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A PROJECT ON DISTRIBUTIVE LATTICES AND CONGRUENCES IN LATTICES

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ON

DISTRIBUTIVE LATTICES AND CONGRUENCES IN LATTICES

BY

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August, 2020

Bahir Dar, Ethiopia

BAHIR DAR UNIVERSITY
COLLEGE OF SCIENCE
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DISTRIBUTIVE LATTICES AND CONGRUENCES IN LATTICES

A Project Submitted to the Department of Mathematics for the Partial Fulfillment of MSc. Degree in Mathematics

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BAHIR DAR UNIVERSITY

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MATHEMATICS DEPARTMENT

Approval of the project for defense

I hereby certify that I have supervised, read and evaluate this project entitled “Distributive Lattices and Congruences in Lattices” by Baye worku prepared under my guidance. I recommend that the project is submitted for oral defense.

Advisors name: ______________________ Sign. __________ Date__________

BAHIR DAR UNIVERSITY

COLLEGE OF SCIENCE

MATHEMATICS DEPARTMENT

Approval of the project for defense result

We hereby certify that we have examined this project entitled “Distributive lattices and Congruences in lattices” by Baye Worku. We recommend that Mr. Baye Worku is approved for the degree of “Master of science in Mathematics”.

Board of Examiners

Advisor name:_________________________________Sign _________ Date _________

Internal examiner name:_________________________Sign _________ Date _________

External examiner name: _________________________Sign _________ Date _________

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ABSTRACT

This project aims to develop a better understanding of Distributive Lattices and congruence in Lattices. We present the definition of Distributive Lattice and Congruences in Lattice,

Finaly state and proof important properties that will be used in developing further theory.

Contents page

SYMBOLS

1. $\in$ *Element*
2. $=$ *Is equal to*
3. $\leq$ *Less than or equal to*
4. $\subseteq$ *Subset*
5. $\rightarrow$ *Mapping*
6. $|$ *Divides*
7. $\vee$, $\wedge$ *Join and Meet respectively*
8. $\Rightarrow$ *Imply*
9. $\Leftrightarrow$ *If and only if*
10. $\uparrow, \downarrow$ *Up-set and Down-set respectively*
11. $\cup, \cap$ *Union and Intersection respectively*
12. φ *Phi*
13. $\leftrightarrow$ *Corresponding*
14. $\equiv$ *Congruence*
15. $\lhd$ *Normal subgroup*
16. $\sim$ *Equivalence relation*

Chapter One

1 Introduction and Preliminaries

1.1 Introduction

The origin of the lattice concept dates back to the nineteenth-century attempts to formalise logic [3]. In the first half of the nineteenth-century, George Boole discovered Boolean Algebras. While investigating the axiomatics of Boolean algebras, Charles S.pierce and Ernst Schroder introduce the concept of lattice in the late the nineteenth-century. Lattices especially distributive lattices and Boolean algebras arise naturally in logic, and thus some of the elementary theory of lattice had been worked out earlier by Ernst Schroder in his book Die Algebra de logik. Richard Dedekind also independently discovered Lattices. In the early 1890's, Richard Dedekind was working on a revised and enlarged edition of Dirichlet's Vorlesungen iiber zahlentheorie, and asked himself the following question: Given three subgroups A, B, C of an abelian group G, how many different subgroups can you get by taking intersections and sums, e.g., $A+B, (A+B)\cap C$,etc. the answer is 28. In looking at this and related questions, Dedekind was led to develop the basic theory of lattices, which he called dualgruppen. The publication of two fundamental papers iiber zerlegungen von zahlen durch ihre grobten gemeinsamen Teiler (1897) and iiber die von drei moduln erzeuget Dualgruppe (1900) on the subject of Richard Dedekind brought the theory to life well over one hundred years ago. These two papers are classical and have inspired many later mathematicians

Richard Dedekind defined modular lattices which are weakened form of distributive lattices [11]. He recognized the connection between modern algebra and lattice theory which provided the impetus for the development of lattice theory as a subject. Later Jonsson, Kurosh, Maclev, Ore, von Neumann, Tarski, and Garrett Birkhoff contributed prominently to the development of lattice theory.it was Garrett Birkhoff's work in the mid-thirties that started the general development of the subject [1]. In a series of papers he demonstrated the importance of a lattice theory and showed that it provides a unifying framework for unrelated development of in many mathematical disciplines. After that Valere Glivenko, Karl Menger, John Von Neumann, Oystein Ore and others developed this field. In the development of lattice theory, distributive lattices have played a vital role. These lattices have provided the motivation for many results in general lattice theory. Many conditions on lattices are weakened form of distributivity. In many applications the condition of distributivity is imposed on lattices arising in various areas of mathematics, especially algebras.

The important current research on lattice theory has been initiated by *G.Birkhoff, R.P. Dilworth* and *G.Gratzer*. They are primarily concerned with the systematic development of results which lie at the heart of the subject.

In this project, we discussed the notion of distributive lattices and congruence in lattice. We will study some properties of distributive lattice and congruence in lattice and we will see some definitions, theorems, lemmas about distributive lattice and congruence in a lattice. We discuss a set of equivalent class of distributive lattices which leads to the characterization of distributive lattice. We will also discuss a set of equivalent conditions for every equivalence relation to become congruence relation, which leads to a characterization of congruence in lattice.

In part 1.2, we discuss some preliminary results of distributive lattices and congruence in lattice. We give the definition of distributive lattices in section 2.1. In section 2.2, set of equivalent conditions will be established to characterize distributive lattices. In section3.1 we give the definition of congruence. In section 3.2 definition of congruence relation in lattice. In section3.3 will be established to characterize congruence in lattice. Finally, we give the conclusion.

1.2 Preliminary

Definition 1.2.1:-[8] For any two sets X and Y, a subset R of $X \times Y$ is called a **relation** on X to Y (or, a relation "between" X and Y). If $(x, y) \in R$ and we usually write as xRy and read: "x stands in the relation R to y''.

Example1.2.2 If X={1,3,5} and Y={0,2,4},then the set R={(1,2),(1,4),(3,4)} consists of all pairs (x , y) with $x \leq y$, $x \in X$ and $y \in Y$,so is the relation '' $\leq$'' between X and Y.

Example 1.2.3 Set inclusion S $\subseteq$ T is a binary relation on a power set $P(U)$, for any set U.

Definition 1.2.4:-[8] **Binary operation** $*$ on a set X is a function mapping $X \times X \rightarrow X$, for each $(a, b) \in X \times X$, we will denote the element $*\big($(a, b)$\big)$ of X by $a * b$.

Definition 1.2.5:-[3] A **binary relation** ϑ defined on a non-empty set P is called an **ordering** relation or **partial ordering relation** on a set P if, for all $a, b, c \in P$, it satisfies the following axioms:

I. $(a , a) \in \vartheta$ **(reflexivity)**
II. $(a , b) \in \vartheta$ and $(b, a) \in \vartheta$ imply $a = b$ **(anti-symmetry)**
III. $(a , b) \in \vartheta$ and $(b , c) \in \vartheta$ imply $(a , c) \in \vartheta$ (**transitivity**)

- A non-empty set P equipped with an ordering relation is called **partial order set** or simply **poset** in short. We write $(P;\ \vartheta)$ when we want to specify the poset.
- If ϑ is an **ordering relation**, it will usually denoted by $\leq$ and we write $a \leq b$ or $b \leq a$ instead of $(a, b) \in \leq$ or $(b, a) \in \leq$. Also $a \geq b$ will mean $b \leq a$.
- If P is a **poset** and if for every $a, b \in P$ either $a \leq b$ or $b \leq a$, then $(P; \leq)$ is called **totally ordered set** or a **chain**.

Example 1.2.6:-Let $\mathbb{R}$ be the set of real numbers, and let $x \leq y$ have its usual meaning for real numbers, then $(\mathbb{R}; \leq)$ is a poset.

Example 1.2.7:-Let $\mathbb{N}$ be the set of natural numbers, and let $x|y$ mean that x divides y, then $(\mathbb{N};|)$ is a poset.

Example 1.2.8:-The set $P(X)$ of all subset of a non-empty set X with relation $\subseteq$ of set inclusion $(P(X);\subseteq)$ is a poset.

Definition 1.2.9:-[11] If $(P;\leq)$ is a poset, Then $\geq$ can also be regarded as a **binary relation** on P defined by $a \geq b$ iff $b \leq a$ and satisfies axioms (1) - (3). Then $(P;\geq)$ is also a poset and is called the **dual** of a poset $(P;\leq)$. More precisely if ϑ is a statement about a posets and if in ϑ we replace all occurrences of $\leq$ by $\geq$, we get the dual of ϑ.

Definition 1.2.10: [1] A partial order set P is complete if for every subset H of P both $sup\, H$ and $inf\ H$ exist (in P).

