu(b).$$

Then there exist μ-atoms $i,j \in Q$ with respect to $\alpha = \inf\{\mu(x) : x \in (a,b)^l\}$ such that $i \leq a$ and $j \leq b$. Then, by hypothesis, we have $i = j$ and hence $i \in (a,b)^l$. Therefore $\alpha = \inf\{\mu(x) : x \in (a,b)^l\} \leq \mu(i)$, which is a contradiction to the fact that i is a μ-atom with respect to $\alpha = \inf\{\mu(x) : x \in (a,b)^l\}$. Therefore μ is an L-fuzzy prime ideal. □

Lemma 5.2.16. *Let μ be a proper L-fuzzy ideal of a poset Q satisfying DCC and $A = \{i \in Q : i \text{ is a } \mu - atom\}$. Then $\mu = \bigcap_{i\in A} \mu : i$.*

Proof. We show that $\bigcap_{i\in A} \mu : i \subseteq \mu$ as the converse inclusion always holds. Suppose that $\bigcap_{i\in A} \mu : i \nsubseteq \mu$. This implies that there exists $a \in Q$ such that $(\bigcap_{i\in A} \mu : i)(a) \nleq \mu(a)$. Thus there exists a μ-atom $j \in Q$ with respect to $\alpha = (\bigcap_{i\in A} \mu : i)(a)$ such that $j \leq a$. Then we have $j \in A$ and hence

$$\begin{aligned}
(\bigcap_{i\in A} \mu : i)(a) &\leq (\mu : j)(a) \\
&= \inf\{\mu(x) : x \in (j,a)^l\} \\
&= \inf\{\mu(x) : x \in j^l\} \\
&= \mu(j),
\end{aligned}$$

which is a contradiction to the fact that j is a μ-atom with respect to $\alpha = (\bigcap_{i\in A}\mu : i)(a)$. Hence $\bigcap_{i\in A}\mu : i \subseteq \mu$. Therefore $\bigcap_{i\in A}\mu : i = \mu$. □

Lemma 5.2.17. *The intersection of any non empty family of L-fuzzy prime ideals of Q is an L-fuzzy semi-prime ideal Q.*

Proof. Let $\{\mu_i : i \in \Delta\}$ be a non empty family of L-fuzzy prime ideals of Q. Put $\mu = \bigcap_{i\in\Delta}\mu_i$. Then clearly μ is an L-fuzzy ideal of Q. Let $a,b,c \in Q$ and $z \in \{a,(b,c)^u\}^l$. Now

$$\begin{aligned}
& inf\{\mu(x)\wedge\mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \\
= \;& inf\{\mu(x) : x \in (a,b)^l\} \wedge \inf\{\mu(y) : y \in (a,c)^l\} \\
\leq \;& inf\{\mu_i(x) : x \in (a,b)^l\} \wedge \inf\{\mu_i(y) : y \in (a,c)^l\} \text{for each } i \in \Delta. \\
= \;& \mu_i(a) \text{ or } \; \mu_i(b)\wedge\mu_i(c) \text{ for each } i \in \Delta. \\
\leq \;& \mu_i(z) \text{ for each } i \in \Delta.
\end{aligned}$$

This implies that

$$inf\{\mu(x)\wedge\mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \leq (\bigcap_{i\in\Delta}\mu_i)(z) = \mu(z) \text{ for all } z \in \{a,(b,c)^u\}^l.$$

Therefore $\mu = \bigcap_{i\in\Delta}\mu_i$ is an L-fuzzy semi-prime ideal Q. □

As an immediate consequence of Theorem 5.2.12, Lemma 5.2.16 and Lemma 5.2.17 in the case of posets satisfying DCC we obtain the following result.

Theorem 5.2.18. *Let μ be a proper L-fuzzy ideal of a poset Q satisfying DCC. Then μ is an L-fuzzy semi-prime ideal of Q if and only if μ is expressed as an intersection of L-fuzzy prime ideals of Q.*

In the following we characterize the distributive posets in terms of L-fuzzy semi-prime ideals in the following results.

Theorem 5.2.19. *A poset Q is distributive if and only if $\chi_{(x]}$ of Q is an L-fuzzy semi-prime ideal of Q, for each $x \in Q$.*

Proof. Suppose that Q is a distributive poset and $x \in Q$. Now to show $\chi_{(x]}$ is an L-fuzzy semi-prime ideal of Q, by Corollary 5.2.2, it is enough to show that $(x]$ is a semi-prime ideal of Q. Let $a,b,c \in Q$ such that $(a,b)^l \subseteq (x]$ and $(a,c)^l \subseteq (x]$. Let $z \in \{a,(b,c)^u\}^l$. Then $z \leq a$ and $z \in (b,c)^{ul}$. This implies that

$$z^l = \{z,(b,c)^u\}^l = \{(z,b)^l,(z,c)^l\}^{ul}$$

Since $z \leq a$, we have $(z,b)^l \subseteq (a,b)^l \subseteq (x]$ and $(z,c)^l \subseteq (a,c)^l \subseteq (x]$.
This implies that $(z,a)^l \cup (z,b)^l \subseteq (x]$. Thus $z \in z^l = \{(z,b)^l,(z,c)^l\}^{ul} \subseteq (x]^{ul} = (x]$ and hence $\{a,(b,c)^u\}^l \subseteq (x]$. Therefore $\chi_{(x]}$ of Q is an L-fuzzy semi-prime ideal of Q.

Conversely suppose that $\chi_{(x]}$ is an L-fuzzy semi- prime ideal of Q for each $x \in Q$. Then, by Corollary 5.2.2, it is clear that $(x]$ is semi- prime ideal of Q for each $x \in Q$. Let $a,b,c \in Q$. It is enough to prove that $\{a,(b,c)^u\}^l \subseteq \{(a,b)^l,(a,c)^l\}^{ul}$, as the converse inclusion is always true. Now let $x \in \{a,(b,c)^u\}^l$ and $y \in \{(a,b)^l,(a,c)^l\}^u$. We claim that $x \leq y$. Indeed, since $\{(a,b)^l,(a,c)^l\}^{ul} \subseteq y^l$ we have

$$(a,b)^l \subseteq y^l = (y] \text{ and } (a,c)^l \subseteq y^l = (y].$$

Then. by semi-primeness of $(y]$, we conclude that $x \in \{a,(b,c)^u\}^l \subseteq (y]$. Hence $x \leq y$ for all $y \in \{(a,b)^l,(a,c)^l\}^u$. Therefore $x \in \{(a,b)^l,(a,c)^l\}^{ul}$ and hence

$$\{a,(b,c)^u\}^l \subseteq \{(a,b)^l,(b,c)^l\}^{ul}.$$

This proves that Q is a distributive poset. □

Note that in a distributive poset every L-fuzzy ideal need not be an L-fuzzy semi-prime ideal. Consider the distributive poset Q depicted in the Fig 5.2 below

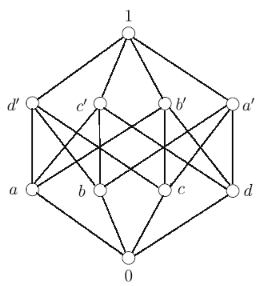

Fig. 5.2

Define a fuzzy subset $\mu : Q \longrightarrow [0,1]$ by:

$$\mu(0) = \mu(a) = 1,\ \mu(a') = \mu(b') = \mu(c') = \mu(d') = \mu(1) = 0.2,\ \mu(b) = 0.6,\ \mu(c) = 0.5$$
$$\text{and } \mu(d) = 0.7.$$

Then μ is an L-fuzzy ideal but not an L-fuzzy semi-prime ideal as $d', c', b' \in Q$ and $d' \in \{d', (b', c')^u\}^l$ but

$$inf\{\mu(x) \wedge \mu(y) : x \in (d', c')^l, y \in (d', b')^l\} = 0.5 \nleq 0.2 = \mu(d').$$

An immediate consequence of Theorem 5.2.18 and Theorem 5.2.19, we have the following corollary.