Definition 1.2.11:-An algebra $(L;\vee,\wedge)$ of type $(2,2)$ is called a **lattice** if, for any $a,b,c \in L$, it satisfies the following lattice axioms [9].

1) $a \wedge b = b \wedge a$ (commutative law of "$\wedge$")
2) $a \vee b = b \vee a$ (commutative law of "$\vee$")
3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (associative law of "$\wedge$")
4) $(a \vee b) \vee c = a \vee (b \vee c)$ (associative law of "$\vee$")
5) $a \wedge (a \vee b) = a$ (absorption law of "$\wedge$")
6) $a \vee (a \wedge b) = a$ (absorption law of "$\vee$")

Definition 1.2.12:-[11] The dual of any statement in a lattice $(L;\wedge,\vee)$ is defined to be the statement that is obtained by interchanging $\wedge$ and $\vee$.

For example; the dual of $a \wedge (b \vee a) = a \vee a$ is given by $a \vee (b \wedge a) = a \wedge a$.

Theorem 1.2.13:-[6] **Idempotent law,** Let $(L;\vee,\wedge)$ be any lattice. Then for every $a \in L$, the following properties hold:

1) $a \wedge a = a$
2) $a \vee a = a$

Example 1.2.14:- Let L be a collection of sets closed under intersection and union. Then $(L;\cup,\cap)$ form a lattice.

Definition 1.2.15:-We define a partial order $\leq$ on a lattice L by $a \leq b$ if $a \vee b = b$. Analogously we can define $a \leq b$ if $a \wedge b = a$.

An alternative way to define a lattice as a poset is in the following way:

Definition 1.2.16:- [3] A lattice L is a poset $(L;\leq)$ where any two of whose elements are greatest lower bound (glb) and least upper bound (lub) exist in L. We shall use the notations;

$$a \wedge b = inf\ (a, b)$$

$$a \vee b = sup\ (a, b)$$

and call $\wedge$ the meet and $\vee$ the join. In lattices, they are both binary operations, which means that they can be applied to a pair of elements a, b of L to yield again an element of L.Thus $\wedge$ is a map of $L \times L$ into L and so is $\vee$.

Note:

- For example, any totally ordered set $(C;\leq)$ is a lattice. Since $inf\ (a, b) = min\,\{a, b\}$ and $sup\ (a, b) = max\,\{a, b\}$ for any $a, b \in C$.
- To show that a partial order set is not a lattice, it suffices to find a pair that doesn't have a glb or lub.

Definition 1.2.17:- [11] Let $(L;\vee,\wedge)$ be a lattice and has an element 0 and 1 such that for any $x \in L$ it satisfies the inequality $0 \leq x$ and $x \leq 1$. Then 0 and 1 are the **least** and **greatest** element of a lattice respectively, and are called **bounded element**. Such types of lattices are called **bounded lattices**, denoted by $(L;\vee,\wedge,0,1)$.

Example 1.2.18:-Let $L = \{a_1, a_2, a_3, \dots, a_n\}$ be a finite lattice. Then $a_1 \wedge a_2 \wedge \dots \wedge a_n$ and $a_1 \vee a_2 \vee \dots \vee a_n$ are the least and greatest elements of L, respectively. Hence L is a bounded lattice.

Definition 1.2.19:-[11] A lattice L is said to be complete if $\bigwedge H$ and $\bigvee H$ exist for any subset H of L.

Definition 1.2.20:-[11] Let $(L;\vee,\wedge)$ be a lattice. Then we define the following:

1) A non-empty subset H of L is said to be **sub-lattice** of L if for any $a, b \in H, a \wedge b$ and $a \vee b$ exist in H.
2) A non-empty subset I of L is called an **ideal** of L if it satisfies
   i) $a, b \in I \Rightarrow a \vee b \in I$.
   ii) For any $x \in L$ and $a \in I, a \wedge x \in I$.

Note: - Let I be an ideal of a lattice L. Then $x \in L, a \in I$ and $x \leq a \Rightarrow x \in I$.

Example 1.2.21:- The power set on some set ordered by set inclusion is an ideal.

Proposition 1.2.22:- Every ideal I of a lattice L is **sub-lattice** of L.

Notation1.2.23:-The set of all ideals I of a lattice L is denoted by $I(L)$.

Definition 1.2.24:- [1] Let I be an ideal of a lattice L. Then we can define the following:

i) I is said to be **principal ideal** if for any a in L, $I = \{x \in L: x \leq a\} = (a]$. In this case a is called a principal element of an ideal I. It is the smallest ideal that contains the element a.
ii) I is called **prime ideal** if it is proper and $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Definition 1.2.25:-[11] Let $(L_1;\vee,\wedge)$ and $(L_2;\vee,\wedge)$ be two lattices. A single-valued mapping φ of L_1 into (especially onto) L_2 is a **homomorphism** (homomorphic mapping) if, for every $a, b \in L_1$, then the following condition holds:

1) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$
2) $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$

A homomorphism, which is both one-to-one and onto is called an **isomorphism**.

Definition 1.2.26:-[3] Let L_1 and L_2 be two lattices and let $\varphi: L_1 \to L_2$ be a homomorphism. If L_2 has a least element 0_2, then the set of the elements $x \in L_1$ satisfying the equation $\varphi(x) = 0_2$ is called the **kernel** of the homomorphism φ, denoted by Ker φ. That is Ker $\varphi = \{x \in L_1: \varphi(x) = 0_2\}$.

Theorem 1.2.27:- [3] Let L_1 and L_2 be two lattices and let φ**:** $L_1 \to L_2$ be a homomorphism. Then the Ker φ is an ideal.

Definition 1.2.28:-[11] Let $(L;\vee,\wedge,0,1)$ be a bounded lattice. A complement of an element a is an element x such that $a \wedge x = 0$ and $a \vee x = 1$. A bounded lattice L in which every element has at least one complement is called a **complemented lattice**, and is denoted by $(L;\vee,\wedge,',0,1)$.

Definition 1.2.29:-[10] An element a of a lattice L is said to be **join- irreducible** iff a is not a zero element and whenever $a = b \vee c$, then either $a = b$ or $a = c$. Dually an element a in a lattice L is said to be meet-irreducible iff a is not a unit element and whenever $a = b \wedge c$, then either $a = b$ or $a = c$. If a is both **join** and **mee**t-irreducible, then a is said to be **irreducible**

Example 1.2.30:-In the lattice diagram below

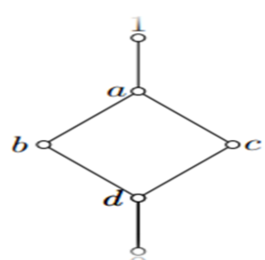

Fig 1.1

a is meet-irreducible but not join-irreducible, d is join-irreducible but not meet-irreducible, While b, c are irreducible.

Definition 1.2.31:-[10] By a **ring** we mean a non-empty set R with two binary operations $+$ and $\cdot$, called addition and multiplication(also called product),respectively, such that,

i) $(R;\ +)$ is an additive abelian group.
ii) $(R;\cdot)$ is a multiplicative semi group.
iii) Multiplication is distributive(on both sides) over addition; that is, for all $a, b, c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

(The two distributive laws are respectively called the left distributive law and the right distributive law.)

Note:-We usually write ab instead of $a \cdot b$

The identity of the additive abelian group is called a zero element of the ring and is unique.

We denote the zero element of a ring R by 0.

Example 1.2.32

$\mathbb{Z}$: The ring of all integers,

$\mathbb{Q}$: The ring of all rational numbers,

$C\ [0, 1]$: The ring of all continuous functions from the interval $[0, 1]$ to $\mathbb{R}$,

Definition 1.2.33:-[3] A binary relation β on the nonempty set A satisfying the following properties: for all $a, b, c\ \in A$

- $(a, a) \in \beta$ **Reflexivity**
- $(a, b) \in \beta$ Implies that $(b, a)\ \in \beta$ **symmetry**
- $(a, b), (b, c) \in \beta$ Implies that $(a, c) \in \beta$ **transitivity**

is called an **equivalence relation**. If β is an equivalence relation, the relation $(a, b) \in\ \beta$ is often denoted by $\boldsymbol{a \equiv b(mod\beta)}$.

►For an equivalence relation β on the nonempty set A and for an $a \in A$, we define the block of β containing a(often called, the equivalence class of β containing a) as follow:

$a/\ \beta$=$\{b \in A|(a, b) \in \beta\}$

Note: - If $b \in A/\beta$, then $a/\ \beta$=$b/\ \beta$.

Chapter Two

2 Distributive Lattice

2.1 Definitions, Theorems and Examples of Distributive Lattice

Definition 2.1.1:- [10] A distributive lattice L is a lattice, which satisfies all the lattice axioms $(L_1 - L_6)$ as we have seen (def. 1.2.11), and either of the following distributive laws: for all $a, b, c \in L$

D_1: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

D_2 : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Theorem 2.1.2:-[10] A lattice L satisfies D_1 if and only if it satisfies D_2.

Proof: Suppose D_1 holds. Then

$a \vee (b \wedge c) = (a \vee (a \wedge c)) \vee (b \wedge c)$ by absorption law

$= a \vee ((a \wedge c) \vee (b \wedge c))$ by associativity of "$\vee$"

$= a \vee ((c \wedge a) \vee (c \wedge b))$ by commutativity of "$\wedge$"

$= a \vee (c \wedge (a \vee b))$ by D_1

$= a \vee ((a \vee b) \wedge c)$ by commutativity of "$\wedge$"

$= (a \wedge (a \vee b)) \vee ((a \vee b) \wedge c)$ by absorption law

$= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)$ by commutativity of "$\wedge$"

$= (a \vee b) \wedge (a \vee c)$ by D_1

Thus D_2 also holds.

Conversely, Suppose D_2 holds. Then

$a \wedge (b \vee c) = (a \wedge (a \vee c) \wedge (b \vee c)$ by absorption law

$= a \wedge ((a \vee c) \wedge (b \vee c))$ by associativity of "$\wedge$"

$=a \wedge ((c \vee a) \wedge (c \vee b))$ by commutativity of "∨"

$=a \wedge (c \vee (a \wedge b))$ by D_2

$=a \wedge ((a \wedge b) \vee c)$ by commutativity of "∨"

$=\big(a \vee (a \wedge b)\big) \wedge ((a \wedge b) \vee c)$ by absorption law

$=((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c)$ by commutativity of "∨"

$=(a \wedge b) \vee (a \wedge c)$ by D_2

Therefore D_1 holds.

Note:-1, A distributive lattice of fundamental importance is a two - element chain $(2; \vee, \wedge)$. It is the only two-element lattice.

Note:-2, For any lattice $(L; \vee, \wedge)$, $a \leq b$ iff $a \wedge b = a \Leftrightarrow a \vee b = b$.

Results: [11] For any triplets a, b, c of lattice L the following inequality holds.

(1) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
(2) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Similar to the distributive identities, (1) and (2) are called distributive inequalities.

Proof 1: Since $a \geq a \wedge b$ and $a \geq a \wedge c$

$\Rightarrow a = a \vee a \geq (a \wedge b) \vee (a \wedge c)$ .... $(*)$

Again, we have

$b \geq a \wedge b$ and $c \geq a \wedge c$

$\Rightarrow b \vee c \geq (a \wedge b) \vee (a \wedge c)$ ... ... $(**)$

From $(*)$ and $(**)$, we get $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Proof 2:Since $a \leq a \vee b$ and $a \leq a \vee c$

$\Rightarrow a = a \wedge a \leq (a \vee b) \wedge (a \vee c)$ ⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯ $(***)$

Again, we have

$b \leq a \vee b$ and $c \leq a \vee c$

$\Rightarrow b \wedge c \leq (a \vee b) \wedge (a \vee c)$ ⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯ $(****)$

From $(***)$ and $(****)$, we get $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Conclusion, thus to prove that a lattice $(L;\vee,\wedge)$ is a distributive lattice it is sufficient to prove that

(1*) $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$

(2*) $a \vee (b \wedge c) \geq (a \vee b) \wedge (a \vee C)$

Definition 2.1.3:- A lattice L is called n-distributive if in L the following identity holds:

$$x \wedge \left(\bigvee_{i=1}^{n} y_i\right) = \bigvee_{i=1}^{n}(x \wedge \left(\bigvee_{(j\neq i)=1}^{n} y_j\right))$$

Corollary 2.1.4:[6] Let L be a distributive lattice such that $a, b \in L$ and $a \neq b$. Then there is a **prime ideal** containing exactly one of a and b.

Proof:- Suppose $a \nleq b$. Let $D{=}\uparrow a$ be dual ideal of L with $a \in D$ and therefore, $b \notin D$. Let $I = \downarrow b$ be an ideal of L, and note $a \notin I$ and $b \in I$. Then, $I \cap D = \emptyset$ and there exist prime ideal P of L such that $P \supseteq I$ and $P \cap D = \emptyset$. Since $P \supseteq I$ and $b \in I$ we have b $\in P$. Thus, since $a \in D$ and $P \cap D = \emptyset$, $a \notin P$. A similar argument holds if $b \nleq a$. Therefore there exists prime ideal of L containing exactly one of a and b.

Theorem 2.1.5:- [3] (G. Birkhoff and M. H. Stone) A lattice L is a distributive iff it is isomorphic to a ring of sets.

Proof: - Let $I_p(L)$ denote the set of prime ideals of L.

($\Rightarrow$) Let L be a lattice and let $\varphi: L \to P\ (I_p(L))$, $a \mapsto \{p \mid a \notin p, p \in I_p\ (L)\}$

We need to show that φ is one-to-one and preserves meet and join. If $a \neq b$ in L, by corollary 2.1.4 there exists $Q \in I_p(L)$ for which we may assume $a \in Q$ and $b \notin Q$. Therefore, $Q \in \varphi(b)$ but $Q \notin \varphi(a)$, which implies $\varphi(a) \neq \varphi(b)$. Therefore φ is one-to-one function.

Let $a, b \in L$. To show that φ preserves join we need to show that $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$. Let $P \in I_p(L)$. We first need to show that $a \vee b \notin P \leftrightarrow a \notin P$ or $b \notin P$. The contrapositive of this is $a \vee b \in P \leftrightarrow a \in P$ and $b \in P$. Now if $a \vee b \in P$ and $a \leq a \vee b$ and $b \leq a \vee b$, since p is an ideal and is closed going down, $a \in p$ and $b \in p$. Now assume that $a, b \in p$. Then, since p is closed under join $a \vee b \in P$. Therefore $a \vee b \notin P$ if and only if $a \notin P$ or $b \notin P$. Now: $\varphi(a \vee b) = \{$p$|\ (a \vee b) \notin P, P \in I_p(L)\}$

$=\{Q | a \notin Q, Q \in I_p(L)\} \cap \{R | b \notin R, R \in I_p(L)\}$

$= \varphi(a) \vee \varphi(b)$

To show that φ preserves meet we need to show that $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$. We first need to show that $a \wedge b \notin p \leftrightarrow a \notin P$ and $b \notin P$. The contrapositive of this is: $a \wedge b \in P \leftrightarrow a \in P$ or $b \in P$. Now if $a \wedge b \in P$, since P is a prime ideal, $a \in P$ or $b \in P$. Now assume that a or b is in P. Then, since P is closed going down $a \wedge b \in P$. Therefore $a \wedge b \notin P$ if and only if $a \notin P$ and $b \notin P$.

Now: $\varphi(a \wedge b) = \{p | (a \wedge b) \notin P, P \in I_p(L)\}$

$$=\{Q | a \notin Q, Q \in I_p(L)\} \cap \{R | b \notin R, R \in I_p(L)\}$$

$$= \varphi(a) \wedge \varphi(b)$$

Since φ is an injective homomorphism, whose image is a sublattice of $P(I_p(L))$, L is isomorphic to a ring of sets.

($\Leftarrow$) Any ring of sets is distributive and therefore, any lattice isomorphic to a ring of sets is itself distributive.

Example 2.1.6:-The chain $\mathbb{Z}$ is distributive lattice. Since every chain is a lattice and also every chain is distributive, Since $\mathbb{Z}$ is a chain, it follows that $\mathbb{Z}$ is a distributive lattice.

Example 2.1.7:-The following figure is distributive lattice

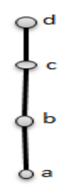

Figure 2.1 distributive lattice

2.2 Characterization Theorems of Distributive Lattice

The two typical examples of non-distributive lattices are $\boldsymbol{N_5}$ and $\boldsymbol{M_3}$. Whose diagrams are given in fig 2.2

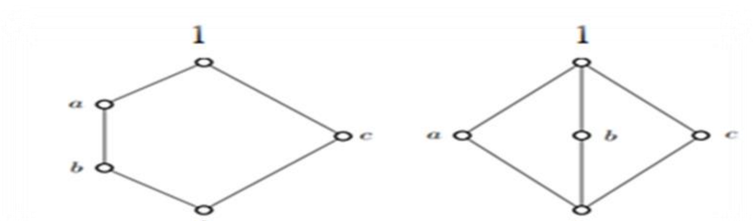

Figure 2.2 the lattices N_5 and M_3

Our next results characterizes the distributivity by absence of these lattice as a sub-lattices.

Definition 2.2.1:- [10] A sub-lattice L' of a lattice L is called a pentagon, respectively, diamond, if L' is isomorphic to $\boldsymbol{N_5}$, respectively, $\boldsymbol{M_3}$.