Corollary 5.2.20. *Let Q is a poset satisfying DCC. Then Q is distributive if and only if for every $x \in Q$, $\chi_{(x]}$ is representable as an intersection of L-fuzzy prime ideals of Q.*

5.3 The Lattice of L-Fuzzy Semi-prime Ideals

In this section we prove that the set of all L-fuzzy semi-prime ideals in a poset forms a complete lattice. The relations between the L-fuzzy semi-prime (respectively,L-fuzzy prime) ideals of a poset and the L-fuzzy semi-prime (respectively,L-fuzzy prime) ideals of the lattice of all ideals of a poset are established.

We begin by proving that the set $\mathscr{F}\mathscr{S}\mathscr{P}(Q)$ of all L-fuzzy semi-prime ideals of a poset Q forms a complete lattice.

Lemma 5.3.1. *Let $\mathscr{F}\mathscr{S}\mathscr{P}(Q)$ be the set of all L-fuzzy semi-prime ideals of a poset Q and μ be an L fuzzy subset of Q. Then the L-fuzzy semi-prime ideal generated by μ is $(\mu] = \cap\{\theta \in \mathscr{F}\mathscr{S}\mathscr{P}(Q) : \mu \subseteq \theta\}$.*

Theorem 5.3.2. *The set $\mathscr{F}\mathscr{S}\mathscr{P}(Q)$ of all L- fuzzy semi- prime ideals of Q forms a complete lattice, in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\mu_i$ of any family*

$\{\mu_i : i \in \Delta\}$ *of L-fuzzy semi-prime ideals of Q respectively are given by:*

$$\sup_{i\in\Delta}\mu_i = \cap\{\theta \in \mathscr{F}\mathscr{S}\mathscr{I}(Q) : \cup_{i\in\Delta}\mu_i \subseteq \theta\} \text{ and } \inf_{i\in\Delta}\mu_i = \cap_{i\in\Delta}\mu_i.$$

Definition 5.3.1. *Let Q be a given poset and $\mathscr{I}(Q)$ be a lattice of all ideals of Q. Then an extension of an L-fuzzy ideal μ of Q, denoted by μ^e, is an L -fuzzy subset $\mathscr{I}(Q)$ defined by:*

$$\mu^e(I) = \inf\{\mu(x) : x \in I\}, \text{ for all } I \in \mathscr{I}(Q).$$

Lemma 5.3.3. *Let μ be an L-fuzzy ideal of a poset Q and $\alpha \in L$. Then $(\mu^e)_\alpha = (\mu_\alpha)^e$.*

Proof. Since μ is an L-fuzzy ideal of a poset Q and $\alpha \in L$ μ_α is an ideal of Q. Recall that $(\mu_\alpha)^e = \{I \in \mathscr{I}(Q) : I \subseteq \mu_\alpha\}$. Now since

$$\begin{aligned} I \in (\mu_\alpha)^e &\Leftrightarrow I \subseteq \mu_\alpha \\ &\Leftrightarrow \mu(a) \geq \alpha, \text{ for all } a \in I \\ &\Leftrightarrow \mu^e(I) = \inf\{\mu(a) : a \in I\} \geq \alpha \\ &\Leftrightarrow I \in (\mu^e)_\alpha \end{aligned}$$

we have $(\mu_\alpha)^e = (\mu^e)_\alpha$. □

Lemma 5.3.4. *Let μ be an L-fuzzy ideal of a poset Q. Then μ^e is an L-fuzzy ideal of the lattice $\mathscr{I}(Q)$.*

Proof. Now $\mu^e((0]) = \inf\{\mu(x) : x \in (0]\} = \mu(0) = 1$. Let $I, J \in \mathscr{I}(Q)$. Then

$$\begin{aligned} \mu^e(I) &= \inf\{\mu(x) : x \in I\} \\ &\geq \inf\{\mu(x) : x \in I \vee J\} \text{ (..since } I \subseteq I \vee J) \\ &= \mu^e(I \vee J) \end{aligned}$$

and similarly we have $\mu^e(J) \geq \mu^e(I \vee J)$. Thus $\mu^e(I) \wedge \mu^e(J) \geq \mu^e(I \vee J)$. Again to show the other inequality put $\alpha = \mu^e(I) \wedge \mu^e(J)$. Now

$$\begin{aligned}
\alpha = \mu^e(I) \wedge \mu^e(J) \quad &\Rightarrow \quad \alpha \leq \mu^e(I) = \inf\{\mu(x) : x \in I\} \text{ and} \\
&\qquad \alpha \leq \mu^e(J) = \inf\{\mu(y) : y \in I\} \\
&\Rightarrow \quad \alpha \leq \mu(x) \text{ for all } x \in I \text{ and } \alpha \leq \mu(y) \text{ for all } y \in J \\
&\Rightarrow \quad I \subseteq \mu_\alpha \text{ and `} J \subseteq \mu_\alpha \\
&\Rightarrow \quad I \cup J \subseteq \mu_\alpha \\
&\Rightarrow \quad I \vee J \subseteq \mu_\alpha \\
&\Rightarrow \quad I \vee J \in (\mu_\alpha)^e = (\mu^e)_\alpha \\
&\Rightarrow \quad \mu^e(I \vee J) \geq \alpha = \mu^e(I) \wedge \mu^e(J)
\end{aligned}$$

Therefore $\mu^e(I \vee J) = \mu^e(I) \wedge \mu^e(J)$. Hence μ^e is an L-fuzzy ideal $\mathscr{I}(Q)$. □

In order to study the relations between the L-fuzzy semi-prime ideals of a poset Q and the L-fuzzy semi-prime ideals of the lattice $\mathscr{I}(Q)$, we consider the following sets that are studied in [29]. For any ideals I and J of a poset Q, define subsets of Q by:

$C_1(I,J) = \bigcup\{(a,b)^{ul} : a,b \in I \cup J\}$ and $C_{n+1}(I,J) = \bigcup\{(a,b)^{ul} : a,b \in C_n(I,J)\}$ for each $n \in \mathbb{N}$, inductively.

It is easy to observe that the set $\{C_n(I,J) : n \in \mathbb{N}\}$ forms a chain and each $C_n(I,J)$ is a semi-ideal or a down set of Q.

We use the following Lemma in the result followed by it which is a relation between the L-fuzzy semi-prime ideals of a poset Q and the L-fuzzy semi-prime ideals of the lattice $\mathscr{I}(Q)$.

Lemma 5.3.5. *[29] Let Q be a poset and $I,J \in \mathscr{I}(Q)$. Then*

$$I \vee J = \bigcup\{\{C_n(I,J) : n \in \mathbb{N}\}$$

Theorem 5.3.6. *Let μ be an L-fuzzy semi-prime ideal of a poset Q. Then μ^e is an L-fuzzy semi-prime ideal of the lattice $\mathscr{I}(Q)$.*

Proof. Let $I,J,K \in \mathscr{I}(Q)$. Now we prove that

$$\mu^e(I\cap J)\wedge\mu^e(I\cap K) = \mu^e(I\cap(J\vee K).$$

Since μ^e is an L-fuzzy ideal of $\mathscr{I}(Q)$ and $I\cap J \subseteq I\cap(J\vee K)$ and $I\cap K \subseteq I\cap(J\vee K)$ we clearly have

$$\mu^e(I\cap J)\wedge\mu^e(I\cap K) \geq \mu^e(I\cap(J\vee K)).$$

Again to show the other inequality it is enough to show that for each $n\in\mathbb{N}$

$$\mu^e(I\cap J)\wedge\mu^e(I\cap K) \leq \mu(x) \text{ for all } x\in I\cap C_n(J,K),$$

in view of Lemma 5.3.5

1. Let $n=1$ and $x\in I\cap C_1(J,K)$. Then $x\in I$ and $x\in(a,b)^{ul}$ for some $a,b\in J\cup K$. If $a,b\in J$ or K, then obviously $\mu^e(I\cap J)\wedge\mu^e(I\cap K)\leq\mu(x)$. So, let us suppose, without loss of generality, that $a\in J$ and $b\in K$. Then $(x,a)^l\subseteq I\cap J$ and $(x,b)^l\subseteq I\cap K$. By L-fuzzy semi-primness of μ, we have

$$\begin{aligned}\mu^e(I\cap J)\wedge\mu^e(I\cap K) &= \inf\{\mu(y): y\in I\cap J\}\wedge\inf\{\mu(y): y\in I\cap K\}\\ &\leq \inf\{\mu(y): y\in(x,a)^l\}\wedge\inf\{\mu(z): z\in(x,b)^l\}\\ &= \inf\{\mu(y)\wedge\mu(z): y\in(x,a)^l, z\in(x,b)^l\}\\ &\leq \mu(x) \text{ .... (since } x\in\{x,(a,b)^u\}^l).\end{aligned}$$

   Thus the statement is true for $n=1$.