Note: - If we say that e_0, e_1, e_2, e_3, e_4 is a pentagon (respectively, a diamond), we also assume that

$e_0 \longmapsto 0, e_1 \longmapsto a, e_2 \longmapsto b, e_3 \longmapsto c, e_4 \longmapsto 1$is an isomorphism of L' with N_5(respectively, with M_3).

The characterization theorem will be stated in two forms. Theorem 2.2.2 is a striking and useful characterization of distributive lattice; theorem 2.2.3 is a more detailed version of theorem 2.2.2 with some additional information.

Theorem 2.2.2:-[10**]** A lattice L is distributive iff L does not contain a pentagon or a diamond Proof:-Suppose L is distributive lattice, then for $a, b, c \in L$, which is prove by Theorem 2.1.2.

Conversely:- Suppose either a pentagon or a diamond embedded into L ,then L cannot distributive lattice, since the distributive laws do not holds in L, there must be elements a, b, c from L such that $a \wedge(b \vee c) < (a \wedge b) \vee (a \wedge c)$.Let us define

$$d = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$$

$$e = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$$

$$a_1 = (a \wedge e) \vee d$$

$$b_1 = (b \wedge e) \vee d$$

$$c_1 = (c \wedge e) \vee d$$

Then it is easily seen that $d \ \leq a_1 \ , b_1, c_1 \ \leq \ e$. Now from

$a \wedge e = a \wedge(b \vee c)$ (by absorption of "$\wedge$") and (applying the modular law to switch the underlined terms)

$$a \wedge d = \underline{a} \wedge(\underline{(a \wedge b) \vee (a \wedge c)} \vee(b \wedge c))$$

$$= ((a \wedge b) \vee (a \wedge c) \vee (a \wedge (b \wedge c)) \qquad \text{(by modular M)}$$

$$=(a \wedge b) \vee (a \wedge c)$$

It follows that $d < e$. Therefor if L does contains a pentagon or a diamond it is not distributive lattice.

Definition 2.2.3:-[11] Let (L;∨,∧) be a lattice and Let $a, b, c \in L$, then for $a \leq c$ the following identity satisfying the modular identity is called modular lattice

a$\vee$ $(b \wedge c) = (a \vee b) \wedge c$.

Theorem 2.2.4:-[11] (i) A lattice L is modular iff it does not contain a pentagon.

(ii) A modular lattice L is distributive iff it does not contain a diamond.

Proof: (i) If L is modular, then every sub-lattice of L is also modular; $\boldsymbol{N}_5$ is not modular, thus it cannot be isomorphic to a sub-lattice of L.

Conversely:Let L be non-modular, let $a, b, c \in L$ with $a \geq b$ and let $(a \wedge c) \vee b \neq a \wedge (c \vee b)$ the free lattice generated by a, b, c $with$ $a \geq b$ is shown in fig 2.3. Therefore, the sub-lattice of L generated by a, b, c must be homomorphic image of the lattice of fig 2.3. Observe that if two of the five elements

$$a \wedge c, (a \wedge c) \vee b, a \wedge (b \vee c), b \vee c, c$$

are identified under a homomorphism, then so are $(a \wedge c) \vee b$ and $a \wedge (b \vee c)$. Consequently, these five elements are distinct in L, and they form a pentagon.

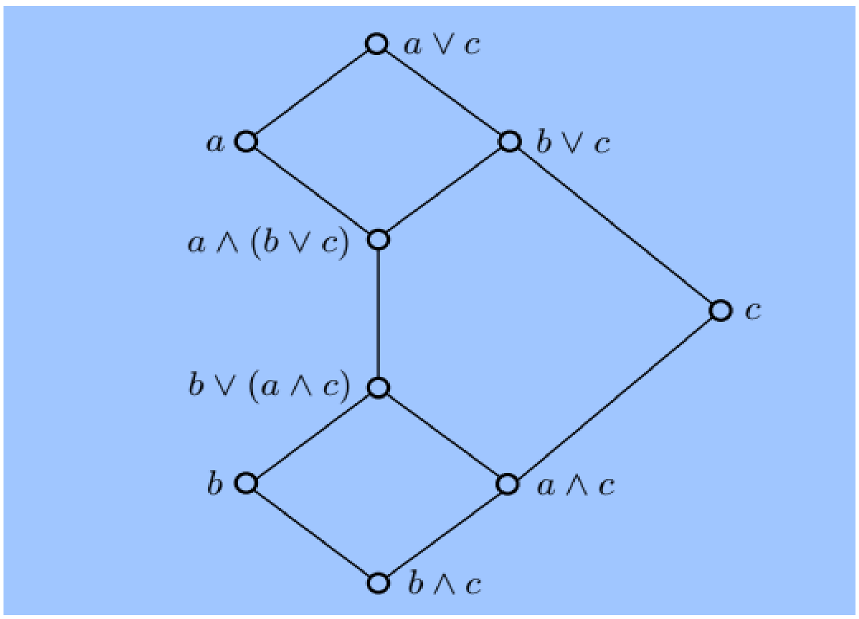

Figure 2.3 the most general lattice generated by $b \leq a$ and c

(ii) Let L be modular, but non-distributive, and choose $x, y, z \in L$ such that

$$x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)$$

the free modular lattice generated by x, y, z. Thus in any modular lattice, they form a sublattice isomorphic to the quotient lattice of M_3. But M_3 has only two quotient lattice, M_3 and the one-element lattice. In the former case, we have finished the proof. In the later case, note that if u and v collapse, then so do $x \wedge (y \vee z)$ and $(x \wedge y) \vee (x \wedge z)$, contrary to our assumption

Lemma 2.2.5:-[3] A lattice L is distributive iff for any two ideals I, J of L:

$$I \vee J = \{i \vee j : i \in I, j \in J\}$$

Proof: Suppose L is distributive and let us take $t \in I \vee J$, then $t = i \vee j$. Then by distributivity, $t = t \wedge (i \vee j) = (t \wedge i) \vee (t \wedge j) = i_1 \vee j_1$ where $i_1 = t \wedge i \in I, j_1 = t \wedge j \in J$, since I, J are ideals of L. Thus, $t = i_1 \vee j_1$ for $i_1 \in I, j_1 \in J$. This implies that $I \vee J = \{i \vee j : i \in I, j \in J\}$.

Conversely: Suppose that $I \vee J = \{i \vee j : i \in I, j \in J\}$ and suppose, if possible, L is non-distributive. Then there exist three elements a, b, c (as in the lattice M_3). Now let us consider the principal ideals $I = (b], J = (c]$. (Keeping in mind the figure M_3), $a \leq b \vee c$ and so $a \in I \vee J$. We claim that a cannot be written as $a = i \vee j$ because if it so then $i \leq a, j \leq a$. Then as $j \in J = (c], j \leq c$. Now combing $j \leq a, j \leq c$ gives us $j \leq a \wedge c = 0 < b \in (b] = I$. Thus $j \leq a = i \vee j \in I = (b] = \{0, b\}$, a contradiction. Hence L is distributive.

Lemma2.2.6:-[11] In a bounded distributive lattice an element can have only one complement.

Proof: let L be a distributive lattice and suppose, if possible, an element $x \in L$ has two complements y_1 and y_2. Then using distributivity,

$$\begin{aligned} y_1 &= 1 \wedge y_1 \\ &= (x \vee y_2) \wedge y_1 \\ &= (x \wedge y_1) \vee (y_2 \wedge y_1) \\ &= 0 \vee (y_2 \wedge y_1) \\ &= y_2 \wedge y_1 \\ &= y_1 \wedge y_2. \end{aligned}$$

Similarly,

$$\begin{aligned} y_2 &= 1 \wedge y_2 \\ &= (x \vee y_1) \wedge y_2 \\ &= (x \wedge y_2) \vee (y_1 \wedge y_2) \\ &= 0 \vee (y_1 \wedge y_2) \\ &= y_1 \wedge y_2. \end{aligned}$$

These two give us $y_1 = y_2$. Hence the complement is unique.

Here we mention a nice characterization of distributive lattice due to Oystein Ore (1938). Consider the lattice of all subgroup of a group G. Oystein Ore prove that the group G is locally cyclic iff the lattice of subgroups of G is distributive.

Definition 2.2.7:-[5] Let D be a distributive lattice and $J(D)$ denote the collection of all nonzero join-irreducible elements of D. Then $J(D)$ is a poset under the partial ordering inherited from D. For $a \in D$, let us define $r(a) = \{x | x \leq a, x \in J(D)\} = (a] \wedge J(D)$

i.e. $r\ (a)$ is a set of join-irreducible elements below a

Definition 2.2.8:-Let P be a poset and $A \subseteq P$. We call A **hereditary** iff $x \in A$ and $y \leq x$ imply that y$\in A$. Let $H(P)$ be denote the set of all hereditary subsets of P partially ordered by set inclusion. The $H(P)$ is a lattice in which meet and join are intersection and union , respectively and hence $H(P)$ is a distributive lattice.

Theorem 2.2.9:-[5] Let D be finite distributive lattice. Then D is **isomorphic** to $H(J(D))$.

Proof: let us define the map $\Phi : D \longrightarrow H(J(D))$ by $a\Phi = r(a)$. Then we prove that

Φ is an isomorphism.

One-to-**one**: Take $a, b \in D$ such that $a\Phi = b\Phi$. Then we have $r(a) = r(b)$. This means the two sets

$$r(a) = \{x | x \leq a, x \in J(D)\}$$

And

$$r(b) = \{y | y \leq b, y \in J(D)\}$$

are equal. This is possible only when $a = b$.This prove that $\mathbf{\Phi}$ is **one-to-one**

Onto: We have to show that for every $A \in H(J(D))$,there exists a $\in D$ such that $a\Phi =$ A. Let us set $a = \bigvee A$(which exists because A is finite). Then as $A's$ elements are join-irreducible and $a \leq a$, for every $a \in A$, we get by definition $r(a) \supseteq A$. For reverse inclusion, we take any $x \in r(a)$.Then by definition $x \leq a$. Then we can write $x = x \wedge a = x \wedge \bigvee A$=$\bigvee\{x \wedge y | y \in A\}$. Now since x is join –irreducible so we will have $x = x \wedge y$, for some $y \in A$.This means $x \leq y$. But since A is hereditary, so it is follow that $x \in A$.Therefore $r(a) \subseteq A$. The two containments together give us $r(a) = A$. Thus the pre-image of $A \in H(J(D))$ is the join of A. Hence $\mathbf{\Phi}$ is **onto**.

$\mathbf{\Phi}$ is a homomorphism**:** By definition $(a \wedge b)\Phi = r(a \wedge b)$.

Now we show that, $r(a \wedge b) = r(a) \cap (b)$.

We note that $x \in r(a \wedge b) \Leftrightarrow x \in a \wedge b$

$\Leftrightarrow x \leq a$ and $x \leq b$

$$\Leftrightarrow x \in r(a) \text{ and } x \in r(b)$$

$$\Leftrightarrow x \in r(a) \cap r(b)$$

Hence we get $r(a \wedge b) = r(a) \cap r(b)$.

Therefore $(a \wedge b)\Phi = \mathrm{r(a \wedge b) = r(a) \cap r(b) = a\Phi \wedge b\Phi}$.

Next, by definition $(a \vee b)\Phi = r(a \vee b)$.

We prove that $r(a \vee b) = r(a) \cup r(b)$.

It is trivial that $r(a) \cup r(b) \subseteq r(a \vee b)$.

For reverse containment, let us take any $x \in r(a \vee b)$.

Then by definition, $x \leq a \vee b$.From this we can write $x = x \wedge (a \vee b)$. Applying distributivity, this can be written as $x = (x \wedge a) \vee (x \wedge b)$. Now since x is join- irreducible, we shall get $x = x \wedge a$ or $x = x \wedge b$ and this implies that $x \leq a$ or $x \leq b$. Then $x \in r(a)$ or $x \in r(b)$ which means $x \in r(a) \cup r(b)$, proving that $r(a \vee b) \subseteq r(a) \cup r(b)$. Thus the two containments together imply that

$r(a \vee b) = r(a) \cup r(b)$.

So we have $(a \vee b)\Phi = r(a \vee b) = r(a) \cup r(b) = a\Phi \vee b\Phi$.

Therefore, Φ is a homomorphism.

Hence Φ is an isomorphism.

These prove the theorem.

Definition 2.2.10:-[6] A modular lattices $(L;\vee,\wedge)$ are lattices that satisfy the following identity (called the modular identity), described by Dedekind: if $a \leq c, b \in L$,

$$a \vee (b \wedge c) = (a \vee b) \wedge c \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (*\cdot)$$

Remark: In the equality $(*\cdot)$, it is trivial that,

$$a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

So to prove that a lattice to be modular, it is sufficient to show that

$$a \vee (b \wedge c) \geq (a \vee b) \wedge c$$

Theorem 2.2.11:-[6] A lattice L is modular, if and only if, every triplet a, b, c of L satisfies the equation

$a \vee \big(b \wedge (a \vee c)\big) = (a \vee b) \wedge (a \vee c)$ (Jordan [105]).

Proof: if L is modular, then by $(*\cdot)$ and because $a \leq a \vee c$, the above equality holds. Conversely if the above equality is true for any triplet a, b, c of L, then, in particular, $(*\cdot)$ is true as will, since $a \leq c$ implies $a \vee c = c$

Example 2.2.12:- By taking $a \leq c$ in the distributive identity, we get a modular identity. Thus it implies that **every distributive lattice** is modular.

Example 2.2.13:-The lattice of all ideals of a ring is a modular lattice **but not** distributive, in general.

Example 2.2.14:-The lattice of all **subgroup** of a group is not modular, in general.

Theorem 2.2.15:-[6] The lattice of normal subgroups $\mathcal{N}\text{-}sub(G)$ of a group G is modular.

Proof: It is trivial to show that $\mathcal{N}\text{-}sub(G)$ is a poset under set containment. Now for subgroups G_1, G_2 in $\mathcal{N}\text{-}sub(G)$, let us define $G_1 \wedge G_2 = G_1 \cap G_2$ and $G_1 \vee G_2 = \{g_1 g_2 | g_1 \in G_1, g_2 \in G_2\}$, subgroup generated by G_1, G_2 which we shall denote by $G_1 G_2$. Then it is easy to check that $G_1 \cap G_2$and $G_1 G_2$ are members of $\mathcal{N}\text{-}sub(G)$.To prove $\mathcal{N}\text{-}sub(G)$ to be a modular lattice, we shall show that for G_1, G_2 in $\mathcal{N}\text{-}sub(G)$ such that $G_2 \subseteq G_1, G_1 \cap (G_2 G_3) = G_2(G_1 \cap G_3)$. For this we take $x \in G_1 \cap (G_2 G_3)$. Then $x \in G_1$ and $x \in (G_2 G_3)$. Thus $x = g_1$ and $x = g_2 g_3$.

For some $g_1 \in G_1, g_2 \in G_2, g_3 \in G_3$.From these we can write $g_3 = g_2^{-1} g_1 \in G_1$. Thus $g_3 \in G_1 \cap G_2$ and then $g_2 g_3 \in G_2(G_1 \cap G_3)$ which implies that $x \in G_2(G_1 \cap G_3)$.

Therefore we get

$G_1 \cap (G_2 G_3) \subseteq \; G_2(G_1 \cap G_3)$. Now as the reverse containment holds. These together yield the modular identity.

This prove that $\mathcal{N}\text{-}sub(G)$ is indeed a modular lattice.

Theorem 2.2.16:-[3] The dual**, every sub-lattice** and **every homomorphic image** of a distributive lattice is likewise a distributive lattice.

Proof: (i) Let $(\boldsymbol{L};\vee,\wedge)$ be a distributive lattice and $\boldsymbol{H}$ be a **sub-lattice** of $\boldsymbol{L}$.

Now, let $a, b, c \in \boldsymbol{H}$. Then $a, b, c \in \boldsymbol{L}$.

Therefore $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$................ in $\boldsymbol{L}$

Then, it holds also for $\boldsymbol{H}$. Hence $\boldsymbol{H}$ is a distributive lattice

(ii) Let $\varphi : L \longrightarrow L'$ be a homomorphism and L be a distributive lattices, where L' is a homomorphic image of L.

Suppose $\varphi(L) = L^* \subseteq L'$.Now, let $a^*, b^*, c^* \in L^* = \varphi(L)$

This implies that, there exist $a, b, c \in L$ such that $\varphi(a) = a^*, \varphi(b) = b^*, \varphi(c) = c^*$.

Now, $a^* \wedge (b^* \vee c^*) = \varphi(a) \wedge (\varphi(b) \vee \varphi(c))$

$= \varphi(a) \wedge [\varphi(b \vee c)]$ ……… φ is a homomorphism

$= \varphi([a \wedge (b \vee c)])$ …………. φ is a homomorphism

$= \varphi[(a \wedge b) \vee (a \wedge c)]$ ……….. L is a distributive lattice

$= \varphi[(a \wedge b)] \vee \varphi[(a \wedge c)]$……… φ is a homomorphism

$= [\varphi(a) \wedge \varphi(b)] \vee [\varphi(a) \wedge \varphi(c)]$………. φ is a homomorphism

$= (a^* \wedge b^*) \vee (a^* \wedge c^*)$

Therefore $L^* = \varphi(L)$ is a distributive lattice.

Example 2.2.17:-Every chain is a distributive lattice.

Example 2.2.18:-A group G is called a generalized cyclic group if every finite subset of G generates a cyclic subgroup. The subgroup lattice of every group of this type is distributive (Ore [152]).