2. Suppose that $\mu^e(I\cap J)\wedge\mu^e(I\cap K)\leq\mu(x)$ for all $x\in I\cap C_n(I,J)$ holds for some $n\in\mathbb{N}$. We will prove that it also holds for n+1. Now $x\in I\cap C_{n+1}(I,J)$ implies that $x\in I$ and $x\in(a,b)^{ul}$ for some $a,b\in C_n(I,J)$. This implies that $(x,a)^l,\subseteq I\cap C_n(I,J)$ and $(x,b)^l\subseteq I\cap C_n(I,J)$. Thus, by induction hypothesis, we have

$$\mu^e(I\cap J)\wedge\mu^e(I\cap K)\leq\mu(y) \text{ for all } y\in(x,a)^l$$

and

$$\mu^e(I)\wedge\mu^e(J)\leq\mu(z) \text{ for all } z\in(x,b)^l$$

and since μ is L-fuzzy semi-prime and $x\in\{x,(a,b)^u\}^l$ we have

$$\begin{aligned}\mu^e(I\cap J)\wedge\mu^e(I\cap K) &\leq \inf\{\mu(y)\wedge\mu(z): y\in(x,a)^l, z\in(x,b)^l\\ &\leq \mu(x).\end{aligned}$$

Therefore $\mu^e(I\cap J)\wedge\mu^e(I\cap K)\leq\mu(x)$ for all $x\in I\cap C_n(I,J)$ for each $n\in\mathbb{N}$. Thus we have

$$\mu^e(I\cap J)\wedge\mu^e(I\cap K)\leq\inf\{\mu(x): x\in I\cap(J\vee K)\}=\mu^e(I\cap(J\vee K)).$$

Therefore $\mu^e(I\cap J)\wedge\mu^e(I\cap K)=\mu^e(I\cap(J\vee K))$ and hence μ^e is an L-fuzzy semi-prime ideal of the lattice $\mathscr{I}(Q)$. □

Definition 5.3.2. *Let Q be a given poset and $\mathscr{I}(Q)$ be the lattice of all ideals of Q. Then a contraction of an L-fuzzy ideal Φ of $\mathscr{I}(Q)$, denoted by Φ^c, is an L-fuzzy subset of Q given by:*

$$\Phi^c(x)=\sup\{\Phi(I): x\in I\}, \text{ for all } x\in Q.$$

Lemma 5.3.7. *Let Φ be an L-fuzzy ideal of the lattice $\mathscr{I}(Q)$ of all ideals of Q. Then Φ^c is an L-fuzzy ideal of the poset Q.*

Proof. Now since

$$\begin{aligned}\Phi^c(0) &= sup\{\Phi(I): 0\in I\}\\ &\geq \Phi((0]) \text{ ...(since } 0\in(0])\\ &= 1\end{aligned}$$

we have $\Phi^c(0) = 1$. Again let $a, b \in Q$ and $x \in (a,b)^{ul}$. Now

$$\begin{aligned}
\Phi^c(a) \wedge \Phi^c(b) &= \sup\{\Phi(I) : a \in I\} \wedge \sup\{\Phi(J) : b \in J\} \\
&= \sup\{\Phi(I) \wedge \Phi(J) : a \in I, b \in J\} \\
&= \sup\{\Phi(I \vee J) : a \in I, b \in J\} \\
&\leq \sup\{\Phi(I \vee J) : x \in (a,b)^{ul} \subseteq I \vee J\} \\
&\leq \Phi^c(x)
\end{aligned}$$

Therefore Φ^c is an L-fuzzy ideal of the poset Q. □

Lemma 5.3.8. *Let Φ be an L-fuzzy ideal of $\mathscr{I}(Q)$ with sup property and $\alpha \in L$. Then $(\Phi^c)_\alpha = (\Phi_\alpha)^c$*

Proof. Let Φ be an L-fuzzy ideal of $\mathscr{I}(Q)$ with sup property and $\alpha \in L$. Now

$$\begin{aligned}
x \in (\Phi_\alpha)^c &\Rightarrow x \in I_0 \text{ for some } I_0 \in \Phi_\alpha \\
&\Rightarrow \Phi(I_0) \geq \alpha \text{ and } x \in I_0 \\
&\Rightarrow \Phi^c(x) = \sup\{\Phi(I) : x \in I\} \geq \Phi(I_0) \geq \alpha \\
&\Rightarrow x \in (\Phi^c)_\alpha
\end{aligned}$$

Thus we have $(\Phi_\alpha)^c \subseteq (\Phi^c)_\alpha$.

To show the other inclusion, let $x \in (\Phi^c)_\alpha$. Then $\Phi^c(x) = \sup\{\Phi(I) : x \in I\} \geq \alpha$. Since Φ is an L-fuzzy subset of $\mathscr{I}(Q)$ with sup property, there exists $I_0 \in \mathscr{I}(Q)$ with $x \in I_0$ such that $\Phi(I_0) = \sup\{\Phi(I) : x \in I\} \geq \alpha$. This implies that $I_0 \in \Phi_\alpha$ and $x \in I_0$ and so $x \in (\Phi_\alpha)^c$. Therefore $(\Phi^c)_\alpha \subseteq (\Phi_\alpha)^c$ and hence $(\Phi^c)_\alpha = (\Phi_\alpha)^c$. □

Theorem 5.3.9. *Let Q be a finite poset and let Φ be an L-fuzzy semi-prime ideal of $\mathscr{I}(Q)$ with sup property. Then Φ^c is an L-fuzzy semi-prime ideal of a poset Q.*

Proof. Clearly Φ^c is an L-fuzzy ideal of Q ,by Lemma 5.3.7. Now we show that Φ^c is an L-fuzzy semi-prime ideal Q. Let $a,b,c \in Q$ and $z \in \{a,(b,c)^u\}^l$. Now put

$$\alpha = \inf\{\Phi^c(x) \wedge \Phi^c(y) : x \in (a,b)^l, y \in (a,c)^l\}.$$

Then it is clear that

$$\Phi^c(x) \geq \alpha \text{ for all } x \in (a,b)^l \text{ and } \Phi^c(y) \geq \alpha \text{ for all } y \in (a,c)^l.$$

This implies that $(a,b)^l \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c$ and $(a,c)^l \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c$. Since Q is finite and $(\Phi_\alpha)^c = \bigcup\{I : I \in \Phi_\alpha\}$, there exist $I_1, I_2, \cdots, I_n$ and $J_1, J_2, \cdots, J_m$ in Φ_α such that

$$(a] \cap (b] = (a,b)^l \subseteq \bigcup_{i=1}^{n} I_i \subseteq \bigvee_{i=1}^{n} I_i \in \Phi_\alpha$$

and

$$(a] \cap (c] = (a,c)^l \subseteq \bigcup_{j=1}^{m} J_j \subseteq \bigvee_{j=1}^{m} J_j \in \Phi_\alpha.$$