2.3 Infinitely Distributive and Completely Distributive Lattices

From the distributive identities D_1,D_2 follow at once by complete induction on n the identity

$$a \wedge \bigvee_{k=1}^{n} b_k = \bigvee_{k=1}^{n} (a \wedge b_k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1)$$

And

$$a \vee \bigwedge_{k=1}^{n} b_k = \bigwedge_{k=1}^{n} (a \vee b_k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (2)$$

Quite naturally the question arises whether the equations

$$a \wedge \bigvee_{\beta \in B} b_\beta = \bigvee_{\beta \in B} (a \wedge b_\beta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

And

$$a \vee \bigwedge_{\beta \in B} b_\beta = \bigwedge_{\beta \in B} (a \vee b_\beta) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (4)$$

Which can be considered as generalization of (1) and (2), respectively, are valid for any subset R=$\{b_\beta\}\beta \in B$ of a distributive complete lattice.

- It can be shown by a simple counter example that the answer to the question is negative, in the general case.

Consider for instance the set N_0 of all non-negative integers. N_0 ordered by divisibility, form a complete lattice the least element of which is 1, the greatest 0, and in which the meet of two elements is their greatest common divisor, the join of two elements their least common multiple. By the identities concerning the least common multiple and the greatest common divisor, as affirmed by the number theory, the lattice N_0 is distributive lattice as well.

Therefore N_0 is a **distributive complete lattice**; nevertheless, (3) fails to hold in it. Consider, for example, the set $\{a_1, a_3, \dots\}$, (a_k =2k-1) of all odd positive integers; then

$$2 \wedge \bigvee_{k=1}^{\infty} a_k = 2 \wedge 0 = 2$$

But

$$\bigvee_{k=1}^{\infty} (2 \wedge a_k) = \bigvee_{k=1}^{\infty} 1 = 1$$

but, by making use of representation of the greatest common divisor and the least common multiple by their prime factors, it is easy to see that (4) holds in N_0.

Of course, in the dual of lattice N_0, (3) is satisfied and (4) is not.

From the above, the first conclusion to be drawn is that (3) and (4) do not hold in any distributive complete lattice.

Definition 2.3.1:-[4] A lattice L is said to be **infinitely meet-distributive** if it is join-complete and (3) holds foe every subset R= $\{b_\beta\}\beta \in B$ of the lattice.

Definition 2.3.2:-[11] A lattice L is said to be **infinitely join-distributive** if it is meet-complete and (4) holds for every subset R= $\{b_\beta\}\beta \in B$ of the lattice.

Definition 2.3.3:-[11] A lattice L is said to be **infinitely distributive** if it is both infinitely meet-distributive and join-distributive.

Note:-by applying (1) twice we have for any finite number of elements of a distributive lattice

$$\bigvee_{j=1}^{m} a_{1j} \wedge \bigvee_{k=1}^{n} a_{2k} = \bigvee_{j=1}^{m}\left(a_{1j} \wedge \bigvee_{k=1}^{n} a_{2k}\right) = \bigvee_{j=1}^{m}\bigvee_{k=1}^{n}(a_{1j} \wedge a_{2k})$$

and hence, by induction on r,

$$\bigwedge_{j=1}^{r}\bigvee_{k=1}^{n_j} a_{jk} = \bigvee_{j_1=1}^{n_1} \dots \bigvee_{j_r=1}^{n_r}(a_{1j_1} \wedge \dots \wedge a_{rj_r}) \dots\dots\dots\dots\dots\dots\dots\dots (5)$$

The identity (5) can be stated in a form that is more concise, and better suited to generalization. Let us introduce the notation $A = \{1,2,\dots,r\}$,$B_1 = \{1,\dots,n_1\}$,…,$B_r = \{1,\dots,n_r\}$. Furthermore, let γ be some choice function defined on the setsB_1 . . ., B_r (that is, le t γ be a function which assigns to each of the sets B_1, . . ., B_r one, and only one, of their respective elements). Let $\gamma(a)$ denote the element selected from B (a=1, . . . , r). Then

$$a_{1\gamma(1)} \wedge \dots \wedge a_{r\gamma(r)} \dots\dots\dots\dots\dots\dots\dots\dots\dots (6)$$

is one of the terms of the right side of (5) and if γ runs through the set Γ of all choice functions definable on the sets B_1, . . ., B_r, expressions of the form (6) give the meet expressions figuring on the right side of (5). Hence (5) can be rewritten as follows:

$$\bigwedge_{\alpha\in A}\bigvee_{\beta\in B} a_{\alpha\beta} = \bigvee_{\gamma\in\Gamma}\bigwedge_{\alpha\in A} a_{\alpha\gamma(\alpha)} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (7)$$

Hence, formula (7) and its dual formula

$$\bigvee_{\alpha\in A}\bigwedge_{\beta\in B} a_{\alpha\beta} = \bigwedge_{\gamma\in\Gamma}\bigvee_{\beta\in B} a_{\alpha\gamma(\alpha)} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (8)$$

Hold for any finite system of elements of a distributive lattice; whereas, for all infinite A or B_α, these formula are not generally true.

Definition 2.3.4:-[4] A lattice L is said to be completely meet-distributive if it is complete and satisfies (7) without restriction.

Definition 2.3.5:-[11] A lattice L is said to be completely join-distributive if it is complete and satisfies (8) without restriction.

Definition 2.3.6:-[11**]** A lattice L is said to be **completely- distributive** if it is both completely meet- and join-distributive.

Chapter Three

3 Congruences in Lattices

Congruence relations play a central role in lattice theory. In this section we introduce the congruence relations on groups. Then we will see the concept of congruence relation in lattices. Some examples and properties are given to illustrate these concepts.

3.1 Congruence

We know that, If A is an algebraic structure, the equivalence relation $\equiv$ is a binary relation that is at the same time reflexive, symmetric and transitive relation. We write $a \equiv b$ or $a\beta b$ to indicated that a and b are related under the relation β. The relation "is equal to" on the set of real numbers is a prime example of an equivalence relation. For example, $\frac{2}{3}$ is equal to $\frac{4}{6}$.

Definition 3.1.1:-[5] If A is an algebraic structure, then an equivalence relation on A that also preserves the algebraic operations of A is called a **congruence relation** on A. For example if G is a group with operation $*$, a congruence relation on G is an equivalence relation $\equiv$ on the element of G satisfying

$$g_1 \equiv g_2 \text{ and } h_1 \equiv h_2 \Rightarrow g_1 * h_1 \equiv g_2 * h_2, \text{ for all } g_1, g_2, h_1, h_2 \in G.$$

Example 3.1.2:-The prototypical example of a congruence relation is congruence modulo n on the set of integers. For a given positive integer n, two integers a and b are called congruent modulo n, written $a \equiv b(mod\ n)$ if $a - b$ divisible by n (or equivalently if a and b have the same remainder when divided by n).

Theorem 3.1.3:-[12] Every normal subgroup has corresponding congruence relation and vice versa.

Proof:- ($\Rightarrow$) Let G be a group where $H \lhd G$ is a normal subgroup in G, define relation a relation $\rho \subseteq$ G×G as follows:

$$g_1 \sim g_2 \text{ Under } \rho \text{ if and only if } g_1 g_2^{-1} \in H.$$

1. $\sim$ is an equivalence relation
   - **Reflexivity**: It is easy to see that $g \sim g$ since $gg^{-1}=e \in H$.
   - **Symmetry:** Suppose $g_1 \sim g_2$, so by definition we have $g_1 g_2^{-1} \in H$, so $g_1 g_2^{-1}$ =h for some h $\in H$, and then $g_1 g_2^{-1}$ =h $\Rightarrow$ $g_2 g_1^{-1} = h^{-1} \in H \Rightarrow g_2 \sim g_1$.

- **Transitivity:** Suppose $g_1 \sim g_2$ and $g_2 \sim g_3$, then: $g_1 g_2^{-1} = h$ and $g_2 g_3^{-1} = k$, where h, k ∈ H, now from the last identity we have $g_2^{-1} = g_3^{-1} k^{-1}$, and then $g_1 g_3^{-1} k^{-1} = h \Rightarrow g_1 g_3^{-1} = hk \Rightarrow g_1 \sim g_3$.

2. The relation preserves the group structure, since if $g_1 \sim g_2$ and $g_3 \sim g_4$, then

$$g_1 \sim g_2 \Rightarrow g_1 g_2^{-1} = h_1 \in H,$$

$$g_3 \sim g_4 \Rightarrow g_3 g_4^{-1} = h_2 \in H$$

$$\Rightarrow g_1 g_3 (g_2 g_4)^{-1} = g_1 g_3 g_4^{-1} g_2^{-1} = g_1 h_2 g_2^{-1} = g_1 g_2^{-1} h' = h_1 h' \in H$$

Because normality implies that for all $x \in G$, and all h ∈ H there exist $h' \in H$,

With $xh = xh'$.

(⇐) Now let ~ be a congruence relation on a group G. Define the set H = $\{g \in G : g \sim e\}$. Firstly, we prove H is a subgroup.

(i) Of course, $e \sim e$, thus $e \in H$

(ii) Suppose $h_1, h_2 \in H$, then $h_1 \sim e$ and $h_2 \sim e$. Since ~ is a congruence relation, we have $h_1 h_2 \sim ee \Rightarrow h_1 h_2 \sim e \Rightarrow h_1 h_2 \in H$

(iii) Suppose $h \in$ H, then $h \sim e$ since ~ is an equivalence, we have $h^{-1} \sim h^{-1}$ and since it's also a congruence relation, we get $hh^{-1} \sim eh^{-1} \Rightarrow e \sim h^{-1} \Rightarrow h^{-1} \in H$ we want to show that H is normal, that is , ∀g ∈ G and ∀$h \in H$, we have $ghg^{-1} \in H$
Since ~ is an equivalence relation, we have $g \sim g$ and $g^{-1} \sim g^{-1}$. Furthermore, as ~ is congruence, and $h \in H \Rightarrow h \sim$e and so $ghg^{-1} \sim geg^{-1}$=e, for any $g \in G$ and $h \in H$.

3.2 Congruence Relation in Lattice

We begin with definition of a congruence relation in a lattice.

Definition 3.2.1:-[7] An equivalence relation θ (that is, reflexive, symmetry and transitive binary relation) on a lattice ***L*** is called congruence relation iff $a \equiv b(\theta)$ and $c \equiv d(\theta)$ for some a, b, c, d ∈L imply that;

(i) $a \wedge c \equiv b \wedge d(\theta)$

(ii) $a \vee c \equiv b \vee d(\theta)$

Note:

►The equivalence classes under a congruence relation θ are called **congruence classes** or **blocks.**

►The congruence class containing $a \in \boldsymbol{L}$ is denoted by $[a]$ that is, $[a] = \{x : x\theta a\}$.

►The set of all congruence relations on $\boldsymbol{L}$ is denoted by $\mathrm{Con}(\boldsymbol{L})$.

►Congruence relations on an arbitrary lattice have an interesting connection with the **distributive lattices.**

Example 3.2.2:-In any lattice there are always two trivial congruence relations, the congruence relation θ_1 where each element is it's own equivalence class (block), this is called the smallest congruence relation, and the congruence relation θ_2 with a single block.

i.e. $a \equiv b(\theta_1)$ if and only if $a = b$

$a \equiv b(\theta_2)$ for all $a, b \in L$

Example 3.2.3:- Let L be a lattice with the Hasse diagram in figure 3.1

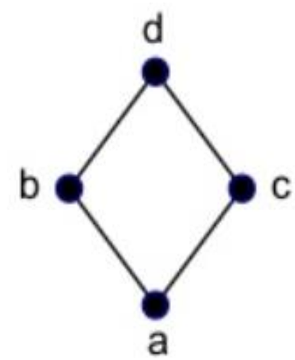

Figure 3.1

The following are all congruence of a lattice L:

$\theta_1 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$

$\theta_2=\{a, b, c, d\},\{\{a, b\},\{c, d\}\},\{\{a, c\},\{b, d\}\},\{\{a\},\{c\},\{b, d\}\},\{\{a\},\{b\},\{c, d\}\}$.

Example 3.2.4:- In a finite chain C, a congruence relation is any decomposition of C into disjoint closed interval as shown in the figure 3.2

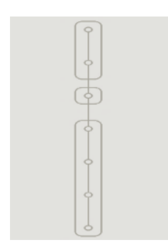

Figure 3.2 A congruence of a finite chain C

Example 3.2.5:-A congruence relation of a lattice is shown in figure 3.3

Figure 3.3 Congruence of lattice

Examples 3.2.6 ;(i) In the integer $\mathbb{Z}$, a congruence relation is the same as congruence mod n, for some n. The case n=0 gives the equality relation.

(ii) In a group G, a congruence relation is the same thing as the coset Decomposition for some normal subgroup and in a commutative ring it is the some thing as the coset decomposition for an ideal.

Theorem 3.2.7:-[4] An equivalence relation $\equiv$ on a lattice $\boldsymbol{L}$ is a congruence relation if and only if for all $a, b, x \in \boldsymbol{L}$,

$$a \equiv b \Longrightarrow a \wedge x \equiv b \wedge x \text{ and } a \vee x \equiv b \vee x.$$

Proof :-($\Longrightarrow$) Assume that $\equiv$ is congruence on a lattice L. If $a \equiv b(\theta)$ then since $x \equiv x(\theta)$, we have

$$a \vee x \equiv b \vee x \text{ and } a \wedge x \equiv b \wedge x$$

($\Longleftarrow$) If the stated property holds, then $a \equiv b, x \equiv y \Rightarrow a \wedge x \equiv b \wedge x, b \wedge x \equiv b \wedge y$

$$\Rightarrow a \wedge x \equiv b \wedge y$$

And similarly $a \equiv b, x \equiv y \Rightarrow a \vee x \equiv b \vee x, b \vee x \equiv b \vee y$

$$\Rightarrow a \vee x \equiv b \vee y$$

Lemma 3.2.8:-[4] Let β be a congruence relation of a lattice L. Then, for any pair a, b of elements of a lattice L, the following conditions (i)-(iii) are equivalent:

(i) $a \equiv b(\beta)$

(ii) $a \wedge b \equiv a \vee b(\beta)$

(iii) $x, y \in [a \wedge b, a \vee b] \Longrightarrow x \equiv y(\beta)$

Proof: - (i) $\Longrightarrow$ (ii), since, if $a \equiv b(\beta)$, then by the substitution property

$a \wedge b \equiv a \wedge a = a = a \vee a \equiv a \vee b(\beta)$

(ii) $\Longrightarrow$ (iii), since if $x, y \in [a \wedge b, a \vee b]$, then $x = x \vee (a \wedge b) \equiv x \vee (a \vee b) = a \vee b(\beta)$ and similarly, $y \equiv a \vee b(\beta)$; but then, $x \equiv y(\beta)$. Finally, (iii) $\Longrightarrow$ (i) under the substitution $x = a, y = b$

Definition 3.2.9:-Let L be an arbitrary lattice. Then the collection Con **(**L**)** of all congruence relations of L form a lattice [6] with the meet and join defined as:

For $\beta_1, \beta_2 \in L$, $\beta_1 \wedge \beta_2 = \beta_1 \cap \beta_2$, that is

$a \equiv b(\beta_1 \wedge \beta_2)$, iff $a \equiv b(\beta_1)$, and $a \equiv b(\beta_2)$.

The join $\beta_1 \vee \beta_2$ is defined as $a \equiv b(\beta_1 \vee \beta_2)$ iff there is a sequence $c_0 = a \wedge b, c_1, \dots, c_{n-1} = a \vee b$ of elements of L such that $c_0 \le c_1 \le, \dots, \le c_{n-1}$and for each i, $0 \le i \le n-1$, $C_i \equiv c_{i+1}(\beta_1)$ or $c_i \equiv c_{i+1}(\beta_2)$.

Theorem 3.2.10 (N.Funayama and T.Nakayama) Con **(**L**)** is distributive lattice [2].

Proof: Let us take three congruences $\theta, \phi, \psi \in$ con **(**L**)**. By distributive inequality of chapter two, we have

$\theta \wedge (\phi \vee \psi) \ge (\theta \wedge \phi) \vee (\theta \wedge \psi)$ So we show reverse inequality.

i.e. $(\theta \wedge \phi) \vee (\theta \wedge \psi) \ge \theta \wedge (\phi \vee \psi)$**.**Taking $a \equiv b(\theta \wedge (\phi \vee \psi))$, we have $a \equiv b(\theta)$ and $a \equiv b(\phi \vee \psi)$. Then by the above lemma 3.2.8 $a \equiv b(\theta)$ implies that $a \wedge b \equiv a \vee b(\theta)$. Now consider $a \equiv b(\phi \vee \psi)$.By the definition of join of congruences, $a \equiv b(\phi \vee \psi)$ implies that there exist $z_0, z_1, \dots, z_{n-1}$such that

$a \wedge b = z_0, z_1, \dots, z_{n-1} = a \vee b$ Such that for all $0 \le i \le n-2, z_i \equiv z_{i+1}(\phi)$or $z_i \equiv z_{i+1}(\psi)$ and so $z_i \equiv z_{i+1}(\theta)$ for each $0 \le i \le n-2$, therefore we have

$(z_i \equiv z_{i+1}(\theta))$ and $(z_i \equiv z_{i+1}(\phi)$ or $z_i \equiv z_{i+1}(\psi))$

$(z_i \equiv z_{i+1}(\theta))$ and $(z_i \equiv z_{i+1}(\phi)$ or $z_i \equiv z_{i+1}(\theta))$ and $z_i \equiv z_{i+1}(\psi)$

$\big(z_i \equiv z_{i+1}(\theta \wedge \phi)\big)$ or $\big(z_i \equiv z_{i+1}(\theta \wedge \psi)\big)$ for all $0 \le i \le n-2$.