Since Φ_α a semi-prime ideal of $\mathscr{I}(Q)$, we have $(a] \cap ((b] \vee (c]) \in \Phi_\alpha$. Now

$$\begin{aligned} z \in \{a,(b,c)^u\}^l &\Rightarrow z \in (a] \cap ((b] \vee (c]) \in \Phi_\alpha \\ &= z \in (\Phi_\alpha)^c = (\Phi^c)_\alpha \\ &= \Phi^c(z) \geq \alpha \\ &= \Phi^c(z) \geq \inf\{\Phi^c(x) \wedge \Phi^c(y) : x \in (a,b)^l, y \in (a,c)^l\} \end{aligned}$$

Hence Φ^c is an L-fuzzy semi-prime ideal of a poset Q. □

Remark 5.3.1. *The finiteness conditions in the statement of the Theorem 5.3.9 is necessary. For example consider the infinite poset depicted in the Fig. 5.3 and its ideal lattice $\mathscr{I}(Q)$ in Fig. 5.4 below.*

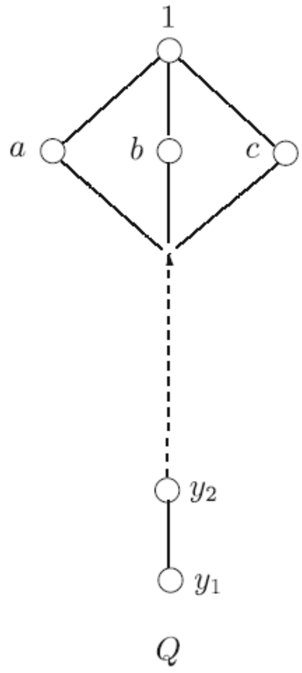

Fig. 5.3

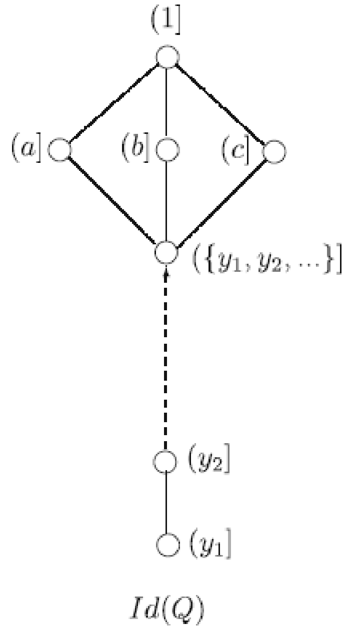

Fig. 5.4

Let $L = [0,1]$. Then consider the L- fuzzy subset Φ of $\mathscr{I}(Q)$ given by:

$$\Phi(I) = \begin{cases} 1 & if\ I = (y_1] \\ 1 - \dfrac{\frac{1}{2}i}{1+i} & if\ I = (y_i]\ i = 2,3,\cdots, \\ 0 & if \text{ otherwise} \end{cases}$$

for all $I \in \mathscr{I}(Q)$. which is an L-fuzzy semi-prime ideal of $\mathscr{I}(Q)$. Then its contraction, Φ^c, is given by:

$$\Phi^c(x) = \begin{cases} 1 & if\ x = y_1 \\ 1 - \dfrac{\frac{1}{2}i}{1+i} & if\ x = y_i\ i = 2,3,\cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $x \in Q$. But Φ^c is not an L-fuzzy semi-prime ideal of a poset Q as $a \in a^l \cap (a,b)^{ul} = \{a,(b,c)^u\}^l$ but $\Phi^c(a) = 0 \ngeq \frac{1}{2} = \inf\{\Phi^c(x) \wedge \Phi^c(y) : x \in (a,b)^l, y \in (a,c)^l\}$.

However, if Q is a meet semi-lattice, then we have

Theorem 5.3.10. *Let Q be a meet semi-lattice and Φ be an L-fuzzy semi-prime ideal of $\mathscr{I}(Q)$ with sup property. Then Φ^c is an L-fuzzy semi-prime ideal of Q.*

Proof. Let Φ be an L-fuzzy semi-prime ideal of $\mathscr{I}(Q)$. Let $a,b,c \in Q$ and $z \in \{a,(b,c)^u\}^l$. Since Q is a meet semi-lattice it is clear that

$$\inf\{\Phi^c(x) \wedge \Phi^c(y) : x \in (a,b)^l, y \in (a,c)^l\} = \Phi^c(a \wedge b) \wedge \Phi^c(a \wedge c).$$

Now put $\alpha = \Phi^c(a \wedge b) \wedge \Phi^c(a \wedge c)$. Then we have

$$\Phi^c(a \wedge b) \geq \alpha \text{ and } \Phi^c(a \wedge c) \geq \alpha\ .$$

This implies that

$$a \wedge b \in (\Phi^c)_\alpha = (\Phi_\alpha)^c \text{ and } a \wedge c \in (\Phi^c)_\alpha = (\Phi_\alpha)^c$$

Thus there exist $I,J \in \Phi_\alpha$ such that $a \wedge b \in I$ and $a \wedge c \in J$. This implies that

$$(a] \cap (b] = (a \wedge b] \subseteq I \in \Phi_\alpha \text{ and } (a] \cap (c] = (a \wedge c] \subseteq J \in \Phi_\alpha.$$

Since Φ_α a semi-prime ideal of $\mathscr{I}(Q)$, $(a] \cap ((b] \vee (c]) \in \Phi_\alpha$. Now

$$\begin{aligned} z \in \{a, (b,c)^u\}^l &\Rightarrow z \in (a] \cap ((b] \vee (c]) \in \Phi_\alpha \\ &= z \in (\Phi_\alpha)^c = (\Phi^c)_\alpha \\ &= \Phi^c(z) \geq \alpha = \Phi^c(a \wedge b) \wedge \Phi^c(a \wedge c) \end{aligned}$$

Therefore Φ^c is an L-fuzzy semi-prime ideal of a meet semi-lattice Q. □

In the next two theorems, we investigate the relationships between L-fuzzy prime ideal of a poset Q and L-fuzzy prime ideal of the lattice $\mathscr{I}(Q)$.

Theorem 5.3.11. *Let μ be an L-fuzzy prime ideal of a poset Q. Then μ^e is an L-fuzzy prime ideal of the lattice $\mathscr{I}(Q)$.*

Proof. Let μ be an L-fuzzy prime ideal of a poset Q. Now we show that μ^e is an L-fuzzy prime ideal of the lattice $\mathscr{I}(Q)$. Since μ is proper, there exists $a \in Q$ such that $\mu(a) \neq 1$. As $(a] \in \mathscr{I}(Q)$ and $\mu^e((a]) = \inf\{\mu(x) : x \in (a]\} = \mu(a) \neq 1$, μ^e is proper. Again let $I, J \in \mathscr{I}(Q)$. We need to show that

$$\mu^e(I \cap J) = \mu^e(I) \text{ or } \mu^e(J).$$

Indeed, on the contrary if both $\mu^e(I \cap J) \neq \mu^e(I)$ and $\mu^e(I \cap J) \neq \mu^e(J)$, then there exist $a \in I$ and $b \in J$ such that $\mu^e(I \cap J) \nleq \mu(a)$ and $\mu^e(I \cap J) \nleq \mu(b)$. Since $(a,b)^l \subseteq I \cap J$ and hence $\mu^e(I \cap J) \leq \inf\{\mu(x) : x \in (a,b)^l\}$, we have

$$\inf\{\mu(x) : x \in (a,b)^l\} \nleq \mu(a) \text{ and } \inf\{\mu(x) : x \in (a,b)^l\} \nleq \mu(b),$$

which contradicts the fact that μ is an L-fuzzy prime ideal of a poset Q. Therefore μ^e is an L-fuzzy prime ideal of the lattice $\mathscr{I}(Q)$. □

Theorem 5.3.12. *Let Q be a finite poset and Φ be an L-fuzzy prime ideal of $\mathscr{I}(Q)$ with sup property. Then Φ^c is an L-fuzzy prime ideal of Q.*

Proof. Suppose that Φ is an L-fuzzy prime ideal of $\mathscr{I}(Q)$ with sup property, where Q is a finite poset. Then, since Φ is proper, there exists $I_0 \in \mathscr{I}(Q)$ such that $\Phi(I_0) \neq 1$. Since Q is finite and hence satisfying ACC, there exists $a \in Q$ such that $I_0 = (a]$. Let $X = \{I \in \mathscr{I}(Q) : a \in I\}$. Then it is clear that X is a finite set and $(a] \in X$ and hence $\Phi^c(a) = \sup\{\Phi(I) : I \in X\} = \Phi((a]) \neq 1$ Thus Φ^c is a proper L-fuzzy ideal of Q.

Let $a, b \in Q$ and put $\alpha = \inf\{\Phi^c(x) : x \in (a,b)^l\}$. This implies that

$$\Phi^c(x) \geq \alpha \text{ for all } x \in (a,b)^l.$$

Thus we have $(a,b)^l \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c$. Since Q is finite and $(\Phi_\alpha)^c = \bigcup\{I : I \in \Phi_\alpha\}$, there exist $I_1, I_2, \cdots, I_n$ such that

$$(a] \cap (b] = (a,b)^l \subseteq \bigcup_{i=1}^n I_i \subseteq \bigvee_{i=1}^n I_i \in \Phi_\alpha$$

Since Φ_α a prime ideal of $\mathscr{I}(Q)$, we have either $(a] \in \Phi_\alpha$ or $(b] \in \Phi_\alpha$. Consequently $a \in (\Phi_\alpha)^c = (\Phi^c)_\alpha$ or $b \in (\Phi_\alpha)^c = (\Phi^c)_\alpha$ and therefore

$$\Phi^c(a) \geq \alpha = \inf\{\Phi^c(x) : x \in (a,b)^l\} \geq \Phi^c(a)$$

or

$$\Phi^c(b) \geq \alpha = \inf\{\Phi^c(x) : x \in (a,b)^l\} \geq \Phi^c(b),$$

that is, $\inf\{\Phi^c(x) : x \in (a,b)^l\} = \Phi^c(a)$ or $\Phi^c(b)$. Hence Φ^c is an L-fuzzy prime ideal of a poset Q. □

Remark 5.3.2. *The statement of Theorem 5.3.12 is not necessarily true if the poset Q is not finite. Consider the infinite poset Q depicted in Fig.5.3 and its ideal lattice Id(Q) in Fig. 5.4 on page 149. Observe that the L- fuzzy subset Φ of $\mathscr{I}(Q)$ into $L = [0,1]$ defined by:*

$$\Phi(I) = \begin{cases} 1 & if\ I = (y_1] \\ 1 - \dfrac{\frac{1}{3}i}{1+i} & if\ I = (y_i],\ i = 2,3,\cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $I \in \mathscr{I}(Q)$. is an L-fuzzy prime ideal of $\mathscr{I}(Q)$.

Also see that Φ^c is given by:

$$\Phi^c(x) = \begin{cases} 1 & if\ x = y_1 \\ 1 - \dfrac{\frac{1}{3}i}{1+i} & if\ x = y_i,\ i = 2,3,\cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $x \in Q$. is an L-fuzzy ideal of Q but not L-fuzzy prime ideal , as $\inf\{\Phi^c(x) : x \in (a,b)^l\} = \frac{2}{3}$ *and neither equal to* $\Phi^c(a)$ nor $\Phi^c(b)$.

However, if the poset is a meet semilattice, then we have

Theorem 5.3.13. *Let Q be a meet semi-lattice and Φ be an L-fuzzy prime ideal of $\mathscr{I}(Q)$ with sup property. Then Φ^c is an L-fuzzy prime ideal of Q.*

Proof. Suppose that Φ is an L-fuzzy prime ideal of $\mathscr{I}(Q)$ with sup property where Q is a meet semi-lattice. Now we claim that Φ^c is an L-fuzzy prime ideal of Q. Let$a,b \in q$ and put $\alpha = \inf\{\Phi^c(x) : x \in (a,b)^l\}$. This implies that

$$a \wedge b \in (a,b)^l \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c.$$

Then there exists $I \in (\Phi_\alpha)^c$ such that $a \wedge b \in I$. This implies that

$$(a] \cap (b] = (a \wedge b] \subseteq I \in \Phi_\alpha$$

and hence $(a] \cap (b] \in \Phi_\alpha$. Now, by primeness of Φ_α, we must have $(a] \in \Phi_\alpha$ or $(b] \in \Phi_\alpha$ and so $a \in (\Phi_\alpha)^c = (\Phi^c)_\alpha$ or $b \in (\Phi_\alpha)^c = (\Phi^c)_\alpha$. Therefore

$$\Phi^c(a) \geq \alpha = \Phi^c(a \wedge b) \geq \Phi^c(a) \text{ or } \Phi^c(b) \geq \alpha = \Phi^c(a \wedge b) \geq \Phi^c(b),$$

i.e. $\Phi^c(a \wedge b) = \Phi^c(a)$ or $\Phi^c(b)$. Hence Φ^c is an L-fuzzy prime ideal of Q. □

Lemma 5.3.14. *Let μ be an L-fuzzy ideal of a poset Q. Then $\mu^{ec} = \mu$.*

Proof. Let μ be an L-fuzzy ideal of a poset Q. Now we claim that $\mu^{ec} = \mu$. Let $x \in Q$. put $\mathscr{S}_x = \{I \in \mathscr{I}(Q) : x \in I\}$. As $(x] \in \mathscr{S}_x$, $\mathscr{S}_x$ is non empty and it is clear that $(x] \subseteq I$ for all $I \in \mathscr{S}_x$and hence

$$\begin{aligned}(\mu^{ec})(x) &= (\mu^e)^c(x)\\ &= \sup\{\mu^e(I) : x \in I, I \in \mathscr{I}(Q)\}\\ &= \mu^e((x])\\ &= \inf\{\mu(y) : y \in (x]\} = \mu(x).\end{aligned}$$

Therefore $\mu^{ec} = \mu$. □

5.4 Separation Theorems

In this section, extend and prove an analogue of Stone's Theorem for finite posets which has been studied by V. S. Kharat and K. A. Mokbel[35] using L-fuzzy semi-prime ideals. Some counter examples are also given. Now we obtain an L-fuzzy filter μ_F in a poset Q with the help of an L-fuzzy filter Φ_F in the lattice $\mathscr{I}(Q)$ of all L-fuzzy ideals of Q and study the L-fuzzy semi-primeness connection between them.

Definition 5.4.1. *Let Q be a poset with 1 and Φ_F be an L-fuzzy filter of $\mathscr{I}(Q)$. Define an L-fuzzy subset μ_F of Q by:*

$$\mu_F(x) = \Phi_F((x]) \text{ for all } x \in Q.$$

We have the following Lemma.

Lemma 5.4.1. *Let Q be a poset with 1. Let μ_F is the L-fuzzy subset of Q given as in the Definition 5.4.1 above. Then μ_F is an L-fuzzy filter of Q .*

Proof. Clearly $\mu_F(1) = 1$. Let $a, b \in Q$ and $x \in (a,b)^{lu}$. This implies that $(a] \cap (b] = (a,b)^l \subseteq x^l = (x]$. Now

$$\begin{aligned}\mu_F(a)\wedge\mu_F(b) &= \Phi_F((a])\wedge\Phi_F((b]) \\ &= \Phi_F((a]\cap(b]) \\ &\leq \Phi_F((x]) \\ &= \mu_F(x)\end{aligned}$$

Therefore μ_F is an L-fuzzy filter of Q. □

In the case of finite posets we have the following.