So by the definition of the join $a \equiv b(\theta \wedge \phi) \vee (\theta \wedge \psi)$ and therefore

$(\theta \wedge \phi) \vee (\theta \wedge \psi) \ge \theta \wedge (\phi \vee \psi)$

Hence the two inequalities together yields

$(\theta \wedge \phi) \vee (\theta \wedge \psi) = \theta \wedge (\phi \vee \psi$

This proves that **con (L)** is distributive lattice.

Definition 3.2.11:- [6] Let L be a lattice and θ be congruence on L. Let L/θ denote the collection of all congruence class induced by the congruence, that is $L/\theta = \{[a]\theta : a \in L\}$ then it form a lattice under

$[a]\theta \wedge [b]\theta = [a \wedge b]\theta$ and $[a]\theta \vee [b]\theta = [a \vee b]\theta$ this lattice is called the **factor lattice** of **modulo $\boldsymbol{\theta}$**.

Lemma 3.2.12:-[6] For the congruence θ of a lattice L, the map $\boldsymbol{\varphi}$**:** $L \longrightarrow L/\theta$ defined by $x \longmapsto [x]\theta$ is a **homomorphism** of L onto L/ θ.

Proof: It is clear that $\boldsymbol{\varphi}$ is surjective. Also

$\boldsymbol{\varphi}(a \wedge b) = [a \wedge b] = [a] \wedge [b] = [a]\,\theta \wedge [b]\,\theta$

And similarly for join. Hence $\boldsymbol{\varphi}$ is homomorphism.

3.3 Characterization of Congruence Lattice

Definition 3.3.1:-[3] Kernel of homomorphism, unlike the group theory or ring theory, there are three kernel concepts in lattice theory. They are defined as follow

(i) Let $\boldsymbol{\varphi}$**:** $L \longrightarrow L_1$ be a homomorphism of L onto L_1**.** Define the congruence relation $\boldsymbol{\theta}$ as $x \equiv y(\boldsymbol{\theta})$ iff $x\boldsymbol{\varphi} = \boldsymbol{y\varphi}$**.** Then this relation $\boldsymbol{\theta}$ is called the congruence kernel of the homomorphism $\boldsymbol{\varphi}$**.**

(ii) If L_1 has a zero, 0, the set of preimages of 0 forms an ideal of L. This ideal is called ideal kernel of the homomorphism $\boldsymbol{\varphi}$**.**

(iii) If for a congruence $\boldsymbol{\theta}$ of $L, L/\theta$ has a zero, $[a]\boldsymbol{\theta}$, then $[a]\boldsymbol{\theta}$ is an ideal of L called the ideal kernel of the congruence relation $\boldsymbol{\theta}$**.**

Definition 3.3.2:-[6] Regular lattice, let L be a lattice. A congruence relation $\boldsymbol{\theta}$ of L is called a regular, if any congruence class of $\boldsymbol{\theta}$ determines the congruence. The lattice L is called regular if all congruences of L are regular.

Example 3.3.3

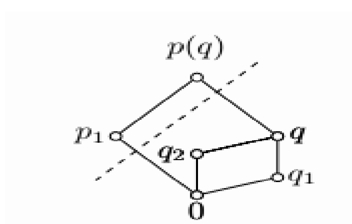

Figure 3.4: regular lattice $N_6 = N(p, q)$

Remark: The lattice N_6 has three congruence relations: The identity congruence relation ω, the universal congruence relation τ, and a non-trivial congruence relation ψ, with the congruence classes $\{0, q_1, q_2, q\}$ and $\{p_1, p(q)\}$.

Claim: Every non-trivial congruence relation of N_6 coincides with the congruence relation ψ, with the above congruence classes.

Proof of the claim: For example, let us determine the congruence relation $\boldsymbol{\theta}$=**con** (q_1, q_2) generated by the pair (q_1, q_2). Then by the definition the elements $(q_1, q_2), (q_2, q_1), (q_1, q_1), (q_2, q_2)$ belong to **con** (q_1, q_2). Now as (q_1, q_2) belong to $\boldsymbol{\theta}$ so $q_1 \wedge q_2 = q_1 \vee q_2 \boldsymbol{\theta}$, i. e (0, q) belong to $\boldsymbol{\theta}$**.** So the elements (q, 0), (0, 0), (q, q) should also be in $\boldsymbol{\theta}$. Next we note that q_2 is such that $0 \leq q_2 \leq q$ so we find that $(q_2, 0)$, (q_2, q), $(q_1, 0)$, (q_1, q) are elements of $\boldsymbol{\theta}$**.** Then we should also have $(0, q_2)$, (q, q_2), $(0, q_1)$, (q, q_1) in $\boldsymbol{\theta}$**.** Next we note that $q_1 \wedge p_1$=0≡ q_1 under $\boldsymbol{\theta}$ and p (q) is the join of p_1 and q_1, so we get $p_1 \equiv$p (q)($\boldsymbol{\theta}$). Then it follows that (p(q),q_1) also belong to $\boldsymbol{\theta}$, and (p_1,p_1),(p(q_1),p(q_1))should also belong to $\boldsymbol{\theta}$. Produces the pairs which have already been obtained. Thus we get **con** (q_1, q_2) = $\{(0,0), (p_1, p_1), (p(q), p(q)), (q, q), (q_2, q_2),$

$(q_1, q_1), (q_1, q_2), (q_2, q_1), (0, q), (q, 0), (q, q_1), (q_1, q), (q, q_2), (q_2, q)$,($p_1$,p(q)),(p(q),$p_1$),($q_1$,0),(0, q_1),(q_2,0),(0,q_2)},containing 20 elements and its congruence classes are {0,q_1,q_2,q} and {p_1,p(q)}

Similarly, if we consider any other non-trivial congruence relation of the lattice N_6, we shall get the same congruence classes. Thus the claim is proved. Hence **con** (p_1, 0) = ι. In other words, $p_1 \equiv 0$ implies that $q_1 \equiv 0$, but $q_1 \equiv 0$ does not implies that $p_1 \equiv 0$

Definition 3.3.4:-[6] **Uniform lattice:** Let L be a lattice. A congruence relation $\boldsymbol{\theta}$ of L is called uniform, if any two congruence classes of $\boldsymbol{\theta}$ are of the same size (cardinality). The lattice L is called uniform if all congruences of L are uniform.

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Conclusion

In this project we discuss the notion of distributive lattice and congruence in lattice. We define distributive lattice with binary operation on the given lattice. Distributive lattices have played a very important role in the development of lattice theory. Lattice theory started with distributive lattices. Many great results in general lattice theory are provided by the work on distributive lattices. Congruences in lattice has a central role in lattice theory. Finally I conclude that every lattice cannot a distributive lattice.

References

[1] B. Davey and Priestly, *Introduction to Lattice and Order*, New York. U.S.A. Australia.Madrid.Spain.South Africa, Printed In the united kingdom at the university press, Cambridge, 1990.

[2] E.T.schmidt *Every Finite distributive lattice is the congruence lattice of some modular lattice*, Algebra Universalis **4** (1974), 49-57.

[3] G. Birkhoff, *lattice theory*, *Amer.Math.Soc*, *colloq.publications*, U.S.A, 1967.

[4] G. Gratzer and H. Lakser *Extension theorem on congruences lattice of Partial lattices. Notices Amer. Math. Soc.***15** (1968), 1122-2234.

[5] G. Gratzer and H. Lakser, *homomorphisms of distributive lattices as restriction of congruences*, *Con. J. Math*. **38**(1986), 1122-1134.

[6] G. Gratzer, *General lattice theory*, Academic press, New York, Sanfrasisco, Birkhauser verlag.Boston, 1998.

[7] N. Funayama and T. Nakayama, *on the congruence relations on lattices*, Proc. Imp. Acad. Tokyo **18** (1942), 530-531.

[8] P.Erdos, A.Hajnal, A.Mate, R.Rado, *Combinatorial set theory: Partion relation for cardinals*, North- Holland, 1984, 282-285.

[9] SaundersMac Lane, G. Birkhoff, *Algebra*, AMS Chelsea, 1999.

[10] Stanley Burris and H.P. Sankappanavar, *A course in universal algebra.*

[11] Szasz, *Introduction to lattice theory*, Academic Press, New York and London, 1963.

[12] U. Swamy, C. Santhi and R. Chudamani, *On Almost Boolean Algebras and Rings*, P.V.P Siddhartha Institute of Technology, Vijayawada, Andhra Pradesh, India, 2016.