Lemma 5.4.2. *Let Q be a finite poset and Φ_F be an L-fuzzy filter of $\mathscr{I}(Q)$ and μ_F be an L-fuzzy filter given as in Definition 5.4.1 above. Then the following statements hold.*

1. *$\Phi_F((a]\vee(b]) = \inf\{\mu_F(x) : x\in(a,b)^u\}$ for any $a,b\in Q$.*

2. *if Φ_F is an L-fuzzy semi-prime filter, then μ_F is an L-fuzzy semi-prime filter.*

Proof. 1. Let $a,b\in Q$. Let$x\in(a,b)^u$. Then $a\leq x$ and $b\leq x$ and so $(a]\subseteq(x]$ and $(b]\subseteq(x]$. This implies that $(a]\vee(b]\subseteq(x]$ for all $x\in(a,b)^u$ and hence $(a]\vee(b]\subseteq\bigcap_{x\in(a,b)^u}(x]$. Again let $t\in\bigcap_{x\in(a,b)^u}(x]$. Then $t\leq x$ for all $x\in(a,b)^u$. This implies that $t\in(a,b)^{ul}\subseteq(a]\vee(b]$. Therefore $(a]\vee(b]=\bigcap_{x\in(a,b)^u}(x]$. Since $(a,b)^u$ is finite and Φ_F is an L-fuzzy filter of $\mathscr{I}(Q)$ we have

$$\begin{aligned}\Phi_F((a]\vee(b]) &= \Phi_F\Big(\bigcap_{x\in(a,b)^u}(x]\Big) \\ &= \inf\{\Phi_F((x]) : x\in(a,b)^u\} \\ &= \inf\{\mu_F(x) : x\in(a,b)^u\}\end{aligned}$$

Hence (1) holds.

2. Let $a,b,c \in Q$ and $z \in \{a,(b,c)^l\}^u$. Then it is clear that $(a] \subseteq (z]$ and $(b] \cap (c] \subseteq (z]$. Thus we have $(a] \vee ((b] \cap (c]) \subseteq (z]$.

$$\begin{aligned}
& \inf\{\mu_F(x) \wedge \mu_F(y) : x \in (a,b)^u, y \in (a,c)^u\} \\
= & \inf\{\mu_F(x) : x \in (a,b)^u\} \wedge \inf\{\mu_F(y) : y \in (a,c)^u\} \\
= & \Phi_F((a] \vee (b]) \wedge \Phi_F((a] \vee (c]) \cdots \text{(by 1)} \\
= & \Phi_F((a] \vee ((b] \cap (c])) \\
\leq & \Phi_F((z]) = \mu_F(z)
\end{aligned}$$

Hence (2) holds.

□

Remark 5.4.1. *We give an example to show that the assertion of Lemma 5.4.2 is not necessarily true if we drop the finiteness condition. Consider the dual of the infinite poset Q that is depicted in Fig 5.3, say Q^d and its ideal lattice $\mathscr{I}(Q^d)$ which is the dual of the ideal lattice $\mathscr{I}(Q)$ depicted in Fig 5.4. Consider the L- fuzzy filter Φ_F of $\mathscr{I}(Q^d)$ into $L=[0,1]$ which is given by:*

$$\Phi_F(I) = \begin{cases} 1 & if\ I = (y_1] \\ 1 - \dfrac{\frac{1}{3}i}{1+i} & if\ I = (y_i]\ for\ i = 2,3,\cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $I \in \mathscr{I}(Q^d)$. Observe that the L-fuzzy subset μ_F of Q^d into $L=[0,1]$ which is given by:

$$\mu_F(x) = \begin{cases} 1 & if\ x = y_1 \\ 1 - \dfrac{\frac{1}{3}i}{1+i} & if\ x = y_i\ for\ i = 2,3,\cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $x \in Q$ is an L-fuzzy filter of Q^d. But

$$\Phi_F((a]\vee(b]) = \Phi_F((\{y_1,y_2,\cdots\}]) = 0 \neq \tfrac{2}{3} = \inf\{\mu_F(z) : z \in (a,b)^u\}.$$

Moreover, Φ_F is an L-fuzzy semi-prime filter of $\mathscr{I}(Q^d)$. But μ_F is not an L-fuzzy semi-prime filter, as $a \in a^u = \{a,(b,c)^l\}^u$ and

$$\mu_F(a) = 0 \ngeq \tfrac{2}{3} = \inf\{\mu_F(x)\wedge\mu_F(y) : x \in (a,b)^u, y \in (a,c)^u\}.$$

However, in the case of join semi-lattices we have

Corollary 5.4.3. *Let Q be a join semi-lattice with 1, Φ_F be an L-fuzzy filter of $\mathscr{I}(Q)$ and μ_F be an L-fuzzy filter defined as in in Definition 5.4.1. Then the following statements hold.*

1. *$\Phi_F((a]\vee(b]) = \inf\{\mu_F(x) : x \in (a,b)^u\} = \mu_F(a\vee b)$ for any $a,b \in Q$.*

2. *if Φ_F is an L-fuzzy semi-prime filter, then μ_F is an L-fuzzy semi-prime filter.*

Proof. 1. Let $a,b \in Q$. Then it is clear that $(a]\vee(b] = (a\vee b]$ and hence

$$\Phi_F((a]\vee(b]) = \Phi_F((a\vee b]) = \mu_F(a\vee b) = \inf\{\mu_F(x) : x \in (a,b)^u\}$$

2. Let $a,b,c \in Q$ and $z \in \{a,(b,c)^l\}^u$. Then it is clear that $(a] \subseteq (z]$ and $(b]\cap(c] \subseteq (z]$. Thus we have $(a]\vee((b]\cap(c]) \subseteq (z]$. Now

$$\begin{aligned}
& \inf\{\mu_F(x)\wedge\mu_F(y) : x \in (a,b)^u, y \in (a,c)^u\} \\
= \ & \inf\{\mu_F(x) : x \in (a,b)^u\} \wedge \inf\{\mu_F(y) : y \in (a,c)^u\} \\
= \ & \mu_F((a\vee(b]) \wedge \mu_F((a\vee c]) \\
= \ & \Phi_F((a\vee(b]) \wedge \Phi_F((a\vee c]) \\
= \ & \Phi_F((a]\vee(b]) \wedge \Phi_F((a]\vee(c]) \\
= \ & \Phi_F((a]\vee((b]\cap(c])) \\
\leq \ & \Phi_F((z]) = \mu_F(z)
\end{aligned}$$

Hence (2) holds.

□

Definition 5.4.2. *Let σ be an l-L-fuzzy filter of a poset Q with 1, Define an L-fuzzy subset Ω of $\mathscr{I}(Q)$ as follows:*

$$\Omega(I) = \sup\{\sigma(x) : x \in I\} \text{ for all`} I \in \mathscr{I}(Q).$$

We establish the following result.

Lemma 5.4.4. *Let σ be an l-L-fuzzy filter of a poset Q with 1 and Ω be an L-fuzzy subset of $\mathscr{I}(Q)$ as given in Definition 5.4.2 above. Then Ω is an L-fuzzy filter of $\mathscr{I}(Q)$.*

Proof. Let σ be an l-L-fuzzy filter of a poset Q. Then clearly $\Omega((1]) = 1$.
Let $I, J \in \mathscr{I}(Q)$. Then

$$\begin{aligned} \Omega(I) \wedge \Omega(J) &= \sup\{\sigma(x) : x \in I\} \wedge \sup\{\sigma(y) : y \in J\} \\ &= \sup\{\sigma(x) \wedge \sigma(y) : x \in I, y \in J\} \\ &\leq \sup\{\sigma(x) \wedge \sigma(y) : (x,y)^l \subseteq I \cap J\} \end{aligned}$$

Since σ is an l-L-fuzzy filter of Q and $x, y \in Q$, there there exists $z \in (x,y)^l$ such that $\sigma(z) = \sigma(x) \wedge \sigma(y)$. Therefore

$$\Omega(I) \wedge \Omega(J) \leq \sup\{\sigma(z) : z \in I \cap J\} = \Omega(I \cap J)$$

Again

$$\begin{aligned} \Omega(I \cap J) &= \sup\{\sigma(x) : x \in I \cap J\} \\ &\leq \sup\{\sigma(x) : x \in I\} \\ &= \Omega(I) \end{aligned}$$

Therefore $\Omega(I \cap J) \subseteq \Omega(I)$. Similarly we can show that $\Omega(I \cap J) \subseteq \Omega(J)$ and hence $\Omega(I \cap J) \subseteq \Omega(I) \wedge \Omega(J)$. Therefore

$$\Omega(I \cap J) = \Omega(I) \wedge \Omega(J)$$

and hence Ω is an L-fuzzy filter of $\mathscr{I}(Q)$. $\square$

We prove the following Lemma, which is analogous to Rav's Separation Theorem for semi-prime ideals in Lattice Theory.[42]

Lemma 5.4.5. *Let α be a prime element in L, μ be an L-fuzzy semi-prime ideal and σ be an L-fuzzy filter of a lattice X such that $\mu \cap \sigma \subseteq \overline{\alpha}$. Then there exists an L-fuzzy semi-prime filter σ^F such that $\sigma \subseteq \sigma^F$ and $\mu \cap \sigma^F \subseteq \overline{\alpha}$.*

Proof. Let μ be an L-fuzzy semi-prime ideal and σ be an L-fuzzy filter of the lattice X such that $\mu \cap \sigma \subseteq \overline{\alpha}$. Now put

$$I = \{x \in X : \mu(x) \nleq \alpha\} \text{ and } K = \{x \in X : \sigma(x) \nleq \alpha\}.$$

Then, clearly I is a semi-prime ideal and K is a filter of X such that $I \cap K = \emptyset$.
Therefore by Rav's Separation Theorem for semi-prime ideals in Lattice, there exists a semi-prime filter F such that $K \subseteq F$ and $I \cap F = \emptyset$. Then, note that the L-fuzzy subset σ^F of X defined by:

$$(\sigma^F)(x) = \begin{cases} 1 & if\ x \in F \\ \alpha & if\ x \notin F \end{cases}$$

for all $x \in X$ is an l-fuzzy semi-prime filter. Now we claim that $\sigma \subseteq \sigma^F$ and $\mu \cap \sigma^F \subseteq \overline{\alpha}$. Let $x \in X$. Now if $x \in F$, then $\sigma(x) \leq 1 = \sigma^F(x)$ and, if $x \notin F$, then $x \notin K$, so that $\sigma(x) \leq \alpha = \alpha^F(x)$. Hence $\sigma \subseteq \alpha^F$. Again if $x \in F$, then $x \notin I$, so that $\mu(x) \leq \alpha$. Thus

$$(\mu \cap \sigma^F)(x) = \mu(x) \wedge \sigma^F(x) = \mu(x) \wedge 1 = \mu(x) \leq \alpha = \overline{\alpha}(x)$$

and if $x \notin F$, then

$$(\mu \cap \sigma^F)(x) \leq \mu(x) \wedge \alpha \leq \alpha = \overline{\alpha}(x).$$

Hence $\mu \cap \sigma^F \subseteq \overline{\alpha}$. Therefore the claim is true. $\square$

Now we extend an analogue of Stone's Theorem for finite posets which has been studied by V. S. Kharat and K. A. Mokbel[35] using L-fuzzy semi-prime ideals as given in Theorem 5.4.6 below.

Theorem 5.4.6. *Let Q be a finite poset and α be a prime element in L. Let μ be an L-fuzzy semi-prime ideal and σ be an l-L-fuzzy filter of Q for which $\mu \cap \sigma \subseteq \overline{\alpha}$. Then there exists an L-fuzzy semi-prime filter σ^F of Q such that $\sigma \subseteq \sigma^F$ and $\mu \cap \sigma^F \subseteq \overline{\alpha}$.*

Proof. Suppose that μ is an L-fuzzy semi-prime ideal and σ is an l-L-fuzzy filter of a finite poset Q such that $\mu \cap \sigma \subseteq \overline{\alpha}$, where α is a prime element in L. By Theorem 5.3.6, μ^e is an L-fuzzy semi-prime ideal of $\mathscr{I}(Q)$. Since σ is an l-L-fuzzy filter, the L-fuzzy subset Ω of $\mathscr{I}(Q)$ given in Definition 5.4.2 is an L-fuzzy filter of $\mathscr{I}(Q)$. (See Lemma 5.4.4. Now we claim that $\mu^e \cap \Omega \subseteq \overline{\alpha}$. Suppose not. Then there exists $I \in \mathscr{I}(Q)$ such that $\mu^e(I) \nleq \alpha$ and $\Omega(I) \nleq \alpha$. This implies that

$$\mu(x) \nleq \alpha \text{ for all } x \in I \text{ and } \sigma(x) \nleq \alpha \text{ for some } \ x \in I.$$

This contradicts the hypothesis $\mu \cap \sigma \subseteq \overline{\alpha}$. Hence the claim holds.

Now, since $\mathscr{I}(Q)$ is a lattice, by Lemma 5.4.5, there exists an L-fuzzy semi-prime filter, say Φ_F of $\mathscr{I}(Q)$ such that $\Omega \subseteq \Phi_F$ and $\mu^e \cap \Phi_F \subseteq \overline{\alpha}$. Consider the L-fuzzy subset μ_F of Q given in definition 5.4.1 which is an L-fuzzy semi-filter of Q. (See Lemma 5.4.1). Put $\sigma^F = \mu_F$ and observe that $\sigma \subseteq \sigma^F$; for, if $x \in Q$, then

$$\sigma(x) \leq \sup\{\sigma(y) : y \in (x]\} = \Omega((x]) \leq \Phi_F((x]) = \mu_F(x) = \sigma^F(x).$$

Further, we must have $\mu \cap \sigma^F \subseteq \overline{\alpha}$. Otherwise if $\mu \cap \sigma^F \nsubseteq \overline{\alpha}$, there exists $x \in Q$ such that $\mu(x) \nleq \alpha$ and $\sigma^F(x) = \mu_F(x) \nleq \alpha$. This implies that $\mu^e((x]) \nleq \alpha$ and $\Phi_F((x]) \nleq \alpha$, which is a contradiction to the fact that $\mu^e \cap \Phi_F \subseteq \overline{\alpha}$. This proves the theorem. □

Remark 5.4.2. *The statement of Theorem 5.4.6 is not necessarily true if we remove the finiteness conditions or if σ is not an an l-L-fuzzy filter. .*

1. *Consider the poset Q^d that is dual of the infinite poset Q depicted in Fig. 5.3. Define an L fuzzy subset $\mu : Q^d \longrightarrow [0,1]$ by:*

$$\mu(x) = \begin{cases} 0 & if\ x = y_i\ for\ i = 1,2,3,\cdots \\ 1 & if\ \text{otherwise} \end{cases}$$

for all $x \in Q^d$. Let σ be an L-fuzzy subset of Q^d given by:

$$\sigma(x) = \begin{cases} 1 & if\ x = y_1 \\ 1 - \dfrac{i}{1+2i} & if\ x = y_i\ for\ i = 2,3,\cdots \\ 0 & if\ \text{otherwise} \end{cases}$$

for all $x \in Q$.

Observe that 0 is a prime element in $L = [0,1]$, μ is an L-fuzzy semi-prime ideal and σ is an l-L-fuzzy filter of Q^d for which $\mu \cap \sigma \subseteq \overline{0}$. But there does not exist an L-fuzzy semi-prime filter σ^F for which $\sigma \subseteq \sigma^F$ and $\mu \cap \sigma^F \subseteq \overline{0}$.

2. *Consider the finite poset depicted in Fig. 5.5 below. Define L fuzzy subsets μ : $Q \longrightarrow [0,1]$ by:*

$$\mu(0) = 1,\ \mu(a) = 0.8,\ \mu(b) = \mu(c) = \mu(d) = 0.5$$
$$\mu(a') = \mu(b') = \mu(c') = \mu(d') = \mu(1) = 0.$$

and $\sigma : Q \longrightarrow [0,1]$ by:

$$\sigma(1) = 1,\ \sigma(a') = \sigma(b') = \sigma(c') = \sigma(d') = 0.8,$$
$$\sigma(a) = \sigma(b) = \sigma(c) = \sigma(d) = \sigma(0) = 0$$

Then μ is an L-fuzzy semi-prime ideal and σ is an L-fuzzy filter, which is not an l-L-fuzzy filter of Q. Observe that 0 is a prime element in $L = [0,1]$ such t $\mu \cap \sigma = \overline{0}$

but there does not exist an L-fuzzy semi-prime filter σ^F for which $\sigma \subseteq \sigma^F$ and $\mu \cap \sigma^F = \overline{0}$.

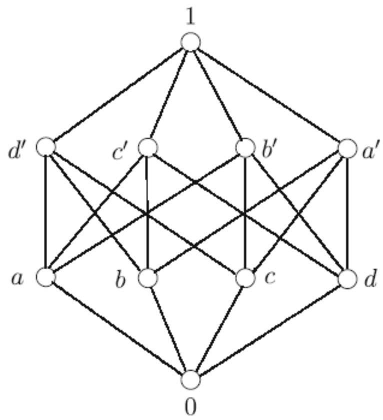

Figure 5.5

However, in the case of join semi-lattices we have

Theorem 5.4.7. *Let Q be a join semi-lattice and $\alpha \in L$ be a prime element. Let μ be an L-fuzzy semi-prime ideal and σ be an l-L-fuzzy filter of Q for which $\mu \cap \sigma \subseteq \overline{\alpha}$. Then there exists an L-fuzzy semi-prime filter σ^F of Q such that $\sigma \subseteq \sigma^F$ and $\mu \cap \sigma^F \leq \overline{\alpha}$.*

The following Lemma is from [35], which is immediately follows from the definition of distributive poset.

Lemma 5.4.8. *Let Q be a finite poset such that $\mathscr{I}(Q)$ is a distributive lattice, then Q is distributive.*

Remark 5.4.3. *The converse of Lemma 5.4.8 is not true in general. The poset Q depicted in Fig 5.5 is distributive but $\mathscr{I}(Q)$ is not so. For, let $I = (d']$, $J = \{0, a, b, c\}$ and $K = (c']$ and observe that $I \cap (J \vee K) = I \cap Q = I \neq J = J \vee \{0, a, b\} = (I \cap J) \vee (I \cap K)$.*

The following theorem is the extension of the celebrated theorem of M. H. Stone [45] on prime ideals of distributive lattices to L-fuzzy prime ideals.

Theorem 5.4.9 ([46])**.** *Let X be a distributive lattice and α be a prime element of L. If μ be an L-fuzzy ideal and σ be an L-fuzzy filter of X such that $\mu \cap \sigma \subseteq \overline{\alpha}$, then there exists a L-fuzzy prime ideal of μ_P such that $\mu \subseteq \mu_P$ and $\mu_P \cap \sigma \subseteq \overline{\alpha}$.*

Now, we extend Theorem 5.4.9 for a finite poset whose ideal lattice is distributive.

Theorem 5.4.10. *Let Q be a finite poset such that $\mathscr{I}(Q)$ is a distributive lattice and $\alpha \in L$ be a prime element. Let μ be an L-fuzzy ideal and σ be an l-L-fuzzy filter of Q for which $\mu \cap \sigma \subseteq \overline{\alpha}$. Then there exists an L-fuzzy prime ideal μ_I of Q such that $\mu \subseteq \mu_I$ and $\mu_I \cap \sigma \subseteq \overline{\alpha}$.*

Proof. Suppose that μ is an L-fuzzy ideal, σ is an l-L-fuzzy filter of a finite poset Q of which $\mathscr{I}(Q)$ is distributive such that $\mu \cap \sigma \subseteq \overline{\alpha}$, where α is a prime element in L. Observe that μ^e is an L-fuzzy ideal of $\mathscr{I}(Q)$ and also the L-fuzzy subset of $\mathscr{I}(Q)$ defined by:

$$\Omega(I) = \sup\{\sigma(x) : x \in I\} \text{ for all`} I \in \mathscr{I}(Q)$$

is an L-fuzzy filter of $\mathscr{I}(Q)$ by Lemma 5.4.4. Note that $\mu^e \cap \Omega \subseteq \overline{\alpha}$ as in the proof of Theorem 5.4.6. Since $\mathscr{I}(Q)$ is a distributive lattice, by Theorem 5.4.9, there exists an L-fuzzy prime ideal Φ_P of $\mathscr{I}(Q)$ such that $\mu^e \subseteq \Phi_P$ and $\Phi_P \cap \Omega \subseteq \overline{\alpha}$. By Theorem 5.3.12, $(\Phi_P)^c$ is an L-fuzzy prime ideal of Q, where $(\Phi_P)^c(x) = \sup\{\Phi_P(I) : x \in I\}$ for all $x \in Q$. Further $\mu \subseteq (\Phi_P)^c$ as for any $x \in Q$

$$\begin{aligned}
(\Phi_P)^c(x) &= \sup\{\Phi_P(I) : x \in I\} \\
&\geq \Phi_P((x]) \cdots \text{ as } x \in (x] \\
&= \inf\{\mu(y) : y \in (x]\} = \mu(x).
\end{aligned}$$

Also, we must have $(\Phi_P)^c \cap \sigma \subseteq \overline{\alpha}$. Otherwise, if $(\Phi_P)^c \cap \sigma \nsubseteq \overline{\alpha}$, then there exists $x \in Q$ such that $(\Phi_P)^c(x) \nleq \alpha$ and $\sigma(x) \nleq \alpha$. This implies that $\mu^e((x]) \nleq \alpha$ and $\Phi_P((x]) \nleq \alpha$, which is a contradiction to the fact that $\mu^e \cap \Phi_P \subseteq \overline{\alpha}$. $\square$

Chapter 6

Conclusion and suggestions for further research work

6.1 Conclusion

In this study, we have introduced several generalizations of L-fuzzy ideals and filters of a lattice to an arbitrary poset whose truth values are in a complete lattice satisfying the infinite meet distributive law and we have given several characterizations of them.

Next we have studied the notions of L-fuzzy prime ideals, prime L-fuzzy ideals, maximal L-fuzzy ideals, L-fuzzy maximal ideals by choosing the L-fuzzy ideal and filter of a poset in the sense of Halaš as an L-fuzzy ideal and filter of a poset. We have also studied and have given sufficient conditions for the existence of L-fuzzy prime ideals and prime L-fuzzy ideals in the lattice of all L-fuzzy ideals of a poset.

Lastly, we have introduced and characterized the concept of an L-fuzzy semi-prime ideal and filter in a general poset. We have also obtained characterizations of an L-fuzzy semi-prime ideal to be L-fuzzy prime ideal. We have also established the relations between the L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of a poset and the L-fuzzy semi-prime ideal (respectively, L-fuzzy prime) of the lattice of all ideals of a poset. Moreover we have extended and proved an analogue of Stone's Theorem for finite posets which has been studied by V. S. Kharat and K. A. Mokbel[35], using L-fuzzy semi-prime ideals of a poset. Further,the fuzzy version of a generalization of Stone's Separation Theorem for posets has obtained in respect of prime *L*-fuzzy ideals of a poset.

6.2 Suggestions for further research work

We ought to mention here our further direction of research as follows.

• The space of prime *L*-fuzzy ideal and maximal *L*-fuzzy ideals of a poset and their topological properties like compactness, connectedness and separation axioms.

• *L*-fuzzy Baer ideal for posets and its characterization.

• The concept of a radical and primary *L*-fuzzy ideal of a poset as a generalization of prime *L*-fuzzy ideals of a poset.

• *L*-fuzzy congruences of a poset and its characterization in terms of *L*-fuzzy ideals of a poset and their correspondence between the lattice of *L*-fuzzy ideals into the lattice of *L*-fuzzy congruences of a boolean poset.

• *L*-fuzzy annihilator ideal of a poset.

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